Real Integrals

Definite Integrals

- If $F(x)$ is an antiderivative of a continuous function f, i.e., F is a function for which $F'(x) = f(x)$, then the definite integral of f on the interval $[a, b]$ is the number

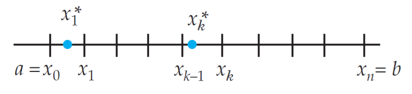
$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

- Example: $\int_{-1}^{2} x^2 dx = \frac{1}{3}x^3\Big|_{-1}^{2} = \frac{8}{3} - \frac{-1}{3} = 3.$
- The fundamental theorem of calculus is a method of evaluating $\int_a^b f(x)dx$; it is not the definition of $\int_a^b f(x)dx$.
- We next define:
  - The definite (or Riemann) integral of a function f;
  - Line integrals in the Cartesian plane.

  Both definitions rest on the limit concept.

Steps Leading to the Definition of the Definite Integral

1. Let f be a function of a single variable x defined at all points in a closed interval $[a, b]$.
2. Let P be a partition:
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
of $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$.
3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$ of $[a, b]$.



5. Form n products $f(x_k^*)\Delta x_k$, $k = 1, 2, \ldots, n$, and then sum these products:
$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

The Definition of the Definite Integral

Definition (Definite Integral)

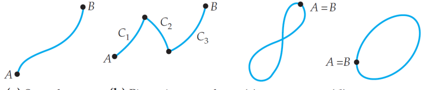
The **definite integral** of f on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

- Whenever the limit exists we say that f is **integrable** on the interval $[a, b]$ or that the definite integral of f **exists**.
- It can be proved that if f is continuous on $[a, b]$, then the integral exists.

Terminology About Curves

- Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the initial and terminal points of C $(x(a), y(a))$, $(x(b), y(b))$ be denoted by A, B. We say that:
  - (i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).
  - (ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, i.e., the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.
  - (iii) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.
  - (iv) C is a **closed curve** if $A = B$.
  - (v) C is a **simple closed curve** if the curve C does not cross itself and $A = B$, i.e., C is simple and closed.



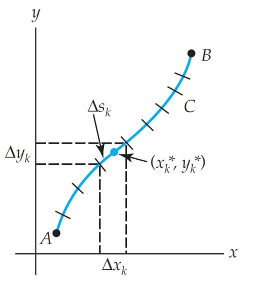
(a) Smooth curve and simple **(b)** Piecewise smooth curve and simple **(c)** Closed but not simple **(d)** Simple closed curve

Steps Leading to the Definition of Line Integrals

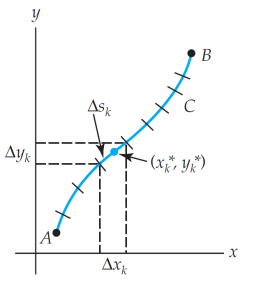
1. Let G be a function of two real variables x and y, defined at all points on a smooth curve C that lies in some region of the xy-plane. Let C be defined by the parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The partition P induces a partition of the curve C into n subarcs of length Δs_k. Let the projection of each subarc onto the x- and y-axes have lengths Δx_k and Δy_k, respectively.



Steps Leading to the Definition of Line Integrals (Cont'd)



3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, that is, the length of the longest subinterval.
4. Choose a point (x_k^*, y_k^*) on each subarc of C.
5. Form n products $G(x_k^*, y_k^*)\Delta x_k$, $G(x_k^*, y_k^*)\Delta y_k$, $G(x_k^*, y_k^*)\Delta s_k$, $k = 1, 2, \ldots, n$, and then sum these products

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

The Definition of Line Integrals

Definition (Line Integrals in the Plane)

(i) The **line integral of G along C with respect to x** is

$$\int_C G(x,y)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k.$$

(ii) The **line integral of G along C with respect to y** is

$$\int_C G(x,y)dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k.$$

(iii) The **line integral of G along C with respect to arc length s** is

$$\int_C G(x,y)ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

- If G is continuous on C, then the three types of line integrals exist.
- The curve C is referred to as the **path of integration**.

