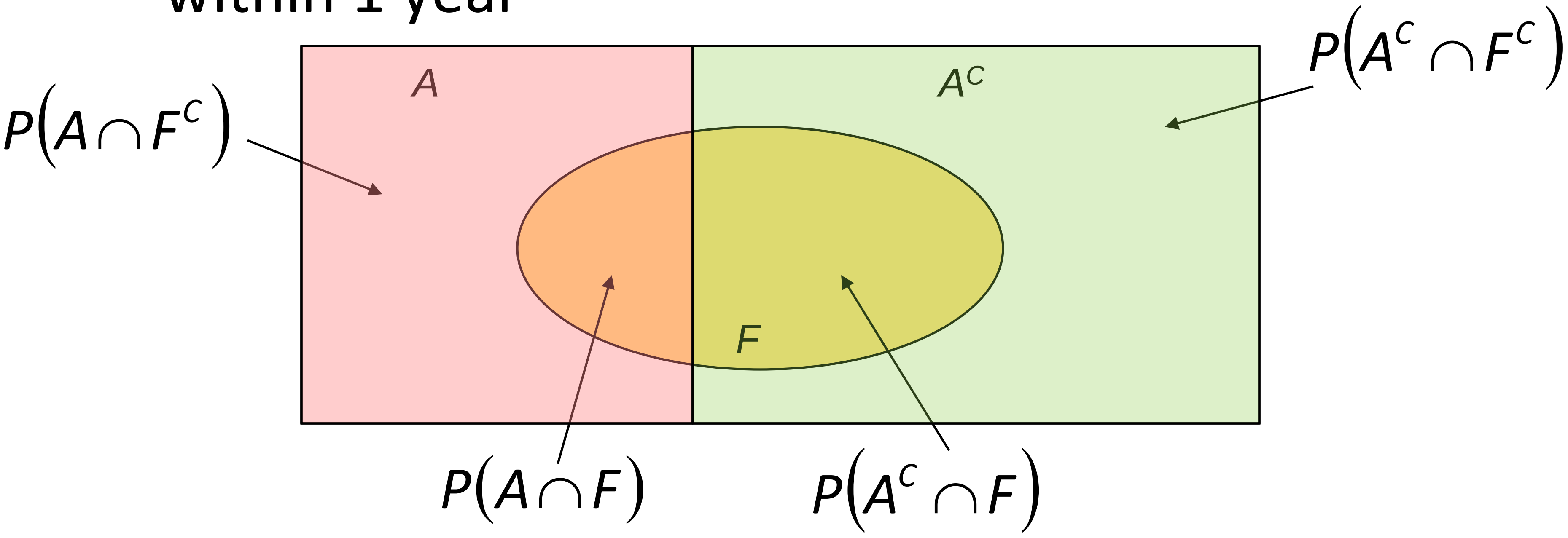
Bayes’ Theorem

Bayes' Theorem

- An insurance company divides its clients into two categories: those who are accident prone and those who are not. Statistics show there is a 40% chance an accident prone person will have an accident within 1 year whereas there is a 20% chance non-accident prone people will have an accident within the first year.
- If 30% of the population is accident prone, what is the probability that a new policyholder has an accident within 1 year?

Bayes' Theorem

- Let A be the event a person is accident prone
- Let F be the event a person has an accident within 1 year



Bayes' Theorem

- Notice that $(A \cap F)$ and $(A^c \cap F)$ are mutually exclusive events and that

$$(A \cap F) \cup (A^c \cap F) = F$$

- Therefore $P(F) = P(A \cap F) + P(A^c \cap F)$

- We need to find $P(A \cap F)$ and $P(A^c \cap F)$

- How?

