MATHEMATICAL ANALYSIS - II (MAT 315, CID: 00107)

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

1. Differential Equations and Mathematical Models

Process of *mathematical modeling* (Fig.1.) involves the following steps:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation; for example, answering the question originally posed.

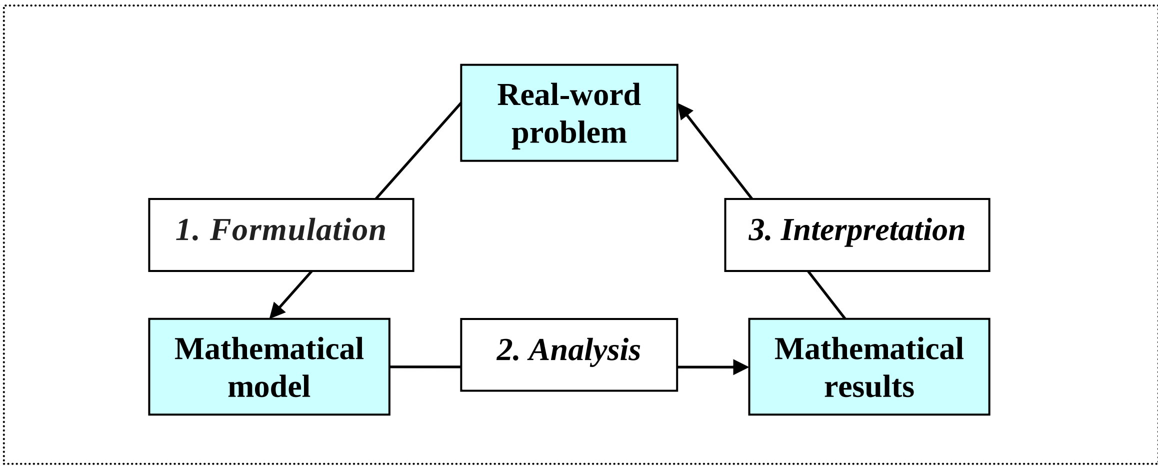

Fig. 1.

Population Models

Suppose that $P(t)$ is the number of individuals in a population (of humans, or insects, or bacteria), and population changes only by the occurrence of births and deaths—there is no immigration or emigration from outside the country or environment under consideration. It is customary to track the growth or decline of a population in terms of its *birth rate* and *death rate* functions defined as follows:

- $\beta(t)$ is the number of births per unit of population per unit of time at time *t;*
- $\delta(t)$ is the number of deaths per unit of population per unit of time at time *t.*

Then the numbers of births and deaths that occur during the time interval $[t, t+\Delta t]$ is given by (approximately for a small Δt):

- births: $\beta(t)\cdot P(t)\cdot\Delta t$;
- deaths: $\delta(t)\cdot P(t)\cdot\Delta t$.

Hence the change ΔP in the population during the time interval $[t, t+\Delta t]$ of length Δt is

$$\Delta P = \{birth\} - \{deaths\} \approx \beta(t)\cdot P(t)\cdot\Delta t - \delta(t)\cdot P(t)\cdot\Delta t$$

so

$$\frac{\Delta P}{\Delta t} \approx \left[\beta(t) - \delta(t)\right] \cdot P(t).$$

The error in this approximation should approach zero as $\Delta t \to 0$, so-taking the limit-we get the differential equation:

$$\frac{dP}{dt} = (\beta - \delta) \cdot P, \tag{1}$$

in which we write $\beta = \beta(t), \delta = \delta(t), P = P(t)$ for brevity. Equation (1) is the ***general population equation.*** If β and δ are constant, Eq.(1) reduces to the ***natural growth*** (**or** ***natural decay***) ***equation*** (2) with $k = \beta - \delta$:

$$\frac{dP}{dt} = k \cdot P. \tag{2}$$

In situations as diverse as the human population of a nation and a fruit fly population in a closed container, it is often observed that the birth rate decreases as the population itself increases. The reasons may range from increased scientific or cultural sophistication to a limited food supply. Suppose, for example, that the birth rate β is a *linear* decreasing function of the population size *P*, so that $\beta = \beta_0 - \beta_1 P$, where β_0 and β_1 are positive constants. If the death rate $\delta = \delta_0$ remains constant, then Eq. (1) takes the form

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) \cdot P$$

that is,

$$\frac{dP}{dt} = aP - bP^2 \tag{3}$$

where $a = \beta_0 - \delta_0$ and $b = \beta_1$. If the coefficients *a* and *b* are both positive, then Eq. (3) is called the ***logistic equation.*** For the purpose of relating the behavior of the population $P(t)$ to the values of the parameters in the equation, it is useful to rewrite the logistic equation in the form

$$\frac{dP}{dt} = kP(M - P), \tag{4}$$

Where $k = b$ and $M = a/b$ are constants.

