Basic concepts from probability

Learning Objectives

- sample space
- probability space
- sample point
- null subset
- impossible event
- complement
- mutually exclusive events
- informal definition of probability
- classical definition of probability
- frequency definition of probability
- method of computing probability

Probability

*How **likely** something is to happen.*

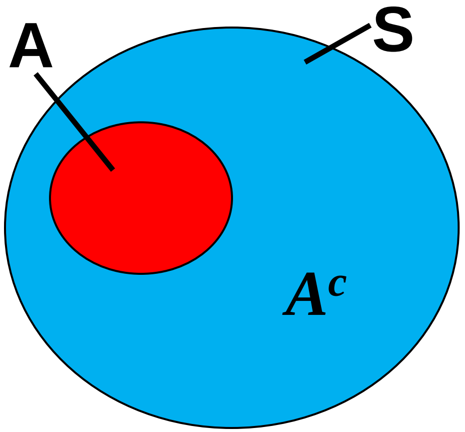
Many events can't be predicted with total certainty. The best we can say is how **likely** they are to happen, using the idea of probability.

Tossing a Coin

When a coin is tossed, there are two possible outcomes: heads (H) or tails (T)

We say that the probability of the coin landing H is ½ or 50%.
And the probability of the coin landing T is ½ or 50%.

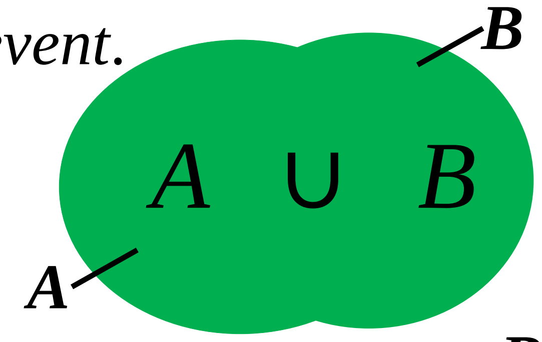
BASIC DEFINITIONS



1. The *sample space* associated with an experiment is the set consisting of all possible outcomes and is called the sure event in the experiment. A sample space is also referred to as a *probability space*. A sample space will be denoted by S.

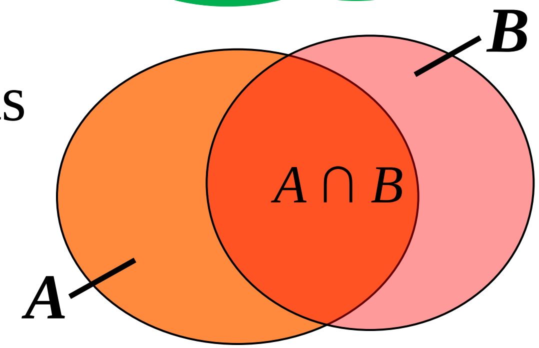
2. An outcome in S is also called a *sample point*. An event A is a subset of outcomes in S, that is, $A \subset S$. We say that an event A occurs if the outcome of the experiment is in A.

3. The *null subset* $\varnothing$ of S is called an *impossible event*.



4. The event $A \cup B$ consists of all outcomes that are in A or in B or in both.

5. The event $A \cap B$ consists of all outcomes that are both in A and B.



6. The event A^c (the *complement* of A in S) consists of all outcomes not in A, but in S.

Probability: Complement

Complement of an Event: All outcomes that are **NOT** the event.

When the event is **Heads**, the complement is **Tails**

When the event is **{Monday, Wednesday, Friday}** the complement is **{Tuesday, Thursday, Saturday, Sunday}**

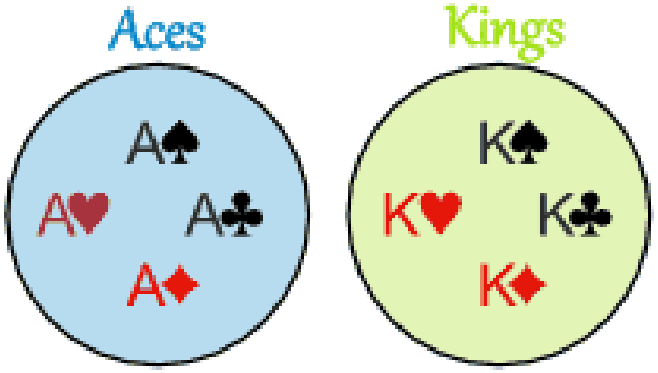
When the event is **{Hearts}** the complement is **{Spades, Clubs, Diamonds, Jokers}**

So the Complement of an event is all the **other** outcomes (**not** the ones you want).

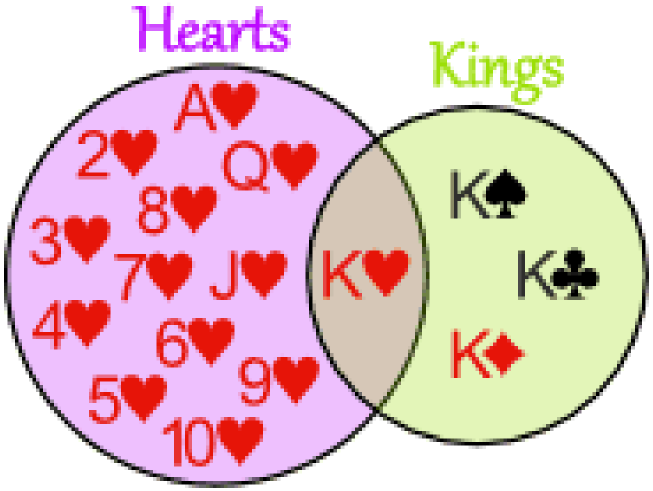
And together the Event and its Complement make all possible outcomes.

Definition 1 *Two events A and B are said to be* **mutually exclusive** *or* **disjoint** *if* $A \cap B = \varnothing$. *Mutually exclusive events cannot happen together.*

Examples: Turning left and turning right are Mutually Exclusive (you can't do both at the same time)
Tossing a coin: Heads and Tails are Mutually Exclusive
Cards: Kings and Aces are Mutually Exclusive



Aces and Kings are Mutually Exclusive (can't be both)



Hearts and Kings are not Mutually Exclusive (can be both)

INFORMAL DEFINITION OF PROBABILITY

Definition 2 The **probability** of an event is a measure (number) of the chance with which we can expect the event to occur. We assign a number between 0 and 1 inclusive to the probability of an event. A probability of 1 means that we are 100% sure of the occurrence of an event, and a probability of 0 means that we are 100% sure of the nonoccurrence of the event. The probability of any event A in the sample space *S* is denoted by ***P(A)*.**

CLASSICAL DEFINITION OF PROBABILITY

Definition 3 If there are ***n*** equally likely possibilities, of which one must occur, and ***m*** of these are regarded as favorable to an event, or as "success," then the **probability** of the event or a "success" is given by ***m/n***.

METHOD OF COMPUTING PROBABILITY BY THE CLASSICAL APPROACH

A. When all outcomes are equally likely

1. Count the number of outcomes in the sample space; say this is $\boldsymbol{n}$.
2. Count the number of outcomes in the event of interest, A, and say this is $\boldsymbol{m}$.
3. $\boldsymbol{P(A) = m/n}$.

B. When all outcomes are not equally likely

1. Let $O_1, O_2, \ldots, O_n$ be the outcomes of the sample space S. Let $P(O_i) = p_i$, $i = 1, 2, \ldots, n$. In this case, the probability of each outcome, p_i, is assumed to be known.
2. List all the outcomes in A, say, $O_i, O_j, \ldots, O_m$.
3. $P(A) = P(O_i) + P(O_j) + \cdots + P(O_m) = p_i + p_j + \cdots + p_m$, the sum of the probabilities of the outcomes in A.

