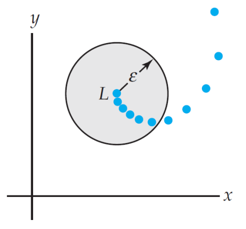
Sequences and Series

Sequences

- A **sequence** $\{z_n\}$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $\mathbb{C}$.
- Example: The sequence $\{1+i^n\}$ is $\underset{n=1}{1+i}, \underset{n=2}{0}, \underset{n=3}{1-i}, \underset{n=4}{2}, \underset{n=5}{1+i}, \ldots$.
- If $\lim_{n\to\infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent**, i.e., $\{z_n\}$ converges to the number L if, for each positive real number ε, an N can be found, such that $|z_n - L| < \varepsilon$, whenever $n > N$.
- Since $|z_n - L|$ is distance, the terms z_n of a sequence that converges to L can be made arbitrarily close to L. In a different way, when a sequence $\{z_n\}$ converges to L, then all but a finite number of the terms of the sequence are within every ε-neighborhood of L.



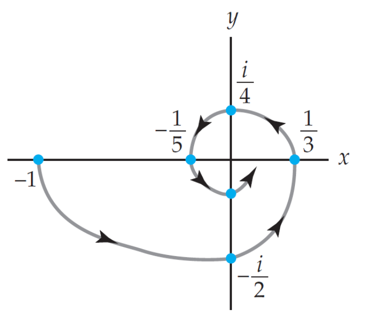
- A sequence that is not convergent is said to be **divergent**. Example: The sequence $\{1+i^n\}$ is divergent since the general term $z_n = 1 + i^n$ does not approach a fixed complex number as $n \to \infty$.

An Example of a Convergent Sequence

- The sequence $\left\{\dfrac{i^{n+1}}{n}\right\}$ converges since $\lim\limits_{n\to\infty}\dfrac{i^{n+1}}{n}=0$. As we see from

$$-1, -\frac{i}{2}, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5}, \ldots,$$

the terms of the sequence spiral in toward the point $z = 0$ as n increases.



Criterion for Convergence

Theorem (Criterion for Convergence)

A sequence $\{z_n\}$ converges to a complex number $L = a + ib$ if and only if $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(L) = a$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(L) = b$.

- Example: Consider the sequence $\left\{\dfrac{3+ni}{n+2ni}\right\}$.

$$z_n = \frac{3+ni}{n+2ni} = \frac{(3+ni)(n-2ni)}{n^2+4n^2} = \frac{2n^2+3n}{5n^2} + i\frac{n^2-6n}{5n^2}.$$

Thus, we get

$$\mathrm{Re}(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \to \frac{2}{5}$$

$$\mathrm{Im}(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \to \frac{1}{5}.$$

By the theorem, the given sequence converges to $a + ib = \frac{2}{5} + \frac{1}{5}i$.

Series and Geometric Series

- An **infinite series** or **series** of complex numbers $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$ is **convergent** if the sequence of partial sums $\{S_n\}$, where $S_n = z_1 + z_2 + z_3 + \cdots + z_n$ converges. If $S_n \to L$ as $n \to \infty$, we say that the series **converges to** L or that the **sum** of the series is L.
- Geometric Series: A **geometric series** is any series of the form $\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots$. The n-th term of the sequence of partial sums is $S_n = a + az + az^2 + \cdots + az^{n-1}$. To get a formula for S_n, multiply by z: $zS_n = az + az^2 + az^3 + \cdots + az^n$. Subtract this from S_n: $S_n - zS_n = (a + az + az^2 + \cdots + az^{n-1}) - (az + az^2 + az^3 + \cdots + az^{n-1} + az^n) = a - az^n$. Thus,

  $(1-z)S_n = a(1-z^n)$, and, hence, $S_n = \dfrac{a(1-z^n)}{1-z}$.
  - If $|z| < 1$, $z^n \to 0$ as $n \to \infty$. So $S_n \to \frac{a}{1-z}$. I.e., for $|z| < 1$, $\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots$.
  - If $|z| \geq 1$, a geometric series diverges.

Special Geometric Series

- Recall the sum formulas
$$S_n = \frac{a(1-z^n)}{1-z}, \quad \frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots .$$
- If we set $a = 1$, we get
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots .$$
- If we then replace z by $-z$:
$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots .$$
- For the finite sum, we have $\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$. Rewriting the left side of the above equation as $\frac{1-z^n}{1-z} = \frac{1}{1-z} + \frac{-z^n}{1-z}$, we get
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1-z}.$$

A Convergent Geometric Series

- The infinite series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series.

