The $3n+1$-Problem and Holomorphic Dynamics

Simon Letherman, Dierk Schleicher, and Reg Wood

CONTENTS

The 3n+1-problem is the following iterative procedure on the positive integers: the integer n maps to n/2 or 3n+1, depending on whether n is even or odd. It is conjectured that every positive integer will be eventually periodic, and the cycle it falls onto is $1 \mapsto 4 \mapsto 2 \mapsto 1$. We construct entire holomorphic functions that realize the same dynamics on the integers and for which all the integers are in the Fatou set. We show that no integer is in a Baker domain (domain at infinity). We conclude that any integer that is not eventually periodic must be in a wandering domain.

1. INTRODUCTION

The following problem is well known under the name of $3n+1$-problem and has infected many people over the years: *Start with any positive integer n. If it is even, divide it by two; if it is odd, replace it by $3n+1$. Show that, after finitely many such steps, this process reaches the number* 1.

The integer 1 itself is periodic under this map: $1 \mapsto 4 \mapsto 2 \mapsto 1$. A priori, three possibilities are conceivable for any positive integer n:

- it falls, after finitely many steps, onto the periodic cycle $1 \mapsto 4 \mapsto 2 \mapsto 1$;
- it falls, after finitely many steps, onto a periodic cycle other than $1 \mapsto 4 \mapsto 2 \mapsto 1$;
- it never falls onto a periodic cycle; it will then necessarily diverge to $+\infty$.

It is conjectured that every positive integer realizes the first possibility: it eventually reaches 1. Extensive numerical experiments have been performed, all in support of this conjecture: Roosendaal [1999] has checked it for numbers up to 1.26×10^{16}. More precisely, the second case has never been observed for positive integers. (However, it is well known that there are extra cycles of negative integers. Two examples are $-1 \mapsto -2 \mapsto -1$ and $-5 \mapsto -14 \mapsto -7 \mapsto -20 \mapsto -10 \mapsto -5$, and the only further known cycle goes through -17.) When any positive

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initial number n does not land at 1 or another cycle after some large number of iterations, this might either be an indication that it indeed converges to $+\infty$, or that it takes much longer to settle onto a periodic orbit. Indeed, some relatively small numbers are known to take surprisingly long until they finally reach 1: as an example, the number 27 takes 111 steps to reach 1, and the largest number it visits along its way is 9232. Therefore, it is not quite clear how the third case could ever be "observed" numerically. For an entertaining discussion of this question, see [Hofstadter 1979, Aria XII]. In any case, no integer is known to diverge to ∞, and no cycle other than the ones above has been found. By a recent theorem of Halbeisen and Hungerbühler [1997], any extra cycle of positive integers must contain more than 10^8 numbers (even when counting any $3n+1$-step and the subsequent division by 2 as one; see below). Surveys of the history and the variety of known results about the $3n+1$-problem can be found in [Lagarias 1985; Wirsching 1998]. Among the gems to be found there is a theorem of John H. Conway to the effect that a simple generalization of the $3n+1$-problem is "algorithmically undecidable" because it encodes the halting problem for Turing machines, and a comment of Erdős saying that "Mathematics is not yet ready for such problems".

The idea of this paper is to interpret the $3n+1$-problem as an iterative procedure and to find an entire holomorphic map that extends the given map on the integers. This idea has been circulated by the third author for a decade or two. In this paper, we consider the problem from the point of view of holomorphic dynamics and show that the existence of any integer that is not eventually periodic implies the existence of a "wandering domain" for our entire holomorphic map. For a large class of entire holomorphic maps, it is known that there are no wandering domains. Unfortunately, this class does not include our maps. Nonetheless, we hope that this point of view might stimulate further progress.

First observe that the image $3n+1$ of an odd integer n is even and will be replaced by $(3n+1)/2$ in the next step. We can therefore equivalently consider the iteration $n \mapsto n/2$ if n is even and $n \mapsto (3n+1)/2$ if n is odd. The orbit $1 \mapsto 4 \mapsto 2 \mapsto 1$ turns into the orbit $1 \mapsto 2 \mapsto 1$, while 0 and -1 are fixed points and $-5 \mapsto -7 \mapsto -10 \mapsto -5$. It is for this map that Halbeisen and Hungerbühler showed that the period of any extra cycle of positive integers must be more than 10^8.

We consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \tfrac{1}{2}z + \tfrac{1}{2}\big(1 - \cos \pi z\big)\big(z + \tfrac{1}{2}\big) + \tfrac{1}{\pi}\big(\tfrac{1}{2} - \cos \pi z\big) \sin \pi z + h(z) \sin^2 \pi z, \tag{1–1}$$

for any entire holomorphic function h. The construction is chosen in such a way that many dynamical properties of interest to us will not depend on h. Therefore, although f of course depends on h, we will suppress this from the notation. The function f is displayed in Figure 1 for $h \equiv 0$. We will often be interested in the special case that f preserves the reals, which happens if and only if h preserves the reals, or if and only if $f(\bar z) = \overline{f(z)}$ for all z.