Cooling and Heating Models

According to Newton's law of cooling, the time rate of *change* of the temperature $T(t)$ of a body (the rate of change with respect to time *t*) is proportional to the difference between *T* and the temperature *A* of the surrounding medium (Fig. 2.). That is,

Fig. 2.

$$\frac{dT}{dt} = -k \cdot (T - A), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then

$$\frac{dT}{dt} < 0,$$

so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then

$$\frac{dT}{dt} > 0,$$

so that T is increasing. Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body.

<u>Torricelli's Law (Model)</u>

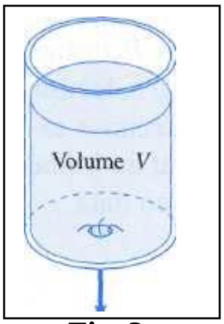

Fig. 3.

Suppose that a water tank has a hole with area "*a*" at its bottom, from which water is leaking (Fig. 3.). Denote by $y(t)$ the depth of water in the tank at time *t*, and by $V(t)$ the volume of water in the tank then. It is plausible—and true, under ideal conditions— that the velocity of water exiting through the hole is

$$v = \sqrt{2gy} \tag{4}$$

Torricelli's law implies that the *time rate of change* of the volume *V* of water in a draining tank is proportional to the velocity (4):

$$\frac{dV}{dt} = -av = -a\sqrt{2gy},$$

Equivalently,

$$\frac{dV}{dt} = -k\sqrt{y} \quad \text{where } k = a\sqrt{2g}. \tag{5}$$

If $A(y)$ denotes the horizontal cross-sectional area of the tank at height y above the hole, the method of volume by cross sections gives:

$$V = \int_0^y A(y)\,dy,$$

so the fundamental theorem of calculus implies that

$$\frac{dV}{dy} = A(y)$$

and therefore that

$$\frac{dV}{dt} = \frac{dV}{dy}\cdot\frac{dy}{dt} = A(y)\cdot\frac{dy}{dt}. \tag{6}$$

From Eqs. (5) and (6) we finally obtain:

$$A(y)\frac{dy}{dt} = -k\sqrt{y}\,, \tag{7}$$

an alternative form of Torricelli's law.

Motion of a particle (mass point)

The motion of a particle along a straight line (the x-axis) is describe by its ***position function:***

$$x = f(t)$$

Giving its x-coordinate at time t. The ***velocity*** of the particle is defined to be

$$v(t) = f'(t) = \frac{dx}{dt}\,.$$

Its ***acceleration*** $a(t)$ is

$$a(t) = v'(t) = \frac{d^2x}{dt^2}\,.$$

Newton's second law of motion implies that if a force $F(t)$ acts on the particle and is directed along its line of motion, then

$$ma(t) = F(t)\,,$$

where m is the mass of the particle. If the force $F(t)$ is known, then the equation

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m}$$

can be integrated twice to find the position function $x(t)$.

1. (Population model) A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000? (51,840)

2. (Population model) In a certain culture of bacteria, the number of bacteria increased six fold in 10 h. How long did it take for the population to double? (3.869 h)

3. (Population model) The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable? (Ignore the probable presence of other radioactive isotopes).

4. (Population model) A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28 x 10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

5. When sugar is dissolved in water, the amount P that remains undissolved after t minutes satisfies the differential equation (2) with $k<0$. If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

6. The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation

$$\frac{dI}{dx} = (-1.4) \cdot I$$

(a) At what depth is the intensity half the intensity I at the surface (where $x = 0$)? (b) What is the intensity at a depth of 10 m (as a fraction of I_0)? (c) At what depth will the intensity be 1% of that at the surface?

7. (Cooling and Heating Model) A pitcher of milk initially at 25 °C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the milk has dropped to 15°C after 20 min. When will it be at 5° C?

8. (Cooling and Heating Model) A cake is removed from an oven at 210° F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

9. (Cooling and Heating Model) A roast, initially at 50°F, is placed in a 375°F oven at 5:00 P.M. After 75 minutes it is found that the temperature of the roast is 125°F. When will the roast be 150°F (medium rare)?

10. According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the "big bang." At present there are 137.7 atoms of ^{238}U for each atom of ^{235}U. Using the half-lives 4.51 x 10^9 years for ^{238}U and 7.10 x 10^8 years for ^{235}U, calculate the age of the universe.

11. A certain piece of dubious information about phenyl ethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week, 10000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

12. (Torricelli's Model) A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

13. (Torricelli's Model) At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft high is removed. After 1 h the water in the tank is 9 ft deep. When will the tank be empty?