Bayes' Theorem

Thus:

- $P(A \cap F) = \boxed{P(F \mid A) \cdot P(A)}$
- $P(A^c \cap F) = \underbrace{P(F \mid A^c)}\cdot\underbrace{P(A^c)}$

$P(A)$ = 0.30 since 30% of population is accident prone

$P(F|A)$ = 0.40 since if a person is accident prone, then his chance of having an accident within 1 year is 40%

$P(F|A^C)$ = 0.2 since non-accident prone people have a 20% chance of having an accident within 1 year

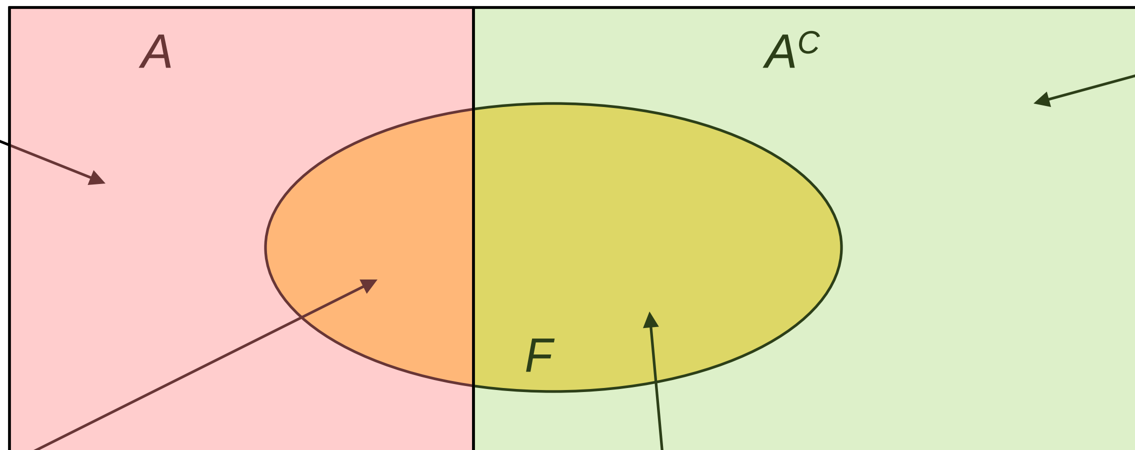
$P(A^C)$ = 1- $P(A)$ = 0.70

Bayes' Theorem

Updating our Venn Diagram

$P\left(A^c \cap F^c\right)$

$P\left(A \cap F^c\right)$



$P\left(A \cap F\right) = P\left(F \mid A\right)P\left(A\right)$

$P\left(A^c \cap F\right) = P\left(F \mid A^c\right)P\left(A^c\right)$

Notice again that $P\left(F\right) = P\left(A \cap F\right) + P\left(A^c \cap F\right)$

