# Special Power Functions

# Complex Polynomial and Principal Root Functions

- A **complex polynomial function** is a function of the form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $n$ is a positive integer and $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex constants.
- In general, a complex polynomial mapping can be quite complicated, but in many special cases the action of the mapping is easily understood.
- We now study complex polynomials of the form $f(z) = z^n, n \geq 2$.
- Unlike the linear mappings, the mappings $w = z^n, n \geq 2$, do not preserve the basic shape of every figure in the complex plane.
- Associated to the function $z^n, n \geq 2$, we also have the *principal nth root function* $z^{1/n}$.

  The principal $n$th root functions are inverse functions of the functions $z^n$ defined on a sufficiently restricted domain.

# Power Functions

- A real function of the form $f(x) = x^a$, where $a$ is a real constant, is called a **power function**.
- We form a complex power function by allowing the input or the exponent $a$ to be a complex number.
- A **complex power function** is a function of the form

$$f(z) = z^{\alpha}, \quad \alpha \text{ a complex constant.}$$

- If $\alpha$ is an integer, then the power function $z^{\alpha}$ can be evaluated using the algebraic operations on complex numbers seen earlier:
  Example: $z^2 = z \cdot z$ and $z^{-3} = \frac{1}{z \cdot z \cdot z}$.
- We can also use the formulas for taking roots of complex numbers to define power functions with fractional exponents of the form $\frac{1}{n}$.
- We restrict attention to special complex power functions of the form $z^n$ and $z^{1/n}$, where $n \geq 2$ and $n$ is an integer.
- More complicated complex power functions such as $z^{\sqrt{2}-i}$, will be discussed after the introduction of the complex logarithmic function.

# The Power Function $z^n$

- We consider complex power functions of the form $z^n$, $n \geq 2$.
- We begin with the simplest of these functions, the **complex squaring function** $z^2$.
- Values of the complex power function $f(z) = z^2$ are easily found using complex multiplication.

  Example: At $z = 2 - i$, we have

$$f(2-i) = (2-i)^2 = (2-i) \cdot (2-i) = 3 - 4i.$$

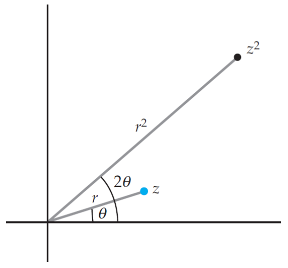
- We express $w = z^2$ in exponential notation by replacing $z$ with $re^{i\theta}$:

$$w = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}.$$

  - The modulus $r^2$ of the point $w$ is the square of the modulus $r$ of the point $z$;
  - The argument $2\theta$ of $w$ is twice the argument $\theta$ of $z$.

# The Complex Squaring Function $z^2$

- If we plot both $z$ and $w$ in the same copy of the complex plane, then $w$ is obtained by magnifying $z$ by a factor of $r$ and then by rotating the result through the angle $\theta$ about the origin.




- The figure shows $z$ and $w = z^2$, when $r > 1$ and $\theta > 0$.
- If $0 < r < 1$, then $z$ is contracted by a factor of $r$, and if $\theta < 0$, then the rotation is clockwise.

# Magnification and Rotation in Complex Squaring

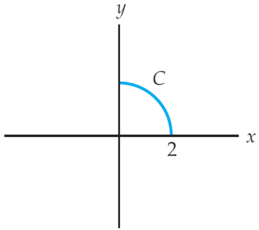
- The magnification factor and the rotation angle associated to $w = f(z) = z^2$ depend on where $z$ is located in the complex plane.
  Example: Since $f(2) = 4$ and $f(\frac{i}{2}) = -\frac{1}{4}$, the point $z = 2$ is magnified by 2 but not rotated, whereas the point $z = \frac{i}{2}$ is contracted by $\frac{1}{2}$ and rotated through $\frac{\pi}{2}$.
- The function $z^2$ does not magnify the modulus of points on the unit circle $|z| = 1$ and it does not rotate points on the positive real axis.
- Consider a ray emanating from the origin and making an angle of $\phi$ with the positive real axis.
  - The images of all points have an argument of $2\phi$. Thus, they lie on a ray emanating from the origin and making an angle of $2\phi$ with the positive real axis.
  - The modulus $\rho$ of a point on the ray can be any value in $[0, \infty]$. So the modulus $\rho^2$ of a point in the image can also be any value in $[0, \infty]$.

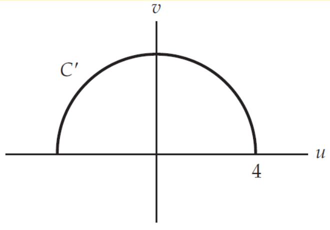
  Hence, the ray is mapped onto a ray emanating from the origin making an angle $2\phi$ with the positive real axis.

# Image of a Circular Arc under $w = z^2$

- Find the image of the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^2$.
  Let $C$ be the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, and let $C'$ denote the image of $C$ under $w = z^2$.
  - Since each point in $C$ has modulus 2, each point in $C'$ has modulus $2^2 = 4$. Thus, the image $C'$ must be contained in the circle $|w| = 4$.
  - Since the arguments of the points in $C$ take on every value in $[0, \frac{\pi}{2}]$, the points in $C'$ have arguments that take on every value in $[0, \pi]$.

  So $C'$ is the semicircle defined by $|w| = 4, 0 \leq \arg(w) \leq \pi$.



# Alternative Solution

- An alternative way to find the image of the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^2$ is to use a parametrization.

  The circular arc $C$ can be parametrized by $z(t) = 2e^{it}, 0 \leq t \leq \frac{\pi}{2}$. Its image $C'$ is given by $w(t) = f(z(t)) = 4e^{i2t}, 0 \leq t \leq \frac{\pi}{2}$. By replacing the parameter $t$ with $s = 2t$, we obtain $W(s) = 4e^{is}, 0 \leq s \leq \pi$. This is a parametrization of the semicircle $|w| = 4, 0 \leq \arg(w) \leq \pi$.

- Similarly, the squaring function maps a semicircle

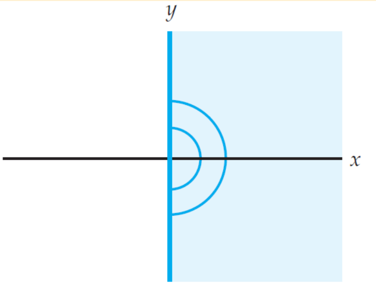
$$|z| = r, -\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2},$$

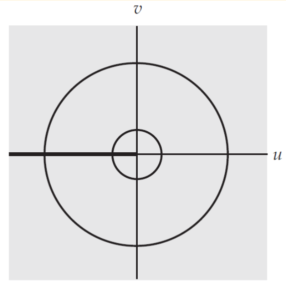
  onto a circle $|w| = r^2$.