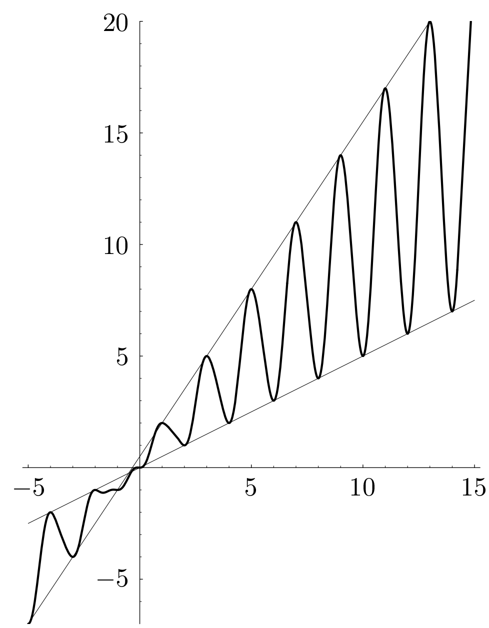

FIGURE 1. A graph of the map f on the reals, for $h \equiv 0$. Also indicated are the lines $x \mapsto x/2$ and $x \mapsto (3x+1)/2$.

To begin with, $\sin \pi z$ vanishes on the integers, so the values of f on integers do not depend on h. Since $\cos \pi z$ is $+1$ on even integers and -1 on odd integers, we have

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is an even integer,} \\ (3n+1)/2 & \text{if } n \text{ is an odd integer.} \end{cases}$$

That is, our function f agrees on all integers with the given iteration function. Therefore, the problem can equivalently be formulated thus:

Conjecture 1.1 (The holomorphic 3n+1-problem). *Iterating the function f on any positive integer will land at the number 1 after finitely many steps.*

A related function has independently been constructed by Chamberland [1996]: his map is equal to the first line in (1–1). The same map has independently been discovered and investigated by Sasha Gajfullin. Below, we will discuss similarities and differences between this map and ours.

2. HOLOMORPHIC DYNAMICS

In this section we provide some fundamental background on holomorphic dynamics. Details can be found in [Milnor 1999], for example. The dynamics of a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ starts with a dichotomy of the complex plane as follows: a point $z \in \mathbb{C}$ is in the *Fatou set* of f if there is a neighborhood U of z such that the sequence $f^{\circ n}$ of iterates on U forms a normal family in the sense of Montel (that is, every infinite subfamily contains a subsequence that converges compactly to a holomorphic limit function from U to the Riemann sphere; in particular, a constant limit function with value ∞ is allowed). The Fatou set is evidently open. Its complement is known as the *Julia set.* Fatou and Julia sets are forward and backward invariant under the dynamics. By Montel's Theorem, every neighborhood U of any point z in the Julia set has the property that the collection of iterates of f on U will cover all of $\mathbb{C}$ with at most one exception.

Intuitively, the dynamics of f on the Fatou set is tame and well-behaved, while the dynamics on the Julia set is wild and chaotic. Depending on the map f, the Fatou set may or may not be empty, while the Julia set is never empty (unless f is a rational map of degree at most 1). Any connected component of the Fatou set is a *Fatou component.* Since the Fatou set is open, any Fatou component is always path connected.

A periodic point z is a point for which $f^{\circ n}(z) = z$; the period of z is the least positive integer n for which this equation holds. Associated to such a periodic point is the *multiplier* $\mu := (d/dz)f^{\circ n}(z)$, which is the same for all the points on any periodic orbit. If $|\mu| < 1$, the orbit is *attracting* and a neighborhood of the orbit will converge to this orbit under iteration. Attracting orbits are in the Fatou set. The special case $\mu = 0$ is called *superattracting.* An orbit with $|\mu| > 1$ is known as *repelling* and contained in the Julia set. Orbits with $|\mu| = 1$ are known as *indifferent* and further subdivided according to the rotation angle θ satisfying $\mu = e^{2\pi i\theta}$. If θ is rational, the orbit is *rationally indifferent* and contained in the Julia set; there are associated Fatou components in which the dynamics converges locally uniformly to the rationally indifferent orbit. In the *irrationally indifferent* case (θ irrational), the orbit may or may not be in the Fatou set. If it is, the dynamics in these Fatou components is conformally conjugate to a rigid rotation of a Euclidean disk about the angle θ, and the center of the disk corresponds to the indifferent orbit. Such Fatou components are known as *Siegel disks.* Finally, irrationally indifferent periodic points in the Julia set are known as *Cremer points*; they are not associated to Fatou components.

The *basin of attraction* of an attracting orbit is the open set of points converging to this orbit. An *immediate basin* is a connected component of this basin containing a point on the attracting orbit. Every rationally indifferent orbit has a basin of points converging to this orbit; an immediate basin in this case is a connected component that contains a point of the rationally indifferent orbit on its boundary.

