1. Solution by Separating Variables. Use of Fourier Series

We continue our work from Sec. 12.2, where we modeled a vibrating string and obtained the one-dimensional wave equation. We now have to complete the model by adding additional conditions and then solving the resulting model.

The model of a vibrating elastic string (a violin string, for instance) consists of the **one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho} \tag{1}$$

for the unknown deflection $u(x, t)$ of the string, a PDE that we have just obtained, and some ***additional conditions***, which we shall now derive.

Since the string is fastened at the ends $x = 0$ and $x = L$ (see Sec. 12.2), we have the two **boundary conditions**

$$\begin{aligned} &\text{(a)} \quad u(0, t) = 0, \\ &\text{(b)} \quad u(L, t) = 0, \\ &\text{for all } t \geq 0. \end{aligned} \tag{2}$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$ and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

$$\begin{aligned} &\text{(a)} \quad u(x, 0) = f(x), \\ &\text{(b)} \quad u_t(x, 0) = g(x) \\ &(0 \leq x \leq L) \end{aligned} \tag{3}$$

where $u_t = \partial u / \partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3). This will be the solution of our problem. We shall do this in three steps, as follows.

Step 1. By the **method of separating variables** or *product method*, setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3. Finally, using **Fourier series**, we compose the solutions found in Step 2 to obtain a solution of (1) satisfying both (2) and (3), that is, the solution of our model of the vibrating string.

1.1. Step 1. Two ODEs from the Wave Equation (1).

In the **method of separating variables**, or *product method*, we determine solutions of the wave equation (1) of the form

$$u(x, t) = F(x)G(x) \tag{4}$$

which are a product of two functions, each depending on only one of the variables x and t. This is a powerful general method that has various applications in engineering mathematics, as we shall see in this chapter. Differentiating (4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dots denote derivatives with respect to t and primes derivatives with respect to x. By inserting this into the wave equation (1) we have

$$F\ddot{G} = c^2 F'' G.$$

Dividing by $c^2 FG$ and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The variables are now separated, the left side depending only on t and the right side only on x. Hence both sides must be constant because, if they were variable, then changing t or x would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

Multiplying by the denominators gives immediately two ***ordinary*** DEs

$$F'' - kF = 0 \tag{5}$$

and

$$\ddot{G} - c^2 kG = 0. \tag{6}$$

Here, the **separation constant** k is still arbitrary.

1.2. **Step 2. Satisfying the Boundary Conditions (2).** We now determine solutions F and G of (5) and (6) so that $u = FG$ satisfies the boundary conditions (2), that is,

$$\begin{aligned} &u(0,t) = F(0)G(t) = 0, \\ &u(L,r) = F(L)G(t) = 0 \\ &\text{for all } t. \end{aligned} \tag{7}$$

We first solve (5). If $G = 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \neq 0$ and then by (7),

$$\text{(a)} \quad F(0) = 0, \qquad \text{(b)} \quad F(L) = 0. \tag{8}$$

We show that k must be negative. For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest. For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} = Be^{-\mu x}$$

and from (8) we obtain $F \equiv 0$ as before (verify!). Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then (5) becomes $F'' + p^2 F = 0$ and has as a general solution

$$F(x) = A \cos px + B \sin px.$$

From this and (8) we have

$$\begin{aligned} &F(0) = A = 0 \qquad \text{and then} \\ &F(L) = B \sin pL = 0 \end{aligned}$$

We must take $B \neq 0$ since otherwise $F \equiv 0$. Hence $\sin pL = 0$. Thus

$$\begin{aligned} pL &= n\pi, \qquad \text{so that} \\ p &= \frac{n\pi}{L} \qquad (n \text{ integer}). \end{aligned} \tag{9}$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots). \tag{10}$$

These solutions satisfy (8). [For negative integer n we obtain essentially the same solutions, except for a minus sign, because $\sin(-\alpha) = -\sin \alpha$.]

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cp = \frac{cn\pi}{L}. \tag{11*}$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^\star \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are $u_n(x,t) = F_n(x)G_n(t) = G_n(t)F_n(x)$, written out

$$\begin{aligned} u_n(x,t) = &(B_n \cos \lambda_n t + B_n^\star \sin \lambda_n t) \\ &\sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots). \end{aligned} \tag{11}$$

These functions are called the **eigenfunctions**, or *characteristic functions*, and the values $\lambda_n = cn\pi/L$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string. The set $\{\lambda_1, \lambda_n, \cdots\}$ is called the **spectrum**.

1.2.1. *Discussion of Eigenfunctions.* We see that each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* ($n = 1$), and the others are known as *overtones*; musically they give the octave, octave plus fifth, etc. Since in (11)

$$\sin\frac{n\pi x}{L} = 0 \qquad \text{at}$$
$$x = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n}L,$$

the nth normal mode has $n - 1$ **nodes**, that is, points of the string that do not move (in addition to the fixed endpoints); see Fig. 287.

Figure 288 shows the second normal mode for various values of t. At any instant the string has the form of a sine wave. When the left part of the string is moving down, the other half is moving up, and conversely. For the other modes the situation is similar.

Tuning is done by changing the tension T. Our formula for the frequency $\lambda_n/2\pi = cn/2L$ of u_n with $c = \sqrt{T/p}$ [see (3), Sec. 12.2] confirms that effect because it shows that the frequency is proportional to the tension. T cannot be increased indefinitely, but can you see what to do to get a string with a high fundamental mode? (Think of both L and ρ.) Why is a violin smaller than a double-bass?