The $3x + 1$ Problem

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Overview

- The $3x + 1$ Problem and Collatz Conjecture
- What Makes This Problem Interesting?
- History of the Collatz Conjecture
- Interesting Attributes of the $3x + 1$ Problem

Interesting Attributes

- Cycles of the Function
- Stochastic Approximations
- Stopping Time of the Function

What is the $3x + 1$ Problem?

The Function

- based on the Collatz function [3]

$$C(x) = \begin{cases} 3x + 1 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

- equivalent to the $3x + 1$ function [3]

$$T(x) = \begin{cases} (3x + 1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

Details

- it is conjectured that for some $x, k \in \mathbb{N} + 1$ we attain $T^{(k)}(x) = 1$ [1]
- the $3x + 1$ function $T(x)$ maps $\mathbb{N} + 1 \to \mathbb{N} + 1$ [4]
- the function has a *stopping time*, *total stopping time*, and a *trajectory* for each m

Stopping Time for x

- check that every positive integer up to $x - 1$ iterates to one [1]
- if $T^{(k)}(x) < x$, we know it will iterate to 1
- thus the stopping time is

$$\sigma(x) = \inf\{k : T^{(k)}(x) < x\}$$

Total Stopping Time for x

- total stopping time is the number of steps needed to iterate to 1 [1]

$$\sigma_\infty(x) = \inf\{k : T^{(k)}(x) = 1\}$$

Trajectory of x Under T

- also called the *forward orbit* of x under T
- defined as the sequence of its forward iterates [3]

$$\{x, T(x), T^{(2)}(x), T^{(3)}(x), \dots\}$$

The Collatz Conjecture

Possible behaviors of T

The function $T(x)$ can

1. enter the trivial cycle $\{2, 1, 2, \dots\}$ (reach 1)
2. enter a non-trivial cycle
3. diverge to infinity, have a divergent orbit [1]

The Conjecture

- beginning at any positve integer x, iterations of $T(x)$ will eventually reach 1 and enter the trivial cycle [3]
- equivalent to stating that the total stopping time $\sigma_\infty(x)$ is finite [1]
- every trajectory of $T(x)$ contains 1 and is finite [2]

What Makes This Problem Interesting?

Mathematics is not ready for such problems.

— *Paul Erdös* [1]

- it is simple to state but hard to prove
- part of the difficulty comes from its pseudorandom nature of iterations of $T(x)$
- the problem itself is not really important, it has no immediate applications
- represents a class of iterative mappings that are interesting [3]

History of the Collatz Conjecture

Beginnings

- named after Lothar Collatz who formulated similar problems in the 1930s
- also known as Syracuse Problem, Hasse's Algorithm, Kakutani's Problem, and Ulam's Problem after other people that studied it
- academic publishing about it began in the 1970s [3]

Recent Developments

- $> 10^{20}$ numbers have been verified to fit the conjecture [4]
- a September 2019 paper by Terence Tao “Almost All Orbits of the Collatz Map Attain Almost Bounded Values” made progress
- research is still actively ongoing

Interesting Attributes of the $3x + 1$ Problem

Cycles of the Function

- the $3x + 1$ function has a trivial cycle $\{2, 1, 2, \dots\}$ at 1 [1]
- if $T(x)$ is applied to all integers, three more cycles emerge at -1, -5, and -17
- these cycles are conjectured to be the only ones [1]
- if non-trivial cycles of the $3x + 1$ problem exist, they have been proven to be at least 10,439,860,591 numbers long [3]

Stochastic Approximations

- number of odd and even integers in an orbit is approximately equal [3]
- behavior is seen as pseudorandom, if the numbers are large enough they almost behave like random variables [2]
- probabilistic models describe the behavior of the $3x + 1$ problem
- models describe groups of trajectories, not individual ones [3]

Stopping Time of the Function

- stopping time for odd numbers is ≈ 9.477955 for $C(x)$ [1]
- total stopping time for most trajectories is about $6.95212 \log n$ steps
- number of even integers in an orbit equal to stopping time
- upper bound for total stopping time $41.677647 \log n$, suggests all sequences are finite [3]

Conclusion

The $3x + 1$ Problem and Collatz Conjecture

For every $x \in \mathbb{N} + 1$ and the function

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

there is some $k \in \mathbb{N} + 1$ such that $T^{(k)}(x) = 1$.

What Makes This Problem Interesting?

- simple to state but hard to prove
- represents a class of iterative mappings that are interesting [3]
- maybe mathematics right now cannot solve that problem

History of the Collatz Conjecture

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- the $3x + 1$ function has a trivial cycle $\{2, 1, 2, \dots\}$ at 1 [1]
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Any questions?

References

1. Marc Chamberland, *An Update on the $3x+1$ Problem*, `http://www.math.grinnell.edu/~chamberl/papers/3x_survey_eng.pdf`, 2005.
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3. Jeffrey C. Lagarias, *The $3x+1$ Problem: An Overview*, `https://pdfs.semanticscholar.org/1000/46dd8b4ee901bc71043da7d42f5d87ca0224.pdf`, 2010
4. Terence Tao, *Almost All Orbits of the Collatz Map Attain Almost Bounded Values*, arXiv:1909.03562v2 [math.PR], 2019