1 The $3x + 1$ Problem

1.0.1

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1.0.2 Overview

- The $3x + 1$ Problem and Collatz Conjecture
- What Makes This Problem Interesting?
- History of the Collatz Conjecture
- Interesting Attributes of the $3x + 1$ Problem

1.0.3 Interesting Attributes

- Cycles of the Function
- Stochastic Approximations
- Height of the Function
- Stopping Time of the Function

1.1 What is the $3x + 1$ Problem?

1.1.1 The Function

- based on the Collatz function **06**

$$C(x) = \begin{cases} 3x + 1 & \text{if } x \equiv 1 \pmod 2, \\ x/2 & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

- equivalent to the $3x + 1$ function

$$T(x) = \begin{cases} (3x + 1)/2 & \text{if } x \equiv 1 \pmod 2, \\ x/2 & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

1.1.2 Details

- it is conjectured that for some $x, k \in \mathbb{N} + 1$ we attain $T^{(k)}(x) = 1$ **05**
- the $3x + 1$ function $T(x)$ maps $\mathbb{N} + 1 \to \mathbb{N} + 1$ **01**
- the function has a *stopping time*, *total stopping time*, *trajectory*, and a *height* for each m

1.1.3 Stopping Time for x

- check that every positive integer up to $x - 1$ iterates to one **05**
- if $T^{(k)}(x) < x$, we know it will iterate to 1
- thus the stopping time is

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\}$$

1.1.4 Total Stopping Time for x

- total stopping time is the number of steps needed to iterate to 1

$$\sigma_\infty(x) = \inf\{k : T^{(k)}(x) = 1\}$$

05

1.1.5 Trajectory of x Under T

- also called the *forward orbit* of x under T
- defined as the sequence of it forward iterates

$$\{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\}$$

06

1.1.6 Height of x

- is the highes number part of the trajectory of $T(x)$

$$h(x) = \sup\{T^{(k)}(x) : k \in \mathbb{N} + 1\}$$

05

1.2 The Collatz Conjecture

1.2.1 Possible behaviors of T

1. the trivial cycle $\{4, 2, 1, 4, 2, 1, \dots\}$ (reaching 1)
2. a non-trivial cycle
3. infinity, having a divergent orbit **05**

1.2.2 The Conjecture

- beginning at any positve integer x, iterations of $T(x)$ will eventually reach 1 and enter the trivial cycle **06**
- equivalent to stating that height $h(x)$ and total stopping time $\sigma_\infty(x)$ are finite **05**
- if a trajectory of $T(x)$ does *not* contain 1 it is infinite **09**

1.3 What Makes This Problem Interesting?

1.3.1

Mathematics is not ready for such problems.

— Paul Erdös

05

1.3.2

- the problem itself is not important, it has no immediate applications
- represents a class of iterative mappings that are interesting
- it is simple to state but hard to prove
- part of the difficulty comes from its pseudorandom nature of iterations of $T(x)$ **06**

1.4 History of the Collatz Conjecture

1.4.1 Beginnings

- also known as Syracuse Problem, Hasse's Algorithm, Kakutani's Problem, and Ulam's Problem after other people that studied it
- named after Lothar Collatz who formulated similar problems in the 1930s
- academic publishing about it began in the 1970s

1.4.2 Recent Developments

- $> 10^{20}$ numbers have been verified to fit the conjecture **01**
- a September 2019 paper by Terence Tao "Almost All Orbits of the Collatz Map Attain Almost Bounded Values" made progress
- research is still actively ongoing

1.5 Interesting Attributes of the $3x + 1$ Problem

1.5.1 Cycles of the Function

- the $3x + 1$ function has a trivial cycle $\{4, 2, 1, 4, 2, \ldots\}$ at 1 **05**
- if $T(x)$ is applied to all integers, three more cycles emerge at -1, -5, and -17
- these cycles are conjectured to be the only ones **05**
- if non-trivial cycles of the $3x + 1$ problem exist, they have been proven to be at least 10,439,860,591 numbers long **06**

1.5.2 Stochastic Approximations

- number of odd and even integers in an orbit is approximately equal
- behavior is seen as pseudorandom
- probabilistic models describe the behavior of the $3x + 1$ problem
- models describe groups of trajectories, not individual ones

1.5.3

- interesting because stochastic models are used to approach deterministic systems
- we assume that the number of add iterated and even iterates is about the same
- because it seems random people are using probability distributions to describe groups of these functions

1.5.4 Height of the Function

- height can be called the cardinality of the trajectory?
- how many approximation of the height of a function
- graph actual height of the function vs the approximation?

1.5.5 Stopping Time of the Function

- most ints have large stopping times, even though they can be very large
- average stopping time for odd integers should be around 9.477955
- general total stopping time estimation
- total stopping time is equal to the number of even iterates in the sequence
- upper bound for the total stopping time is $41....$ log n – suggests that all sequences are finite
- graph the stopping times for some functions vs their approximations?

1.5.6

- logarithmically the slope of the function is equal x
- most trajectories follow that shape
- some are split and more interesting
- iterates can be arbitrarily larger than the starting values
- sum of even ints equals the sum of odd ints plus the number of odd ints?

1.6 Conclusion

1.6.1

- what have I told you?

1.7 Any questions?

1.8 References

1.8.1

- lagarias and friends