It has the standard form, with $a = \frac{1}{5}(1+2i)$ and $z = \frac{1}{5}(1+2i)$. Since $|z| = \frac{\sqrt{5}}{5} < 1,$ the series is convergent and its sum is given by:

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1-\frac{1+2i}{5}} = \frac{1+2i}{4-2i} = \frac{1}{2}i.$$

Necessary Condition for Convergence

- We turn to some important theorems about convergence and divergence of an infinite series:

Theorem (A Necessary Condition for Convergence)

If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = 0$.

- Let L denote the sum of the series. Then $S_n \to L$ and $S_{n-1} \to L$ as $n \to \infty$. By taking the limit of both sides of $S_n - S_{n-1} = z_n$ as $n \to \infty$, we obtain the desired conclusion.

Theorem (The n-th Term Test for Divergence)

If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{k=1}^{\infty} z_k$ diverges.

- Example: The series $\sum_{k=1}^{\infty} \frac{ik+5}{k}$ diverges, since $z_n = \frac{in+5}{n} \to i \neq 0$ as $n \to \infty$.
  The geometric series $\sum_{k=1}^{\infty} az^k$ diverges if $|z| \geq 1$ because even in the case when $\lim_{n\to\infty} |z^n|$ exists, the limit is not zero.

Absolute and Conditional Convergence

Definition (Absolute and Conditional Convergence)

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

- In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a *p*-**series** and
  - converges for $p > 1$;
  - diverges for $p \leq 1$.
- Example: The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent: The series $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right|$ is the same as the real convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.
- As in real calculus, absolute convergence implies convergence.
- Example: The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2} = i - \frac{1}{2^2} - \frac{i}{3^2} + \cdots$ converges, because it is was shown to be absolutely convergent.

Tests for Convergence

Theorem (The Ratio Test)

Let $\sum_{k=1}^{\infty} z_k$ be a series of nonzero terms, with $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$.

(i) If $L < 1$, then the series converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then the series diverges.

(iii) If $L = 1$, the test is inconclusive.

Theorem (The Root Test)

Let $\sum_{k=1}^{\infty} z_k$ be a series of complex terms, with $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$.

(i) If $L < 1$, then the series converges absolutely.

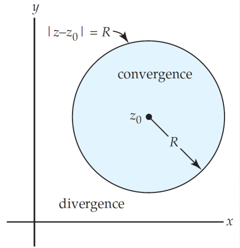
(ii) If $L > 1$ or $L = \infty$, then the series diverges.

(iii) If $L = 1$, the test is inconclusive.

- We are interested primarily in applying these tests to power series.

Power Series and Circle of Convergence

- An infinite series of the form $\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$, where the coefficients a_k are complex constants, is called a **power series** in $z - z_0$.
- The power series is said to be **centered at** z_0 and the complex point z_0 is referred to as the **center** of the series.
- It is also convenient to define $(z-z_0)^0 = 1$ even when $z = z_0$.
- Every complex power series has a **radius of convergence** and a **circle of convergence**: It is the circle centered at z_0 of largest radius $R > 0$ for which the series converges at every point within the circle $|z - z_0| = R$.



A power series converges absolutely at all points z satisfying $|z - z_0| < R$, and diverges at all points z, with $|z - z_0| > R$.

Possibilities for Radius of Convergence

- The radius of convergence can be:
  (i) $R = 0$ (series converges only at its center $z = z_0$);
  (ii) R a finite positive number (series converges in interior of $|z - z_0| = R$);
  (iii) $R = \infty$ (series converges for all z).

  A power series may converge at some, all, or at none of the points on the actual circle of convergence.
- Example: Consider $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test, $\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| =$ $\lim_{n\to\infty} \frac{n}{n+1}|z| = |z|$. Thus, the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$. On the circle $|z| = 1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the well-known divergent harmonic series. This does not mean that the series diverges on the circle of convergence. In fact, at $z = -1$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is the convergent alternating harmonic series. It can be shown that the series converges at all points on the circle $|z| = 1$ except at $z = 1$.

