

An Update on the $3x+1$ Problem

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1 Introduction

The $3x+1$ Problem is perhaps today's most enigmatic unsolved mathematical problem: it can be explained to child who has learned how to divide by 2 and multiply by 3, yet there are relatively few strong results toward solving it. Paul Erdős was correct when he stated, "Mathematics is not ready for such problems."

The problem is also referred to as the $3n+1$ problem and is associated with the names of Collatz, Hasse, Kakutani, Ulam, Syracuse, and Thwaites. It may be stated in a variety of ways. Defining the *Collatz function* as

$$C(x) = \begin{cases} 3x+1 & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2}, \end{cases}$$

the conjecture states that for each $m \in \mathbf{Z}^+$, there is a $k \in \mathbf{Z}^+$ such that $C^{(k)}(m) = 1$, that is, any positive integer will eventually iterate to 1. Note that an odd number m iterates to $3m+1$ which then iterates to $(3m+1)/2$. One may therefore "compress" the dynamics by considering the map

$$T(x) = \begin{cases} \frac{3x+1}{2} & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2}. \end{cases}$$

The map T is usually favored in the literature.

To a much lesser extent some authors work with the most dynamically streamlined $3x+1$ function, $F: \mathbf{Z}_{\text{odd}}^+ \rightarrow \mathbf{Z}_{\text{odd}}^+$, defined by

$$F(x) = \frac{3x+1}{2^{m(3x+1)}}$$

where $m(x)$ equals the number of factors of 2 contained in $3x+1$. While working with F allows one to work only on the odd positive integers, the variability of m seems to prohibit any substantial analysis.

This survey reflects the author's view of how work on this problem can be structured. I owe a huge debt to Jeff Lagarias and Günther Wirsching for the important work they have done in bringing this problem forward. The paper of Lagarias[45](1985) thoroughly catalogued earlier results, made copious

connections, and developed many new lines of attack; it has justly become the classical reference for this problem. Wirsching's book [87](1998) begins with a strong survey, followed by several chapters of his own noteworthy analysis. Lagarias has also maintained an annotated bibliography [48](1998) of work on this problem, another valuable resource. This current survey would have been much more difficult to write in the absence of these significant contributions.

This survey is not meant to be exhaustive, but rather is complementary to the work of Lagarias and Wirsching. Where I believed there was significant new work in a given area, I included earlier contributions for the sake of completeness. Some areas which have not seen recent development, such as the interesting work connected to functional equations, cellular automata, and the origin of the problem, have not been mentioned; the reader may consult Wirsching's book [87](1998).

2 Numerical Investigations and Stopping Time

The structure of the positive integers forces any orbit of T to iterate to one of the following:

1. the trivial cycle $\{1, 2\}$
2. a non-trivial cycle
3. infinity (the orbit is divergent)

The $3x + 1$ Problem claims that option 1 occurs in all cases. Oliveira e Silva[61, 62](1999,2000) proved that this holds for all numbers $n < 100 \times 2^{50} \approx 1.12 \times 10^{17}$. This was accomplished with two 133MHz and two 266MHz DEC Alpha computers and using 14.4 CPU years. This computation ended in April 2000. Roosendaal[65](2003) claims to have improved this to $n = 195 \times 2^{50} \approx 2.19 \times 10^{17}$. His calculations continue, with the aid of many others in this distributed-computer project.

The record for proving the non-existence of non-trivial cycles is that any such cycle must have length no less than 272,500,658. This was derived with

the help of numerical results like those in the last paragraph coupled with the theory of continued fractions – see a section 5 for more on cycles.

There is a natural algorithm for checking that all numbers up to some N iterate to one. First, check that every positive integer up to $N - 1$ iterates to one, then consider the iterates of N . Once the iterates go below N , you are done. For this reason, one considers the so-called *stopping time* of n , that is, the number of steps needed to iterate below n :

$$\sigma(n) = \inf\{k : T^{(k)}(n) < n\}.$$

Related to this is the *total stopping time*, the number of steps needed to iterate to 1:

$$\sigma_\infty(n) = \inf\{k : T^{(k)}(n) = 1\}.$$

One considers the *height* of n , namely, the highest point to which n iterates:

$$h(n) = \sup\{T^{(k)}(n) : k \in \mathbb{Z}^+\}.$$

Note that if n is in a divergent trajectory, then $\sigma_\infty(n) = h(n) = \infty$. These functions may be surprisingly large even for small values of n . For example,

$$\sigma(27) = 59, \quad \sigma_\infty(27) = 111, \quad h(27) = 9232.$$

The orbit of 27 is depicted in Figure 1.

Roosendaal[65](2003) has computed various “records” for these functions¹. Various results about consecutive numbers with the same height are catalogued by Wirsching[87, pp.21–22](1998).

2.1 Stopping Time

The natural algorithm mentioned earlier can be rephrased: the $3x + 1$ problem is true if and only if every positive integer has a finite stopping time. Terras[77,

¹Roosendaal has different names for these functions, and he applies them to the map $C(x)$.

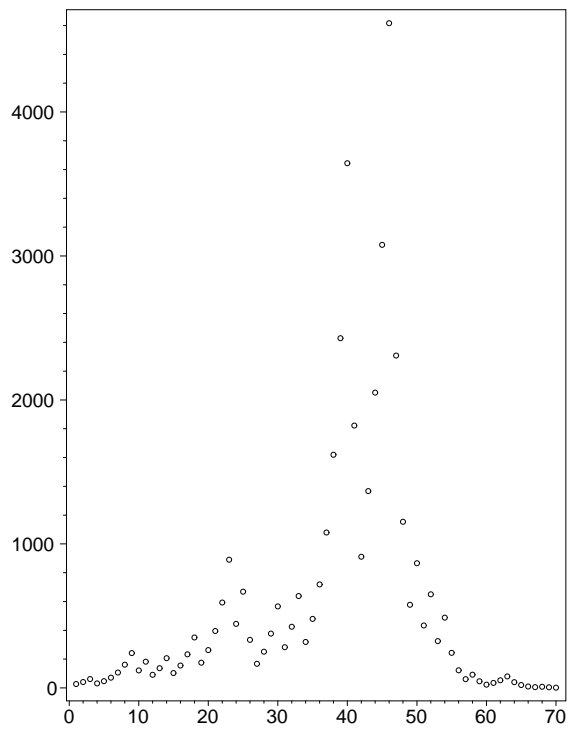


Figure 1: Orbit starting at 27.