# Mapping of a Half-Plane onto the Entire Plane

- Since the right half-plane $\mathrm{Re}(z) \geq 0$ consists of the collection of semicircles $|z| = r$, $-\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2}$, where $r$ takes on every value in the interval $[0, \infty)$, the image of this half-plane consists of the collection of circles $|w| = r^2$ where $r$ takes on any value in $[0, \infty)$.

  This implies that $w = z^2$ maps the right half-plane $\mathrm{Re}(z) \geq 0$ onto the entire complex plane.



# Image of a Vertical Line under $w = z^2$

- Find the image of the vertical line $x = k$ under the mapping $w = z^2$.

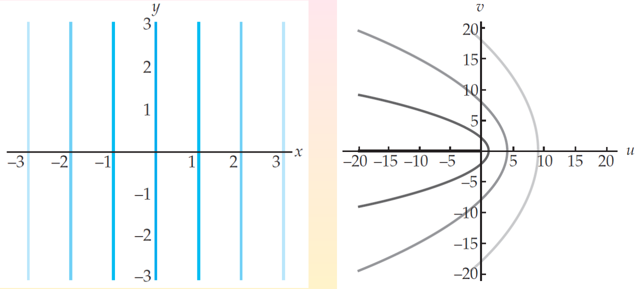
  In this example it is convenient to work with real and imaginary parts of $w = z^2$ which are

  $$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

  Since the vertical line $x = k$ consists of the points $z = k + iy, -\infty < y < \infty,$ it follows that the image of this line consists of all points $w = u + iv$, where $u = k^2 - y^2, v = 2ky.$ If $k \neq 0$, we get $y = \frac{v}{2k}$ and then $u = k^2 - \frac{v^2}{4k^2}, -\infty < v < \infty.$ Thus, the image of the line $x = k$ (with $k \neq 0$) under $w = z^2$ is a parabola that opens in the direction of the negative $u$-axis, has its vertex at $(k^2, 0)$, and has $v$-intercepts at $(0, \pm 2k^2).$ Since the image is unchanged if $k$ is replaced by $-k$, if $k \neq 0$, the pair of vertical lines $x = k$ and $x = -k$ are both mapped onto the parabola $u = k^2 - \frac{v^2}{4k^2}.$

# Image of a Vertical Line under $w = z^2$ (Cont'd)

- The action of the mapping $w = z^2$ on vertical lines is depicted below:



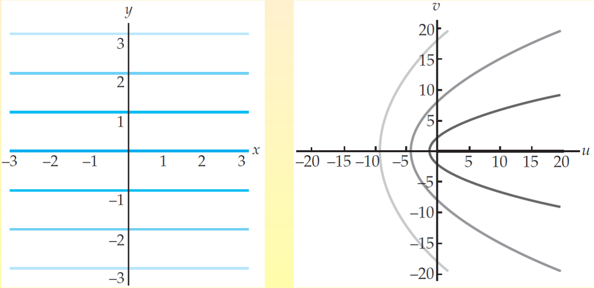

- The lines $x = 3$ and $x = -3$ are mapped onto the parabola with vertex at $(9, 0)$.
- Similarly, the lines $x = \pm 2$ are mapped onto the parabola with vertex at $(4, 0)$, and the lines $x = \pm 1$ onto the parabola with vertex at $(1, 0)$.
- In the case when $k = 0$, the image of the line $x = 0$ (the imaginary axis) is given by: $u = -y^2, v = 0, -\infty < y < \infty$. Therefore, the imaginary axis is mapped onto the negative real axis.

# Image of a Horizontal Line under $w = z^2$

- The same method can be used to show that a horizontal line $y = k$, $k \neq 0$, is mapped by $w = z^2$ onto the parabola

$$u = \frac{v^2}{4k^2} - k^2.$$

- The image is unchanged if $k$ is replaced by $-k$. So the pair $y = k$ and $y = -k$, $k \neq 0$, are both mapped onto the same parabola.
- If $k = 0$, then the horizontal line $y = 0$ (the real axis) is mapped onto the positive real axis.



# Image of a Triangle under $w = z^2$

- Find the image of the triangle with vertices $0, 1+i$ and $1-i$ under the mapping $w = z^2$.

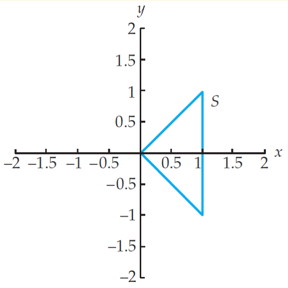
  Let $S$ denote the triangle with vertices at $0, 1+i$ and $1-i$, and let $S'$ denote its image under $w = z^2$.

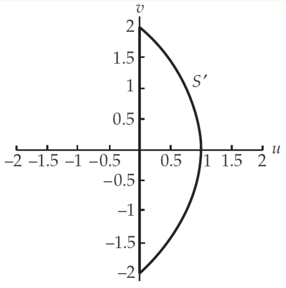
  - The side of $S$ containing the vertices 0 and $1+i$ lies on a ray emanating from the origin and making an angle of $\frac{\pi}{4}$ radians with the positive x-axis. The image of this segment must lie on a ray making an angle of $2\frac{\pi}{4} = \frac{\pi}{2}$ radians with the positive $u$-axis. Since the moduli of the points on the edge containing 0 and $1+i$ vary from 0 to $\sqrt{2}$, the moduli of the images of these points vary from 0 to 2. Thus, the image of this side is a vertical line segment from 0 to $2i$ contained in the $v$-axis.
  - In a similar manner, we find that the image of the side of $S$ containing the vertices 0 and $1-i$ is a vertical line segment from 0 to $-2i$ contained in the $v$-axis.

# Image of a Triangle under $w = z^2$ (Cont'd)

- We continue with the image of the triangle with vertices $0, 1+i$ and $1-i$ under the mapping $w = z^2$:
  - The remaining side of $S$ contains the vertices $1-i$ and $1+i$. This side consists of the set of points $z = 1+iy$, $-1 \le y \le 1$. Because this side is contained in the vertical line $x = 1$, its image is a parabolic segment given by: $u = 1 - \frac{v^2}{4}, -2 \le v \le 2$.

  Thus, we have shown that the image of triangle $S$ is the figure $S'$ shown below.



