Conformal Mapping and its Applications

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Conformal (Same form or shape) mapping is an important technique used in complex analysis and has many applications in different physical situations.If the function is harmonic (ie it satisfies Laplace's equation $\nabla^2 f = 0$)then the transformation of such functions via conformal mapping is also harmonic. So equations pertaining to any field that can be represented by a potential function (all conservative fields) can be solved via conformal mapping.If the physical problem can be represented by complex functions but the geometric structure becomes inconvenient then by an appropriate mapping it can be transferred to a problem with much more convenient geometry. This article gives a brief introduction to conformal mappings and some of its applications in physical problems.

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I. INTRODUCTION

A conformal map is a function which preserves the angles.Conformal map preserves both angles and shape of infinitesimal small figures but not necessarily their size.More formally, a map

$$w = f(z) \tag{1}$$

is called conformal (or angle-preserving) at z_0 if it preserves oriented angles between curves through z_0, as well as their orientation, i.e. direction.

An important family of examples of conformal maps comes from complex analysis. If U is an open subset of the complex plane, , then a function

f :U → C

is conformal if and only if it is holomorphic and its derivative is everywhere non-zero on U. If f is antiholomorphic (that is, the conjugate to a holomorphic function), it still preserves angles, but it reverses their orientation.

The Riemann mapping theorem, states that any nonempty open simply connected proper subset of C admits a bijective conformal map to the open unit disk(the open unit disk around P (where P is a given point in the plane), is the set of points whose distance from P is less than 1) in complex plane C ie if U is a simply connected open subset in complex plane C, which is not all of C,then there exists a bijective ie one-to-one mapping f from U to open unit disk D.

f :U → D

where

D = {z ∈ C : $| z | < 1$ }

As f is a bijective map it is conformal.

A map of the extended complex plane (which is conformally equivalent to a sphere) onto itself is conformal if and only if it is a Mobius transformation ie a transformation leading to a rational function of the form f(z) = $\frac{az+b}{cz+d}$. Again, for the conjugate, angles are preserved, but orientation is reversed.

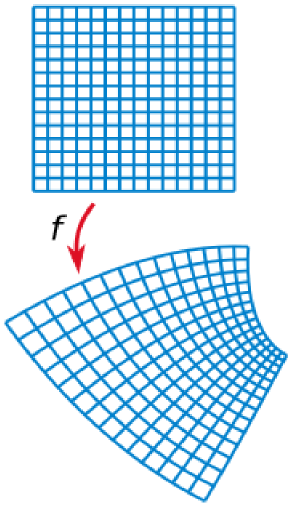


FIG. 1: Mapping of graph

II. BASIC THEORY

Let us consider a function

$$w = f(z) \tag{2}$$

where $z = x + iy$ and $w = u + iv$ We find that $dz = dx + idy$, and $dw = du + idv$,

$$| dz |^2 = dx^2 + dy^2, \tag{3}$$

and

$$| dw |^2 = du^2 + dv^2 \tag{4}$$

Then the square of the length element in (x,y) plane is

$$ds^2 = dx^2 + dy^2 \tag{5}$$

and square of the length element in (u,v) plane is

$$dS^2 = du^2 + dv^2 \tag{6}$$

From equations (3) , (4), (5), (6) we find that,

$$dS/ds = \mid dw/dz \mid . \tag{7}$$

ie the ratio of arc lengths of two planes remains essentially constant in the neighborhood of each point in z plane provided $w(z)$ is analytic and have a nonzero or finite slope at that point.This implies the linear dimensions in two planes are proportional and the net result of this transformation is to change the dimensions in equal proportions and rotate each infinitesimal area in the neighborhood of that point. Thus the angle (which is represented as the ratio of linear dimensions) is preserved although shape in a large scale will not be preserved in general as the value of $\mid dw/dz \mid$ will vary considerably at different points in the plane. Due to this property such transformations are called conformal. This leads to the following theorem.

Theorem : *Assume that f (z) is analytic and not constant in a domain D of the complex z plane. For any point $z \in D$ for which $\acute{f}\ (z) \neq 0$, this mapping is conformal, that is, it preserves the angle between two differentiable arcs.*

Example: Let D be the rectangular region in the z plane bounded by x = 0, y = 0, x = 2 and y = 1. The image of D under the transformation w = $(1 + \imath)$z $+(1 + 2\imath)$ is given by the rectangular region D of the w plane bounded by u + v = 3, u - v = -1, u + v = 7 and u - v = -3.

