微分・差分・q−差分

微分

$$f(x+h) = f(x) + f'(x)h + o(h)$$

$f \in \mathbb{C}[[x]]$ の場合

$$f(x+h) \equiv f(x) + f'(x)\varepsilon \quad (\text{modulo } \varepsilon^2)$$

差分

$$f(x+1) = f(x) + (\Delta_+ f)(x)$$
$$f(x) = f(x-1) + (\Delta_- f)(x)$$

q−差分

$$f(qx) = f(x) + (\delta_q f)(x)\cdot(q-1)x \qquad (q>1)$$
$$f(x) = f(qx) + (\delta_q f)(x)\cdot(1-q)x \qquad (q<1)$$

x^n の微分と q−差分

$(x+\varepsilon)^n \equiv x^n + nx^{n-1}\varepsilon \quad (\text{modulo } \varepsilon^2)$ より $(x^n)' = nx^{n-1}$

$\nearrow q\to 1$

$x^n = (qx)^n + \dfrac{1-q^{-n}}{1-q}x^{n-1}(1-q)x$ より $\delta_q(x^n) = \dfrac{1-q^n}{1-q}x^{n-1}$

q-pochhammer 記号

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$$

レポート

$$\boxed{\frac{1-q^n}{1-q}x^{n-1} \to nx^{n-1} \quad (q\to 1)}$$

超幾何関数

$x^{-1}\theta(\theta+\gamma-1)F = (\theta+\alpha)(\theta+\beta)F \qquad F(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!}x^n$

↓ q−差分方程式

$x^{-1}[\theta]_q[\theta+\gamma-1]_q\varphi = [\theta+\alpha]_q[\theta+\beta]_q\varphi \qquad \varphi(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(q^{2\alpha};q^2)_n(q^{2\beta};q^2)_n}{(q^{2\gamma};q^2)_n(q^2;q^2)_n}x^n$

Aomoto-Gelfandの超幾何微分方程式系

$$\Phi : \mathrm{Mat}'(2,4;\mathbb{C}) \longrightarrow \mathbb{C}$$

2×4行列で
全ての小行列式≠0 ∴ Grassmann多様体 $G_{2,4}$ 上の関数

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix} \mapsto \Phi\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix}$$

$$\sum_{j=1}^{4} t_{ij} \frac{\partial}{\partial t_{kj}} \Phi = -\delta_{ik} \Phi \quad \leftarrow \quad \Phi\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix}$$

$$= \Delta_{12}^{-1} \Phi \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix}$$

$$\sum_{i=1}^{2} t_{ij} \frac{\partial}{\partial t_{ij}} \Phi = \lambda_j \Phi$$

$$x_{13} := -\Delta_{12}^{-1}\Delta_{23} \qquad x_{14} := -\Delta_{12}^{-1}\Delta_{24}$$

$$x_{23} := \Delta_{12}^{-1}\Delta_{13} \qquad x_{24} := \Delta_{12}^{-1}\Delta_{14}$$

$$\left(\frac{\partial}{\partial t_{ij}}\frac{\partial}{\partial t_{kl}} - \frac{\partial}{\partial t_{kj}}\frac{\partial}{\partial t_{il}}\right)\Phi = 0$$

$$\downarrow$$

$$\Psi\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} := \Phi\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix}$$ として書換え.

$$\left(x_{13}\frac{\partial}{\partial x_{13}} + x_{14}\frac{\partial}{\partial x_{14}}\right)\Psi = -(\lambda_1+1)\Psi$$

$$\left(x_{23}\frac{\partial}{\partial x_{23}} + x_{24}\frac{\partial}{\partial x_{24}}\right)\Psi = -(\lambda_2+1)\Psi \quad \cdots Ⓐ$$

$$\varphi(t) := \Psi\begin{pmatrix} x_{13} & x_{14} \\ t x_{23} & t x_{24} \end{pmatrix}$$

$$t\varphi'(t) = -(\lambda_2+1)\varphi(t)$$

を導く

$$\left(x_{13}\frac{\partial}{\partial x_{13}} + x_{23}\frac{\partial}{\partial x_{23}}\right)\Psi = \lambda_3\Psi \quad \cdots Ⓑ$$

$$\left(x_{14}\frac{\partial}{\partial x_{14}} + x_{24}\frac{\partial}{\partial x_{24}}\right)\Psi = \lambda_4\Psi \quad \cdots Ⓒ$$

$$\left(\frac{\partial}{\partial x_{13}}\frac{\partial}{\partial x_{24}} - \frac{\partial}{\partial x_{23}}\frac{\partial}{\partial x_{14}}\right)\Psi = 0 \quad \cdots Ⓓ$$

Ⓐ, Ⓑ, Ⓒ より $f(x) = \psi\begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} \left(= \Phi\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{pmatrix} \right.$ とおくと $\left. \right)$

$$\psi\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} = x_{13}^{\lambda_3} x_{14}^{\lambda_4} \left(\frac{x_{23}}{x_{13}}\right)^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)$$

$$= x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)$$

$$\frac{\partial}{\partial x_{13}}\frac{\partial}{\partial x_{24}}\left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\right)$$

$$= \frac{\partial}{\partial x_{13}}\left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} \frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\cdot\frac{dx}{dx_{24}}\right) \qquad x := \frac{x_{13}x_{24}}{x_{14}x_{23}}$$

