What is q, and what is q-Hypergeometric Functions

～微分方程式の量子化～

・超幾何級数とは何か．　$F\left(\begin{matrix}\alpha\ \beta\\ \gamma\end{matrix};x\right)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!}x^n$

① 2項定理，幾何級数，指数関数を含む関数

2項定理：$F\left(\begin{matrix}-m\ \beta\\ \beta\end{matrix};-x\right)=\sum_{n=0}^{m}\frac{(-m)_n}{n!}(-x)^n=(1+x)^m$

幾何級数：$F\left(\begin{matrix}1\ \beta\\ \beta\end{matrix};x\right)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$

指数関数：$\lim_{\beta\to\infty}F\left(\begin{matrix}\alpha\ \beta\\ \alpha\end{matrix};\frac{x}{\beta}\right)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

② $F(x)=\sum_{n=0}^{\infty}a_n x^n$ のとき $\frac{a_{n+1}}{a_n}$ がnの有理式になるもの．

③ 表現論的視点（Lie環の作用）‥ $U(gl(4))$ の作用．

$$\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}=x_{14}^{\lambda_4}x_{13}^{\lambda_3+\lambda_2+1}x_{23}^{-\lambda_2-1}F\left(\begin{matrix}\lambda_2+1\ \ -\lambda_4\\ \lambda_2+\lambda_3+2\end{matrix};\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)$$

とすると　$(E_{ij})_{1\leq i\leq j\leq 4}$：$i,j$ 成分が1の行列に対する作用を考えて．

$$E_{33}\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}=\lambda_3\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}$$

$$E_{33}E_{44}\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}+E_{44}\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}-E_{43}E_{34}\Phi\begin{pmatrix}1&0&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\end{pmatrix}=0$$

とすることができる．

q-アナログの存在

$$F\left(\begin{matrix}\alpha & \beta \\ \gamma & \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} x^n \qquad x^{-1}\theta(\theta+\gamma-1)F = (\theta+\alpha)(\theta+\beta)F$$

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$$

$$\downarrow$$

Heine の超幾何級数

$$\varphi\left(\begin{matrix}\alpha & \beta \\ \gamma & \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(q^\alpha;q)_n (q^\beta;q)_n}{(q^\gamma;q)_n (q;q)_n} x^n$$

$$(\alpha;q)_n = \prod_{k=0}^{n-1}(1-\alpha q^k)$$

$$x^{-1}[\theta]_q[\theta+\gamma-1]_q\varphi = [\theta+\alpha]_q[\theta+\beta]_q\varphi$$

$$[\theta+c]_q x^n = (1-q^{n+c})x^n$$

Remark

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

として.

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

とすることもできる.

Heineの超幾何級数の表現論的視点

Gaussの超幾何

$$\Phi\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24}\end{pmatrix} = x_{14}^{\lambda_4} x_{13}^{\lambda_2+\lambda_3+1} x_{23}^{-\lambda_2-1} \sum_{n=0}^{\infty} \frac{(\lambda_2+1)_n(-\lambda_4)_n}{(\lambda_2+\lambda_3+2)_n\, n!}\left(\frac{x_{13}x_{24}}{x_{23}x_{14}}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda_2+1)_n(-\lambda_4)_n}{(\lambda_2+\lambda_3+2)_n\, n!}\, \underline{x_{14}^{\lambda_4-n} x_{13}^{\lambda_2+\lambda_3+1+n} x_{23}^{-\lambda_2-1-n} x_{24}^{n}}$$

$U(\mathfrak{gl}(4))$（4次行列）の作用

$E_{jj}\Phi = \lambda_j \Phi \quad (j=1,2,3,4)$

$E_{33}E_{44}\Phi + E_{44}\Phi - E_{43}E_{34}\Phi = 0.$

量子化のイメージ

Quantum Grassmannian Algebra $\mathbb{C}\langle x_{14}, x_{13}, x_{24}, x_{23}, \xi_{12}, \xi_{12}^{-1}\rangle$

以下のcommutation relationを満たす非可換環.

