A CONTINUOUS EXTENSION OF THE 3X+1 PROBLEM TO THE REAL LINE

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Abstract. The $3x+1$ problem is a long-standing problem in number theory concerning iteration of the map $T : \mathbb{Z} \to \mathbb{Z}$ given by $T(n) = \frac{3n+1}{2}$ if n is odd, and $T(n) = \frac{n}{2}$ if n is even. This paper studies iteration of the function $f(x) = \frac{x}{2}\left(\cos\frac{\pi x}{2}\right)^2 + \frac{3x+1}{2}\left(\sin\frac{\pi x}{2}\right)^2$ which interpolates the function T on $\mathbb{R}$, and applies methods of discrete dynamical systems to f. We show that the function f has negative Schwarzian derivative on the positive reals $\mathbb{R}^+$, and that all periodic orbits of f on the positive integers are attracting. The dynamics of f on most points of $\mathbb{R}^+$ is closely related to that of T, but there is a nonempty closed subset U of $\mathbb{R}^+$ whose iterates are unbounded. We conjecture that the set U has measure zero.

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1 Introduction

The $3x+1$ Conjecture is an easily-stated conjecture which has circulated in mathematical circles for fifty years. Let $T : \mathbf{Z}^+ \to \mathbf{Z}^+$ be defined as

$$T(x) = \begin{cases} x/2, & \text{x even} \\ (3x+1)/2, & \text{x odd.} \end{cases}$$

The conjecture is that for each positive integer n there exists a k such that $T^k(n) = 1$, where $T^k(n)$ is the k^{th} iterate of n under the map T. This has been verified numerically for all $n < 5.6 \times 10^{13}$ [11]. One easily sees that $\{1, 2\}$ forms a two-cycle, which is sometimes referred to as the trivial cycle. It has been shown that if another cycle exists, its period must be at least 17,087,915 [7]. Lagarias[8] gives an excellent overview of the problem until 1985, making connections with topics such as continued fractions, ergodic theory and theoretical computer science. There has been a great deal of work on the problem since 1985; see Lagarias[9].

In this paper, we study the iterates of an extension of the map T to an entire function f, defined by

$$\begin{aligned} f(x) &= \frac{x}{2}\cos^2\left(\frac{\pi}{2}x\right) + \frac{3x+1}{2}\sin^2\left(\frac{\pi}{2}x\right) \\ &= x + \frac{1}{4} - \frac{2x+1}{4}\cos(\pi x). \end{aligned}$$

The non-negative real axis $\mathbb{R}^+ = [0, \infty)$ is an invariant set for f and we study its iterates there. Clearly $f(x) = T(x)$ for $x \in \mathbb{Z}^+$. The non-negative real axis $\mathbb{R}^+ = [0, \infty)$ is an invariant set for f, and we shall primarily consider iterates of f on $\mathbb{R}^+$. A key factor in our analysis is that the function f has negative Schwarzian derivative on $\mathbb{R}^+$. The function $f(x)$ is displayed in Figure 1.

By studying the map f, we shall determine not only some aspects of the asymptotic behaviour on $\mathbb{R}^+$, but also on $\mathbb{Z}^+$. Other work has been done in extending the map T to the rationals with odd denominators and to the 2-adic integers (see for example [1], [13], [14], [15]), but these extensions are not compatible with the function f.

One reason which seems to make the $3x + 1$ problem so hard is the apparent randomness in the iterates for "large" initial values; whether an iterate bounces up or down seems difficult to determine. If this randomness was true, it would

