Painlevé 方程式の Bäcklund 変換

Painlevé 第2方程式

$$P_{II}(b):\ y'' = 2y^3 + ty + b - \frac{1}{2} \quad (b \in \mathbb{C})$$

$\Updownarrow$ 同等 $\quad q := y,\ p := q' + q^2 + \frac{t}{2}$

$$H_{II}(b):\ q' = p - q^2 - \frac{t}{2} \qquad p' = 2qp + b$$

簡単な解

$b = \frac{1}{2}$

$q = 0,\ p = \frac{t}{2}$

$b = 0$

$q, p \in \mathbb{C}(\varphi_1, \varphi_2, \varphi_1', \varphi_2')$

Bäcklund 変換 $\quad H_{II}(b)$ の解から $H_{II}(-b), H_{II}(1-b), \cdots$ を得る変換

$s:\ H_{II}(b) \longrightarrow H_{II}(-b)$

$$q \mapsto \tilde{q} = q + \frac{b}{p}$$

$$p \mapsto \tilde{p} = p$$

$\tilde{q}, \tilde{p}$ は以下を満たす.

$$\tilde{q}' = \tilde{p} - \tilde{q}^2 - \frac{t}{2},\quad \tilde{p} = 2\tilde{q}\tilde{p} - b$$

$r:\ H_{II}(b) \longrightarrow H_{II}(1-b)$

$$q \mapsto \tilde{q} = -q$$

$$p \mapsto \tilde{p} = -p + 2q^2 + t$$

$\tilde{q}, \tilde{p}$ は以下を満たす

$$\tilde{q}' = \tilde{p} - \tilde{q}^2 - \frac{t}{2},\quad \tilde{p} = 2\tilde{q}\tilde{p} + 1 - b.$$

$H_{II}(b)$: $q' = p - q^2 - \frac{t}{2}$, $p' = 2qp + b$

$$s\quad \begin{aligned} q &\mapsto q + \frac{b}{p} \\ p &\mapsto p \\ b &\mapsto -b \end{aligned} \qquad\qquad r\quad \begin{aligned} q &\mapsto -q \\ p &\mapsto -p + 2q^2 + t \\ b &\mapsto 1-b \end{aligned}$$

定数の変換に注目

対称形式 : $p = f_1$ $f_0 = r(p) = -p + 2q^2 + t$

$\alpha_1 = b$ $\alpha_0 = 1 - b$

$$f_0' = -2qf_0 + \alpha_0$$

$$f_1' = 2qf_1 + \alpha_1$$

$$q' = \frac{1}{2}(f_1 - f_0)$$

こうして

$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$

$(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\pi(\alpha_0) = \alpha_1$
$\pi(\alpha_1) = \alpha_0$

とすると

(s_i, π は Weyl 群の Generator)

$s = s_i$

$r = \pi$

になる.