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# Parallel Numerics

Exercise 6: Domain Decomposition

## 1 Domain Decomposition

i) Name the two main types of domain decomposition approaches.
Overlapping and Non-overlapping domain decomposition.

ii) Which direct solver for the decomposed system do you know (assume that our problem is linear)?
One can use LU-decomposition for the solvers in the subdomains and the Schur-complement, as shown here.

Alternatively it would also be possible to use QR-decomposition.

iii) Give the Schur-complement and the right-hand-side for the following system:

$$\begin{pmatrix} A & 0 & C \\ 0 & B & D \\ C^T & D^T & E \end{pmatrix} \begin{pmatrix} u_A \\ u_B \\ u_E \end{pmatrix} = \begin{pmatrix} b_A \\ b_B \\ b_E \end{pmatrix}$$

From the first two line-blocks, we get:

$$u_A = A^{-1}(b_A - Cu_E), \quad u_B = B^{-1}(b_B - D^{-1}u_E)$$

Inserting this into the third line-block gives:

$$\begin{aligned} C^T A^{-1}(b_A - Cu_E) + D^T B^{-1}(b_B - Du_E) + Eu_E = b_E, \\ \underbrace{(E - C^T A^{-1} C - D^T B^{-1} D)u_E}_{\text{Schur-complement}} = \underbrace{b_E - C^T A^{-1} b_A - D^T B^{-1} b_B}_{\text{right-hand side}}. \end{aligned}$$

iv) List the steps of the algorithm solving the decomposed system based on the Schur-complement and LU-decomposition.

a) LU-decomposition of $A$ and $B$ (can be done simultaneously)

$$A = L_A U_A, \quad B = L_B U_B.$$

b) Compute the Schur-complement:

   i. Compute $\bar{A} = A^{-1}C$ and $\bar{B} = B^{-1}D$ via forward and backward substitution:

$$\begin{aligned}&\text{Solve } \; L_A Y = C \; \text{ and } \; L_B Z = D,\\ &\text{solve } \; U_A \bar{A} = Y \; \text{ and } U_B \bar{B} = Z.\end{aligned}$$

   (can be done simultaneously for all columns of $Y$ and $Z$ or $\bar{A}$ and $\bar{B}$, respectively).

   ii. Compute the Schur-complement using BLAS-3:

$$S = E - C^T \bar{A} - D^T \bar{B}.$$

c) LU-decomposition of the Schur-complement

$$S = L_S U_S.$$

d) Compute the right-hand side of the Schur system via forward and backward substitution:

$$\begin{aligned}&\text{Solve } \; L_A y = b_A \; \text{ and } \; L_B z = b_B,\\ &\text{solve } \; U_A \bar{b}_A = y \; \text{ and } U_B \bar{b}_B = z.\end{aligned}$$

   Set (BLAS-2)

$$b_S = b_E - C^T \bar{b}_A - D^T \bar{b}_B.$$

e) Solve the Schur-system via forward and backward substitution:

$$\begin{aligned}&\text{Solve } \; L_S y = b_S,\\ &\text{solve } \; U_S u_E = y.\end{aligned}$$

f) Compute (simultaneously) the right-hand sides for the systems for $u_A$ and $u_B$ using BLAS-2:

$$\tilde{b}_A = b_A - C u_E, \quad \tilde{b}_B = b_B - D u_E.$$

g) Solve (simultaneously) the systems for $u_A$ and $u_B$:

$$\begin{aligned}&\text{Solve } \; L_A y = \tilde{b}_A \; \text{ and } \; L_B z = \tilde{b}_B,\\ &\text{solve } \; U_A u_A = y \; \text{ and } \; U_B u_B = z.\end{aligned}$$

## 2 Robin-Robin Coupling for Non-Overlapping DD

In the lecture, we have learned that solving the Poisson equation

$$\begin{aligned}-\Delta u &= f && \text{in } \; \Omega,\\ u &= 0 && \text{on } \; \partial\Omega\end{aligned}$$

on $\Omega$ is equivalent to solving the decomposed system

$$\begin{aligned}
-\Delta u_1 &= f && \text{in} \quad \Omega_1,\\
u_1 &= 0 && \text{on} \quad \partial\Omega_1 \backslash \Gamma,\\
u_1 &= u_2 && \text{on} \quad \Gamma,\\[1em]
-\Delta u_2 &= f && \text{in} \quad \Omega_2,\\
u_2 &= 8 && \text{on} \quad \partial\Omega_2 \backslash \Gamma,\\
\frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} && \text{on} \quad \Gamma.
\end{aligned}$$

with two non-overlapping subdomains $\Omega_1$ and $\Omega_2$.

