Linear Mappings

Real and Complex Linear Functions

- A real function of the form $f(x) = ax + b$, where a and b are any real constants, is called a linear function.
- By analogy, we define a **complex linear function** to be a function of the form

$$f(z) = az + b,$$

  where a and b are any complex constants.
- Just as real linear functions are the easiest types of real functions to graph, complex linear functions are the easiest types of complex functions to visualize as mappings of the complex plane.
- We will show that every nonconstant complex linear mapping can be described as a composition of three basic types of motions:
  - a translation,
  - a rotation, and
  - a magnification.

Translations

- We use the symbols T, R and M to represent mapping by translation, rotation, and magnification, respectively.

Definition (Translation)

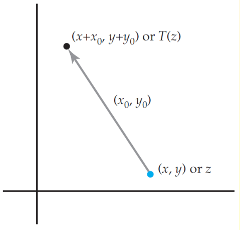
A complex linear function

$$T(z) = z + b, \quad b \neq 0,$$

is called a **translation**.

- If we set $z = x + iy$ and $b = x_0 + iy_0$, then we obtain:

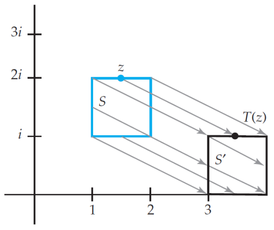
$$T(z) = (x + iy) + (x_0 + iy_0) = x + x_0 + i(y + y_0).$$

- 

The linear mapping $T(z) = z + b$ can be visualized in a single copy of the complex plane as the process of translating the point z along the vector representation (x_0, y_0) of b to the point $T(z)$.

Image of a Square under Translation

- Find the image S' of the square S with vertices at $1+i, 2+i, 2+2i$ and $1+2i$ under the linear mapping $T(z) = z + 2 - i$.
  We will represent S and S' in the same copy of the complex plane.
  The mapping T is a translation. Identify $b = x_0 + iy_0 = 2 + i(-1)$.
  Plot the vector $(2, -1)$ originating at each point in S.



The set of terminal points of these vectors is S'. S' is a square with vertices at: $T(1+i) = 3$, $T(2+i) = 4$, $T(2+2i) = 4+i$, $T(1+2i) = 3+i$. Therefore, the blue square S is mapped onto the black square S' by the translation $T(z) = z + 2 - i$.

Rotations

- A translation does not change the shape or size of a figure in the complex plane, i.e., the image of a line, circle, or triangle under a translation will also be a line, circle, or triangle, respectively. A mapping with this property is sometimes called a **rigid motion**.

Definition (Rotation)

A complex linear function

$$R(z) = az, \quad |a| = 1,$$

is called a **rotation**.

- If α is any nonzero complex number, then $a = \frac{\alpha}{|\alpha|}$ is a complex number for which $|a| = 1$.
- So, for any nonzero complex number α, we have that $R(z) = \frac{\alpha}{|\alpha|}z$ is a rotation.

Description of Rotations

- Consider the rotation $R = az$ and assume that $\text{Arg}(a) > 0$. Since $|a| = 1$ and $\text{Arg}(a) > 0$, we can write a in exponential form as $a = e^{i\theta}$, with $0 < \theta \leq \pi$. If we set $a = e^{i\theta}$ and $z = re^{i\phi}$, then we obtain the following description of R:

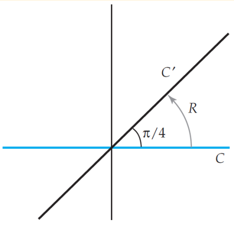
$$R(z) = e^{i\theta} re^{i\phi} = re^{i(\theta+\phi)}.$$

  - The modulus of $R(z)$ is r, which is the same as the modulus of z. Therefore, if z and $R(z)$ are plotted in the same copy of the complex plane, then both points lie on a circle centered at 0 with radius r.
  - An argument of $R(z)$ is $\theta + \phi$, which is θ radians greater than an argument of z. Therefore, $R(z) = az$ rotates z counterclockwise through an angle of θ radians about the origin to $R(z)$.
- If $\text{Arg}(a) < 0$, then the linear mapping $R(z) = az$ can be visualized in a single copy of the complex plane as the process of rotating points clockwise through an angle of θ radians about the origin.
- The angle $\theta = \text{Arg}(a)$ is called an **angle of rotation** of R.