$x_{13}x_{14} = q\, x_{14}x_{13}$, $\quad x_{23}x_{24} = q\, x_{24}x_{23}$

$q\, x_{13}x_{23} = x_{23}x_{13}$, $\quad q\, x_{14}x_{24} = x_{24}x_{14}$

$x_{13}x_{24} = x_{24}x_{13}$, $\quad x_{23}x_{14} - x_{14}x_{23} = (q-q^{-1})x_{13}x_{24}$

$\xi_{12}^{l}\, x_{ij} = q^{l} x_{ij}\, \xi_{12}^{l}$

$U_q(\mathfrak{gl}(4))$ の generator $q^{\varepsilon_2}, q^{\varepsilon_3}, q^{\varepsilon_4}, e_3, f_3$ の作用

$q^{\varepsilon_2}\xi_{12}^{\pm1} = q^{\pm1}\xi_{12}^{\pm1}$, $\quad q^{\varepsilon_3}\xi_{12}^{\pm1} = \xi_{12}^{\pm1}$, $\quad q^{\varepsilon_4}\xi_{12}^{\pm1} = \xi_{12}^{\pm1}$, $\quad e_3\xi_{12}^{\pm1} = f_3\xi_{12}^{\pm1} = 0$

$q^{\varepsilon_2}x_{13} = x_{13}$, $\quad q^{\varepsilon_2}x_{14} = x_{14}$, $\quad q^{\varepsilon_2}x_{23} = q^{-1}x_{23}$, $\quad q^{\varepsilon_2}x_{24} = q^{-1}x_{24}$

$q^{\varepsilon_3}x_{13} = q x_{13}$, $\quad q^{\varepsilon_3}x_{14} = x_{14}$, $\quad q^{\varepsilon_3}x_{23} = q x_{23}$, $\quad q^{\varepsilon_3}x_{24} = x_{24}$

$q^{\varepsilon_4}x_{13} = x_{13}$, $\quad q^{\varepsilon_4}x_{14} = q x_{14}$, $\quad q^{\varepsilon_4}x_{23} = x_{23}$, $\quad q^{\varepsilon_4}x_{24} = q x_{24}$

$e_3x_{13} = 0$, $\quad e_3x_{14} = x_{13}$, $\quad e_3x_{23} = 0$, $\quad e_3x_{24} = x_{23}$

$f_3x_{13} = x_{14}$, $\quad f_3x_{14} = 0$, $\quad f_3x_{23} = x_{24}$, $\quad f_3x_{24} = 0$

"Left $U_q(\mathfrak{gl}(4))$-symmetry" $\varphi, \psi \in \mathbb{C}\langle x_{14}, x_{13}, x_{24}, x_{23}, \xi_{12}, \xi_{12}^{-1} \rangle$

（論文では $\mathcal{A}\langle \xi_{12}^{-1} \rangle$）

$$q^{\varepsilon_j}(\varphi\psi) = (q^{\varepsilon_j}\varphi)(q^{\varepsilon_j}\psi) \quad \leftrightarrow \quad \Delta(q^{\varepsilon_j}) = q^{\varepsilon_j} \otimes q^{\varepsilon_j}$$

$$e_j(\varphi \cdot \psi) = (e_j\varphi)\psi + (q^{\varepsilon_j - \varepsilon_{j+1}}\varphi)(e_j\psi) \quad \leftrightarrow \quad \Delta(e_j) = e_j \otimes 1 + q^{\varepsilon_j - \varepsilon_{j+1}} \otimes e_j$$

$$f_j(\varphi\psi) = (q^{-\varepsilon_j + \varepsilon_{j+1}}\varphi)(f_j\psi) + (f_j\varphi)\psi \quad \leftrightarrow \quad \Delta(f_j) = q^{-\varepsilon_j + \varepsilon_{j+1}} \otimes f_j + f_j \otimes 1$$

可換変数の monomial $\bigoplus_v \mathbb{C} z^v$ への $U_q(\mathfrak{gl}(4))$ の作用

$\rho : U_q(\mathfrak{gl}(4)) \longrightarrow \mathrm{End}_{\mathbb{C}}(\bigoplus_v \mathbb{C} z^v)$

$v \in \mathbb{N}^4$ の場合：以下の可換図式を考える。 $\rho(a) z^v = \psi^{-1}(a\psi(z^v))$

$$\begin{tikzcd}
\bigoplus_v \mathbb{C}\, z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}} \arrow[r, "\rho(a)"] \arrow[d, "\psi"'] & \bigoplus_v \mathbb{C}\, z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}} \\
\left(\bigoplus_v \mathbb{C}\, \underbrace{x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}}}\right) \xi_{12}^{-1} \arrow[r, "a"] & \left(\bigoplus_v \mathbb{C}\, x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}}\right) \xi_{12}^{-1} \arrow[u, "\psi^{-1}"']
\end{tikzcd}$$

isomorphism of vector sp（論文中では $\psi_{(-1,\phi)}$）

$\psi : \bigoplus_v \mathbb{C} z^v \xrightarrow{\sim} \bigoplus_v \mathbb{C} x^v$ (isomorphism of vector sp)

