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Individual Project 2

This report is split into two parts, one for the first two assigned problems, as they are very similar, and one for the last problem. The problems are 9, 11, and 22. For each of the problems I will consider the following methods: the formulas for rectangles, and Simpson's Rule. For both methods I will examine the magnitude of the error for certain numbers of nodes and parameters and compare the calculations with the graphs of the functions.

It should be noted that both variants for rectangles give the same results if the parameter theta is 0. This means that there is no reason to distinguish between them. Out of interest, I will still compare the error for a couple chosen values of theta, to see its influence on both variants. Trapezia can be created by using the second variant of rectangles with theta = 0.5, they will also be considered.

Magnitude of the Error for Problems 9 and 11

The following tables show the error of the approximation of the integrals for numbers of nodes from 3 to 129. The error is calculated according to the lab task. For the rectangles, two values have been chosen for theta, 0.25 and 0.75. The reason for this is that they fall in between the points 0 and 1, where both variants of rectangles have the same value, and 0.5, which creates trapezia for variant 2.

Problem 9

Number of Nodes	Numerical Error						
	Rectangle	Rectangle V1 (Theta)		Rectangle V2 (Theta)		Trapezia	Simpson's Rule
		0.25	0.75	0.25	0.75		
3	3.81E-01	1.74E-01	1.81E-01	2.04E-01	1.49E-01	2.77E-02	1.86E-04
5	1.84E-01	8.77E-02	8.94E-02	9.52E-02	8.16E-02	6.78E-03	1.06E-05
9	9.01E-02	4.40E-02	4.44E-02	4.59E-02	4.25E-02	1.69E-03	6.47E-07
17	4.46E-02	2.21E-02	2.22E-02	2.25E-02	2.17E-02	4.21E-04	4.03E-08
33	2.22E-02	1.10E-02	1.11E-02	1.12E-02	1.09E-02	1.05E-04	2.52E-09
65	1.11E-02	5.52E-03	5.53E-03	5.55E-03	5.50E-03	2.63E-05	1.57E-10
129	5.53E-03	2.76E-03	2.76E-03	2.77E-03	2.76E-03	6.58E-06	9.82E-12

Problem 11

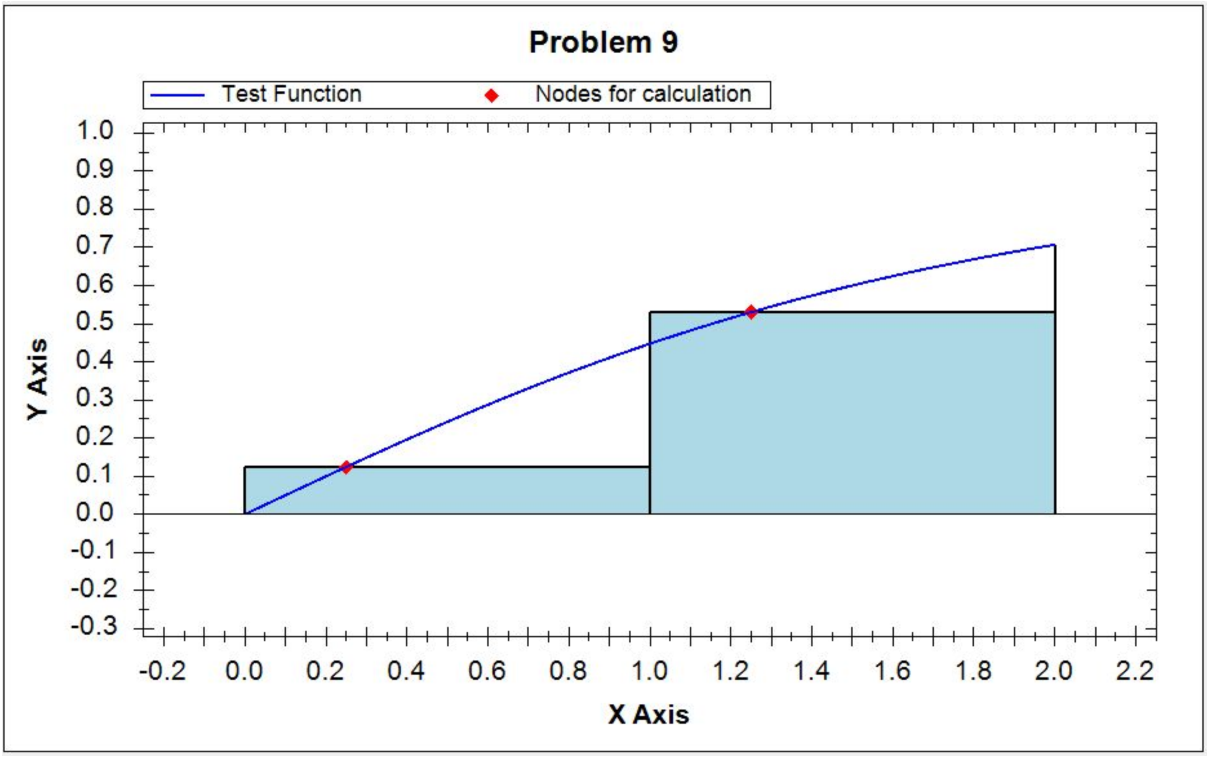
Number of Nodes	Numerical Error						
	Rectangle	Rectangle V1 (Theta)		Rectangle V2 (Theta)		Trapezia	Simpson's Rule
		0.25	0.75	0.25	0.75		
3	6.03E-01	2.62E-01	2.75E-01	3.28E-01	2.21E-01	5.34E-02	5.15E-04
5	2.88E-01	1.35E-01	1.38E-01	1.51E-01	1.24E-01	1.37E-02	3.85E-05
9	1.41E-01	6.81E-02	6.90E-02	7.21E-02	6.52E-02	3.46E-03	2.56E-06
17	6.95E-02	3.42E-02	3.44E-02	3.52E-02	3.35E-02	8.67E-04	1.62E-07
33	3.46E-02	1.71E-02	1.72E-02	1.74E-02	1.70E-02	2.17E-04	1.02E-08
65	1.72E-02	8.58E-03	8.59E-03	8.64E-03	8.53E-03	5.43E-05	6.38E-10
129	8.60E-03	4.29E-03	4.29E-03	4.31E-03	4.28E-03	1.36E-05	3.99E-11