78](1976,1979) has proven that the set of positive integers with finite stopping time has density one. Specifically, he showed that the limit

$$F(k) := \lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : \sigma(n) \leq k\}|$$

exists for each $k \in \mathbf{Z}^+$ and $\lim_{k \rightarrow \infty} F(k) = 1$. A shorter proof was provided by Everett[31](1977). Lagarias[45](1985) proved a result regarding the speed of convergence:

$$F(k) \geq 1 - 2^{-\eta k}$$

for all $k \in \mathbf{Z}^+$, where $\eta = 1 - H(\theta)$, $H(x) = -x \log(x) - (1-x) \log(1-x)$ and $\theta = (\log_2 3)^{-1}$. He uses this to constrain any possible divergent orbits:

$$|\{n \in \mathbf{Z}^+ : n \leq x, \sigma(n) = \infty\}| \leq c_1 x^{1-\eta}$$

for some positive constant c_1 . This implies that any divergent trajectory cannot diverge too slowly. Along these lines, Garcia and Tal[33](1999) have recently proven that the density of any divergent orbit is zero. This result holds for more general sets and more general maps.

Along different lines, Venturini[81](1989) showed that for every $\rho > 0$, the set $\{n \in \mathbf{Z}^+ : T^{(k)}(n) < \rho n \text{ for some } k\}$ has density one. Allouche[1](1979) showed that $\{n \in \mathbf{Z}^+ : T^{(k)}(n) < n^c \text{ for some } k\}$ has density one for $c > 3/2 - \log_2 3 \approx 0.869$. This was improved by Korec[39](1994) who obtained the same result with $c > \log_4 3 \approx 0.7924$.

Not surprisingly, results of a probabilistic nature abound concerning these $3x+1$ functions. An assumption often made is that after many iterations, the next iterate has an equal chance of being either even or odd. Wagon[84](1985) argues that the average stopping time for odd n (with the function $C(x)$) approaches a constant, specifically, the number

$$\sum_{i=1}^{\infty} [1 + 2i + i \log_2 3] \frac{c_i}{2^{[i \log_2 3]}} \approx 9.477955$$

where c_i is the number of sequences containing $3/2$ and $1/2$ with exactly i $3/2$'s such that the product of the whole sequence is less than 1, but the product

of any initial sequence is greater than 1. Note that this formula bypasses the pesky “+1”, perhaps justified asymptotically. This seems to be borne out in Wagon’s numerical testing of stopping times for odd numbers up to $n = 10^9$, which matches well with the approximation given above.

2.2 Total Stopping Time

Results regarding the total stopping time are also plentiful. Applegate and Lagarias[9](2002) make use of two new auxiliary functions: the *stopping time ratio*

$$\gamma(n) := \frac{\sigma_\infty(n)}{\log n},$$

and the *ones-ratio* $\rho(n)$ for convergent sequences, defined as the ratio of the number of odd terms in the first $\sigma_\infty(n)$ iterates divided by $\sigma_\infty(n)$. It is easy to see that $\gamma(n) \geq 1/\log 2$ for all n , with equality only when $n = 2^k$. Stronger inequalities are

$$\gamma(n) \geq \frac{1}{\log 2 - \rho(n) \log 3}$$

for any convergent trajectory, while if $\rho(n) \leq 0.61$, then for any positive ϵ ,

$$\gamma(n) \leq \frac{1}{\log 2 - \rho(n) \log 3} + \epsilon$$

for all sufficiently large n . If one assumes that the ones-ratio equals $1/2$ (which is equivalent to the equal probability of encountering an even or an odd after many iterates), we have that the average value of $\gamma(n)$ is $2/\log(4/3) \approx 6.95212$, as has been observed by Shanks[69](1965), Crandall[26](1978), Lagarias[45](1985), Rawsthorne[64](1985), Lagarias and Weiss[47](1992), and Borovkov and Pfeifer[15](2000). Experimental evidence supports this observation. Compare this with the upper bounds suggested by the stochastic models of Lagarias and Weiss[47](1992):

$$\limsup_{n \rightarrow \infty} \gamma(n) \approx 41.677647.$$

Applegate and Lagarias[9](2002) note from records of Roosendaal[65](2003) what is apparently the largest known value of γ :

$$n = 7, 219, 136, 416, 377, 236, 271, 195$$

produces $\sigma_\infty(n) = 1848$ and $\gamma(n) \approx 36.7169$. Applegate and Lagarias seek a lower bound for γ which holds infinitely often. Using the well-known fact that $T^{(k)}(2^k - 1) = 3^k - 1$ (see, for example, Kuttler[44](1994)), they note that

$$\gamma(2^k - 1) \geq \frac{\log 2 + \log 3}{(\log 2)^2} \approx 3.729.$$

They go on to show that there are infinitely many converging n whose ones-ratio is at least $14/29$, hence giving the lower bound

$$\gamma(n) \geq \frac{29}{29 \log 2 - 14 \log 3} \approx 6.14316$$

for infinitely many n . Though the proof involves an extensive computational search on the Collatz tree to depth 60, the authors note with surprise that the probabilistic average of $\gamma(n)$ – approximately 6.95212 – was not attained.

Roosendaal[65](2003) defines a function which he calls the *residue* of a number $x \in \mathbf{Z}^+$, denoted $\text{Res}(x)$. Suppose x is a convergent C -trajectory with E even terms and O odd terms before reaching one. Then the residue is defined as

$$\text{Res}(x) := \frac{2^E}{x3^O}.$$

Roosendaal notes that $\text{Res}(993) = 1.253142\dots$, and that this is the highest residue attained for all $x < 2^{32}$. He conjectures that this holds for all $x \in \mathbf{Z}^+$.