Image of a Line under Rotation

- Find the image of the real axis $y = 0$ under the linear mapping $R(z) = (\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i)z$.
- Let C denote the real axis $y = 0$ and let C' denote the image of C under R. Since $|\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i| = 1$, the complex mapping $R(z)$ is a rotation. In order to determine the angle of rotation, we write $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$ in exponential form $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i = e^{i\pi/4}$.

  If z and $R(z)$ are plotted in the same copy of the complex plane, then the point z is rotated counterclockwise through $\frac{\pi}{4}$ radians about the origin to the point $R(z)$. The image C' is, therefore, the line $v = u$, which contains the origin and makes an angle of $\frac{\pi}{4}$ radians with the real axis.



Magnifications

- Rotations will not change the shape or size of a figure in the complex plane either.

Definition (Magnification)

A complex linear function

$$M(z) = az, \quad a > 0,$$

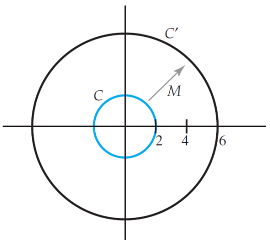
is called a **magnification**.

- It is implicit in $a > 0$ that the symbol a represents a real number.
- If $z = x + iy$, then $M(z) = az = ax + iay$. So the image of the point (x, y) is the point (ax, ay). If $z = re^{i\theta}$, $M(z) = a(re^{i\theta}) = (ar)e^{i\theta}$, so that the magnitude of $M(z)$ is ar.
  - If $a > 1$, then the complex points z and $M(z)$ have the same argument θ, but different moduli $r \neq ar$. $M(z)$ is the unique point on the ray emanating from 0 and containing z whose distance from 0 is a times further than z. a is called the **magnification factor** of M.
  - If $0 < a < 1$, then the point $M(z)$ is a times closer to the origin than the point z. This case of a magnification is a **contraction**.

Image of a Circle under Magnification

- Find the image of the circle C given by $|z| = 2$ under the linear mapping $M(z) = 3z$.
  Since M is a magnification with magnification factor of 3, each point on the circle $|z| = 2$ will be mapped onto a point with the same argument but with modulus magnified by 3.
  - Thus, each point in the image will have modulus $3 \cdot 2 = 6$.
  - The image points can have any argument since the points z in the circle $|z| = 2$ can have any argument.

  Therefore, the image C' is the circle $|w| = 6$, centered at the origin with radius 6.



Composition and Mappings

- A magnification mapping will change the size of a figure in the complex plane, but it will not change its basic shape.
- We will now show that a general linear mapping $f(z) = az + b$ is a composition of a rotation, a magnification, and a translation.
- Recall that if f and g are two functions, then the **composition of** f **and** g is the function $f \circ g$ defined by

$$f \circ g(z) = f(g(z)).$$

- The value $w = f \circ g(z)$ is determined by
    - first evaluating the function g at z;
    - and, then, evaluating the function f at $g(z)$.
- In a similar manner, the image, S'', of set S under a composition $w = f \circ g(z)$ is determined by
    - first finding the image S' of S under g;
    - and, then, finding the image S'' of S' under f.

Linear Mappings

- Suppose that $f(z) = az + b$ is a complex linear function, with $a \neq 0$ (if $f(z) = b$, every point is mapped onto the single point b).
- We can express f as:

$$f(z) = az + b = |a| \cdot \frac{a}{|a|} z + b.$$

- Consider a point z_0:
  - First, z_0 is multiplied by the complex number $\frac{a}{|a|}$. Since $\left|\frac{a}{|a|}\right| = \frac{|a|}{|a|} = 1$, the complex mapping $w = \frac{a}{|a|} z$ is a rotation that rotates the point z_0 through an angle of $\theta = \text{Arg}(\frac{a}{|a|})$ radians about the origin. The angle of rotation can also be written as $\theta = \text{Arg}(a)$, since $\frac{1}{|a|}$ is a real number. Let z_1 be the image of z_0 under this rotation by $\text{Arg}(a)$.
  - Then z_1 is multiplied by $|a|$. Because $|a| > 0$ is a real number, the complex mapping $w = |a|z$ is a magnification with a magnification factor $|a|$. Let z_2 be the image of z_1 under magnification by $|a|$.
  - The last step is to add b to z_2. The complex mapping $w = z + b$ translates z_2 by b onto the point $w_0 = f(z_0)$.