Dependence of the Radius on the Coefficients

- For a power series
$$\sum_{k=0}^{\infty} a_k(z-z_0)^k,$$
the limit depends only on the coefficients a_k. Thus:
  (i) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, the radius of convergence is $R = \frac{1}{L}$;
  (ii) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, the radius of convergence is $R = \infty$;
  (iii) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, the radius of convergence is $R = 0$.
- Similar conclusions can be made for the root test by utilizing $\lim_{n\to\infty}\sqrt[n]{|a_n|}$. E.g., if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L \neq 0$, then $R = \frac{1}{L}$.

Finding Radius of Convergence Using Ratio Test

- Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^k$.

  With the identification $a_n = \frac{(-1)^{n+1}}{n!}$, we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

  Hence, the radius of convergence is ∞. The power series with center $z_0 = 1 + i$ converges absolutely for all z, i.e., for $|z-1-i| < \infty$.

Finding Radius of Convergence Using Root Test

- Consider the power series $\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5} \right)^k (z-2i)^k$.
- With $a_n = \left(\frac{6n+1}{2n+5} \right)^n$, the root test gives

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left(\frac{6n+1}{2n+5} \right) = 3.$$

We conclude that the radius of convergence of the series is $R = \frac{1}{3}$. The circle of convergence is $|z-2i| = \frac{1}{3}$; the power series converges absolutely for $|z-2i| < \frac{1}{3}$.

The Arithmetic of Power Series

- Some facts concerning power-series stated informally:
  - A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be multiplied by a nonzero complex constant c without affecting its convergence or divergence.
  - A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely within its circle of convergence. As a consequence, within the circle of convergence the terms of the series can be rearranged and the rearranged series has the same sum L as the original series.
  - Two power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ can be added and subtracted by adding or subtracting like terms:

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \pm \sum_{k=0}^{\infty} b_k(z-z_0)^k = \sum_{k=0}^{\infty}(a_k \pm b_k)(z-z_0)^k.$$

    - If both series have the same nonzero radius R of convergence, the radius of convergence of $\sum_{k=0}^{\infty}(a_k \pm b_k)(z-z_0)^k$ is R.
    - If one series has radius of convergence $r > 0$ and the other $R > 0$, where $r \neq R$, then $\sum_{k=0}^{\infty}(a_k \pm b_k)(z-z_0)^k$ has radius of convergence the smaller of r and R.
  - Two power series can (with care) be multiplied and divided.

Final Remarks on Series and Power Series

- If $z_n = a_n + ib_n$ then the n-th term of the sequence of partial sums for $\sum_{k=1}^{\infty} z_k$ is $S_n = \sum_{k=1}^{n}(a_k + ib_k) = \sum_{k=1}^{n} a_k + i\sum_{k=1}^{n} b_k$. Thus, $\sum_{k=1}^{\infty} z_k$ converges to $L = a + ib$ if and only if $\mathrm{Re}(S_n) = \sum_{k=1}^{n} a_k$ converges to a and $\mathrm{Im}(S_n) = \sum_{k=1}^{n} b_k$ converges to b.
- In summation notation a geometric series need not start at $k = 1$ nor does the general term have to appear precisely as az^{k-1}.
- Example: Consider $\sum_{k=3}^{\infty} 40\frac{i^{k+2}}{2^{k-1}}$. It does not appear to match the form $\sum_{k=1}^{\infty} az^{k-1}$ of a geometric series. By writing out three terms, $\sum_{k=3}^{\infty} 40\frac{i^{k+2}}{2^{k-1}} = 40\frac{i^5}{2^2} + 40\frac{i^6}{2^3} + 40\frac{i^7}{2^4} + \cdots$ we see $a = 40\frac{i^5}{2^2}$ and $z = \frac{i}{2}$. Since $|z| = \frac{1}{2} < 1$, the sum is $\sum_{k=3}^{\infty} 40\frac{i^{k+2}}{2^{k-1}} = \dfrac{40\frac{i^5}{2^2}}{1-\frac{i}{2}} = -4 + 8i$.
- A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ *always* possesses a radius of convergence R. The ratio and root tests lead to $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ and $\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{|a_n|}$ assuming the appropriate limit exists.

Taylor Series

Differentiation of Power Series

Theorem (Continuity)

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ represents a continuous function f within its circle of convergence $|z-z_0| = R$.

Theorem (Term-by-Term Differentiation)

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be differentiated term by term within its circle of convergence $|z-z_0| = R$.

- Differentiating a power series term-by-term gives,
$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dz}(z-z_0)^k = \sum_{k=1}^{\infty} a_k k(z-z_0)^{k-1}.$$
- Using the ratio test, it can be shown that the original series and the differentiated series have the same circle of convergence.
- Since the derivative of a power series is another power series, the first series can be differentiated as many times as we wish.