For entire holomorphic maps, any Fatou component is of one of the following types (for a general reference, see [Eremenko and Lyubich 1989] or [Bergweiler 1993]):

- (periodic) immediate basins of (super-)attracting periodic points;
- (periodic) immediate basins of rationally indifferent periodic points;
- (periodic) Siegel disks;
- (periodic) domains at infinity, also called Baker domains, in which the dynamics converges to ∞ locally uniformly;
- preperiodic components, those which eventually map onto a periodic component of one of the types above;
- wandering components, those whose forward orbits never repeat.

For entire meromorphic maps, there is one extra possibility: Arnold–Herman rings, which are doubly connected domains on which the dynamics is conformally conjugate to a rigid rotation of an annulus about an irrational angle. Moreover, Baker domains for such maps can be adjacent to singularities other than ∞, and when the period is greater than one, the periodic components can be based at different singularities. In this paper, we will be concerned only with entire holomorphic maps.

A *critical point* of a holomorphic map f is a point where the derivative vanishes. Its image under f is a *critical value*. A point $w \in \mathbb{C}$ is an *asymptotic value* if there is a curve $\gamma \in \mathbb{C}$ tending to ∞ such that, along this curve, the values $f(z)$ converge to w. The closure of the set of critical and asymptotic values is known as the set of *singular values*. It is a well known observation in holomorphic dynamics that the fates of singular values under iteration (the *singular orbits*) determine many dynamical features. For example, any periodic cycle of Fatou components corresponding to attracting or rationally indifferent periodic orbits must contain at least one singular value, and every boundary point of a Siegel disk and every Cremer point must be on the closure of some singular orbit.

We will now discuss whether Fatou components may be multiply connected. As a consequence of the Riemann–Hurwitz formula, periodic Fatou components of holomorphic maps are simply connected, doubly connected or infinitely connected. Doubly connected components are always Arnold–Herman rings; they must surround a pole of the map, so they cannot occur for entire maps. For a simply or infinitely connected Fatou component, all its preperiodic preimages must also be simply respectively infinitely connected.

In many cases, Fatou components of entire maps must be simply connected; the following lemma contains known results.

Lemma 2.1 (Fatou components are simply connected). *Any periodic or preperiodic Fatou component of an entire transcendental map is simply connected.*

Proof. For a Fatou component corresponding to an attracting or rationally indifferent periodic orbit, this is quite easy to see (compare [Eremenko and Lyubich 1989, Theorem 4.4]): any loop within this component must converge to the attracting or rationally indifferent orbit, so it visits only a compact subset of $\mathbb{C}$. By the maximum principle, the same is true for the region surrounded by this loop, which is thus entirely contained in the Fatou set. It then follows that the corresponding preperiodic Fatou components are also simply connected. (The statement is false for rational maps and even for polynomials: the basin of the superattracting fixed point ∞ is infinitely connected if and only if it contains a critical point in $\mathbb{C}$. The proof above also applies to Fatou components of polynomials around attracting or rationally indifferent orbits in $\mathbb{C}$, but it does not apply to the basin of the superattracting fixed point ∞ because it uses the maximum principle in $\mathbb{C}$.)

Siegel disks are always simply connected, because they are conformally equivalent to the unit disk. Baker domains are also simply connected: in fact, any multiply connected Fatou component of an entire holomorphic map must be bounded [Baker 1975, Theorem 1] (also to be found as [Bergweiler 1993, Theorem 9] or [Eremenko and Lyubich 1989, Theorem 4.3]). $\square$

For certain choices of h, it can be shown that every Fatou component, including any wandering domains, is simply connected: see [Bergweiler 1993, Theorem 10]. We will give an argument below that is custom-tailored to our maps.

For rational maps, as well as for entire holomorphic maps with only finitely many singular values (known as "entire maps of finite type"), there are no domains at infinity and no wandering domains; this is Sullivan's Theorem [McMullen and Sullivan 1998] in the extension of Eremenko and Lyubich [1992]. Unfortunately, the entire maps we are looking at cannot possibly be of finite type. We will be able to exclude domains at infinity, and we will show that a diverging integer for the $3n+1$-problem must sit in a simply connected wandering domain.

3. THE HOLOMORPHIC $3n+1$ MAP

We now begin to investigate the dynamics of our maps f interpolating the $3n+1$ problem. In view of the discussion above, we start by looking at critical points. A little calculation yields

$$f'(z) = \Big(\frac{\pi}{2}\big(z+\tfrac{1}{2}\big) + 2\sin \pi z + 2\pi h(z)\cos \pi z + h'(z)\sin \pi z\Big)\sin \pi z, \quad (3\text{–}1)$$

so all integers are critical points of f. Further critical points depend on h. Our function f is built up as