Image of a Point under a Linear Mapping

- Let $f(z) = az + b$ be a linear mapping with $a \neq 0$ and let z_0 be a point in the complex plane.
- If the point $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by
  (i) rotating z_0 through an angle of $\text{Arg}(a)$ about the origin;
  (ii) magnifying the result by $|a|$, and
  (iii) translating the result by b.
- The image S' of a set S under $f(z) = az + b$ is the set of points obtained by
  - rotating S through $\text{Arg}(a)$,
  - magnifying by $|a|$,
  - and, then, translating by b.
- Thus, every nonconstant complex linear mapping is a composition of at most one rotation, one magnification, and one translation.

Example and Remarks

- Example: The linear mapping $f(z) = 3z + i$ involves
  - a magnification by 3,
  - and a translation by i.
- If $a \neq 0$ is a complex number, and if
  - $R(z)$ is a rotation through $\mathrm{Arg}(a)$,
  - $M(z)$ is a magnification by $|a|$, and
  - $T(z)$ is a translation by b,

  then the composition $f(z) = T \circ M \circ R(z) = T(M(R(z)))$ is a complex linear function.
- Since the composition of any finite number of linear functions is again a linear function, it follows that the composition of finitely many rotations, magnifications, and translations is a linear mapping.

Preservation of Shapes and Order of Composition

- Since translations, rotations, and magnifications all preserve the basic shape of a figure in the complex plane, a linear mapping will also preserve the basic shape of a figure in the complex plane.
- A complex linear mapping $w = az + b$, with $a \neq 0$, can distort the size of a figure, but it cannot alter the basic shape of the figure.
- When writing a linear function as a composition of a rotation, a magnification and a translation, the order is important.

  Example: The mapping $f(z) = 2z + i$ magnifies by 2, then translates by i. Thus, 0 maps onto i. If we reverse the order of composition, i.e., translate by i, then magnify by 2, the effect is 0 maps onto $2i$.
- A complex linear mapping can always be represented as a composition in more than one way.

  Example: $f(z) = 2z + i$ can also be expressed as $f(z) = 2(z + i/2)$. Therefore, a magnification by 2 followed by translation by i is the same mapping as translation by $\frac{i}{2}$ followed by magnification by 2.

Image of a Rectangle under a Linear Mapping

- Find the image of the rectangle with vertices $-1+i$, $1+i$, $1+2i$, and $-1+2i$ under the linear mapping $f(z) = 4iz + 2 + 3i$.

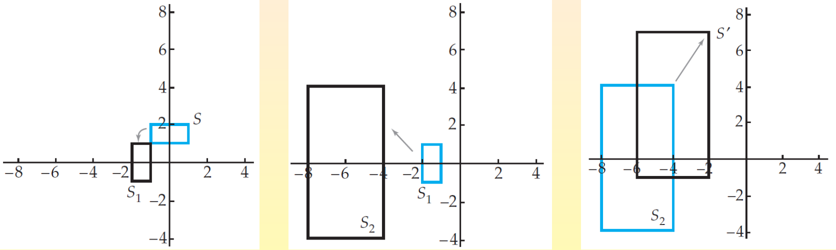
  Let S be the rectangle with the given vertices and let S' denote the image of S under f. Because f is a linear mapping, S' has the same shape as S, i.e., it is a rectangle. Thus, in order to determine S', we need only find its vertices, which are the images of the vertices of S under f:

$$\begin{aligned} f(-1+i) &= -2-i & \qquad f(1+i) &= -2+7i \\ f(1+2i) &= -6+7i & \qquad f(-1+2i) &= -6-i. \end{aligned}$$

  Therefore, S' is the rectangle with vertices $-2-i, -2+7i, -6+7i$ and $-6-i$.

Alternative Point of View

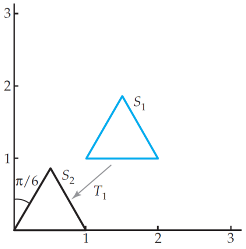
- The linear mapping $f(z) = 4iz + 2 + 3i$ is a composition of
  - a rotation through $\text{Arg}(4i) = \frac{\pi}{2}$ radians;
  - a magnification by $|4i| = 4$ and
  - a translation by $2 + 3i$.



A Linear Mapping of a Triangle I

- Find a complex linear function that maps the equilateral triangle with vertices $1+i, 2+i$ and $\frac{3}{2}+(1+\frac{1}{2}\sqrt{3})i$ onto the equilateral triangle with vertices $i, \sqrt{3}+2i$ and $3i$.