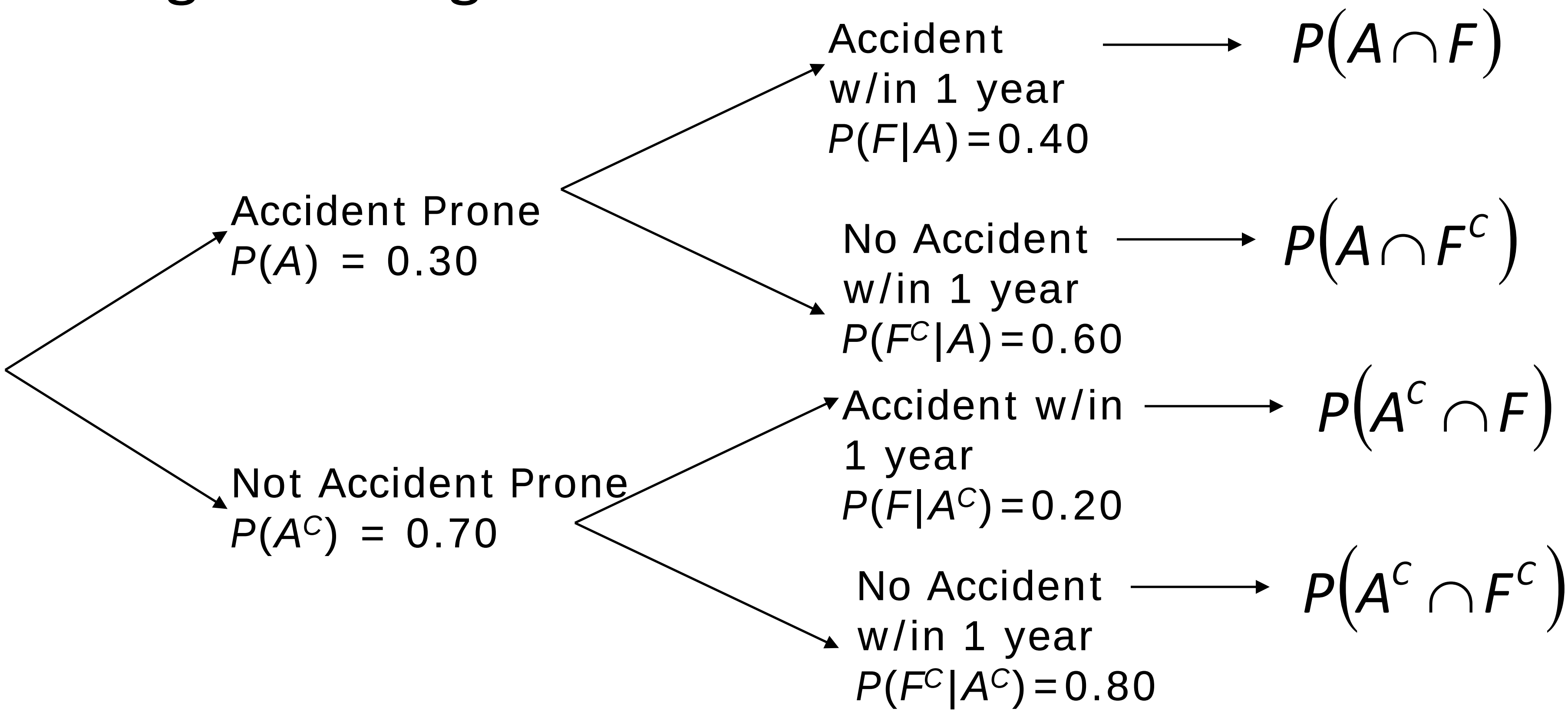
Bayes' Theorem

So the probability of having an accident within 1 year is:

$$
\begin{aligned}
P(F) &= P(A \cap F) + P(A^C \cap F) \\
&= P(F \mid A)P(A) + P(F \mid A^C)P(A^C) \\
&= 0.40 \times 0.30 + 0.20 \times 0.70 = 0.26
\end{aligned}
$$

Bayes' Theorem

Using Tree Diagrams:



Bayes' Theorem

Notice you can have an accident within 1 year by following branch *A* until *F* is reached

- The probability that *F* is reached via branch *A* is given by $P(F \mid A) \cdot P(A)$
- In other words, the probability of being accident prone and having one within 1 year is

$$P(A \cap F) = P(F \mid A) \cdot P(A)$$

Bayes' Theorem

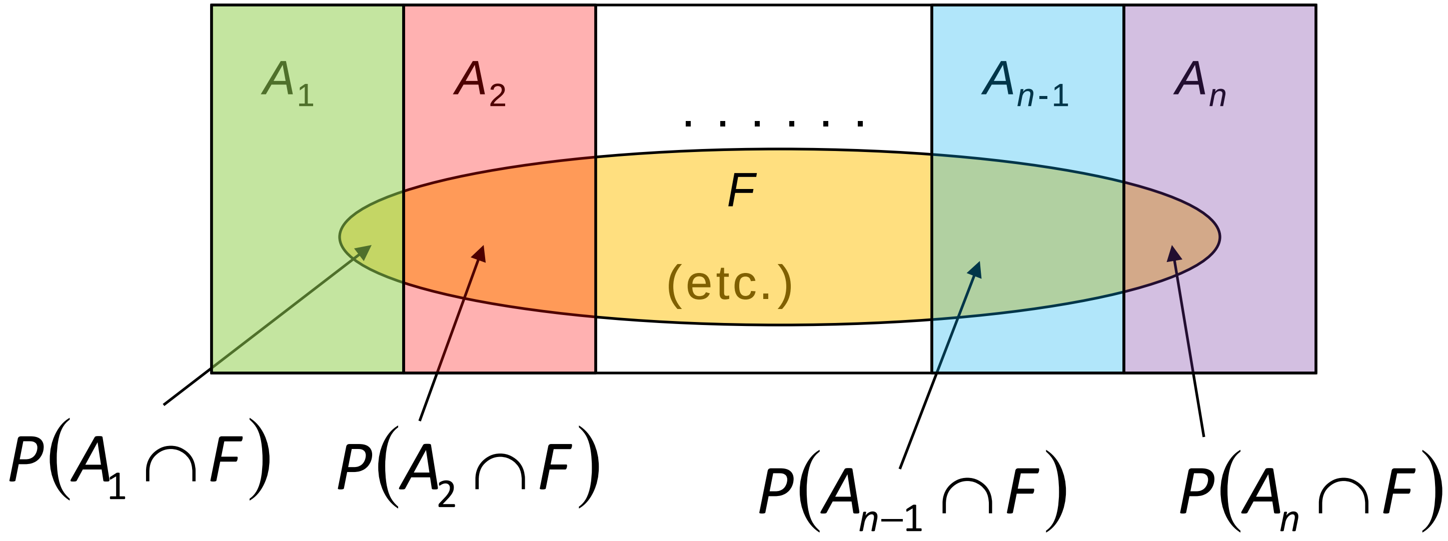
You can also have an accident within 1 year by following branch A^C until F is reached

