Functions

Real versus Complex Functions

- A **function** f **from** a set A **to** a set B is a rule of correspondence that assigns to each element in A one and only one element in B.
- We often think of a function as a rule or a machine that accepts inputs from the set A and returns outputs in the set B.
- In calculus we studied functions whose inputs and outputs were real numbers. Such functions are called **real-valued functions of a real variable**.
- Now we study functions whose inputs and outputs are complex numbers. We call these functions **complex functions of a complex variable**, or **complex functions** for short.
- Many interesting complex functions are simply generalizations of well-known functions from calculus.

Domain and Range

- Suppose that f is a function from the set A to the set B.
- If f assigns to a in A the element b in B, then we say that b is the **image** of a under f, or the **value** of f at a, and we write $b = f(a)$.
- The set A, the set of inputs, is called the **domain** of f and the set of images in B, the set of outputs, is called the **range** of f.
- We denote the domain of f by $\mathrm{Dom}(f)$ and the range of f by $\mathrm{Range}(f)$.

  Example: Consider the "squaring" function $f(x) = x^2$ defined for the real variable x.

  Since any real number can be squared, the domain of f is the set $\mathbb{R}$ of all real numbers, i.e., $\mathrm{Dom}(f) = A = \mathbb{R}$. The range of f consists of all real numbers x^2, where x is a real number. Of course, $x^2 \geq 0$, for all real x, and one can see from the graph of f that $\mathrm{Range}(f) = [0, \infty)$.
- The range of f need not be the same as the set B. For instance, because the interval $[0, \infty)$ is a subset of $\mathbb{R}$, f can be viewed as a function from $A = \mathbb{R}$ to $B = \mathbb{R}$, so the range of f is not equal to B.

Complex Functions

Definition (Complex Function)

A **complex function** is a function f whose domain and range are subsets of the set $\mathbb{C}$ of complex numbers.

- A complex function is also called a **complex-valued function of a complex variable**.
- Ordinarily, the usual symbols f, g and h will denote complex functions.
- Inputs to a complex function f will typically be denoted by the variable z and outputs by the variable $w = f(z)$.
- When referring to a complex function we will use three notations interchangeably: E.g.,

$$f(z) = z - i, \quad w = z - i, \quad \text{or, simply, the function } z - i.$$

- The notation $w = f(z)$ will always denote a complex function; the notation $y = f(x)$ will represent a real-valued function of a real variable x.

Examples of Complex Functions

(a) The expression $z^2 - (2+i)z$ can be evaluated at any complex number z and always yields a single complex number, and so

$$f(z) = z^2 - (2+i)z$$

defines a complex function.
Values of f are found by using the arithmetic operations for complex numbers. For instance, at the points $z = i$ and $z = 1 + i$ we have:

$$\begin{aligned} f(i) &= (i)^2 - (2+i)(i) = -1 - 2i + 1 = -2i; \\ f(1+i) &= (1+i)^2 - (2+i)(1+i) = 2i - 1 - 3i = -1 - i. \end{aligned}$$

(b) The expression $g(z) = z + 2\mathrm{Re}(z)$ also defines a complex function. Some values of g are:

$$\begin{aligned} g(i) &= i + 2\mathrm{Re}(i) = i + 2(0) = i; \\ g(2-3i) &= 2 - 3i + 2\mathrm{Re}(2-3i) = 2 - 3i + 2(2) = 6 - 3i. \end{aligned}$$

Natural Domains

- When the domain of a complex function is not explicitly stated, we assume the domain to be the set of all complex numbers z for which $f(z)$ is defined. This set is sometimes referred to as the **natural domain** of f.
- Example: The functions

$$f(z) = z^2 - (2+i)z \quad \text{and} \quad g(z) = z + 2\text{Re}(z)$$

are defined for all complex numbers z, and so, $\text{Dom}(f) = \mathbb{C}$ and $\text{Dom}(g) = \mathbb{C}$. The complex function $h(z) = \dfrac{z}{z^2+1}$ is not defined at $z = i$ and $z = -i$ because the denominator $z^2 + 1$ is equal to 0 when $z = \pm i$. Therefore, $\text{Dom}(h)$ is the set of all complex numbers except i and $-i$, written $\text{Dom}(h) = \mathbb{C} - \{-i, i\}$.
- Since $\mathbb{R}$ is a subset of $\mathbb{C}$, every real-valued function of a real variable is also a complex function. We will see that real-valued functions of two real variables x and y are also special types of complex functions.