Throwing Dice

When a single die is thrown, there are six possible outcomes: **1, 2, 3, 4, 5, 6**.
The probability of any one of them is 1/6 .

Example 1. The chances of rolling a "4" with a die

Number of ways it can happen: *m=1* (there is only 1 face with a "4" on it)

Total number of outcomes: *n=6* (there are 6 faces altogether)

$$P(A)=\frac{1}{6}$$

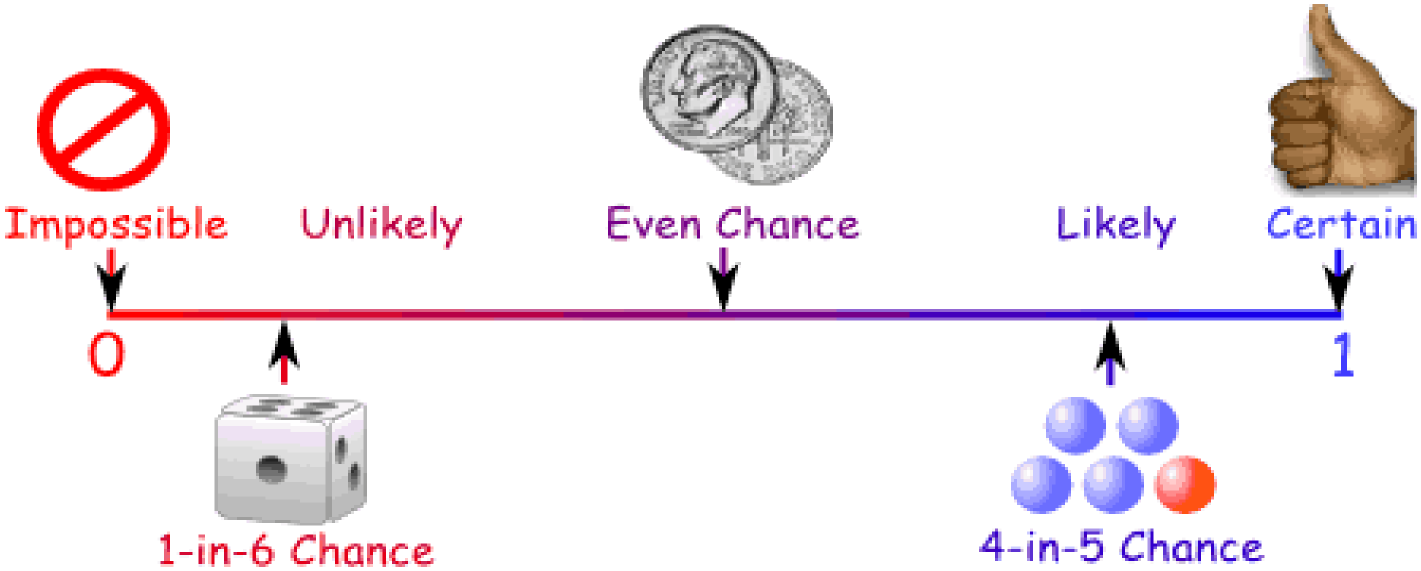
Example 2. There are 5 marbles in a bag: 4 are blue, and 1 is red. What is the probability that a blue marble will be picked?

Number of ways it can happen: *m=4* (there are 4 blues)

Total number of outcomes: *n=5* (there are 5 marbles in total)

$$P(A)=\frac{4}{5}=0.8=80\%$$

Probability Line



Probability is always between 0 and 1

Probability is Just a Guide

Probability does not tell us exactly what will happen, it is just a guide!
Example 3. Toss a coin 100 times, how many Heads will come up?
Probability says that heads have a ½ chance, so we would **expect 50 Heads**. But when you actually try it out you might get 48 heads, or 55 heads ... or anything really, but in most cases it will be a number near 50.

Buffon's coin experiment is a very old and famous random experiment, named after **Compte De Buffon**.

The French naturalist Georges-Louis Leclerc, Comte de Buffon (1707 - 1788) tossed a coin 4040 times. The result was 2048 heads or **2048/4040 = 0.5069** for heads. Suppose, like Buffon, you would like to flip a coin 4040 times.

FREQUENCY DEFINITION OF PROBABILITY

Definition 4 The probability of an outcome (event) is the proportion of times the outcome (event) would occur in a long run of repeated experiments.

Example 4. Alex decide to see how many times a "double" would come up when throwing 2 dice. Each time Alex throws the 2 dice is an **Experiment.** It is an Experiment because the result is uncertain.

The **Event** Alex is looking for is a "double", where both dice have the same number. It is made up of these **6 Sample Points**:

{1,1} {2,2} {3,3} {4,4} {5,5} and {6,6}

The **Sample Space** is all possible outcomes **(36 Sample Points):**

{1,1} {1,2} {1,3} {1,4} ... {6,3} {6,4} {6,5} {6,6}

These are Alex's Results:

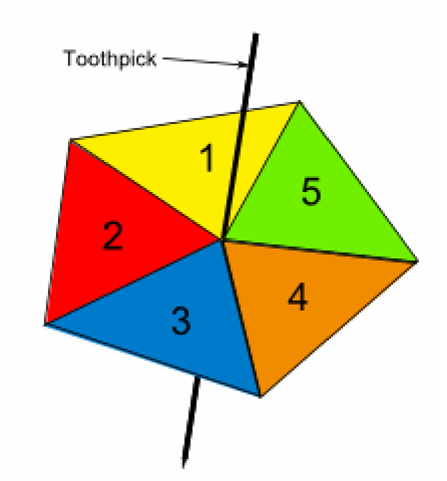
Experiment	Is it a Double?
{3,4}	No
{5,1}	No
{2,2}	**Yes**
{6,3}	No
...	...

After 100 **Experiments**, Alex had 19 "double" **Events** ... is that close to what you would expect?

Questions

1. A die is thrown once. What is the probability that the score is a factor of 6?

2. The diagram shows a spinner made up of a piece of card in the shape of a regular pentagon, with a toothpick pushed through its center. The five triangles are numbered from 1 to 5.
The spinner is spun until it lands on one of the five edges of the pentagon. What is the probability that the number it lands on is odd?



3. Each of the letters of the word MISSISSIPPI are written on separate pieces of paper that are then folded, put in a hat, and mixed thoroughly. One piece of paper is chosen (without looking) from the hat. What is the probability it is an I?

Answers

1. Number of ways it can happen: *m=4* (the factor of 6 are 1,2,3 and 6)

Total number of outcomes: *n=6* (there are 6 faces altogether)

$$P(A) = \frac{4}{6} = \frac{2}{3}$$

2. Number of ways it can happen: *m=3* (the are 3 odd number 1,3 and 5)

Total number of outcomes: *n=5* (there are 5 numbers altogether)

$$P(A) = \frac{3}{5}$$

3. Number of ways it can happen: *m=4* (the are 4 I's in word MISSISSIPPI)

Total number of outcomes: *n=11* (there are 11 letters altogether)

$$P(A) = \frac{4}{11}$$

Questions

4. A card is chosen at random from a deck of 52 playing cards. There are 4 Queens and 4 Kings in a deck of playing cards. What is the probability it is a Queen or a King?