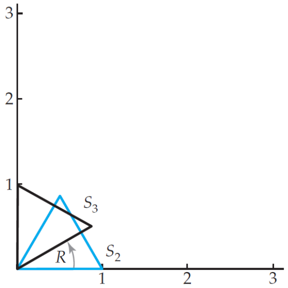
  Let S_1 denote the triangle with vertices $1+i, 2+i$ and $\frac{3}{2}+(1+\frac{1}{2}\sqrt{3})i$ and let S' represent the triangle with vertices $i, 3i$ and $\sqrt{3}+2i$.
  - We first translate S_1 to have one of its vertices at the origin. If $1+i$ should be mapped onto 0, then this is accomplished by the translation $T_1(z)=z-(1+i)$. Let S_2 be the image of S_1 under T_1.



Note that the angle between the imaginary axis and the edge of S_2 containing the vertices 0 and $\frac{1}{2}+\frac{1}{2}\sqrt{3}i$ is $\frac{\pi}{6}$.

A Linear Mapping of a Triangle II

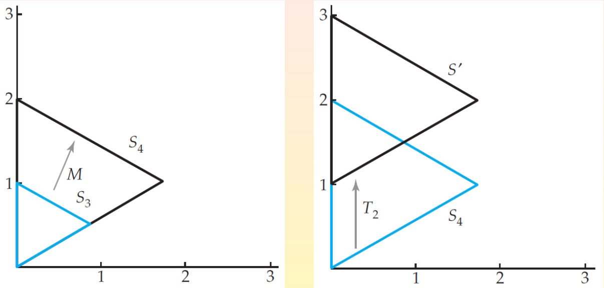
- A rotation through an angle of $\frac{\pi}{6}$ radians counterclockwise about the origin will map S_2 onto a triangle with two vertices on the imaginary axis. This rotation is given by $R(z) = (e^{i\pi/6})z = (\frac{1}{2}\sqrt{3} + \frac{1}{2}i)z$. The image of S_2 under R is the triangle S_3 with vertices at 0, $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ and i:



It is easy to verify that each side of the triangle S_3 has length 1.

A Linear Mapping of a Triangle III

- Because each side of the desired triangle S' has length 2, we next magnify S_3 by a factor of 2. The magnification $M(z) = 2z$ maps the triangle S_3 onto the triangle S_4 with vertices 0, $\sqrt{3}+i$, and $2i$:



- Finally, we translate S_4 by i using the mapping $T_2(z) = z + i$. This maps triangle S_4 onto the triangle S' with vertices i, $\sqrt{3}+2i$, and $3i$.
- Thus, the linear mapping: $f(z) = T_2 \circ M \circ R \circ T_1(z) = (\sqrt{3}+i)z + 1 - \sqrt{3} - \sqrt{3}i$ maps the triangle S_1 onto the triangle S'.

Linear Approximations

- The study of differential calculus is based on the principle that real linear functions are the easiest types of functions to understand. One of the many uses of the derivative is to find a linear function that approximates f in a neighborhood of a point x_0.
- The linear approximation of a differentiable function $f(x)$ at $x = x_0$ is the linear function $\ell(x) = f(x_0) + f'(x_0)(x - x_0)$. Geometrically, the graph of $\ell(x)$ is the tangent line to the graph of f at $(z_0, f(z_0))$.
- The linear approximation formula can be applied to complex functions once an appropriate definition of the derivative of complex function is given. If $f'(z_0)$ represents the derivative of the complex function $f(z)$ at z_0, then the linear approximation of f in a neighborhood of z_0 is the complex linear function $\ell(z) = f(z_0) + f'(z_0)(z - z_0)$. Geometrically, $\ell(z)$ approximates how $f(z)$ acts as a complex mapping near the point z_0.

An Example

- The derivative of the complex function $f(z) = z^2$ is $f'(z) = 2z$.
- Therefore, the linear approximation of $f(z) = z^2$ at $z_0 = 1 + i$ is

$$\ell(z) = 2i + 2(1+i)(z - 1 - i) = 2\sqrt{2}(e^{i\pi/4}z) - 2i.$$

  Near the point $z_0 = 1 + i$ the mapping $w = z^2$ can be approximated by the linear mapping consisting of the composition of:
  - rotation through $\frac{\pi}{4}$,
  - magnification by $2\sqrt{2}$,
  - and translation by $-2i$.
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  The image of the circle $|z - (1+i)| = 0.25$ under both f and ℓ are shown on the right.