として chain rule

$$= \frac{\partial}{\partial x_{13}}\left(x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\right)$$

$$= (2+\lambda_2+\lambda_3)\, x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)$$

$$+ x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \underline{\frac{d^2f}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\frac{x_{24}}{x_{14}x_{23}}}$$ ← chain rule

$$= (2+\lambda_2+\lambda_3)\, x_{12}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx}(x)$$

$$+ x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2}\, x \frac{d^2f}{dx^2}(x)$$

$$\frac{\partial}{\partial x_{23}}\frac{\partial}{\partial x_{14}}\left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\right)$$

$$= \frac{\partial}{\partial x_{23}}\left(\lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right) + x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1}\frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\left(-\frac{x_{13}x_{24}}{x_{14}^2x_{23}}\right)\right)$$

$$= \frac{\partial}{\partial x_{23}}\left(\lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-1} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right) - x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2} x_{24}\frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\right.$$

$$= -\lambda_4(\lambda_2+1)\, x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} f\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right) + \lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2}\frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)(-x)$$

$$+ \underline{(\lambda_2+2)\, x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-3} x_{24}\frac{df}{dx}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)} \rightarrow x_{13}^{1+\lambda_2+\lambda_3} x_{24}^{\lambda_4-2} x_{23}^{-\lambda_2-2}\frac{df}{dx}\cdot x$$

$$- x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2} x_{24}\cdot\frac{d^2f}{dx^2}\left(\frac{x_{13}x_{24}}{x_{14}x_{23}}\right)\left(-\frac{x_{13}x_{24}}{x_{14}x_{23}^2}\right)$$

$$(\mathrm{LHS})-(\mathrm{RHS}) = x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2}\left(x(1-x)\frac{d^2f}{dx^2} + (2+\lambda_2+\lambda_3 - (\lambda_2+2-\lambda_4)x)\frac{df}{dx} - \lambda_4(\lambda_2+1)f\right) = 0$$

Lie代数の幾何と代数

$\mathfrak{gl}(4)$：4次行列全体（$U(\mathfrak{gl}(4))$）

Lie群 $GL(2)$ $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \simeq (x_{11}, x_{12}, x_{21}, x_{22})$

代数群 $GL(2)$ $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$

$x_{11}x_{22} - x_{12}x_{21} \neq 0$

幾何的意味

$GL(4)$：行列式 $\neq 0$ の4次行列

$\{ X \in Mat(4) \mid \exp tX \in GL(4) \}$

$\det(\exp tX) = \exp t(\mathrm{Tr}\, X)$
より $\det(\exp tX) \neq 0$

$\psi = \Phi \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix}$ に対して　$g \in GL(4)$ の作用を．

$$g\psi = (g\Phi)\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} := \Phi\left(\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} g\right)$$

として

$X\psi$ を $\Phi\left(\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix}(1 + hX + o(h))\right)$ の h の1次の項とすると..

Ⓑ

$$E_{33}\psi = \lambda_3 \psi$$

Ⓒ

$$E_{44}\psi = \lambda_4 \psi$$

Ⓓ'

$$((E_{33} + 1)E_{44} - E_{43}E_{34})\psi = 0$$

量子化への準備.

$\mathfrak{gl}(4)$ の代数 (Lie代数)

generator E_{ii} $(i=1,2,3,4)$, $\underset{}{e_k}$, f_k $(k=1,2,3)$

$$e_k = E_{k,k+1}, \quad f_k = E_{k+1,k}$$

relation $h = \sum c_i h_i$, $\langle , \rangle$ bilinear form $\alpha_k = E_{kk} - E_{k+1,k+1}$

$$[h, e_k] = \langle \alpha_k, h \rangle e_k$$

$$[h, f_k] = -\langle \alpha_k, h \rangle f_k$$

$$[e_i, f_j] = \delta_{ij} (\varepsilon_i - \varepsilon_{i+1})$$

$\downarrow$

$U_q(\mathfrak{gl}(4))$

generator : $q^{\pm \varepsilon_i}$ $(i=1,2,3,4)$, e_k, f_k $(k=1,2,3)$

relation : $q^h e_k = q^{\langle \alpha_k, h \rangle} e_k q^h$

$$\left.\begin{aligned} q^h e_k &= q^{\langle \alpha_k, h \rangle} e_k q^h \\ q^h f_k &= q^{-\langle \alpha_k, h \rangle} f_k q^h \end{aligned}\right\} \text{使うのはこれ.}$$

$$[e_i, f_k] = \delta_{ik} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}$$

(cf) $U_q(\mathfrak{sl}(4))$

generator q^{α_k}, e_k, f_k $(k=1,2,3)$

relation $q^{\alpha_k} e_l = q^{\langle \alpha_k, \alpha_l \rangle} e_l q^{\alpha_k}$

$\vdots$

$$\langle \alpha_k, \alpha_l \rangle \leftrightarrow \underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{\text{Cartan matrix}}$$