Method of Evaluation: C Defined Parametrically

- Convert a line integral to a definite integral in a single variable.
- If C is a smooth curve parametrized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, then replace
  - x and y in the integral by the functions $x(t)$ and $y(t)$;
  - the appropriate differential dx, dy, or ds by

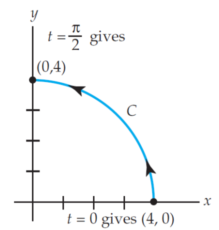
$$x'(t)dt, \quad y'(t)dt, \quad \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

- The term $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$ is called the **differential of the arc length**.
- The line integrals become definite integrals in which the variable of integration is the parameter t:

$$\begin{array}{rcl} \int_C G(x,y)dx & = & \int_a^b G(x(t),y(t))x'(t)dt, \\ \int_C G(x,y)dy & = & \int_a^b G(x(t),y(t))y'(t)dt, \\ \int_C G(x,y)ds & = & \int_a^b G(x(t),y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt. \end{array}$$

Evaluation of a Line Integral I

- Evaluate $\int_C xy^2 dx$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \le t \le \frac{\pi}{2}$.



We have

$$dx = -4\sin t dt.$$

Thus,

$$\begin{aligned} \int_C xy^2 dx &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t dt) \\ &= -256 \int_0^{\pi/2} \sin^3 t \cos t dt \\ &= -256[\tfrac{1}{4}\sin^4 t]_0^{\pi/2} \\ &= -64. \end{aligned}$$

Evaluation of a Line Integral II

- Evaluate $\int_C xy^2 dy$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have

  $$dy = 4\cos t dt.$$

  Thus,

  $$\begin{array}{rcl}\int_C xy^2 dy & = & \int_0^{\pi/2} (4\cos t)(4\sin t)^2 (4\cos t dt) \\ & = & 256 \int_0^{\pi/2} \sin^2 t \cos^2 t dt \\ & = & 256 \int_0^{\pi/2} \frac{1}{4} \sin^2 2t dt \\ & = & 64 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4t) dt \\ & = & 32[t - \frac{1}{4}\sin 4t]_0^{\pi/2} = 16\pi.\end{array}$$

Evaluation of a Line Integral III

- Evaluate $\int_C xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have
  $$ds = \sqrt{16(\sin^2 t + \cos^2 t)}dt = 4dt.$$
  Therefore,
  $$\begin{aligned} \int_C xy^2 ds &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2(4dt) \\ &= 256 \int_0^{\pi/2} \sin^2 t \cos t dt \\ &= 256[\tfrac{1}{3}\sin^3 t]_0^{\pi/2} \\ &= \tfrac{256}{3}. \end{aligned}$$

Method of Evaluation: C Defined by a Function

- If the path of integration C is the graph of an explicit function $y = f(x)$, $a \le x \le b$, then we can use x as a parameter:
- The differential of y is $dy = f'(x)dx$, and the differential of arc length is $ds = \sqrt{1 + [f'(x)]^2}dx$.
- We, thus, obtain the definite integrals:

$$\begin{array}{rcl} \int_C G(x,y)dx & = & \int_a^b G(x,f(x))dx, \\ \int_C G(x,y)dy & = & \int_a^b G(x,f(x))f'(x)dx, \\ \int_C G(x,y)ds & = & \int_a^b G(x,f(x))\sqrt{1+[f'(x)]^2}dx. \end{array}$$

- A line integral along a piecewise smooth curve C is defined as the sum of the integrals over the various smooth pieces.
- Example: To evaluate $\int_C G(x,y)ds$ when C is composed of two smooth curves C_1 and C_2, we write

$$\int_C G(x,y)ds = \int_{C_1} G(x,y)ds + \int_{C_2} G(x,y)ds.$$

Notation for Line Integrals

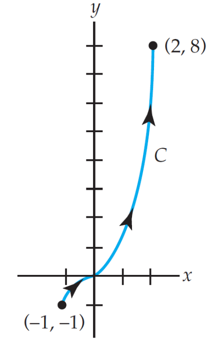
- In many applications, line integrals appear as a sum
$$\int_C P(x,y)dx + \int_C Q(x,y)dy.$$
- It is common practice to write this sum as one integral without parentheses as
$$\int_C P(x,y)dx + Q(x,y)dy$$
or simply
$$\int_C Pdx + Qdy.$$
- A line integral along a closed curve C is usually denoted by
$$\oint_C Pdx + Qdy.$$

C Defined by an Explicit Function

- Evaluate $\int_C xydx + x^2dy$, where C is the graph of $y = x^3$, $-1 \le x \le 2$.

