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Individual Project 1.1

This report is split into five parts, one for each of the test problems that were assigned to me. The problems are: 1, 2, 5, 7, and 14. For each of the problems I will consider the behavior of the function with respect to epsilon and the number of nodes, whether or not the interpolation converges, and the accuracy of the interpolation.

Problem 1

Convergence of Interpolation

On a uniform grid, this function's lagrange interpolation function converges up until a certain number of nodes. In my testing, the interpolation starts to diverge from the test function if N >= 58 because of the Runge Phenomenon, where the interpolation polynomial gets too large and starts to heavily oscillate at both ends of the interpolation interval. For a chebyshev grid, the interpolation is convergent at least up to N >= 513, I have not encountered divergence at all. This means that increasing N leads to ever higher accuracy if one is using a chebyshev grid, and stops increasing accuracy for uniform grids once it gets larger than N >= 58.

Accuracy of Interpolation

The accuracy of the interpolation function of problem 1 is dependent on N and epsilon. The larger N gets, the higher the accuracy of the interpolation gets. N >= 58 is an exception for a uniform grid, where the accuracy starts to decrease once this value is surpassed. Epsilon has the opposite effect of N. The smaller epsilon gets, the more inaccurate the interpolation gets (when N is constant). In problem 1 epsilon dictates the “sharpness” of the “corner” the function makes; the smaller epsilon is, the more the function looks like like a right angle corner. Because of this “sharp corner”, more nodes are needed to accurately represent it and too keep the error as small as possible.

The method used to compute the lagrange polynomial has little influence on its accuracy in most cases. For high numbers of nodes (> 65), Neville's Method is more accurate than the standard lagrange method, but because the polynomial starts to oscillate violently, this difference does not matter, as the error is very large in all cases. The reason for this difference may be the fact that Neville's Method requires less cycles to compute the interpolation function, which may make the error from using floating point arithmetic less pronounced. The most significant difference between the two methods is their speed. Neville's Method, thanks to its recursive properties, is 2 to 3 times faster than the normal lagrange method. If one imagines computing tens of thousands or even more points, this is a significant reduction in time.

Problem 2

Convergence of Interpolation

On a uniform grid, this function's lagrange interpolation function converges up until N >= 60. If N gets larger than that, of Runge's Phenomenon starts to make the interpolation vary inaccurate. For a chebyshev grid, the interpolation is convergent at least up to N >= 513, I have not encountered divergence at all. This means that increasing N leads to ever higher accuracy if one is using a chebyshev grid, and stops increasing accuracy for uniform grids once it gets larger than N >= 58. In case of this problem, the error is smaller than 0.01 when N = 10 (chebyshev grid) or N = 13 (uniform grid), for epsilon = 2^{-9} so any N larger than that is not necessary for decent accuracy.

Accuracy of Interpolation

The accuracy of the interpolation function of problem 2 is dependent on N and epsilon. The larger N gets, the higher the accuracy of the interpolation gets. N >= 60 is an exception for a uniform grid, where the accuracy starts to decrease once this value is surpassed. As mentioned above, vales above N = 13 are not needed for accuracy and can be ignored. Epsilon has the opposite effect of N. The smaller epsilon gets, the more inaccurate the interpolation gets (when N is constant). In problem 2 epsilon dictates the "sharpness" of the "corners" the function makes; the smaller epsilon is, the more the function looks like like a rectangle. Because of these "sharp corners", more nodes are needed to accurately represent it and too keep the error as small as possible. The influence of epsilon is relatively small though, as even when epsilon is 2^{-9}, the corners are still very rounded.

The method used to compute the lagrange polynomial has little influence on its accuracy in most cases. For high numbers of nodes (> 65), Neville's Method is more accurate than the standard lagrange method, but because the polynomial starts to oscillate violently, this difference does not matter, as the error is very large in all cases. The most significant difference between the two methods is again their speed. Neville's Method, thanks to its recursive properties, is 2 to 3 times faster than the normal lagrange method.

Problem 5

Convergence of Interpolation

This cosine function is very interesting because its periodic properties require a high number of nodes to achieve acceptable accuracy, for example to get within 0.01 of the test function at epsilon = 2^{-5} one needs 59 nodes on a chebyshev grid. On a uniform grid, the number, if N < 127, is 4. This does not meet the error criteria, but it is the best approximation with N < 127 (and considering Runge's Phenomenon, probably also for all other values). This is because all values above four do not fit the curve as well and then quickly start to oscillate out of control. Here, Runge's Phenomenon is very pronounced, I assume because the oscillation of the test function adds itself to the oscillation of the polynomial. For a chebyshev grid, the interpolation is convergent at least up to N >= 513, I have not encountered divergence at all. This means that increasing N leads to ever higher accuracy if one is using a chebyshev grid, and stops increasing accuracy for uniform grids. For this problem, one really needs to use chebyshev grids if one wants to have good accuracy, as uniform grids are not feasible considering Runge's Phenomenon.

