Complex Trigonometric Functions

Complex Sine and Cosine Functions

- If x is a real variable, then
$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x.$$
- By adding these equations and simplifying, we get:
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$
- If we subtract the two equations, then we obtain
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$
- The formulas for the real cosine and sine functions can be used to define the complex sine and cosine functions.

Definition (Complex Sine and Cosine Functions)

The **complex sine and cosine functions** are defined by:
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The Complex Tangent, Cotangent, Secant, and Cosecant

- The complex sine and cosine functions agree, by definition, with the real sine and cosine functions for real input.
- Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

- These functions also agree with their real counterparts for real input.

Values of Complex Trigonometric Functions

- Express the value of the given trigonometric function in the form $a + ib$.

$$\text{(a)} \quad \cos i \qquad \text{(b)} \quad \sin(2+i) \qquad \text{(c)} \quad \tan(\pi - 2i).$$

(a) $\cos i = \dfrac{e^{i \cdot i} + e^{-i \cdot i}}{2} = \dfrac{e^{-1} + e}{2}$.

(b) $\sin(2+i) = \dfrac{e^{i(2+i)} - e^{-i(2+i)}}{2i} = \dfrac{e^{-1+2i} - e^{1-2i}}{2i} = \dfrac{e^{-1}(\cos 2 + i \sin 2) - e(\cos(-2) + i \sin(-2))}{2i}$.

(c) $\tan(\pi - 2i) = \dfrac{(e^{i(\pi-2i)} - e^{-i(\pi-2i)})/2i}{(e^{i(\pi-2i)} + e^{-i(\pi-2i)})/2} = \dfrac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{(e^{i(\pi-2i)} + e^{-i(\pi-2i)})i} = \dfrac{e^2 - e^{-2}}{(e^2 + e^{-2})i} = -\dfrac{e^2 - e^{-2}}{e^2 + e^{-2}} i$.

Identities

- We now list some of the more useful of the trigonometric identities:
  - $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$;
  - $\cos^2 z + \sin^2 z = 1$;
  - $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$;
  - $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.
- Because of the sum/difference formulas, we also have the double-angle formulas:

$$\sin 2z = 2 \sin z \cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z.$$

- We only verify $\cos^2 z + \sin^2 z = 1$:

$$\begin{aligned} \cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2iz}+2+e^{-2iz}}{4} - \frac{e^{2iz}-2+e^{-2iz}}{4} = 1. \end{aligned}$$

- Some properties of the real trigonometric functions are not satisfied by their complex counterparts:
  E.g., $|\sin x| \leq 1$ and $|\cos x| \leq 1$, for all real x, but $|\cos i| > 1$ and $|\sin(2+i)| > 1$.

Periodicity

- We know that the complex exponential function is periodic with a pure imaginary period of $2\pi i$, i.e., $e^{z+2\pi i} = e^z$, for all complex z.
- Replacing z with iz, we get $e^{iz+2\pi i} = e^{i(z+2\pi)} = e^{iz}$.
- Thus, e^{iz} is a periodic function with real period 2π.
- Similarly, $e^{-i(z+2\pi)} = e^{-iz}$, i.e., e^{-iz} is periodic with period of 2π.
- It now follows that:
$$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$
- A similar statement also holds for the complex cosine function.
- Thus, the complex sine and cosine are periodic functions with a real period of 2π.
- The periodicity of the secant and cosecant functions follows immediately from the definitions.
- Moreover, the complex tangent and cotangent are periodic with a real period of π.

Trigonometric Equations

- Since the complex sine and cosine functions are periodic, there are always infinitely many solutions to equations of the form $\sin z = w$ or $\cos z = w$.
- One approach to solving such equations is to use the definition in conjunction with the quadratic formula:

  Example: Find all solutions to the equation $\sin z = 5$.