These tables illustrate that rectangles V1 and V2, for these values of theta, have similar errors. They also seem to converge as the number of nodes increases indicating that, for high numbers of nodes, the value of theta and the variant are not as relevant for the error.

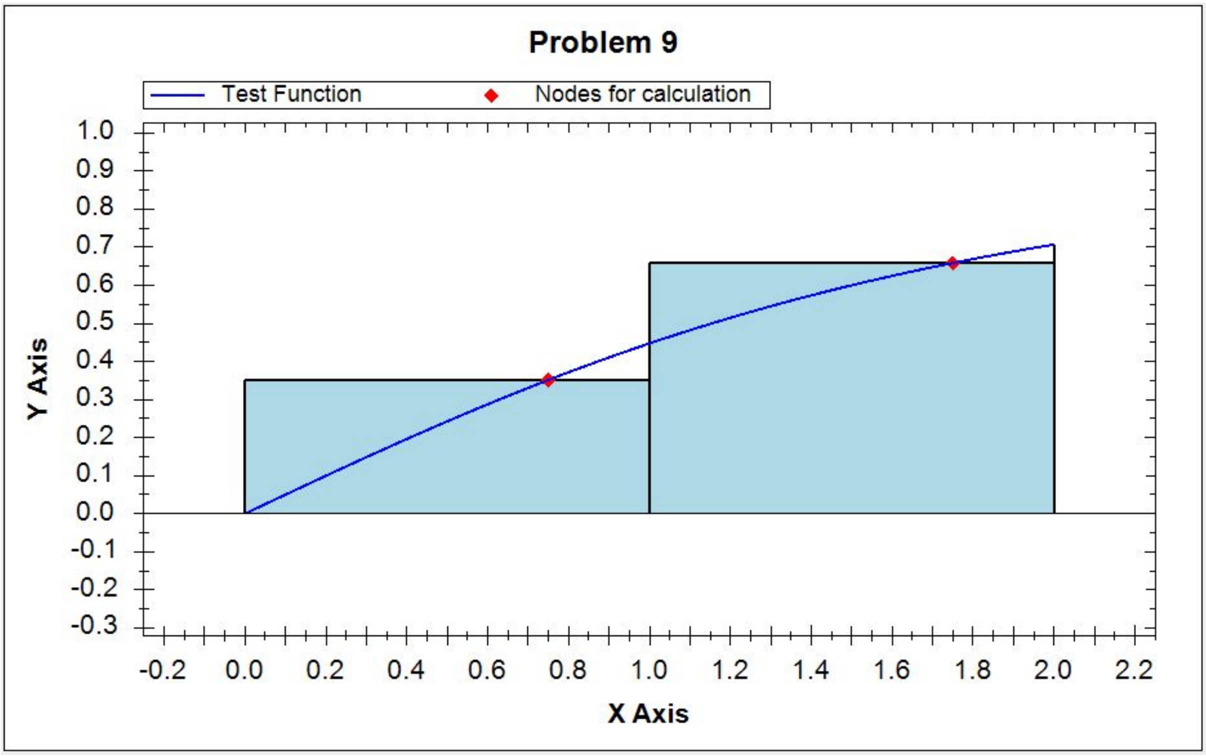
The normal rectangles without theta have a significantly larger error than the ones that are modified by theta. The error is about twice as large in both examples. This shows that modifying normal rectangles with theta will improve the accuracy of the approximation greatly.

