

The $3x + 1$ Problem

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The Function

- ▶ Collatz function $C(x)$

$$C(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 3x + 1 & \text{if } x \text{ is odd.} \end{cases}$$

- ▶ $3x + 1$ function $T(x)$

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 1 \pmod{2} \\ \frac{3x+1}{x} & \text{if } x \equiv 0 \pmod{2} \end{cases}$$

The Conjecture

- ▶ starting at any positive integer, some iterate of these functions takes the value 1
- ▶ for the $3x + 1$ function $T(x)$: every $m \geq 1$ has some $T^{(k)}(m) = 1$
- ▶ repeated applications of this function yield *trajectories* or *forward orbits*

Related Aspects

- ▶ *trajectory or forward orbit:*
 $O^+(m) := \{m, T(m), T^{(2)}(m), \dots\}$
- ▶ *height $h(m)$:* least k for for which the *Collatz function* $C(x)$ has $C^{(k)}(m) = 1$

Examples of Trajectories

- ▶ $x = 13 \rightarrow \{13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \dots\}$
- ▶ $x = 20 \rightarrow \{20, 10, 5, 16, 8, 4, 2, 1, \dots\}$
- ▶ $x = 512 \rightarrow \{512, 256, 128, 64, 32, 16, 8, 4, 2, 1, \dots\}$
- ▶ numbers up to $17 \times 2^{58} \approx 4.89 \times 10^{18}$ have been computationally verified

Related Questions

- ▶ this is part of *number theory*: study of integers and integer valued functions
- ▶ two related questions
 - ▶ is there a divergent trajectory?
 - ▶ are there other cycles besides $\{4, 2, 1, 4, \dots\}$?
- ▶ for negative integers there are cycles starting at $-1, -5, -17$

Background of the $3x + 1$ Problem

- ▶ also known as: Syracuse Problem, Hasse's Algorithm, Kakutani's Problem, Ulam's Problem
- ▶ Collatz started studying the function in the 1930s
- ▶ informally discussed in his lecture at the International Math. Congress in Cambridge, Mass. in 1950
- ▶ first published paper about the problem in 1963
- ▶ starting in 1970 there were a lot more publications

Attempted Proofs

- ▶ Yamada 1981: had mathematical faults
- ▶ Cadogan 2006: had incorrect indices in a function
- ▶ Bruckman's 2008: proof would exclude the cycle starting at 1

Why is the $3x + 1$ Problem Interesting?

- ▶ simple problem but it is not solved yet
- ▶ maybe new areas of mathematics are needed to prove or disprove this conjecture
- ▶ could lead to insights into prime factorization

References

- ▶ Lagarias, J. C. (2011). The $3x + 1$ Problem: An Annotated Bibliography (1963-1999)
- ▶ Lagarias, J. C. (2012). The $3x + 1$ Problem: An Annotated Bibliography, II (2000-2009)