Complex Powers

Complex Powers

- Complex powers, such as $(1+i)^i$, are defined in terms of the complex exponential and logarithmic functions.
- Recall from that $z = e^{\ln z}$, for all nonzero complex numbers z.
- Thus, when n is an integer, z^n can be written as

$$z^n = (e^{\ln z})^n = e^{n \ln z}.$$

- This formula, which holds for integer exponents n, suggests the following definition for the complex power z^α, for any complex exponent α:

Definition (Complex Powers)

If α is a complex number and $z \neq 0$, then the complex power z^α is defined to be:

$$z^\alpha = e^{\alpha \ln z}.$$

Complex Power Function

- $z^{\alpha} = e^{\alpha \ln z}$ gives an infinite set of values because the complex logarithm $\ln z$ is multiple-valued.
- When n is an integer, the expression is single-valued (in agreement with fact that z^n is a function when n is an integer).

  To see this, note $z^n = e^{n \ln z} = e^{n[\log_e |z| + i\arg(z)]} = e^{n \log_e |z|} e^{n\arg(z)i}$. If $\theta = \mathrm{Arg}(z)$, then $\arg(z) = \theta + 2k\pi$, where k is an integer. So $e^{n\arg(z)i} = e^{n(\theta + 2k\pi)i} = e^{n\theta i} e^{2nk\pi i}$. But, by definition, $e^{2nk\pi i} = \cos(2nk\pi) + i \sin(2nk\pi)$. Because n and k are integers, we have $2nk\pi$ is an even multiple of π, and so $\cos(2nk\pi) = 1$ and $\sin(2nk\pi) = 0$. Consequently, $e^{2nk\pi i} = 1$ and we get $z^n = e^{n \log_e |z|} e^{n\mathrm{Arg}(z)i}$, which is single-valued.
- In general, $z^{\alpha} = e^{\alpha \ln z}$ defines a multiple-valued function.
- It is called a **complex power function**.

Computing Complex Powers

- Find the values of the given complex power:

$$\text{(a)} \quad i^{2i} \qquad \text{(b)} \quad (1+i)^i.$$

(a) We have seen that $\ln i = \frac{(4n+1)\pi}{2} i$. Thus, we obtain:

$$i^{2i} = e^{2i \ln i} = e^{2i[(4n+1)\pi i/2]} = e^{-(4n+1)\pi},$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) We have also seen that $\ln(1+i) = \frac{1}{2} \log_e 2 + \frac{(8n+1)\pi}{4} i$, for $n = 0, \pm 1, \pm 2, \ldots$. Thus, we obtain:

$$(1+i)^i = e^{i \ln(1+i)} = e^{i[(\log_e 2)/2 + (8n+1)\pi i/4]},$$

or

$$(1+i)^i = e^{-(8n+1)\pi/4 + i(\log_e 2)/2},$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Properties of Complex Powers

- Complex powers satisfy the following properties that are analogous to properties of real powers:
  - $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$;
  - $\dfrac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1 - \alpha_2}$;
  - $(z^{\alpha})^n = z^{n\alpha}$, for $n = 0, \pm 1, \pm 2, \ldots$.
- Each of these properties can be derived from the definition of complex powers and the algebraic properties of the complex exponential function e^z:
  - For example, by the definition, $z^{\alpha_1} z^{\alpha_2} = e^{\alpha_1 \ln z} e^{\alpha_2 \ln z}$. By using properties of the exponential, $z^{\alpha_1} z^{\alpha_2} = e^{\alpha_1 \ln z + \alpha_2 \ln z} = e^{(\alpha_1 + \alpha_2) \ln z}$. By the definition, $e^{(\alpha_1 + \alpha_2) \ln z} = z^{\alpha_1 + \alpha_2}$. Thus, $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$.

Principal Value of a Complex Power

- The complex power z^α is, in general, multiple-valued because it is defined using the multiple-valued complex logarithm $\ln z$.
- We can assign a unique value to z^α by using the principal value of the complex logarithm $\text{Ln}z$ in place of $\ln z$.
- This value of the complex power is called the **principal value** of z^α.
- Example: Since $\text{Ln}i = \frac{\pi}{2}i$, the principal value of i^{2i} is $i^{2i} = e^{2i\text{Ln}i} = e^{2i\frac{\pi}{2}i} = e^{-\pi}$.

Definition (Principal Value of a Complex Power)

If α is a complex number and $z \neq 0$, then the function defined by:

$$z^\alpha = e^{\alpha \text{Ln}z}$$

is called the **principal value of the complex power** z^α.

- Notation: z^α will be used to denote both the multiple-valued power function $F(z) = z^\alpha$ and the **principal value power function**.

