

平行移動と接続

$M = n$ 次元微分可能多様体; $x, \tilde{x} \in M$
($x \neq \tilde{x}$)

$$\textcircled{1} \quad \tilde{V}_\mu(\tilde{x}) = V_\mu(x) + T_{\mu\rho}^\sigma V_\sigma(x) h^\rho + o(h)$$

$$\left(\tilde{V}_\mu(x(t+\varepsilon)) = V_\mu(x) + T_{\nu\rho}^\sigma V_\sigma(x) h^\rho(\varepsilon) + o(\varepsilon) \right)$$

$$\textcircled{2} \quad g^{\mu\nu}(\tilde{x}) \tilde{V}_\mu(\tilde{x}) \tilde{V}_\nu(\tilde{x}) = g^{\mu\nu}(x) V_\mu(x) V_\nu(x).$$

を満たすとする。

<共変ベクトル V_μ の共変微分>

$$V_\mu(\tilde{x}) - \tilde{V}_\mu(\tilde{x}) = (\nabla_\rho V_\mu(x)) h^\rho + o(h)$$

となる $\nabla_\rho V_\mu$ を V_μ の共変微分といふ。これを定義すると $\textcircled{1}$ より

$$\nabla_\rho V_\mu = \partial_\rho V_\mu - T_{\mu\rho}^\sigma V_\sigma$$

となる。

<スカラーフィールの共変微分>

$$\phi(\tilde{x}) - \phi(x) = (\nabla_\rho \phi(x)) h^\rho + o(h)$$

となる $\nabla_\rho \phi$ を ϕ の共変微分といい $(d\phi)_x$ に一致する。

<反変ベクトルの共変微分>

$$\tilde{V}_\mu(\tilde{x}) \tilde{V}^\mu(\tilde{x}) - V_\mu(x) V^\mu(x) = 0$$

ここで

とおいて

$$\text{LHS} = (V_\mu(x) + T_{\mu\rho}^\sigma V_\sigma(x) h^\rho + o(h)) (V^\mu(x) + \delta V^\mu(x)) - V_\mu(x) V^\mu(x)$$

$$= T_{\mu\rho}^\sigma V_\sigma(x) h^\rho V^\mu(x) + V_\mu(x) \delta V^\mu(x) + o(h)$$

$$\delta V^\mu(x) = -T_{\sigma\rho}^\mu V^\sigma h^\rho$$

$$V^\mu(\tilde{x}) - \tilde{V}^\mu(\tilde{x}) = (\nabla_\rho V^\mu(x)) h^\rho + o(h)$$

を共変微分とす。このとき

$$\begin{aligned} V^\mu(\tilde{x}) - \tilde{V}^\mu(\tilde{x}) &= V^\mu(\tilde{x}) - (V^\mu(x) - T_{\sigma\rho}^\mu V^\sigma h^\rho + o(h)) \\ &\equiv \partial_\rho V^\mu + T_{\sigma\rho}^\mu V^\sigma \end{aligned}$$

より

$$\nabla_\rho V^\mu = \partial_\rho V^\mu + T_{\sigma\rho}^\mu V^\sigma$$

<テンソルの共変微分> Leibnitz-rule

$$\nabla_\rho (A^\mu B^\nu) = (\nabla_\rho A^\mu) B^\nu + A^\mu (\nabla_\rho B^\nu)$$

$$\nabla_\rho (A_\mu B_\nu) = (\nabla_\rho A_\mu) B_\nu + A_\mu (\nabla_\rho B_\nu)$$

$$\nabla_\rho (A_\mu^\nu B^\rho) = (\nabla_\rho A_\mu) B^\nu + A_\mu (\nabla_\rho B^\nu)$$

を考へておき。

$$\tilde{A}_\mu(\tilde{x}) B_\nu(\tilde{x}) - \tilde{A}_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) =$$

$$= A_\mu(\tilde{x}) B_\nu(\tilde{x}) - A_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) + A_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) - \tilde{A}_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) +$$

$$= (\nabla_\rho A_\mu) B_\nu(\tilde{x}) + A_\mu(\tilde{x}) (\nabla_\rho B_\nu) + o(h)$$

