AN OVERVIEW OF THE $3x + 1$ PROBLEM

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Abstract. This paper gives an overview of the Collatz function, a function that takes a positive integer as input and divides it by 2 if is even or multiplies it by 3 and adds 1. The Collatz conjecture states that this function, when repeatedly applied to itself, will eventually yield 1. This conjecture has been computationally verified to be correct for very large numbers, but the proof is still outstanding. The background and reasons to study the conjecture will be discussed. Furthermore, details of the $3x + 1$ problem will be illustrated. Those details are trajectories, cycles, stopping time, and stochastic approximations.

Contents

Date: 01.02.2020.

1. Introduction

The $3x+1$ Problem is a number theoretical problem that has been studied by mathematicians since the 1950s, but still today it remains unsolved [3]. This problem is defined in terms of the *Collatz function*, which, for any positive integer x, returns $x/2$ if x is even and $3x+1$ if x is odd. Mathematicians study the behavior of this function when it is repeatedly iterated, meaning it is applied to itself. Following Lagarias [3], the Collatz function is defined as

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases} \tag{1}$$

When the $3x+1$ Problem is studied, the $3x+1$ *function*

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases} \tag{2}$$

The $3x+1$ function $T(x)$ takes integers as input and yields integers as outputs, making it a number theoretic function like $C(x)$. More specifically, its domain are all positive integers and its range are also the positive integers. As such, $T(x)$ can be seen as the mapping of $\mathbb{N}+1 \to \mathbb{N}+1$, where $\mathbb{N}+1$ represents the positive integers [4]. The $3x+1$ function is, like the Collatz function, repeatedly applied to itself and has a *stopping time*, *total stopping time*, and a *trajectory* for each $x \in \mathbb{N}+1$.

The $3x+1$ function is used instead of the Collatz function because it is more predictable [3]. If x is odd, $3x+1$ yields an even number that number is then immediately divided by 2. This has the effect that an otherwise following iteration of $C(x)$ is not needed. Furthermore, the function cannot grow as quickly because the factor is now $3/2$ instead of 3. These two changes make the function easier to study [3]. Accordingly, $T(x)$ can alternatively be defined in terms of equation (1) as $C(x)$ if x is even and as $C(C(x))$ if x is odd:

$$T(x) = \begin{cases} C(C(x)) & \text{if } x \equiv 1 \pmod{2}, \\ C(x) & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

1.1. **The Collatz Conjecture.** *For all* $x \in \mathbb{N}+1$ *there is a* $k \in \mathbb{N}+1$ *such that* $T^{(k)}(x) = 1$. *For any positive integer* x, k *iterations of the* $3x+1$ *function will yield the result 1.*

This conjecture has not yet been proven and remains of interest to mathematicians who continue to study it to this day [3]. According to Chamberland [2], because $T(x)$ is a number theoretical function, successive applications of $T(x)$ can have the following results:

(1) reach 1, which is equivalent to entering the trivial cycle, $\{2, 1, 2, 1, \dots\}$
(2) enter a non-trivial cycle that does not include 1, or
(3) diverge to infinity and not enter any type of cycle.

The Collatz conjecture states that point (1) is the only possible result and happens regardless of the positive integer that the iteration starts with.

1.2. **Background Information.** The Collatz conjecture and function are named after the German mathematician Lothar Collatz. Collatz worked on problems similar to the $3x+1$ problem in the early 1930s, according to Lagarias [3]. The problem is also known as Hasse's Algorithm, Kakutani's Problem, and Ulam's Problem after other scientists who studied it or similar problems. During the 1950s the problem was circulated in various mathematical circles and the first publication was written about it in 1971 [3].

Today, the conjecture has been verified for over 10^{20} numbers using the help of computers to check if iterations of the $3x+1$ function eventually reach 1 [4]. A number that does not iterate to 1 has not yet been found.

One of the most recent papers on the $3x+1$ problem, "*Almost All Orbits of the Collatz Map Attain Almost Bounded Values*" by Terence Tao [4], was published in September of 2019. He proved that almost all iterations of the Collatz function are bounded, bringing mathematics one step closer to proving the Collatz conjecture. Currently, the problem is being actively researched.

1.3. **Reasons the Study the $3x+1$ Problem.** According to Lagarias [3], the fact that the problem is easy to state but hard to prove makes it an interesting challenge for mathematicians. The fact that the $3x+1$ problem has remained unsolved after over 50 years despite much research being done about it underlines the challenge.