Computing the Principal Value of a Complex Power

- Find the principal value of each complex power:

$$\text{(a)} \quad (-3)^{i/\pi} \qquad \text{(b)} \quad (2i)^{1-i}.$$

(a) For $z = -3$, we have $|z| = 3$ and $\mathrm{Arg}(-3) = \pi$, and so $\mathrm{Ln}(-3) = \log_e 3 + i\pi$. Thus, we obtain:

$$(-3)^{i/\pi} = e^{(i/\pi)\mathrm{Ln}(-3)} = e^{(i/\pi)(\log_e 3 + i\pi)} = e^{-1+i(\log_e 3)/\pi}.$$

Finally, since $e^{-1+i(\log_e 3)/\pi} = e^{-1}[\cos\frac{\log_e 3}{\pi} + i\sin\frac{\log_e 3}{\pi}]$, $(-3)^{i/\pi} = e^{-1}[\cos\frac{\log_e 3}{\pi} + i\sin\frac{\log_e 3}{\pi}]$.

(b) For $z = 2i$, we have $|z| = 2$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$, and so $\mathrm{Ln}2i = \log_e 2 + i\frac{\pi}{2}$. Thus, we obtain:

$$(2i)^{1-i} = e^{(1-i)\mathrm{Ln}2i} = e^{(1-i)(\log_e 2 + i\pi/2)} = e^{\log_e 2 + \pi/2 - i(\log_e 2 - \pi/2)}.$$

Since $(2i)^{1-i} = e^{\log_e 2 + \pi/2}[\cos(\log_e 2 - \frac{\pi}{2}) - i\sin(\log_e 2 - \frac{\pi}{2})]$, we finally get $(2i)^{1-i} = e^{\log_e 2 + \pi/2}[\cos(\log_e 2 - \frac{\pi}{2}) - i\sin(\log_e 2 - \frac{\pi}{2})]$.

Analyticity

- In general, the principal value of a complex power z^α is not a continuous function on the complex plane because the function $\operatorname{Ln} z$ is not continuous on the complex plane.
- The function $e^{\alpha z}$ is continuous on the entire complex plane and the function $\operatorname{Ln} z$ is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, so z^α is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$.
- Using polar coordinates $r = |z|$ and $\theta = \arg(z)$, we have found that $f_1(z) = e^{\alpha(\log_e r + i\theta)}$, $-\pi < \theta < \pi$ is a branch of $F(z) = z^\alpha = e^{\alpha \ln z}$.
- It is called the **principal branch of the complex power** z^α. Its branch cut is the non-positive real axis, and $z = 0$ is a branch point.
- The branch f_1 agrees with the principal value z^α on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$. Consequently, the derivative of f_1 can be found using the chain rule:
$$f_1'(z) = \frac{d}{dz} e^{\alpha \operatorname{Ln} z} = e^{\alpha \operatorname{Ln} z} \frac{d}{dz}[\alpha \operatorname{Ln} z] = e^{\alpha \operatorname{Ln} z} \frac{\alpha}{z}.$$
Using the principal value $z^\alpha = e^{\alpha \operatorname{Ln} z}$, we find $f_1'(z) = \frac{\alpha z^\alpha}{z} = \alpha z^{\alpha - 1}$.

Derivative of a Power Function

- Find the derivative of the principal value z^i at the point $z = 1 + i$.

  Because the point $z = 1 + i$ is in the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows that $\frac{d}{dz} z^i = iz^{i-1}$, and so, $\frac{d}{dz} z^i \big|_{z=1+i} = \; iz^{i-1}\big|_{z=1+i} = i(1+i)^{i-1}$. We can rewrite this value as:

$$i(1+i)^{i-1} = i(1+i)^i(1+i)^{-1} = i(1+i)^i \frac{1}{1+i} = \frac{1+i}{2}(1+i)^i.$$

  Moreover, the principal value of $(1+i)^i$ is: $(1+i)^i = e^{-\pi/4 + i(\log_e 2)/2}$, and so

$$\frac{d}{dz} z^i \bigg|_{z=1+i} = \frac{1+i}{2} e^{-\pi/4 + i(\log_e 2)/2}.$$

Remarks

(i) There are some properties of real powers that are not satisfied by complex powers. One example of this is that for complex powers, $(z^{\alpha_1})^{\alpha_2} \neq z^{\alpha_1 \alpha_2}$ unless α_2 is an integer.

(ii) As with complex logarithms, some properties that hold for complex powers do not hold for principal values of complex powers.

For example, we can prove that $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$, for any nonzero complex numbers z_1 and z_2. However, this property does not hold for principal values of these complex powers:

If $z_1 = -1$, $z_2 = i$, and $\alpha = i$, then the principal value of $(-1 \cdot i)^i$ is $e^{i\mathrm{Ln}(-i)} = e^{\pi/2}$. On the other hand, the product of the principal values of $(-1)^i$ and i^i is $e^{i\mathrm{Ln}(-1)} e^{i\mathrm{Ln}\,i} = e^{-\pi} e^{-\pi/2} = e^{-3\pi/2}$.