書き下すと

$$\nabla_\rho (A_\mu B_\nu) = (\nabla_\rho A_\mu) B_\nu + A_\mu (\nabla_\rho B_\nu)$$

$$= (\partial_\rho A_\mu - T_{\mu\rho}^\sigma A_\sigma) B_\nu + A_\mu (\partial_\rho B_\nu - T_{\nu\rho}^\sigma B_\sigma)$$

$$= (\partial_\rho A_\mu) B_\nu + A_\mu (\partial_\rho B_\nu) - T_{\mu\rho}^\sigma A_\sigma B_\nu - T_{\nu\rho}^\sigma A_\mu B_\sigma$$

$$= \partial_\rho (A_\mu B_\nu) - T_{\mu\rho}^\sigma (A_\sigma B_\nu) - T_{\nu\rho}^\sigma (A_\mu B_\sigma)$$

曲率テンソル

$$R_{\mu\nu\alpha}{}^\beta A_\beta = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\alpha.$$

$$R_{\mu\nu\alpha\beta} = g_{\beta 0} R_{\mu\nu\alpha}{}^0$$

$$R_{\mu\nu\alpha}{}^\beta = \partial_\nu T_{\mu\alpha}^\beta - \partial_\mu T_{\nu\alpha}^\beta + T_{\mu\alpha}^\rho T_{\nu\rho}^\beta - T_{\nu\alpha}^\rho T_{\mu\rho}^\beta$$

$$\therefore \nabla_\mu \nabla_\nu A_\alpha = \partial_\mu (\nabla_\nu A_\alpha) - T_{\mu\nu}^\beta \nabla_\beta A_\alpha - T_{\mu\alpha}^\beta \nabla_\nu A_\beta.$$

- 一般の共変微分の定義: $\nabla_\mu T_{\nu\alpha} = \partial_\mu T_{\nu\alpha} + T_{\mu\nu}^\beta T_{\beta\alpha} - T_{\nu\alpha}^\beta \nabla_\mu A_\beta$

$$\nabla_\nu \nabla_\mu A_\alpha = \partial_\nu (\nabla_\mu A_\alpha) - T_{\nu\mu}^\beta \nabla_\beta A_\alpha - T_{\nu\alpha}^\beta \nabla_\mu A_\beta$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\alpha = (\partial_\mu \nabla_\nu - \partial_\nu \nabla_\mu) A_\alpha + T_{\nu\alpha}^\beta \nabla_\mu A_\beta - T_{\mu\alpha}^\beta \nabla_\nu A_\beta$$

$$= \partial_\mu (\partial_\nu A_\alpha - T_{\alpha\nu}^\beta A_\beta) - \partial_\nu (\partial_\mu A_\alpha - T_{\alpha\mu}^\beta A_\beta)$$

$$\nabla_\nu A_\alpha = \partial_\nu A_\alpha - T_{\alpha\nu}^\beta A_\beta$$

$$+ T_{\nu\alpha}^\beta (\partial_\mu A_\beta - T_{\beta\mu}^\rho A_\rho) - T_{\mu\alpha}^\beta (\partial_\nu A_\beta - T_{\beta\nu}^\rho A_\rho)$$

$$= \cancel{\partial_\mu \nabla_\nu A_\alpha} - \cancel{\partial_\mu T_{\alpha\nu}^\beta A_\beta} - \cancel{T_{\alpha\nu}^\beta \partial_\mu A_\beta} - \cancel{\partial_\nu \nabla_\mu A_\alpha} + \cancel{\partial_\nu T_{\alpha\mu}^\beta A_\beta} + \cancel{T_{\alpha\mu}^\beta \partial_\nu A_\beta}$$

$$+ \cancel{T_{\nu\alpha}^\beta \partial_\mu A_\beta} - \cancel{T_{\nu\alpha}^\beta T_{\beta\mu}^\rho A_\rho} - \cancel{T_{\mu\alpha}^\beta \partial_\nu A_\beta} + \cancel{T_{\mu\alpha}^\beta T_{\rho\nu}^\rho A_\rho}$$

$$R_{\mu\nu\alpha}{}^\beta = \partial_\nu T_{\mu\alpha}^\beta - \partial_\mu T_{\nu\alpha}^\beta + T_{\mu\alpha}^\rho T_{\nu\rho}^\beta - T_{\nu\alpha}^\rho T_{\mu\rho}^\beta$$