5. A fair coin is tossed three times. What is the probability of obtaining one Head and two Tails? (A fair coin is one that is not loaded, so there is an equal chance of it landing Heads up or Tails up.)

6. A committee of three is chosen from five councilors - Adams, Burke, Cobb, Dilby and Evans. What is the probability Burke is on the committee?

7. There are 10 counters in a bag: 3 are red, 2 are blue and 5 are green. The contents of the bag are shaken before Maxine randomly chooses one counter from the bag. What is the probability that she **doesn't** pick a red counter?

Answers

4. Number of ways it can happen: *m=8* (4 Queens and 4 Kings)
Total number of outcomes: *n=52* (there are 52 cards altogether)

$$P\left(A\right)=\frac{8}{52}=\frac{2}{13}$$

5. Represent 'Heads up' by H and 'Tails up' by T.
There are 8 possible ways the coins can land: (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H) and (T, T, T). Of these, 3 have one Head and two Tails: (H, T, T), (T, H, T) and (T, T, H)
Number of ways it can happen: *m=3*
Total number of outcomes: *n=8*

$$P\left(A\right)=\frac{3}{8}$$

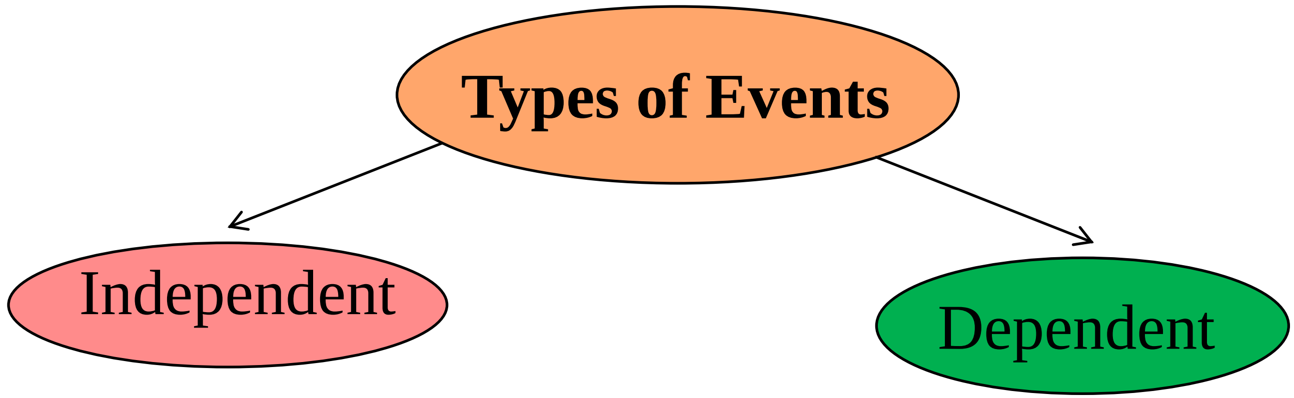
Answers

6. Abbreviate the names of the five councilors with the letters A, B, C, D and E. There are 10 possible committees: (A, B, C), (A, B, D), (A, B, E), (A, C, D), (A, C, E), (A, D, E), (B, C, D), (B, C, E), (B, D, E) and (C, D, E). Of these, Burke is included in 6: (A, B, C), (A, B, D), (A, B, E), (B, C, D), (B, C, E) and (B, D, E)
Number of ways it can happen: $m=6$
Total number of outcomes: $n=10$

$$P(A)=\frac{6}{10}=\frac{3}{5}$$

7. There are 7 counters that are not red: 2 blue and 5 green
Number of ways it can happen: $m=7$
Total number of outcomes: $n=10$

$$P(A)=\frac{7}{10}$$



Events

When we say "Event" we mean one (or more) outcomes.

Example Events:

Getting a Tail when tossing a coin is an event

Rolling a "5" is an event.

An event can include several outcomes:

Choosing a "King" from a deck of cards (any of the 4 Kings) **is also** an event

Rolling an "even number" (2, 4 or 6) is an event

Independent Events

Events can be "Independent", meaning each event is **not affected** by any other events.
This is an important idea! A coin does not "know" that it came up heads before ... each toss of a coin is a perfect isolated thing.

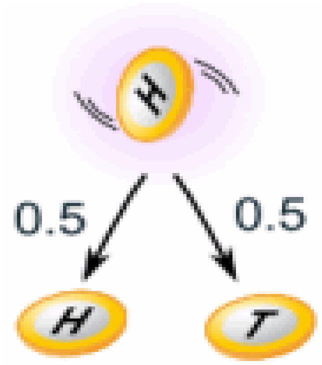
Example: You toss a coin three times and it comes up "Heads" each time ... what is the chance that the next toss will also be a "Head"? The chance is simply 1/2, or 50%, just like ANY OTHER toss of the coin. What it did in the past will not affect the current toss!

Some people think "it is overdue for a Tail", but *really truly* the next toss of the coin is totally independent of any previous tosses. Saying "a Tail is due", or "just one more go, my luck is due" is called **The Gambler's Fallacy**

Independent Events

Independent Events are **not affected** by previous events.
This is an important idea!
A coin does not "know" it came up heads before ...
.... each toss of a coin is a perfect isolated thing.

Example: You toss a coin and it comes up "Heads" three times
... what is the chance that the next toss will also be a "Head"?
The chance is simply ½ (or 0,5) just like ANY toss of the coin.
What it did in the past will not affect the current toss!



Some people think "it is overdue for a Tail", but *really truly* the next toss of the coin is totally independent of any previous tosses.
Saying "a Tail is due", or "just one more go, my luck is due" is called The Gambler's Fallacy. Of course your luck may change, because each toss of the coin has an equal chance.

Independent Events

So how do we calculate probability?

$$P(A)=\frac{m}{n}$$

m - Number of ways it can happen
n - Total number of outcomes

Example: what is the probability of getting a "Head" when tossing a coin?
Number of ways it can happen: 1 (Head)
Total number of outcomes: 2 (Head and Tail)

$$P(A)=\frac{1}{2}$$

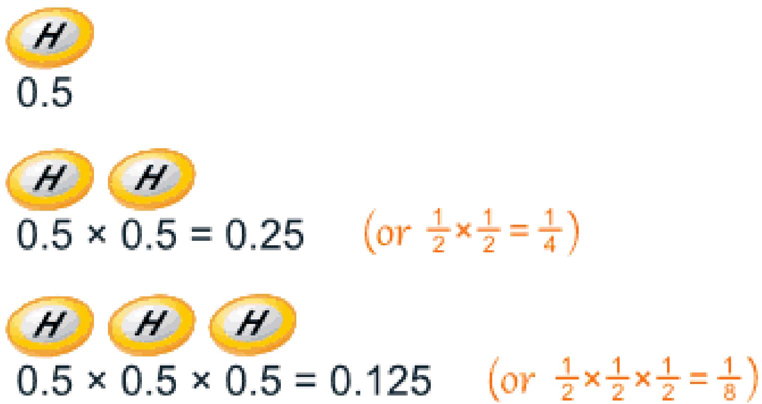
Example: what is the probability of getting a "5" or "6" when rolling a die?
Number of ways it can happen: 2 ("5" and "6")
Total number of outcomes: 6 ("1", "2", "3", "4", "5" and "6")

$$P(A)=\frac{2}{6}=\frac{1}{3}$$

Two or More Events

You can calculate the chances of two or more independent events by multiplying the chances.

Example: Probability of 3 Heads in a Row. For each toss of a coin a "Head" has a probability of 0,5:



And so the chance of getting 3 Heads in a row is 0,125.

