An Overview of the $3x + 1$ Problem

Moritz M. Konarski

Applied Mathematics Department
American University of Central Asia

March 5, 2020

Outline

Introduction

The $3x+1$ Function

The $3x+1$ Problem is based on the **Collatz Function** [2]

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

When the $3x+1$ Problem is studied, the $3x+1$ **Function**

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

is used [2].

The $3x + 1$ Function

- $T(x)$ is a function in **number theory** [1]
- domain of $T(x)$ are positive integers, its range are positive integers
- mathematically, $T(x)$ maps $\mathbb{N} + 1 \to \mathbb{N} + 1$ [3]
- $T(x)$ has a **stopping time**, **total stopping time**, and **trajectory** for each $x \in \mathbb{N} + 1$ [2]
- $T(x)$ is repeatedly applied to an initial x

Collatz Conjecture

Conjecture

For all $x \in \mathbb{N} + 1$ there is a $k \in \mathbb{N} + 1$ such that $T^{(k)}(x) = 1$.

- starting at any positive integer x, k iterations of $T(x)$ will give the result 1 [2]
- the Collatz Conjecture has **not been proven** [2]

Possible Behavior of $T(x)$

$T(x)$ can:

1. reach 1, which is equivalent to entering the **trivial cycle** $\{2, 1, 2, 1, \dots\}$
2. enter a non-trivial cycle that does not include 1
3. diverge to infinity and not enter any type of cycle

The Collatz Conjecture states **1. always happens**. [1]

Background Information

- named after German mathematician Lothar Collatz
- problem circulated since the 1950s
- academic publications started in the 1970s [2]
- conjecture has been verified for over 10^{20} numbers [3]
- most recent progress was in September of 2019 [3]
- problem is still being actively researched

Reasons to Study the Problem

- problem is simple to state, but hard to prove
- remains unsolved after over 50 years of research [2]
- iterative mappings are currently a popular research topic [2]
- verifying numbers is computationally interesting [2]
- could yield results connected to prime factorization using 2 and 3 [2]

Mathematics is not ready for such problems.

— *Paul Erdös*

Details

Trajectory

- the trajectory of x under $T(x)$ is the set of successive iterations of $T(x)$ [1]
- it is also called the forward orbit $O^+(x)$ of x under $T(x)$ [1]
- trajectories can be graphed

$$O^+(x) := \{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\}$$

Example 1: Trajectory of $T(39)$

The trajectory for $T(39)$ graphed for k and $T^{(k)}(39)$

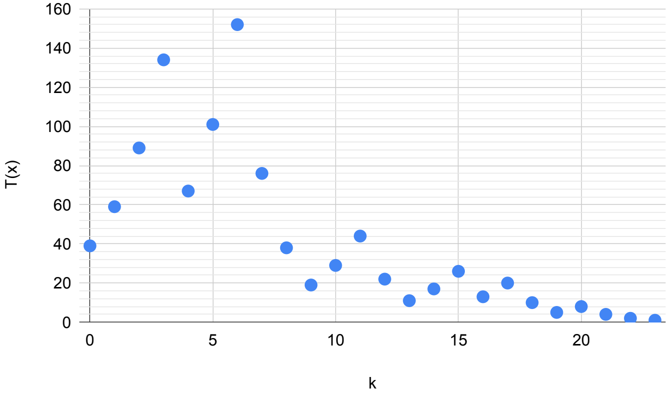

Figure 1: Trajectory of T(39)

Example 2: Trajectory of $T(27)$

The trajectory of $T(27)$ graphed for k and $T^{(k)}(27)$

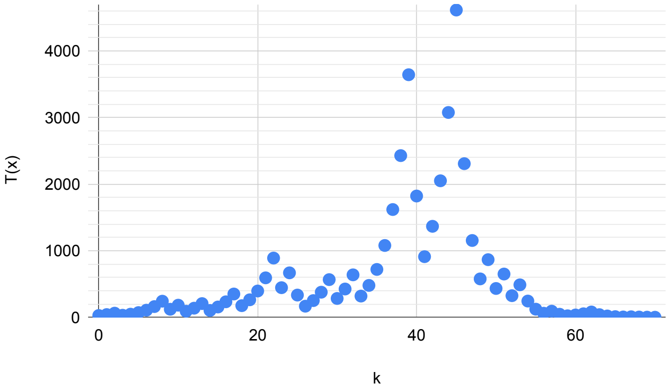

Figure 2: Trajectory of T(27)

Cycles

- $T(x)$ has the **trivial cycle** $\{2, 1, 2, \ldots\}$, which is equivalent to reaching 1 [1]
- the Collatz Conjecture states that **all orbits will eventually enter the trivial cycle** and thus that **it is the only cycle** [1]
- if $T(x)$ has non-trivial cycles, they have been proven to be over 10.4 billion numbers long [2]

Stopping Time

- the number of iterations of $T(x)$ until the result is smaller than x
- first it is checked that every positive integer up to $x-1$ iterates to 1
- then, if $T^{(k)}(x) < x$, we know it will iterate to 1
- if the Collatz Conjecture is true, all $x \in \mathbb{N}+1$ have a finite stopping time [1]

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\}$$

Example: Stopping time of $T(39)$

With the trajectory of $T(39)$

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, \mathbf{38}, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\},$$

we see that 38 is the first number $<$39.
Thus $\sigma(39) = 8$, as 38 is the result of the 8th iteration.

Total Stopping Time

The total stopping time is the number of steps needed for $T(x)$ to iterate to 1. By [1] it is defined as

$$\sigma_\infty(x) = \inf\{k : T^{(k)}(x) = 1\}$$

Example for $\sigma_\infty(39)$

For $T(39)$,

$$O^+(39) := \{39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$$

and we see that $T^{(23)}(39) = 1$, so $\sigma_\infty(39) = 23$.

Stochastic Approximations

- each trajectory has approximately the same number of odd and even elements [2]
- the behavior of $T(x)$ is pseudorandom for large numbers [2]
- thus, probabilistic models describe its behavior
- these models describe groups of trajectories [2]
- e.g., the upper bound for σ_∞ is $41.677647 \log x$ [2]

Example: Stopping Time Approximations

The total stopping time for most trajectories is approximated to be about $6.95212 \log x$ steps [2].

Example for $T(39)$

For $T(39)$ we have the approximation

$$6.95212 \log 39 \approx 25.4952$$

Compared to the known $\sigma_\infty(39) = 23$ this is not bad.

Summary

Summary

- the Collatz Conjecture states that for $x, k \in \mathbb{N} + 1$ $T^{(k)}(x) = 1$
- the conjecture has not been proven, but verified for 10^{20} numbers
- all orbits of $T(x)$ should reach the trivial cycle
- $T(x)$ can be probabilistically described because of pseudo-randomness

References

[1] Marc Chamberland, *An Update on the $3x+1$ Problem*, http://www.math.grinnell.edu/~chamberl/papers/3x_survey_eng.pdf, 2005.

[2] Jeffrey C.Lagarias, *The $3x+1$ Problem: An Overview*, https://pdfs.semanticscholar.org/100046dd8b4ee901bc71043da7d42f5d87ca0224.pdf, 2010.

[3] Terence Tao, *Almost All Orbits of the Collatz Map Attain Almost Bounded Values*, arXiv:1909.03562v2 [math.PR], 2019.