

# The $3x + 1$ Problem

Moritz Konarski, 20.02.2020

# The Function

- ▶ Collatz function  $C(x)$

$$C(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 3x + 1 & \text{if } x \text{ is odd.} \end{cases}$$

- ▶  $3x + 1$  function  $T(x)$

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 1 \pmod{2} \\ \frac{3x+1}{2} & \text{if } x \equiv 0 \pmod{2} \end{cases}$$

# The Conjecture

- ▶ starting at any positive integer, some iterate of these functions takes the value 1
- ▶ for the  $3x + 1$  function  $T(x)$ : every  $m \geq 1$  has some  $T^{(k)}(m) = 1$
- ▶ repeated applications of this function yield *trajectories* or *forward orbits*

# Related Aspects

- ▶ *trajectory or forward orbit:*  
 $O^+(m) := \{m, T(m), T^{(2)}(m), \dots\}$
- ▶ *height  $h(m)$ :* least  $k$  for for which the *Collatz function*  $C(x)$  has  $C^{(k)}(m) = 1$

# Examples of Trajectories

- ▶  $x = 13 \rightarrow \{13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \dots\}$
- ▶  $x = 20 \rightarrow \{20, 10, 5, 16, 8, 4, 2, 1, \dots\}$
- ▶  $x = 512 \rightarrow \{512, 256, 128, 64, 32, 16, 8, 4, 2, 1, \dots\}$
- ▶ numbers up to  $17 \times 2^{58} \approx 4.89 \times 10^{18}$  have been computationally verified

# Why is the $3x + 1$ Problem Interesting?

- ▶ simple problem but it is not solved yet
- ▶ maybe new areas of mathematics are needed to prove or disprove this conjecture
- ▶ could lead to insights into prime factorization

# Related Information

- ▶ part of *number theory*: study of integers and integer valued functions
- ▶ two related questions
  - ▶ is there a divergent trajectory?
  - ▶ are there other cycles besides  $\{4, 2, 1, 4, \dots\}$ ?
- ▶ for negative integers there are cycles starting at  $-1, -5, -17$



# Background of the $3x + 1$ Problem

- ▶ also known as: Syracuse Problem, Hasse's Algorithm, Kakutani's Problem, Ulam's Problem
- ▶ Collatz started studying the function in the 1930s
- ▶ informally discussed in his lecture at the International Math. Congress in Cambridge, Mass. in 1950
- ▶ first published paper about the problem in 1963
- ▶ starting in 1970 there were a lot more publications

# Attempted Proofs

- ▶ Yamada 1981: had mathematical faults
- ▶ Cadogan 2006: had incorrect indices in a function
- ▶ Bruckman's 2008: proof would exclude the cycle starting at 1

# References

- ▶ Lagarias, J. C. (2011). The  $3x + 1$  Problem: An Annotated Bibliography (1963-1999)
- ▶ Lagarias, J. C. (2012). The  $3x + 1$  Problem: An Annotated Bibliography, II (2000-2009)