- The probability that F is reached via branch A^C is given by $P\left(F \mid A^C\right) \cdot P\left(A^C\right)$
- In other words, the probability of NOT being accident prone and having one within 1 year is

$$P\left(A^C \cap F\right) = P\left(F \mid A^C\right) \cdot P\left(A^C\right)$$

Bayes' Theorem

- What would happen if we had partitioned our sample space over more events, say $A_1, A_2, \ldots, A_n$, all them mutually exclusive?
- Venn Diagram



Bayes' Theorem

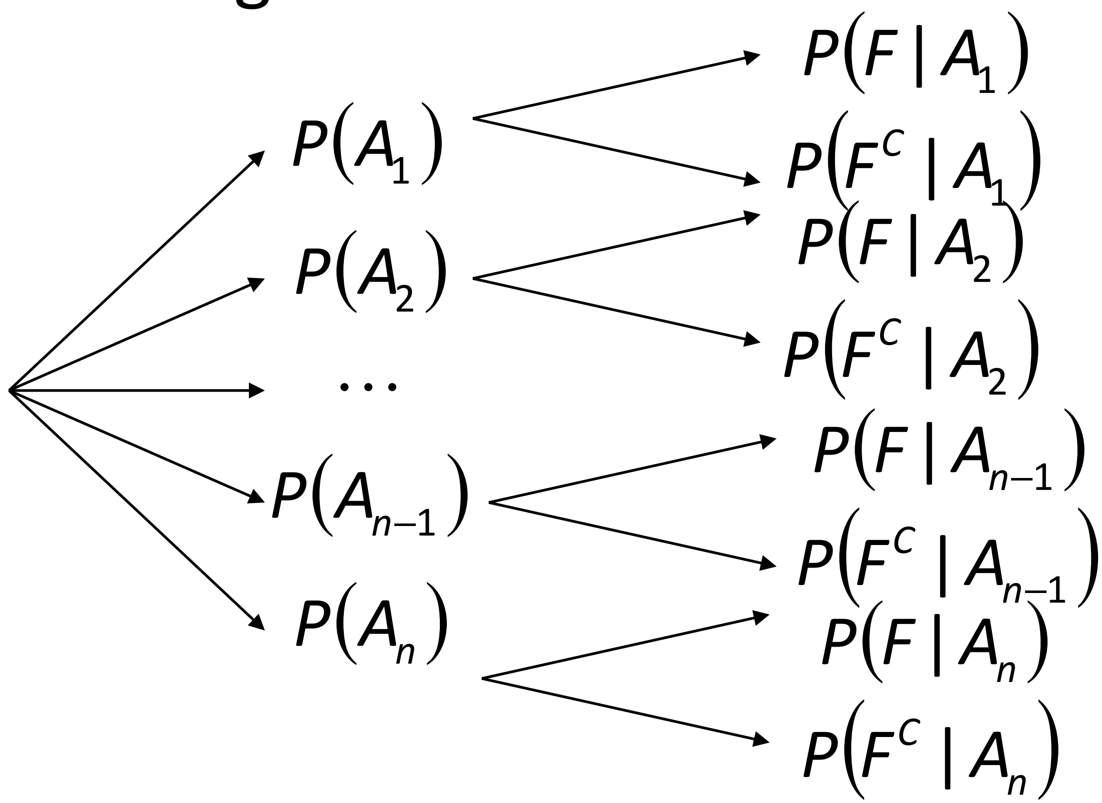
$$P\left(F\right) = P\left(A_1 \cap F\right) + P\left(A_2 \cap F\right) + \cdots + P\left(A_n \cap F\right)$$