If w = u + $\imath$v, where u, v $\in$ R, then u = x - y + 1, v = x + y + 2. Thus the points a, b, c, and d are mapped to the points (0,3), (1,2), (3,4), and (2,5), respectively. The line x = 0 is mapped to u = -y+1, v = y+2, or u+v = 3; similarly for the other sides of the rectangle (Fig 2). The rectangle D is translated by $(1 + 2\imath)$, rotated by an angle $\pi/4$ in the counterclockwise direction, and dilated by a factor $\sqrt(2)$. In general, a linear transformation f(z) = αz + β, translates by β, rotates by arg($/\alpha$), and dilates (or contracts) by $\mid \alpha \mid$. Because f(z) = $\alpha \neq 0$, a linear transformation is always conformal.

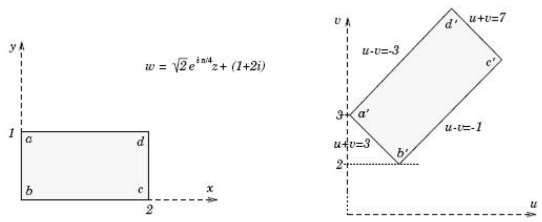


FIG. 2: Mapping of a rectangle

The below theorem (stated without proof), related to inverse mapping, is an important property of conformal mapping as it states that inverse mapping also preserves the angle.

Theorem: *Assume that f (z) is analytic at z_0 and that $f'(z) \neq 0$. Then f (z) is univalent in the neighborhood of z_0. More precisely, f has a unique analytic inverse F in the neighborhood of $w_0 \equiv f(z_0)$; that is, if z is sufficiently near z_0, then $z = F(w)$, where $w \equiv f(z)$. Similarly, if w is sufficiently near w_0 and $z \equiv F(w)$, then $w = f(z)$. Furthermore, $f'(z)F'(w) = 1$, which implies that the inverse map is conformal.*

This uniqueness and conformal property of inverse mapping allows us to map the solution obtained in w-plane to z-plane.

Critical Points: If f$'(z_0) = 0$, then the analytic transformation f(z) ceases to be conformal. Such a point is called a critical point of f . Because critical points are zeroes of the analytic function f$'$, they are isolated.

III. APPLICATIONS

A large number of problems arising in fluid mechanics, electrostatics, heat conduction, and many other physical situations can be mathematically formulated in terms of Laplaces equation. ie, all these physical problems reduce to solving the equation

$$\Phi_{xx} + \Phi_{yy} = 0. \tag{8}$$

in a certain region D of the z plane. The function Φ(x, y), in addition to satisfying this equation also satisfies certain boundary conditions on the boundary C of the region D. From the theory of analytic functions we know that the real and the imaginary parts of an analytic function satisfy Laplaces equation. It follows that solving the above problem reduces to finding a function that is analytic in D and that satisfies certain boundary conditions on C. It turns out that *the solution of this problem can be greatly simplified if the region D is either the upper half of the z plane or the unit disk.*

Example: Consider two infinite parallel flat plates, separated by a distanced and maintained at zero potential. A line of charge q per unit length is located between the two planes at a distance 'a' from the lower plate .The problem is to find the electrostatic potential in the shaded region of the z plane.

The conformal mapping w = exp(πz/d) maps the shaded strip of the z plane onto the upper half of the w plane. So the point z = $\imath$a is mapped to the point w_0 = exp($\imath\pi$a/d); the points on the lower plate, z = x, and on the upper plate, z = x + $\imath$d, map to the real axis w = u for u > 0 and u < 0, respectively. Let us consider a line of charge q at w_0 and a line of charge -q at w_0. Consider the associated complex potential

$$\Omega(w) = -2\log(w-w_0)+2q\log(w-\bar{w}_0) = 2q\log(\frac{w-\bar{w}_0}{w-w_0}) \tag{9}$$

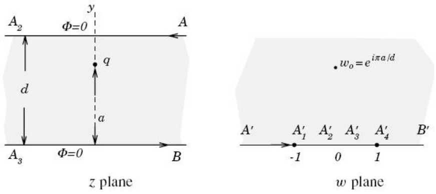


FIG. 3: Mapping of two infinite parallel conducting plate with a charge in between

Calling C_q a closed contour around the charge q, we see that Gauss law is satisfied,

$$\oint_{C_q} E_n ds = Im \oint_{C_q} \bar{E} dz = Im \oint_{\tilde{C}_q} -\Omega\prime(w) = 4\pi q \quad (10)$$

where $\tilde{C}_q$ is the image of C_q in the w-plane. Then, calling $\Omega = \Phi + \imath\Psi$, we see that Φ is zero on the real axis of the w plane. Consequently, we have satisfied the boundary condition $\Phi = 0$ on the plates, and hence the electrostatic potential at any point of the shaded region of the z plane is given by

