

微分・差分・ q -差分

微分

$$f(x+h) = f(x) + f'(x)h + o(h)$$

$f \in C[[x]]$ の場合

$$f(x+h) \equiv f(x) + f'(x)\varepsilon \pmod{\varepsilon^2}$$

差分

$$f(x+1) = f(x) + (\Delta + f)(x)$$

$$f(x) = f(x-1) + (\Delta - f)(x)$$

q -差分

$$f(qx) = f(x) + (\delta_q f)(x) \cdot (q-1)x \quad (q > 1)$$

$$f(x) = f(qx) + (\delta_q f)(x) \cdot (1-q)x \quad (q < 1)$$

x^n の微分と q -差分

$$(x+\varepsilon)^n \equiv x^n + nx^{n-1}\varepsilon \pmod{\varepsilon^2} \quad \text{if } (x^n)' = nx^{n-1}$$

$$x^n = (qx)^n + \frac{1-q^n}{1-q} x^{n-1} \cdot x \quad \text{if } \delta_q(x^n) = \frac{1-q^n}{1-q} x^n$$

q -pochhammer 記号

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)$$

$$\boxed{\frac{1-q^n}{1-q} x^n \rightarrow nx^{n-1}} \quad (q \rightarrow 1)$$

超幾何関数

$$x^{-1} \theta(\theta+r-1) F = (\theta+\alpha)(\theta+\beta) F$$

$$F(x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(r)_n n!} x^n$$

$$x^{-1} [\theta]_q [\theta+r-1]_q \varphi = [\theta+\alpha]_q [\theta+\beta]_q \varphi$$

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{(q^{2\alpha}; q^2)_n (q^{2\beta}; q^2)_n}{(q^{2r}; q^2)_n (q^2; q^2)_n} x^n$$

Aomoto-Gelfand の 超幾何微分方程式系

$$\Phi : \text{Mat}(2,4; \mathbb{C}) \rightarrow \mathbb{C}$$

$\begin{matrix} 2 \times 4 \times 3! \times 2 \\ \text{全2行4列式} \neq 0 \end{matrix} \quad \text{Grassmann 多様体 } G_{2,4} \text{ 上の関数}$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix} \mapsto \bar{\Phi} \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix}$$

$$\sum_{j=1}^4 t_{ij} \frac{\partial}{\partial t_{kj}} \bar{\Phi} = -\delta_{ik} \bar{\Phi} \quad \leftarrow \bar{\Phi} \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix}$$

$$\sum_{i=1}^2 t_{ij} \frac{\partial}{\partial t_{ij}} \bar{\Phi} = \lambda_j \bar{\Phi} \quad = \Delta_{12}^{-1} \bar{\Phi} \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix}$$

$$\left(\frac{\partial}{\partial t_{ij}} \frac{\partial}{\partial t_{kl}} - \frac{\partial}{\partial t_{kj}} \frac{\partial}{\partial t_{il}} \right) \bar{\Phi} = 0$$

$$x_{13} := -\Delta_{12}^{-1} \Delta_{13} \quad x_{14} := -\Delta_{12}^{-1} \Delta_{24} \\ x_{23} := \Delta_{12}^{-1} \Delta_{13} \quad x_{24} := \Delta_{12}^{-1} \Delta_{14}$$

$$\psi \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} := \bar{\Phi} \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} \quad \text{とし} \text{て書換}.$$

$$\left(x_{13} \frac{\partial}{\partial x_{13}} + x_{14} \frac{\partial}{\partial x_{14}} \right) \psi = -(\lambda_1 + 1) \psi$$

$$\left(x_{23} \frac{\partial}{\partial x_{23}} + x_{24} \frac{\partial}{\partial x_{24}} \right) \psi = -(\lambda_2 + 1) \psi$$

$$f(t) := \psi \begin{pmatrix} tx_{13} & x_{14} \\ tx_{23} & x_{24} \end{pmatrix}$$

$$\text{とし} \text{て} \quad tf'(t) = -(\lambda_2 + 1) f(t)$$

Euler

$$\left(x_{13} \frac{\partial}{\partial x_{13}} + x_{23} \frac{\partial}{\partial x_{23}} \right) \psi = \lambda_3 \psi \quad \text{--- ③}$$

$$\left(x_{14} \frac{\partial}{\partial x_{14}} + x_{24} \frac{\partial}{\partial x_{24}} \right) \psi = \lambda_4 \psi \quad \text{--- ④}$$

$$\left(\frac{\partial}{\partial x_{13}} \frac{\partial}{\partial x_{24}} - \frac{\partial}{\partial x_{23}} \frac{\partial}{\partial x_{14}} \right) \psi = 0 \quad \text{--- ⑤}$$

$$\textcircled{A}, \textcircled{B}, \textcircled{C} \text{ かつ } f(x) = \Psi \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} \cdot \left(= \Phi \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{pmatrix} \text{ とかくと} \right)$$

$$\Psi \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} = x_{13}^{\lambda_2} x_{14}^{\lambda_4} \left(\frac{x_{23}}{x_{13}} \right)^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right)$$

$$= x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right)$$

$$\frac{\partial}{\partial x_{13}} \frac{\partial}{\partial x_{24}} \left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \right)$$