$z^v \mapsto x^v \xi_{12}^{-1} q^{v_{13}v_{24} - v_{14}^2 - v_{23}(v_{23}+2)}$

↳ ordered monomial を basis とする vector space

$$\begin{aligned}
\rho(q^{\varepsilon_2}) z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}} &= \psi^{-1}(q^{\varepsilon_2} x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{-1-v_{23}-v_{24}} \psi^{-1}(x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{-1-v_{23}-v_{24}} z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}}
\end{aligned}$$

$$\begin{aligned}
\rho(q^{\varepsilon_3}) z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}} &= \psi^{-1}(q^{\varepsilon_3} x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{v_{13}+v_{23}} \psi^{-1}(x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{v_{13}+v_{23}} z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}}
\end{aligned}$$

$$\begin{aligned}
\rho(q^{\varepsilon_4}) z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}} &= \psi^{-1}(q^{\varepsilon_4} x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{v_{14}+v_{24}} \psi^{-1}(x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{\phi(v)}) \\
&= q^{v_{14}+v_{24}} z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}}
\end{aligned}$$

以下 $z^v := z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}} z_{23}^{v_{23}}$, $x^v := x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}}$, $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ とする.

$e_3 x_{14}^{v_{14}} = x_{14}^{v_{14}-1} x_{13}$ など

$e_3 x_{24}^{v_{24}} = x_{24}^{v_{24}-1} x_{23}$

$$
\begin{aligned}
(q-q^{-1})\rho(e_3) z^v &= (q-q^{-1})\psi^{-1}\left(e_3 x^v \xi_{12}^{-1} q^{\phi(v)}\right) \\
&= (q-q^{-1})\psi^{-1}\left(e_3 (x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}}) \xi_{12}^{-1} q^{\phi(v)}\right) \\
&= (q-q^{-1})\psi^{-1}\Big([v_{14}]_q x_{14}^{v_{14}-1} x_{13}^{v_{13}+1} x_{24}^{v_{24}} x_{23}^{v_{23}} \\
&\qquad + q^{-v_{14}+v_{13}} [v_{24}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}-1} x_{23}^{v_{23}+1}\Big) \xi_{12}^{-1} q^{\phi(v)}\Big) \\
&= (q-q^{-1})\psi^{-1}\Big([v_{14}]_q x_{14}^{v_{14}-1} x_{13}^{v_{13}+1} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{v_{13}v_{24} - v_{14}^2 - v_{23}(v_{23}+2)} \\
&\qquad + q^{-v_{14}+v_{13}} [v_{24}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}-1} x_{23}^{v_{23}+1} \xi_{12}^{-1} q^{v_{13}v_{24} - v_{14}^2 - v_{23}(v_{23}+2)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= (q-q^{-1})\psi^{-1}\left([v_{14}]_q x_{14}^{v_{14}-1} x_{13}^{v_{13}+1} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{(v_{13}+1)v_{24} - (v_{14}-1)^2 - v_{23}(v_{23}+2)}\right) q^{-v_{24} - 2v_{14} + 1} \\
&\quad + (q-q^{-1})\psi^{-1}\left([v_{24}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}-1} x_{23}^{v_{23}+1} \xi_{12}^{-1} q^{v_{13}(v_{24}-1) - v_{14}^2 - (v_{23}+1)(v_{23}+3)}\right) q^{v_{13} + 2v_{23} + 3}
\end{aligned}
$$

$$
\begin{aligned}
&= (q-q^{-1})[v_{14}]_q z_{14}^{v_{14}-1} z_{13}^{v_{13}+1} z_{24}^{v_{24}} z_{23}^{v_{23}} q^{-v_{24} - 2v_{14} + 1} \\
&\quad + (q-q^{-1})[v_{24}]_q z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}-1} z_{23}^{v_{23}+1} q^{v_{13} + 2v_{23} + 3}
\end{aligned}
$$