# The Function $z^n$, $n > 2$

- An analysis similar to that used for the mapping $w = z^2$ can be applied to the mapping $w = z^n, n > 2$.
- By replacing the symbol $z$ with $re^{i\theta}$ we obtain:

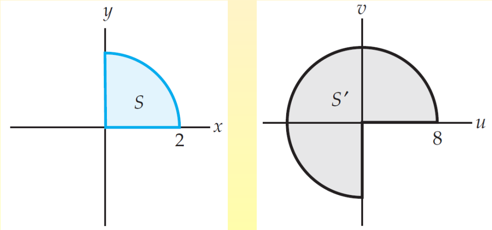
$$w = z^n = r^n e^{in\theta}.$$

- Consequently, if $z$ and $w = z^n$ are plotted in the same copy of the complex plane, then this mapping can be visualized as the process of
  - magnifying or contracting the modulus $r$ of $z$ to the modulus $r^n$ of $w$;
  - rotating $z$ about the origin to increase an argument $\theta$ of $z$ to an argument $n\theta$ of $w$.
- Example: A ray emanating from the origin and making an angle of $\phi$ radians with the positive $x$-axis is mapped onto a ray emanating from the origin and making an angle of $n\phi$ radians with the positive $u$-axis.

# Image of a Circular Wedge under $w = z^3$

- Determine the image of the quarter disk defined by the inequalities $|z| \leq 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^3$.
  Let $S$ denote the quarter disk and let $S'$ denote its image under $w = z^3$.
  - Since the moduli of the points in $S$ vary from 0 to 2 the moduli of the points in $S'$ vary from 0 to 8.
  - In addition, because the arguments of the points in $S$ vary from 0 to $\frac{\pi}{2}$, the arguments of the points in $S'$ vary from 0 to $\frac{3\pi}{2}$.

  Therefore, $S'$ is given by the inequalities $|w| \leq 8$, $0 \leq \arg(w) \leq \frac{3\pi}{2}$:



# The Power Function $z^{1/n}$

- We now investigate complex power functions of the form $z^{1/n}$, where $n$ is an integer and $n \geq 2$. We begin with $n = 2$.
- We have seen that the $n$ $n$-th roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ are given by:

$$\sqrt[n]{r}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right] = \sqrt[n]{r}e^{i(\theta+2k\pi)/n},$$

  for $k = 0, \ldots, n-1$.
- For $n = 2$, we get

$$\sqrt{r}\left[\cos\frac{\theta + 2k\pi}{2} + i\sin\frac{\theta + 2k\pi}{2}\right] = \sqrt{r}e^{i(\theta+2k\pi)/2}, \quad k = 0, 1.$$

- By setting $\theta = \mathrm{Arg}(z)$ and $k = 0$, we can define a function that assigns to $z$ the unique principal square root.

# The Principal Square Root Function

## Definition (The Principal Square Root Function)

The function $z^{1/2}$ defined by

$$z^{1/2} = \sqrt{|z|}e^{i\mathrm{Arg}(z)/2}$$

is called the **principal square root function**.

- If we set $\theta = \mathrm{Arg}(z)$ and replace $z$ with $re^{i\theta}$, then we obtain an alternative description of the principal square root function for $|z| > 0$:

$$z^{1/2} = \sqrt{r}e^{i\theta/2}, \quad r = |z| \text{ and } \theta = \mathrm{Arg}(z).$$

- Note that the symbol $z^{1/2}$, as used in the definition, represents something different from the same symbol as used previously.

# Values of the Principal Square Root Function

- Example: Find the values of the principal square root function $z^{1/2}$ at the following points: (a) $z = 4$ (b) $z = -2i$ (c) $z = -1 + i$.

(a) For $z = 4$, $|z| = |4| = 4$ and $\mathrm{Arg}(z) = \mathrm{Arg}(4) = 0$. Thus, $4^{1/2} = \sqrt{4}e^{i(0/2)} = 2e^{i(0)} = 2$.

(b) For $z = -2i$, $|z| = |-2i| = 2$ and $\mathrm{Arg}(z) = \mathrm{Arg}(-2i) = -\frac{\pi}{2}$, whence $(-2i)^{1/2} = \sqrt{2}e^{i(-\pi/2)/2} = \sqrt{2}e^{-i\pi/4} = 1 - i$.

(c) For $z = -1 + i$, $|z| = |-1 + i| = \sqrt{2}$ and $\mathrm{Arg}(z) = \mathrm{Arg}(-1 + i) = \frac{3\pi}{4}$, and, hence, $(-1 + i)^{1/2} = \sqrt{(\sqrt{2})}e^{i(3\pi/4)/2} = \sqrt[4]{2}e^{i(3\pi/8)}$.

# One-to-One Functions

- The principal square root function $z^{1/2}$ is an inverse function of the squaring function $z^2$.
- A real function must be one-to-one in order to have an inverse function. The same is true for a complex function.
- A complex function $f$ is **one-to-one** if each point $w$ in the range of $f$ is the image of a unique point $z$, called the **pre-image** of $w$, in the domain of $f$. That is, $f$ is one-to-one if whenever $f(z_1) = f(z_2)$, then $z_1 = z_2$. Equivalently, if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

  Example: The function $f(z) = z^2$ is not one-to-one because $f(i) = f(-i) = -1$.
- If $f$ is a one-to-one complex function, then for any point $w$ in the range of $f$ there is a unique pre-image in the $z$-plane, which we denote by $f^{-1}(w)$.
- This correspondence between a point $w$ and its pre-image $f^{-1}(w)$ defines the inverse function of a one-to-one complex function.

# Inverse Functions

## Definition (Inverse Function)

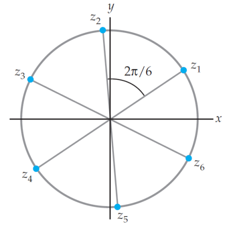
If $f$ is a one-to-one complex function with domain $A$ and range $B$, then the **inverse function** of $f$, denoted by $f^{-1}$, is the function with domain $B$ and range $A$ defined by

$$f^{-1}(z) = w \quad \text{if} \quad f(w) = z.$$

- If a set $S$ is mapped onto a set $S'$ by a one-to-one function $f$, then $f^{-1}$ maps $S'$ onto $S$.
- If $f$ has an inverse function, then $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$. I.e., the two compositions $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identities.
- Example: Show that the complex function $f(z) = z + 3i$ is one-to-one on the entire complex plane and find a formula for its inverse function.
  $f(z_1) = f(z_2)$ implies $z_1 + 3i = z_2 + 3i$ which implies $z_1 = z_2$.
  The inverse function of $f$ can often be found algebraically by solving the equation $z = f(w)$ for the symbol $w$: $z = w + 3i$ implies $w = z - 3i$. Therefore, $f^{-1}(z) = z - 3i$.