$$= \frac{\partial}{\partial x_{13}} \left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \cdot \frac{dx}{dx_{24}} \right)$$

$$= \frac{\partial}{\partial x_{13}} \left(x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \right)$$

$$= (2 + \lambda_2 + \lambda_3) x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right)$$

$$+ x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{d^2f}{dx^2} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \frac{x_{24}}{x_{14} x_{23}}$$

$$= (2 + \lambda_2 + \lambda_3) x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx}(x)$$

$$+ x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} x \frac{d^2f}{dx^2}(x)$$

$$\frac{\partial}{\partial x_{23}} \frac{\partial}{\partial x_{14}} \left(x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \right)$$

$$= \frac{\partial}{\partial x_{23}} \left(\lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) + x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4} x_{23}^{-\lambda_2-1} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \left(-\frac{x_{13} x_{24}}{x_{14}^2 x_{23}} \right) \right)$$

$$= \frac{\partial}{\partial x_{23}} \left(\lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-1} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) - x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2} x_{24} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \right)$$

$$= -\lambda_4(\lambda_2+1) x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} f \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) + \lambda_4 x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right)(-x)$$

$$+ (\lambda_2+2) x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-3} x_{24} \frac{df}{dx} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right)$$

$$x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-1} x_{23}^{-\lambda_2-2} \frac{df}{dx}(-x)$$

$$- x_{13}^{2+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2} x_{24} \cdot \frac{d^2f}{dx^2} \left(\frac{x_{13} x_{24}}{x_{14} x_{23}} \right) \left(-\frac{x_{13} x_{24}}{x_{14} x_{23}^2} \right)$$

$$(LHS) - (RHS) = x_{13}^{1+\lambda_2+\lambda_3} x_{14}^{\lambda_4-2} x_{23}^{-\lambda_2-2} \left(x(1-x) \frac{d^2f}{dx^2} + (2+\lambda_2+\lambda_3 - (\lambda_2+2-\lambda_4)x) \frac{df}{dx} - \lambda_4(\lambda_2+1)f \right) = 0$$

Lie 代数の幾何と代数

$\mathfrak{gl}(4)$: 4次行列全体 ($U(\mathfrak{gl}(4))$)

幾何的意味

$GL(4)$: 行列式 $\neq 0$ の 4 次行列

$$\{ X \in \text{Mat}(4) \mid \exp tX \in GL(4) \}$$

$$\begin{aligned} & \xrightarrow{\text{Lie群 } GL(2)} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ & \cong (x_{11}, x_{12}, x_{21}, x_{22}) \\ & \xrightarrow{\text{代数群 } GL(2)} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ & x_{11}x_{22} - x_{12}x_{21} \neq 0 \end{aligned}$$

$$\det(\exp tX) = \exp t(\text{Tr } X) + \dots$$

$$\text{det}(\exp tX) \neq 0$$

$$\psi = \Phi \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} \quad (\text{これは } g \in GL(4) \text{ の作用})$$

$$g\psi = (\Phi g) \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} := \Phi \left(\left(\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} g \right) \right)$$

LCZ

$$X\psi \in \Phi \left(\left(\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} (I + hX + o(h)) \right) \right) \text{ の } h \text{ の 1 次の 並とすると..}$$

③

$$E_{33}\psi = \lambda_3 \psi$$

$$f(\lambda) = \det(\exp tX)$$

$$f'(t) = \frac{d}{dt} f(t) \Big|_{t=0}$$

④

$$E_{44}\psi = \lambda_4 \psi$$

$$f'(0) = \text{Tr } X$$

⑤

$$((E_{33} + 1)E_{44} - E_{43}E_{34})\psi = 0$$

$$f'(z)$$

$$f'(z) = \frac{d}{dz} f(z) \Big|_{z=0}$$

$gl(4)$ の代数 (Lie 代数)

generator E_{ii} ($i=1, 2, 3, 4$), e_k , f_k ($k=1, 2, 3$)

$$E_{k,k+1} \quad E_{k+1,k}$$

relation $h = \sum c_i h_i$, bilinear form $\alpha_k = E_{kk} - E_{k+1,k+1}$

$$[h, e_k] = \langle \alpha_k, h \rangle e_k$$

$$[h, f_k] = -\langle \alpha_k, h \rangle f_k$$

$$[e_i, f_j] = \delta_{ij} (E_i - E_{i+1})$$

$U_q(gl(4))$

generator : $q^{\pm E_i}$ ($i=1, 2, 3, 4$), e_k , f_k ($k=1, 2, 3$)

$$\text{relation} : q^{h_i} e_k = q^{\langle \alpha_k, h_i \rangle} e_k q^{h_i}$$

$$q^{h_i} f_k = q^{-\langle \alpha_k, h_i \rangle} f_k q^{h_i}$$

$$[e_i, f_k] = \delta_{ik} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}$$

使用回数

(cf) $U_q(sl(4))$

generator q^{α_k} , e_k , f_k ($k=1, 2, 3$)

$$\text{relation} : q^{\alpha_k} e_k = q^{\langle \alpha_k, \alpha_k \rangle} e_k q^{\alpha_k}$$

$$\begin{matrix} (\alpha_k, \alpha_k) \\ (\alpha_k, \alpha_\ell) \end{matrix} \leftrightarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Cartan matrix