$$(ii) R_{\mu\nu\alpha}{}^\beta = -R_{\nu\mu\alpha}{}^\beta \quad R_{\mu\nu\alpha\beta} = -R_{\nu\mu\beta\alpha} \quad (\delta_{\rho\sigma} R_{\mu\nu\alpha}{}^\sigma = -\delta_{\beta\sigma} R_{\mu\nu\alpha}{}^\sigma)$$

$$\therefore R_{\mu\nu\alpha}{}^\beta = \partial_\nu T_{\mu\alpha}{}^\beta - \partial_\mu T_{\nu\alpha}{}^\beta + T_{\mu\alpha}{}^\rho T_{\nu\rho}{}^\beta - T_{\nu\alpha}{}^\rho T_{\mu\rho}{}^\beta$$

$$R_{\nu\mu\alpha}{}^\beta = \partial_\mu T_{\nu\alpha}{}^\beta - \partial_\nu T_{\mu\alpha}{}^\beta + T_{\nu\alpha}{}^\rho T_{\mu\rho}{}^\beta - T_{\mu\alpha}{}^\rho T_{\nu\rho}{}^\beta$$

$$(iii) R_{[\mu\nu\alpha]}{}^\beta := \frac{1}{3!} (R_{\mu\nu\alpha}{}^\beta - R_{\mu\alpha\nu}{}^\beta + R_{\nu\alpha\mu}{}^\beta - R_{\nu\mu\alpha}{}^\beta + R_{\alpha\mu\nu}{}^\beta - R_{\alpha\nu\mu}{}^\beta)$$

$$\text{Ansatz: } R_{[\mu\nu\alpha]}{}^\beta = 0 \quad R_{[\mu\nu\alpha]\beta} = 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T_{\alpha\beta} = R_{\mu\nu\alpha}{}^\sigma T_{\sigma\beta} + R_{\mu\nu\beta}{}^\sigma T_{\alpha\sigma}$$

$$\begin{aligned} \textcircled{1} \quad \nabla_\mu (\nabla_\nu T_{\alpha\beta}) &= \partial_\mu (\nabla_\nu T_{\alpha\beta}) - T_{\nu\mu}^\sigma (\nabla_\sigma T_{\alpha\beta}) - T_{\alpha\mu}^\sigma (\nabla_\nu T_{\sigma\beta}) - T_{\beta\mu}^\sigma (\nabla_\nu T_{\alpha\sigma}) \\ &= \underbrace{\partial_\mu (\partial_\nu T_{\alpha\beta} - T_{\alpha\nu}^\rho T_{\rho\beta} - T_{\beta\nu}^\rho T_{\alpha\rho})}_{\text{symmetric}} - \underbrace{T_{\nu\mu}^\sigma (\nabla_\sigma T_{\alpha\beta})}_{\text{symmetric}} \\ &\quad - T_{\alpha\mu}^\sigma (\partial_\nu T_{\sigma\beta} - T_{\sigma\nu}^\rho T_{\rho\beta} - T_{\beta\nu}^\rho T_{\sigma\rho}) \\ &\quad - T_{\beta\mu}^\sigma (\partial_\nu T_{\alpha\sigma} - T_{\alpha\nu}^\rho T_{\rho\sigma} - T_{\sigma\nu}^\rho T_{\alpha\rho}) \end{aligned}$$

$$\nabla_\mu \nabla_\nu T_{\alpha\beta} - \nabla_\nu \nabla_\mu T_{\alpha\beta} = ?$$