Accuracy of Interpolation

The accuracy of the interpolation function of problem 5 is again dependent on N and epsilon. The larger N gets, the higher the accuracy of the interpolation gets if one is using chebyshev grids and not uniform ones. In this problem, epsilon's influence is really big, as it directly influences the frequency of the oscillation of the cosine wave. This means that if you divide epsilon by two, the number of oscillations of the function double. N is also very important, as you need at least three nodes per period of the cosine wave to represent it, this means that N gets really big if one wants high accuracy.

The method used to compute the lagrange polynomial has little influence on its accuracy. For high numbers of nodes (>= 65), Neville's Method is more accurate than the standard lagrange method, but because the polynomial starts to oscillate violently, this difference does not matter, as the error is very large in all cases. The most significant difference between the two methods is again their speed. Neville's Method, thanks to its recursive properties, is 2 to 3 times faster than the normal lagrange method.

Problem 7

Convergence of Interpolation

This function is very interesting because its shape, the very flat ends and a sharp "spike" in the middle, makes it nearly impossible to approximate it using uniform grids. Runge's Phenomenon is too strong to use anything but chebyshev nodes. To get within 0.01 of the test function at epsilon = 2^{-7} one needs 53 nodes on a chebyshev grid. On a uniform grid, the number, if N < 127, is 10. This is because all values above 10 do not fit the curve as well and then quickly start to oscillate out of control. The minimal error that can be achieved using uniform grids in this way is 0.55 instead of the given 0.01. Runge's Phenomenon is very pronounced in this problem as well, I assume because the shape of the function lends itself to oscillating polynomials. For a chebyshev grid, the interpolation is convergent at least up to N >= 513, I have not encountered divergence at all. This means that increasing N leads to ever higher accuracy if one is using a chebyshev grid, and stops increasing accuracy for uniform grids very quickly and even with small values. For this problem, one really needs to use chebyshev grids if one wants to have good accuracy, as uniform grids are not feasible considering Runge's Phenomenon.

Accuracy of Interpolation

The larger N gets, the higher the accuracy of the interpolation gets if one is using chebyshev grids and not uniform ones. In this problem, epsilon's influence is relatively big, as it dictates how sharp and pronounced the spike of the function is and how close the ends are to the x-axis. N is also very important, as one needs a lot of points close together to represent the ofte sharp transition from the ends of the function to the spike, where the slope is very large.

In this problem, the method used to calculate the interpolation polynomial does not have any measurable influence on the accuracy. The most significant difference between the two methods is again their speed. Neville's Method, thanks to its recursive properties, is 2 to 3 times faster than the normal lagrange method.

Problem 14

Convergence of Interpolation

As my problem 14 I chose the Dirac delta function. It is very similar in shape to problem 7, the only difference being that its max value is not 1 but theoretically infinity. It starts out a lot flatter than problem 7 for large epsilon values and becomes a lot pointier for small values of epsilon. This function is also very interesting because its shape, the very flat ends and a sharp "spike" in the middle, makes it nearly impossible to approximate it using uniform grids. Runge's Phenomenon is too strong to use anything but chebyshev nodes. To get within 0.01 of the test function at epsilon = 2^{-4} one needs 39 nodes on a chebyshev grid. On a uniform grid, the number, if N < 127, is 10. This is because all values above 10 do not fit the curve as well and then quickly start to oscillate out of control. The minimal error that can be achieved using uniform grids in this way is 4 instead of the given 0.01.

Runge's Phenomenon is the most pronounced in this problem out of all the given problems, I assume because the shape of the function lends itself to oscillating polynomials. For a chebyshev grid, the interpolation is convergent at least up to N >= 513, I have not encountered divergence at all. This means that increasing N leads to ever higher accuracy if one is using a chebyshev grid, and stops increasing accuracy for uniform grids very quickly and even with small values. For this problem, one really needs to use chebyshev grids if one wants to have good accuracy, as uniform grids are not feasible considering Runge's Phenomenon.

Accuracy of Interpolation

The larger N gets, the higher the accuracy of the interpolation gets if one is using chebyshev grids and not uniform ones. In this problem, epsilon's influence is relatively big, as it dictates how sharp and pronounced the spike of the function is and how close the ends are to the x-axis, again, this is even more extreme than in problem 7. N is also very important, as one needs a lot of points close together to represent the ofte sharp transition from the ends of the function to the spike, where the slope is very large.

The method used to compute the lagrange polynomial has little influence on its accuracy. The most significant difference between the two methods is again their speed. Neville's Method, thanks to its recursive properties, is 2 to 3 times faster than the normal lagrange method.