$$
\begin{aligned}
&= (1 - q^{-2v_{14}}) q^{1-(v_{14}+v_{24})} z_{14}^{v_{14}-1} z_{13}^{v_{13}+1} z_{24}^{v_{24}} z_{23}^{v_{23}} \\
&\quad + (q^{2v_{24}} - 1) q^{3 + 2(v_{13}+v_{23}) - (v_{14}+v_{24})} z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}-1} z_{23}^{v_{23}+1}
\end{aligned}
$$

$$f_3 x_{13}^{v_{13}} = [v_{13}]_q x_{14} x_{13}^{v_{13}-1}$$
$$f_3 x_{23}^{v_{23}} = [v_{23}]_q x_{24} x_{23}^{v_{23}-1}$$

$$(q-q^{-1})\rho(f_3) z^v = (q-q^{-1})\psi^{-1}(f_3 x^v \xi_{12}^{-1} q^{\phi(v)})$$
$$= (q-q^{-1})\psi^{-1}(f_3(x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}} x_{23}^{v_{23}})\xi_{12}^{-1} q^{\phi(v)})$$
$$= (q-q^{-1})\psi^{-1}([v_{13}]_q q^{v_{24}-v_{23}} x_{14}^{v_{14}+1} x_{13}^{v_{13}-1} x_{24}^{v_{24}} x_{23}^{v_{23}}$$
$$[v_{23}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}+1} x_{23}^{v_{23}-1})\xi_{12}^{-1} q^{\phi(v)}).$$

$$= (q-q^{-1})\psi^{-1}([v_{13}]_q x_{14}^{v_{14}-1} x_{13}^{v_{13}+1} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{v_{13}v_{24}-v_{14}^2-v_{23}(v_{23}+2)}$$
$$+ [v_{23}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}+1} x_{22}^{v_{23}-1} \xi_{12}^{-1} q^{v_{13}v_{24}-v_{14}^2-v_{23}(v_{23}+2)})$$

$$= (q-q^{-1})\psi^{-1}([v_{13}]_q q^{v_{24}-v_{23}} x_{14}^{v_{14}+1} x_{13}^{v_{13}-1} x_{24}^{v_{24}} x_{23}^{v_{23}} \xi_{12}^{-1} q^{(v_{13}-1)v_{24}-(v_{14}+1)^2-v_{23}(v_{23}+2)} q^{v_{24}+2v_{14}+}$$
$$+(q-q^{-1})\psi^{-1}([v_{23}]_q x_{14}^{v_{14}} x_{13}^{v_{13}} x_{24}^{v_{24}+1} x_{23}^{v_{23}-1} \xi_{12}^{-1} q^{v_{13}(v_{24}+1)-v_{14}^2-(v_{23}-1)(v_{23}+1)} q^{-v_{13}+2v_{23}-1}$$

$$= (q-q^{-1})[v_{13}]_q q^{v_{24}-v_{23}} z_{14}^{v_{14}+1} z_{13}^{v_{13}-1} z_{24}^{v_{24}} z_{23}^{v_{23}} \cdot q^{v_{24}+2v_{14}+1}$$
$$+ (q-q^{-1})[v_{23}]_q z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}+1} z_{23}^{v_{23}-1} \cdot q^{-v_{13}+2v_{23}-1}$$

$$= (q^{2v_{13}}-1) q^{1-(v_{13}+v_{23})+2(v_{14}+v_{24})} z_{14}^{v_{14}+1} z_{13}^{v_{13}-1} z_{24}^{v_{24}} z_{23}^{v_{23}}$$
$$+(1-q^{-2v_{23}}) q^{-1-(v_{13}+v_{23})} z_{14}^{v_{14}} z_{13}^{v_{13}} z_{24}^{v_{24}+1} z_{23}^{v_{23}-1}$$