For both problems, the rectangles of variant two are less accurate than the ones of variant one if theta is 0.25, but they are more accurate if theta is 0.75. This is caused by the fact that both problem 9 and 11 have slopes of at most 1 in the interval and the slopes decrease as x increases. For V1, theta moves the node's x value from the start of its interval towards the next one's x value. For V2, the same happens with the y values. Because the slopes are at most 1 and decreasing, most of the height change in an interval happens at the beginning of it. Thus, theta of the width of the interval corresponds to more height than theta of the height of the next node. This leads to the fact that the rectangles of V1 are always at least as big as the rectangles of V2 (for these two problems).

For theta equals 0.25, both approximations are smaller than the real value of the interval. But because the value of the rectangles of V1 is larger than that of V2, the error of V1 is smaller than that of V2. The following graph shows that for theta 0.25, the value of the approximation is smaller than the value of the integral.



For theta equals 0.75, both approximations are larger than the real value of the interval. The value of the rectangles of V1 is larger than that of V2, thus the error of V1 is larger than that of V2. The following graph shows that for theta 0.75, the value of the approximation is larger than the value of the integral.



The approximation using trapezia is far more accurate than the approximation using rectangles (except V2 with theta = 0.5, which gives trapezia). In the presented cases, even for small numbers of nodes the accuracy of trapezia is at least 4 times higher than the accuracy of rectangles.

By far the best approximation for both problems is Simpson's Rule. It's errors are 100 to 1,000,000 times smaller than then ones of trapezia, which is already more accurate than rectangles.

From this we can conclude that the best method for numerical integration is these two problems that was tested is Simpson's Rule. Even for small numbers of nodes it is very accurate. The next most accurate integration method is the method using trapezia. It is the most accurate method that uses simple geometric shapes and is also very accurate, even for low numbers of nodes. If one wants to use rectangles to approximate an integral, one should definitely use some form of theta with the method as well as relatively high numbers of nodes because this method is the most inaccurate of the ones tested and

Magnitude of the Error for Problem 22

Problem 22 is different from the first two problems in multiple ways. First of all, we do not have the analytical result of the integral to compare our calculations to. Thus we need to calculate the integral using different methods and try to find its value that way. Additionally, problem 22 has a very different curve than the first two, as its slope sharply increases as x approaches 1.