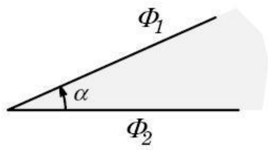
$$\Phi = 2q \log\left(\frac{w - e^{-\imath v}}{w - e^{\imath v}}\right) \quad (11)$$

where $v = \pi$a/d

Conformal mappings are invaluable for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable, but that exhibit inconvenient geometries. By choosing an appropriate mapping, the analyst can transform the inconvenient geometry into a much more convenient one. For example, one may be desirous of calculating the electric field, E(z), arising from a point charge located near the corner of two conducting planes making a certain angle (where z is the complex coordinate of a point in 2-space). This problem is quite clumsy to solve in closed form.

However, by employing a very simple conformal mapping, the inconvenient angle is mapped to one of precisely pi radians, meaning that the corner of two planes is transformed to a straight line. In this new domain, the problem, that of calculating the electric field impressed by a point charge located near a conducting wall, is quite easy to solve.

The solution is obtained in this domain, E(w), and then mapped back to the original domain by noting that w was obtained as a function (viz., the composition of E and w) of z, whence E(w) can be viewed as E(w(z)), which is a function of z, the original coordinate basis.Note that this application is not a contradiction to the fact that conformal mappings preserve angles, they do so only for points in the interior of their domain, and not at the boundary.



$$\Phi = \Phi_2 + \left(\frac{\Phi_1 - \Phi_2}{\alpha}\right)\theta, \quad E_\theta = \frac{\Phi_2 - \Phi_1}{\alpha r}, \quad E_r = 0,$$

where $z = re^{i\theta}, 0 \leq \theta \leq \alpha$.

FIG. 4: Two semiinfinite plane conductors meet at an angle $0 < \alpha < \pi/2$ and are charged at constant potentials Φ_1 and Φ_2

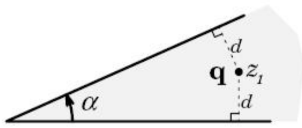


FIG. 5: Two inclined plates with a charge in between

The above fundamental technique is used to obtain closed form expressions of characteristic impedance and dielectric constant of different types of waveguides.A series of conformal mappings are performed to obtain the characteristics for a range of different geometric parameters[1].

Conformal mapping has various applications in the field of medical physics.For example conformal mapping is applied to brain surface mapping. This is based on the fact that any genus zero (The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along closed simple curves without rendering the resultant manifold disconnected; a sphere,disk or annulus have genus zero) surface can be mapped conformally onto the sphere and any local portion thereof onto a disk[2].

Conformal mapping can be used in scattering and diffraction problems.For scattering and diffraction problem of plane electromagnetic waves, the mathematical problem involves finding a solution to scaler wave function which satisfies both boundary condition and radiation condition at infinity. Exact solutions are available for such problems only for a few cases.Conformal mappings are used to study far field expressions of scattered and diffracted waves for more general cases.

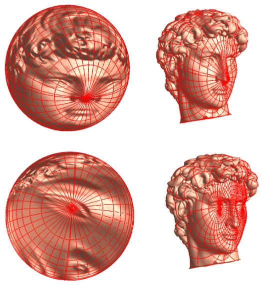


FIG. 6: Mobius transformation.

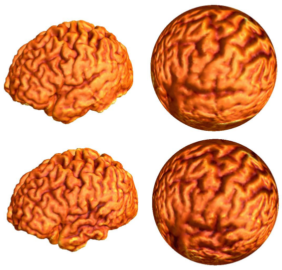


FIG. 7: Reconstruction of brain onto sphere

IV. CONCLUSION

There are different aspects of conformal mapping that can be used for practical applications though the essence remains the same: it preserves the angle and shape locally and mappings of harmonic potentials remains harmonic. These properties of conformal mapping make it advantageous in complex situations , specifically electromagnetic potential problems for general systems. Various conformal techniques such as genus zero conformal mapping is also used to complex surface mapping problems. However the conformal mapping approach is limited to problems that can be reduced to two dimensions and to problems with high degrees of symmetry. It is often impossible to apply this technique when the symmetry is broken.

[1] C.P. Wen, 1969,IEEE.Trans.Microw Theory Tech.,17,1087

[2] X. Gu, Y. Wang, T.F. Chan, P.M. Thompson and S.-T. Yau, "Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping", IEEE Transaction on Medical Imaging, 23(8), Aug. 2004, pp. 949-958