Limits

Real and Complex Limits

- $\lim_{x \to x_0} f(x) = L$ intuitively means that values $f(x)$ of the function f can be made arbitrarily close to the real number L if values of x are chosen sufficiently close to, but not equal to, the real number x_0.
- In real analysis, the concepts of continuity, the derivative, and the definite integral were all defined using the concept of a limit.
- $\lim_{z \to z_0} f(z) = L$ will mean that the values $f(z)$ of the complex function f can be made arbitrarily close to the complex number L if values of z are chosen sufficiently close to, but not equal to, the complex number z_0.
- There is an important difference between these two concepts of limit:
  - In a real limit, there are two directions from which x can approach x_0 on the real line, from the left or from the right.
  - In a complex limit, there are infinitely many directions from which z can approach z_0 in the complex plane. In order for a complex limit to exist, each way in which z can approach z_0 must yield the same limiting value.

Real Limits: From Intuition to Formalism

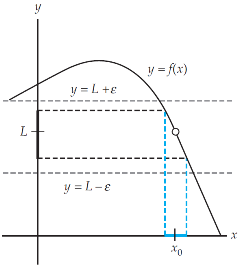
- To rigorously define a real limit, we must formalize what is meant by the phrases "arbitrarily close to" and "sufficiently close to".
- A precise statement should involve the use of absolute values since $|a-b|$ measures the distance between a, b on the real number line.
- The points x and x_0 are close if $|x - x_0|$ is a small positive number.
- Also, $f(x)$ and L are close if $|f(x) - L|$ is a small positive number.
- We let the Greek letters ε and δ represent small positive real numbers.
- The expression "$f(x)$ can be made arbitrarily close to L" can be made precise by stating that for any real number $\varepsilon > 0$, x can be chosen so that $|f(x) - L| < \varepsilon$.
- We require that $|f(x) - L| < \varepsilon$ whenever values of x are "sufficiently close to, but not equal to, x_0".
- This means that there is some distance $\delta > 0$ with the property that, if x is within distance δ of x_0 and $x \neq x_0$, then $|f(x) - L| < \varepsilon$.

Formal Definition of a Real Limit

Definition (Limit of a Real Function $f(x)$)

The **limit of** f **as** x **tends to** x_0 exists and is equal to L if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(x) - L| < \varepsilon$ when $0 < |x - x_0| < \delta$.

- The geometric interpretation is shown:



The graph of the function $y = f(x)$ over the interval $(x_0 - \delta, x_0 + \delta)$, excluding the point x_0, lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. In the terminology of mappings, the interval $(x_0 - \delta, x_0 + \delta)$, excluding the point $x = x_0$, is mapped onto a set in the interval $(L - \varepsilon, L + \varepsilon)$ on the y-axis.

Complex Limits

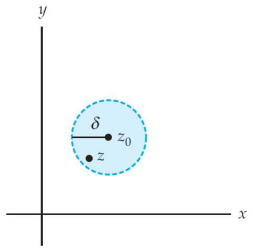
- A complex limit is based on a notion of "close" in the complex plane.
- Because the distance in the complex plane between two points z_1 and z_2 is given by the modulus of the difference of z_1 and z_2, the precise definition of a complex limit will involve $|z_2 - z_1|$.
- E.g., the phrase "$f(z)$ can be made arbitrarily close to the complex number L" can be stated precisely: "for every $\varepsilon > 0$, z can be chosen so that $|f(z) - L| < \varepsilon$.
- Since the modulus of a complex number is a real number, both ε and δ still represent small positive real numbers:

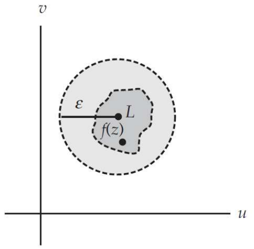
Definition (Limit of a Complex Function)

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The **limit of f as z tends to z_0** exists and is equal to L, written as $\lim_{z\to z_0} f(z) = L$, if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Geometric Representation

- The set of points w satisfying $|w - L| < \varepsilon$ is called a neighborhood of L, and consists of all points in the complex plane lying within, but not on, a circle of radius ε centered at the point L.
- The set of points satisfying $0 < |z - z_0| < \delta$ is called a deleted neighborhood of z_0 and consists of all points in the neighborhood $|z - z_0| < \delta$ excluding the point z_0.
- If $\lim_{z \to z_0} f(z) = L$ and if ε is any positive number, then there is a deleted neighborhood of z_0 of radius δ, such that, for every z in this deleted neighborhood, $f(z)$ is in the ε neighborhood of L:





Real One-Sided Limits

- There is at least one very important difference between real and complex limits.
  - For real functions, $\lim_{x\to x_0} f(x) = L$ if and only if $\lim_{x\to x_0^+} f(x) = L$ and $\lim_{x\to x_0^-} f(x) = L$. Since there are two directions from which x can approach x_0 on the real line, the real limit exists if and only if these two one-sided limits have the same value.
- Example: Consider the real function $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x-1, & \text{if } x \geq 0 \end{cases}$.

  The limit of f as x approaches 0 does not exist:
  - $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0,$ but
  - $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-1) = -1.$

Criterion for the Nonexistence of a Limit

- For limits of complex functions, z is allowed to approach z_0 from any direction in the complex plane, i.e., along any curve or path through z_0.
- For $\lim_{z \to z_0} f(z)$ to exist and to equal L, we require that $f(z)$ approach the same complex number L along every possible curve through z_0.

Criterion for the Nonexistence of a Limit

If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

Example: Nonexistence of a Limit

- Example: Show that $\lim_{z \to 0} \frac{z}{\overline{z}}$ does not exist.
  We show that this limit does not exist by finding two different ways of letting z approach 0 that yield different values for $\lim_{z \to 0} \frac{z}{\overline{z}}$.
  - First, we let z approach 0 along the real axis. That is, we consider complex numbers of the form $z = x + 0i$, where the real number x is approaching 0. For these points we have:
  $$\lim_{z \to 0} \frac{z}{\overline{z}} = \lim_{x \to 0} \frac{x + 0i}{x - 0i} = \lim_{x \to 0} 1 = 1.$$
  - On the other hand, if we let z approach 0 along the imaginary axis, then $z = 0 + iy$, where the real number y is approaching 0. For this approach we have:
  $$\lim_{z \to 0} \frac{z}{\overline{z}} = \lim_{y \to 0} \frac{0 + iy}{0 - iy} = \lim_{y \to 0} (-1) = -1.$$

  Since the two values are not the same, we conclude that $\lim_{z \to 0} \frac{z}{\overline{z}}$ does not exist.

Epsilon-Delta Proofs

- Computing values of $\lim_{z \to z_0} f(z)$ as z approaches z_0 from different directions can prove that a limit does not exist, but cannot be used to prove that a limit does exist.
- To prove that a limit exists we must use the definition directly.
- This requires demonstrating that for every positive real number ε there is an appropriate choice of δ that meets the relevant requirements.
- Such proofs are commonly called "epsilon-delta proofs".
- Even for relatively simple functions, epsilon-delta proofs can be quite complicated.
- We only show some easy examples of such proofs.

Example: An Epsilon-Delta Proof

- Prove that $\lim_{z\to 1+i}(2+i)z = 1+3i$.

  According to the definition, $\lim_{z\to 1+i}(2+i)z = 1+3i$, if, for every $\varepsilon > 0$, there is a $\delta > 0$, such that $|(2+i)z-(1+3i)| < \varepsilon$ whenever $0 < |z-(1+i)| < \delta$. Proving that the limit exists requires that we find an appropriate value of δ for a given value of ε. One way of finding δ is to "work backwards". The idea is to start with the inequality: $|(2+i)z-(1+3i)| < \varepsilon$ and then use properties of complex numbers and the modulus to manipulate this inequality until it involves the expression $|z-(1+i)|$.

  We first factor $(2+i)$ out of the left-hand side: $|2+i| \cdot \left|z - \frac{1+3i}{2+i}\right| < \varepsilon$. Because $|2+i| = \sqrt{5}$ and $\frac{1+3i}{2+i} = 1+i$, we get: $\sqrt{5}\cdot|z-(1+i)| < \varepsilon$ or $|z-(1+i)| < \frac{\varepsilon}{\sqrt{5}}$. This indicates that we should take $\delta = \frac{\varepsilon}{\sqrt{5}}$.