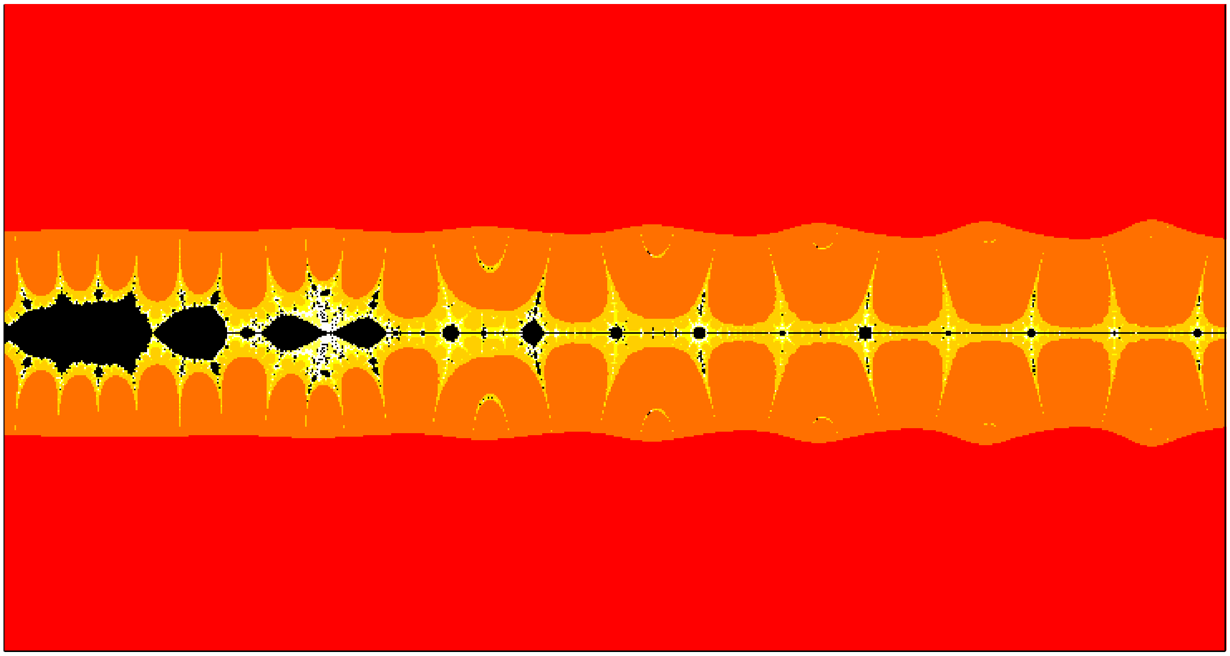

FIGURE 2. The Julia set of the map f for $h \equiv 0$. The real parts in the picture range through $[-2.4, 12.4]$. The black Fatou component to the left contains -1 and -2, the component next to it is the basin of 0. To the right of it, there are the two basins of the 2-cycle $1 \mapsto 2 \mapsto 1$. Further Fatou components around integers are clearly visible, particularly prominently around even integers.

follows: the first term models the behavior on even integers, the second term adds the modification necessary for odd integers, the third term vanishes on the integers but makes them into critical points, and the last term contains whatever freedom is left under these circumstances: the difference between any two such interpolations must have a double zero at every integer. We state this as follows.

Lemma 3.1 (Interpolating the 3n+1-problem). *Any entire holomorphic map that interpolates the* $3n+1$*-problem in such a way that all integers are critical points is of the form* (1–1).

Much of our paper will be concerned with the case that f and equivalently h preserve the reals. In every result, we will state explicitly whether or not this assumption is made.

The periodic cycle $1 \mapsto 2 \mapsto 1$, like any other cycle of integers, is superattracting. Our conjecture can then equivalently be formulated as follows: *every positive integer is in the basin of attraction of the superattracting orbit* $1 \mapsto 2 \mapsto 1$.

We have arranged our maps so that we only have to deal with the Fatou set. Figure 2 shows the Julia set of our map f, again in the case $h \equiv 0$.

Lemma 3.2 (Integers in Fatou set). *If* h *vanishes everywhere, all the integers are in the Fatou set of* f.

Proof. The integer 0 is a superattracting fixed point and thus in the Fatou set. For any non-zero integer n, we define the open neighborhood

$$U_n := \{z \in \mathbb{C} : |z-n| < 1/|2\pi^2 n|\};$$

for completeness, we choose U_0 to be a neighborhood of 0 so that $f(U_0) \subset U_0$. First observe that, for $z = n+\delta \in U_n$, we have $|\sin \pi z| = |\sin \pi\delta| \le \sinh(\pi|\delta|)$; for $|\delta| < 1/|2\pi^2 n| \le 1/(2\pi^2)$, we have $\sinh(\pi|\delta|) \le 1.005\,\pi|\delta|$ and thus

$$\begin{aligned}
|f'(z)| &= \left|\frac{\pi}{2}(z+\tfrac{1}{2})+2\sin\pi z\right| |\sin \pi z| \\
&\le (\pi|n|/2+\pi|\delta|/2+\pi/4+2\cdot 1.005\pi|\delta|) \\
&\qquad \times 1.005\pi|\delta| \\
&\le (\pi|n|/2+1/(4\pi)+\pi/4+1.005/\pi) \\
&\qquad \times 1.005/|2\pi n| \\
&\le 1.005(\tfrac{1}{4}+1/(8\pi)^2+\tfrac{1}{8}+1.005/(2\pi^2)) \\
&< 0.441.
\end{aligned}$$

It now follows that $f(U_n) \subset U_{f(n)}$: the center of any such neighborhood obviously maps to the center of the image neighborhood, and the derivative has absolute value less than $\frac{1}{2}$; thus the image of U_n is contained in a disk around $f(n)$ with radius $1/|4\pi^2 n|$. But since $|f(n)| < 2|n|$ for integers and the neighborhood $U_{f(n)}$ is the disk around $f(n)$ with radius $1/|2\pi^2 f(n)| > 1/|4\pi^2 n|$, we have indeed $f(U_n) \subset U_{f(n)}$.

Consequently, the union $\bigcup U_n$ is mapped into itself under the dynamics. By the Montel criterion above, points in this union cannot be in the Julia set. □

A similar conclusion can be made for any map f that satisfies an appropriate growth condition in a neighborhood of the integers.

In this section, we will investigate the types of Fatou sets that can contain integers.

Lemma 3.3 (Integers in Fatou set). *If a Fatou component of f corresponding to an attracting orbit contains an integer, then this orbit is superattracting. No Fatou component corresponding to a rationally indifferent orbit or to a Siegel disk can contain an integer. If f preserves the reals, then no Siegel disk can intersect the real axis.*

Proof. If the integer n is within the basin of attraction of an attracting or rationally indifferent periodic orbit, it must fall exactly onto this periodic orbit because the orbit of n is contained in $\mathbb{Z}$. This orbit then consists of integers and is superattracting because all integers are critical points of f. The first case in the classification above is thus realized only for superattracting orbits, the second case not at all.

No integer can be in a Siegel disk because no orbit in a Siegel disk is discrete (except at the center, which is an indifferent periodic point and cannot be an integer either). This rules out the third possibility above. Stronger yet: if f preserves the reals, no Siegel disk can intersect the real line because that would require the closure of an orbit to be a smooth simple closed curve. □

We proceed by showing that for all our maps, all Fatou components are simply connected. Except for wandering domains, this is of course the statement of the more general Lemma 2.1 above, but our proof works the same for all kinds of Fatou components.

Proposition 3.4 (Fatou components are simply connected). *Every Fatou component of any map f is simply connected, whether or not f preserves the reals.*

Proof. Suppose there is a multiply connected Fatou component. Its forward images must also be multiply connected. Since every connected component in the basin of the superattracting fixed point 0 or of the superattracting cycle $1 \mapsto 2 \mapsto 1$ is simply connected, we may exclude the possibility that our Fatou component contains a point on the backwards orbit of 0 or 1. The region surrounded by the Fatou component contains points in the Julia set and thus points that will eventually map to the fixed point 0. Therefore, after finitely many iteration steps, every forward image of our Fatou component must surround 0. We thus have a sequence $W_0, W_1, W_2, \ldots$ of Fatou components such that $f : W_n \to W_{n+1}$ is a holomorphic covering, and all these domains surround the origin. Since the point 0 is critical, the mapping degrees of $f : W_n \to W_{n+1}$ must be at least two.

We use the unique normalized hyperbolic metrics on all W_n. The map $f : W_n \to W_{n+1}$ cannot increase hyperbolic metrics. The component W_0 contains a simple smooth curve γ_0 surrounding 0; let l_0 be its hyperbolic length. The image curve $f(\gamma_0)$ is a smooth curve surrounding 0 at least twice and has total length at most l_0. Therefore, there is a smooth curve $\gamma_1 \subset W_1$ of length $l_1 \le l_0/2$ surrounding 0. Continuing, we obtain smooth curves $\gamma_n \subset W_n$ surrounding 0 and having hyperbolic lengths at most $l_0/2^n$. However, since no W_n can contain the Fatou component around 0 or any point 2^k for a positive integer k, the hyperbolic length of any curve surrounding 0 is uniformly bounded below: in fact, for a smooth curve in $\mathbb{C}$ minus the superattracting basin of 0 to be hyperbolically short, the curve must be far away from the origin, so it must separate 0 and 2^k from 2^{k+1} and ∞ for some positive integer k. However, since all domains $V_k := \mathbb{C} - \{0, 2^k, 2^{k+1}\}$ are conformally equivalent, the hyperbolic lengths of such curves are uniformly bounded below within the appropriate V_k. Since any Fatou component containing such a curve γ_n is contained in all V_k, the hyperbolic length of γ_n within this Fatou component is even greater than within V_k. This contradiction finishes the proof. □

We will now exclude domains at infinity (Baker domains) in two ways: for the case that f preserves the reals, we will show that no domain at infinity can intersect the reals. We will then present an argument due to Bergweiler that shows that no integer can be in a domain at infinity even if f does not preserve the reals.

Corollary 3.5 (No domains at infinity intersect reals for real maps). *If f preserves the reals, no domain at infinity can intersect the real line. In particular, no integer can be in a domain at infinity.*

Proof. Suppose that $x \in \mathbb{R}$ is in a domain at infinity and denote its Fatou component by U. Then the orbit of x must tend to ∞ along the real line, and since U is periodic, infinitely many points on this orbit must be in U. Since all the numbers $\pm 2^n$ land at ± 1 and thus have bounded orbits, they cannot be in U. Therefore, U intersects the real line at infinitely many intervals. Since f is real, U must be symmetric with respect to the real line, and it must thus be infinitely connected. This is a contradiction to simple connectivity of any Fatou component. □

A main idea for the following variant has kindly been contributed by Walter Bergweiler. It applies even if f does not preserve the reals, but it is somewhat weaker in that it does not exclude domains at infinity meeting the real line away from integers.

Proposition 3.6 (No integers in domains at infinity). *No domain at infinity can contain an integer, whether or not f or h preserve the reals.*

Proof. We will use the following variant of Koebe's $\frac{1}{4}$-theorem: for any two points a, b in a simply connected domain $U \subset \mathbb{C}$ (with $U \neq \mathbb{C}$) such that the Euclidean distance from a to ∂U is d, the hyperbolic distance in U between a and b is at least

$$\tfrac{1}{4}\log(1 + |a - b|/d).$$

First we show that there is a number $R > 0$ such that, for every integer n, the disk of radius R around n intersects the Julia set. If not, then for every $\varepsilon > 0$ there is an integer n such that n and $n + 1$ are within the same Fatou component U and the hyperbolic distance in U between n and $n+1$ is less than ε. Let d be the Euclidean distance from $f(n)$ or $f(n + 1)$ to $\partial f(U)$, whichever is smaller. Since $0 \notin f(U)$, we have

$$d < \min\{f(n), f(n + 1)\} \leq \tfrac{1}{2}(n + 1).$$

But $|f(n + 1) - f(n)| \geq n$, so the hyperbolic distance in $f(U)$ between $f(n)$ and $f(n+1)$ is at least $\frac{1}{4}\log(1 + n/d)$. This yields a lower bound for ε and is a contradiction.

Now let U be a domain at infinity. We know that it is simply connected by Proposition 3.4 above. Let R be a number as above. Suppose that an integer $n \in U$. Let p be the period of U and let e_k and h_k be the Euclidean respectively hyperbolic distances (in $f^{\circ k}(U)$) between $f^{\circ k}(n)$ and $f^{\circ(k+p)}(n)$. By the Schwarz Lemma, the sequence h_k is monotonically decreasing and thus bounded. We have

$$h_k \geq \tfrac{1}{4}\log(1 + e_k/R),$$

which bounds the e_k as well. But since the $f^{\circ k}(n)$ diverge to ∞, this implies that the pattern of steps in which the orbit of n visits even or odd integers (which determines the image point) must eventually be periodic. This pattern determines the orbit uniquely: if m and m' have the same pattern of even and odd elements in their orbits and the highest power of two dividing $|m - m'|$ is 2^s, then $|f(m) - f(m')|$ is divisible only by 2^{s-1}, and after s steps, the pattern will be different. This implies that the orbit of n is eventually periodic as well, contrary to our assumption. □

We see that every integer sits either in the basin of attraction of a superattracting periodic orbit of integers, or it is in a wandering Fatou component. We conjecture that the latter case does not occur.

Conjecture 3.7 (No wandering domains at integers). *For some entire function h, the corresponding map f contains all the integers in its Fatou set and has no simply connected wandering domain intersecting the integers.*

This conjecture immediately implies that every integer is eventually periodic for the $3n$+1-problem. Notice that it suffices to prove the conjecture for *any* entire function h.

Even if the conjecture is proved, this does not imply that every positive integer will eventually land on the cycle $1 \mapsto 2 \mapsto 1$. However, it does make all integer orbits finite. The arguments do not distinguish between positive and negative integers, and there are three known periodic orbits among negative integers (and the integer 0 forms a cycle by itself).

4. DYNAMICS ON THE REAL LINE

While our approach does not distinguish between positive and negative integers, it does distinguish between integers and non-integers in the case that f preserves the reals and all the integers are in the Fatou set (for example when $h \equiv 0$). In this section, we will discuss the dynamics of maps f restricted to the real line.

Lemma 4.1 (Wandering real numbers). *Consider any real continuous interpolation of the* $3n+1$*-problem (not necessarily analytic). Then between any pair of consecutive positive integers greater than* 1*, there is a Cantor set of points that diverge to* ∞ *strictly monotonically.*

Proof. For any integer $n \geq 3$, the real interval $[2n,\, 2n+1]$ has subintervals that map onto $[2n+2,\, 2n+3]$ and onto $[2n+4,\, 2n+5]$ (not necessarily diffeomorphically). For any one-sided sequence of integers in $\{1, 2\}$, there is thus at least one real number in $[2n,\, 2n+1]$ that diverges to infinity strictly monotonically such that its orbit is restricted to intervals of the type $[2n_i,\, 2n_i+1]$ with integers n_i such that $n_{i+1} - n_i$ is the prescribed sequence of integers. This is (at least) a Cantor set: a non-empty compact completely disconnected set without isolated points. (In fact, for $h \equiv 0$ and n sufficiently large, the derivatives at these subintervals will be strictly larger than 1, and we obtain exactly a Cantor set).