  We have $dy = 3x^2dx$. Therefore,

$$\begin{aligned}
\int_C xydx + x^2dy &= \int_{-1}^{2} xx^3dx + x^2 3x^2 dx \\
&= \int_{-1}^{2} (x^4 + 3x^4)dx \\
&= \int_{-1}^{2} 4x^4 dx \\
&= \left.\tfrac{4}{5}x^5\right|_{-1}^{2} \\
&= \tfrac{4}{5}(32 - (-1)) = \tfrac{132}{5}.
\end{aligned}$$



C a Closed Curve

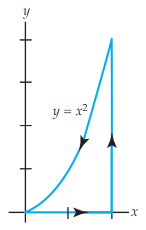
- Evaluate $\oint_C x dx$, where C is the circle defined by $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

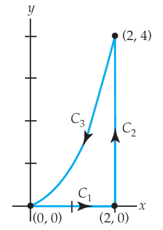
  We have $dx = -\sin t dt$, whence:

$$
\begin{aligned}
\oint_C x dx &= \int_0^{2\pi} \cos t(-\sin t dt) \\
&= \frac{1}{2} \cos^2 t \Big|_0^{2\pi} \\
&= \frac{1}{2}(1-1) \\
&= 0.
\end{aligned}
$$

C Another Closed Curve

- Evaluate $\oint_C y^2 dx - x^2 dy$, where C is the closed curve shown on the left.





C is piecewise smooth. So, the given integral is expressed as a sum of integrals, i.e., we write $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$, with C_1, C_2, C_3 as shown on the right.

- On C_1, with x as a parameter: $\int_{C_1} y^2 dx - x^2 dy = \int_0^2 0dx - x^2(0) = 0$.
- On C_2, with y as a parameter: $\int_{C_2} y^2 dx - x^2 dy = \int_0^4 y^2(0) - 4dy = -\int_0^4 4dy = -16$.
- On C_3, we again use x as a parameter. From $y = x^2$, we get $dy = 2xdx$. Thus, $\int_{C_3} y^2 dx - x^2 dy = \int_2^0 (x^2)^2 dx - x^2(2xdx) = \int_2^0 (x^4 - 2x^3) dx = \left(\frac{1}{5}x^5 - \frac{1}{2}x^4\right)\Big|_2^0 = \frac{8}{5}$.

Hence, $\oint_C y^2 dx - x^2 dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-16) + \frac{8}{5} = -\frac{72}{5}$.

Orientation of a Curve

- If C is not a closed curve, then we say the **positive direction** on C, or that C has **positive orientation**, if we traverse C from its initial point A to its terminal point B, i.e., if $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations for C, then the positive direction on C corresponds to increasing values of the parameter t.
- If C is traversed in the sense opposite to that of the positive orientation, then C is said to have **negative orientation**.
- If C has an orientation (positive or negative), then the **opposite curve**, the curve with the opposite orientation, will be denoted $-C$.
- Then
$$\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy,$$
or, equivalently
$$\int_{-C} Pdx + Qdy + \int_C Pdx + Qdy = 0.$$
- A line integral is independent of the parametrization of C, provided C is given the same orientation.

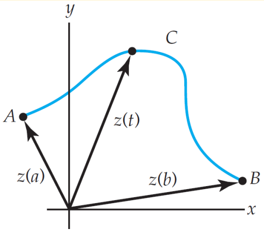
Complex Integrals

Curves Revisited

- Suppose the continuous real-valued functions $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane.
- By considering $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t, called a **parametrization** of C: $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.

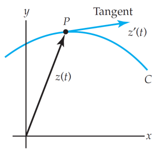
  Example: The parametric equations $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}$, $0 \leq t \leq 2\pi$.
- The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C. and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ the **terminal point**.

  As t varies from $t = a$ to $t = b$, C is being traced out by the moving arrowhead of the vector corresponding to $z(t)$.



Smooth Curves and Contours

- Suppose the derivative of $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is $z'(t) = x'(t) + iy'(t)$.
- We say C is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$.

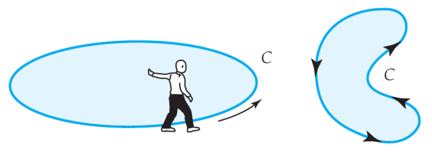


Since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent.