So each toss of a coin has a ½ chance of being Heads, but lots of Heads in a row is unlikely.

Example: Why is it unlikely to get, say, 7 heads in a row, when *each* toss of a coin has a ½ chance of being Heads?

Because you are asking two different questions:

Question 1: What is the probability of 7 heads in a row?

Answer: ½×½×½×½×½×½×½ = 0,0078125 (less than 1%).

Question 2: Given that you have just got 6 heads in a row, what is the probability that the next toss is also a head?

Answer: ½, as the previous tosses don't affect the next toss.

So, for Independent Events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Probability of A and B equals the probability of A times the probability of B.

Example: you are going to a concert, and your friend says it is some time on the weekend between 4 and 12, but won't say more. What are the chances it is on Sunday between 10 and 12?

Day: there are two days on the weekend, so ***P(Sunday) = 0,5***

Time: between 4 and 12 is 8 hours, but you want between 10 and 12 which is only 2 hours:

$$P(\text{Your Time}) = 2/8 = 0{,}25$$

And:

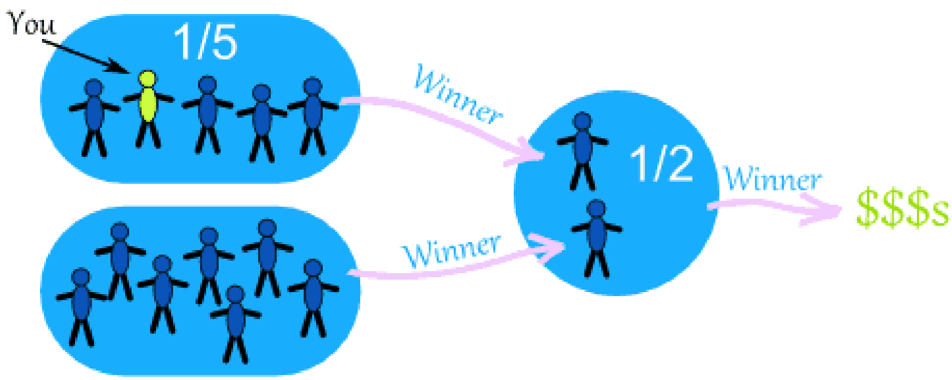
$$\mathbf{P(Sunday\ and\ Your\ Time)} = P(\text{Sunday}) \times P(\text{Your Time}) =$$

$$= 0{,}5 \times 0{,}25 = \mathbf{0{,}125}$$

Or a 12,5% chance

Example: Imagine there are two groups:

- A member of each group gets randomly chosen for the winners circle,
- **then** one of those gets randomly chosen to get the big money prize:



What is your chance of winning the big prize?
there is a **1/5 chance** of going to the winners circle and a **1/2 chance** of winning the big prize

So you have a 1/5 chance followed by a 1/2 chance ... which makes a 1/10 chance overall:

$$P(A)=\frac{1}{5}\times\frac{1}{2}=\frac{1}{10}$$

Or you can calculate using decimals (1/5 is 0,2, and 1/2 is 0,5):

$$\mathit{0{,}2} \text{ x } \mathit{0{,}5} = \boldsymbol{\mathit{0{,}1}}$$

So your chance of winning the big money is **0,1** (which is the same as 1/10).

Dependent Events

But some events can be "dependent" ... which means they **can be affected by previous events** ...

Example: Drawing 2 Cards from a Deck

After taking one card from the deck there are **less cards** available, so the probabilities change!

Let's say you are interested in the chances of getting a King.

For the 1st card the chance of drawing a King is 4 out of 52

But for the 2nd card:

If the 1st card was a King, then the 2nd card is **less** likely to be a King, as only 3 of the 51 cards left are Kings.

If the 1st card was **not** a King, then the 2nd card is slightly **more** likely to be a King, as 4 of the 51 cards left are King.

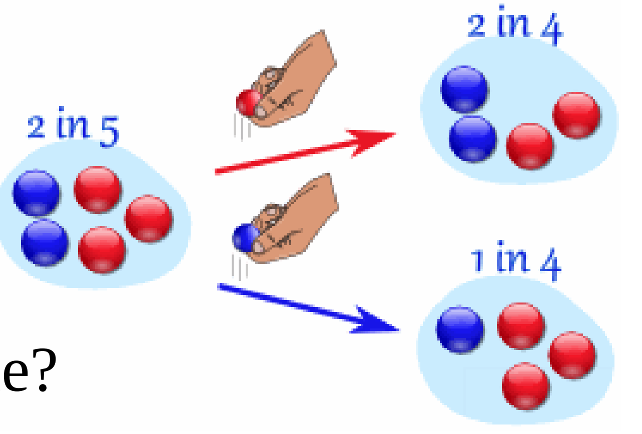
This is because you are **removing cards** from the deck.

Replacement: When you put each card **back** after drawing it the chances don't change, as the events are independent.

Without Replacement: The chances will change, and the events are **dependent**.

Dependent Events

But events can also be "dependent" ... which means they **can be affected by previous events** ...



Example: Marbles in a Bag
2 blue and 3 red marbles are in a bag.

What are the chances of getting a blue marble?
The chance is **2 in 5**
But after taking one out you change the chances!
So the next time:

- if you got a **red** marble before, then the chance of a blue marble next is **2 in 4**
- if you got a **blue** marble before, then the chance of a blue marble next is **1 in 4**

See how the chances change each time? Each event **depends on** what happened in the previous event, and is called **dependent**.

Tree Diagrams

Example: Soccer Game

You are off to soccer, and love being the Goalkeeper, but that depends who is the Coach today:

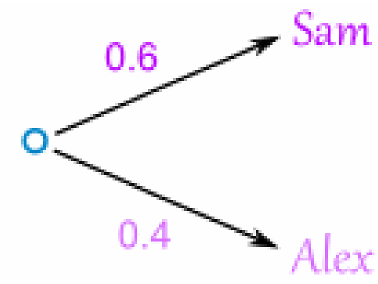
with Coach Sam your probability of being Goalkeeper is **0.5**

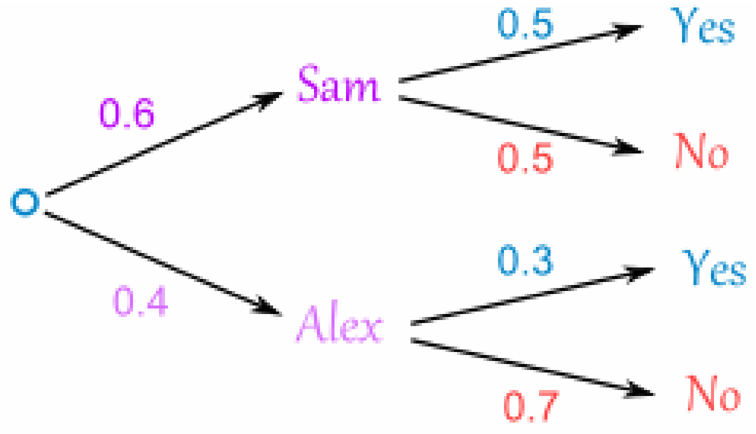
with Coach Alex your probability of being Goalkeeper is **0.3**

Sam is Coach more often ... about 6 of every 10 games (a probability of **0.6**).

Let's build the Tree Diagram!

