平行移動と接続

M：n次元微分可能多様体；$x, \tilde{x} \in M$ $(x \neq \tilde{x})$

① $\tilde{V}_\mu(\tilde{x}) = V_\mu(x) + \Gamma^\sigma_{\mu\rho} V_\sigma(x) h^\rho + o(h)$

$\left(\tilde{V}_\mu(x(t+\varepsilon)) = V_\mu(x) + \Gamma^\sigma_{\nu\rho} V_\sigma(x) h^\rho(\varepsilon) + o(\varepsilon) \right)$

② $g^{\mu\nu}(\tilde{x}) \tilde{V}_\mu(\tilde{x}) \tilde{V}_\nu(\tilde{x}) = g^{\mu\nu}(x) V_\mu(x) V_\nu(x)$

を満たすとする.

＜共変ベクトル V_μ の共変微分＞

$V_\mu(\tilde{x}) - \tilde{V}_\mu(\tilde{x}) = (\nabla_\rho V_\mu(x)) h^\rho + o(h)$

となる $\nabla_\rho V_\mu$ を V_μ の共変微分という. こう定義すると①より

$$\nabla_\rho V_\mu = \partial_\rho V_\mu - \Gamma^\sigma_{\mu\rho} V_\sigma$$

となる.

＜スカラー ϕ の共変微分＞

$\phi(\tilde{x}) - \phi(x) = (\nabla_\rho \phi(x)) h^\rho + o(h)$

となる $\nabla_\rho \phi$ を ϕ の共変微分と言い $(d\phi)_x$ に一致する.

＜反変ベクトルの共変微分＞

$\tilde{V}_\mu(\tilde{x}) \tilde{V}^\mu(\tilde{x}) - V_\mu(x) V^\mu(x) = 0$

ここで

とおくと

$LHS = (V_\mu(x) + \Gamma^\sigma_{\mu\rho} V_\sigma(x) h^\rho + o(h))(V^\mu(x) + \delta V^\mu(x)) - V_\mu(x) V^\mu(x)$

$= \Gamma^\sigma_{\mu\rho} V_\sigma(x) h^\rho V^\mu(x) + V_\mu(x) \delta V^\mu(x) + o(h)$

$\delta V^\mu(x) = - \Gamma^\mu_{\sigma\rho} V^\sigma h^\rho$

$$V^\mu(\tilde{x}) - \tilde{V}^\mu(\tilde{x}) = (\nabla_\rho V^\mu(x)) h^\rho + o(h)$$

を共変微分とする。このとき

$$V^\mu(\tilde{x}) - \tilde{V}^\mu(\tilde{x}) = V^\mu(\tilde{x}) - (V^\mu(x) - \Gamma^\mu_{\sigma\rho} V^\sigma h^\rho + o(h))$$

$$= \partial_\rho V^\mu + \Gamma^\mu_{\sigma\rho} V^\sigma$$

より

$$\nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\sigma\rho} V^\sigma$$

〈テンソルの共変微分〉 Leibnitz-rule

$$\nabla_\rho (A^\mu B^\nu) = (\nabla_\rho A^\mu) B^\nu + A^\mu (\nabla_\rho B^\nu)$$

$$\nabla_\rho (A_\mu B_\nu) = (\nabla_\rho A_\mu) B_\nu + A_\mu (\nabla_\rho B_\nu)$$

$$\nabla_\rho (A_\mu B^\nu) = (\nabla_\rho A_\mu) B^\nu + A_\mu (\nabla_\rho B^\nu)$$

を考えれば o.k.

$$A_\mu(\tilde{x}) B_\nu(\tilde{x}) - \tilde{A}_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) =$$

$$= A_\mu(\tilde{x}) B_\nu(\tilde{x}) - A_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) + A_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x}) - \tilde{A}_\mu(\tilde{x}) \tilde{B}_\nu(\tilde{x})$$

$$= (\nabla_\rho A_\mu) B_\nu(\tilde{x}) + A_\mu(\tilde{x}) (\nabla_\rho B_\nu) + o(h)$$