- A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.
- A curve C in the complex plane is **simple** if $z(t_1) \neq z(t_2)$, for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$.
- C is a **closed curve** if $z(a) = z(b)$.
- C is a **simple closed curve** if it is simple and closed.
- A piecewise smooth curve C is also called a **contour** or **path**.

Positive and Negative Directions

- We define the **positive direction** on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. It is also said that the curve C has **positive orientation**.
- In the case of a *simple closed curve* C, the **positive direction** roughly corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left.



- The **negative direction** on a contour C is the direction opposite the positive direction.
- If C has an orientation, the **opposite curve**, that is, a curve with opposite orientation, is denoted by $-C$.
- On a *simple closed curve*, the **negative direction** corresponds to the clockwise direction.

Steps Leading to the Definition of the Complex Integral I

1. Let f be a function of a complex variable z defined at all points on a smooth curve C that lies in some region of the plane. Suppose C is defined by the parametrization $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

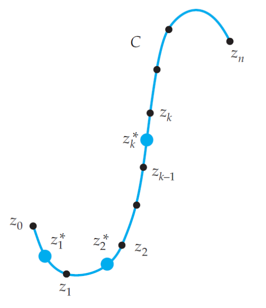
The partition P induces a partition of the curve C into n subarcs whose initial and terminal points are the pairs of numbers

$$\begin{array}{ll} z_0 = x(t_0) + iy(t_0), & z_1 = x(t_1) + iy(t_1), \\ z_1 = x(t_1) + iy(t_1), & z_2 = x(t_2) + iy(t_2), \\ \vdots & \vdots \\ z_{n-1} = x(t_{n-1}) + iy(t_{n-1}), & z_n = x(t_n) + iy(t_n). \end{array}$$

Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.

Steps Leading to the Definition of the Complex Integral II

3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a point $z_k^* = x_k^* + iy_k^*$ on each subarc of C.



5. Form n products $f(z_k^*)\Delta z_k$, $k = 1, 2, \ldots, n$, and then sum these products: $\sum_{k=1}^{n} f(z_k^*)\Delta z_k$.

The Definition of the Complex Integral

Definition (Complex Integral)

The **complex integral** of f on C is

$$\int_C f(z)dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k.$$

- If the limit exists, f is said to be **integrable** on C.
- The limit exists whenever f is continuous at all points on C and C is either smooth or piecewise smooth.
- Thus, we always assume that these conditions are fulfilled.
- By convention, we will use the notation $\oint_C f(z)dz$ to represent a complex integral around a *positively oriented closed curve* C.
- The notations $\ointctrclockwise_C f(z)dz$, $\ointclockwise_C f(z)dz$ denote more explicitly integration in the positive and negative directions, respectively.
- We shall refer to $\int_C f(z)dz$ as a **contour integral**.

Complex-Valued Function of a Real Variable

- Example: If t represents a real variable, then $f(t) = (2t + i)^2$ is a complex number. For $t = 2$, $f(2) = (4 + i)^2 = 16 + 8i + i^2 = 15 + 8i$.
- If f_1 and f_2 are real-valued functions of a real variable t, then $f(t) = f_1(t) + if_2(t)$ is a complex-valued function of a real variable t.
- We are interested in integration of a complex-valued function $f(t) = f_1(t) + if_2(t)$ of a real variable t carried out over a real interval.
- Example: On the interval $0 \le t \le 1$, it seems reasonable for $f(t) = (2t + i)^2$ to write

$$\int_0^1 (2t+i)^2 dt = \int_0^1 (4t^2 - 1 + 4ti)dt = \int_0^1 (4t^2 - 1)dt + i\int_0^1 4tdt.$$

The integrals $\int_0^1 (4t^2 - 1)dt$ and $\int_0^1 4tdt$ are real, and could be called the real and imaginary parts of $\int_0^1 (2t+i)^2 dt$. Each can be evaluated using the fundamental theorem of calculus to get:
$\int_0^1 (2t+i)^2 dt = \left(\frac{4}{3}t^3 - t\right)\Big|_0^1 + i\, 2t^2\Big|_0^1 = \frac{1}{3} + 2i.$

