

An Overview of the $3x + 1$ Problem

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The $3x + 1$ Problem is based on the **Collatz Function**

$$C(x) = \begin{cases} 3x + 1 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

When the $3x + 1$ Problem is studied, the $3x + 1$ **Function**

$$T(x) = \begin{cases} (3x + 1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

is used.

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- ▶ $T(x)$ is a function in **number theory**
- ▶ domain of $T(x)$ are positive integers, its range are positive integers
- ▶ mathematically, $T(x)$ maps $\mathbb{N} + 1 \rightarrow \mathbb{N} + 1$
- ▶ $T(x)$ has a **stopping time**, **total stopping time**, and **trajectory** for each $x \in \mathbb{N} + 1$
- ▶ $T(x)$ is repeatedly applied to an initial x

Conjecture

For all $x \in \mathbb{N} + 1$ there is a $k \in \mathbb{N} + 1$ such that $T^{(k)}(x) = 1$.

- ▶ starting at any positive integer x , k iterations of $T(x)$ will give the result 1
- ▶ the Collatz Conjecture has **not been proven**

$T(x)$ can:

1. reach 1, which is equivalent to entering the **trivial cycle** $\{2, 1, 2, 1, \dots\}$
2. enter a non-trivial cycle that does not include 1
3. diverge to infinity and not enter any type of cycle

The Collatz Conjecture states that **1. always happens**.

- ▶ named after German mathematician Lothar Collatz
- ▶ problem circulated since the 1950s
- ▶ academic publications started in the 1970s
- ▶ conjecture has been verified for over 10^{20} numbers
- ▶ most recent progress was in September of 2019
- ▶ problem is still being actively researched

Reasons to Study the Problem

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- ▶ problem is simple to state, but hard to prove
- ▶ remains unsolved after over 50 years of research
- ▶ iterative mappings are currently a popular research topic
- ▶ verifying large numbers is computationally interesting
- ▶ could yield results connected to prime factorization using 2 and 3

Mathematics is not ready for such problems.

— *Paul Erdős*

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Summary

- ▶ the trajectory of x under $T(x)$ is the set of successive iterations of $T(x)$
- ▶ it is also called the forward orbit $O^+(x)$ of x under $T(x)$
- ▶ trajectories can be graphed

$$O^+(x) := \{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\}$$

Example: Trajectory of $T(39)$

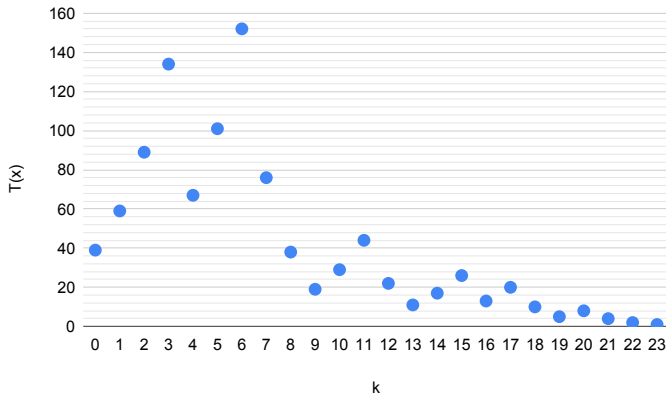
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The trajectory of $T(39)$ is

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29, \\ 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$$

and can be graphed like this for k and $T^{(k)}(39)$



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- ▶ $T(x)$ has the **trivial cycle** $\{2, 1, 2, \dots\}$, which is equivalent to reaching 1
- ▶ the Collatz Conjecture states that **all orbits will eventually enter the trivial cycle** and thus that **it is the only cycle**
- ▶ if $T(x)$ has non-trivial cycles, they have been proven to be over 10.4 billion numbers long

- ▶ the number of iterations of $T(x)$ until the result is smaller than x
- ▶ first it is checked that every positive integer up to $x - 1$ iterates to 1
- ▶ then, if $T^{(k)}(x) < x$, we know it will iterate to 1
- ▶ if the Collatz Conjecture is true, all $x \in \mathbb{N} + 1$ have a finite stopping time

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\}$$

Example: Stopping time of $T(39)$

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With the trajectory of $T(39)$

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, \mathbf{38}, 19, 29, \\ 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\},$$

we see that 38 is the first number < 39 .

Thus $\sigma(39) = 8$, as 38 is the result of the 8th iteration.

Total Stopping Time

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The total stopping time is the number of steps needed for $T(x)$ to iterate to 1. It is defined as

$$\sigma_{\infty}(x) = \inf\{k : T^{(k)}(x) = 1\}$$

Example for $\sigma_{\infty}(39)$

For $T(39)$,

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29, \\ 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$$

and we see that $T^{(23)}(39) = 1$, so $\sigma_{\infty}(39) = 23$.

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- ▶ each trajectory has approximately the same number of odd and even elements
- ▶ the behavior of $T(x)$ is pseudorandom for large numbers
- ▶ thus, probabilistic models describe its behavior
- ▶ these models describe groups of trajectories
- ▶ e.g., the upper bound for σ_∞ is $41.677647 \log x$

Example: Stopping Time Approximations

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The total stopping time for most trajectories is approximated to be about $6.95212 \log x$ steps.

Example for $T(39)$

For $T(39)$ we have the approximation

$$6.95212 \log 39 \approx 25.4952$$

Compared to the known $\sigma_{\infty}(39) = 23$ this is not bad.

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- ▶ the Collatz Conjecture states that for $x, k \in \mathbb{N} + 1$
 $T^{(k)}(x) = 1$
- ▶ the conjecture has not been proven, but verified for 10^{20} numbers
- ▶ all orbits of $T(x)$ should reach the trivial cycle
- ▶ $T(x)$ can be probabilistically described because of pseudo-randomness

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- [4] Terence Tao, *Almost All Orbits of the Collatz Map Attain Almost Bounded Values*, arXiv:1909.03562v2 [math.PR], 2019.