Example: An Epsilon-Delta Proof (Cont'd)

- We now present the formal proof: Given $\varepsilon > 0$, let $\delta = \frac{\epsilon}{\sqrt{5}}$. If $0 < |z - (1+i)| < \delta$, then we have $|z - (1+i)| < \frac{\varepsilon}{\sqrt{5}}$. Multiplying both sides by $|2+i| = \sqrt{5}$ we obtain: $|2+i| \cdot |z - (1+i)| < \sqrt{5} \cdot \frac{\varepsilon}{\sqrt{5}}$ or $|(2+i)z - (1+3i)| < \varepsilon$. Therefore, $|(2+i)z - (1+3i)| < \varepsilon$ whenever $0 < |z - (1+i)| < \delta$. So, by definition, $\lim_{z \to 1+i} (2+i)z = 1+3i$.

Real Multivariable Limits

- We present a practical method for computing complex limits which also establishes an important connection between the complex limit of $f(z) = u(x, y) + iv(x, y)$ and the real limits of the real-valued functions of two real variables $u(x, y)$ and $v(x, y)$.
- Since every complex function is completely determined by the real functions u and v, the limit of a complex function can be expressed in terms of the real limits of u and v.

Definition (Limit of the Real Function $F(x, y)$)

The **limit of F as (x, y) tends to (x_0, y_0)** exists and is equal to the real number L if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $|F(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

- The expression $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ represents the distance between the points (x, y) and (x_0, y_0) in the Cartesian plane.

Properties of Limits

- Using the definitions, we can prove that:
  - $\lim_{(x,y)\to(x_0,y_0)} 1 = 1$,
  - $\lim_{(x,y)\to(x_0,y_0)} x = x_0$,
  - $\lim_{(x,y)\to(x_0,y_0)} y = y_0$.
  - If $\lim_{(x,y)\to(x_0,y_0)} F(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} G(x,y) = M$, then:
    - $\lim_{(x,y)\to(x_0,y_0)} cF(x,y) = cL$, c a real constant,
    - $\lim_{(x,y)\to(x_0,y_0)} (F(x,y) \pm G(x,y)) = L \pm M$,
    - $\lim_{(x,y)\to(x_0,y_0)} F(x,y) \cdot G(x,y) = L \cdot M$,
    - $\lim_{(x,y)\to(x_0,y_0)} \dfrac{F(x,y)}{G(x,y)} = \dfrac{L}{M}$, $M \neq 0$.

Limits Involving Polynomial Expressions

- Example: Limits involving polynomial expressions in x and y can be easily computed using these rules:

$$\begin{aligned}
&\lim_{(x,y)\to(1,2)} (3xy^2 - y) \\
&= 3(\lim_{(x,y)\to(1,2)} x)(\lim_{(x,y)\to(1,2)} y)(\lim_{(x,y)\to(1,2)} y) - \lim_{(x,y)\to(1,2)} y \\
&= 3 \cdot 1 \cdot 2 \cdot 2 - 2 \\
&= 10.
\end{aligned}$$

- In general, if $p(x,y)$ is a two-variable polynomial function, then

$$\lim_{(x,y)\to(x_0,y_0)} p(x,y) = p(x_0,y_0).$$

- If $p(x,y)$ and $q(x,y)$ are two-variable polynomial functions and $q(x_0,y_0) \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{p(x,y)}{q(x,y)} = \frac{p(x_0,y_0)}{q(x_0,y_0)}.$$

Real and Imaginary Parts of a Limit

Theorem (Real and Imaginary Parts of a Limit)

Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

- The theorem reduces the computation of complex limits to the computation of a pair of real limits.
- Example: Compute $\lim_{z\to 1+i} (z^2 + i)$.
  Since $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$, we set $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy + 1$ and $z_0 = 1 + i$, i.e., $x_0 = 1$ and $y_0 = 1$. We next compute the two real limits:

$$\begin{aligned} u_0 &= \lim_{(x,y)\to(1,1)} (x^2 - y^2) = 1^2 - 1^2 = 0, \\ v_0 &= \lim_{(x,y)\to(1,1)} (2xy + 1) = 2 \cdot 1 \cdot 1 + 1 = 3. \end{aligned}$$

  Therefore, $L = u_0 + iv_0 = 0 + i(3) = 3i$, i.e., $\lim_{z\to 1+i} (z^2 + i) = 3i$.