  $\sin z = 5$ is equivalent to the equation $\frac{e^{iz} - e^{-iz}}{2i} = 5$. By multiplying this equation by e^{iz} and simplifying we obtain $e^{2iz} - 10ie^{iz} - 1 = 0$. This equation is quadratic in e^{iz}, i.e., $(e^{iz})^2 - 10i(e^{iz}) - 1 = 0$. By the quadratic formula that the solutions are given by $e^{iz} = \frac{10i + (-96)^{1/2}}{2} = 5i \pm 2\sqrt{6}i = (5 \pm 2\sqrt{6})i$. In order to find the values of z, we must solve the two resulting exponential equations using the complex logarithm.

Trigonometric Equations (Cont'd)

- We must solve $e^{iz} = (5 \pm 2\sqrt{6})i$ using the complex logarithm.
  - If $e^{iz} = (5 + 2\sqrt{6})i$, then $iz = \ln(5i + 2\sqrt{6}i)$ or $z = -i\ln[(5 + 2\sqrt{6})i]$. Because $(5 + 2\sqrt{6})i$ is a pure imaginary number and $5 + 2\sqrt{6} > 0$, we have $\arg[(5 + 2\sqrt{6})i] = \frac{1}{2}\pi + 2n\pi$. Thus, $z = -i\ln[(5 + 2\sqrt{6})i] = -i[\log_e(5 + 2\sqrt{6}) + i(\frac{\pi}{2} + 2n\pi)]$ or $z = \frac{(4n+1)\pi}{2} - i\log_e(5 + 2\sqrt{6})$, for $n = 0, \pm 1, \pm 2, \ldots$.
  - Similarly, if $e^{iz} = (5 - 2\sqrt{6})i$, then $z = -i\ln[(5 - 2\sqrt{6})i]$. Since $(5 - 2\sqrt{6})i$ is a pure imaginary number and $5 - 2\sqrt{6} > 0$, it has an argument of $\frac{\pi}{2}$, and so: $z = -i\ln[(5 - 2\sqrt{6})i] = -i[\log_e(5 - 2\sqrt{6}) + i(\frac{\pi}{2} + 2n\pi)]$ or $z = \frac{(4n+1)\pi}{2} - i\log_e(5 - 2\sqrt{6})$ for $n = 0, \pm 1, \pm 2, \ldots$.

$\sin z$ and $\cos z$ in terms of x and y

- To find a formula in terms of x and y for the modulus of the sine and cosine functions, we replace z by $x + iy$ in $\sin z$:

$$\begin{aligned}\sin z &= \frac{e^{-y+ix} - e^{y-ix}}{2i} \\ &= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \\ &= \sin x\frac{e^{y} + e^{-y}}{2} + i\cos x\frac{e^{y} - e^{-y}}{2}.\end{aligned}$$

- Since the real hyperbolic sine and cosine functions are defined by $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$ and $\cosh y = \dfrac{e^{y} + e^{-y}}{2}$, we can rewrite as

$$\sin z = \sin x\cosh y + i\cos x\sinh y.$$

- A similar computation gives

$$\cos z = \cos x\cosh y - i\sin x\sinh y.$$

Modulus of Sine and Cosine

- By the expression for $\sin z$:
$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}.$$
- Recall $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y = 1 + \sinh^2 y$:
$$\begin{aligned} |\sin z| &= \sqrt{\sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y} \\ &= \sqrt{\sin^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y}, \end{aligned}$$
$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}.$$
- Similarly, for the modulus of the complex cosine function:
$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$$
- Since $\sinh x$ is unbounded, the complex sine and cosine functions are not bounded on the complex plane, i.e., there does not exist a real constant M so that $|\sin z| < M$, for all z in $\mathbb{C}$, nor does there exist a real constant M so that $|\cos z| < M$, for all z in $\mathbb{C}$.

Zeros

- The zeros of the real sine occur at integer multiples of π and the zeros of the real cosine occur at odd integer multiples of $\frac{\pi}{2}$.
- These zeros of the real sine and cosine functions are also zeros of the complex sine and cosine, respectively.
- To find all zeros, we must solve $\sin z = 0$ and $\cos z = 0$.
- $\sin z = 0$ is equivalent to $|\sin z| = 0$, i.e., $\sqrt{\sin^2 x + \sinh^2 y} = 0$, which is equivalent to: $\sin^2 x + \sinh^2 y = 0$.
- Since $\sin^2 x$ and $\sinh^2 y$ are nonnegative real numbers, we must have $\sin x = 0$ and $\sinh y = 0$.
  - $\sin x = 0$ occurs when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.
  - $\sinh y = 0$ occurs only when $y = 0$.