For each $P\left(A_i \cap F\right) = P\left(F \mid A_i\right) P\left(A_i\right)$

$$\begin{aligned} P\left(F\right) &= P\left(A_1 \cap F\right) + P\left(A_2 \cap F\right) + \cdots + P\left(A_n \cap F\right) \\ &= P\left(F \mid A_1\right) P\left(A_1\right) + P\left(F \mid A_2\right) P\left(A_2\right) + \cdots + P\left(F \mid A_n\right) P\left(A_n\right) \\ &= \sum_{i=1}^{n} P\left(F \mid A_i\right) P\left(A_i\right) \end{aligned}$$

Bayes' Theorem

Tree Diagram



Bayes' Theorem

- Notice that F can be reached via $A_1, A_2, \ldots, A_n$ branches

- Multiplying across each branch tells us the probability of the intersection

- Adding up all these products gives:

$$P(F) = \sum_{i=1}^{n} P(F \mid A_i) P(A_i)$$

Bayes' Theorem

Ex: 2 (text tractor example) Suppose there are 3 assembly lines: Red, White, and Blue. Chances of a tractor not starting for each line are 6%, 11%, and 8%. We know 48% are red and 31% are blue. The rest are white. What % don't start?

Bayes' Theorem

Soln.

R: red	$P(R) = 0.48$
W: white	$P(W) = 0.21$
B: blue	$P(B) = 0.31$
N: not starting	

$P(N \mid R) = 0.06$

$P(N \mid W) = 0.11$

$P(N \mid B) = 0.08$

Bayes' Theorem

Soln.

$$\begin{aligned}P(N) &= P(N \mid R)\cdot P(R) + P(N \mid W)\cdot P(W) + P(N \mid B)\cdot P(B)\\ &= (0.06)\cdot(0.48) + (0.11)\cdot(0.21) + (0.08)\cdot(0.31)\\ &= 0.0767\end{aligned}$$

Bayes' Theorem

Main theorem:

- Ex. Suppose B_1 and B_2 partition a space and *A* is some event.

- Use $P(B_1), P(B_2), P(A \mid B_1),$ and $P(A \mid B_2)$ to determine $P(B_1 \mid A)$.

Bayes' Theorem

Recall the formulas:

$$P\left(B_1 \cap A\right) = P\left(A \mid B_1\right) \cdot P\left(B_1\right)$$

$$P\left(B_1 \cap A\right) = P\left(A \cap B_1\right) = P\left(B_1 \mid A\right) \cdot P\left(A\right) \Rightarrow P\left(B_1 \mid A\right) = \frac{P\left(B_1 \cap A\right)}{P\left(A\right)}$$

$$P\left(A\right) = P\left(B_1\right) \cdot P\left(A \mid B_1\right) + P\left(B_2\right) \cdot P\left(A \mid B_2\right)$$

So, $P\left(B_1 \mid A\right) = \dfrac{P\left(B_1 \cap A\right)}{P\left(A\right)} = \dfrac{P\left(A \mid B_1\right) \cdot P\left(B_1\right)}{P\left(B_1\right) \cdot P\left(A \mid B_1\right) + P\left(B_2\right) \cdot P\left(A \mid B_2\right)}$

Bayes' Theorem

Bayes' Theorem:

$$\boxed{P\left(B_k \mid A\right) = \frac{P\left(A \mid B_k\right) \cdot P\left(B_k\right)}{\sum_{i=1}^{n} P\left(A \mid B_i\right) \cdot P\left(B_i\right)}}$$

Bayes' Theorem

Ex. 3 (text tractor example) 3 assembly lines: Red, White, and Blue. Some tractors don't start (see Ex. 2). Find prob. of each line producing a non-starting tractor.