Properties of Complex Limits

Theorem (Properties of Complex Limits)

Suppose that f and g are complex functions. If $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then:

(i) $\lim_{z\to z_0} cf(z) = cL$, c a complex constant;

(ii) $\lim_{z\to z_0} (f(z) \pm g(z)) = L \pm M$;

(iii) $\lim_{z\to z_0} f(z) \cdot g(z) = L \cdot M$, and

(iv) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}$, provided $M \neq 0$.

- We only prove part (i): Let $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, $L = u_0 + iv_0$, and $c = a + ib$. Since $\lim_{z\to z_0} f(z) = L$, $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$. Then $\lim_{(x,y)\to(x_0,y_0)} (au(x,y) - bv(x,y)) = au_0 - bv_0$ and $\lim_{(x,y)\to(x_0,y_0)} (bu(x,y) + av(x,y)) = bu_0 + av_0$.

Properties of Complex Limits (Cont'd)

- We set $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, $L = u_0 + iv_0$, and $c = a + ib$. We then computed
  $\lim_{(x,y)\to(x_0,y_0)} (au(x,y) - bv(x,y)) = au_0 - bv_0$ and
  $\lim_{(x,y)\to(x_0,y_0)} (bu(x,y) + av(x,y)) = bu_0 + av_0$.
  However, note that

$$\begin{aligned} cf(z) &= (a+ib)(u+iv) \\ &= (au - bv) + i(bu + av). \end{aligned}$$

  Thus, $\mathrm{Re}(cf(z)) = au(x,y) - bv(x,y)$ and
  $\mathrm{Im}(cf(z)) = bu(x,y) + av(x,y)$. Therefore,
  $\lim_{z\to z_0} cf(z) = au_0 - bv_0 + i(bu_0 + av_0) = (a+ib)(u_0+iv_0) = cL$.
- Many limits can now be computed starting from:
  - $\lim_{z\to z_0} c = c$, c a complex constant;
  - $\lim_{z\to z_0} z = z_0$.

Computing Limits I

- Compute the limit $\lim_{z\to i} \dfrac{(3+i)z^4 - z^2 + 2z}{z+1}$.

  $\lim_{z\to i} z^2 = \lim_{z\to i} z \cdot z = (\lim_{z\to i} z)(\lim_{z\to i} z) = i \cdot i = -1$. Similarly, $\lim_{z\to i} z^4 = i^4 = 1$. Using these limits, and the properties, we obtain: $\lim_{z\to i}((3+i)z^4 - z^2 + 2z) = (3+i)\lim_{z\to i} z^4 - \lim_{z\to i} z^2 + 2\lim_{z\to i} z = (3+i)(1) - (-1) + 2(i) = 4+3i$, and $\lim_{z\to i}(z+1) = 1+i$. Therefore, finally,
  $$\lim_{z\to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z\to i}((3+i)z^4 - z^2 + 2z)}{\lim_{z\to i}(z+1)} = \frac{4+3i}{1+i}.$$
  After carrying out the division, we obtain
  $$\lim_{z\to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \tfrac{7}{2} - \tfrac{1}{2} i.$$

Computing Limits II

- Compute the limit $\lim_{z\to 1+\sqrt{3}i} \dfrac{z^2-2z+4}{z-1-\sqrt{3}i}$.

  $\lim_{z\to 1+\sqrt{3}i}(z^2-2z+4) = (1+\sqrt{3}i)^2 - 2(1+\sqrt{3}i)+4 = -2+2\sqrt{3}i-2-2\sqrt{3}i+4=0,$ and $\lim_{z\to 1+\sqrt{3}i}(z-1-\sqrt{3}i) = 1+\sqrt{3}i-1-\sqrt{3}i=0$. It appears that we cannot apply the quotient rule since the limit of the denominator is 0. However, in the previous calculation we found that $1+\sqrt{3}i$ is a root of the quadratic polynomial z^2-2z+4. If z_1 is a root of a quadratic polynomial, then $z-z_1$ is a factor of the polynomial. Using long division, we find that $z^2-2z+4=(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)$. Because z is not allowed to take on the value $1+\sqrt{3}i$ in the limit:
  $\lim_{z\to 1+\sqrt{3}i}\dfrac{z^2-2z+4}{z-1-\sqrt{3}i} = \lim_{z\to 1+\sqrt{3}i}\dfrac{(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)}{z-1-\sqrt{3}i} =$
  $\lim_{z\to 1+\sqrt{3}i}(z-1+\sqrt{3}i) = 1+\sqrt{3}i-1+\sqrt{3}i=2\sqrt{3}i.$

Continuity

Continuity of Real Functions

- If the limit of a real function f as x approaches the point x_0 exists and agrees with the value of the function f at x_0, then we say that f is **continuous** at the point x_0.