# Functions of $z^n$, $n \geq 2$, Not One-to-One

- The function $f(z) = z^n, n \geq 2$, is not one-to-one: Consider the points $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+2\pi/n)}$ with $r \neq 0$. Because $n \geq 2$, the points $z_1$ and $z_2$ are distinct. Note $f(z_1) = r^n e^{in\theta}$ and $f(z_2) = r^n e^{i(n\theta+2\pi)} = r^n e^{in\theta} e^{i2\pi} = r^n e^{in\theta}$. Therefore, $f$ is not one-to-one.
- In fact, the $n$ distinct points $z_1 = re^{i\theta}$, $z_2 = re^{i(\theta+2\pi/n)}$, $z_3 = re^{i(\theta+4\pi/n)}, \ldots, z_n = re^{i(\theta+2(n-1)\pi/n)}$ are all mapped onto the single point $w = r^n e^{in\theta}$ by $f(z) = z^n$.
- This fact is illustrated for $n = 6$:



# Restricting the Domain

- Recall that even though the real functions $f(x) = x^2$ and $g(x) = \sin x$ are not one-to-one and, thus, appear not to have inverses, yet we still have the inverse functions $f^{-1}(x) = \sqrt{x}$ and $g^{-1}(x) = \arcsin x$.
- The key is to appropriately restrict the domains of $f(x) = x^2$ and $g(x) = \sin x$ to sets on which the functions are one-to-one.

  Example: Whereas $f(x) = x^2$ defined on $(-\infty, \infty)$ is not one-to-one, the same function defined on $[0, \infty)$ is one-to-one.

  Similarly, $g(x) = \sin x$ is not one-to-one on $(-\infty, \infty)$, but it is one-to-one on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

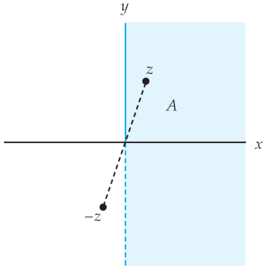
  The function $f^{-1}(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ defined on the interval $[0, \infty)$. Since $\mathrm{Dom}(f) = [0, \infty)$ and $\mathrm{Range}(f) = [0, \infty)$, the domain and range of $f^{-1}(x) = \sqrt{x}$ are both $[0, \infty)$ as well.

  Similarly, $g^{-1}(x) = \arcsin x$ is the inverse function of the function $g(x) = \sin x$ defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The domain and range of $g^{-1}$ are $[-1, 1]$ and $[-\frac{\pi}{2}, \frac{\pi}{2}]$, respectively.

# A Restricted Domain for $f(z) = z^2$

- Show that $f(z) = z^2$ is a one-to-one function on the set $A$ defined by $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$

  We show that $f$ is one-to-one by demonstrating that if $z_1$ and $z_2$ are in $A$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$. If $f(z_1) = f(z_2)$, then $z_1^2 = z_2^2$, or, equivalently, $z_1^2 - z_2^2 = 0$. By factoring this expression, we obtain $(z_1 - z_2)(z_1 + z_2) = 0$. It follows that either $z_1 = z_2$ or $z_1 = -z_2$. By definition of the set $A$, both $z_1$ and $z_2$ are nonzero. The complex points $z$ and $-z$ are symmetric about the origin.




Inspection shows that if $z_2$ is in $A$, then $-z_2$ is not in $A$. This implies that $z_1 \neq -z_2$, since $z_1$ is in $A$. Therefore, we conclude that $z_1 = z_2$, and this proves that $f$ is a one-to-one function on $A$.

# An Alternative Approach

- The preceding technique does not extend to the function $z^n, n > 2$.
- We present an alternative approach.
- We prove that $f(z) = z^2$ is one-to-one on $A$ by showing that if $f(z_1) = f(z_2)$ for two complex numbers $z_1$ and $z_2$ in $A$, then $z_1 = z_2$. Suppose that $z_1$ and $z_2$ are in $A$. Then we may write $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ with $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < \theta_2 \leq \frac{\pi}{2}$. If $f(z_1) = f(z_2)$, then it follows $r_1^2 e^{i2\theta_1} = r_2^2 e^{i2\theta_2}$. We conclude that the complex numbers $r_1^2 e^{i2\theta_1}$ and $r_2^2 e^{i2\theta_2}$ have the same modulus and principal argument: $r_1^2 = r_2^2$ and $\text{Arg}(r_1^2 e^{i2\theta_1}) = \text{Arg}(r_2^2 e^{i2\theta_2})$. Because both $r_1$ and $r_2$ are positive, we get $r_1 = r_2$. Moreover, since $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < \theta_2 \leq \frac{\pi}{2}$, it follows that $-\pi < 2\theta_1 \leq \pi$ and $-\pi < 2\theta_2 \leq \pi$. This means that $\text{Arg}(r_1^2 e^{i2\theta_1}) = 2\theta_1$ and $\text{Arg}(r_2^2 e^{i2\theta_2}) = 2\theta_2$. This fact combined with the second equation implies that $2\theta_1 = 2\theta_2$, or $\theta_1 = \theta_2$. Therefore, $z_1$ and $z_2$ are equal because they have the same modulus and principal argument.

# An Inverse of $f(z) = z^2$

- The squaring function $z^2$ is one-to-one on the set $A$ defined by $-\frac{\pi}{2} < \arg(z) \le \frac{\pi}{2}$. Thus, this function has a well-defined inverse function $f^{-1}$. We show this inverse function is the principal square root function $z^{1/2}$.

  Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, where $\theta$ and $\phi$ are the principal arguments of $z$ and $w$, respectively. Suppose that $w = f^{-1}(z)$. Since the range of $f^{-1}$ is the domain of $f$, the principal argument $\phi$ of $w$ must satisfy: $-\frac{\pi}{2} < \phi \le \frac{\pi}{2}$. On the other hand, $f(w) = w^2 = z$. Hence, $w$ is one of the two square roots of $z$, i.e., either $w = \sqrt{r}e^{i\theta/2}$ or $w = \sqrt{r}e^{i(\theta+2\pi)/2}$. Assume that $w$ is the latter, i.e., assume that $w = \sqrt{r}e^{i(\theta+2\pi)/2}$. Because $\theta = \mathrm{Arg}(z)$, we have $-\pi < \theta \le \pi$, and so, $\frac{\pi}{2} < \frac{\theta+2\pi}{2} \le \frac{3\pi}{2}$. We conclude that the principal argument $\phi$ of $w$ must satisfy either $-\pi < \phi \le -\frac{\pi}{2}$ or $\frac{\pi}{2} < \phi \le \pi$. However, this cannot be true since $-\frac{\pi}{2} < \phi \le \frac{\pi}{2}$. So $w = \sqrt{r}e^{i\pi/2}$, which is the value of the principal square root function $z^{1/2}$.