書き下すと

$$\begin{aligned}
\nabla_\rho (A_\mu B_\nu) &= (\nabla_\rho A_\mu) B_\nu + A_\mu (\nabla_\rho B_\nu) \\
&= (\partial_\rho A_\mu - \Gamma^\sigma_{\mu\rho} A_\sigma) B_\nu + A_\mu (\partial_\rho B_\nu - \Gamma^\sigma_{\nu\rho} B_\sigma) \\
&= (\partial_\rho A_\mu) B_\nu + A_\mu (\partial_\rho B_\nu) - \Gamma^\sigma_{\mu\rho} A_\sigma B_\nu - \Gamma^\sigma_{\nu\rho} A_\mu B_\sigma \\
&= \partial_\rho (A_\mu B_\nu) - \Gamma^\sigma_{\mu\rho} (A_\sigma B_\nu) - \Gamma^\sigma_{\nu\rho} (A_\mu B_\sigma)
\end{aligned}$$

曲率テンソル

$$R_{\mu\nu\alpha}{}^{\beta} A_\beta = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\alpha .$$

$$R_{\mu\nu\alpha\beta} = g_{\beta\sigma} R_{\mu\nu\alpha}{}^{\sigma}$$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\nu \Gamma^\beta_{\mu\alpha} - \partial_\mu \Gamma^\beta_{\nu\alpha} + \Gamma^\rho_{\mu\alpha} \Gamma^\beta_{\nu\rho} - \Gamma^\rho_{\nu\alpha} \Gamma^\beta_{\mu\rho}$$

$$\because \quad \nabla_\mu \underbrace{\nabla_\nu A_\alpha} = \partial_\mu (\nabla_\nu A_\alpha) - \Gamma^\beta_{\mu\nu} \nabla_\beta A_\alpha - \Gamma^\beta_{\mu\alpha} \nabla_\nu A_\beta .$$

一般の共変テンソルの微分 : $\nabla_\mu T_{\nu\alpha} = \partial_\mu T_{\nu\alpha} + \Gamma^\beta_{\mu\nu} T_{\beta\alpha} - \Gamma$

$$\nabla_\nu \nabla_\mu A_\alpha = \partial_\nu \nabla_\mu A_\alpha - \Gamma^\beta_{\nu\mu} \nabla_\beta A_\alpha - \Gamma^\beta_{\nu\alpha} \nabla_\mu A_\beta$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\alpha = (\partial_\mu \nabla_\nu - \partial_\nu \nabla_\mu) A_\alpha + \Gamma^\beta_{\nu\alpha} \nabla_\mu A_\beta - \Gamma^\beta_{\mu\alpha} \nabla_\nu A_\beta$$

$$= \partial_\mu (\partial_\nu A_\alpha - \Gamma^\rho_{\alpha\nu} A_\rho) - \partial_\nu (\partial_\mu A_\alpha - \Gamma^\rho_{\alpha\mu} A_\rho)$$

$\nabla_\nu A_\alpha = \partial_\nu A_\alpha - \Gamma^\rho_{\alpha\nu} A_\rho$

$$+ \Gamma^\beta_{\nu\alpha} (\partial_\mu A_\beta - \Gamma^\rho_{\beta\mu} A_\rho) - \Gamma^\beta_{\mu\alpha} (\partial_\nu A_\beta - \Gamma^\rho_{\beta\nu} A_\rho)$$

$$= \partial_\mu \cancel{\partial_\nu} A_\alpha - \partial_\mu \Gamma^\rho_{\alpha\nu} A_\rho - \cancel{\Gamma^\rho_{\alpha\nu} \partial_\mu A_\rho} - \partial_\nu \cancel{\partial_\mu} A_\alpha + \partial_\nu \Gamma^\rho_{\alpha\mu} A_\rho + \cancel{\Gamma^\rho_{\alpha\mu} \partial_\nu A_\rho}$$

$$+ \cancel{\Gamma^\beta_{\nu\alpha} \partial_\mu A_\beta} - \Gamma^\beta_{\nu\alpha} \Gamma^\rho_{\beta\mu} A_\rho - \cancel{\Gamma^\beta_{\mu\alpha} \partial_\nu A_\beta} + \Gamma^\beta_{\mu\alpha} \Gamma^\rho_{\beta\nu} A_\rho$$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\nu \Gamma^\beta_{\mu\alpha} - \partial_\mu \Gamma^\beta_{\nu\alpha} + \Gamma^\rho_{\mu\alpha} \Gamma^\beta_{\nu\rho} - \Gamma^\rho_{\nu\alpha} \Gamma^\beta_{\mu\rho}$$

(i) $R_{\mu\nu\alpha}{}^{\beta} = -R_{\nu\mu\alpha}{}^{\beta}$　　$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$

$(g_{\beta\sigma} R_{\mu\nu\alpha}{}^{\sigma} = -g_{\beta\sigma} R_{\nu\mu\alpha}{}^{\sigma})$

