THE $3x+1$ PROBLEM

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Abstract. This paper gives an overview of the Collatz function and conjecture. Furthermore, its history and some interesting attributes are discussed.

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1.1. What is the $3x+1$ Problem?

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1.1.1. *The Function.* Based on the Collatz function [3]

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod 2, \\ x/2 & \text{if } x \equiv 0 \pmod 2. \end{cases} \quad (1)$$

Is equivalent to the $3x + 1$ function [3]

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod 2, \\ x/2 & \text{if } x \equiv 0 \pmod 2. \end{cases} \quad (2)$$

1.1.2. *Details.*

- it is conjectured that for some $x, k \in \mathbb{N} + 1$ we attain $T^{(k)}(x) = 1$ [1]
- the $3x + 1$ function $T(x)$ maps $\mathbb{N} + 1 \to \mathbb{N} + 1$ [4]
- the function has a *stopping time*, *total stopping time*, and a *trajectory* for each m

1.1.3. *Stopping Time for x.*

- check that every positive integer up to $x - 1$ iterates to one [1]
- if $T^{(k)}(x) < x$, we know it will iterate to 1
- thus the stopping time is

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\} \quad (3)$$

1.1.4. *Total Stopping Time for x.* Total stopping time is the number of steps needed to iterate to 1 [1]

$$\sigma_\infty(x) = \inf\{k : T^{(k)}(x) = 1\} \quad (4)$$

1.1.5. *Trajectory of x Under T.* Also called the *forward orbit* of x under T, defined as the sequence of it forward iterates [3]

$$\{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\} \quad (5)$$

1.2. **The Collatz Conjecture.**

1.2.1. *Possible behaviors of T.*

(1) the trivial cycle $\{4, 2, 1, 4, 2, 1, \dots\}$ (reaching 1)
(2) a non-trivial cycle
(3) infinity, having a divergent orbit [1]

1.2.2. *The Conjecture.*

- beginning at any positive integer x, iterations of $T(x)$ will eventually reach 1 and enter the trivial cycle [3]
- equivalent to stating that the total stopping time $\sigma_\infty(x)$ are finite [1]
- if a trajectory of $T(x)$ does *not* contain 1 it is infinite [2]

1.3. What Makes This Problem Interesting?

Mathematics is not ready for such problems. — Paul Erdös [1]

- the problem itself is not important, it has no immediate applications
- represents a class of iterative mappings that are interesting
- it is simple to state but hard to prove
- part of the difficulty comes from its pseudorandom nature of iterations of $T(x)$ [3]

2. History of the Collatz Conjecture

2.1. Beginnings.

- also known as Syracuse Problem, Hasse's Algorithm, Kakutani's Problem, and Ulam's Problem after other people that studied it
- named after Lothar Collatz who formulated similar problems in the 1930s
- academic publishing about it began in the 1970s [3]

2.2. Recent Developments.

- $> 10^{20}$ numbers have been verified to fit the conjecture [4]
- a September 2019 paper by Terence Tao "Almost All Orbits of the Collatz Map Attain Almost Bounded Values" made progress
- research is still actively ongoing

3. Interesting Attributes of the $3x + 1$ Problem

3.1. Cycles of the Function.

- the $3x + 1$ function has a trivial cycle $\{4, 2, 1, 4, 2, \dots\}$ at 1 [1]
- if $T(x)$ is applied to all integers, three more cycles emerge at -1, -5, and -17
- these cycles are conjectured to be the only ones [1]
- if non-trivial cycles of the $3x + 1$ problem exist, they have been proven to be at least 10,439,860,591 numbers long [3]

3.2. Stochastic Approximations.

- number of odd and even integers in an orbit is approximately equal
- behavior is seen as pseudorandom, if the numbers are large enough they almost behave like random variables [2]
- probabilistic models describe the behavior of the $3x + 1$ problem
- models describe groups of trajectories, not individual ones [3]

3.3. **Stopping Time of the Function.**

- stopping time for odd numbers is ≈ 9.477955 for $C(x)$ [1]
- total stopping time for most trajectories is about $6.95212 \log n$ steps
- number of even integers in an orbit equal to stopping time
- upper bound for total stopping time $41.677647 \log n$, suggests all sequences are finite [3]

3.4. **Conclusion.**

3.4.1. *The $3x+1$ Problem and Collatz Conjecture.* For every $x \in \mathbb{N}+1$ and the function

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases} \tag{6}$$

there is some $k \in \mathbb{N}+1$ such that $T^{(k)}(x) = 1$.

3.4.2. *What Makes This Problem Interesting?*

- simple to state but hard to prove
- represents a class of iterative mappings that are interesting [3]
- maybe mathematics right now cannot solve that problem

3.4.3. *History of the Collatz Conjecture.*

- named after Lothar Collatz, from the 1930s
- academic publishing began in the 1970s [3]
- $> 10^{20}$ numbers have been verified to fit the conjecture [4]
- research is still actively ongoing

3.4.4. *Interesting Attributes of the $3x+1$ Problem.*

- the $3x+1$ function has a trivial cycle $\{2, 1, 2, \dots\}$ at 1 [1]
- non-trivial cycles of the $3x+1$ problem have been proven to be at least 10,439,860,591 numbers long [3]
- behavior is seen as pseudorandom, if the numbers are large enough they almost behave like random variables [2]
- total stopping time for most trajectories is about $6.95212 \log n$ steps

$$\pi(x) \approx \rho \times \dagger c^2 \tag{7}$$

The above equation eq01 refers to some mathematical concept that I have made up.

References

[1] Marc Chamberland, *An Update on the $3x + 1$ Problem*, 2005.

[2] R. E. Crandall, *On the "$3x+1$" Problem*, Mathematics of Computation, **32** (1978), no. 144, 1281-1292

[3] Jeffrey C.Lagarias, *The $3x + 1$ Problem: An Overview*, `https://pdfs.semanticscholar.org/100046dd8b4ee901bc71043da7d42f5d87ca0224.pdf`, 2010

[4] Terence Tao, *Almost All Orbits of the Collatz Map Attain Almost Bounded Values*, arXiv:1909.03562v2 [math.PR], 2019