Zarnowski[89](2001) recasts the $3x + 1$ problem as one with Markov chains. Let g be the slightly altered function

$$g(x) = \begin{cases} \frac{3x+1}{2} & x \equiv 1 \pmod{2}, x > 1 \\ \frac{x}{2} & x \equiv 0 \pmod{2} \\ 1 & x = 1, \end{cases}$$

\mathbf{e}_n the column vector whose n^{th} entry is one and all other entries zero, and the transition matrix P be defined by

$$P_{ij} = \begin{cases} 1 & \text{if } g(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a correspondence between $g^{(k)}(n)$ and $\mathbf{e}_n^T P^k$, and the $3x + 1$ Problem is true if

$$\lim_{k \rightarrow \infty} P^k = [\mathbf{1} \ \mathbf{0} \ \mathbf{0} \ \dots],$$

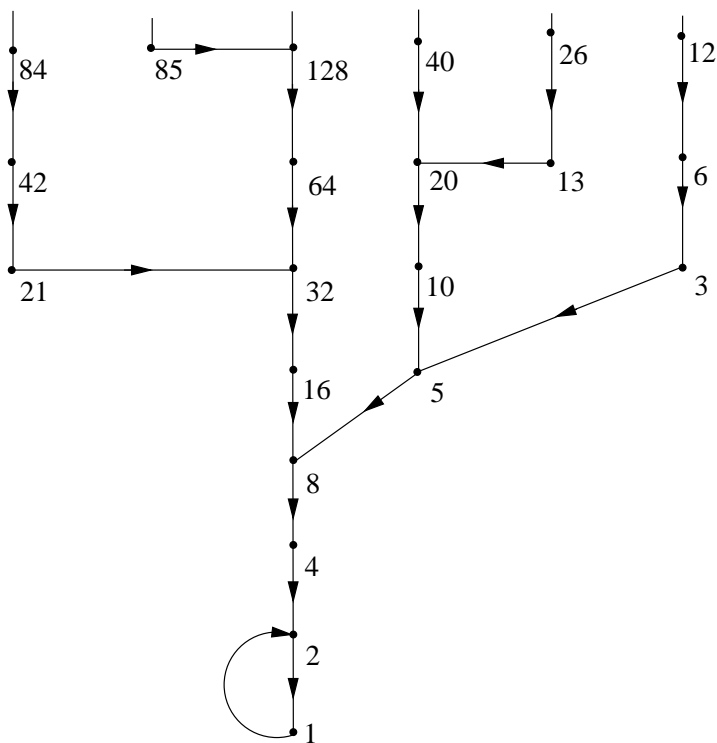


Figure 2: The Collatz graph.

where $\mathbf{1}$ and $\mathbf{0}$ represent constant column vectors. Zarnowski goes on to show that the structure of a certain generalized inverse X of $I - P$ encodes the total stopping times of \mathbb{Z}^+ .

For pictorial representations of stopping times for an extension of T , see Dumont and Reiter[29].

2.3 The Collatz Graph and Predecessor Sets

The topic of stopping times is intricately linked with the *Collatz tree* or *Collatz graph*, the directed graph whose vertices are predecessors of one via the map T . It is depicted in Figure 2.3.

The structure of the Collatz graph has attracted some attention. Andaloro[4](2002) studied results about the connectedness of subsets of the Collatz graph. Urvoy[79](2000)

proved that the Collatz graph is non-regular, in the sense that it does not have a decidable monadic second order theory; this is related to the work of Conway mentioned in Section 4.

Instead of analyzing how fast iterates approach one, one may consider the set of numbers which approach a given number a , that is, the *predecessor set* of a :

$$P_T(a) := \{b \in \mathbb{Z}^+ : T^{(k)}(b) = a \text{ for some } k \in \mathbb{Z}^+\}$$

Since such sets are obviously infinitely large, one may measure those terms in the predecessor set not exceeding a given bound x , that is,

$$Z_a(x) := |\{n \in P_T(a) : n \leq x\}|.$$

The size of $Z_a(x)$ was first studied by Crandall[26](1979), who proved the existence of some $c > 0$ such that

$$Z_1(x) > x^c, \quad x \text{ sufficiently large.}$$

Wirsching[87, p.4](1998) notes that this result extends to $Z_a(x)$ for all $a \not\equiv 0 \pmod{3}$. Using the tree-search method of Crandall, Sander[67](1990) gave a specific lower bound, $c = 0.25$, and Applegate and Lagarias[6](1995) extended this to $c = 0.643$. Using functional difference inequalities, Krasikov[41](1989) introduced a different approach and obtained $c = 3/7 \approx 0.42857$. Wirsching[85](1993) used the same approach to obtain $c = 0.48$. Applegate and Lagarias[7](1995) superseded these results with $Z_1(x) > x^{0.81}$, for sufficiently large x , by enhancing Krasikov's approach with nonlinear programming. Recently, Krasikov and Lagarias[42](2002) streamlined this approach to obtain

$$Z_1(x) > x^{0.84}, \quad x \text{ sufficiently large.}$$

Wirsching[87](1998) has pushed these types of results in a different direction. On a seemingly different course, for $j, k \in \mathbb{Z}^+$ let $R_{j,k}$ denote the set of all sums of the form

$$2^{\alpha_0} + 2^{\alpha_1}3 + 2^{\alpha_2}3^2 + \dots + 2^{\alpha_j}3^j$$

where $j + k \geq \alpha_0 > \cdots > \alpha_j \geq 0$. The size of $R_{j,k}$ satisfies

$$|R_{j,k}| = \binom{j+k+1}{j+1}.$$

To maximize the size, one has $|R_{j-1,j}| = \binom{2j}{j}$. Wirsching posed the following “covering conjecture”: there is a constant $K > 0$ such that for every $j, l \in \mathbf{Z}^+$ we have the implication

$$|R_{j-1,j}| \geq K \cdot 2 \cdot 3^{l-1} \implies R_{j-1,j} \text{ covers the prime residue classes to modulus } 3^l.$$

This conjecture implies a stronger version of the earlier inequalities:

$$\liminf_{x \rightarrow \infty} \left(\inf_{a \not\equiv 0 \pmod{3}} \frac{Z_a(ax)}{x^\delta} \right) > 0 \quad \text{for any } \delta \in (0, 1).$$

Wirsching argues that, intuitively, the sums of mixed powers can be seen as accumulated non-linearities occurring when iterating the map T .

A large part of Wirsching’s book [87](1998) concerns a finer study of the predecessor sets using the functions

$$e_l(k, a) = |\{b \in \mathbf{Z}^+ : T^{(k+l)}(b) = a, \text{ } k \text{ even iterates, } l \text{ odd iterates}\}|.$$

Defining the n^{th} *estimating series* as

$$s_n(a) := \sum_{l=1}^{\infty} e_l(n + \lfloor l \log_2(3/2) \rfloor, a),$$

Wirsching proves the implication

$$\liminf_{n \rightarrow \infty} \frac{s_n(a)}{\beta^n} > 0 \implies Z_a(x) \geq C \left(\frac{x}{a} \right)^{\log_2 \beta} \text{ for some constant } C > 0 \text{ and large } x.$$

Since $e_l(k, a)$ depends on a only through its residue class to modulo 3^l , Wirsching considers the functions $e_l(k, \cdot)$ whose domain is \mathbf{Z}_3 , the group of 3-adic integers. Since $e_l(k, a) = 0$ whenever $l \geq 1$ and $3|a$, the set $\{e_l(k, \cdot) : l \geq 1, k \geq 0\}$ is a family of functions on the compact topological group \mathbf{Z}_3^* of invertible 3-adic integers. This application of 3-adic integers to the $3x + 1$ Problem was first seen in Wirsching[86](1994), and similar analysis was also done by Applegate

and Lagarias[8](1995). The functions $e_l(k, \cdot)$ are integrable with respect to \mathbf{Z}_3^* 's unique normalized Haar measure, yielding the 3-adic average

$$\bar{e}_l(k) := \int_{\mathbf{Z}_3^*} e_l(k, a) da = \frac{1}{2 \cdot 3^{l-1}} \binom{k+l}{l}.$$

Several results follow with this approach, including

$$\liminf_{n \rightarrow \infty} \frac{1}{2^n} \int_{\mathbf{Z}_3^*} s_n(a) da > 0,$$

implying that every predecessor set $P_T(a)$ with $a \not\equiv 0 \pmod{3}$ has positive density.