Continuity of a Real Function $f(x)$

A function f is **continuous at a point** x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.

- In order for the equation $\lim_{x\to x_0} f(x) = f(x_0)$ to hold:
  - The limit $\lim_{x\to x_0} f(x)$ must exist;
  - f must be defined at x_0;
  - the two values must be equal.

  If anyone of these three conditions fail, then f is not continuous at x_0.
- Example: The function $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x-1, & \text{if } x \geq 0 \end{cases}$ is not continuous at the point $x = 0$ since $\lim_{x\to 0} f(x)$ does not exist.
- Example: Even though $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$, the function $f(x) = \frac{x^2-1}{x-1}$ is not continuous at $x = 1$ because $f(1)$ is not defined.

Continuity of Complex Functions

- A complex function f is continuous at a point z_0 if the limit of f as z approaches z_0 exists and is the same as the value of f at z_0.

Definition (Continuity of a Complex Function)

A complex function f is **continuous at a point** z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Criteria for Continuity at a Point

A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(i) $\lim_{z \to z_0} f(z)$ exists;

(ii) f is defined at z_0;

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$.

- If a complex function f is not continuous at a point z_0, then we say that f is **discontinuous** at z_0.
- Example: The function $f(z) = \frac{1}{1+z^2}$ is discontinuous at $z = i$ and $z = -i$.

Checking Continuity at a Point

- Consider the function $f(z) = z^2 - iz + 2$.

  To determine if f is continuous at the point $z_0 = 1 - i$, we must find
  - $\lim_{z \to z_0} f(z)$;
  - $f(z_0)$,
  - and then check to see whether these two complex values are equal.

  We obtain:
  $\lim_{z \to z_0} f(z) = \lim_{z \to 1-i} (z^2 - iz + 2) = (1-i)^2 - i(1-i) + 2 = 1 - 3i.$
  Furthermore, for $z_0 = 1 - i$, we have:
  $f(z_0) = f(1 - i) = (1 - i)^2 - i(1 - i) + 2 = 1 - 3i.$
  Since $\lim_{z \to z_0} f(z) = f(z_0)$, we conclude that $f(z) = z^2 - iz + 2$ is continuous at the point $z_0 = 1 - i$.

Discontinuity of Principal Square Root Function

- Show that the principal square root function $f(z) = z^{1/2} = \sqrt{|z|}e^{i\mathrm{Arg}(z)/2}$ is discontinuous at the point $z_0 = -1$.

  We show that the limit $\lim_{z\to z_0} f(z) = \lim_{z\to -1} z^{1/2}$ does not exist. We let z approach -1 via two different paths.

  - Consider z approaching -1 along the quarter of the unit circle lying in the second quadrant, i.e., $|z| = 1$, $\frac{\pi}{2} < \arg(z) < \pi$. In exponential form $z = e^{i\theta}$, $\frac{\pi}{2} < \theta < \pi$, with θ approaching π. By setting $|z| = 1$ and letting $\mathrm{Arg}(z) = \theta$ approach π, we obtain: $\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|}e^{i\mathrm{Arg}(z)/2} = \lim_{\theta\to\pi} \sqrt{1}e^{i\theta/2} = \lim_{\theta\to\pi} (\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$.
  - Let z approach -1 along the quarter of the unit circle lying in the third quadrant, i.e., $z = e^{i\theta}$, $-\pi < \theta < -\frac{\pi}{2}$, with θ approaching $-\pi$. By setting $|z| = 1$ and letting $\mathrm{Arg}(z) = \theta$ approach $-\pi$ we find: $\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|}e^{i\mathrm{Arg}(z)/2} = \lim_{\theta\to -\pi} e^{i\theta/2} = \lim_{\theta\to -\pi} (\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) = -i$.

  We conclude that $\lim_{z\to -1} z^{1/2}$ does not exist. Therefore, $f(z) = z^{1/2}$ is discontinuous at the point $z_0 = -1$.