One of the challenging aspects of this problem is the pseudo-random nature of the $T(x)$ function. The function is difficult to predict when it comes to how iterations of it will behave. Because mathematical proofs tend to rely on patterns, this pseudo-randomness makes a proof of the conjecture very challenging.

Another reason for studying the $3x + 1$ problem is that it can be studied as an iterative mapping of positive integers to positive integers. These kinds of mappings are currently a popular research topic, according to Lagarias [3].

Computer scientists also have an interest in the Collatz conjecture because the computational testing of its correctness is an important part of current research. Since the 1960s, computers have been used to verify if positive integers follow the Collatz conjecture. Because these calculations have verified the conjecture for over 10^{20} numbers [4], finding and optimizing computer algorithms is important. Furthermore, whether or not a given positive integer will iterate to 1 can be studied as a decision problem in computer science [4].

The study of prime numbers is also connected to the $3x + 1$ problem and the Collatz conjecture. If it were to be proven to be current, the proof may further the understanding of prime factorization of integers. This is because the Collatz function, in a way, factors integers when dividing by 2. When odd numbers are multiplied by 3, another prime factor is introduced to the number. The addition of 1 then can drastically change the way an integer can be factored and accordingly this behavior is interesting to study [3].

Finally, when talking about this problem, mathematician Paul Erdös said that

> Mathematics is not ready for such problems.

If he is right and mathematics is indeed not ready for problems such as the $3x+1$ problem, that could mean that new areas of mathematics are required to prove or disprove it. A proof of the Collatz conjecture may thus involve new areas of mathematics or maybe require old areas to be applied in new and different ways. Whatever the case may be, the $3x + 1$ problem is an interesting and challenging problem in mathematics and has connections to multiple other areas of mathematics and computer science. If it were to be proven or disproven, the effects could have wide-ranging implications [3].

2. The $3x+1$ Problem in Detail

2.1. **Trajectories.** The trajectory of x under $T(x)$ is the set of the successive iterations of $T(x)$ that ends when $T^{(k)}(x) = 1$. Accordingly, the trajectory of $T(x)$ has the length $k+1$ because the first element of the trajectory is x whose iteration number is 0. Trajectories are also called forward orbits $O^{+}(x)$ of x under $T(x)$. The trajectory is defined as

$$O^{+}(x) := \{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\}$$

These trajectories can be graphed to illustrate what $T(x)$ does when repeatedly iterated.

2.1.1. *The trajectory of* $T(39)$. The trajectory of $T(39)$ is

$$O^{+}(39) := \{39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29,\\ 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$$

When graphed, one can see that $T^{(k)}(39)$ increases and decreases in a seemingly random manner.