∵ $R_{\mu\nu\alpha}{}^{\beta} = \partial_\nu \Gamma^{\beta}_{\mu\alpha} - \partial_\mu \Gamma^{\beta}_{\nu\alpha} + \Gamma^{\rho}_{\mu\alpha}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\beta}_{\mu\rho}$

$R_{\nu\mu\alpha}{}^{\beta} = \partial_\mu \Gamma^{\beta}_{\nu\alpha} - \partial_\nu \Gamma^{\beta}_{\mu\alpha} + \Gamma^{\rho}_{\nu\alpha}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\rho}_{\mu\alpha}\Gamma^{\beta}_{\nu\rho}$

(ii) $R_{[\mu\nu\alpha]}{}^{\beta} := \frac{1}{3!}(R_{\mu\nu\alpha}{}^{\beta} - R_{\mu\alpha\nu}{}^{\beta} + R_{\nu\alpha\mu}{}^{\beta} - R_{\nu\mu\alpha}{}^{\beta} + R_{\alpha\mu\nu}{}^{\beta} - R_{\alpha\nu\mu}{}^{\beta})$

のとき $R_{[\mu\nu\alpha]}{}^{\beta} = 0$　　$R_{[\mu\nu\alpha]\beta} = 0$

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)T_{\alpha\beta} = R_{\mu\nu\alpha}{}^{\sigma}T_{\sigma\beta} + R_{\mu\nu\beta}{}^{\sigma}T_{\alpha\sigma}$$

∵ $\nabla_\mu(\nabla_\nu T_{\alpha\beta}) = \partial_\mu(\nabla_\nu T_{\alpha\beta}) - \Gamma^{\sigma}_{\nu\mu}(\nabla_\sigma T_{\alpha\beta}) - \Gamma^{\sigma}_{\alpha\mu}(\nabla_\nu T_{\sigma\beta}) - \Gamma^{\sigma}_{\beta\mu}(\nabla_\nu T_{\alpha\sigma})$

$$= \partial_\mu(\partial_\nu T_{\alpha\beta} - \Gamma^{\rho}_{\alpha\nu}T_{\rho\beta} - \Gamma^{\rho}_{\beta\nu}T_{\alpha\rho}) - \Gamma^{\sigma}_{\nu\mu}(\nabla_\sigma T_{\alpha\beta})$$

$$- \Gamma^{\sigma}_{\alpha\mu}(\partial_\nu T_{\sigma\beta} - \Gamma^{\rho}_{\sigma\nu}T_{\rho\beta} - \Gamma^{\rho}_{\beta\nu}T_{\sigma\rho})$$

$$- \Gamma^{\sigma}_{\beta\mu}(\partial_\nu T_{\alpha\sigma} - \Gamma^{\rho}_{\alpha\nu}T_{\rho\sigma} - \Gamma^{\rho}_{\sigma\nu}T_{\alpha\rho})$$

↓

$$\nabla_\mu\nabla_\nu T_{\alpha\beta} - \nabla_\nu\nabla_\mu T_{\alpha\beta} =$$

$$= -\underset{②}{(\partial_\mu\Gamma^{\rho}_{\alpha\nu})}T_{\rho\beta} - \Gamma^{\rho}_{\alpha\nu}(\partial_\mu T_{\rho\beta}) - \underset{②'}{(\partial_\mu\Gamma^{\rho}_{\beta\nu})}T_{\alpha\rho} - \Gamma^{\rho}_{\beta\nu}(\partial_\mu T_{\alpha\rho})$$

$$- \Gamma^{\sigma}_{\alpha\mu}(\partial_\nu T_{\sigma\beta}) + \underset{③}{\Gamma^{\sigma}_{\alpha\mu}\Gamma^{\rho}_{\sigma\nu}}T_{\rho\beta} - \Gamma^{\sigma}_{\beta\mu}(\partial_\nu T_{\alpha\sigma}) + \underset{③'}{\Gamma^{\sigma}_{\beta\mu}\Gamma^{\rho}_{\sigma\nu}}T_{\alpha\rho}$$

$$\underset{①}{\partial_\nu\Gamma^{\rho}_{\alpha\mu}}T_{\rho\beta} + \Gamma^{\rho}_{\alpha\mu}(\partial_\nu T_{\rho\beta}) + \underset{①'}{(\partial_\nu\Gamma^{\rho}_{\beta\mu})}T_{\alpha\rho} + \Gamma^{\rho}_{\beta\mu}(\partial_\nu T_{\alpha\rho})$$