Continuity on a Set of Points

- Besides continuity of a complex function f at a single point z_0 in the complex plane, we are often also interested in the continuity of a function on a set of points in the complex plane.
- A complex function f is **continuous on a set** S if f is continuous at z_0, for each z_0 in S.
- Example: Using the properties, we can show that $f(z) = z^2 - iz + 2$ is continuous at any point z_0 in the complex plane. Therefore, we say that f is continuous on $\mathbb{C}$.
- Example: The function $f(z) = \frac{1}{z^2+1}$ is continuous on the set consisting of all complex z such that $z \neq \pm i$.

Real and Imaginary Parts of a Continuous Function

- Various properties of complex limits can be translated into statements about continuity.
- E.g., a preceding theorem described the connection between the complex limit of $f(z) = u(x, y) + iv(x, y)$ and the real limits of u, v:

Definition (Continuity of a Real Function $F(x, y)$)

A function F is **continuous at** (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} F(x, y) = F(x_0, y_0)$.

Theorem (Real and Imaginary Parts of a Continuous Function)

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point (x_0, y_0).

Proof of the Theorem

- Assume that the complex function $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$. Then $\lim_{z \to z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$. This implies: $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u(x_0, y_0)$, $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v(x_0, y_0)$. Therefore, both u and v are continuous at (x_0, y_0).

  Conversely, if u and v are continuous at (x_0, y_0), then $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u(x_0, y_0)$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v(x_0, y_0)$. It then follows that $\lim_{z \to z_0} f(z) = u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$. Therefore, f is continuous.

Checking Continuity Using the Theorem

- Show that the function $f(z) = \overline{z}$ is continuous on $\mathbb{C}$.

  According to the theorem, $f(z) = \overline{z} = \overline{x+iy} = x - iy$ is continuous at $z_0 = x_0 + iy_0$ if both $u(x,y) = x$ and $v(x,y) = -y$ are continuous at (x_0, y_0).
  Because u and v are two-variable polynomial functions, it follows that: $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = x_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = -y_0$.
  This implies that u and v are continuous at (x_0, y_0). Therefore, f is continuous at $z_0 = x_0 + iy_0$ by the preceding theorem.
  Since $z_0 = x_0 + iy_0$ was arbitrary, we conclude that the function $f(z) = \overline{z}$ is continuous on $\mathbb{C}$.

Properties of Continuous Functions

Theorem (Properties of Continuous Functions)

If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

(i) cf, c a complex constant;

(ii) $f \pm g$;

(iii) $f \cdot g$;

(iv) $\frac{f}{g}$, provided $g(z_0) \neq 0$.

- We only prove (ii). Since f and g are continuous at z_0, we have that $\lim_{z \to z_0} f(z) = f(z_0)$ and $\lim_{z \to z_0} g(z) = g(z_0)$. It follows that $\lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = f(z_0) + g(z_0)$. Therefore, $f + g$ is continuous at z_0.

Continuity of Polynomial Functions

Theorem (Continuity of Polynomial Functions)

Polynomial functions are continuous on the entire complex plane $\mathbb{C}$.

- Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial function and let z_0 be any point in the complex plane $\mathbb{C}$. The identity function $f(z) = z$ is continuous at z_0, whence, by repeated application of the product rule, the power function $f(z) = z^n$, where n is an integer and $n \geq 1$, is continuous at this point as well. Moreover, every complex constant function $f(z) = c$ is continuous at z_0, so it follows by the theorem that each of the functions $a_n z^n$, $a_{n-1} z^{n-1}$, $\ldots$, $a_1 z$, and a_0 are continuous at z_0. Finally, from repeated application of the sum rule, $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is continuous at z_0. Since z_0 was allowed to be any point in the complex plane, we have shown that the polynomial function p is continuous on the entire complex plane $\mathbb{C}$.

Continuity of Rational Functions

Continuity of Rational Functions

Rational functions are continuous on their domains.

- Since a rational function $f(z) = \frac{p(z)}{q(z)}$ is quotient of the polynomial functions p and q, it follows from the theorem and the quotient rule that f is continuous at every point z_0 for which $q(z_0) \neq 0$.