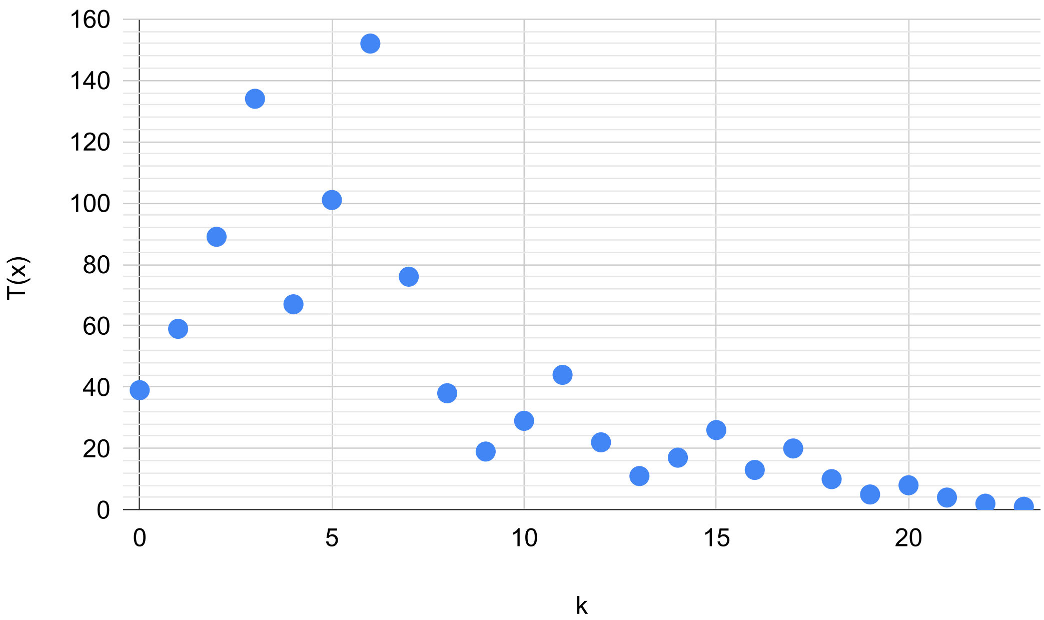

Figure 1. The trajectory of $T^{(k)}(39)$ by number of iterations

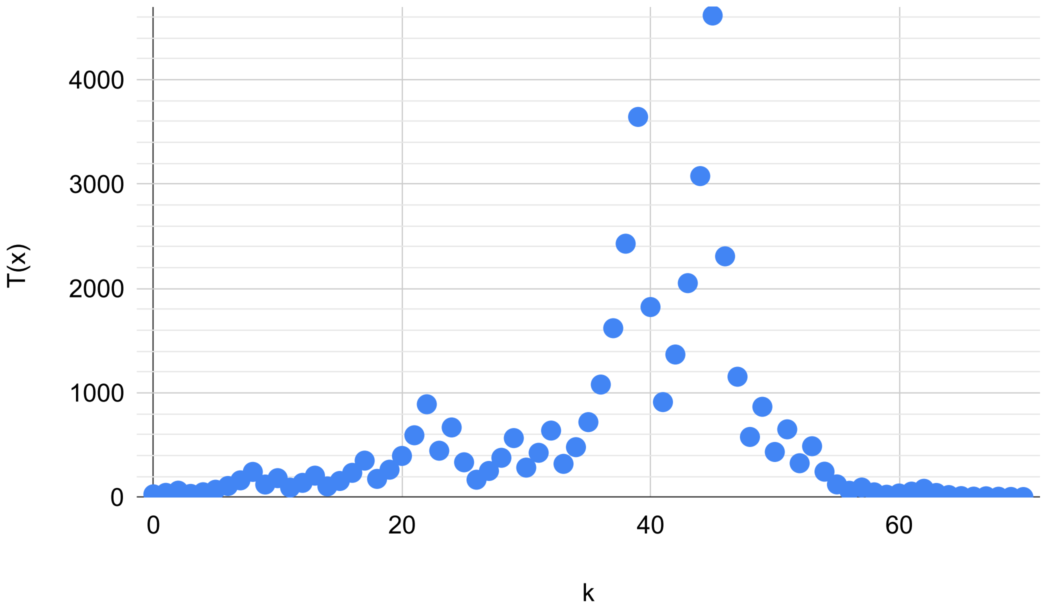

FIGURE 2. The trajectory of $T^{(k)}(27)$ by number of iterations

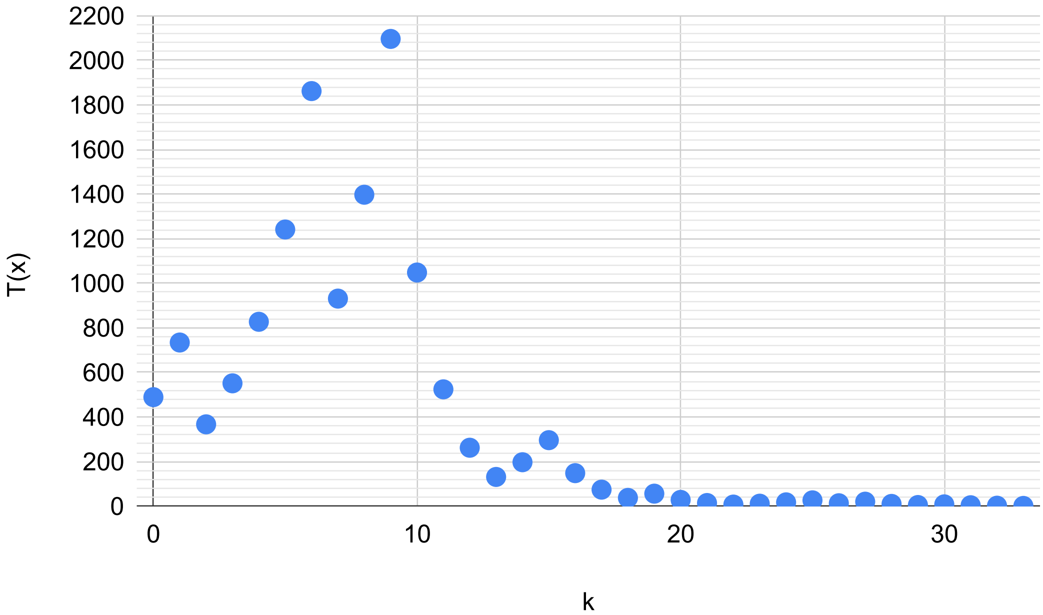

FIGURE 3. The trajectory of $T^{(k)}(489)$ by number of iterations

2.2. **Cycles.** Cycles of $T(x)$ are parts of trajectories that periodically repeat. It is known that $T(x)$ has the trivial cycle $\{2, 1, 2, \dots\}$, which is equivalent to reaching 1. The Collatz conjecture states that any initial positive integer x will, after k iterations of $T(x)$, enter this trivial cycle. This means that the trivial cycle is the only cycle that exist for the positive integers. It has additionally been proven that if non-trivial cycles exist, they must be more than 10.4 billion numbers long.
When applied to all integers, we get three more cycles, which are also conjectured to be the only ones. These cycles start at -1, -5, and -17. -5, -7, -10, -5 -1, -1, -1, -1 -17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34, -17, -25

2.3. **Stopping Time.** The stopping time of $T(x)$ is the number of iterations of $T(x)$ that it takes for the result to be less than x. To do this, first all numbers up to and including $x - 1$ have to be verified to iterate to 1. Then, if $T^{(k)}(x) < x$, it is obvious that $T(x)$ will also iterate to 1. This abbreviation makes it simple to check if a particular number iterates to 1. If the Collatz Conjecture is true, all $x \in \mathbb{N} + 1$ have a finite stopping time, meaning they iterate to 1 eventually. The stopping time of x is called $\sigma(x)$ and defined as

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\}.$$

2.3.1. *The stopping time of* $T(39)$. With the trajectory of $T(39)$ being

$$\begin{aligned} O^+(39) := \{&39, 59, 89, 134, 67, 101, 152, 76, \mathbf{38}, 19, 29, \\ &44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}, \end{aligned}$$

we see that 38 is the first number <39. If we have previously shown that $T(38)$ iterates to 1, we can infer that $T(39)$ does so too. Thus $\sigma(39) = 8$, as 38 is the result of the 8th iteration.