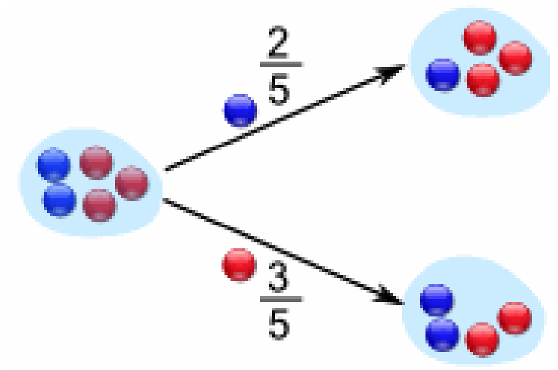
Start with the Coaches. We know 0.6 for Sam, so it must be 0.4 for Alex (the probabilities must add to 1):





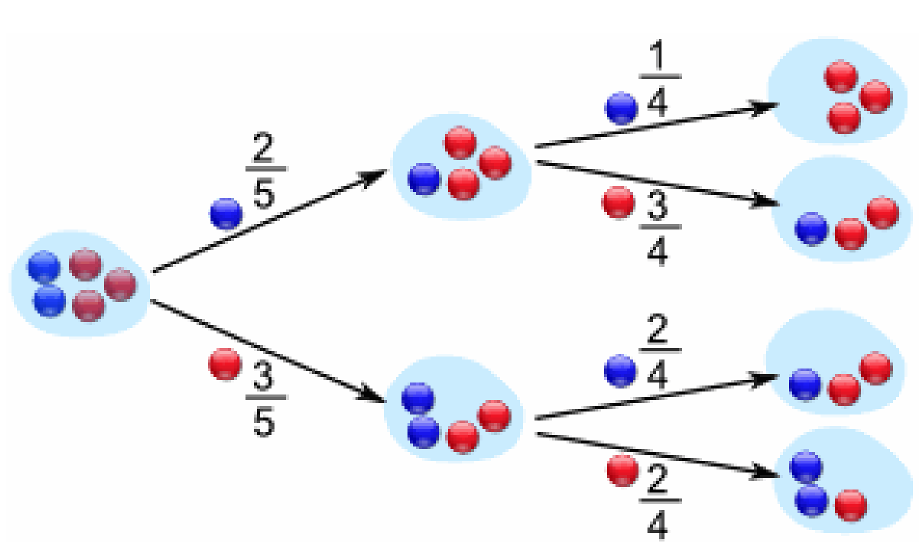
Then fill out the branches for Sam (0.5 Yes and 0.5 No), and then for Alex (0.3 Yes and 0.7 No):

Example: Marbles in a Bag 2 blue and 3 red marbles are in a bag. What are the chances of getting a blue marble?



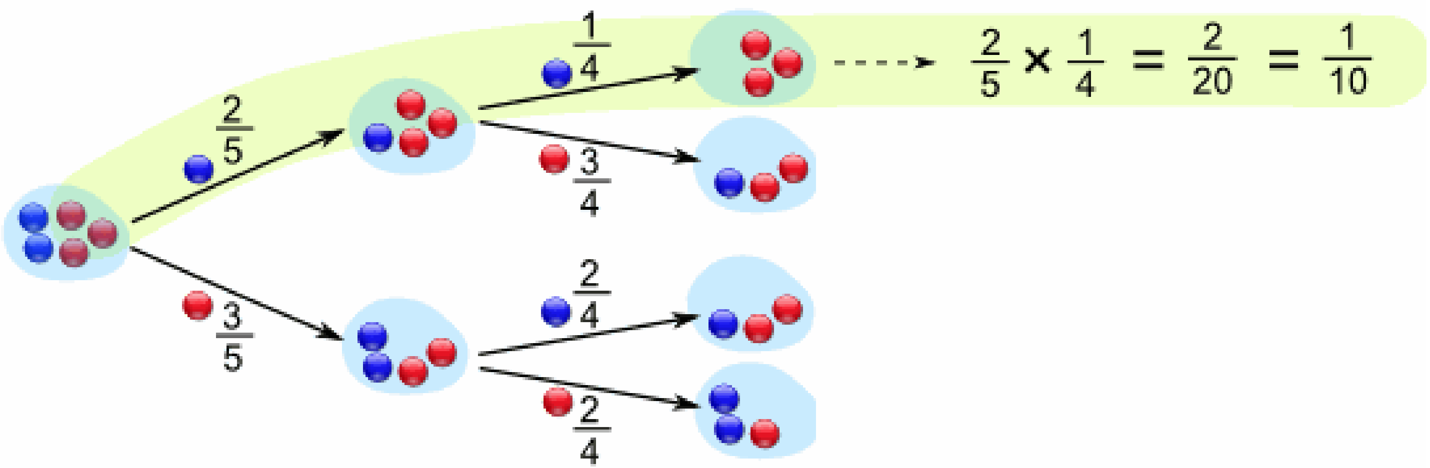
There is a 2/5 chance of pulling out a Blue marble, and a 3/5 chance for Red.

We can even go one step further and see what happens when we select a second marble:



If a blue marble was selected first there is now a 1/4 chance of getting a blue marble and a 3/4 chance of getting a red marble.
If a red marble was selected first there is now a 2/4 chance of getting a blue marble and a 2/4 chance of getting a red marble.

Now we can answer questions like **"What are the chances of drawing 2 blue marbles?"**
Answer: it is a **2/5 chance** followed by a **1/4 chance**:



Did you see how we multiplied the chances? And got 1/10 as a result.
The chances of drawing 2 blue marbles is 1/10

$P(A)$ means "Probability Of Event A"

In our marbles example Event A is "get a Blue Marble first" with a probability of 2/5:

$$P(A) = 2/5$$

And Event B is "get a Blue Marble second" ... but for that we have 2 choices:

- If we got a **Blue Marble first** the chance is now **1/4**
- If we got a **Red Marble first** the chance is now **2/4**

So we have to say **which one we want**, and use the symbol "|" to mean "given":

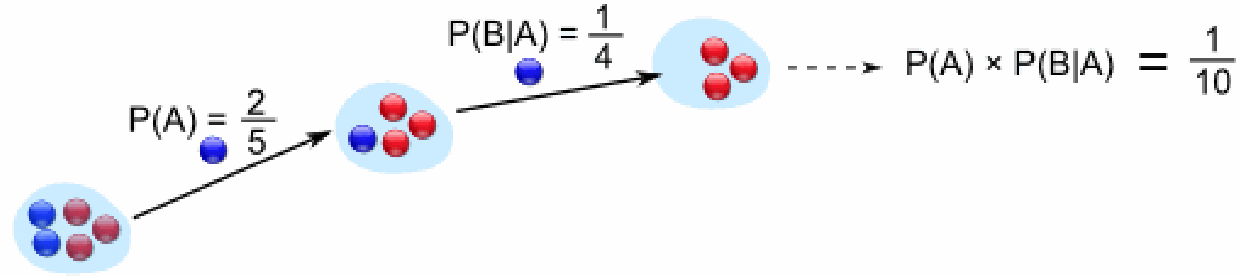
$P(B|A)$ means "Event B given Event A"

In other words, event *A* has already happened, now what is the chance of event *B*?