$$\begin{aligned} &= \underbrace{-(\partial_\mu P_{\alpha\nu}^\rho)}_{\textcircled{2}} T_{\rho\beta} - T_{\alpha\nu}^\rho (\partial_\mu T_{\rho\beta}) - (\partial_\mu P_{\rho\mu}^\rho) T_{\alpha\beta} - T_{\rho\nu}^\rho (\partial_\mu T_{\alpha\rho}) \\ &\quad - T_{\alpha\mu}^\rho (\partial_\nu T_{\rho\beta}) + \underbrace{T_{\alpha\mu}^\sigma T_{\sigma\nu}^\rho T_{\rho\beta}}_{\textcircled{3}} - T_{\beta\mu}^\rho (\partial_\nu T_{\alpha\sigma}) + \underbrace{T_{\beta\mu}^\sigma T_{\sigma\nu}^\rho T_{\alpha\rho}}_{\textcircled{3}'} \\ &\quad \underbrace{\partial_\nu T_{\alpha\mu}^\rho T_{\rho\beta} + T_{\alpha\mu}^\rho (\partial_\nu T_{\rho\beta})}_{\textcircled{1}'} + \underbrace{(\partial_\nu P_{\rho\mu}^\rho) T_{\alpha\beta} + T_{\beta\mu}^\rho (\partial_\nu T_{\alpha\rho})}_{\textcircled{1}''} \\ &\quad + \underbrace{P_{\alpha\nu}^\rho (\partial_\mu T_{\rho\beta}) - T_{\alpha\nu}^\rho T_{\sigma\mu}^\rho T_{\rho\beta}}_{\textcircled{4}} + \underbrace{T_{\beta\nu}^\rho (\partial_\mu T_{\alpha\sigma}) - T_{\beta\mu}^\rho T_{\sigma\nu}^\rho T_{\alpha\sigma}}_{\textcircled{4}'} \end{aligned}$$

$$R_{\mu\nu\alpha}{}^\sigma = \underbrace{\partial_\nu T_{\mu\alpha}^\sigma}_{\textcircled{1}} - \underbrace{\partial_\mu T_{\nu\alpha}^\sigma}_{\textcircled{2}} + \underbrace{T_{\mu\alpha}^\rho T_{\nu\rho}^\sigma}_{\textcircled{3}} - \underbrace{T_{\nu\alpha}^\rho T_{\mu\rho}^\sigma}_{\textcircled{4}}$$

$$R_{\mu\nu\beta}{}^\sigma = \underbrace{\partial_\nu T_{\mu\beta}^\sigma}_{\textcircled{1}'} - \underbrace{\partial_\mu T_{\nu\beta}^\sigma}_{\textcircled{2}'} + \underbrace{T_{\mu\beta}^\rho T_{\nu\rho}^\sigma}_{\textcircled{3}'} - \underbrace{T_{\nu\beta}^\rho T_{\mu\rho}^\sigma}_{\textcircled{4}'}$$

$$= R_{\mu\nu\alpha}{}^\sigma T_{\sigma\beta} + R_{\mu\nu\beta}{}^\sigma T_{\alpha\sigma}$$

従って $T_{\alpha\beta} = \nabla_\alpha w_\beta$ となる。

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(\nabla_\alpha w_\beta) = R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma w_\beta + R_{\mu\nu\beta}{}^\sigma \nabla_\alpha w_\sigma$$

ビアンキの恒等式

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(\nabla_\alpha w_\beta) = R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma w_\beta + R_{\mu\nu\beta}{}^\sigma \nabla_\sigma w_\alpha$$

IXFの場合で見てみる

$$\mu < \begin{matrix} \mu\nu - \mu\nu \\ \mu\alpha - \mu\alpha \end{matrix} \quad +1$$

$$\nabla_\mu (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) w_\beta = \nabla_\mu (R_{\nu\alpha\beta}{}^\sigma w_\sigma) \textcircled{1}$$

$$\mu < \begin{matrix} \mu\nu - \mu\nu \\ \mu\alpha - \mu\alpha \end{matrix} \quad -1$$

$$\nabla_\mu (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) w_\beta = \nabla_\mu (R_{\alpha\nu\beta}{}^\sigma w_\sigma) \textcircled{2}$$

$$\nu < \begin{matrix} \nu\mu - \nu\mu \\ \nu\alpha - \nu\alpha \end{matrix} \quad -1$$

$$\nabla_\nu (\nabla_\mu \nabla_\alpha - \nabla_\alpha \nabla_\mu) w_\beta = \nabla_\nu (R_{\mu\alpha\beta}{}^\sigma w_\sigma) \textcircled{3}$$

$$\nu < \begin{matrix} \nu\mu - \nu\mu \\ \nu\alpha - \nu\alpha \end{matrix} \quad +1$$

$$\nabla_\nu (\nabla_\mu \nabla_\alpha - \nabla_\alpha \nabla_\mu) w_\beta = \nabla_\nu (R_{\alpha\mu\beta}{}^\sigma w_\sigma) \textcircled{4}$$

$$\alpha < \begin{matrix} \alpha\mu - \alpha\mu \\ \alpha\nu - \alpha\nu \end{matrix} \quad +1$$

$$\nabla_\alpha (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) w_\beta = \nabla_\alpha (R_{\nu\mu\beta}{}^\sigma w_\sigma) \textcircled{5}$$

$$\alpha < \begin{matrix} \alpha\mu - \alpha\mu \\ \alpha\nu - \alpha\nu \end{matrix} \quad -1$$

$$\nabla_\alpha (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) w_\beta = \nabla_\alpha (R_{\mu\nu\beta}{}^\sigma w_\sigma) \textcircled{6}$$