3 Representations of Iterates of a $3x + 1$ Map

The oft-cited result in this area is due to Böhm and Sontacchi[14](1978): the $3x + 1$ problem is equivalent to showing that each $n \in \mathbf{Z}^+$ may be written as

$$n = \frac{1}{3^m} \left(2^{v_m} - \sum_{k=0}^{m-1} 3^{m-k-1} 2^{v_k} \right)$$

where $m \in \mathbf{Z}^+$ and $0 \leq v_0 < v_1 < \dots < v_m$ are integers. Similar results were obtained by Amigó[2, Prop.4.2](2001).

Sinai[70](2003) studies the function F on the set Π defined as

$$\Pi = \{1\} \cup \Pi^+ \cup \Pi^-, \quad \Pi^{\pm 1} = 6\mathbf{Z}^+ \pm 1.$$

Let $k_1, k_2, \dots, k_m \in \mathbf{Z}^+$ and $\epsilon = \pm 1$. Then the set of $x \in \Pi^\epsilon$ to which one can apply $F^{(k_1)} F^{(k_2)} \dots F^{(k_m)}$ is an arithmetic progression

$$\sigma^{(k_1, k_2, \dots, k_m, \epsilon)} = \{6 \cdot (2^{k_1+k_2+\dots+k_m} + q_m) + \epsilon\}$$

for some q_m such that $1 \leq q_m \leq 2^{k_1+k_2+\dots+k_m}$. Moreover,

$$F^{(k_1)} F^{(k_2)} \dots F^{(k_m)}(\sigma^{(k_1, k_2, \dots, k_m, \epsilon)}) = \{6 \cdot (3^m p + r) + \delta\}$$

for some r and δ satisfying $1 \leq r \leq 3^m$, $\delta = \pm 1$. Results in this spirit may be found in Andrei *et al.*[5](2000) and Kuttler[44](1994). Sinai uses this result

to obtain some distributional information regarding trajectories of F . Let $1 \leq m \leq M$, $x_0 \in \square$ and

$$\omega\left(\frac{m}{M}\right) = \frac{\log(F^{(m)}(x_0)) - \log(x_0) + m(2 \log 2 - \log 3)}{\sqrt{M}}.$$

If $0 \leq t \leq 1$, then $\omega(t)$ behaves like a Wiener trajectory.

Related to these results is the connection made in Blecksmith *et al.*[13](1998) between the $3x+1$ problem and 3-smooth representations of numbers. A number $n \in \mathbf{Z}^+$ has a 3-smooth representation if and only if there exist integers $\{a_i\}$ and $\{b_i\}$ such that

$$n = \sum_{i=1}^k 2^{a_i} 3^{b_i}, \quad a_1 > a_2 > \cdots > a_k \geq 0, \quad 0 \leq b_1 < b_2 < \cdots < b_k.$$

Note the similarity with the terms considered by Wirsching in the last section. These numbers were studied at least as early as Ramanujan. A 3-smooth representation of n is *special of level k* if

$$n = 3^k + 3^{k-1}2^{a_1} + \cdots + 3 \cdot 2^{a_{k-1}} + 2^{a_k}$$

in which every power of 3 appears up to 3^k . For a fixed k , each n has at most one such representation – see Lagarias[46](1990), who credits the proof to Don Coppersmith. Blecksmith *et al.* then offer the reader to prove that $m \in \mathbf{Z}^+$ iterates to 1 under C if and only if there are integers e and f such that the positive integer $n = 2^e - 3^f m$ has a special 3-smooth representation of level $k = f - 1$. The choice of e and f is not unique, if it exists.

4 Reduction to Residue Classes and Other Sets

Once one is convinced that the $3x+1$ problem is true, a natural approach is to find subsets $S \subset \mathbf{Z}^+$ such that proving the conjecture on S implies it is true on \mathbf{Z}^+ . It is obvious this holds if $S = \{x : x \equiv 3 \pmod{4}\}$ since numbers in the other residue classes decrease after one or two iterations. Puddu[63](1986) and Cadogan[19](1984) showed that this works if $S = \{x : x \equiv 1 \pmod{4}\}$. The work of Böhm and Sontacchi[14](1978) implies that m odd T -iterates of x



Figure 3: Dynamics of Integers (a) *mod* 3 and (b) *mod* 4.

equals $(3/2)^m(x+1) - 1$, hence odds must eventually become even. Coupling this with the dynamics of integers in \mathbf{Z}_4 (see Figure 3b) implies both $S = \{x : x \equiv 1 \pmod{4}\}$ and $S = \{x : x \equiv 2 \pmod{4}\}$ are sufficient. Similarly, Figure 3a indicates $S = \{x : x \equiv 1 \pmod{3}\}$ or $S = \{x : x \equiv 2 \pmod{3}\}$ suffice. Andaloro[3](2000) has improved these to $S = \{x : x \equiv 1 \pmod{16}\}$. All of these sets have an easily computed positive density, the lowest being Andaloro's set S at $1/16$. Korec and Znam[40](1987) significantly improve this by showing the sufficiency of the set $S = \{x : x \equiv a \pmod{p^n}\}$ where p is an odd prime, a is a primitive root mod p^2 , $p \nmid a$, and $n \in \mathbf{Z}^+$. This set has density p^{-n} which can be made arbitrarily small. In a similar vein, Yang[88](1998) proved the sufficiency of the set