$$+ \Gamma^{\sigma}_{\alpha\nu}(\partial_\mu T_{\sigma\beta}) - \underset{④}{\Gamma^{\sigma}_{\alpha\nu}\Gamma^{\rho}_{\sigma\mu}}T_{\rho\beta} + \Gamma^{\sigma}_{\beta\nu}(\partial_\mu T_{\alpha\sigma}) - \underset{④'}{\Gamma^{\sigma}_{\beta\nu}\Gamma^{\rho}_{\sigma\mu}}T_{\alpha\rho}$$

$$R_{\mu\nu\alpha}{}^{\sigma} = \underset{①}{\partial_\nu\Gamma^{\sigma}_{\mu\alpha}} - \underset{②}{\partial_\mu\Gamma^{\sigma}_{\nu\alpha}} + \underset{③}{\Gamma^{\rho}_{\mu\alpha}\Gamma^{\sigma}_{\nu\rho}} - \underset{④}{\Gamma^{\rho}_{\nu\alpha}\Gamma^{\sigma}_{\mu\rho}}$$

$$R_{\mu\nu\beta}{}^{\sigma} = \underset{①'}{\partial_\nu\Gamma^{\sigma}_{\mu\beta}} - \underset{②'}{\partial_\mu\Gamma^{\sigma}_{\nu\beta}} + \underset{③'}{\Gamma^{\rho}_{\mu\beta}\Gamma^{\sigma}_{\nu\rho}} - \underset{④'}{\Gamma^{\rho}_{\nu\beta}\Gamma^{\sigma}_{\mu\rho}}$$

$$= R_{\mu\nu\alpha}{}^{\sigma}T_{\sigma\beta} + R_{\mu\nu\beta}{}^{\sigma}T_{\alpha\sigma}$$

従って $T_{\alpha\beta} = \nabla_\alpha\omega_\beta$ とおくと、

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)(\nabla_\alpha\omega_\beta) = R_{\mu\nu\alpha}{}^{\rho}\nabla_\rho\omega_\beta + R_{\mu\nu\beta}{}^{\rho}\nabla_\alpha\omega_\rho$$

ビアンキの恒等式

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(\nabla_\alpha \omega_\beta) = R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma \omega_\beta + R_{\mu\nu\beta}{}^\sigma \nabla_\alpha \omega_\sigma$$

以下の場合で和をとる

μ	$\mu\nu - \mu\nu\alpha$	$+1$	$\nabla_\mu(\nabla_\nu\nabla_\alpha - \nabla_\alpha\nabla_\nu)\omega_\beta = \nabla_\mu(R_{\nu\alpha\beta}{}^\sigma \omega_\sigma)$	①
	$\mu\alpha - \mu\alpha\nu$	-1	$\nabla_\mu(\nabla_\alpha\nabla_\nu - \nabla_\nu\nabla_\alpha)\omega_\beta = \nabla_\mu(R_{\alpha\nu\beta}{}^\sigma \omega_\sigma)$	②
ν	$\nu\mu - \nu\mu\alpha$	-1	$\nabla_\nu(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu)\omega_\beta = \nabla_\nu(R_{\mu\alpha\beta}{}^\sigma \omega_\sigma)$	③
	$\nu\alpha - \nu\alpha\mu$	$+1$	$\nabla_\nu(\nabla_\alpha\nabla_\mu - \nabla_\mu\nabla_\alpha)\omega_\beta = \nabla_\nu(R_{\alpha\mu\beta}{}^\sigma \omega_\sigma)$	④
α	$\alpha\mu - \alpha\mu\nu$	$+1$	$\nabla_\alpha(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\omega_\beta = \nabla_\alpha(R_{\mu\nu\beta}{}^\sigma \omega_\sigma)$	⑤
	$\alpha\nu - \alpha\nu\mu$	-1	$\nabla_\alpha(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu)\omega_\beta = \nabla_\alpha(R_{\nu\mu\beta}{}^\sigma \omega_\sigma)$	⑥