As an example, we have studied the 1D equation

$$u_{xx} = 0 \;\text{ in }\; ]0;8[, \quad u(0) = 0,\, u(8) = 8, \tag{1}$$

started with the initial approximations

$$u_1(x) = 0 \;\text{ in }\; \Omega_1 = ]0;4[, \quad u_2(x) = 2\,x - 8 \;\text{ in }\; \Omega_2 = ]4;8[, \tag{2}$$

and tried to solve the problem iteratively with a block-Gauss-Seidel iteration, which did not converge.

i) Show that the solution of the modified decomposed system

$$\begin{aligned}
-\Delta u_1 &= f && \text{in} \quad \Omega_1,\\
u_1 &= 0 && \text{on} \quad \partial\Omega_1 \backslash \Gamma,\\
\frac{\partial u_1}{\partial n_1} + \theta u_1 &= -\frac{\partial u_2}{\partial n_2} + \theta u_2 && \text{on} \quad \Gamma,\\[1em]
-\Delta u_2 &= f && \text{in} \quad \Omega_2,\\
u_2 &= 8 && \text{on} \quad \partial\Omega_2 \backslash \Gamma,\\
\frac{\partial u_1}{\partial n_1} - \theta u_1 &= -\frac{\partial u_2}{\partial n_2} - \theta u_2 && \text{on} \quad \Gamma.
\end{aligned}$$

with $\theta > 0$ has the same solution as the decomposed system given above.
Adding the boundary conditions in $\Omega_1$ and $\Omega_2$ at $\Gamma$ gives the Dirichlet-boundary condition from our original decomposition, subtracting the boundary conditions in $\Omega_1$ and $\Omega_2$ at $\Gamma$ gives $\theta$ times the Neumann boundary condition. With $\theta > 0$, we immediately see that both decompositions are equivalent since a solution fulfils both Dirichlet and Neumann compatibility conditions at $\Gamma$.

ii) Perform (by hand) one block-Gauss-Seidel iteration for the modified decomposed system for problem (1) with initial approximations from (2).

1st iteration:

$$
\begin{aligned}
\text{Solve } u_{1,xx} &= 0 \text{ in } ]0;4[, u_1(0)=0, u_{1,x}(4)+\theta u_1(4) = u_{2,x}(4)+\theta u_2(4) = 2\\
\Rightarrow u_1(x) &= a\,x \;\text{ and }\; a + 4a\theta = 2\\
\Rightarrow u_1(x) &= \frac{2}{1+4\theta}x,\\
\text{Solve } u_{2,xx} &= 0 \text{ in } ]4;8[, u_{2,x}(4)-\theta u_2(4) = u_{1,x}(4)-\theta u_1(4) = 0, u_2(8)=8\\
\Rightarrow u_2(x) &= a(x-8)+8 \;\text{ and }\; a - \theta\,(-4a+8) = \frac{2}{1+4\theta} - 4\theta\frac{2}{1+4\theta}\\
\Rightarrow u_2(x) &= \frac{2(1-16\theta)}{(1+4\theta)^2}(x-8)+8.
\end{aligned}
$$

iii) For which values of $\theta$ do you get convergence?
We examine iterations for perform starting from a more general initial guess for $u_1$ and $u_2$:

$$u_1(x) = a_1^i\,x, \quad u_2(x) = a_2^i\,(x-8)+8.$$

Our initial guesses fulfil the boundary conditions at $x=0$ and $x=8$ and all iterates are of this form. To show this and examine convergence properties, we perform one iteration:

$$
\begin{aligned}
u_1(x) &= a_1^{i+1}\,x \;\text{ and }\; a_1^{i+1} + 4a_1^{i+1}\theta = a_2^i - 4\theta\,a_2^i + 8\theta\\
\Rightarrow a_1^{i+1} &= \frac{1-4\theta}{1+4\theta}a_2^i + \frac{8\theta}{1+4\theta},\\
u_2(x) &= a_2^{i+1}(x-8)+8 \;\text{ and }\; a_2^{i+1} + \theta\,4a_2^{i+1} - 8\theta = a_1^{i+1} - 4\theta a_1^{i+1}\\
\Rightarrow a_2^{i+1} &= \frac{1-4\theta}{1+4\theta}a_1^{i+1} + \frac{8\theta}{1+4\theta}.
\end{aligned}
$$

$a_1 = a_2 = 1$ is the exact solution. For the errors $e_1 = a_1 - 1$ and $e_2 = a_2 - 1$, thus,

$$
\begin{aligned}
e_1 &= \frac{1-4\theta}{1+4\theta}e_2,\\
e_2 &= \frac{1-4\theta}{1+4\theta}e_1 = \left(\frac{1-4\theta}{1+4\theta}\right)^2 e_2.
\end{aligned}
$$

Thus, we get convergence if

$$\left|\frac{1-4\theta}{1+4\theta}\right| < 1 \;\Leftrightarrow\; \theta > 0.$$

**Remark:** The optimal value ist $\theta = \frac{1}{4}$. In this case, we get the exact solution in one iteration.