$$\left\{ n : n \equiv 3 + \frac{10}{3}(4^k - 1) \pmod{2^{2k+2}} \right\}$$

for any fixed $k \in \mathbf{Z}^+$. Korec and Znam also claim to have a sufficient set with density zero, but details were not provided. To this end, a recent result of Monks[57](2002) is noteworthy. The last section showed how difficult it is to find a usable closed form expression for $T^{(k)}(x)$. If the “+1” was omitted from the iterations, this would yield $T^{(k)}(x) = 3^m x / 2^k$, where m is the number of odd terms in the first k iterations of x . Monks[57](2002) has proven that there are infinitely many “linear versions” of the $3x+1$ problem. An example given

is the map

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11|n \\ \frac{136}{15}n & \text{if } 15|n \text{ and } NOTA \\ \frac{5}{17}n & \text{if } 17|n \text{ and } NOTA \\ \frac{4}{5}n & \text{if } 5|n \text{ and } NOTA \\ \frac{26}{21}n & \text{if } 21|n \text{ and } NOTA \\ \frac{7}{13}n & \text{if } 13|n \text{ and } NOTA \\ \frac{1}{7}n & \text{if } 7|n \text{ and } NOTA \\ \frac{33}{4}n & \text{if } 4|n \text{ and } NOTA \\ \frac{5}{2}n & \text{if } 2|n \text{ and } NOTA \\ 7n & \text{otherwise} \end{cases}$$

where *NOTA* means “none of the above” conditions hold. Monks shows the $3x+1$ problem is true if and only if for every positive integer n the R -orbit of 2^n contains 2. The proof uses Conway’s *Fractran* language[25](1987) which Conway used [24](1972) to prove the existence of a similar map whose long-term behavior on the integers was algorithmically undecidable. Connecting this material back to sufficient sets, one notes that the density of the set $\{2^n : n \in \mathbb{Z}^+\}$ is zero (albeit, one has a much more complicated map).

5 Cycles

Studying the structure of any possible cycles of T has received much attention.

Letting Ω be a cycle of T , and Ω_{odd} , Ω_{even} denote the odd and even terms in Ω , one may rearrange the equation

$$\sum_{x \in \Omega} x = \sum_{x \in \Omega} T(x)$$

to obtain

$$\sum_{x \in \Omega_{even}} x = \sum_{x \in \Omega_{odd}} x + |\Omega_{odd}|.$$

This was noted by Chamberland[21](1999) and Monks[57](2002).

Using modular arithmetic, Figure 3 indicates the dynamics of integers under T both in \mathbb{Z}_3 and \mathbb{Z}_4 . Since no cycles may have all of its elements of the form

0 mod 3, 0 mod 4, or 3 mod 4, the figures imply that no integer cycles (except $\{0\}$) have elements of the form 0 mod 3. Also, the number of terms in a cycle congruent to 1 mod 4 equals the number congruent to 2 mod 4. This analysis was presented by Chamberland[21](1999).

An early cycles result concerns a special class of T -cycles called circuits. A *circuit* is a cycle which may be written as k odd elements followed by l even elements. Davison[27](1976) showed that there is a one-to-one correspondence between circuits and solutions (k, l, h) in positive integers such that

$$(2^{k+l} - 3^k)h = 2^l - 1. \quad (1)$$

It was later shown using continued fractions and transcendental number theory (see Steiner[71](1977), Rozier[66](1990)) that equation (1) has only the solution $(1, 1, 1)$. This implies that $\{1, 2\}$ is the only circuit.

For general cycles of T , Böhm and Sontacchi[14](1978) showed that $x \in \mathbf{Z}^+$ is in an n -cycle of T if and only if there are integers $0 \leq v_0 \leq v_1 \leq \dots \leq v_m = n$ such that

$$x = \frac{1}{2^n - 3^m} \sum_{k=0}^{m-1} 3^{m-k} 2^{v_k}$$

A similar result is derived by B. Seifert[68](1988).

Eliahou[30](1993) has given some strong results concerning any non-trivial cycle Ω of T . Letting Ω_0 denote the odd terms in Ω , he showed

$$\log_2 \left(3 + \frac{1}{M} \right) \leq \frac{|\Omega|}{|\Omega_0|} \leq \log_2 \left(3 + \frac{1}{m} \right) \quad (2)$$

where m and M are the smallest and largest terms in Ω . Note that for “large” cycles this implies

$$\frac{|\Omega|}{|\Omega_0|} \approx \log_2 3.$$

Eliahou uses this in conjunction with the Diophantine approximation of $\log_2 3$ and the numerical bound $m > 2^{40}$ to show that

$$|\Omega| = 301994a + 17087915b + 85137581c$$

where a, b, c are nonnegative integers, $b \geq 1$ and $ac = 0$. Similar results were found by Chisala[22](1994) and Halbeisen and Hungerbühler[35](1997). This

approach was pushed the farthest by Tempkin and Arteaga[75](1997). They tighten the relations in (2) and use a better lower bound on m to obtain

$$|\Omega| = 187363077a + 272500658b + 357638239c$$

where a, b, c are nonnegative integers, $b \geq 1$ and $ac = 0$.

It is not apparent how the even-odd dissection of a cycle used in these results may be extended to a finer dissection of the terms in a cycle, say, mod 4. Related to this, Brox[17](2000) has proven that there are finitely many cycles such that

$$\sigma_1 < 2 \log(\sigma_1 + \sigma_3)$$

where σ_i equals the number of terms in a cycle congruent to $i \bmod 4$.

6 Extending T to Larger Spaces

A common problem-solving technique is to imbed a problem into a larger class of problems and use techniques appropriate to that new space. Much work has been done along these lines for the $3x + 1$ problem. The subsequent subsections detail work done in increasingly larger spaces.

6.1 The Integers \mathbf{Z}

The first natural extension of T is to all of \mathbf{Z} . The definition of T suffices to cover this case. One soon finds three new cycles: $\{0\}$, $\{-5, -7, -10\}$, and the long cycle

$$\{-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34\}.$$

These new cycles could also be obtained (without minus signs) if one considered the map define $T'(n) = -T(-n)$, which corresponds to the “ $3x - 1$ ” problem. It is conjectured that these cycles are all the cycles of T on \mathbf{Z} . This problem was considered by Seifert[68](1988).

6.2 Rational Numbers with Odd Denominators

Just as the study of the “3x-1” problem of the last subsection is equivalent to the $3x + 1$ problem on \mathbf{Z}^+ , Lagarias[46](1990) has extended the $3x + 1$ to the rationals by considering the class of maps

$$T_k(x) = \begin{cases} \frac{3x+k}{2} & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2} \end{cases}$$

positive $k \equiv \pm 1 \pmod{6}$ with $(x, k) = 1$. He has shown that cycles for T_k correspond to rational cycles x/k for the $3x + 1$ function T . Lagarias proves there are integer cycles of T_k for infinitely many k (with estimates on the number of cycles and bounds on their lengths) and conjectures that there are integer cycles for all k . Halbeisen and Hungerbühler[35](1997) derived similar bounds as Eliahou[30](1993) on cycle lengths for rational cycles.