Leibnitz-rule

$$\nabla_\mu (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) w_\beta = \nabla_\mu (R_{\nu\alpha\beta}{}^\sigma w_\sigma) = (\nabla_\mu R_{\nu\alpha\beta}{}^\sigma) w_\beta + R_{\nu\alpha\beta}{}^\sigma (\nabla_\mu w_\sigma)$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(\nabla_\alpha w_\beta) = R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma w_\beta + R_{\mu\nu\beta}{}^\sigma \nabla_\sigma w_\alpha \quad \textcircled{1}-1 \quad \textcircled{2}-1$$

$$-(\nabla_\mu \nabla_\alpha - \nabla_\alpha \nabla_\mu)(\nabla_\nu w_\beta) = -(R_{\mu\alpha\nu}{}^\sigma \nabla_\sigma w_\beta + R_{\mu\alpha\beta}{}^\sigma \nabla_\nu w_\beta) \quad \textcircled{1}-2 \quad \textcircled{2}-1$$

$$-(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)(\nabla_\alpha w_\beta) = -(R_{\nu\mu\alpha}{}^\sigma \nabla_\sigma w_\beta + R_{\nu\mu\beta}{}^\sigma \nabla_\alpha w_\sigma) \quad \textcircled{1}'-2 \quad \textcircled{2}'-1$$

$$(\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu)(\nabla_\mu w_\beta) = R_{\nu\alpha\mu}{}^\sigma \nabla_\sigma w_\beta + R_{\nu\alpha\beta}{}^\sigma \nabla_\mu w_\sigma \quad \textcircled{1}'-1 \quad \textcircled{2}'-2$$

$$(\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha)(\nabla_\nu w_\beta) = R_{\alpha\mu\nu}{}^\sigma \nabla_\sigma w_\beta + R_{\alpha\mu\beta}{}^\sigma \nabla_\nu w_\sigma \quad \textcircled{1}'-1 \quad \textcircled{2}'-2$$

$$-(\nabla_\alpha \nabla_\nu - \nabla_\nu \nabla_\alpha)(\nabla_\mu w_\beta) = -(R_{\alpha\nu\mu}{}^\sigma \nabla_\sigma w_\beta + R_{\alpha\nu\beta}{}^\sigma \nabla_\mu w_\sigma) \quad \textcircled{1}'-2 \quad \textcircled{2}'-1$$

$$(R_{\mu\nu\alpha}{}^\rho + R_{\mu\alpha\nu}{}^\rho - R_{\nu\mu\alpha}{}^\rho + R_{\nu\alpha\mu}{}^\rho + R_{\alpha\nu\mu}{}^\rho - R_{\alpha\mu\nu}{}^\rho) \nabla_\rho w_\beta$$

$$= (\nabla_\mu R_{\nu\alpha\rho}{}^\rho - \nabla_\mu R_{\alpha\nu\rho}{}^\rho + \nabla_\nu R_{\mu\alpha\rho}{}^\rho + \nabla_\nu R_{\alpha\mu\rho}{}^\rho + \nabla_\alpha R_{\mu\nu\rho}{}^\rho - \nabla_\alpha R_{\nu\mu\rho}{}^\rho) w_\beta$$