2.4. **Total Stopping Time.** The total stopping time of x is the total number of steps needed for $T(x)$ to iterate to 1. It is defined as

$$\sigma_\infty(x) = \inf\{k : T^{(k)}(x) = 1\}$$

For $T(39)$, which we considered above,

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29,\\ 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$$

and we see that $T^{(23)}(39) = 1$, this means that it takes 23 iterations of $T(x)$ to iterate to one. Accordingly, $\sigma_\infty(39) = 23$.

2.5. **Stochastic Approximations.** When looking at the values that individual iterations of $T(x)$ yielded, mathematicians noticed that for the trajectories of large numbers, the number of odd and even numbers is approximately the same. Furthermore, they observed that $T(x)$ behaves pseudo-randomly, meaning that it is hard to predict which value the function will take on in one or two iterations. Mathematicians use this property of $T(x)$ to make probabilistic statements about groups of trajectories and study their behaviors in this way. For example, the upper bound of for the total stopping time has been proven to be $41.677647 \log x$

2.5.1. *Stopping Time Approximation.* The total stopping time for most trajectories is approximated to be $6.95212 \log x$ steps. For $T(39)$ we have the approximation

$$6.95212 \log 39 \approx 25.4952$$

Compared to the known $\sigma_\infty(39) = 23$ this is not bad.

3. Conclusion

- the Collatz Conjecture states that for $x, k \in \mathbb{N} + 1$ $T^{(k)}(x) = 1$
- the conjecture has not been proven, but verified for 10^{20} numbers
- all orbits of $T(x)$ should reach the trivial cycle
- $T(x)$ can be probabilistically described because of pseudorandomness

References

[1] Marc Chamberland, *An Update on the $3x + 1$ Problem*, `http://www.math.grinnell.edu/~chamberl/papers/3x_survey_eng.pdf`, 2005.

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[3] Jeffrey C.Lagarias, *The* $3x + 1$ *Problem: An Overview*, `https://pdfs.semanticscholar.org/100046dd8b4ee901bc71043da7d42f5d87ca0224.pdf`, 2010.

[4] Terence Tao, *Almost All Orbits of the Collatz Map Attain Almost Bounded Values*, arXiv:1909.03562v2 [math.PR], 2019.