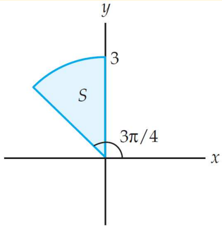
# Domain and Range of $f^{-1}(z) = z^{1/2}$

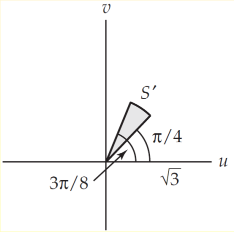
- Since $z^{1/2}$ is an inverse function of $f(z) = z^2$ defined on the set $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$, it follows that the domain and range of $z^{1/2}$ are the range and domain of $f$, respectively. In particular, $\text{Range}(z^{1/2}) = A$, that is, the range of $z^{1/2}$ is the set of complex $w$ satisfying $-\frac{\pi}{2} < \arg(w) \leq \frac{\pi}{2}$. In order to find $\text{Dom}(z^{1/2})$ we need to find the range of $f$. We saw that $w = z^2$ maps the right half-plane $\text{Re}(z) \geq 0$ onto the entire complex plane. The set $A$ is equal to the right half-plane $\text{Re}(z) \geq 0$ excluding the set of points on the ray emanating from the origin and containing the point $-i$. That is, $A$ does not include the point $z = 0$ or the points satisfying $\arg(z) = -\frac{\pi}{2}$. However, we have seen that the image of the set $\arg(z) = \frac{\pi}{2}$, the positive imaginary axis, is the same as the image of the set $\arg(z) = -\frac{\pi}{2}$. Both sets are mapped onto the negative real axis. Since the set $\arg(z) = \frac{\pi}{2}$ is contained in $A$, it follows that the only difference between the image of the set $A$ and the image of the right half-plane $\text{Re}(z) \geq 0$ is the image of the point $z = 0$, which is the point $w = 0$. Since $A$ is mapped onto the entire complex plane excluding the point $w = 0$, the domain of $f^{-1}(z) = z^{1/2}$ is the entire complex plane $\mathbb{C}$ excluding $0$.

# The Mapping $w = z^{1/2}$

- As a mapping, $z^2$ squares the modulus of $z$ and doubles its argument.
- Thus, the mapping $w = z^{1/2}$ takes the square root of the modulus of a point and halves its principal argument, i.e., if $w = z^{1/2}$, then we have $|w| = \sqrt{|z|}$ and $\text{Arg}(w) = \frac{1}{2}\text{Arg}(z)$.
- Example (Image of a Circular Sector under $w = z^{1/2}$): Find the image of the set $S$ defined by $|z| \leq 3$, $\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{4}$, under $w = z^{1/2}$.

  Let $S'$ denote the image of $S$ under $w = z^{1/2}$.
  - Since $|z| \leq 3$ for points in $S$, we have that $|w| \leq \sqrt{3}$ for points $w$ in $S'$.
  - Since $\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{4}$ for points in $S$, $\frac{\pi}{4} \leq \arg(w) \leq \frac{3\pi}{8}$ for points $w$ in $S'$.



# Principal $n$-th Root Function

- The complex power function $f(z) = z^n, n > 2$, is one-to-one on the set defined by $-\frac{\pi}{n} < \arg(z) \le \frac{\pi}{n}$.
- It can be seen that the image of this set under the mapping $w = z^n$ is the entire complex plane $\mathbb{C}$ excluding $w = 0$.
- Therefore, there is a well-defined inverse function for $f$.
- Analogous to the case $n = 2$, this inverse function of $z^n$ is called the **principal $n$-th root function** $z^{1/n}$.
- The domain of $z^{1/n}$ is the set of all nonzero complex numbers, and the range of $z^{1/n}$ is the set of $w$ satisfying $-\frac{\pi}{n} < \arg(w) \le \frac{\pi}{n}$.

**Definition (Principal $n$-th Root Functions)**

For $n \ge 2$, the function $z^{1/n}$ defined by

$$z^{1/n} = \sqrt[n]{|z|}e^{i\mathrm{Arg}(z)/n}$$

is called the **principal $n$-th root function**.

- By setting $z = re^{i\theta}$, with $\theta = \mathrm{Arg}(z)$, we have $z^{1/n} = \sqrt[n]{r}e^{i\theta/n}$.

# Values of $z^{1/n}$

- Find the value of the given principal $n$th root function $z^{1/n}$ at the given point $z$: (a) $z^{1/3}$; $z = i$ (b) $z^{1/5}$; $z = 1 - \sqrt{3}i$.

(a) For $z = i$, $|z| = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$. Thus, we obtain:

$$i^{1/3} = \sqrt[3]{1}e^{i(\pi/2)/3} = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

(b) For $z = 1 - \sqrt{3}i$, we have $|z| = 2$ and $\text{Arg}(z) = -\frac{\pi}{3}$. Thus, we get

$$(1 - \sqrt{3}i)^{1/5} = \sqrt[5]{2}e^{i(-\pi/3)/5} = \sqrt[5]{2}e^{-i(\pi/15)}.$$

# Multiple-Valued Functions

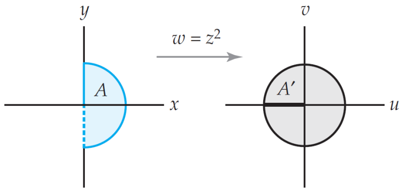
- A nonzero complex number $z$ has $n$ distinct $n$-th roots in the complex plane. Thus, the process of "taking the $n$-th root" of a complex number $z$ does not define a complex function. We introduced the symbol $z^{1/n}$ to represent the set consisting of the $n$ $n$-th roots of $z$.
- Similarly, $\arg(z)$ represents an infinite set of values.
- These types of operations on complex numbers are examples of **multiple-valued functions**.

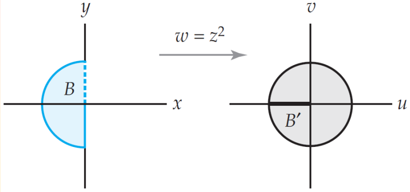
  When representing multiple-valued functions with functional notation, we will use uppercase letters such as $F(z) = z^{1/2}$ or $G(z) = \arg(z)$.

