Complex Functions as Mappings

Complex Mappings

- The graph of a complex function lies in four-dimensional space, and so we cannot use graphs to study complex functions.
- The concept of a complex mapping gives a geometric representation of a complex function:
  - The basic idea is that every complex function describes a correspondence between points in two copies of the complex plane.
  - The point z in the z-plane is associated with the unique point $w = f(z)$ in the w-plane.
- The alternative term complex mapping in place of "complex function" is used when considering the function as this correspondence between points in the z-plane and points in the w-plane.
- The geometric representation of a complex mapping $w = f(z)$ consists of two figures:
  - the first, a subset S of points in the z-plane;
  - the second, the set S' of the images of points in S under $w = f(z)$ in the w-plane.

Mappings

- If $y = f(x)$ is a real-valued function of a real variable x, then the **graph** of f is defined to be the set of all points $(x, f(x))$ in the two-dimensional Cartesian plane.
- If $w = f(z)$ is a complex function, then both z and w lie in a complex plane, whence the set of all points $(z, f(z))$ lies in four-dimensional space.

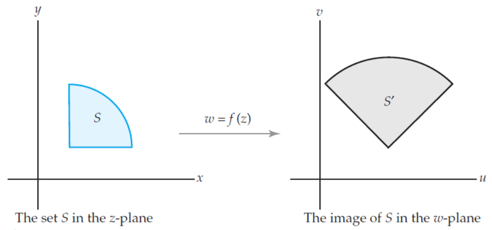
  A subset of four-dimensional space cannot be easily illustrated and, thus, the graph of a complex function cannot be drawn.
- The term **complex mapping** refers to the correspondence determined by a complex function $w = f(z)$ between points in a z-plane and images in a w-plane.
- If the point z_0 in the z-plane corresponds to the point $w_0 = f(z_0)$ in the w-plane, then we say that f **maps** z_0 onto w_0 or that z_0 is **mapped** onto w_0 by f.

Example (Physical Motion)

- Consider the real function $f(x) = x + 2$.
- The known representation of this function is a line of slope 1 and y-intercept $(0, 2)$.
- Another representation shows how one copy of the real line (the x-line) is mapped onto another copy of the real line (the y-line) by f: Each point on the x-line is mapped onto a point two units to the right on the y-line.
- You can visualize the action of this mapping by imagining the real line as an infinite rigid rod that is physically moved two units to the right.

Representing a Complex Mapping

- To create a geometric representation of a complex mapping, we begin with two copies of the complex plane, the z-plane and the w-plane.
- A complex mapping is represented by drawing a set S of points in the z-plane and the corresponding set of images of the points in S under f in the w-plane.

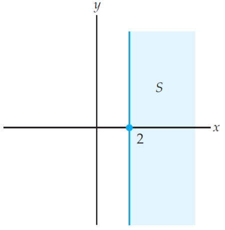


- If $w = f(z)$ is a complex mapping and if S is a set of points in the z-plane, then we call the set of images of the points in S under f the **image** of S under f, denoted S'.
- If S is a domain or a curve, we also use symbols such as D and D' or C and C', in place of S and S'.
- Sometimes $f(C)$ is used to denote the image of C under $w = f(z)$.

Image of a Half-Plane under $w = iz$

- Find the image of the half-plane $\mathrm{Re}(z) \geq 2$ under the complex mapping $w = iz$ and represent the mapping graphically.

  Let S be the half-plane consisting of all complex points z with $\mathrm{Re}(z) \geq 2$. Consider first the vertical boundary line $x = 2$ of S:

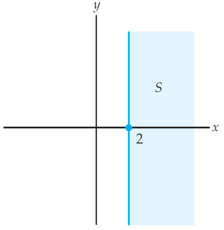


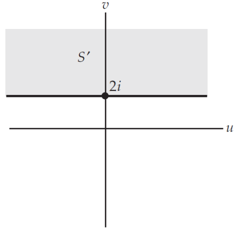
For any point z on this line we have $z = 2 + iy$, where $-\infty < y < \infty$. The value of $f(z) = iz$ at a point on this line is $w = f(2 + iy) = i(2 + iy) = -y + 2i$. The set of points $w = -y + 2i$, $-\infty < y < \infty$, is the line $v = 2$ in the w-plane.

Hence, the vertical line $x = 2$ in the z-plane is mapped onto the horizontal line $v = 2$ in the w-plane by the mapping $w = iz$.

Image of a Half-Plane under $w = iz$ (Cont'd)

- Therefore, the vertical line on the left is mapped onto the horizontal line shown on the right.





Now consider the entire half-plane S. This set can be described by the two simultaneous inequalities, $x \geq 2$ and $-\infty < y < \infty$. In order to describe the image of S:

  - We express $w = iz$ in terms of its real and imaginary parts u and v.
  - Then we use the bounds on x and y in the z-plane to determine bounds on u and v in the w-plane.

We have $w = i(x + iy) = -y + ix$. So the real and imaginary parts of $w = iz$ are $u(x, y) = -y$ and $v(x, y) = x$. We conclude that $v \geq 2$ and $-\infty < u < \infty$. That is, the set S' is the half-plane lying on or above the horizontal line $v = 2$.

Image of a Line under $w = z^2$

- Find the image of the vertical line $x = 1$ under the complex mapping $w = z^2$ and represent the mapping graphically.

  Let C be the set of points on the vertical line $x = 1$, i.e., the set of points $z = 1 + iy$ with $-\infty < y < \infty$. The real and imaginary parts of $w = z^2 = (x + iy)^2$ are

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

  For a point $z = 1 + iy$ in C, we have

$$u(1, y) = 1 - y^2 \quad \text{and} \quad v(1, y) = 2y.$$

  Thus, the image of S is the set of points $w = u + iv$ satisfying $u = 1 - y^2$ and $v = 2y$, for $-\infty < y < \infty$.

Image of a Line under $w = z^2$ (Cont'd)

- We found $w = u + iv$, with $u = 1 - y^2$, $v = 2y$, $-\infty < y < \infty$.

  Note that these are parametric equations in the real parameter y, and they define a curve in the w-plane. By eliminating the parameter y, we find
  $$u = 1 - \left(\frac{v}{2}\right)^2 = 1 - \frac{v^2}{4}.$$
  Since y can take on any real value and since $v = 2y$, it follows that v can take on any real value. Consequently, C' is a parabola in the w-plane with vertex at $(1, 0)$ and u-intercepts at $(0, \pm 2)$: