宿題 1

$q(\boldsymbol{u}_j)$ について次の比例式が成立する。

$$q(\boldsymbol{u}_j) \propto p(\boldsymbol{u}_j|\boldsymbol{\Lambda}_u)\exp\left\{\sum_{j,\ i\in\mathscr{O}}\int q(\boldsymbol{v}_i)\log p(r_{j,\ i}|\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{v}_i,\ \sigma^2)\,\mathrm{d}\boldsymbol{v}_i\right\} \tag{1}$$

ここで，$p(\boldsymbol{u}_j|\boldsymbol{\Lambda}_u)$ について，

$$p(\boldsymbol{u}_j|\boldsymbol{\Lambda}_u) \propto \exp\left(-\frac{1}{2}\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{\Lambda}_u\boldsymbol{u}_j\right) \tag{2}$$

が成立する。また，

$$\begin{aligned}
l_{u,\ i,\ j} &\equiv \int q(\boldsymbol{v}_i)\log p(r_{j,\ i}|\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{v}_i,\ \sigma^2)\,\mathrm{d}\boldsymbol{v}_i & (3)\\
&= -\frac{1}{2\sigma^2}\int q(\boldsymbol{v}_i)(r_{j,\ i}-\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{v}_i)^2\,\mathrm{d}\boldsymbol{v}_i\\
&= -\frac{1}{2\sigma^2}\int q(\boldsymbol{v}_i)\left(r_{j,\ i}^2-2r_{j,\ i}\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{v}_i+\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{u}_j\right)\mathrm{d}\boldsymbol{v}_i\\
&= -\frac{1}{2\sigma^2}\left(r_{j,\ i}^2-2r_{j,\ i}\boldsymbol{u}_j^{\mathrm{T}}\mathbb{E}_{q(\boldsymbol{v}_i)}[\boldsymbol{v}_i]+\boldsymbol{u}_j^{\mathrm{T}}\mathbb{E}_{q(\boldsymbol{v}_i)}\left[\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\right]\boldsymbol{u}_j\right) & (4)
\end{aligned}$$

となる。式 (1)，(2)，(4) から，

$$\begin{aligned}
q(\boldsymbol{u}_j) &\propto \exp\left(-\frac{1}{2}\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{\Lambda}_u\boldsymbol{u}_j\right)\exp\left\{\sum_{j,\ i\in\mathscr{O}} l_{u,\ i,\ j}\right\}\\
&= \exp\left(-\frac{1}{2}\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{\Lambda}_u\boldsymbol{u}_j\right)\exp\left\{\sum_{j,\ i\in\mathscr{O}}-\frac{1}{2\sigma^2}\left(r_{j,\ i}^2-2r_{j,\ i}\boldsymbol{u}_j^{\mathrm{T}}\mathbb{E}_{q(\boldsymbol{v}_i)}[\boldsymbol{v}_i]+\boldsymbol{u}_j^{\mathrm{T}}\mathbb{E}_{q(\boldsymbol{v}_i)}\left[\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\right]\boldsymbol{u}_j\right)\right\}\\
&= \exp\left\{-\frac{1}{2\sigma^2}\left(-2\boldsymbol{u}_j\left(\sum_{j,\ i\in\mathscr{O}} r_{j,\ i}\mathbb{E}_{q(\boldsymbol{v}_i)}[\boldsymbol{v}_i]\right)+\boldsymbol{u}_j^{\mathrm{T}}\left(\sum_{j,\ i\in\mathscr{O}}\mathbb{E}_{q(\boldsymbol{v}_i)}\left[\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\right]+\sigma^2\boldsymbol{\Lambda}_u\right)\boldsymbol{u}_j\right)\right\} & (5)
\end{aligned}$$

いま，多次元正規分布に従う確率変数 $\boldsymbol{x}$ について次が成立する。

$$q(\boldsymbol{x}) = \mathscr{N}(\boldsymbol{x}|\boldsymbol{\mu},\ \boldsymbol{\Sigma}) \propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{x}-2\boldsymbol{x}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)\right\} \tag{6}$$

これと式 (5) を比べると，

$$q(\boldsymbol{u}_j) = \mathscr{N}(\boldsymbol{u}_j|\boldsymbol{\mu}_{u,\ j},\ \boldsymbol{V}_{u,\ j}) \tag{7}$$

$$\boldsymbol{V}_{u,\ j}^{-1}\boldsymbol{\mu}_{u,\ j} = \frac{1}{\sigma^2}\sum_{j,\ i\in\mathscr{O}} r_{j,\ i}\mathbb{E}_{q(\boldsymbol{v}_i)}[\boldsymbol{v}_i] \tag{8}$$

$$\boldsymbol{V}_{u,\ j}^{-1} = \frac{1}{\sigma^2}\left(\sum_{j,\ i\in\mathscr{O}}\mathbb{E}_{q(\boldsymbol{v}_i)}\left[\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\right]+\sigma^2\boldsymbol{\Lambda}_u\right) \tag{9}$$