  Lowercase letters such as $f$ and $g$ will be reserved for functions.
- Example: $g(z) = z^{1/3}$ refers to the principal cube root function whereas $G(z) = z^{1/3}$ represents the multiple-valued function that assigns the three cube roots of $z$ to the value of $z$. Thus, $g(i) = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$ and $G(i) = \{\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -i\}$.

# Riemann Surface of $f(z) = z^2$

- $f(z) = z^2$ is not one-to-one. $f(z) = z^2$ is one-to-one on $A$ defined by $|z| \leq 1, -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$.

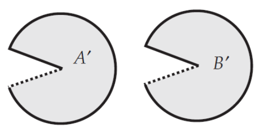


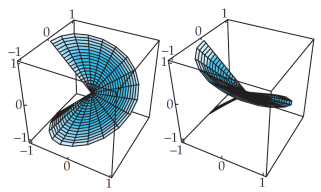



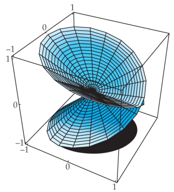

- $w = z^2$ is a one-to-one mapping of the set $B$ defined by $|z| \leq 1$, $\frac{\pi}{2} < \arg(z) \leq \frac{3\pi}{2}$, onto the closed unit disk $|w| \leq 1$.
- Since the unit disk $|z| \leq 1$ is the union of the sets $A$ and $B$, the image of the disk $|z| \leq 1$ under $w = z^2$ covers the disk $|w| \leq 1$ twice (once by $A$ and once by $B$).
- We visualize this "covering" by considering two image disks for $w = z^2$.

# Riemann Surface of $f(z) = z^2$ (Cont'd)

- Let $A'$ denote the image of $A$ under $f$ and $B'$ the image of $B$ under $f$.
- Imagine the disks $A'$ and $B'$ cut open along the negative real axis:



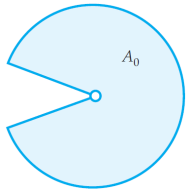


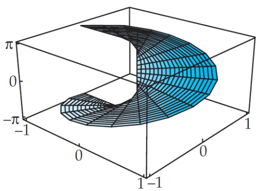
- We construct a Riemannn surface for $f(z) = z^2$ by stacking the cut disks $A'$ and $B'$ one atop the other in $xyz$-space and attaching them by gluing together their edges.

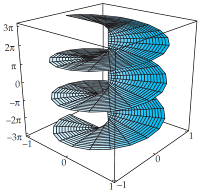
  After attaching in this manner we obtain the **Riemann surface**:
- Although $w = z^2$ is not a one-to-one mapping of the closed unit disk $|z| \leq 1$ onto the closed unit disk $|w| \leq 1$, it is a one-to-one mapping of the closed unit disk $|z| \leq 1$ onto the Riemann surface.

# Riemann Surface of $G(z) = \arg(z)$

- Another interesting Riemann surface is one for the multiple valued function $G(z) = \arg(z)$ defined on $0 < |z| \leq 1$. We take a copy $A_0$ of the punctured disk $0 < |z| \leq 1$ and cut it open along the negative real axis. Let $A_0$ represent the points $re^{i\theta}$, $-\pi < \theta \leq \pi$.





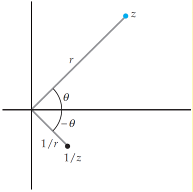



Take another copy $A_1$ and let it represent $re^{i\theta}$, $\pi < \theta \leq 3\pi$. Let $A_{-1}$ represent the points $re^{i\theta}$, $-3\pi < \theta \leq -\pi$. We have an infinite set of cut disks $\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots$. Place $A_n$ in $xyz$-space so that $re^{i\theta}$, with $(2n-1)\pi < \theta \leq (2n+1)\pi$, lies at height $\theta$ above the point $re^{i\theta}$ in the $xy$-plane. The collection of all the cut disks in $xyz$-space forms the Riemann surface for the multiple-valued function $G(z)$.

# Reciprocal Function

# The Reciprocal Function

- Analogous to real functions, we define a **complex rational function** to be a function of the form $f(z) = \frac{p(z)}{q(z)}$ where both $p(z)$ and $q(z)$ are complex polynomial functions.
- The most basic complex rational function is the reciprocal function.
- The function $\frac{1}{z}$, whose domain is the set of all nonzero complex numbers, is called the **reciprocal function**.
- Given $z \neq 0$, if we set $z = re^{i\theta}$, we obtain: $w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.
  - The modulus of $w$ is the reciprocal of the modulus of $z$;
  - The argument of $w$ is the negative of the argument of $z$.
- Therefore, the reciprocal function maps a point in the $z$-plane with polar coordinates $(r, \theta)$ onto a point in the $w$-plane with polar coordinates $(\frac{1}{r}, -\theta)$.

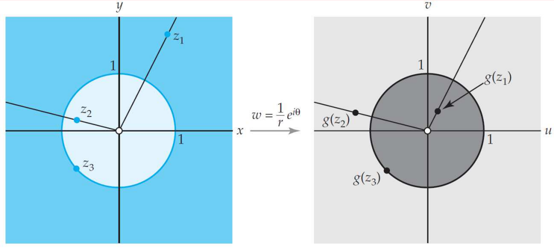


# Inversion in the Unit Circle

- The function $g(z) = \frac{1}{r}e^{i\theta}$, whose domain is the set of all nonzero complex numbers, is called **inversion in the unit circle**.
- We consider separately the images of points on the unit circle, points outside the unit circle, and points inside the unit circle.
  - Consider, first, a point $z$ on the unit circle. Since $z = 1 \cdot e^{i\theta}$, $g(z) = \frac{1}{1}e^{i\theta} = z$. So each point on the unit circle is mapped onto itself by $g$.
  - If, on the other hand, $z$ is a nonzero complex number that does not lie on the unit circle, then $z = re^{i\theta}$, with $r \neq 1$.
    - When $r > 1$ ($z$ is outside of the unit circle), we have that $|g(z)| = |\frac{1}{r}e^{i\theta}| = \frac{1}{r} < 1$. So, the image under $g$ of a point $z$ outside the unit circle is a point inside the unit circle.
    - Conversely, if $r < 1$ ($z$ is inside the unit circle), then $|g(z)| = \frac{1}{r} > 1$. Thus, if $z$ is inside the unit circle, then its image under $g$ is outside the unit circle.