6.3 The Ring of 2-adic Integers \mathbf{Z}_2

The next extension is to the ring \mathbf{Z}_2 of 2-adic integers consisting of infinite binary sequences of the form

$$a = a_0 + a_1 2 + a_2 2^2 + \cdots = \sum_{j=0}^{\infty} a_j 2^j$$

where $a_j \in \{0, 1\}$ for all non-negative integers j . Congruence is defined by $a \equiv a_0 \pmod{2}$.

Chisala[22](1994) extends C to \mathbf{Z}_2 but then restricts his attention to the rationals. He derives interesting restrictions on rational cycles, for example, if m is the least element of a positive rational cycle, then

$$m > 2^{\frac{\lceil m \log_2 3 \rceil}{m}} - 3.$$

A more developed extension of T , initiated by Lagarias[45](1985), defines

$$T : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2, \quad T(a) := \begin{cases} a/2 & \text{if } a \equiv 0 \pmod{2} \\ (3a + 1)/2 & \text{if } a \equiv 1 \pmod{2} \end{cases}$$

It has been shown (see Mathews and Watts[52](1984) and Müller[59](1991)) that the extended T is surjective, not injective, infinitely many times differentiable, not analytic, measure-preserving with respect to the Haar measure, and strongly mixing. Similar results concerning iterates of T may be found in Lagarias[45](1985), Müller[59](1991),[60](1994) and Terras[77](1976). Defining the shift map $\sigma : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$ as

$$\sigma(x) = \begin{cases} \frac{x-1}{2} & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2} \end{cases},$$

Lagarias[45](1985) proved that T is conjugate to σ via the parity vector map $\Phi^{-1} : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$ defined by

$$\Phi^{-1}(x) = \sum_{k=0}^{\infty} 2^k (T^k(x) \bmod 2),$$

that is, $\Phi \circ T \circ \Phi^{-1} = \sigma$. Bernstein[11](1994) gives an explicit formula for the inverse conjugacy Φ , namely

$$\Phi(2^{d_0} + 2^{d_1} + 2^{d_2} + \dots) = -\frac{1}{3}2^{d_0} - \frac{1}{9}2^{d_1} - \frac{1}{27}2^{d_2} - \dots$$

where $0 \leq d_0 < d_1 < d_2 < \dots$. He also shows that the $3x+1$ problem is equivalent to having $\mathbf{Z}^+ \subset \Phi(\frac{1}{3}\mathbf{Z})$. Letting \mathbb{Q}_{odd} denote the set of rationals whose reduced form has an odd denominator, Bernstein and Lagarias[12](1996) proved that the $3x+1$ problem has no divergent orbits if $\Phi^{-1}(\mathbb{Q}_{odd}) \subset (\mathbb{Q}_{odd})$.

Recent developments along these lines have been made by Monks and Yazinski[58](2002). Since Hedlund[36](1969) proved that the automorphism group of σ is simply $\text{Aut}(\sigma) = \{id, V\}$, where id is the identity map and $V(x) = -1 - x$, Monks and Yazinski define a function Ω as

$$\Omega = \Phi \circ V \circ \Phi^{-1}.$$

We then have that Ω is the unique nontrivial autoconjugacy of the $3x+1$ map, i.e. $\Omega \circ T = T \circ \Omega$ and $\Omega^2 = id$. Coupled with the results of Bernstein and Lagarias, the authors show that three statements are equivalent: $\Phi^{-1}(\mathbb{Q}_{odd}) \subset (\mathbb{Q}_{odd})$, $\Omega(\mathbb{Q}_{odd}) \subset (\mathbb{Q}_{odd})$, and no rational 2-adic integer has a divergent T -trajectory.

Monks and Yazinski also extend the results of Eliahou[30](1993) and Lagarias[45](1985) concerning the density of “odd” points in an orbit. Let $\kappa_n(x)$ denote the number of ones in the first n -digits of the parity vector x . If $x \in \mathbb{Q}_{odd}$ eventually enters an n -periodic orbit, then

$$\frac{\ln(2)}{\ln(3 + 1/m)} \leq \lim_{n \rightarrow \infty} \frac{\kappa_n(x)}{n} \leq \frac{\ln(2)}{\ln(3 + 1/M)}$$

where m, M are the least and greatest cyclic elements in the eventual cycle. If $x \in \mathbb{Q}_{odd}$ diverges, then

$$\frac{\ln(2)}{\ln(3)} \leq \liminf_{n \rightarrow \infty} \frac{\kappa_n(x)}{n}.$$

Monks and Yazinski define another extension of T , namely $\xi : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$ as

$$\xi(x) = \begin{cases} \Omega(x) & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2} \end{cases}$$

and prove that the $3x + 1$ problem is equivalent to having the number one in the ξ -orbit of every positive.

6.4 The Gaussian Integers and $\mathbf{Z}_2[i]$

Joseph[38](1998) extends T further to $\mathbf{Z}_2[i]$, defining \tilde{T} as

$$\tilde{T}(\alpha) = \begin{cases} \alpha/2, & \text{if } \alpha \in [0] \\ (3\alpha + 1)/2, & \text{if } \alpha \in [1] \\ (3\alpha + i)/2, & \text{if } \alpha \in [i] \\ (3\alpha + 1 + i)/2, & \text{if } \alpha \in [1 + i] \end{cases}$$

where $[x]$ denotes the equivalence class of x in $\mathbf{Z}_2[i]/2\mathbf{Z}_2[i]$. Joseph shows that \tilde{T} is not conjugate to $T \times T$ via a \mathbf{Z}_2 -module isomorphism, but is topologically conjugate to $T \times T$. Arguing akin to results of the last subsection, Joseph shows that \tilde{T} is chaotic (in the sense of Devaney). Kucinski[43](2000) studies cycles of Joseph’s extension \tilde{T} restricted to the Gaussian integers $\mathbf{Z}[i]$.