となることがわかる。これを整理すると，

$$q(\boldsymbol{u}_j) = \mathcal{N}(\boldsymbol{u}_j | \boldsymbol{\mu}_{u,\ j},\ \boldsymbol{V}_{u,\ j}) \tag{10}$$

$$\boldsymbol{\mu}_{u,\ j} = \frac{1}{\sigma^2} \boldsymbol{V}_{u,\ j} \sum_{j,\ i \in \mathcal{O}} r_{j,\ i} \mathbb{E}_{q(\boldsymbol{v}_i)}[\boldsymbol{v}_i] \tag{11}$$

$$\boldsymbol{V}_{u,\ j} = \sigma^2 \left(\sum_{j,\ i \in \mathcal{O}} \mathbb{E}_{q(\boldsymbol{v}_i)} \left[\boldsymbol{v}_i \boldsymbol{v}_i^{\mathrm{T}} \right] + \sigma^2 \boldsymbol{\Lambda}_u \right)^{-1} \tag{12}$$

となる。

同様に，$\boldsymbol{v}_i$ について，

$$q(\boldsymbol{v}_i) \propto p(\boldsymbol{v}_i | \boldsymbol{\Lambda}_v) \exp \left\{ \sum_{j,\ i \in \mathcal{O}} \int q(\boldsymbol{u}_j) \log p(r_{j,\ i} | \boldsymbol{u}_j^{\mathrm{T}} \boldsymbol{v}_i,\ \sigma^2) \, \mathrm{d}\boldsymbol{u}_j \right\} \tag{13}$$

$$\begin{aligned} &= \exp\left(-\frac{1}{2} \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{\Lambda}_v \boldsymbol{v}_i \right) \exp\left\{ \sum_{j,\ i \in \mathcal{O}} -\frac{1}{2\sigma^2} \left(r_{j,\ i}^2 - 2 r_{j,\ i} \boldsymbol{v}_i^{\mathrm{T}} \mathbb{E}_{q(\boldsymbol{u}_j)}[\boldsymbol{u}_j] + \boldsymbol{v}_i^{\mathrm{T}} \mathbb{E}_{q(\boldsymbol{u}_j)} \left[\boldsymbol{u}_j \boldsymbol{u}_j^{\mathrm{T}} \right] \boldsymbol{v}_i \right) \right\} \\ &= \exp\left\{ -\frac{1}{2\sigma^2} \left(-2 \boldsymbol{v}_i \left(\sum_{j,\ i \in \mathcal{O}} r_{j,\ i} \mathbb{E}_{q(\boldsymbol{u}_j)}[\boldsymbol{u}_j] \right) + \boldsymbol{v}_i^{\mathrm{T}} \left(\sum_{j,\ i \in \mathcal{O}} \mathbb{E}_{q(\boldsymbol{u}_j)} \left[\boldsymbol{u}_j \boldsymbol{u}_j^{\mathrm{T}} \right] + \sigma^2 \boldsymbol{\Lambda}_v \right) \boldsymbol{v}_i \right) \right\} \end{aligned} \tag{14}$$

となることから，

$$q(\boldsymbol{v}_i) = \mathcal{N}(\boldsymbol{v}_i | \boldsymbol{\mu}_{v,\ i},\ \boldsymbol{V}_{v,\ i}) \tag{15}$$

$$\boldsymbol{V}_{v,\ i}^{-1} \boldsymbol{\mu}_{v,\ i} = \frac{1}{\sigma^2} \sum_{j,\ i \in \mathcal{O}} r_{j,\ i} \mathbb{E}_{q(\boldsymbol{u}_j)}[\boldsymbol{u}_j] \tag{16}$$

$$\boldsymbol{V}_{v,\ i}^{-1} = \frac{1}{\sigma^2} \left(\sum_{j,\ i \in \mathcal{O}} \mathbb{E}_{q(\boldsymbol{u}_j)} \left[\boldsymbol{u}_j \boldsymbol{u}_j^{\mathrm{T}} \right] + \sigma^2 \boldsymbol{\Lambda}_v \right) \tag{17}$$

を得て，結局，

$$q(\boldsymbol{v}_i) = \mathcal{N}(\boldsymbol{v}_i | \boldsymbol{\mu}_{v,\ i},\ \boldsymbol{V}_{v,\ i}) \tag{18}$$

$$\boldsymbol{\mu}_{v,\ i} = \frac{1}{\sigma^2} \boldsymbol{V}_{v,\ i} \sum_{j,\ i \in \mathcal{O}} r_{j,\ i} \mathbb{E}_{q(\boldsymbol{u}_j)}[\boldsymbol{u}_j] \tag{19}$$

$$\boldsymbol{V}_{v,\ i} = \sigma^2 \left(\sum_{j,\ i \in \mathcal{O}} \mathbb{E}_{q(\boldsymbol{u}_j)} \left[\boldsymbol{u}_j \boldsymbol{u}_j^{\mathrm{T}} \right] + \sigma^2 \boldsymbol{\Lambda}_v \right)^{-1} \tag{20}$$

を得る。