可換変数の monomial z^{ν} $(= z_{14}^{\nu_{14}} z_{13}^{\nu_{13}} z_{24}^{\nu_{24}} z_{23}^{\nu_{23}} \quad \nu_{ij} \in \mathbb{C})$
に対して $e_3, f_3, q^{\varepsilon_2}, q^{\varepsilon_3}, q^{\varepsilon_4} \in U_q(\mathfrak{gl}(4))$ の作用を

$$\rho(q^{\varepsilon_2}) z^{\nu} = q^{-1-(\nu_{23}+\nu_{24})} z^{\nu}$$

$$\rho(q^{\varepsilon_3}) z^{\nu} = q^{\nu_{13}+\nu_{23}} z^{\nu}$$

$$\rho(q^{\varepsilon_4}) z^{\nu} = q^{\nu_{14}+\nu_{24}} z^{\nu}$$

$$\rho(e_3) z^{\nu} = (1-q^{-2\nu_{14}}) q^{1-\nu_{14}-\nu_{24}} z_{14}^{\nu_{14}-1} z_{13}^{\nu_{13}+1} z_{24}^{\nu_{24}} z_{23}^{\nu_{23}} + (q^{2\nu_{24}}-1) q^{3+2(\nu_{13}+\nu_{23})-(\nu_{14}+\nu_{24})} z_{14}^{\nu_{14}} z_{13}^{\nu_{13}} z_{24}^{\nu_{24}-1} z_{23}^{\nu_{23}+1}$$

$$\rho(f_3) z^{\nu} = (q^{2\nu_{13}}-1) q^{1-(\nu_{13}+\nu_{23})+2(\nu_{14}+\nu_{24})} z_{14}^{\nu_{14}+1} z_{13}^{\nu_{13}-1} z_{24}^{\nu_{24}} z_{23}^{\nu_{23}} + (1-q^{-2\nu_{23}}) q^{-1-(\nu_{13}+\nu_{23})} z_{14}^{\nu_{14}} z_{13}^{\nu_{13}} z_{24}^{\nu_{24}+1} z_{23}^{\nu_{23}-1}$$

$$\psi_\lambda(z) := z_{13}^{\lambda_2+\lambda_3+1} z_{23}^{-\lambda_2-1} z_{14}^{\lambda_4} \sum_{n=0}^{\infty} \frac{(q^{2(\lambda_2+1)};q^2)_n (q^{-2\lambda_4};q^2)_n}{(q^{2(\lambda_2+\lambda_3+2)};q^2)_n (q^2;q^2)_n} \left(\frac{z_{13}z_{24}}{z_{23}z_{14}}\right)^n$$

とおくと ψ_λ は以下の方程式の解である.

$$\rho(q^{\varepsilon_2})\psi_\lambda = q^{\lambda_2}\psi_\lambda, \quad \rho(q^{\varepsilon_3})\psi_\lambda = q^{\lambda_3}\psi_\lambda, \quad \rho(q^{\varepsilon_4})\psi_\lambda = q^{\lambda_4}\psi_\lambda$$

$$(q-q^{-1})^2 \rho(C_{44}^{33})\psi_\lambda = 0$$

(ただし $(q-q^{-1})^2 C_{44}^{33} := (q^{\varepsilon_3+1}-q^{-\varepsilon_3-1})(q^{\varepsilon_4}-q^{-\varepsilon_4}) - (q-q^{-1})^2 f_3 e_3$
であり, $(q^{\varepsilon_3+1}-q^{-\varepsilon_3-1})(q^{\varepsilon_4}-q^{-\varepsilon_4})\psi_\lambda = (q-q^{-1})^2[\lambda_3+1]_q[\lambda_4]_q\psi_\lambda$ となる
$(q-q^{-1})^2 f_3 e_3 \psi_\lambda = (q-q^{-1})^2[\lambda_3+1]_q[\lambda_4]_q\psi_\lambda$)

Aomoto-Gelfand の超幾何関数の量子化 (Noumi)

$E_{22}\Phi = \lambda_2\Phi, \quad E_{33}\Phi = \lambda_3\Phi, \quad E_{44}\Phi = \lambda_4\Phi$

$\hookrightarrow q^{\varepsilon_2}\psi_\lambda = q^{\lambda_2}\psi_\lambda, \quad q^{\varepsilon_3}\psi_\lambda = q^{\lambda_3}\psi_\lambda, \quad q^{\varepsilon_4}\psi_\lambda = q^{\lambda_4}\psi_\lambda$

$E_{33}E_{44}\Phi - E_{44}\Phi - E_{43}E_{34}\Phi = 0 \quad \cdots \quad (E_{33}+1)E_{44}\Phi - E_{43}E_{34}\Phi = 0$

$\hookrightarrow (q^{\varepsilon_3+1}-q^{-\varepsilon_3-1})(q^{\varepsilon_4}-q^{-\varepsilon_4})\psi_\lambda - (q-q^{-1})^2 f_3 e_3 \psi_\lambda = 0.$