Real and Complex Bounded Functions

- Recall that if a real function f is continuous on a closed interval I on the real line, then f is bounded on I, i.e., there is a real number $M > 0$ such that $|f(x)| \leq M$, for all x in I.
- An analogous result for real functions $F(x, y)$ states that, if $F(x, y)$ is continuous on a closed and bounded region R of the Cartesian plane, then there is a real number $M > 0$, such that $|F(x, y)| \leq M$, for all (x, y) in R, and we say F is bounded on R.
- Suppose that the function $f(z) = u(x, y) + iv(x, y)$ is defined on a closed and bounded region R in the complex plane. As with real functions, we say that the complex function f is **bounded** on R if there exists a real constant $M > 0$, such that $|f(z)| \leq M$, for all z in R.

Bounded Property for Complex Functions

Theorem (A Bounding Property)

If a complex function f is continuous on a closed and bounded region R, then f is bounded on R. That is, there is a real constant $M > 0$, such that $|f(z)| \leq M$, for all z in R.

- If f is continuous on R, then u and v are continuous real functions on R. Since the square root function is continuous, it follows that the real function $F(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$ is also continuous on R. Because F is continuous on the closed and bounded region R, F is bounded on R, i.e., there is a real constant $M > 0$, such that $|F(x, y)| \leq M$, for all (x, y) in R. However, since $|f(z)| = F(x, y)$, we have that $|f(z)| \leq M$, for all z in R. Thus, the complex function f is bounded on R.

Branches

- We have discussed the concept of a multiple valued function $F(z)$ that assigns a set of complex numbers to the input z.
- Examples of multiple valued functions include $F(z) = z^{1/n}$, which assigns to the input z the set of n n-th roots of z, and $G(z) = \arg(z)$, which assigns to the input z the infinite set of arguments of z.
- In practice, it is often the case that we need a consistent way of choosing just one of the values of a multiple-valued function.
- If we make this choice of value with the concept of continuity in mind, then we obtain a function that is called a branch of a multiple-valued function.
- A **branch** of a multiple-valued function F is a function f_1 that is continuous on some domain and that assigns exactly one of the multiple values of F to each point z in that domain.
- Notation for Branches: When representing branches of a multiple valued function F with functional notation, we will use lowercase letters with a numerical subscript such as f_1, f_2, and so on.

Discontinuities of the Square Root Function

- The requirement that a branch be continuous means that the domain of a branch is different from the domain of the multiple valued function.
- Example: The multiple-valued function $F(z) = z^{1/2}$ that assigns to each input z the set of two square roots of z is defined for all nonzero complex numbers z. Even though the principal square root function $f(z) = z^{1/2}$ does assign exactly one value of F to each input z (the principal square root of z), f is not a branch of F. The reason is that the principal square root function is not continuous on its domain. E.g., we showed that $f(z) = z^{1/2}$ is not continuous at $z_0 = -1$. We can also show that $f(z) = z^{1/2}$ is discontinuous at every point on the negative real axis.

  In order to obtain a branch of $F(z) = z^{1/2}$ that agrees with the principal square root function, we must restrict the domain to exclude points on the negative real axis.

The Principal Branch of the Square Root Function

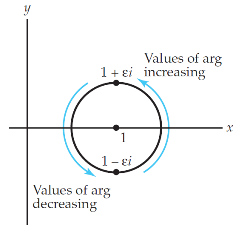
- We define the **principal branch** of $F(z) = z^{1/2}$ by $f_1(z) = \sqrt{r}e^{i\theta/2}$, $-\pi < \theta < \pi$. The function f_1 is a branch of $F(z) = z^{1/2}$. The domain $\mathrm{Dom}(f_1)$ of f_1 is defined by $|z| > 0$, $-\pi < \arg(z) < \pi$. The function f_1 agrees with the principal square root function f on this set. Thus, f_1 does assign to the input z exactly one of the values of $F(z) = z^{1/2}$. To show that f_1 is a continuous, let z be a point with $|z| > 0$, $-\pi < \arg(z) < \pi$. If $z = x + iy$ and $x > 0$, then $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Since $-\frac{\pi}{2} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{2}$, the inequality $-\pi < \theta < \pi$ is satisfied. Thus, substituting the expressions for r and θ: $f_1(z) = \sqrt[4]{x^2 + y^2}e^{i\tan^{-1}(y/x)/2} = \sqrt[4]{x^2 + y^2}\cos\left(\frac{\tan^{-1}(y/x)}{2}\right) + i\sqrt[4]{x^2 + y^2}\sin\left(\frac{\tan^{-1}(y/x)}{2}\right)$. Because the real and imaginary parts of f_1 are continuous real functions for $x > 0$, we conclude that f_1 is continuous for $x > 0$. A similar argument can be made for points with $y > 0$ using $\theta = \cot^{-1}\left(\frac{x}{y}\right)$ and for points with $y < 0$ using $\theta = -\cot^{-1}\left(\frac{x}{y}\right)$. So f_1 is continuous.