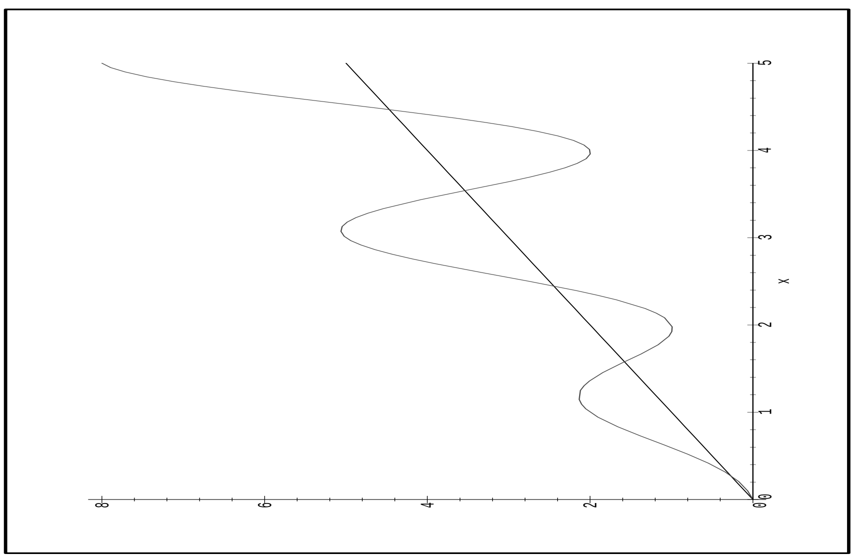

Figure 1: The function $f(x)$.

support the conjecture, for then on average, one goes down by a factor of

$$\left(\frac{3}{2}\right)^{1/2}\left(\frac{3}{4}\right)^{1/4}\left(\frac{3}{8}\right)^{1/8}\cdots=\frac{3}{4}.$$

This randomness is true in the sense that for any block of 2^k consecutive integers n, the 2^k vectors

$$v(n) := (n(\,mod\,2), T(n)(\,mod\,2), \ldots, T^{(k-1)}(n)(\,mod\,2))$$

enumerate every pattern of zeroes and ones exactly once.

The conjecture is discussed in the papers of Crandall[4], Lagarias and Weiss[10] and Wagon[17].

2 Negative Schwarzian Derivative

The Schwarzian derivative of a function is defined as

$$Sf(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Let $Orb(x)$ denote the positive orbit of x, namely $Orb(x)=\{f^n(x)|n\in\mathbf{Z}^+\}$. The immediate basin of attraction of an attracting periodic point p is the union of the connected components of its basin of attraction which contain $Orb(p)$. Singer[16] showed that the condition of everywhere negative Schwarzian derivative $Sf<0$ puts very strong constraints on the iterates of the function f.

Theorem 2.1 *(Singer[16]) Let J be an interval in $\mathbb{R}$. If $g: J\to J$ is a C^3 function with $Sg(x)<0$ for all $x\in J$, then*

1. *The immediate basin of attraction of any attracting periodic orbit contains either a critical point of g or a boundary point of the interval J;*
2. *Each neutral periodic point is attracting;*
3. *There exists no interval of periodic points.*

Later Blokh and Lyubich[3] determined the possible limiting behaviour of iterates of almost all initial points x of a function g on an interval with $Sg < 0$.

Theorem 2.2 *(Blokh and Lyubich[3]) Let J be a compact interval. Suppose $g : J \to J$ is piecewise monotone, and on each interval of monotonicity, g is C^3, has negative Schwarzian derivative and no other critical points. Also suppose that g is C^1 in the neighbourhood of each extrema. Then for almost every (Lebesgue) $x \in J$, $\omega(x)$ is an indecomposable attractor A of one of the following forms:*

1. *A is a periodic orbit;*
2. *A is a periodic interval;*
3. *$A = \omega(c)$, where c is a critical point.*

It should be noted that when case 1 occurs, the immediate basin of attraction of the periodic orbit A contains a critical point of g. Similarly, one may prove that at least one subinterval of the periodic interval of case 2 also contains a critical point of g. This implies almost all of the behaviour on J may be determined by considering the ω-limit sets of the critical points of g in J.

Now we prove the main result of this section.