$P(R) = 0.48$	$P(N \mid R) = 0.06$
$P(W) = 0.21$	$P(N \mid W) = 0.11$
$P(B) = 0.31$	$P(N \mid B) = 0.08$

Bayes' Theorem

<u>Soln.</u>

Find $P(R \mid N)$, $P(W \mid N)$, and $P(B \mid N)$

$P(R) = 0.48$	$P(N \mid R) = 0.06$
$P(W) = 0.21$	$P(N \mid W) = 0.11$
$P(B) = 0.31$	$P(N \mid B) = 0.08$

Bayes' Theorem

Soln.

$$P(R\mid N)=\frac{P(R\cap N)}{P(N)}$$

$$=\frac{P(N\mid R)\cdot P(R)}{P(N\mid R)\cdot P(R)+P(N\mid W)\cdot P(W)+P(N\mid B)\cdot P(B)}$$

$$=\frac{(0.06)\cdot(0.48)}{(0.06)\cdot(0.48)+(0.11)\cdot(0.21)+(0.08)\cdot(0.31)}$$

$$\approx 0.3755$$

Bayes' Theorem

Soln.

$$P(W\mid N)=\frac{P(N\mid W)\cdot P(W)}{P(N\mid R)\cdot P(R)+P(N\mid W)\cdot P(W)+P(N\mid B)\cdot P(B)}$$

$$=\frac{(0.11)\cdot(0.21)}{(0.06)\cdot(0.48)+(0.11)\cdot(0.21)+(0.08)\cdot(0.31)}$$

$$\approx 0.3012$$

$$P(B\mid N)\approx 0.3233$$

Ex. 4 Hazel thinks she may be allergic to eating peanuts, and takes a test that gives the following results:

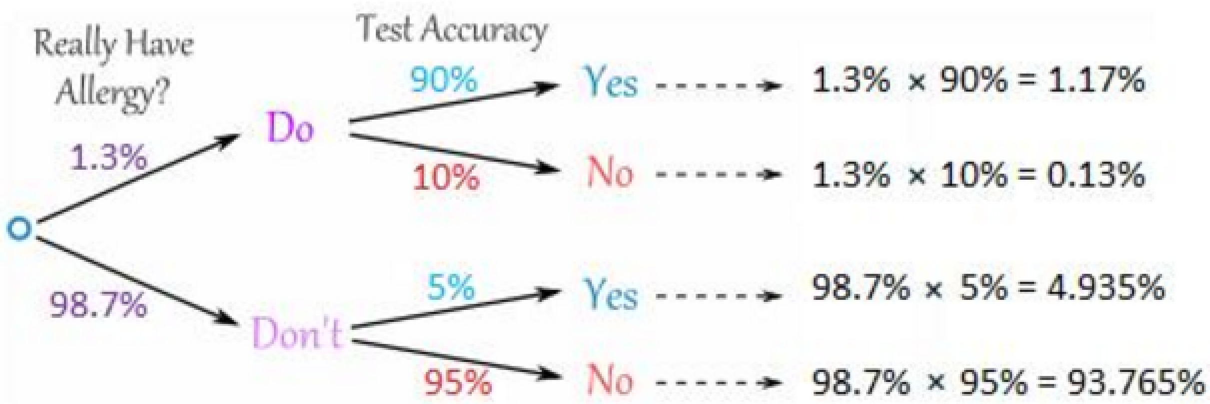
- For people that really do have the allergy, the test says "Yes" 90% of the time
- For people that do not have the allergy, the test says "Yes" 5% of the time ("false positive")

If 1.3% of the population have the allergy, and Hazel's test says "Yes", what are the chances that Hazel really does have the allergy?

The following table shows the percents:

	Test says "Yes"	Test says "No"
Have allergy	**90%**	**10% "False Negative"**
Don't have it	**5% "False Positive"**	**95%**

Drawing a tree diagram can really help:



First of all, let's check that all the percentages add up:
1.17% + 0.13% + 4.935% + 93.765% = 100% (good!)

And the two "Yes" answers add up to 1.17% + 4.935% = 6.105%, but only 1.17% are correct.

