Exponential Functions

Complex Exponential Function

- We repeat the definition of the complex exponential function:

Definition (Complex Exponential Function)

The function e^z defined by

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function**.

- This function agrees with the real exponential function when z is real: in fact, if $z = x + 0i$,
$$e^{x+0i} = e^x(\cos 0 + i \sin 0) = e^x(1 + i \cdot 0) = e^x.$$
- The complex exponential function also shares important differential properties of the real exponential function:
  - e^x is differentiable everywhere;
  - $\frac{d}{dx} e^x = e^x$, for all x.

Analyticity of e^z

Theorem (Analyticity of e^z)

The exponential function e^z is entire and its derivative is $\frac{d}{dz}e^z = e^z$.

- We use the criterion based on the real and imaginary parts. $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are continuous real functions and have continuous first-order partial derivatives, for all (x, y). In addition, the Cauchy-Riemann equations in u and v are easily verified: $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$. Therefore, the exponential function e^z is entire. The derivative of an analytic function f is given by $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. So the derivative of e^z is: $\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z$.
- Since the real and imaginary parts of an analytic function are harmonic conjugates, we can show the only entire function f that agrees with the real exponential function e^x for real input and that satisfies $f'(z) = f(z)$ is the complex exponential function e^z.

Derivatives of Exponential Functions

- Find the derivative of each of the following functions:
  (a) $iz^4(z^2 - e^z)$
  (b) $e^{z^2-(1+i)z+3}$
- We use the various rules for complex derivatives:
  (a)
$$\begin{aligned}\frac{d}{dz}(iz^4(z^2 - e^z)) &= \frac{d}{dz}(iz^4)(z^2 - e^z) + iz^4\frac{d}{dz}(z^2 - e^z)\\ &= 4iz^3(z^2 - e^z) + iz^4(2z - e^z)\\ &= 6iz^5 - iz^4e^z - 4iz^3e^z.\end{aligned}$$
  (b)
$$\begin{aligned}\frac{d}{dz}(e^{z^2-(1+i)z+3}) &= e^{z^2-(1+i)z+3} \cdot \frac{d}{dz}(z^2 - (1+i)z + 3)\\ &= e^{z^2-(1+i)z+3} \cdot (2z - 1 - i).\end{aligned}$$

Modulus, Argument, and Conjugate

- If we express the complex number $w = e^z$ in polar form:
$$w = e^x \cos y + ie^x \sin y = r(\cos\theta + i\sin\theta),$$
we see that $r = e^x$ and $\theta = y + 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$.
- Because r is the modulus and θ is an argument of w, we have:
$$|e^z| = e^x, \quad \arg(e^z) = y + 2n\pi, n = 0, \pm 1, \pm 2, \ldots.$$
- We know from calculus that $e^x > 0$, for all real x, whence $|e^z| > 0$. This implies that $e^z \neq 0$, for all complex z, i.e., $w = 0$ is not in the range of $w = e^z$.
- Note, however, that e^z may be a negative real number: E.g., if $z = \pi i$, then $e^{\pi i}$ is real and $e^{\pi i} < 0$.
- A formula for the conjugate of the complex exponential e^z is found using the even-odd properties of the real cosine and sine functions: $\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\overline{z}}$. Therefore, for all complex z, $\overline{e^z} = e^{\overline{z}}$.

Algebraic Properties

Theorem (Algebraic Properties of e^z)

If z_1 and z_2 are complex numbers, then:

(i) $e^0 = 1$;

(ii) $e^{z_1} e^{z_2} = e^{z_1+z_2}$;

(iii) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.

(iv) $(e^{z_1})^n = e^{nz_1}$, $n = 0, \pm 1, \pm 2, \ldots$.

(i) Clearly, $e^{0+0i} = e^0(\cos 0 + i \sin 0) = 1$.

(ii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence $e^{z_1} e^{z_2} = (e^{x_1} \cos y_1 + ie^{x_1} \sin y_1)(e^{x_2} \cos y_2 + ie^{x_2} \sin y_2) = e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + ie^{x_1+x_2}(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)$. Using the addition formulas for the real cosine and sine functions, we get $e^{z_1} e^{z_2} = e^{x_1+x_2} \cos(y_1 + y_2) + ie^{x_1+x_2} \sin(y_1 + y_2)$. The right-hand side is $e^{z_1+z_2}$.

- The proofs of (iii) and (iv) are similar.

Periodicity

- The most striking difference between the real and complex exponential functions is the periodicity of e^z.
- We say that a complex function f is **periodic** with **period** T if

$$f(z+T) = f(z), \quad \text{for all complex } z.$$

- The real exponential function is not periodic, but the complex exponential function is because it is defined using the real cosine and sine functions, which are periodic.
- We have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z.$$

The complex exponential function e^z is periodic with a pure imaginary period $2\pi i$.

That is, for $f(z) = e^z$, we have $f(z+2\pi i) = f(z)$, for all z.