EXERCISES 3.1

In Problems 1–6, find $f'(z)$ forthe given function.

1. $f(z) = 9iz + 2 - 3i$

2. $f(z) = 15z^2 - 4z + 1 - 3i$

3. $f(z) = iz^3 - 7z^2$

4. $f(z) = \dfrac{1}{z}$

5. $f(z) = z - \dfrac{1}{z}$

6. $f(z) = -z^{-2}$

In Problems 7–10, use the alternative definition to find $f'(z)$ f or the given function.

7. $f(z) = 5z^2 - 10z + 8$

8. $f(z) = z^3$

9. $f(z) = z^4 - z^2$

10. $f(z) = \dfrac{1}{2iz}$

In Problems 11–18, use the rules of differentiation to find $f'(z)$ for the given function.

11. $f(z) = (2 - i)z^5 + iz^4 - 3z^2 + i^6$

12. $f(z) = 5(iz)^3 - 10z^2 + 3 - 4i$

13. $f(z) = (z^6 - 1)(z^2 - z + 1 - 5i)$

14. $f(z) = (z^2 + 2z - 7i)^2(z^4 - 4iz)^3$

15. $f(z) = \dfrac{iz^2 - 2z}{3z + 1 - i}$

16. $f(z) = -5iz^2 + \dfrac{2+i}{z^2}$

17. $f(z) = \left(z^4 - 2iz^2 + z\right)^{10}$

18. $f(z) = \left(\dfrac{(4+2i)z}{(2-i)z^2 + 9i}\right)^3$

19. The function $f(z) = |z|^2$ is continuous at the origin.

(**a**) Show that f *is* differentiable at the origin.

(**b**) Show that f *is not* differentiable at any point $z \neq 0$.

20. Show that the function

$$f(z) = \begin{cases} 0, & z = 0 \\ \dfrac{x^3 - y^3}{x^2 + y^2} + i\dfrac{x^3 + y^3}{x^2 + y^2}, & z \neq 0 \end{cases}$$

is not differentiable at $z = 0$ by letting $\Delta z \to 0$ first along the x-axis and then along the line $y = x$.

In Problems 21 and 22, show that the given function is nowhere differentiable.

21. $f(z) = \bar{z}$

22. $f(z) = |z|$

In Problems 23–26, use L'Hôpital's rule to compute the given limit.

23. $\lim\limits_{z \to i} \dfrac{z^7 + i}{z^{14} + 1}$

24. $\lim\limits_{z \to \sqrt{2} + \sqrt{2}i} \dfrac{z^4 + 16}{z^2 - 2\sqrt{2}z + 4}$

25. $\lim\limits_{z \to 1+i} \dfrac{z^5 + 4z}{z^2 - 2z + 2}$

26. $\lim\limits_{z \to \sqrt{2}i} z\dfrac{z^3 + 5z^2 + 2z + 10}{z^5 + 2z^3}$

In Problems 27–30, determine the points at which the given function is not analytic.

27. $f(z) = \dfrac{iz^2 - 2z}{3z + 1 - i}$

28. $f(z) = -5iz^2 + \dfrac{2+i}{z^2}$

29. $f(z) = \left(z^4 - 2iz^2 + z\right)^{10}$

30. $f(z) = \left(\dfrac{(4+2i)z}{(2-i)z^2 + 9i}\right)^3$

EXERCISES 3.2

In Problems 1 and 2, the given function is analytic for all z. Show that the Cauchy-Riemann equations are satisfied at every point.

1. $f(z) = z^3$

2. $f(z) = 3z^2 + 5z - 6i$

In Problems 3–8, show that the given function is not analytic at any point.

3. $f(z) = \text{Re}(z)$

4. $f(z) = y + ix$

5. $f(z) = 4z - 6\bar{z} + 3$

6. $f(z) = \bar{z}^2$

7. $f(z) = x^2 + y^2$

8. $f(z) = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

In Problems 9–16, use Theorem to show that the given function is analytic in an appropriate domain.

9. $f(z) = e^{-x}\cos y - ie^{-x}\sin y$

10. $f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$

11. $f(z) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$

12. $f(z) = 4x^2 + 5x - 4y^2 + 9 + i(8xy + 5y - 1)$

13. $f(z) = \dfrac{x-1}{(x-1)^2+y^2} - i\dfrac{y}{(x-1)^2+y^2}$

14. $f(z) = \dfrac{x^3+xy^2+x}{x^2+y^2} + i\dfrac{x^2y+y^3-y}{x^2+y^2}$

15. $f(z) = \dfrac{\cos\theta}{r} - i\dfrac{\sin\theta}{r}$

16. $f(z) = 5r\cos\theta + r^4\cos 4\theta + i(5r\sin\theta + r^4\sin 4\theta)$

In Problems 17 and 18, find real constants a, b, c, and d so that the given function is analytic.

17. $f(z) = 3x - y + 5 + i(ax + by - 3)$

18. $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

In Problems 19–22, show that the given function is not analytic at any point but is differentiable along the indicated curve(s).

19. $f(z) = x^2 + y^2 + 2ixy$; x-axis

20. $f(z) = 3x^2y^2 - 6ix^2y^2$; coordinate axes

21. $f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y)$; coordinate axes

22. $f(z) = x^2 - x + y + i(y^2 - 5y - x)$; $\quad y = x + 2$

EXERCISES 3.3

In Problems 1–8, verify that the given function u is harmonic in an appropriate domain D.

1. $u(x, y) = x$

2. $u(x, y) = 2x - 2xy$

3. $u(x, y) = x^2 - y^2$

4. $u(x, y) = x^3 - 3xy^2$

5. $u(x, y) = \log_e(x^2 + y^2)$

6. $u(x, y) = \cos x \cosh y$

7. $u(x, y) = e^x(x \cos y - y \sin y)$

8. $u(x, y) = -e^{-x} \sin y$

9. For each of the functions $u(x, y)$ in Problems 1, 3, 5, and 7, find $v(x, y)$, the harmonic conjugate of u. Form the corresponding analytic function $f(z) = u + iv$.

10. Repeat Problem 9 for each of the functions $u(x, y)$ in Problems 2, 4, 6, and 8.

In Problems 11 and 12, verify that the given function u is harmonic in an appropriate domain D. Find its harmonic conjugate v and find analytic function $f(z) = u + iv$ satisfying the indicated condition.

11. $u(x, y) = xy + x + 2y$; $f(2i) = -1 + 5i$

12. $u(x, y) = 4xy^3 - 4x^3y + x$; $f(1 + i) = 5 + 4i$