Integration of Power Series

Theorem (Term-by-Term Integration)

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be integrated term-by-term within its circle of convergence $|z-z_0|=R$, for every contour C lying entirely within the circle of convergence.

- The theorem states that
$$\int_C \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k dz,$$
whenever C lies in the interior of $|z-z_0|=R$.
- Indefinite integration can also be carried out term by term:
$$\int \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int (z-z_0)^k dz = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1}+K.$$
- The ratio test can be used to prove that both series have the same circle of convergence.

Analyticity

- Suppose a power series represents a function f within $|z - z_0| = R$, i.e., $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k =$
  $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$.
- Then, the derivatives of f are the series

$$
\begin{aligned}
f'(z) &= \sum_{k=1}^{\infty} a_k k(z - z_0)^{k-1} = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots \\
f''(z) &= \sum_{k=2}^{\infty} a_k k(k-1)(z - z_0)^{k-2} = 2 \cdot 1 a_2 + 3 \cdot 2 a_3(z - z_0) + \cdots \\
f'''(z) &= \sum_{k=3}^{\infty} a_k k(k-1)(k-2)(z - z_0)^{k-3} = 3 \cdot 2 \cdot 1 a_3 + \cdots \\
&\vdots
\end{aligned}
$$

- Since the power series represents a differentiable function f within its circle of convergence $|z - z_0| = R$, it represents an analytic function within its circle of convergence.

Taylor Series and Maclaurin Series

- Evaluating the derivatives at $z = z_0$ gives

$$f(z_0) = a_0, \; f'(z_0) = 1!a_1, \; f''(z_0) = 2!a_2, \; f'''(z_0) = 3!a_3.$$

- In general, $f^{(n)}(z_0) = n!a_n$, or $a_n = \frac{f^{(n)}(z_0)}{n!}$, $n \geq 0$.
- When $n = 0$, we interpret the zero-order derivative as $f(z_0)$ and $0! = 1$, so that the formula gives $a_0 = f(z_0)$.
- Substituting into the series yields

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

  This series is called the **Taylor series** for f centered at z_0.
- A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

  is referred to as a **Maclaurin series**.

Taylor's Theorem

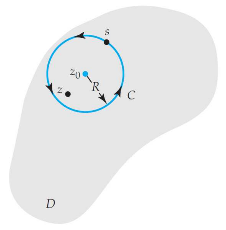
- Since a power series converges in a circular domain, and a domain D is generally not circular, the following question arises:

  > Can we expand f in one or more power series that are valid, i.e., a power series that converges at z and the number to which the series converges is $f(z)$, in circular domains that are all contained in D?

Theorem (Taylor's Theorem)

Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ valid for the largest circle C with center at z_0 and radius R that lies entirely within D.

- Let z be a fixed point within the circle C and let s denote the variable of integration. The circle C is then described by $|s - z_0| = R$. We use the Cauchy integral formula to obtain the value of f at z:



Proof of Taylor's Theorem I

- $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)-(z-z_0)}ds =$
$\frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z_0}\left(\frac{1}{1-\frac{z-z_0}{s-z_0}}\right)ds$. By the power series for $\frac{1}{1-z}$, we get
$\frac{1}{1-\frac{z-z_0}{s-z_0}} = 1 + \frac{z-z_0}{s-z_0} + \left(\frac{z-z_0}{s-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{s-z_0}\right)^{n-1} + \frac{(z-z_0)^n}{(s-z)(s-z_0)^{n-1}}$,
whence, we get
$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z_0}ds + \frac{z-z_0}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)^2}ds + \frac{(z-z_0)^2}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)^3}ds +$
$\cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)^n}ds + \frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(s)}{(s-z)(s-z_0)^n}ds$. By Cauchy's integral formula for derivatives, $f(z) =$
$f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1}+R_n(z)$,
where $R_n(z) = \frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(s)}{(s-z)(s-z_0)^n}ds$. This is called **Taylor's formula with remainder** R_n. The goal now is to show that $R_n(z) \to 0$ as $n \to \infty$.