Real and Imaginary Parts of a Complex Function

- If $w = f(z)$ is a complex function, then the image of a complex number $z = x + iy$ under f is a complex number $w = u + iv$. By simplifying the expression $f(x + iy)$, we can write the real variables u and v in terms of the real variables x and y.
  Example: By replacing the symbol z with $x + iy$ in the complex function $w = z^2$, we obtain:
  $$w = u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi.$$
  Thus, $u = x^2 - y^2$ and $v = 2xy,$ respectively.
- If $w = u + iv = f(x + iy)$ is a complex function, then both u and v are real functions of the two real variables x and y, i.e., by setting $z = x + iy$, we can express any complex function $w = f(z)$ in terms of two real functions as:
  $$f(z) = u(x, y) + iv(x, y).$$
- The functions $u(x, y)$ and $v(x, y)$ are called the **real** and **imaginary parts** of f, respectively.

Examples

- Find the real and imaginary parts of the functions:
  (a) $f(z) = z^2 - (2+i)z$;
  (b) $g(z) = z + 2\text{Re}(z)$.

  In each case, we replace the symbol z by $x + iy$, then simplify.
  (a) $f(z) = (x+iy)^2 - (2+i)(x+iy) = x^2 - 2x + y - y^2 + (2xy - x - 2y)i$. So,

$$u(x, y) = x^2 - 2x + y - y^2 \quad \text{and} \quad v(x, y) = 2xy - x - 2y.$$

  (b) Since $g(z) = x + iy + 2\text{Re}(x+iy) = 3x + iy$, we have

$$u(x, y) = 3x \quad \text{and} \quad v(x, y) = y.$$

Specifying w via u and v

- Every complex function is completely determined by the real functions $u(x, y)$ and $v(x, y)$.
- Thus, a complex function $w = f(z)$ can be defined by arbitrarily specifying two real functions $u(x, y)$ and $v(x, y)$, even though $w = u + iv$ may not be obtainable through familiar operations performed solely on the symbol z.

  Example: If we take $u(x, y) = xy^2$ and $v(x, y) = x^2 - 4y^3$, then

  $$f(z) = xy^2 + i(x^2 - 4y^3)$$

  defines a complex function. In order to find the value of f at the point $z = 3 + 2i$, we substitute $x = 3$ and $y = 2$:

  $$f(3 + 2i) = 3 \cdot 2^2 + i(3^2 - 4 \cdot 2^3) = 12 - 23i.$$
- Of course, complex functions defined in terms of $u(x, y)$ and $v(x, y)$ can always be expressed in terms of operations on the symbols z and $\bar{z}$.

Exponential Function

- The complex exponential function e^z is an example of a function defined by specifying its real and imaginary parts.

Definition (Complex Exponential Function)

The function e^z defined by

$$e^z = e^x \cos y + i e^x \sin y$$

is called the **complex exponential function**.

- The real and imaginary parts of the complex exponential function are

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

- Thus, values of the complex exponential function $w = e^z$ are found by expressing the point z as $z = x + iy$ and then substituting the values of x and y in $u(x, y)$ and $v(x, y)$.

Values of the Complex Exponential Function

- Find the values of the complex exponential function e^z at:

$$(a) \quad z = 0 \qquad (b) \quad z = i \qquad (c) \quad z = 2 + \pi i.$$

In each part we substitute $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ in $e^z = e^x \cos y + ie^x \sin y$ and then simplify:

(a) For $z = 0$, we have $x = 0$ and $y = 0$, and so $e^0 = e^0 \cos 0 + ie^0 \sin 0 = 1 \cdot 1 + i1 \cdot 0 = 1$.

(b) For $z = i$, we have $x = 0$ and $y = 1$, and so: $e^i = e^0 \cos 1 + ie^0 \sin 1 = \cos 1 + i \sin 1$.

(c) For $z = 2 + \pi i$, we have $x = 2$ and $y = \pi$, and so $e^{2+\pi i} = e^2 \cos \pi + ie^2 \sin \pi = e^2 \cdot (-1) + ie^2 \cdot 0 = -e^2$.

Exponential Form of a Complex Number

- The exponential function enables us to express the polar form of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ in a particularly convenient and compact form:

$$z = re^{i\theta}.$$

- This form is called the **exponential form** of the complex number z.
- Example: A polar form of the complex number $3i$ is $3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, whereas an exponential form of $3i$ is $3e^{i\pi/2}$.
- In the exponential form of a complex number, the value of $\theta = \arg(z)$ is not unique.