ここで

$$R_{\mu\nu\alpha}{}^\rho + R_{\nu\alpha\mu}{}^\rho + R_{\alpha\nu\mu}{}^\rho = 0$$

$$\left. \begin{aligned} R_{\mu\nu\alpha}{}^\rho &= \partial_\nu T_{\mu\alpha} - \partial_\mu T_{\nu\alpha} + T_{\mu\alpha}^\sigma T_{\nu\sigma} - T_{\nu\alpha}^\sigma T_{\mu\sigma} \\ R_{\nu\alpha\mu}{}^\rho &= \partial_\alpha T_{\nu\mu} - \partial_\nu T_{\alpha\mu} + T_{\nu\mu}^\sigma T_{\alpha\sigma} - T_{\alpha\mu}^\sigma T_{\nu\sigma} \\ R_{\alpha\nu\mu}{}^\rho &= \partial_\mu T_{\alpha\nu} - \partial_\alpha T_{\mu\nu} + T_{\alpha\nu}^\sigma T_{\mu\sigma} - T_{\mu\nu}^\sigma T_{\alpha\sigma} \end{aligned} \right\} \text{を満たす。}$$

$$(\nabla_\mu R_{\nu\alpha\rho}{}^\rho - \nabla_\mu R_{\alpha\nu\rho}{}^\rho + \nabla_\nu R_{\mu\alpha\rho}{}^\rho + \nabla_\nu R_{\alpha\mu\rho}{}^\rho + \nabla_\alpha R_{\mu\nu\rho}{}^\rho - \nabla_\alpha R_{\nu\mu\rho}{}^\rho) w_\beta = 0$$

→ 実は $\nabla_\mu R_{\nu\alpha\rho}{}^\rho + \nabla_\nu R_{\alpha\mu\rho}{}^\rho + \nabla_\alpha R_{\mu\nu\rho}{}^\rho = 0$

Remark Jacobi 律。

$$[\nabla_\mu, [\nabla_\nu, \nabla_\alpha]] + [\nabla_\nu, [\nabla_\alpha, \nabla_\mu]] + [\nabla_\alpha, [\nabla_\mu, \nabla_\nu]] = 0$$

[かくも確認可能]

縮約された七元恒等式。

$$\nabla^\sigma (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 \quad R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta}, \quad R = g^{\mu\nu} R_{\mu\nu}$$

(1)

$$g^{v\beta} g^{\alpha\rho} (\nabla_\mu R_{\nu\alpha\rho} + \nabla_\nu R_{\alpha\mu\rho} + \nabla_\alpha R_{\mu\nu\rho}) = 0$$

$$\downarrow g^{v\beta} g^{\alpha\rho} \nabla_\mu R_{\nu\alpha\rho} = \nabla_\mu g^{v\beta} g^{\alpha\rho} R_{\nu\alpha\rho} \text{などが成り立つ}$$

$$\nabla_\mu g^{v\beta} g^{\alpha\rho} R_{\nu\alpha\rho} = \nabla_\mu g^{v\beta} R_{\nu\beta} = \nabla_\mu R = \nabla^\sigma g_{\mu\sigma} R$$

$$\nabla_\nu g^{v\beta} g^{\alpha\rho} R_{\alpha\mu\rho} = -\nabla_\nu g^{v\beta} g^{\alpha\rho} R_{\alpha\mu\rho}$$

$$= -\nabla_\nu g^{v\beta} g^{\alpha\rho} R_{\mu\alpha\rho} = -\nabla_\nu g^{v\beta} R_{\mu\beta} = -\nabla^\beta R_{\mu\beta}$$

$$\nabla_\alpha g^{v\beta} g^{\alpha\rho} R_{\mu\nu\rho} = -\nabla_\alpha g^{\alpha\rho} g^{v\beta} R_{\mu\nu\rho}$$

$$= -\nabla_\alpha g^{\alpha\rho} g^{\beta\sigma} R_{\beta\sigma\mu\nu} = -\nabla_\alpha g^{\alpha\rho} R_{\mu\beta} = -\nabla^\beta R_{\mu\beta}$$

$$g^{\nu\beta} g^{\alpha\rho} (\nabla_\mu R_{\nu\alpha\rho\beta} + \nabla_\nu R_{\alpha\mu\beta\beta} + \nabla_\alpha R_{\mu\nu\beta\beta}) = 0$$

$$\Leftrightarrow \nabla^\sigma g_{\alpha\mu} R - \nabla^\beta R_{\beta\mu} - \nabla^\rho R_{\rho\mu} = 0$$

$$\Leftrightarrow \nabla^\sigma g_{\alpha\mu} R - 2\nabla^\sigma R_{\alpha\mu} = 0 \Leftrightarrow \nabla^\sigma (R_{\alpha\mu} - \frac{1}{2}g_{\alpha\mu} R) = 0$$