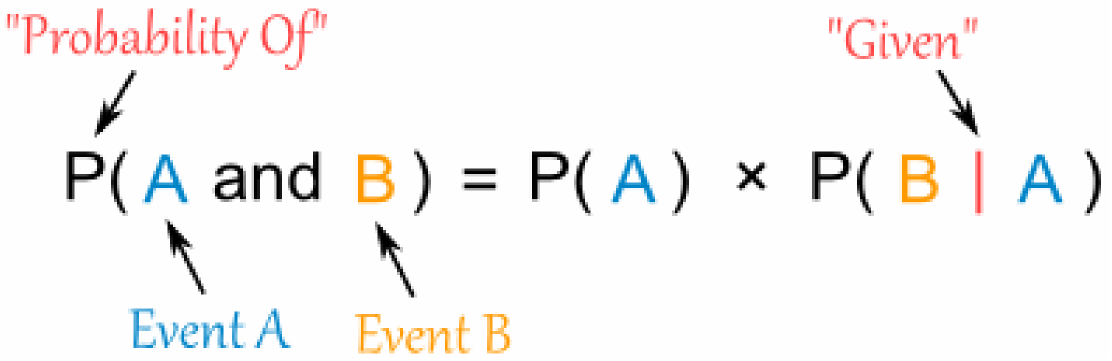
P(B|A) is also called the "**Conditional Probability**" of B given A.
And in our case:

P(B|A) = 1/4

So the probability of getting **2 blue marbles** is:



And we write it as



*"Probability of **event A and event B** equals the probability of **event A** times the probability of **event B given event A**"*

Example: Drawing 2 Kings from a Deck.

Event A is drawing a King first, and Event B is drawing a King second.

For the first card the chance of drawing a King is 4 out of 52

$$P(A) = 4/52.$$

But after removing a King from the deck the probability of the 2nd card drawn is **less** likely to be a King (only 3 of the 51 cards left are Kings):

$$P(B|A) = 3/51.$$

And so:

$$P(A \text{ and } B) = P(A) \times P(B|A) = (4/52) \times (3/51) =$$
$$= 12/2652 = \mathbf{1/221}.$$

So the chance of getting 2 Kings is 1 in 221, or about 0.5%

Using Algebra we can also "change the subject" of the formula, like this:

Start with:	*P(A and B) = P(A) x P(B|A)*
Swap sides:	*P(A) x P(B|A) = P(A and B)*
Divide by P(A):	*P(B|A) = P(A and B) / P(A)*

And we have another useful formula:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

The probability of ***event B given event A*** *equals the probability of* ***event A and event B*** *divided by the probability of* ***event A***

Example: Ice Cream

70% of your friends like Chocolate, and 35% like Chocolate AND like Strawberry.
What percent of those who like Chocolate also like Strawberry?

P(Strawberry|Chocolate) = P(Chocolate and Strawberry) / P(Chocolate)

0.35 / 0.7 = 50%

50% of your friends who like Chocolate also like Strawberry

Example: Soccer Game

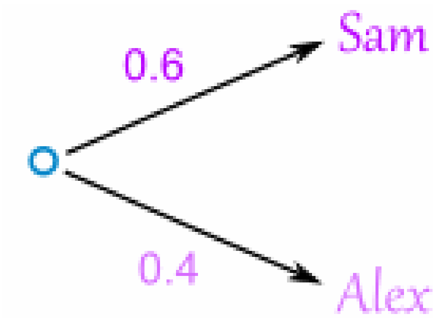
You are off to soccer, and want to be the Goalkeeper, but that depends who is the Coach today:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**

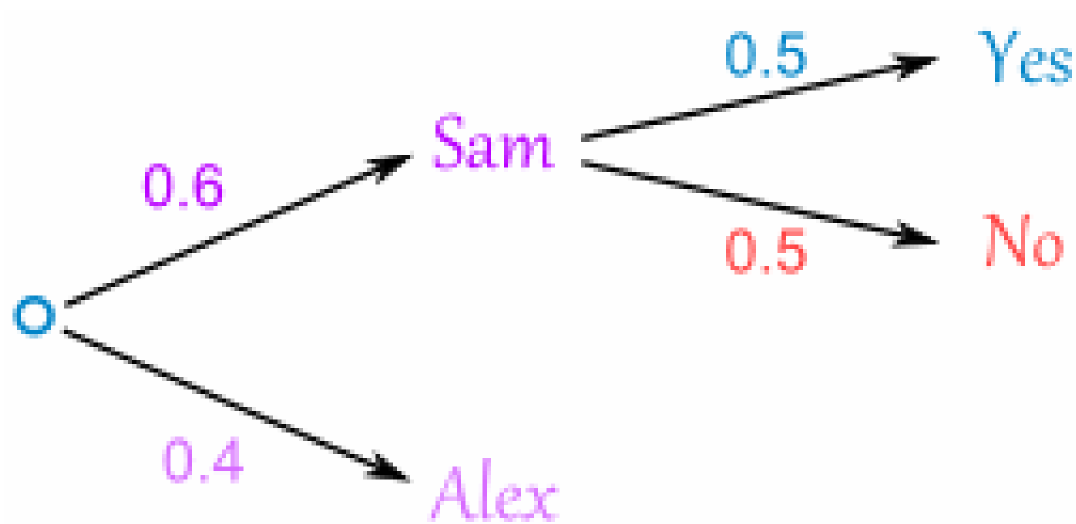
Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).

So, what is the probability you will be a Goalkeeper today?

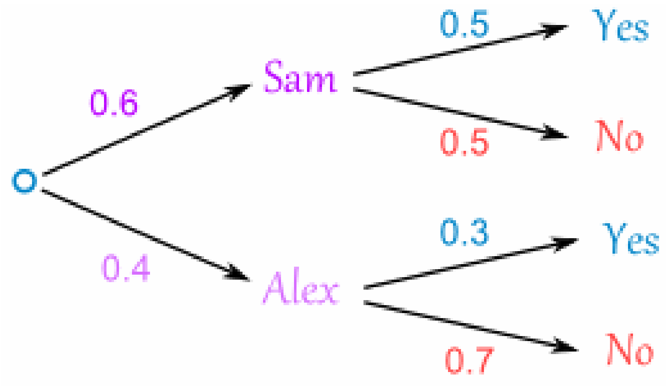
Let's build a tree diagram. First we show the two possible coaches: Sam or Alex:



The probability of getting Sam is 0.6, so the probability of Alex must be 0.4 (together the probability is 1)
Now, if you get Sam, there is 0.5 probability of being Goalie (and 0.5 of not being Goalie):



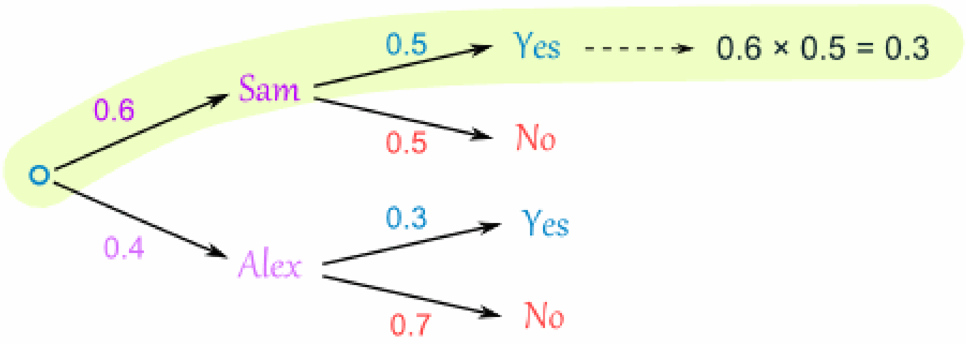
If you get Alex, there is 0.3 probability of being Goalie (and 0.7 not):



The tree diagram is complete, now let's calculate the overall probabilities. Remember that:

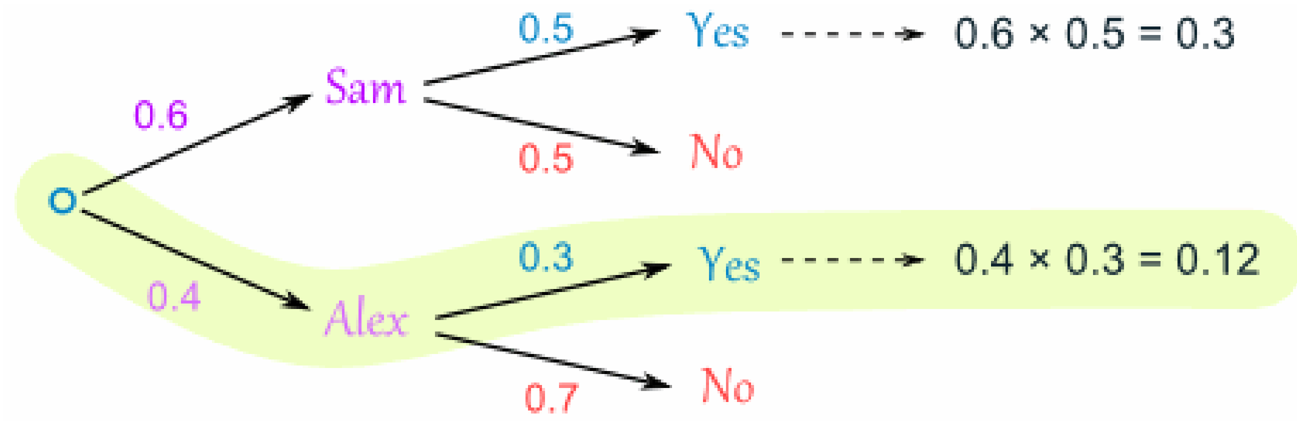
$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Here is how to do it for the "Sam, Yes" branch:



(When we take the 0.6 chance of Sam being coach and include the 0.5 chance that Sam will let you be Goalkeeper we end up with an 0.3 chance.)

But we are not done yet! We haven't included Alex as Coach:



An 0.4 chance of Alex as Coach, followed by an 0.3 chance gives 0.12

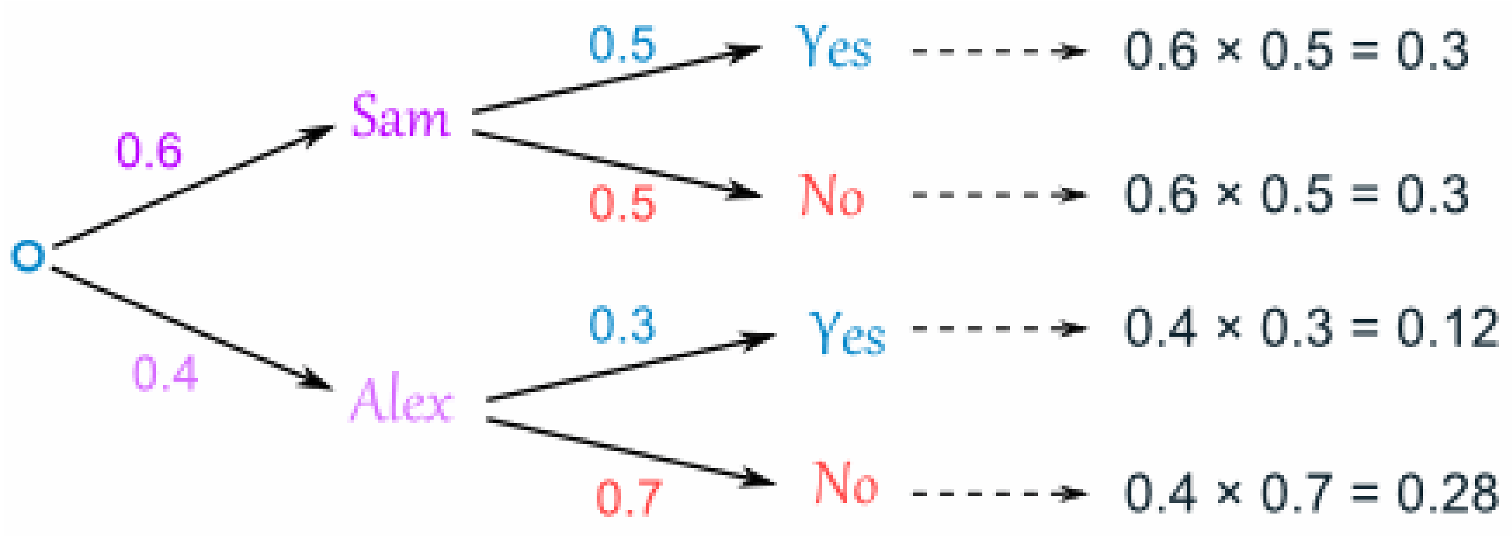
And the two "Yes" branches of the tree together make:

0.3 + 0.12 = **0.42 probability** of being a Goalkeeper today

(That is a 42% chance)

Check

One final step: complete the calculations and make sure they add to **1**:

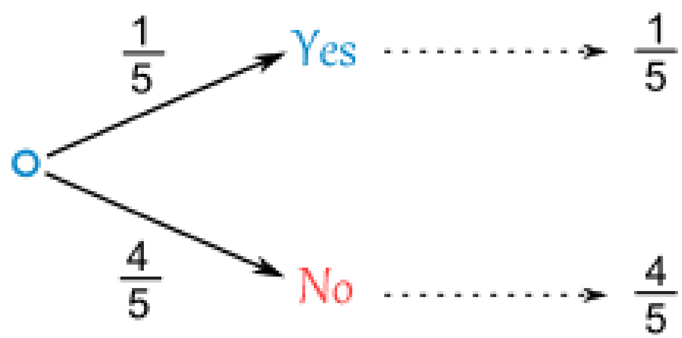


0.3 + 0.3 + 0.12 + 0.28 = 1

Yes, they add to **1**, so that looks right.

Example: 4 friends (Alex, Blake, Chris and Dusty) each choose a random number between 1 and 5. What is the chance that any of them chose the same number?
Let's add our friends one at a time ..

First, what is the chance that Alex and Blake have the same number?
Blake compares his number to Alex's number. There is a 1 in 5 chance of a match.
As a tree diagram:



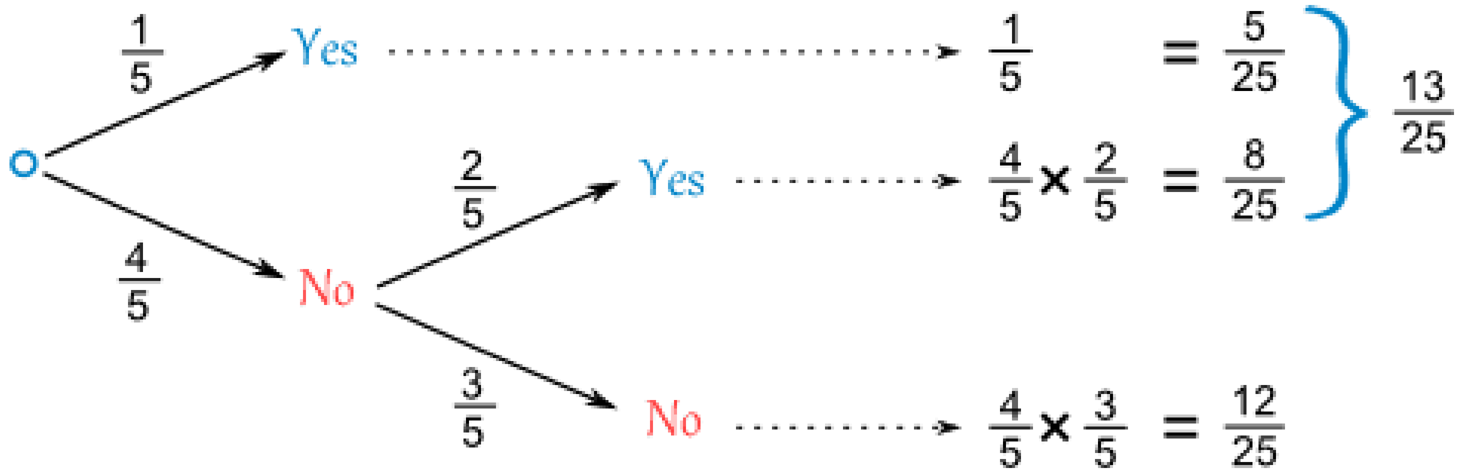
*Note: "**Yes**" and "**No**"*
*together makes **1***
(1/5 + 4/5 = 5/5 = 1)

Now, let's include Chris ...

But there are now two cases to consider:

- If Alex and Billy **did** match, then Chris has only **one number** to compare to.
- But if Alex and Billy **did not** match then Chris has **two numbers** to compare to.

And we get this:



For the top line (Alex and Billy **did** match) we already have a match (a chance of 1/5).

But for the "Alex and Billy **did not** match" there is now a **2/5** chance of Chris matching (because Chris gets to match his number against both Alex and Billy).

And we can work out the combined chance by **multiplying the chances** it took to get there:

- Following the "No, Yes" path ... there is a 4/5 chance of No, followed by a 2/5 chance of Yes:

$$\mathbf{(4/5) \times (2/5) = 8/25}$$

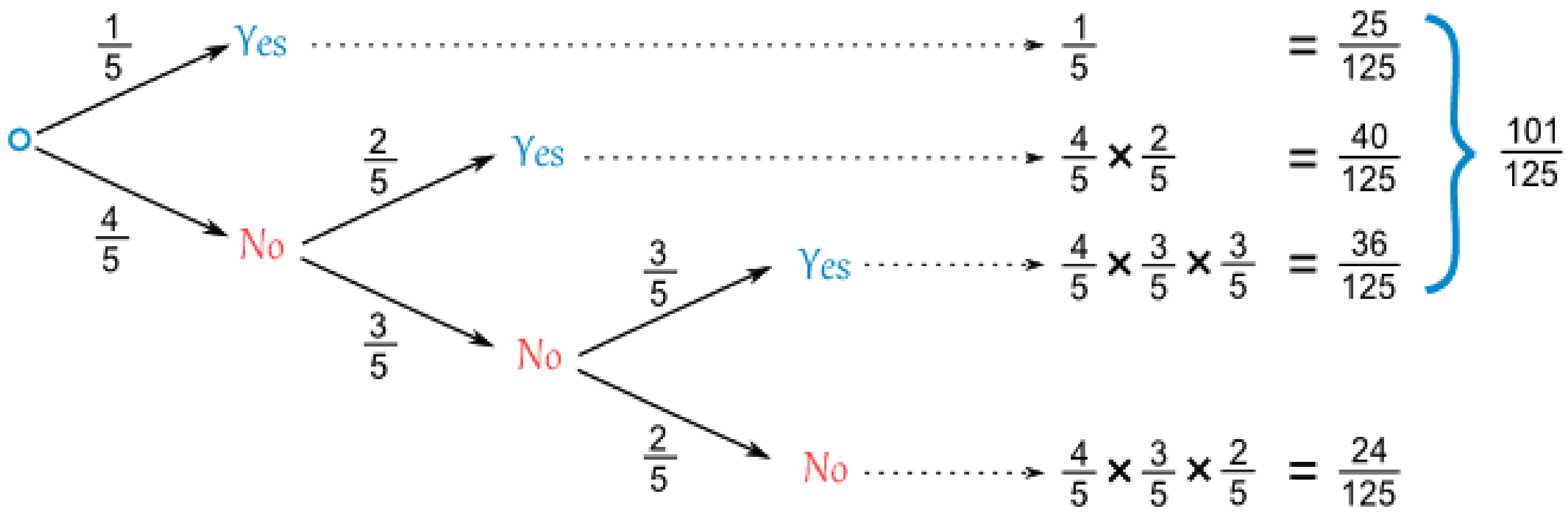
- Following the "No, No" path ... there is a 4/5 chance of No, followed by a 3/5 chance of No:

$$\mathbf{(4/5) \times (3/5) = 12/25}$$

Also notice that when you add all chances together you still get 1 (a good check that we haven't made a mistake):

$$\mathbf{(5/25) + (8/25) + (12/25) = 25/25 = 1}$$

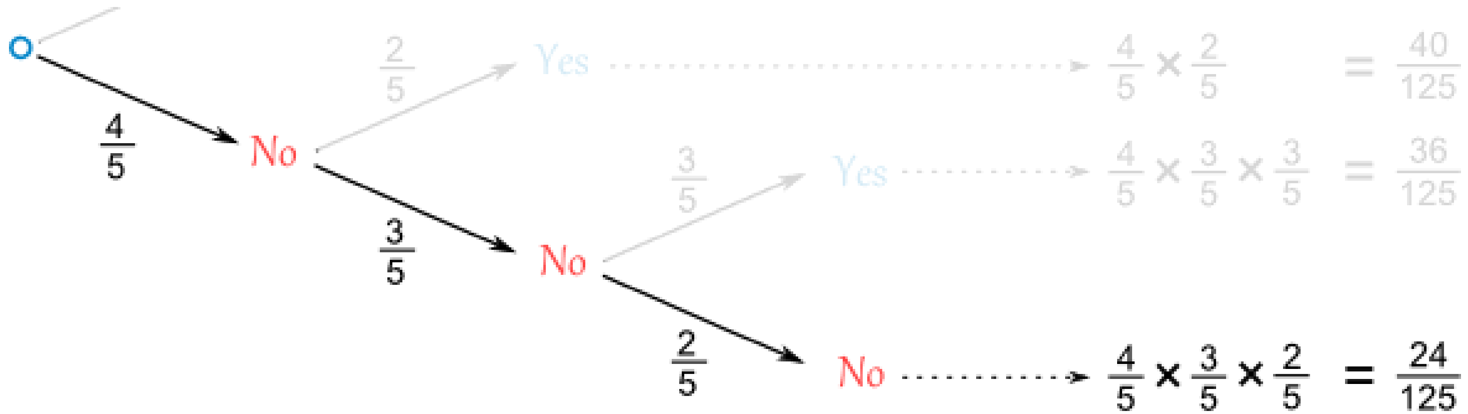
Now what happens when we include Dusty?
It is the same idea, just more of it:



OK, that is all 4 friends, and the "Yes" chances together make 101/125:

Answer: 101/125

But notice something interesting ... if we had followed the "No" path we could have **skipped all the other calculations** and made our life easier:



The chances of **not matching** are:

$$\mathbf{(4/5) \times (3/5) \times (2/5) = 24/125}$$

So the chances of **matching** are:

$$\mathbf{1 - (24/125) = 101/125}$$

(And we didn't really need a tree diagram for that!)