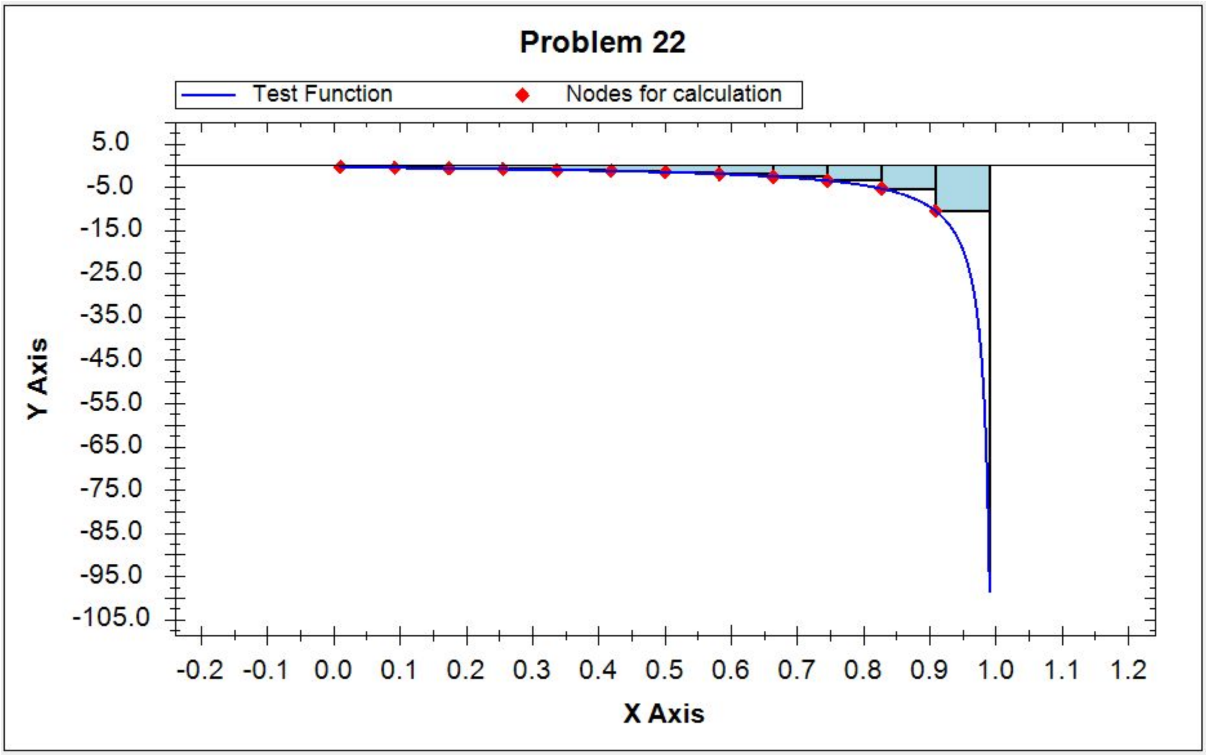
The below table shows the calculated value of the integral in problem 22 for different numbers of nodes and the different methods used in this project.

Problem 22

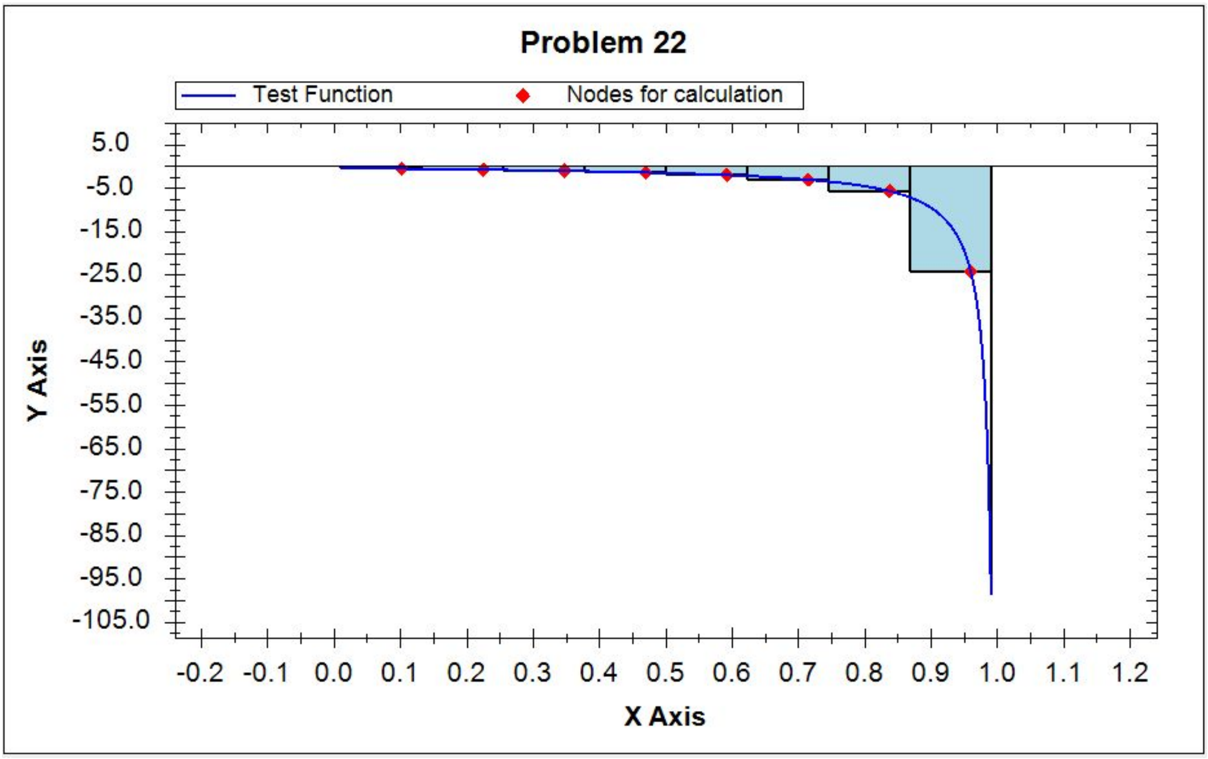
Number of Nodes	Approximate Value of Integral						
	Rectangle	Rectangle V1 (Theta)		Rectangle V2 (Theta)		Trapezia	Simpson's Rule
		0.25	0.75	0.25	0.75		
3	-0.81	-1.28	-3.95	-12.98	-37.30	-25.14	-9.73
5	-1.42	-1.87	-4.41	-7.50	-19.66	-13.58	-6.27
9	-2.02	-2.44	-4.47	-5.06	-11.14	-8.10	-4.79
17	-2.58	-2.96	-4.68	-4.10	-7.14	-5.62	-4.24
33	-3.06	-3.38	-4.54	-3.82	-5.34	-4.58	-4.07
65	-3.44	-3.67	-4.35	-3.82	-4.58	-4.20	-4.04
129	-3.70	-3.84	-4.21	-3.89	-4.27	-4.08	-4.03