Lemma 2.1 *The function f has negative Schwarzian derivative Sf on $\mathbb{R}^+$.*

Proof: To prove that $Sf < 0$ for $x \geq 0$ is equivalent to showing

$$g(x) = \frac{2}{\pi^2}(f'(x))^2 Sf < 0$$

for $x \geq 0$ when $f'(x) \neq 0$. It is simple to see that $f'(x) \neq 0$ when $f''(x) = 0$, hence we need only consider the case when $f'(x) \neq 0$. We may expand $g(x)$ to obtain

$$g(x) \quad = \quad \frac{1}{\pi^2}\left[2f'f''' - 3(f'')^2\right]$$

$$
\begin{aligned}
&= 2\left[1-\frac{1}{2}\cos(\pi x)+\pi\left(\frac{2x+1}{4}\right)\sin(\pi x)\right] \\
&\qquad \times\left[\frac{3}{2}\cos(\pi x)-\pi\left(\frac{2x+1}{4}\right)\sin(\pi x)\right] \\
&\quad -3\left[\sin(\pi x)+\pi\left(\frac{2x+1}{4}\right)\cos(\pi x)\right]^2 \\
&= A\left(\frac{2x+1}{4}\right)^2+B\left(\frac{2x+1}{4}\right)+C,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -\pi^2\left(2+\cos^2(\pi x)\right), \\
B &= -2\pi\sin(\pi x)\left[1+\cos(\pi x)\right], \\
C &= \frac{3}{2}\cos^2(\pi x)+3\cos(\pi x)-3.
\end{aligned}
$$

If $g(x)=0$, then

$$
\frac{2x+1}{4}=h(x)=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.
$$

Now we obtain some inequalities for the different terms in this expression. First, we have

$$
B^2-4AC=k(z)=4\pi^2\left[\frac{1}{2}z^4+z^3+8z-5\right]
$$

where $z=\cos(\pi x)$. Since $k'(z)>0$ on $z\in[-1,1]$, we have $k(z)$ maximized at $z=1$. The term B has its minimum at $x=-1/3$ and A is maximized with value $-2\pi^2$. Putting this all together gives

$$
\begin{aligned}
h(x) &\le \frac{2\pi 3\sqrt{3}/4+2\pi\sqrt{9/2}}{2\pi^2} \\
&< 1.0887.
\end{aligned}
$$

Thus $(2x+1)/4<1.0887$, implying $x<1.6774$.

One point we may exploit is that $k(z)$ must be positive. Since $k(z)$ is an increasing function and $k(0.55)<0$, we require $\cos(\pi x)>0.55$, hence

$$
x\in(2n-0.3146,2n+0.3146),\quad \text{for some } n\in\mathbf{Z}.
$$

Thus, if $g(x) = 0$, then $x < 0.3146$.

Now suppose $g(x) = 0$ and $0 < x < 0.3146$. Then $B < 0$, giving

$$h(x) < \frac{2\pi\sqrt{9/2}}{2\pi^2(2 + (0.55)^2)} < 0.29325.$$

Thus $(2x + 1)/4 < 0.29325$ implying $x < 0.0865$. Repeating this process again with $0 < x < 0.0865$, we obtain $x < 0$, a contradiction. This gives the desired result. □

A simple calculation shows that $Sf(-0.1) > 0$, so Lemma 2.1 is fairly sharp. In view of Theorems 2.1 and 2.2 we determine the critical points of f on $\mathbb{R}^+$.

Lemma 2.2 *1. The fixed points of f on $[0, \infty)$ are $\mu_0 = 0 < \mu_1 < \mu_2 < \dots$ and satisfy $n - 1 \leq \mu_n \leq n$ and*

$$\mu_n = n - \frac{1}{2} + O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty. \tag{1}$$

2. The critical points of f on $[0, \infty)$ are $c_1 < c_2 < c_3 < \dots$ and satisfy $\mu_n < c_n < \mu_{n+1}$ for $n = 1, 2, \dots$ and

$$c_n = n - \frac{2}{\pi^2 n}\left((-1)^n - \frac{1}{2}\right) + O\left(\frac{1}{n^2}\right) \quad \text{as} \quad n \to \infty. \tag{2}$$

Proof: The fixed points of f satisfy $f(x) = x$, which implies

$$\cos(\pi x) = \frac{1}{2x + 1}.$$

Using $\cos(\pi(x + n - \frac{1}{2})) = \sin(\pi(x + n)) = x + O(x^3)$, we obtain the asymptotic formula (1). Since

$$f'(x) = 1 - \frac{1}{2}\cos(\pi x) + \pi\left(\frac{2x + 1}{4}\right)\sin(\pi x),$$

this implies $c_n = n + O(1/n)$ as $n \to \infty$. Expanding $\sin(\pi(x + n))$ and $\cos(\pi(x + n))$ to order $O(x^3)$, and letting $c_n = n + c/n + O(1/n^2)$, we obtain (2). □

Some numerical values may be found in the table in section 3.