1.17/6.105 = 19.2%

Ex. 5 The Nomorebugs Antivirus Software Company tests software for viruses with the following results:

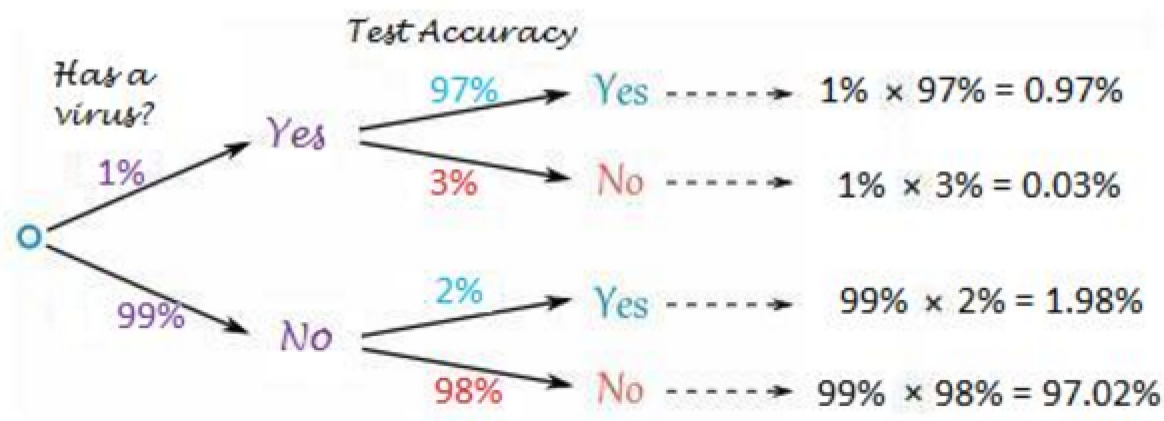
• For software that really has a virus, the test says "Yes" 97% of the time
• For software that is really is virus-free, the test says "Yes" 2% of the time ("false positive")

If 1% of all software has a virus, and the virus test for randomly selected software says "Yes", what are the chances that the software really has a virus?

The following table shows the percents:

	Test says "Yes"	Test says "No"
Has a virus	**97%**	**3% "False Negative"**
Is virus-free	**2% "False Positive"**	**98%**

Drawing a tree diagram can really help:



First of all, let's check that all the percentages add up:

0.97% + 0.03% + 1.98% + 97.02% = 100% (good!)

And the two "Yes" answers add up to 0.97% + 1.98% = 2.95%, but 0.97% are correct.

0.97/2.95 = 32.9%

Ex. 6 In Bard College, 60% of the boys play football and 36% of the boys play ice hockey. Given that 40% of those that play football also play ice hockey, what percent of those that play ice hockey also play football?

Let A = Play football and B = Play ice hockey

Use Bayes' Theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

P(A) = 60% = 0.6

P(B) = 36% = 0.36

P(B|A) = 40% = 0.4

Therefore $P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{0.6 \times 0.4}{0.36} = \frac{0.24}{0.36} = 66\frac{2}{3}\%$

Therefore, $66\frac{2}{3}\%$ of those that play ice hockey also play football.

Ex. 7 In AUCA, 40% of the girls like music and 24% of the girls like dance.
Given that 30% of those that like music also like dance, what percent of those that like dance also like music?

Let A = Like music and B = Like dance

Use Bayes' Theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

P(A) = 40% = 0.4

P(B) = 24% = 0.24

P(B|A) = 30% = 0.3

Therefore $P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{0.4 \times 0.3}{0.24} = \frac{0.12}{0.24} = 50\%$

Therefore 50% of those that like dance also like music.

<u>Ex. 8</u> 35% of the students in AUCA have a tablet, and 24% have a smart phone. Given that 42% of those that have smart phone also have a tablet, what percent of those that have a tablet also have a smart phone?

Let A = Have a smart phone and B = Have a tablet

Use Bayes' Theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

P(A) = 24% = 0.24
P(B) = 35% = 0.35
P(B|A) = 42% = 0.42

Therefore $P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{0.24 \times 0.42}{0.35} = \frac{0.1008}{0.35} = 28.8\%$

Therefore 28.8% of those that have a tablet also have a smart phone.

Ex. 9 A test for a disease gives a correct positive result with a probability of 0.95 when the disease is present, but gives an incorrect positive result (false positive) with a probability of 0.15 when the disease is not present.

If 5% of the population has the disease, and Jean tests positive to the test, what is the probability Jean really has the disease?

Let A = A patient really has the disease and B = A patient tests positive

Use Bayes' Theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

P(A) = 5% = 0.05

P(B) = 5% $\times$ 0.95 + 95% $\times$ 0.15 = 0.0475 + 0.1425 = 0.19

P(B|A) = 0.95

Therefore $P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{0.05 \times 0.95}{0.19} = \frac{0.0475}{0.19} = 0.25$

Therefore, the probability Jean really has the disease $= 0.25$

Ex. 10 Wire manufactured by a company is tested for strength.
The test gives a correct positive result with a probability of 0.85 when the wire is strong, but gives an incorrect positive result (false positive) with a probability of 0.04 when in fact the wire is not strong.