6.5 The Real Line \mathbf{R}

A further extension of T to the real line \mathbf{R} is interesting in that it allows tools from the study of iterating continuous maps. In an unpublished paper,

Tempkin[76](1993) studies what is geometrically the simplest such extension: the straight-line extension to the Collatz function C :

$$\begin{aligned} L(x) &= \begin{cases} C(x), & x \in \mathbf{Z} \\ C(\lfloor x \rfloor) + (x - \lfloor x \rfloor)(C(\lceil x \rceil) - C(\lfloor x \rfloor)), & x \notin \mathbf{Z} \end{cases} \\ &= \begin{cases} -(5n-2)x + n(10n-3), & x \in [2n-1, 2n] \\ (5n+4)x - n(10n+7), & x \in [2n, 2n+1] \end{cases}. \end{aligned}$$

Tempkin proves that on each interval $[n, n+1]$, $n \in \mathbf{Z}^+$, L has periodic points of every possible period. Capitalizing on the fact that iterates of piecewise linear functions are piecewise linear, he also shows that every eventually periodic point of L is rational, every rational is either eventually periodic or divergent, and rationals of the form $k/5$ with $k \not\equiv 0 \pmod{5}$ are divergent.

Tempkin also mentions a smooth extension to C , namely

$$E(x) := \frac{7x+2}{4} + \frac{5x+2}{4} \cos(\pi(x+1))$$

but conducts no specific analysis with it. Chamberland[20](1996) studies a similar extension to T :

$$\begin{aligned} f(x) &:= \frac{x}{2} \cos^2\left(\frac{\pi x}{2}\right) + \frac{3x+1}{2} \sin^2\left(\frac{\pi x}{2}\right) \\ &= x + \frac{1}{4} - \frac{2x+1}{4} \cos(\pi x). \end{aligned}$$

Chamberland shows that any cycle on \mathbf{Z}^+ must be locally attractive. By also showing that the Schwarzian derivative of f is negative on \mathbf{R}^+ , this implies the long-term dynamics of almost all points coincides with the long-term dynamics of the critical points. One quickly finds that there are two attracting cycles,

$$A_1 := \{1, 2\}, \quad A_2 := \{1.192531907\dots, 2.138656335\dots\}.$$

Chamberland conjectures that these are the only two attractive cycles of f on \mathbf{R}^+ . This is equivalent to the $3x+1$ problem. It is also shown that a monotonically increasing divergent orbit exists. Chamberland compactifies the map via the homeomorphism $\sigma(x) = 1/x$ on $[\mu_1, \infty)$ (μ_1 is the first positive

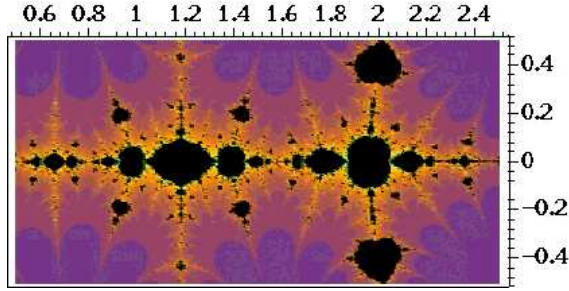


Figure 4: Julia set of Chamberland's map extended to the complex plane

fixed point of the map f), yielding a dynamically equivalent map h defined on $[0, \mu_1]$ as

$$h(x) = \begin{cases} \frac{4x}{4+x-(2+x)\cos(\pi x)}, & x \in (0, \mu_1] \\ 0, & x = 0 \end{cases}.$$

Lastly, Chamberland makes statements regarding any general extension of f : it must have 3-cycle, a homoclinic orbit (snap-back repeller), and a monotonically increasing divergent trajectory. To illustrate how chaos figures into this extension, Ken Monks replaced x with z in Chamberland's map and numerically generated the filled-in Julia set. A portion of the set is indicated in Figure 4.

In a more recent paper, Dumont and Reiter[28](2003) produce similar results for the real extension

$$f(x) := \frac{1}{2} \left(3^{\sin^2(\pi x/2)} x + \sin^2(\pi x/2) \right).$$

The authors have produced several other extensions, as well as figures representing stopping times; see Dumont and Reiter[29](2001).

Borovkov and Pfeifer[15](2000) also show that any continuous extension of T has periodic orbits of every period by arguing that the map is turbulent: there exist compact intervals A_1 and A_2 such that $A_1 \cup A_2 = f(A_1) \cap f(A_2)$.

6.6 The Complex Plane \mathbb{C}

Letherman, Schleicher and Wood[50](1999) offer the next refinement of this approach: they extend T to the complex plane with

$$f(z) := \frac{z}{2} + \frac{1}{2}(1 - \cos(\pi z)) \left(z + \frac{1}{2} \right) + \frac{1}{\pi} \left(\frac{1}{2} - \cos(\pi z) \right) \sin(\pi z) + h(z) \sin^2(\pi z).$$

Note that the first two terms (with z replaced by x) match Chamberland's extension. The clear dynamic advantage of this new function is that the set of critical points on the real line is exactly \mathbf{Z} . They improve Chamberland's result by showing that on $[n, n+1]$, with $n \in \mathbf{Z}^+$, there is a Cantor set of points that diverge monotonically. On the complex plane, Letherman *et al.* use techniques of complex dynamics to derive results about Fatou components. In particular, they show that no integer is in a Baker domain (domain at infinity). One concludes then that any integer either belongs to a super-attracting periodic orbit or a wandering domain.

7 Generalizations of $3x + 1$ Dynamics

There have been many investigations of maps which have similar dynamics to T but are not extensions of T . Here one is usually changing the *function*, as opposed to last section where the *space* was extended.

Belaga and Mignotte[10](1998) considered the “ $3x + d$ ” problem, and conjecture that for any odd $d \geq -1$ and not divisible by three, all integer orbits enter a finite set (hence every orbit is eventually periodic). Note that the “ $3x - 1$ ” problem is equivalent to the $3x + 1$ problem on the negative integers, considered in the previous section.

Another obvious generalization is the class of “ $qx + 1$ ” problems. Steiner[72, 73](1981) extended his cycle results and showed that for $q = 5$, there is only one non-trivial circuit ($13 \rightarrow 208 \rightarrow 13$), while $q = 7$ has no non-trivial circuits. Franco and Pomerance[32](1995) showed that if q is a Wieferich number², then some $x \in \mathbf{Z}^+$ never iterates to one. Crandall[26](1978) conjectured that this is

²An odd integer q is called a *Wieferich number* if $2^{l(q)} \equiv 1 \pmod{q^2}$, where $l(q)$ is the order

true for *any* odd $q \geq 5$. Wirsching[87](1998) offers a heuristic argument using p -adic averages that for $q \geq 5$, the $qx + 1$ problem admits either a divergent trajectory or infinitely-many different periodic orbits.