# Illustration of the Inversion in the Unit Circle

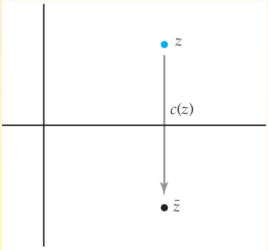
- The mapping $w = \frac{1}{r}e^{i\theta}$ is represented below:




- The arguments of $z$ and $g(z)$ are equal. So, if $z_1 \neq 0$ is a point with modulus $r$ in the $z$-plane, then $g(z_1)$ is the unique point in the $w$-plane with modulus $\frac{1}{r}$ lying on a ray emanating from the origin making an angle of $\arg(z_1)$ with the positive $u$-axis.
- The moduli of $z$ and $g(z)$ are inversely proportional: the farther a point $z$ is from 0 in the $z$-plane, the closer its image $g(z)$ is to 0 in the $w$-plane, and, the closer $z$ is to 0, the farther $g(z)$ is from 0.

# Complex Conjugation

- The second complex mapping that is helpful for describing the reciprocal mapping is a reflection across the real axis.
- Under this mapping the image of the point $(x, y)$ is $(x, -y)$.
- This complex mapping is given by the function $c(z) = \overline{z}$, called the **complex conjugation function**.
- The relationship between $z$ and its image $c(z)$ is shown below:




- If $z = re^{i\theta}$, then $c(z) = \overline{re^{i\theta}} = \overline{r}\,\overline{e^{i\theta}} = re^{-i\theta}$.

# Reciprocal Mapping

- The reciprocal function $f(z) = \frac{1}{z}$ can be written as the composition of inversion in the unit circle and complex conjugation.
- Since $c(z) = re^{-i\theta}$ and $g(z) = \frac{1}{r}e^{i\theta}$, we get

$$c(g(z)) = c(\frac{1}{r}e^{i\theta}) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}.$$

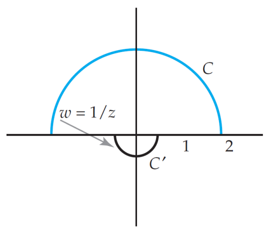
- Thus, as a mapping, the reciprocal function
  - first inverts in the unit circle,
  - then reflects across the real axis.
- In summary: Given $z_0$ a nonzero point in the complex plane the point $w_0 = f(z_0) = \frac{1}{z_0}$ is obtained by:
  (i) inverting $z_0$ in the unit circle, then
  (ii) reflecting the result across the real axis.

# Image of a Semicircle under $w = \frac{1}{z}$

- Find the image of the semicircle $|z| = 2$, $0 \leq \arg(z) \leq \pi$, under the reciprocal mapping $w = \frac{1}{z}$.

  Let $C$ denote the semicircle and let $C'$ denote its image under $w = \frac{1}{z}$. In order to find $C'$, we first invert $C$ in the unit circle, then we reflect the result across the real axis.

  - Under inversion in the unit circle, points with modulus 2 have images with modulus $\frac{1}{2}$. Moreover, inversion in the unit circle does not change arguments. The image is the semicircle $|w| = \frac{1}{2}$, $0 \leq \arg(w) \leq \pi$.
  - Reflecting this set across the real axis negates the argument of a point but does not change its modulus. Hence, the image is the semicircle given by $|w| = \frac{1}{2}$, $-\pi \leq \arg(w) \leq 0$.

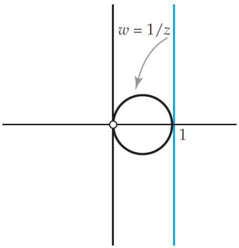


# Image of a Line under $w = \frac{1}{z}$

- Find the image of the vertical line $x = 1$ under the mapping $w = \frac{1}{z}$.

  The vertical line $x = 1$ consists of $z = 1 + iy$, $-\infty < y < \infty$. After replacing $z$ with $1 + iy$ in $w = \frac{1}{z}$ and simplifying, we obtain: $w = \frac{1}{1+iy} = \frac{1}{1+y^2} - \frac{y}{1+y^2} i$. It follows that the image of $x = 1$ under $w = \frac{1}{z}$ consists of all points $u + iv$ satisfying: $u = \frac{1}{1+y^2}$, $v = -\frac{y}{1+y^2}$, $-\infty < y < \infty$. We eliminate $y$: We have $v = -yu$. The first equation implies that $u \neq 0$, so we get $y = -\frac{v}{u}$. Thus, we obtain the quadratic equation $u^2 - u + v^2 = 0$.

  Complete the square to get $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$, $u \neq 0$. It defines a circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$. However, because $u \neq 0$, the point $(0, 0)$ is not in the image. Using the complex variable $w = u + iv$, we can describe this image by $|w - \frac{1}{2}| = \frac{1}{2}$, $w \neq 0$.



# Reverting to the Extended Complex Number System

- The image of $x = 1$ is not the entire circle $|w - \frac{1}{2}| = \frac{1}{2}$ because points on the line $x = 1$ with extremely large modulus map onto points on the circle $|w - \frac{1}{2}| = \frac{1}{2}$ that are extremely close to 0, but there is no point on the line $x = 1$ that actually maps onto 0.
- To obtain the entire circle as the image, we must consider the reciprocal function defined on the extended complex number system.
- The extended complex number system consists of all the points in the complex plane adjoined with the ideal point $\infty$.
- In the context of mappings this set of points is commonly referred to as the **extended complex plane**.
- The important property of the extended complex plane is the correspondence between points on the extended complex plane and the points on the complex plane.
  - In particular, points in the extended complex plane that are near the ideal point $\infty$ correspond to points with extremely large modulus in the complex plane.

# Extending the Reciprocal Function

- We use this correspondence to extend the reciprocal function to a function whose domain and range are the extended complex plane.
- Since $w = \frac{1}{r}e^{-i\theta}$ already defines the reciprocal function for all points $z \neq 0$ or $\infty$ in the extended complex plane, we extend this function by specifying the images of 0 and $\infty$.
  - If $z = re^{i\theta}$ is a point close to 0, then $r$ is small, whence $w$ is a point whose modulus $\frac{1}{r}$ is large. In the extended complex plane, if $z$ is a point that is near 0, then $w = \frac{1}{z}$ is a point that is near the ideal point $\infty$. So we define the reciprocal function $f(z) = \frac{1}{z}$ on the extended complex plane so that $f(0) = \infty$.
  - If $z$ is a point that is near $\infty$, in the extended complex plane, then $f(z)$ is a point that is near 0. Thus, we define the reciprocal function on the extended complex plane so that $f(\infty) = 0$.