Leibnitz-rule

$$\nabla_\mu(\nabla_\nu\nabla_\alpha - \nabla_\alpha\nabla_\nu)\omega_\beta = \nabla_\mu\left(R_{\nu\alpha\beta}{}^\sigma \omega_\sigma\right) = (\nabla_\mu R_{\nu\alpha\beta}{}^\sigma)\omega_\beta + R_{\nu\alpha\beta}{}^\sigma(\nabla_\mu \omega_\sigma)$$

$$\underbrace{(\nabla_\mu\nabla_\nu}_{①-1} - \underbrace{\nabla_\nu\nabla_\mu)(\nabla_\alpha\omega_\beta)}_{④'-2} = \underbrace{R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-1} + \underbrace{R_{\mu\nu\beta}{}^\sigma \nabla_\alpha\omega_\sigma}_{Ⓑ-1}$$

$$-\underbrace{(\nabla_\mu\nabla_\alpha}_{①-2} - \underbrace{\nabla_\alpha\nabla_\mu)(\nabla_\nu\omega_\beta)}_{⑤-1} = -(\underbrace{R_{\mu\alpha\nu}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-2} + \underbrace{R_{\mu\alpha\beta}{}^\sigma \nabla_\nu\omega_\beta}_{Ⓑ-2})$$

$$-\underbrace{(\nabla_\nu\nabla_\mu}_{④'-2} - \underbrace{\nabla_\mu\nabla_\nu)(\nabla_\alpha\omega_\beta)}_{①'-1} = -(\underbrace{R_{\nu\mu\alpha}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-3} + \underbrace{R_{\nu\mu\beta}{}^\sigma \nabla_\alpha\omega_\sigma}_{Ⓑ-3})$$

$$\underbrace{(\nabla_\nu\nabla_\alpha}_{④-1} - \underbrace{\nabla_\alpha\nabla_\nu)(\nabla_\mu\omega_\beta)}_{⑤-2} = \underbrace{R_{\nu\alpha\mu}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-4} + \underbrace{R_{\nu\alpha\beta}{}^\sigma \nabla_\mu\omega_\sigma}_{Ⓑ-4}$$

$$\underbrace{(\nabla_\alpha\nabla_\mu}_{⑤'-1} - \underbrace{\nabla_\mu\nabla_\alpha)(\nabla_\nu\omega_\beta)}_{①'-2} = \underbrace{R_{\alpha\mu\nu}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-5} + \underbrace{R_{\alpha\mu\beta}{}^\sigma \nabla_\alpha\omega_\sigma}_{Ⓑ-5}$$

$$-\underbrace{(\nabla_\alpha\nabla_\nu}_{⑤'-2} - \underbrace{\nabla_\nu\nabla_\alpha)(\nabla_\mu\omega_\beta)}_{④-2} = -(\underbrace{R_{\alpha\nu\mu}{}^\sigma \nabla_\sigma\omega_\beta}_{Ⓐ-6} + \underbrace{R_{\alpha\nu\beta}{}^\sigma \nabla_\alpha\omega_\sigma}_{Ⓑ-6})$$

$$(R_{\mu\nu\alpha}{}^{\rho} + R_{\mu\alpha\nu}{}^{\rho} - R_{\nu\mu\alpha}{}^{\rho} + R_{\nu\alpha\mu}{}^{\rho} + R_{\alpha\mu\nu}{}^{\rho} - R_{\alpha\nu\mu}{}^{\rho})\nabla_{\rho}\omega_{\beta}$$

$$= (\nabla_{\mu} R_{\nu\alpha\beta}{}^{\rho} - \nabla_{\mu} R_{\alpha\nu\beta}{}^{\rho} - \nabla_{\nu} R_{\mu\alpha\beta}{}^{\rho} + \nabla_{\nu} R_{\alpha\mu\beta}{}^{\rho} + \nabla_{\alpha} R_{\mu\nu\beta}{}^{\rho} - \nabla_{\alpha} R_{\nu\mu\beta}{}^{\rho})\omega_{\rho}$$

ここで

$R_{\mu\nu\alpha}{}^{\rho} + R_{\nu\alpha\mu}{}^{\rho} + R_{\alpha\mu\nu}{}^{\rho} = 0$,

$$\because \left.\begin{aligned} R_{\mu\nu\alpha}{}^{\rho} &= \partial_{\nu}\Gamma^{\rho}_{\mu\alpha} - \partial_{\mu}\Gamma^{\rho}_{\nu\alpha} + \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\sigma}_{\nu\alpha}\Gamma^{\rho}_{\mu\sigma} \\ R_{\nu\alpha\mu}{}^{\rho} &= \partial_{\alpha}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\alpha\mu} + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\rho}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\mu}\Gamma^{\rho}_{\nu\sigma} \\ R_{\alpha\mu\nu}{}^{\rho} &= \partial_{\mu}\Gamma^{\rho}_{\alpha\nu} - \partial_{\alpha}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\alpha\sigma} \end{aligned}\right\} \text{を満たす.}$$