Mignosi[55](1995) looked at a related generalization, namely

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & n \equiv 1 \pmod{2} \\ \frac{x}{2} & n \equiv 0 \pmod{2} \end{cases}$$

for any $\beta > 1$ and $r \in \mathbb{R}$. The case $\beta = 3/2$ is equivalent to T . Mignosi conjectures that for any $\beta > 1$, there are finitely many periodic orbits. This is proven for $\beta = \sqrt{2}$ and a heuristic argument is made that this conjecture holds for almost all $\beta \in (1, 2)$. Brocco[16](1995) modifies the map to

$$T_{\alpha,r}(n) = \begin{cases} \lceil \alpha n + r \rceil & n \equiv 1 \pmod{2} \\ \frac{x}{2} & n \equiv 0 \pmod{2} \end{cases}$$

for $1 < \alpha < 2$. He shows for his map that the Mignosi's conjecture is false if the interval $((r-1)/(\alpha-1), r/(\alpha-1))$ contains an odd integer and α is a Salem number or a PV number³.

Matthews and Watts[52, 53](1984,1985) deal with multiple-branched maps of the form

$$T(x) := \frac{m_i x - r_i}{d}, \text{ if } x \equiv i \pmod{d}$$

where $d \geq 2$ is an integer, m_0, m_1, \dots, m_{d-1} are non-zero integers, and $r_0, r_1, r_{d-1} \in \mathbb{Z}$ such that $r_i \equiv im_i \pmod{d}$. They mirror their earlier results (seen in the last section) by extending this map to the ring of d -adic integers and show it is measure-preserving with respect to the Haar measure. Information about divergent trajectories may be found in Leigh[49](1986) and Buttsworth and Matthews[18](1990). Ergodic properties of these maps has been studied by Venturini[80, 82, 83](1982,1992,1997). An encapsulating result of [83](1997) of 2 in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})$. Wiefereich numbers have density one in the odd numbers.

³A *Salem number* is a real algebraic number greater than 1 all of whose conjugates z satisfy $|z| \leq 1$, with at least one conjugate satisfying $|z| = 1$. A *Pisot-Vijayaraghavan (PV) number* is a real algebraic number greater than 1 all of whose conjugates z satisfy $|z| < 1$.

essentially claims that the condition $|m_0 m_1 \cdots m_{d-1}| < 1$ gives “converging” behavior, while divergent orbits may occur if $|m_0 m_1 \cdots m_{d-1}| > 1$. A special class of these maps – known as *Hasse functions*, which are “closer” to the $3x+1$ map T – have been studied by Allouche[1](1979), Heppner[37](1978), Garcia and Tal[33](1999) and Möller[56](1978), with similar results. A recent survey of these generalized mappings – with many examples – has been written by Matthews[54](2002).

Seemingly farther removed from the $3x+1$ problem are results due to Stolarsky[74](1998) who completely solves a problem of “similar appearance.” First, recall Beatty’s Theorem which states that if $\alpha, \beta > 1$ are irrational and satisfy $1/\alpha + 1/\beta = 1$, then the sets

$$A = \{\lfloor n\alpha \rfloor : n \in \mathbf{Z}^+\}, \quad B = \{\lfloor n\beta \rfloor : n \in \mathbf{Z}^+\}$$

form a partition of \mathbf{Z}^+ . If $\alpha = \phi := (1 + \sqrt{5})/2$, we have $\beta = \phi^2 = (3 + \sqrt{5})/2$. Stolarsky considers the map

$$f(m) = \begin{cases} \lceil \lceil \frac{m}{\phi^2} \rceil \phi^2 \rceil + 1, & m \in A \\ \lceil \frac{m}{\phi^2} \rceil, & m \in B \end{cases}.$$

He proves that f admits a unique periodic orbit, namely $\{3, 7\}$. Defining $b(n) = \lfloor n\beta \rfloor$, the set $\{b^{(k)}(3) : k \in \mathbf{Z}^+\}$ – which has density zero – characterizes the eventually periodic points. All other positive integers have divergent orbits. The symbolic dynamics are simple: any trajectory has an itinerary of either $B^l(AB)^\infty$ or $B^l(AB)^k A^\infty$ for some integers $k, l \geq 0$.

8 Miscellaneous

Margenstern and Matiyasevich[51](1999) have encoded the $3x+1$ problem as a logical problem using one universal quantifier and several existential quantifiers. Specifically, they showed that $T^{(m)}(2a) = b$ for some $m \in \mathbf{Z}^+$ if and only if there exist $w, p, r, s \in \mathbf{Z}^+$ such that $a, b \leq w$ and

$$\begin{pmatrix} 4(w+1)(p+r)+1 \\ p+r \end{pmatrix} \begin{pmatrix} pw \\ s \end{pmatrix} \begin{pmatrix} rw \\ t \end{pmatrix} \times$$

$$\begin{pmatrix} 2w+1 \\ w \end{pmatrix} \begin{pmatrix} 2s+2t+r+b((4w+3)(p+r)+1) \\ 3a+(4w+4)(3t+2r+s) \end{pmatrix} \times \\ \begin{pmatrix} p+r \\ p \end{pmatrix} \begin{pmatrix} 3a+(4w+4)(3t+2r+s) \\ 2s+2t+r+b((4w+3)(p+r)+1) \end{pmatrix} \equiv 1 \pmod{2}.$$

Gluck and Taylor[34](2002) have studied another “global statistic” besides the total stopping time function $\sigma_\infty(n)$. If $\sigma_\infty(a_1) = p$ under the map C , the authors define a function⁴ A as

$$A(a_1) = \frac{a_1 a_2 + a_2 a_3 + \cdots + a_p a_1}{a_1^2 + a_2^2 + \cdots + a_p^2}.$$

where a_2, a_3, \dots, a_p are the consecutive C -iterates of a_1 . For any odd $m > 3$, they show that

$$\frac{9}{13} < A(m) < \frac{5}{7}.$$

Moreover, these bounds are sharp since

$$\lim_{n \rightarrow \infty} A\left(\frac{4^n - 1}{3}\right) = \frac{9}{13}$$

and

$$\lim_{k \rightarrow \infty} A\left(\frac{2^k(2^{3^{k-1}} + 1)}{3^k}\right) = \frac{5}{7}.$$

Gluck and Taylor also produce a normalized histogram of A for values between 2^{20} and $2^{20} + 20001$, and its randomized counterpart. These two histograms bear a resemblance, hinting at the stochastic/probabilistic approach to this problem.

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⁴Gluck and Taylor used the symbol C for their new function; to avoid confusion with the Collatz function C , I am using the symbol A .

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