A reference for some of the basics of discrete dynamical systems is Devaney[6]. More advanced treatments are given in the recent works of Block and Coppel[2] and de Melo and van Strien[5].

3 Iterates on Positive Reals

The function f has many invariant subsets on $\mathbb{R}^+$, for example $\mathbf{Z}^+$. We consider iteration of f on certain subintervals of $[0, \infty)$, namely $I_1 = [0, \mu_1]$, $I_2 = [\mu_1, \mu_3]$ and $I_3 = [\mu_3, \infty)$. The first two of these intervals are invariant sets. On the first interval the behaviour of f is extremely simple.

Theorem 3.1 *The invariant set $I_1 = [0, \mu_1]$ is invariant under f. It contains a single attracting fixed point $x = \mu_0 = 0$, and every point in I_1 is attracted to it except the repelling fixed point at $x = \mu_1$.*

Proof: The function f is increasing on I_1, therefore I_1 is invariant. Since there are no fixed points on $(0, \mu_1)$ and $f'(0) = 0.5$, $f(x) < x$ for $x \in (0, \mu_1)$, hence the fixed point attracts every point in $(0, \mu_1)$. $\square$

Next we turn to iterates on the invariant set $I_2 = [\mu_1, \mu_3]$, which contains the trivial cycle $\{1, 2\}$ of the function T. The main result on this interval is

Theorem 3.2 *The interval $I_2 = [\mu_1, \mu_3]$ is invariant under f. It contains exactly two attracting periodic orbits, which are*

$$A_1 := \{1, 2\}$$

$$A_2 := \{1.192531907\ldots, 2.138656335\ldots\}.$$

There is a partition

$$I_2 = \Omega(A_1) \cup \Omega(A_2) \cup \Gamma$$

in which $\Omega(A_i)$ is the basin of attraction for the set A_i, $i = 1, 2$, and Γ is a set of measure zero.

Proof: The interval (μ_1, μ_3) contains exactly two critical points, c_1 and c_2, with $f(c_i) \in (\mu_1, \mu_3)$, $i = 1, 2$, hence I_2 is invariant. By Theorem 2.1, there are at most two attracting periodic orbits in I_2 (we actually have $\omega(c_1) = A_2$ $\omega(c_2) = A_1$.) One may verify that A_1 and A_2 are both attracting two-cycles.

From the remark following Theorem 2.2, we have that the basins of attraction for A_1 and A_2 cover I_2 except for a set of measure zero. $\square$

The interval $I_3 = [\mu_3, \infty)$ is not an invariant set for f, because some of the points eventually escape to the set $[\mu_1, \mu_3)$. The interval I_3 has a partition

$$I_3 = E_f \cup R_f,$$

consisting of the *escape set*

$$E_f := \{x \in I_3 : \sup \omega(x) < \mu_3\}$$

which is the open set of iterates that eventually leave I_3, and the *residual set*

$$R_f := \{x \in I_3 : \inf \omega(x) \geq \mu_3\}$$

which is a closed invariant subset of I_3. The $3x+1$ Conjecture asserts that every integer $n \geq 3$ eventually iterates to 1, hence escapes from I_3, so we may reformulate it as:

$3x+1$ **Conjecture.** *The set* $\mathbf{Z}^+ \cap R_f = \emptyset$.

We partition the residual set R_f as

$$R_f = S_f \cup U_f,$$

in which S_f is an open set consisting of the complete basins of attraction of all attracting periodic orbits of f in I_3, and U_f is the remaining closed set. We call S_f the *stable set* and U_f the *unstable set.* We study S_f in the remainder of this section; we study U_f in the next section.

The following result shows that the stable set S_f includes all nontrivial cycles of the $3x+1$ function on $\mathbf{Z}^+$.

Theorem 3.3 *Any cycle of the* $3x+1$ *function on* $\mathbf{Z}^+$ *is an attracting periodic orbit of* f.