$$(\nabla_{\mu} R_{\nu\alpha\beta}{}^{\rho} - \nabla_{\mu} R_{\alpha\nu\beta}{}^{\rho} - \nabla_{\nu} R_{\mu\alpha\beta}{}^{\rho} + \nabla_{\nu} R_{\alpha\mu\beta}{}^{\rho} + \nabla_{\alpha} R_{\mu\nu\beta}{}^{\rho} - \nabla_{\alpha} R_{\nu\mu\beta}{}^{\rho})\omega_{\rho} = 0$$

↳ 実は $\nabla_{\mu} R_{\nu\alpha\beta}{}^{\rho} + \nabla_{\nu} R_{\alpha\mu\beta}{}^{\rho} + \nabla_{\alpha} R_{\mu\nu\beta}{}^{\rho} = 0$

Remark Jacobi律.

$$[\nabla_{\mu}, [\nabla_{\nu}, \nabla_{\alpha}]] + [\nabla_{\nu}, [\nabla_{\alpha}, \nabla_{\mu}]] + [\nabla_{\alpha}, [\nabla_{\mu}, \nabla_{\nu}]] = 0$$

からも確認可能.

縮約されたビアンキ恒等式.

$$\nabla^{\sigma}\left(R_{\sigma\mu} - \frac{1}{2}g_{\sigma\mu}R\right) = 0 \qquad R_{\mu\nu} = g^{\alpha\beta}R_{\mu\alpha\nu\beta},\ R = g^{\mu\nu}R_{\mu\nu}$$

$\because$

$$g^{\nu\beta}g^{\alpha\rho}(\nabla_{\mu}R_{\nu\alpha\beta\rho} + \nabla_{\nu}R_{\alpha\mu\beta\rho} + \nabla_{\alpha}R_{\mu\nu\beta\rho}) = 0$$

↓ $g^{\nu\beta}g^{\alpha\beta}\nabla_{\mu}R_{\nu\alpha\beta\rho} = \nabla_{\mu}g^{\nu\beta}g^{\alpha\beta}R_{\nu\alpha\beta\rho}$ などが成り立つ

$$\nabla_{\mu}g^{\nu\beta}g^{\alpha\rho}R_{\nu\alpha\beta\rho} = \nabla_{\mu}g^{\nu\beta}R_{\nu\beta} = \nabla_{\mu}R = \nabla^{\sigma}g_{\sigma\mu}R$$

$$\begin{aligned} \nabla_{\nu}g^{\nu\beta}g^{\alpha\rho}R_{\alpha\mu\beta\rho} &= -\nabla_{\nu}g^{\nu\beta}g^{\alpha\rho}R_{\mu\alpha\beta\rho} \\ &= -\nabla_{\nu}g^{\nu\beta}g^{\rho\alpha}R_{\beta\rho\mu\alpha} = -\nabla_{\nu}g^{\nu\beta}R_{\beta\mu} = -\nabla^{\beta}R_{\beta\mu} \end{aligned}$$

$$\begin{aligned} \nabla_{\alpha}g^{\nu\beta}g^{\alpha\rho}R_{\mu\nu\beta\rho} &= -\nabla_{\alpha}g^{\alpha\rho}g^{\nu\beta}R_{\mu\nu\rho\beta} \\ &= -\nabla_{\alpha}g^{\alpha\rho}g^{\beta\nu}R_{\rho\beta\mu\nu} = -\nabla_{\alpha}g^{\alpha\rho}R_{\rho\mu} = -\nabla^{\rho}R_{\rho\mu} \end{aligned}$$

$$g^{\nu\beta} g^{\alpha\rho} (\nabla_\mu R_{\nu\alpha\beta\rho} + \nabla_\nu R_{\alpha\mu\beta\rho} + \nabla_\alpha R_{\mu\nu\beta\rho}) = 0$$

$$\Leftrightarrow \nabla^\sigma g_{\sigma\mu} R - \nabla^\beta R_{\beta\mu} - \nabla^\rho R_{\rho\mu} = 0$$

$$\Leftrightarrow \nabla^\sigma g_{\sigma\mu} R - 2\nabla^\sigma R_{\sigma\mu} = 0 \Leftrightarrow \nabla^\sigma \left(R_{\sigma\mu} - \frac{1}{2} g_{\sigma\mu} R \right) = 0 ,$$