If 98% of the wires are strong, and a wire chosen at random fails the test, what is the probability it really is not strong enough?

Let A = A wire really is not strong enough and B = A wire fails the test

Use Bayes' Theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

P(A) = 2% = 0.02

P(B) = 98% × 0.15 + 2% × 0.96 = 0.147 + 0.0192 = 0.1662

P(B|A) = 0.96

Therefore $P(A|B) = \dfrac{P(A)P(B|A)}{P(B)} = \dfrac{0.02 \times 0.96}{0.1662} = \dfrac{0.0192}{0.1662} = 0.1155$

Therefore, the probability a wire that fails the test really is not strong enough = 0.12 correct to two decimal places

Ex. 11 A supermarket buys light globes (light bulbs) from three different manufacturers - Brightlight (35%), Glowglobe (20%) and Shinewell (45%).

In the past, the supermarket has found that 1% of Brightlight's globes are faulty, and that 1.5% of Glowglobe's and Shinewell's globes are faulty.
A customer buys a globe without looking at the manufacturer's name - in other words, it's a random choice. When she gets home, she finds the globe is faulty.
What is the probability she chose a Shinewell's globe?

Let

A_1 = The globe was a Brightlight's.

A_2 = The globe was a Glowglobe's.

A_3 = The globe was a Shinewell's.

and B = A globe chosen at random is faulty.

Use Bayes' Theorem:

$$P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$$

$P(A_1) = 35\% = 0.35$

$P(A_2) = 20\% = 0.2$

$P(A_3) = 45\% = 0.45$

$P(B|\ A_1) = 1\% = 0.01$

$P(B|\ A_2) = 1.5\% = 0.015$

$P(B|\ A_3) = 1.5\% = 0.015$

Therefore $P(A_3|B) = \dfrac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$

$$= \frac{0.45 \times 0.015}{0.35 \times 0.01 + 0.2 \times 0.015 + 0.45 \times 0.015}$$

$$= \frac{0.00675}{0.0035 + 0.003 + 0.00675}$$

$$= \frac{0.00675}{0.01325}$$

$$= 0.509\ldots$$

Therefore, the probability she chose a Shinewellś globe $= 0.51$ correct to two decimal places

Ex. 12 A glazier buys his glass from four different manufacturers - Clearglass (10%), Strongpane (25%), Mirrorglass (30%) and Reflection (35%).

In the past, the glazier has found that 1% of Clearglass' product is cracked, 1.5% of Strongpane's product is cracked, and 2% of Mirrorglass' and Reflection's products are cracked.

The glazier removes the protective covering from a sheet of glass without looking at the manufacturer's name - in other words, it's a random choice. He finds the glass is cracked. What is the probability it was made by Mirrorglass?

Let

A_1 = The glass was from Clearglass.

A_2 = The glass was from Strongpane.

A_3 = The glass was from Mirrorglass

A_4 = The glass was from Reflection.

and B = A glass chosen at random is cracked.

Use Bayes' Theorem:

$$P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)}$$

$P(A_1) = 10\% = 0.1$

$P(A_2) = 25\% = 0.25$

$P(A_3) = 30\% = 0.3$

$P(A_3) = 35\% = 0.35$

$P(B|\ A_1) = 1\% = 0.01$

$P(B|\ A_2) = 1.5\% = 0.015$

$P(B|\ A_3) = 2\% = 0.02$

$P(B|\ A_4) = 2\% = 0.02$

Therefore $P(A_3|B) = \dfrac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)}$

$$= \frac{0.3 \times 0.02}{0.1 \times 0.01 + 0.25 \times 0.015 + 0.3 \times 0.02 + 0.35 \times 0.02}$$

$$= \frac{0.006}{0.001 + 0.00375 + 0.006 + 0.007}$$

$$= \frac{0.006}{0.01775}$$

$$= 0.3380\ldots$$

Therefore, the probability the glass was made by Mirrorglass $= 0.34$ correct to two decimal places