  So, the only solutions of $\sin z = 0$ in the complex plane are the real numbers $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, i.e., the zeros of the complex sine function are the same as the zeros of the real sine.
- Similarly, the only zeros of the complex cosine function are the real numbers $z = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$.

Analyticity

- The derivatives of the complex sine and cosine functions are found using the chain rule:

$$\begin{aligned} \frac{d}{dz}\sin z &= \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{ie^{iz}+ie^{-iz}}{2i} \\ &= \frac{e^{iz}+e^{-iz}}{2} = \cos z. \end{aligned}$$

  Since this derivative is defined for all complex z, $\sin z$ is entire.
- Similarly, $\frac{d}{dz}\cos z = -\sin z$.
- The derivatives of $\sin z$ and $\cos z$ can then be used to compute the derivatives of all of the complex trigonometric functions:

$$\begin{array}{lll} \frac{d}{dz}\sin z = \cos z & \frac{d}{dz}\cos z = -\sin z & \frac{d}{dz}\tan z = \sec^2 z \\ \frac{d}{dz}\cot z = -\csc^2 z & \frac{d}{dz}\sec z = \sec z \tan z & \frac{d}{dz}\csc z = -\csc z \cot z \end{array}$$

- The sine and cosine functions are entire, but the tangent, cotangent, secant, and cosecant functions are only analytic at those points where the denominator is nonzero.

Trigonometric Mapping

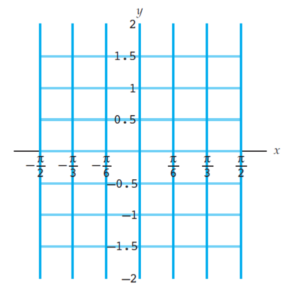
- Since $\sin z$ is periodic with a real period of 2π, it takes on all values in any infinite vertical strip $x_0 < x \leq x_0 + 2\pi$, $-\infty < y < \infty$.
- This allows us to study the mapping $w = \sin z$ on the entire complex plane by analyzing it on any one of these strips.
- Consider the strip $-\pi < x \leq \pi$, $-\infty < y < \infty$.
- Observe that $\sin z$ is not one-to-one on this region, e.g., $z_1 = 0$ and $z_2 = \pi$ are in this region and $\sin 0 = \sin \pi = 0$.
- From $\sin(-z + \pi) = \sin z$, it follows that the image of the strip $-\pi < x \leq -\frac{\pi}{2}$, $-\infty < y < \infty$, is the same as the image of the strip $\frac{\pi}{2} < x \leq \pi$, $-\infty < y < \infty$, under $w = \sin z$.
- Therefore, we need only consider the mapping $w = \sin z$ on the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, to gain an understanding of this mapping on the entire z-plane.
- One can show that the complex sine function is one-to-one on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\infty < y < \infty$.

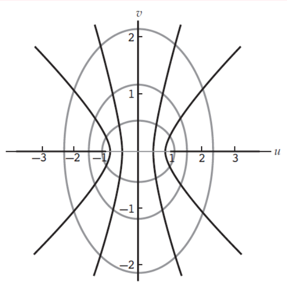
The Mapping $w = \sin z$

- Describe the image of the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, under the complex mapping $w = \sin z$.
  We determine the image of vertical lines $x = a$ with $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$.
  - Assume that $a \neq -\frac{\pi}{2}, 0, \frac{\pi}{2}$. The image of the vertical line $x = a$ is given by: $u = \sin a \cosh y$, $v = \cos a \sinh y$, $-\infty < y < \infty$. Since $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, it follows that $\sin a \neq 0$ and $\cos a \neq 0$, whence $\cosh y = \frac{u}{\sin a}$ and $\sinh y = \frac{v}{\cos a}$. The identity $\cosh^2 y - \sinh^2 y = 1$ gives: $(\frac{u}{\sin a})^2 - (\frac{v}{\cos a})^2 = 1$. It represents a hyperbola with vertices at $(\pm \sin a, 0)$ and slant asymptotes $v = \pm(\frac{\cos a}{\sin a})u$.
    Because the point $(a, 0)$ is on the line $x = a$, the point $(\sin a, 0)$ must be on the image of the line. Therefore, the image of the vertical line $x = a$, with $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, under $w = \sin z$ is the branch of the hyperbola that contains the point $(\sin a, 0)$.
    Because $\sin(-z) = -\sin z$, for all z, the image of the line $x = -a$ is a branch of the hyperbola containing the point $(-\sin a, 0)$.
    Therefore, the pair $x = a$ and $x = -a$, with $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, are mapped onto the full hyperbola.