One interesting occurence is this case is that for rectangles of V2, Trapezia, and Simpson's Rule, the values start large and negative and then become smaller as the all approach a value of about -4. The normal rectangles and rectangles of V1 with theta 0.25 start small and negative and approach -4 from the other direction. Rectangles of V1 for theta 0.75 start small, increase in magnitude and then decrease again as they approach -4.

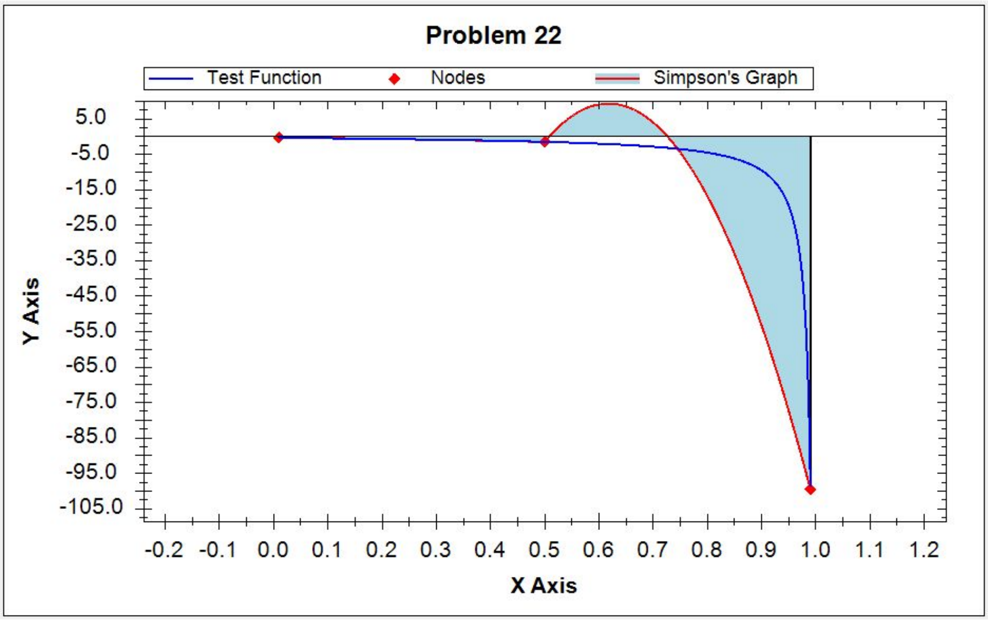
The behavior of the normal rectangles can be explained by showing the following graph. Because of the shape of this graph, the normal rectangle's sum is always smaller than the value of the integral, but as the number of rectangles increases, this error becomes smaller and the value approaches -4. For rectangles of V1 with theta 0.25 the behavior is very similar, theta of 0.25 means a shift of 0.25 of the interval towards the next interval and this shift is too small to make a difference is these measurements. Below see normal rectangles.