14. Suppose that a cylindrical tank initially containing V_o gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leq T$ minutes is $V = V_o\,[1 - (t/T)]^2$.

15. (Torricelli's Model) A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius 1 in. is opened. How long will be required for all the gasoline to drain from the tank?

16. (Motion of a particle) The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second (m/s^2). How far does the car travel before coming to a stop?

17. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dy}{dx} = (0.12)t^2 + (0.6)t \quad \left(ft/s^2 \right).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

18. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

19. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

In Problems 20 through 24, a function y = *g(x)* is described by some geometric property of its graph. Write a differential equation having the function *g* as its solution (or as one of its solutions) and find *a general solution* of this equation.

20. The slope of the graph of *g* at the point (x, y) is the sum of x and y.

21. The line tangent to the graph of *g* at the point (x, y) intersects the x-axis at the point (x/2, 0).

22. Every straight line normal to the graph of *g* passes through the point (0, 1).

23. The graph of *g* is normal to every curve of the form $y = x^2 + c$ (c-is a constant), where they meet.

24. The line tangent to the graph of *g* at (x, y) passes through the point (—y, x).

25. The time rate of change of a rabbit population *P* is proportional to the square root of *P*. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

26. Suppose that the fish population *P(t)* in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

27. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$. (a) Show that

$$P(t)=\left(\frac{kt}{2}+\sqrt{P}\right)^2$$

where k is a constant, (b) If $P_0=100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

28. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

29. Consider a population $P(t)$ satisfying the logistic equation (3), where $B = aP$ is the time rate at which births occur and $D = bP^2$ is, the rate at which deaths occur. If the initial population is $P(0) = Po$, and B_o births per month and D_o deaths per month are occurring at time $t = 0$, show that the limiting population is $M = B_oPo/D_o$.

30. Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 29. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

31. Consider a rabbit population $P\{t)$ satisfying the logistic equation as in Problem 29. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

2. Separable Equations

Find *general solutions* (implicit if necessary, explicit if convenient) of the differential equations in Problems 32 through 41. Primes denote derivatives with respect to x.

32. $\dfrac{dy}{dx}+2xy=0$;

33. $\dfrac{dy}{dx}+2xy^2=0$;

34. $\dfrac{dy}{dx}-y\sin x=0$;

35. $2\sqrt{x}\dfrac{dy}{dx}-\sqrt{1-y^2}=0$;

36. $\dfrac{dy}{dx}=3\sqrt{xy}$;

37. $xy' + y = y^2$**;**

38. $y' = e^{x+y}$**;**

39. $(x+1)y' + xy = 0$**;**

40. $(1-x^2)\dfrac{dy}{dx} = 2y$**;**

41. $yy' = x(1+y^2)$**.**

Find *explicit particular solutions* of the *initial value problems* in Problems 42 through 48.

42. $\dfrac{dy}{dx} = ye^x,\ y(0) = 2e$**;**

43. $\dfrac{dy}{dx} = 3x^2(y^2+1),\ y(0) = 1$**;**

44. $\dfrac{dy}{dx} tgx = y,\ y(\pi/2) = 1$**;**

45. $\dfrac{dy}{dx} = 4x^3y - y,\ y(1) = -3$**;**

46. $\dfrac{dy}{dx} + 1 = 2y,\ y(1) = 1$**;**

47. $\dfrac{dy}{dx} = 6e^{2x-y},\ y(0) = 0$**;**

48. $2\sqrt{x}\dfrac{dy}{dx} = cos^2 y,\ y(4) = \pi/4$**.**

3. Linear First-Order Equations

Find general solutions of the differential equations in Problems 49 through 66. If an initial condition is given, find the corresponding particular solution.

49. $y' + y = 2,\ y(0) = 0$**;**

50. $y' - 2y = 3e^{2x},\ y(0) = 0$**;**

51. $y' - 2xy = e^{x^2}$**;**

52. $xy' + 2y = 3x,\ y(1) = 5$**;**

53. $2xy' + y = 10\sqrt{x}$ **;**

54. $xy' + y = 3xy,\ y(1) = 0$ **;**

55. $y' + 2xy = x,\ y(0) = -2$ **;**

56. $y' = (1-y)\cos x,\ y(\pi) = 2$ **;**

57. $(1+x)y' + y = \cos x,\ y(0) = 1$ **;**

58. $y' = (1-y)\cos x$ **;**

59. $y' = 1 + x + y + xy,\ y(0) = 0$ **;**

60. $y' + 2y = e^{3x}$ **;**

61. $(x^2+4)y' + 3xy = x,\ y(0) = 1$ **;**

62. $(1+x^2)y' = 2xy + (1+x^2)^2$ **;**

63. $xy' = y + x^2\cos x$ **;**

64. $xy' = e^x + xy$ **;**

65. $y' + y\,tgx = 1/\cos x,\ y(0) = 0$ **;**

66. $y' = 2y + e^x - x,\ y(0) = 1/4$ **.**

4. Bernoulli Equations

Find general solutions of the differential equations in Problems 67 through 85.