  Example: All forms $\sqrt{2}e^{i\pi/4}$, $\sqrt{2}e^{i9\pi/4}$, and $\sqrt{2}e^{i17\pi/4}$ are all valid exponential forms of the complex number $1 + i$.

Some Additional Properties

- If z is a real number, that is, if $z = x + 0i$, then

$$e^z = e^x \cos 0 + ie^x \sin 0 = e^x.$$

  Thus, the complex exponential function agrees with the usual real exponential function for real z.

- Many well-known properties of the real exponential function are also satisfied by the complex exponential function: If z_1 and z_2 are complex numbers, then:
  - $e^0 = 1$;
  - $e^{z_1} e^{z_2} = e^{z_1 + z_2}$;
  - $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$;
  - $(e^{z_1})^n = e^{nz_1}$, for $n = 0, 1, 2, \ldots$.

Periodicity of e^z

- The most unexpected difference between the real and complex exponential functions is:

Proposition (Periodicity of e^z)

The complex exponential function is periodic; Indeed, we have

$$e^{z+2\pi i} = e^z, \text{ for all complex numbers } z.$$

$$\begin{aligned} e^{z+2\pi i} &= e^{x+iy+2\pi i} \\ &= e^{x+i(y+2\pi)} \\ &= e^x \cos(y+2\pi) + ie^x \sin(y+2\pi) \\ &= e^x \cos y + ie^x \sin y \\ &= e^{x+iy} = e^z. \end{aligned}$$

Corollary

The complex exponential function has a pure imaginary period $2\pi i$.

Polar Coordinates

- It is often more convenient to express the complex variable z using either the polar form $z = r(\cos\theta + i\sin\theta)$ or, equivalently, the exponential form $z = re^{i\theta}$.
- Given a complex function $w = f(z)$, if we replace the symbol z with $r(\cos\theta + i\sin\theta)$, then we can write this function as:

$$f(z) = u(r,\theta) + iv(r,\theta).$$

We still call the real functions $u(r,\theta)$ and $v(r,\theta)$ the **real** and **imaginary parts** of f, respectively.
- Example: Replacing z with $r(\cos\theta + i\sin\theta)$ in $f(z) = z^2$ yields

$$f(z) = (r(cos\theta + i\sin\theta))^2 = r^2\cos 2\theta + ir^2\sin 2\theta.$$

Thus, the real and imaginary parts of $f(z) = z^2$ are

$$u(r,\theta) = r^2\cos 2\theta \quad \text{and} \quad v(r,\theta) = r^2\sin 2\theta.$$

Note that u and v are not the same as the functions u and v previously computed using $z = x + iy$.

Definition in Polar Coordinates

- A complex function can be defined by specifying its real and imaginary parts in polar coordinates.
- Example: The expression

$$f(z) = r^3 \cos\theta + (2r\sin\theta)i$$

defines a complex function.
To find the value of this function at, say, the point $z = 2i$, we first express $2i$ in polar form $2i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$. We then set $r = 2$ and $\theta = \frac{\pi}{2}$ in the expression for f:

$$f(2i) = (2)^3\cos\frac{\pi}{2} + (2\cdot 2\sin\frac{\pi}{2})i = 8\cdot 0 + (4\cdot 1)i = 4i.$$

Remarks

(i) The complex exponential function provides a good example of how complex functions can be similar to and, at the same time, different from their real counterparts.

(ii) Every complex function can be defined in terms of two real functions $u(x, y)$ and $v(x, y)$ as $f(z) = u(x, y) + iv(x, y)$. Thus, the study of complex functions is closely related to the study of real multivariable functions of two real variables.

(iii) Real-valued functions of a real variable and real-valued functions of two real variables are special types of complex functions. Other types include:

- Real-valued functions of a complex variable are functions $y = f(z)$ where z is a complex number and y is a real number. The functions $x = \text{Re}(z)$ and $r = |z|$ are both examples of this type of function.
- Complex-valued functions of a real variable are functions $w = f(t)$ where t is a real number and w is a complex number. It is customary to express such functions in terms of two real-valued functions of the real variable t, $w(t) = x(t) + iy(t)$. An example is $w(t) = 3t + i\cos t$.