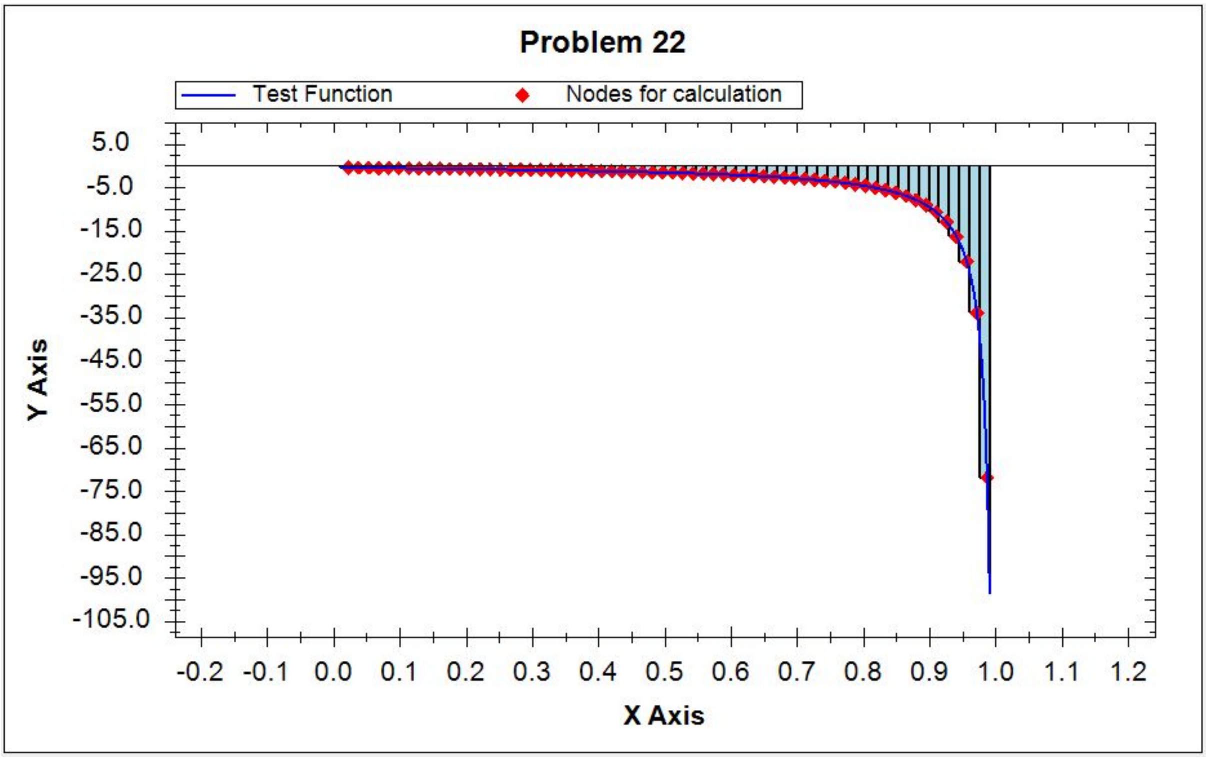
The behavior of the rectangles of variant one with theta 0.75 can be explained by the fact that the slope is >1 for the last part of the function, which also makes up most of the value of it. Theta of 0.75 means that the node is shifted by 0.75 of the interval length towards the next interval. From a certain amount of nodes on, this results in a larger computed value than the actual value of the integral, As the number of nodes increases, the last node gets shifted more and more towards the upper limit of the integral, where the values are the largest. This causes the increase in magnitude. This process continues when more nodes are used, but the intervals get smaller, which means that the last nodes matter less and the approximation starts to converge again. For an example, see the graphics below. The first one shows nine nodes and one can see the rectangles are larger than the area under the function. The second graph shows the graph for 65 nodes, where it starts to approach -4 again.



The other methods of integration all start at a value larger than the actual value of the integral and then start to converge towards it as the number of nodes increases. This is the result of the accuracy of the approximation increasing. The values get so large because the slope towards the end is so steep. This causes all three types of approximations to come up with larger values than the functions actually have. The following graph shows Simpson's Rule with three nodes for problem 22.



As described, the area is larger that the actual area under the curve. This is the case in all three approximations. As the number of nodes increase, this error becomes smaller.



Still, only the method of trapezia and Simpson's Rule approach a value that is close to the actual value of the integral, and even they need 129 nodes to get close to it. To conclude, to find an accurate value for this integral, one needs to use the method of trapezia or Simpson's Rule to achieve satisfying results. All other methods need really high numbers of nodes, to get even close to the actual result.