# The Reciprocal Function on the Extended Complex Plane

## Definition (The Reciprocal Function on the Extended Complex Plane)

The **reciprocal function on the extended complex plane** is the function defined by

$$f(z) = \begin{cases} \frac{1}{z}, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty \end{cases}$$

- We use the notation $\frac{1}{z}$ to represent both the reciprocal function and the reciprocal function on the extended complex plane.
- Whenever the ideal point $\infty$ is mentioned, it will be assumed that $\frac{1}{z}$ represents the reciprocal function defined on the extended complex plane.

# Image of a Line under $w = \frac{1}{z}$

- Find the image of the vertical line $x = 1$ under the reciprocal function on the extended complex plane.

  Since the line $x = 1$ is an unbounded set in the complex plane, the ideal point $\infty$ is on the line in the extended complex plane.

  - We already saw that the image of the points $z \neq \infty$ on the line $x = 1$ is the circle $|w - \frac{1}{2}| = \frac{1}{2}$ excluding the point $w = 0$.
  - We have that $f(\infty) = 0$, and so $w = 0$ is the image of the ideal point. This "fills in" the missing point in the circle $|w - \frac{1}{2}| = \frac{1}{2}$.

  Therefore, the vertical line $x = 1$ is mapped onto the entire circle $|w - \frac{1}{2}| = \frac{1}{2}$ by the reciprocal mapping on the extended complex plane.

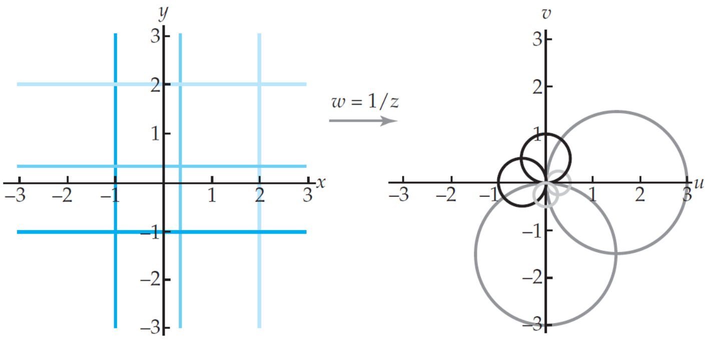
# Mapping Lines to Circles with $w = \frac{1}{z}$

## Mapping Lines to Circles with $w = \frac{1}{z}$

The reciprocal function on the extended complex plane maps:

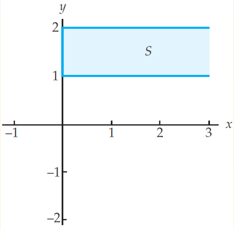
(i) The vertical line $x = k$ with $k \neq 0$ onto the circle $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$;

(ii) The horizontal line $y = k$ with $k \neq 0$ onto the circle $\left|w + \frac{1}{2k}i\right| = \left|\frac{1}{2k}\right|$.



# Mapping of a Semi-infinite Strip

- Find the image of the semi-infinite horizontal strip defined by $1 \leq y \leq 2$, $x \geq 0$, under $w = \frac{1}{z}$.



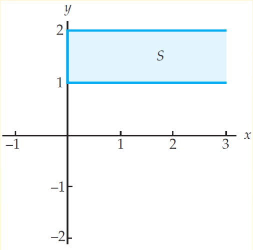

Let $S$ denote the semi-infinite horizontal strip defined by $1 \leq y \leq 2$, $x \geq 0$. The boundary of $S$ consists of the line segment $x = 0$, $1 \leq y \leq 2$, and the two half-lines $y = 1$ and $y = 2$, $0 \leq x < \infty$. We first determine the images of these boundary curves.

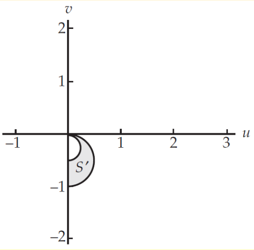
  - The line segment $x = 0$, $1 \leq y \leq 2$, can also be described as the set $1 \leq |z| \leq 2$, $\arg(z) = \frac{\pi}{2}$. Since $w = \frac{1}{z}$, $\frac{1}{2} \leq |w| \leq 1$. In addition, we have that $\arg(w) = \arg(1/z) = -\arg(z)$, and so, $\arg(w) = -\frac{\pi}{2}$. Thus, the image of $x = 0$, $1 \leq y \leq 2$, is the line segment on the $v$-axis from $-\frac{1}{2}i$ to $-i$.

# Mapping of a Semi-infinite Strip (Cont'd)

- Now consider $y = 1$, $0 \leq x < \infty$. The image is an arc in $|w + \frac{1}{2}i| = \frac{1}{2}$. The arguments satisfy $0 < \arg(z) \leq \frac{\pi}{2}$, so $-\frac{\pi}{2} \leq \arg(w) < 0$. Moreover, $\infty$ is on the half-line, and so $w = 0$ is in its image. Thus, the image of $y = 1$, $0 \leq x < \infty$, is $|w + \frac{1}{2}i| = \frac{1}{2}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$.
- Similarly, the image of $y = 2$, $0 \leq x < \infty$, is the circular arc $|w + \frac{1}{4}i| = \frac{1}{4}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$.

Every half-line $y = k$, $1 \leq k \leq 2$, between the boundary half-lines maps onto $|w + \frac{1}{2k}i| = \frac{1}{2k}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$, between these circular arcs:



# The Inverse Mapping of $\frac{1}{z}$

- The reciprocal function $f(z) = \frac{1}{z}$ is one-to-one.
- Thus, $f$ has a well-defined inverse function $f^{-1}$.
- Solving the equation $z = f(w)$ for $w$, we get $f^{-1}(z) = \frac{1}{z}$.
- This observation extends our understanding of the complex mapping $w = \frac{1}{z}$.
  - We have seen that the image of the line $x = 1$ under $\frac{1}{z}$ is the circle $|w - \frac{1}{2}| = \frac{1}{2}$ . Since $f^{-1}(z) = \frac{1}{z} = f(z)$, the image of the circle $|z - \frac{1}{2}| = \frac{1}{2}$ under $\frac{1}{z}$ is the line $u = 1$.
  - Similarly, we see that the circles $|w - \frac{1}{2k}| = |\frac{1}{2k}|$ and $|w + \frac{1}{2k}i| = |\frac{1}{2k}|$ are mapped onto the lines $x = k$ and $y = k$, respectively.