67. $y' + 4xy = 2xe^{-x^2}\sqrt{y}$ **;**

68. $y' = y^2e^x - y$ **;**

69. $(x+y)y' = x - y$ **;**

70. $2xyy' = x^2 + 2y^2$ **;**

71. $xy' = y + 2\sqrt{xy}$;

72. $(x-y)y' = x+y$;

73. $xy^2y' = x^3 + y^3$;

74. $y' = (4x+y)^2$;

75. $x^2y' + 2xy = 5y^3$;

76. $x^2y' + 2xy = 5y^4$;

77. $3y^2y + y^3 = e^{-x}$;

78. $xy' + 6y = 3xy^{4/3}$;

79. $y' = y\left(y^3 \cos x + tgx\right)$;

80. $y' = yctgx + \dfrac{y^3}{\sin x}$;

81. $y' = \dfrac{3y}{2x} + \dfrac{2x}{y}$;

82. $\left(x+e^y\right)y' = xe^{-y} - 1$.

83. $y' = \sqrt{x+y+1}$;

84. $yy' + x = \sqrt{x^2+y^2}$;

85. $xy' = y + \sqrt{x^2+y^2}$.

5. Linear Second-Order Equations

Find general solutions of the homogeneous differential equations given in Problems 86 through 98. Primes denote derivatives with respect to *x*.

86. $y'' - 3y' + 2y = 0$;

87. $y'' + 2y' - 15y = 0$;

88. $y'' + 5y' = 0$;

89. $2y'' + 3y' = 0$;

90. $2y'' - y' - y = 0$;

91. $4y'' + 8y' + 3y = 0$;

92. $9y'' - 12y' + 4y = 0$;

93. $4y'' - 4y' + y = 0$;

94. $25y'' + 10y' + y = 0$;

95. $y'' - 2y' + 2y = 0$;

96. $y'' - 4y' + 13y = 0$;

97. $y'' + 4y = 0$;

98. $9y'' - 54y' + 82y = 0$

Solve the initial and boundary value problems given in Problems 99 through 105.

99. $y'' - 4y' + 3y = 0;\ y(0) = 7,\ y'(0) = 11$;

100. $9y'' + 6y' + 4y = 0;\ y(0) = 3,\ y'(0) = 4$;

101. $y'' - 6y' + 25y = 0;\ y(0) = 3,\ y'(0) = 1$;

102. $y'' + 2y' + y = 0;\ y(0) = 1,\ y'(0) = 0$;

103. $y'' - 6y' + 9y = 0;\ y(0) = 1,\ y(1) = 2e^3$;

104. $4y'' + y = 0;\ y(0) = 1,\ y(\pi) = 2$;

105. $y'' - y = 0;\ y(0) = 0,\ y(1) = 1$.

Find general solutions of the nonhomogeneous differential equations given in Problems 106 through 125.

106. $y'' - 4y' + 3y = 1$;

107. $y'' - y' = 1$;

108. $y'' - 4y = 1$;

109. $y'' - 6y' + 9y = 1$;

110. $y'' + y = 1$

111. $y'' - y' - 2y = x$ **;**

112. $y'' + y' = x$ **;**

113. $y'' + 2y' + y = x;$

115. $y'' - 2y' + 2y = x;$

116. $y'' - 4y' + 3y = e^{2x}$ **;**

117. $y'' - 4y' + 3y = e^{x}$ **;**

118. $y'' + 2y' + y = e^{-x};$

119. $y'' + 2y' + y = e^{x};$

120. $y'' + a^2 y = e^{x}$ **;**

121. $y'' - 7y' + 6y = sin\, x;$

122. $y'' - 2y' + y = cos\, x;$

123. $y'' + a^2 y = sin\, x$ **;**

124. $y'' - 2y' + 5y = cos\, x;$

125. $y'' - 2y' + 5y = sin\, 2x;$

Solve the initial and boundary value problems given in Problems 126 through.

126. $y'' + y' = 1 + x,\ y(0) = 0,\ y'(0) = 1$ **;**

127. $y'' - y = (1/2 + x)e^{x},\ y(0) = 0,\ y'(0) = 1$ **;**