Proof: Suppose there is a periodic orbit Ω of f on $\mathbb{R}^+$. Following Eliahou[7], we have

$$\begin{aligned} 1 &= \prod_{x\in\Omega} \frac{f(x)}{x} \\ &= \prod_{x\in\Omega} \left[1 - \frac{1}{2}\cos(\pi x) + \frac{1}{4x}(1-\cos(\pi x))\right] \\ &> \prod_{x\in\Omega} \left[1 - \frac{1}{2}\cos(\pi x)\right]. \end{aligned}$$

The inequality is strict since there are no periodic orbits strictly on the even integers. Note also that

$$\prod_{x\in\Omega} f'(x) = \prod_{x\in\Omega} \left[1 - \frac{1}{2}\cos(\pi x) + \pi\frac{2x+1}{4}\sin(\pi x)\right]. \tag{3}$$

If $\Omega \subset \mathbf{Z}^+$, we have

$$0 < \prod_{x\in\Omega} f'(x) = \prod_{x\in\Omega} \left[1 - \frac{1}{2}\cos(\pi x)\right] < 1 \tag{4}$$

thus the periodic orbit is an attractor. □

In pursuing the existence of attractive periodic orbits, note that equation (3) shows that if there is an attracting periodic orbit on $\mathbb{R}^+$, it must have at least one element "close" to an integer. In view of the conjecture that the $3x+1$ function T has no nontrivial cycle on $\mathbf{Z}^+$, we extend this conjecture to propose:

Stable Set Conjecture. $S_f = \emptyset$.

Since the Schwarzian derivative is negative, we know that the immediate basin of attraction of Ω contains a critical point[1]. By considering the ω-limit sets of the critical points, numerical evidence indicates that few critical points approach A_2.

[1] Singer's Theorem 2.1 applies not only to maps on compact intervals, but also when the immediate basin of attraction is bounded, which clearly holds for the function f.

n	μ_n	c_n	$\omega(c_n)$
1	0.277337662	1.180938709	A_2
2	1.577373244	1.958312216	A_1
3	2.445707694	3.084794846	A_2
4	3.539500961	3.977293183	A_1
5	4.467909060	5.054721917	A_2
6	5.526439840	5.984349314	A_1
7	6.477169613	7.040311859	A_1
8	7.519857986	7.988051868	A_1
9	8.482272052	9.031889137	A_1
10	9.515896841	9.990334437	A_1

The only values of n with $\omega(c_n) = A_2$ for $n \leq 1000$ are $\{1,3,5,502,656\}$. For all other $n \leq 1000$ we have $\omega(c_n) = A_1$.

Critical Point Conjecture. *Every critical point c_n has either $\omega(c_n) = A_1$ or $\omega(c_n) = A_2$.*

The Critical Point Conjecture implies the Stable Set Conjecture, which itself implies that the $3x + 1$ problem has no nontrivial cycles.

4 The Unstable Set

The unstable set U_f has a complicated structure. We may partition it as

$$U_f = U_f^0 \cup U_f^\infty,$$

in which U_f^0 and U_f^∞ consist of the bounded and unbounded orbits in U_f, respectively. Note that $\mathbf{Z}^+ \cap U_f^0 = \emptyset$, because a bounded orbit in $\mathbf{Z}^+$ is eventually periodic, and all such orbits are stable by Theorem 3.3.

Note that

$$U_f^\infty = \left\{ x \in R_f : \limsup_{k \to \infty} f^{(k)}(x) = \infty \right\},$$

that is, U_f^∞ is the set of "divergent trajectories." The non-existence of divergent trajectories for the $3x+1$ function on $\mathbf{Z}^+$ becomes:

Divergent Trajectories Conjecture. $\mathbf{Z}^+ \cap U_f^\infty = \emptyset$.

If this conjecture is true, then $\mathbf{Z}^+ \cap U_f = \emptyset$, and the dynamics of T on $\mathbf{Z}^+$ is unrelated to the dynamics of f om U_f.

Now we study the structure of the invariant set U_f, showing that U_f is uncountable. For this purpose we recall the theorem of Li and Yorke in "Period Three implies Chaos".