Integral of Complex Valued Function of a Real Variable

- If f_1 and f_2 are real-valued functions of a real variable t continuous on a common interval $a \leq t \leq b$, then we define the **integral** of the complex-valued function $f(t) = f_1(t) + if_2(t)$ on $a \leq t \leq b$ by

$$\int_a^b f(t)dt = \int_a^b f_1(t)dt + i\int_a^b f_2(t)dt.$$

- The continuity of f_1 and f_2 on $[a, b]$ guarantees that both integrals on the right exist.
- If $f(t) = f_1(t) + if_2(t)$ and $g(t) = g_1(t) + ig_2(t)$, are complex-valued functions of a real variable t continuous on $a \leq t \leq b$, then
  - $\int_a^b kf(t)dt = k\int_a^b f(t)dt$, k a complex constant;
  - $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$;
  - $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$, if $c \in [a, b]$;
  - $\int_b^a f(t)dt = -\int_a^b f(t)dt$.

Evaluation of Contour Integrals

- If we use $u + iv$ for f, $\Delta x + i\Delta y$ for Δz, lim for $\lim_{\|P\|\to 0}$ and $\sum$ for $\sum_{k=1}^{n}$, we get $\int_C f(z)dz = \lim \sum (u+iv)(\Delta x + i\Delta y) = \lim [\sum (u\Delta x - v\Delta y) + i \sum (v\Delta x + u\Delta y)]$.
- Thus, we have
$$\int_C f(z)dz = \int_C udx - vdy + i \int_C vdx + udy.$$
- If $x = x(t)$, $y = y(t)$, $a \le t \le b$, are parametric equations of C, then $dx = x'(t)dt$, $dy = y'(t)dt$.
- Now we obtain $\int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt + i \int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]dt$.
- This is the same as $\int_a^b f(z(t))z'(t)dt$ when the integrand $f(z(t))z'(t) = [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]$ is multiplied out and $\int_a^b f(z(t))z'(t)dt$ is expressed in terms of its real and imaginary parts.

Evaluating of a Contour Integral

Theorem (Evaluation of a Contour Integral)

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

- Example: Evaluate $\int_C \overline{z}dz$, where C is given by $x = 3t$, $y = t^2$, $-1 \leq t \leq 4$.

  A parametrization of the contour C is $z(t) = 3t + it^2$. Thus, since $f(z) = \overline{z}$, we have $f(z(t)) = \overline{3t + it^2} = 3t - it^2$. Also, $z'(t) = 3 + 2it$. Now, we have

$$\begin{aligned} \int_C \overline{z}dz &= \int_{-1}^4 (3t - it^2)(3 + 2it)dt \\ &= \int_{-1}^4 (2t^3 + 9t)dt + i\int_{-1}^4 3t^2 dt \\ &= \left(\tfrac{1}{2}t^4 + \tfrac{9}{2}t^2\right)\Big|_{-1}^4 + i\ t^3\Big|_{-1}^4 = 195 + 65i. \end{aligned}$$

Another Evaluation of a Contour Integral

- Evaluate $\oint_C \frac{1}{z} dz$, where C is the circle $x = \cos t, y = \sin t$, $0 \le t \le 2\pi$.

  In this case $z(t) = \cos t + i \sin t = e^{it},\ z'(t) = ie^{it},$ and $f(z(t)) = \frac{1}{z(t)} = e^{-it}$. Hence,

$$\begin{aligned} \oint_c \frac{1}{z} dz &= \int_0^{2\pi} (e^{-it}) ie^{it} dt \\ &= i \int_0^{2\pi} dt \\ &= 2\pi i. \end{aligned}$$

Using x as a Parameter

- For some curves the real variable x itself can be used as the parameter.
- Example: Evaluate $\int_C (8x^2 - iy)dz$ on the line segment $y = 5x$, $0 \leq x \leq 2$.

  We write $z = x + 5xi$, whence $dz = (1 + 5i)dx$. Therefore,

$$\begin{aligned}
\int_C (8x^2 - iy)dz &= (1+5i)\int_0^2 (8x^2 - 5ix)dx \\
&= (1+5i)\,\tfrac{8}{3}x^3\Big|_0^2 - (1+5i)i\,\tfrac{5}{2}x^2\Big|_0^2 \\
&= \tfrac{214}{3} + \tfrac{290}{3}i.
\end{aligned}$$

- If x and y are related by means of a continuous real function $y = f(x)$, then the corresponding curve C can be parametrized by $z(x) = x + if(x)$.