The Mapping $w = \sin z$ (Cont'd)

- Summary:



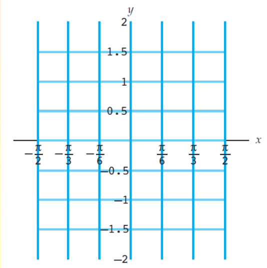


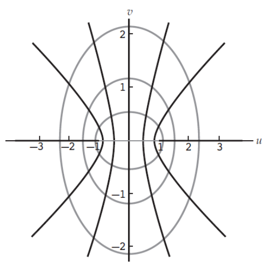
  - The image of the line $x = -\frac{\pi}{2}$ is the set of points $u \leq -1$ on the negative real axis.
  - The image of the line $x = \frac{\pi}{2}$ is the set of points $u \geq 1$ on the positive real axis.
  - The image of the line $x = 0$ is the imaginary axis $u = 0$.

In summary, the image of the infinite vertical strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, under $w = \sin z$, is the entire w-plane.

Following Horizontal Line Segments

- The image could also be found using horizontal line segments $y = b$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, instead of vertical lines.
- The images are: $u = \sin x \cosh b$, $v = \cos x \sinh b$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
  - When $b \neq 0$, we get $(\frac{u}{\cosh b})^2 + (\frac{v}{\sinh b})^2 = 1$, which is an ellipse with u-intercepts at $(\pm \cosh b, 0)$ and v-intercepts at $(0, \pm \sinh b)$.
  - If $b > 0$, then the image of the segment $y = b$ is the upper-half of the ellipse and the image of the segment $y = -b$ the bottom-half.

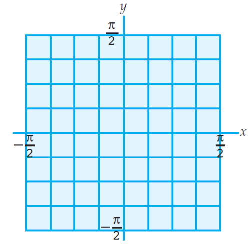


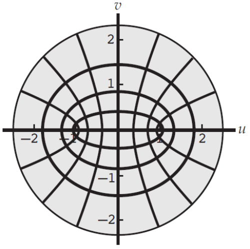


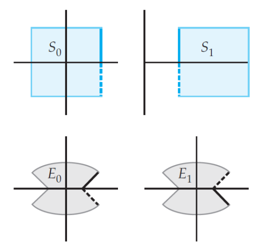
  - Observe that if $b = 0$, then the image of the line segment $y = 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is the line segment $-1 \leq u \leq 1$, $v = 0$ on the real axis.

Riemann Surface I

- Since the complex sine function is periodic, the mapping $w = \sin z$ is not one-to-one on the complex plane. A Riemann surface helps visualize $w = \sin z$.
- Consider the mapping on the square S_0 defined by $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.





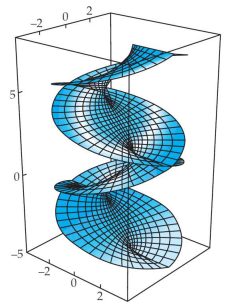


S_0 is mapped onto the elliptical region E.

- Similarly, the adjacent square S_1 defined by $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, also maps onto E.