Theorem 4.1 *(Li and Yorke[12]) Let J be an interval and let $F : J \to J$ be continuous. Assume there is a three-cycle. Then there is a periodic orbit of every period, and there is an uncountable set $S \subset J$ containing no periodic orbits which satisfies the following conditions:*

1. *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{5}$$

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0 \tag{6}$$

2. *For every $p \in S$ and periodic point $q \in J$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{7}$$

We use this result to prove

Theorem 4.2 *The set U_f^0 is uncountable.*

Proof: The function f has an orbit of period 3 in $[\mu_3, \mu_5]$, namely

$$\Omega := \{2.61376758\cdots, 3.408443979\cdots, 4.212821646\ldots\}.$$

Form the new function $\bar{f} : [\mu_3, \mu_5] \to [\mu_3, \mu_5]$ by

$$\bar{f}(x) = \begin{cases} \mu_3 & \text{if } f(x) \leq \mu_3, \\ f(x) & \text{if } \mu_3 < f(x) < \mu_5, \\ \mu_5 & \text{if } f(x) \geq \mu_5. \end{cases}$$

Since $\bar{f}$ has a periodic orbit of period 3, Theorem 4.1 produces an uncountable set $S \subset [\mu_3, \mu_5]$ containing no periodic points which satisfies

$$\limsup_{n\to\infty} \left|\bar{f}^n(p) - \bar{f}^n(q)\right| > 0$$

for any $p \in S$ and periodic point $q \in S$. Applying this to $q = \mu_3$ and $q = \mu_5$ shows that neither of these points is in the orbit of any $p \in S$, thus $\bar{f}(S) = f(S)$. Thus $S \subset U_f^0$, gving the desired result. □

The set U_f^0 contains μ_3. In fact the point μ_3 is a "homoclinic point" in the following sense.

Theorem 4.3 *Given any $\epsilon > 0$, let $J = (\mu_3, \mu_3 + \epsilon)$. Then we have*

$$(1, \infty) \subset \bigcup_{x \in J} Orb(x).$$

Proof: Since $\mu_4 = 3.5395\cdots > 3$, there must be some $x_1 \in J$ and $k_1 \in Z^+$ such that $f^{k_1}(x) = 3$. This implies $f^{k_1+2}(x_1) = 8$. Since f is continuous, intervals map to intervals, thus there must exist $x_2 \in J$ and $k_2 \in Z^+$ such that $f^{k_2}(x_2) = 7$. If $x > 3$ is an odd integer, then

$$f(x) = \frac{3x+1}{2} > x + \lfloor x/2 \rfloor > x + 2,$$

so by continuing this process, we can find values $x \in J$ such that $Orb(x)$ contains values arbitrarily large, thus I_3 is contained in $\cup_{x\in J} Orb(x)$. Since $1 \in Orb(3)$, it is clear that the open interval $(1, \mu_3)$ is also in this set. □

A similar method can be applied to gain information on the set U_f^∞.

Theorem 4.4 *The set U_f^∞ contains a monotonically increasing divergent trajectory. In particular, U_f^∞ is infinite.*

Proof: The continuity of f guarantees there exists an interval $J_1 \subset [3, 4]$ such that $f(J_1) \subset [4, 5]$. Similarly, there is an interval $J_2 \subset J_1$ such that

$f^2(J_2) \subset [7,8]$. This process may be continued to give a sequence of intervals $J_1 \supset J_2 \supset J_3 \supset \cdots$ such that

$$\inf_{x \in J_k} f^k(x) > \sup_{x \in J_{k-1}} f^{k-1}(x)$$

and

$$\lim_{k \to \infty} \inf_{x \in J_k} f^k(x) = \infty.$$

Since

$$\bigcap_{k=1}^{\infty} J_k$$

is non-empty, there exists a divergent trajectory. $\square$

Although both U_f^0 and U_f^∞ are both infinite sets, the preimages of integers $\mathbf{Z}^+$ appear to be dense in I_3, so the $3x+1$ Conjecture suggests:

Unstable Set Conjecture. *The unstable set U_f has Lebesgue measure zero.*

Theorem 4.2 shows that U_f^0 contains an uncountable "scrambled set". The same proof may be used to show the existence of such sets on other intervals between some periodic points of f. Indeed, a computer plot reveals that the period three points of f become increasingly dense for greater values of x, inducing very tight scrambled sets which come arbitrarily close to the integers. Since the dynamics on scrambled sets is associated with randomness, the chaotic dynamics on the scrambled sets strongly influences the dynamics on the large positive integers. This gives a heuristic argument, in terms of chaos, for the apparent randomness in the $3x+1$ problem.