Properties of Contour Integrals

Theorem (Properties of Contour Integrals)

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then:

(i) $\int_C kf(z)dz = k\int_C f(z)dz$, k a complex constant.

(ii) $\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$.

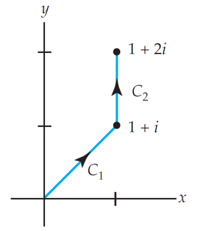
(iii) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of C.

- The four parts of the theorem also hold if C is a *piecewise smooth* curve in D.

C a Piecewise Smooth Curve

- Evaluate $\int_C (x^2+iy^2)dz$, where C is the contour shown:



We write $\int_C (x^2+iy^2)dz = \int_{C_1}(x^2+iy^2)dz + \int_{C_2}(x^2+iy^2)dz$.
Since the curve C_1 is defined by $y=x$, we use x as a parameter: $z(x)=x+ix$, $z'(x)=1+i$, $f(z)=x^2+iy^2$, $f(z(x))=x^2+ix^2$,

whence, finally, $\int_{C_1}(x^2+iy^2)dz = \int_0^1 (x^2+ix^2)(i+1)dx = (1+i)^2\int_0^1 x^2 dx = \frac{(1+i)^2}{3} = \frac{2}{3}i$.
The curve C_2 is defined by $x=1$, $1\le y\le 2$. If we use y as a parameter, then $z(y)=1+iy$, $z'(y)=i$, $f(z(y))=1+iy^2$, and $\int_{C_2}(x^2+iy^2)dz = \int_1^2 (1+iy^2)idy = -\int_1^2 y^2 dy + i\int_1^2 dy = -\frac{7}{3}+i$.
Therefore $\int_C (x^2+iy^2)dz = \frac{2}{3}i + (-\frac{7}{3}+i) = -\frac{7}{3}+\frac{5}{3}i$.

A Bounding Theorem

- We find an upper bound for the modulus of a contour integral.
- Recall the length of a plane curve $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$. If $z'(t) = x'(t) + iy'(t)$, then $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$, whence $L = \int_a^b |z'(t)|dt$.

Theorem (A Bounding Theorem)

If f is continuous on a smooth curve C and if $|f(z)| \leq M$, for all z on C, then $|\int_C f(z)dz| \leq ML$, where L is the length of C.

- By triangle inequality, $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq \sum_{k=1}^n |f(z_k^*)||\Delta z_k|$ $\leq M \sum_{k=1}^n |\Delta z_k|$. Because $|\Delta z_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, we can interpret $|\Delta z_k|$ as the length of the chord joining the points z_k and z_{k-1} on C. Moreover, since the sum of the lengths of the chords cannot be greater than L, we get $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq ML$. Finally, the continuity of f guarantees that $\int_C f(z)dz$ exists. Thus, letting $\|P\| \to 0$, the last inequality yields $|\int_C f(z)dz| \leq ML$.

A Bound for a Contour Integral

- Find an upper bound for the absolute value of $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z| = 4$.

  First, the length L (circumference) of the circle of radius 4 is 8π. Next, for all points z on the circle, we have that $|z+1| \geq |z| - 1 = 4 - 1 = 3$. Thus, $\left|\dfrac{e^z}{z+1}\right| \leq \dfrac{|e^z|}{|z|-1} = \dfrac{|e^z|}{3}$. In addition, $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. For points on the circle $|z| = 4$, the maximum that $x = \mathrm{Re}(z)$ can be is 4, whence $\left|\dfrac{e^z}{z+1}\right| \leq \dfrac{e^4}{3}$. From the theorem, we have

$$\left|\int_C \frac{e^z}{z+1} dz\right| \leq \frac{8\pi e^4}{3}.$$

Single Contour: Many Parametrizations

- There is no unique parametrization for a contour C.
- Example: All of the following:

$$z(t) = e^{it} = \cos t + i \sin t, \quad 0 \le t \le 2\pi,$$
$$z(t) = e^{2\pi it} = \cos 2\pi t + i \sin 2\pi t, \quad 0 \le t \le 1,$$
$$z(t) = e^{\pi it/2} = \cos \frac{\pi t}{2} + i \sin \frac{\pi t}{2}, \quad 0 \le t \le 4,$$

are all parametrizations, oriented in the positive direction, for the unit circle $|z| = 1$.