The remaining intervals are easy to deal with: $[2, 3]$ has a subinterval covering $[4, 5]$, which itself has a subinterval covering $[6, 7]$, and the latter interval contains a Cantor set as just described. Finally, any interval $[2n-1,\, 2n]$ with $n \geq 2$ will cover the interval $[2n,\, 2n+1]$ in an orientation reversing way. □

For our map f with $h \equiv 0$, the interval $[1, 2]$ maps over itself in an orientation reversing way. It contains a repelling fixed point, and everything else will converge to the orbit $1 \mapsto 2 \mapsto 1$.

Remark. The dynamics of f is still richer: one can label the orbits of all the real numbers by the integer parts of the points they visit, and when h is not too wild, it is not difficult to describe the allowed sequences of integer parts: that is, the map f is easy to describe by *symbolic dynamics*.

In the holomorphic case, all the reals in this escaping Cantor set will usually be in the Julia set. The only other possibility is that such points are in wandering Fatou components, and we get entire intervals of monotonically escaping points, rather than only a Cantor set. This is impossible in cases like $h \equiv 0$ when the derivative is bounded below by 1. In any case, we will now show that the Julia set separates almost all integers, at least on the real line.

Lemma 4.2 (Julia set between integers). *For any real continuous interpolation of the* $3n+1$*-problem, there is a real fixed point between any pair of consecutive non-zero integers except possibly* $\{-2, -1\}$, $\{-1, 0\}$, $\{0, 1\}$. *In the holomorphic case when f preserves the reals, then between any pair of consecutive even integers, there is a fixed point in the Julia set, and there is a point in the Julia set between any pair of consecutive integers except possibly* $\{-1, -2\}$.

Proof. Since $|f(n)| < |n|$ for non-zero even integers, while $|f(n)| > |n|$ for odd integers except -1, the intermediate value theorem yields the existence of a fixed point between any pair of consecutive non-zero integers except for the three specified pairs. Between any non-zero even integer and the adjacent odd integer with greater absolute value, the graph of a real holomorphic $f(z)$ has to cross the graph of the identity from below, and the derivative at such a fixed point has to be at least $+1$. Such fixed points are thus repelling or rationally indifferent and hence in the Julia set.

For similar reasons, there are fixed points in the Julia set between the superattracting fixed points 0 and -1 and between 0 and 1.

It remains to consider the case of two adjacent integers such that the odd one has smaller absolute value: $\pm 2n$ and $\pm(2n-1)$. For $n \geq 2$, such an interval maps over two adjacent even integers, and there is a point in the Julia set in between. Since there is a superattracting 2-cycle $1 \mapsto 2 \mapsto 1$, there must be a point in the Julia set between these two points, and the only pair of adjacent integers left is $\{-1, -2\}$. They both map to -1 and could be in the same Fatou component. □

Corollary 4.3 (Integers in different Fatou components). *If the holomorphic map f preserves the reals, then no two integers are in the same Fatou component, except possibly for -1 and -2.*

Proof. Between any pair of integers, other than possibly -1 and -2, there is a real number in the Julia set, and any Fatou component containing these two integers must be multiply connected. However, all Fatou components are simply connected. □

As mentioned earlier, a related function has independently been constructed by Chamberland [1996]: he considers the single map

$$z \mapsto \tfrac{1}{2}z + \tfrac{1}{2}(1 - \cos \pi z)\big(z + \tfrac{1}{2}\big)$$

consisting of the first two terms in (1–1), restricted to the real line. It may be interesting to compare our results to his. Of course, his map has the same dynamics on the integers, but his critical points are different. His map is not included in our family because we insisted in the integers being critical points. This implies that any cycles on the integers are automatically superattracting, while Chamberland proves that they are attracting in his setting and he finds estimates for critical points near integers. He also shows that his map has negative Schwarzian derivative: this is an analytical condition that has several useful consequences. Among iterated real maps, those with negative Schwarzian derivative are special and much easier to deal with, in a similar sense as holomorphic maps are special among differentiable maps on the plane. For example, in both settings there are certain families of maps known for which there are no wandering domains; however, these are quite remote from the maps at hand. Chamberland observes that his map has negative arguments where the Schwarzian fails to be negative, and he notes that it "seems unlikely that a general extension will have this property" (indeed, it is easy to construct plenty of counterexamples).

Moreover, Chamberland obtains monotonically increasing diverging trajectories on the reals (similar to our Lemma 4.1), as well as an uncountable set of "unstable" bounded orbits (the Julia set on the real axis), and he applies a special case of Sharkovskii's theorem [Melo and van Strien 1993] to show that there are periodic orbits of any period. This is true for any real interpolation of the problem. He also shows that any (real) neighborhood of some particular fixed point will under iteration cover every real $x > 1$; the complex analog of this statement is that any neighborhood of any point in the Julia set will eventually cover all of $\mathbb{C}$ (minus at most one point), and this is virtually built into the definition of the Julia set via normal families and Montel's theorem.