5 Compactification of the Map f

We study the iteration of f on the invariant interval $I^* = [\mu_1, \infty)$. This domain is not compact, but we can smoothly conjugate f to a function h which has a continuous extension to a compact domain as follows. Consider the homeomorphism $\sigma(x) = 1/x$ which maps the domain I^* to $(0, 1/\mu_1]$ and define $h(x)$ by

$\sigma(h(x)) = f(\sigma(x))$, with

$$h(x) = \frac{4x}{4 + x - (2 + x)\cos(\pi/x)}$$

for $x \in (0, 1/\mu_1]$. Now $h(x)$ extends to a continuous function $\tilde{h} : [0, 1/\mu_1] \to [0, 1/\mu_1]$ by setting $\tilde{h}(0) = 0$ and $\tilde{h}(x) = h(x)$ for $x \in (0, 1/\mu_1]$. Since the conjugating function $1/x$ has zero Schwarzian derivative, it follows that $h(x)$ has negative Schwarzian derivative on $(0, 1/\mu_1]$. The map $\tilde{h}$ has an infinite number of critical points. Theorem 2.2 unfortunately does not apply to $\tilde{h}$ because it is not C^1 at the extreme point $x = 0$. This compactification of f allows another approach to studying the iterates of f on $\mathbb{R}^+$.

6 Conclusion

These results show that the map f has some "extra" dynamics on $\mathbb{R}^+$ which are not accounted for by its dynamics on Z^+, even if the $3x + 1$ conjecture is true. However, the Stable Set Conjecture and the Unstable Set Conjecture together would say that almost every orbit in $\mathbb{R}^+$ is "explained" by the dynamics of f on $\mathbf{Z}^+$.

One can also consider the $3x + 1$ problem on $\mathbf{Z}^+$, the negative integers; this is essentially the same as studying the $3x - 1$ problem on $\mathbf{Z}^+$. Three periodic orbits are known for T on $\mathbf{Z}^-$, which start at $n = -1, -5$ and -17 and have periods $1, 3$ and 11 respectively. For the function f any periodic orbit on $\mathbf{Z}^-$ is repelling; this can be proved similarly to Theorem 3.3, noting that inequality (4) reverses.

Studying iterates of the map f for complex values would be very different. One natural line of interest would be to determine properties of the Julia set $J(f)$.

Lastly, it should be noted that some of the results given are dependent on the particular interpolation function f. Any continuous interpolation of the function T can be proven to have a three-cycle, a homoclinic orbit and divergent trajectories.

Theorem 6.1 *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function which interpolates the function T. Then g has a three-cycle, a homoclinic orbit and a divergent trajectory.*

Proof: To prove that a three-cycle exists, define M as

$$M = \max\{x \leq 3 : g(x) = x\}.$$

Note that $g(M) = M$. Since $g(2) = 1$ and $g(3) = 5$, we have $2 < M < 3$. The continuity of g also implies there exists an $a \in (M, 3)$ such that $g^3(x) = 1$. Since $g(x) > x$ on $(M, 3)$, $g^3(x) > x$ for $x \in (M, 3)$ sufficiently close to M. Putting this together implies there exists $z \in (M, 3)$ such that $g^3(z) = z$. Since z cannot be a fixed point and 3 is a prime number, z must be a periodic point of period 3. Similarly, there must exist some $y \in (M, 3)$ such that $g^3(y) = M$, thus y is a homoclinic point. The existence of a divergent trajectory follows as in Theorem 4.4. $\square$

If we can compactify the map g, then we would obtain chaos by Theorem 4.1. The function f chosen seems to be the extension which permits the "simplest" analysis. The fact that this function admits a negative Schwarzian derivative on $\mathbb{R}^+$ greatly helps our understanding of the dynamics; it seems unlikely that a general extension will have this property.

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