Permutation and Combination

Learning Objectives

- Permutations with Repetition
- Permutations without Repetition
- Combination with Repetition
- Combination without Repetition

What's the Difference?

In English we use the word "combination" loosely, without thinking if the order of things is important. In other words:

- *"My fruit salad is a combination of apples, grapes and bananas"* We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.
- *"The combination to the safe was 472"*. Now we do care about the order. "724" would not work, nor would "247". It has to be exactly 4-7-2.

So, in Mathematics we use more *precise* language:

- **If the order doesn't matter, it is a Combination.**
- **If the order does matter it is a Permutation.**

Permutation

Definition. All possible arrangements of a collection of things, where the order is important.

Example: You want to visit the homes of three friends Alex ("a"), Betty ("b") and Chandra ("c"), but haven't decided in what order. What choices do you have?

Answer: {a,b,c} {a,c,b} {b,a,c} {b,c,a} {c,a,b} {c,b,a}

If the order does **not** matter, it is a **Combination**

Permutations

There are basically two types of permutation:

Repetition is Allowed: such as the lock above. It could be "333".

No Repetition: for example the first three people in a running race. You can't be first *and* second.

Permutations with Repetition

These are the easiest to calculate. When we have ***n*** things to choose from ... we have ***n*** choices each time!
When choosing ***r*** of them, the permutations are:

$$n \times n \times \ldots (r \text{ times}) = n^r$$

Example: in the lock above, there are 10 numbers to choose from (0,1,...9) and we choose 3 of them:

$$10 \times 10 \times \ldots (3 \text{ times}) = 10^3 = 1.000 \text{ permutations}$$

So, the formula is simply:

$$n^r$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them (Repetition allowed, order matters).

Permutations without Repetition

In this case, we have to **reduce** the number of available choices each time.

For example, what order could 16 pool balls be in?
After choosing, say, number "14" we can't choose it again.

So, our first choice would have 16 possibilities, and our next choice would then have 15 possibilities, then 14, 13, etc. And the total permutations would be:

$$\mathbf{16 \times 15 \times 14 \times 13 \times ... = 20{,}922{,}789{,}888{,}000}$$

But maybe we don't want to choose them all, just 3 of them, so that would be only:

$$\mathbf{16 \times 15 \times 14 = 3{,}360}$$

In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

But how do we write that mathematically? Answer: we use the **"factorial function"**

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers.

Examples:

$4! = 4 \times 3 \times 2 \times 1 = 24$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

$1! = 1$

Note: it is generally agreed that **$0! = 1$**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, if we wanted to select **all** of the billiard balls the permutations would be:

$$\mathbf{16! = 20{,}922{,}789{,}888{,}000}$$

But if we wanted to select just 3, then we have to stop the multiplying after 14. How do we do that? There is a neat trick ... we divide by **13!** ...

$$\frac{\mathbf{16 \times 15 \times 14 \times 13 \times 12 \ldots}}{\mathbf{13 \times 12 \ldots}} \mathbf{= 16 \times 15 \times 14 = 3{,}360}$$

Do you see? **16! / 13! = 16 × 15 × 14**

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them (No repetition, order matters).

Examples:

Our "order of 3 out of 16 pool balls example" would be:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{\mathbf{20,922,789,888,000}}{\mathbf{6,227,020,800}} = 3,360$$

(which is just the same as: **16 × 15 × 14 = 3,360**)

How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{\mathbf{3,628,800}}{\mathbf{40,320}} = 90$$

(which is just the same as: **10 × 9 = 90**)

Notation

Instead of writing the whole formula, people use different notations such as these:

$$P(n,r) = {}^{n}P_{r} = {}_{n}P_{r} = \frac{n!}{(n-r)!}$$

Example: ***P*(10,2) = 90**

Combinations

Definition. A collection of things, in which the order does not matter.

Example: If you are making a sandwich, how many different combinations of 2 ingredients could you make with cheese, mayo and turkey?

Answer: {cheese, mayo}, {cheese, turkey} or {mayo, turkey}

If the order **does** matter, such as a secret code, it is a **Permutation**

Combinations

There are also two types of combinations (remember the order does **not** matter now):

Repetition is Allowed: such as coins in your pocket (5,5,5,10,10)

No Repetition: such as lottery numbers (2,14,15,27,30,33)

Combinations without Repetition

This is how lotteries work. The numbers are drawn one at a time, and if we have the lucky numbers (no matter what order) we win!
The easiest way to explain it is to:
assume that the order does matter (i.e. permutations),
then alter it so the order does **not** matter.
Going back to our pool ball example, let's say we just want to know which 3 pool balls were chosen, not the order.
We already know that 3 out of 16 gave us 3,360 permutations.
But many of those will be the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3	
1 3 2	
2 1 3	
2 3 1	1 2 3
3 1 2	
3 2 1	

So, the permutations will have 6 times as many possibilities.
In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

$$\mathbf{3! = 3 \times 2 \times 1 = 6}$$

(Another example: 4 things can be placed in **4! = 4 × 3 × 2 × 1 = 24** different ways, try it for yourself!)

So we adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in their order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them (No repetition, order doesn't matter).

It is often called "n choose r" (such as "16 choose 3").
And is also known as the "Binomial Coefficient".

Notation

As well as the "big parentheses", people also use these notations:

$$C(n,r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

It is interesting to also note how this formula is nice and **symmetrical**:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

Example

So, our pool ball example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!\times 13!} = \frac{\mathbf{20,922,789,888,000}}{\mathbf{6\times 6,227,020,800}} = 560$$

Or we could do it this way:

$$\frac{16!}{3!(16-3)!} = \frac{16\times 15\times 14}{3\times 2\times 1} = \frac{3360}{6} = 560$$