Branch Cuts

- Although $F(z) = z^{1/2}$ is defined for all nonzero complex numbers $\mathbb{C}$, the principal branch f_1 is defined only on $|z| > 0$, $-\pi < \arg(z) < \pi$.
- In general, a **branch cut** for a branch f_1 of a multiple-valued function F is a portion of a curve that is excluded from the domain of F so that f_1 is continuous on the remaining points.
- Therefore, the non-positive real axis is a branch cut for the principal branch f_1 of the multiple-valued function $F(z) = z^{1/2}$.
- A different branch of F with the same branch cut is given by $f_2(z) = \sqrt{r}e^{i\theta/2}$, $\pi < \theta < 3\pi$. These are distinct since for, e.g., $z = i$, $f_1(i) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$, but $f_2(i) = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i$.
- If we set $\phi = \theta - 2\pi$, then the branch f_2 can be expressed as $f_2(z) = \sqrt{r}e^{i(\phi+2\pi)/2} = \sqrt{r}e^{i\phi/2}e^{i\pi}$, $-\pi < \phi < \pi$. Since $e^{i\pi} = -1$, $f_2(z) = -\sqrt{r}e^{i\phi/2}$, $-\pi < \phi < \pi$. This shows that $f_2 = -f_1$.
- These two branches of $F(z) = z^{1/2}$ are analogous to the positive and negative square roots of a positive real number.

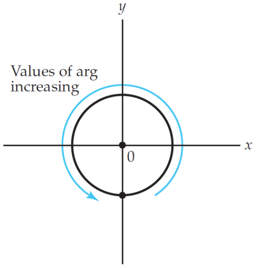
Branch Points

- The point $z = 0$ must be on the branch cut of every branch of the multiple-valued function $F(z) = z^{1/2}$.
- A point with the property that it is on the branch cut of every branch is called a **branch point** of F.
- Alternatively, a branch point is a point z_0 with the following property: If we traverse any circle centered at z_0 with sufficiently small radius starting at a point z_1, then the values of any branch do not return to the value at z_1.
- Example: Consider any branch of $G(z) = \arg(z)$.
  At the point, say, $z_0 = 1$, if we traverse the small circle $|z - 1| = \varepsilon$ counterclockwise from the point $z_1 = 1 - \varepsilon i$, then the values of the branch increase until we reach the point $1 + \varepsilon i$. Then the values of the branch decrease back down to the value of the branch at z_1. Thus, $z_0 = 1$ is not a branch point.



Example (Cont'd)

- Consider again any branch of $G(z) = \arg(z)$. Suppose the process is repeated for the point $z_0 = 0$. For the small circle $|z| = \varepsilon$, the values of the branch increase along the entire circle. By the time we have returned to our starting point, the value of the branch is no longer the same, but has increased by 2π.



Therefore, $z_0 = 0$ is a branch point of $G(z) = \arg(z)$.

Infinite Limits and Limits at Infinity

- In analogy with real analysis, we can also define the concepts of infinite limits and limits at infinity for complex functions.
- Intuitively, the limit $\lim_{z\to\infty} f(z) = L$ means that values $f(z)$ of the function f can be made arbitrarily close to L if values of z are chosen so that $|z|$ is sufficiently large.
- The **limit of f as z tends to ∞** exists and is equal to L if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(z) - L| < \varepsilon$ whenever $|z| > 1/\delta$.
- Using this definition it is not hard to show that: $\lim_{z\to\infty} f(z) = L$ if and only if $\lim_{z\to 0} f(\frac{1}{z}) = L$.
- Similarly, the infinite limit $\lim_{z\to z_0} f(z) = \infty$ is defined by: The **limit of f as z tends to z_0 is ∞** if, for every $\varepsilon > 0$, there is a $\delta > 0$, such that $|f(z)| > 1/\varepsilon$ whenever $0 < |z - z_0| < \delta$.
- From this definition we obtain: $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.