There is a general tendency that the extension of a real-analytic map to the complex plane enriches the available tools considerably: for iteration of continuous real maps, almost anything is possible, and many dynamical properties of real continuous maps can be recovered by C^∞-approximations. However, it is a severe restriction for a real map to extend as a holomorphic map to the complex plane; if this is possible, then the dynamics gains a lot of structure and many dynamical properties become almost obvious. From the point of view of real dynamics, it is not clear how to tell whether a given map can or cannot be extended to the complex plane (or whether this map is in some sense equivalent to one that can be extended).

Chamberland states that his map "seems to be the extension [of the $3n$+1-problem] which permits the 'simplest' analysis". Since the dynamics is influenced by the critical points, we rather want to have at least the real critical points under control, and this leads to the study of Equation (1–1).

5. CRITICAL POINTS

Since critical points control the dynamics, it is desirable to have as few critical points as possible. Even in the special case that h vanishes everywhere, it seems difficult to control all the critical points. The freedom in the choice of h has been introduced in order to have more flexibility to get rid of critical points. Ideally, the only critical points of our maps should be the integers. In view of Equation (3–1) for the derivative, it would be useful to find an entire function h such that

$$2 \sin \pi z + 2\pi h(z) \cos \pi z + h'(z) \sin \pi z = 0;$$

for such a function h, the only critical points of f would be the integers, as well as $z = -\frac{1}{2}$ (on the right hand side, we might allow some function with as few critical points as possible). We will now argue that such a function h does not exist. We can rewrite the desired condition as

$$h'(z) = -2\pi h(z)\frac{\cos \pi z}{\sin \pi z} - 2. \tag{5–1}$$

Since h should be an entire holomorphic function, h' must also be such a function, and thus h must have zeroes at all the integers. Near zero, we can write

$$h'(\varepsilon) \approx -2\pi\varepsilon h'(0)\frac{\cos \pi\varepsilon}{\pi\varepsilon} - 2 \approx -2h'(0) - 2,$$

so it follows that $h'(0) = -2h'(0) - 2$ or $h'(0) = -\frac{2}{3}$. The differential equation (5–1) is real with real initial conditions, so we can look at it over the reals. A periodic differential equation with periodic boundary values will have periodic solutions. For all integers n, we have $h(n) = 0$ and thus $h'(n) = -\frac{2}{3}$. If $h(x) = 0$ for any non-integer value x, then $h'(x) = -2$. Therefore, whenever $h(x) = 0$ for any real number x, we have $h'(x) < 0$, and no real solution curve with $h(x) < 0$ can ever reach the value zero again in positive time.

The boundary condition $h(0) = 0$ will specify a real solution to the differential equation. For small positive x, we will have $h(x) < 0$ because $h'(0) < 0$ and h is analytic. Periodicity of the required solution needs $h(1) = 0$, but this is impossible. Therefore, there is no entire function h solving the differential equation (5–1), and consequently it is not possible to reduce the set of critical points to the set $\mathbb{Z} \cup \{-\frac{1}{2}\}$.

In fact, any solution curve with $h(x) < 0$ for any real value x will tend to $-\infty$ for finite values of x: first consider the homogeneous differential equation

$$\tilde{h}'(x) = -2\pi\tilde{h}(x)\frac{\cos \pi x}{\sin \pi x}.$$

Its solutions are $\tilde{h}(x) = \alpha/\sin^2(\pi x)$ for arbitrary real constants α, and every negative solution at any non-integer will tend to $-\infty$ when x approaches the next integer. In our differential equation for h, the derivative is even more negative than in this homogeneous example, and every negative solution will diverge to $-\infty$ even faster. However, the value of x at which the solution diverges will still be the same integer, not a real number before. This same problem not only exists at the origin, but at all the integers.

This discussion shows that there is no entire function h so that f has no critical points besides the integers and $z = -\frac{1}{2}$. It therefore seems unlikely that the number of extra critical points of f can be reduced significantly by a real entire function h.

A complex candidate map would be

$$z \mapsto z/2 + (2z+1)(1 - e^{i\pi z})/4,$$

possibly added to any entire holomorphic map vanishing on the integers, for example one that makes all the integers again into critical points (such maps are of course included in our family above). As it stands without extra added holomorphic map, this map has an interesting dynamical feature: for points z with large positive imaginary parts, the dynamics is very nearly addition of $\frac{1}{4}$. It is thus conceivable that all points with sufficiently large imaginary parts are contained in a single Fatou component, which would then be a domain at infinity and could help to describe the dynamics.

One could contemplate waiving the condition that f and h must be entire holomorphic functions, allowing entire meromorphic functions. In the real case, we must still have $h(n) = 0$ for all (positive) integers n, and the same problem remains because the considerations above yield another pole exactly at an integer.

We are still hoping that it might be possible to find a holomorphic interpolating function for the $3n+1$-problem for which all the integers are in different Fatou components and for which it is possible to show that there are no wandering Fatou components. We would like to hear suggestions from the readers.

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Simon Letherman, Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, Great Britain

Dierk Schleicher, Fakultät für Mathematik, Technische Universität, Barer Straße 23, D-80290 München, Germany (dierk@mathematik.tu-muenchen.de)

Reg Wood, Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, Great Britain (reg@ma.man.ac.uk)

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