Riemann Surface II

- A Riemann surface is constructed by starting with two copies of E, E_0 and E_1, representing the images of S_0 and S_1, respectively. We cut E_0 and E_1 open along the line segments in the real axis from 1 to $\cosh\left(\frac{\pi}{2}\right)$ and from -1 to $-\cosh\left(\frac{\pi}{2}\right)$.
- Part of the Riemann surface consists of the two elliptical regions E_0 and E_1 with the black segments glued together and the dashed segments glued together.
- To complete the Riemann surface, we take for every integer n an elliptical region E_n representing the image of the square S_n defined by $\frac{(2n-1)\pi}{2} \leq x \leq \frac{(2n+1)\pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Each region E_n is cut open, as E_0 and E_1 were, and E_n is glued to E_{n+1} along their boundaries in a manner analogous to that used for E_0 and E_1.



Complex Hyperbolic Functions

Complex Hyperbolic Sine and Cosine

- The real hyperbolic sine and hyperbolic cosine functions are defined using the real exponential by $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Definition (Complex Hyperbolic Sine and Cosine)

The **complex hyperbolic sine** and **hyperbolic cosine functions** are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

- These agree with the real hyperbolic functions for real input.
- Unlike the real hyperbolic functions, the complex hyperbolic functions are periodic and have infinitely many zeros.
- The **complex hyperbolic tangent**, **cotangent**, **secant**, and **cosecant**:

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$
$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Derivatives of Complex Hyperbolic Functions

- The hyperbolic sine and cosine functions are entire because the functions e^z and e^{-z} are entire.
- Moreover, we have:

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$$

- A similar computation for $\cosh z$ yields

$$\frac{d}{dz}\cosh z = \sinh z.$$

- Derivatives of Complex Hyperbolic Functions

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\tanh z = \mathrm{sech}^2 z,$$
$$\frac{d}{dz}\coth z = -\mathrm{csch}^2 z, \quad \frac{d}{dz}\mathrm{sech} z = -\mathrm{sech} z\tanh z, \quad \frac{d}{dz}\mathrm{csch} z = -\mathrm{csch} z\coth z.$$

Relation To Sine and Cosine

- The real trigonometric and the real hyperbolic functions share many similar properties, e.g., $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\sinh x = \cosh x$.
- There is a simple connection between the complex trigonometric and hyperbolic functions: Replace z with iz in the definition of $\sinh z$:

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{2i} = i\sin z,$$

or $-i\sinh(iz) = \sin z$.

- Substituting iz for z in $\sin z$, we find $\sinh z = -i\sin(iz)$.
- After repeating this process for $\cos z$ and $\cosh z$, we obtain:

$$\begin{array}{lcl} \sin z = -i\sinh(iz) & \text{and} & \cos z = \cosh(iz), \\ \sinh z = -i\sin(iz) & \text{and} & \cosh z = \cos(iz). \end{array}$$

- Other relations can be similarly derived:

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = i\frac{\sinh z}{\cosh z} = i\tanh z.$$

Obtaining Hyperbolic Identities

- Some of the more commonly used hyperbolic identities:
  - $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$;
  - $\cosh^2 z - \sinh^2 z = 1$;
  - $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$;
  - $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$.
- Example: Verify that $\cosh(z_1 + z_1) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$, for all complex z_1 and z_2.

  We have $\cosh(z_1 + z_2) = \cos(iz_1 + iz_2)$. So by a trigonometric identity and additional applications of the preceding identities,

$$\begin{aligned}\cosh(z_1 + z_2) &= \cos(iz_1 + iz_2) \\ &= \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 \\ &= \cos iz_1 \cos iz_2 + (-i \sin iz_1)(-i \sin iz_2) \\ &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.\end{aligned}$$

Real versus Complex Trig and Hyperbolic Trig Functions

(i) In real analysis, the exponential function was just one of a number of apparently equally important elementary functions. In complex analysis, however, the complex exponential function assumes a much greater role: All of the complex elementary functions can be defined solely in terms of the complex exponential and logarithmic functions. The exponential and logarithmic functions can be used to evaluate, differentiate, integrate, and map using elementary functions.

(ii) As functions of a real variable x, $\sinh x$ and $\cosh x$ are not periodic. In contrast, the complex functions $\sinh z$ and $\cosh z$ are periodic. Moreover, $\cosh x$ has no zeros and $\sinh x$ has a single zero at $x = 0$. The complex functions $\sinh z$ and $\cosh z$, on the other hand, both have infinitely many zeros.