Proof of Taylor's Theorem II

- To see that $R_n(z) = \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z)(s-z_0)^n} ds \to 0$, it suffices to show that $|R_n(z)| \to 0$ as $n \to \infty$. Since f is analytic in D, we know that $|f(z)|$ has a maximum value M on the contour C. In addition, since z is inside C, $|z - z_0| < R$ and, consequently, $|s - z| = |s - z_0 - (z - z_0)| \geq |s - z_0| - |z - z_0| = R - d$, where $d = |z - z_0|$ is the distance from z to z_0. The ML-inequality then gives $|R_n(z)| = \left| \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z)(s-z_0)^n} ds \right| \leq \frac{d^n}{2\pi} \cdot \frac{M}{(R-d)R^n} \cdot 2\pi R = \frac{MR}{R-d} \left(\frac{d}{R}\right)^n$. Because $d < R$, $\left(\frac{d}{R}\right)^n \to 0$ as $n \to \infty$, we conclude that $|R_n(z)| \to 0$ as $n \to \infty$. It follows that the infinite series $f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$ converges to $f(z)$.

Isolated Singularities and Important Maclaurin Series

- An **isolated singularity** of a function f is a point at which f fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

  Example: $f(z) = \frac{1}{z-5i}$ has an isolated singularity at $z = 5i$.
- The radius of convergence R of a Taylor series for f is the distance from the center z_0 of the series to the nearest isolated singularity of f.
- Thus, if the function f is entire, then the radius of convergence of a Taylor series centered at any point z_0 is necessarily $R = \infty$.
- We summarize some Important Maclaurin Series:

$$\begin{aligned} e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \end{aligned}$$

Finding Radius of Convergence

- Suppose the function $f(z) = \frac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4 - 2i$. What is its radius of convergence R?

  Observe that the function is analytic at every point except at $z = -1 + i$, which is an isolated singularity of f. The distance from $z = -1 + i$ to $z_0 = 4 - 2i$ is

$$|z - z_0| = \sqrt{(-1-4)^2 + (1-(-2))^2} = \sqrt{34}.$$

  Thus, the radius of convergence for the Taylor series centered at $4 - 2i$ is $R = \sqrt{34}$.

Uniqueness of the Series Expansion

- If two power series with center z_0,

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k(z-z_0)^k$$

  represent the same function f and have the same nonzero radius R of convergence, then $a_k = b_k = \frac{f^{(k)}(z_0)}{k!}$, $k = 0, 1, 2, \ldots$.
- Stated in another way, the power series expansion of a function, with center z_0, is unique.
- Thus, a power series expansion of an analytic function f centered at z_0, irrespective of the method used to obtain it, is the Taylor series expansion of the function.

Finding a Maclaurin Series

- Find the Maclaurin expansion of $f(z) = \frac{1}{(1-z)^2}$.

  Recall that for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots .$$

If we differentiate both sides of the last result with respect to z,

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}1 + \frac{d}{dz}z + \frac{d}{dz}z^2 + \frac{d}{dz}z^3 + \cdots$$

or

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}.$$

The radius of convergence of the last power series is the same as the original series $R = 1$.

Finding a Taylor Series

- Expand $f(z) = \frac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

  We use again $\frac{1}{1-z} = 1 + z + z^2 + \cdots$. By adding and subtracting $2i$ in the denominator, $\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{1-2i-(z-2i)} = \frac{1}{1-2i} \cdot \frac{1}{1-\frac{z-2i}{1-2i}}$.

  We now write $\frac{1}{1-\frac{z-2i}{1-2i}}$ as a power series:

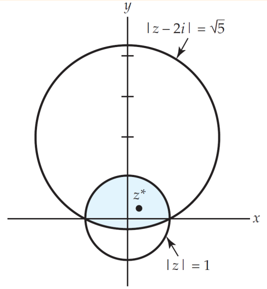
  $\frac{1}{1-z} = \frac{1}{1-2i}\left[1 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^2 + \left(\frac{z-2i}{1-2i}\right)^3 + \cdots\right]$ or

  $\frac{1}{1-z} = \frac{1}{1-2i} + \frac{1}{(1-2i)^2}(z-2i) + \frac{1}{(1-2i)^3}(z-2i)^2 + \frac{1}{(1-2i)^4}(z-2i)^3 + \cdots$.

  Because the distance from the center $z_0 = 2i$ to the nearest singularity $z = 1$ is $\sqrt{5}$, we conclude that the circle of convergence is $|z-2i| = \sqrt{5}$.

Power Series for the Same Function

- We have represented the same function $f(z) = \frac{1}{1-z}$ by two different power series; one with center $z_0 = 0$ and radius of convergence $R = 1$; another with center $z_0 = 2i$ and radius of convergence $R = \sqrt{5}$.



The interior of the intersection of the two circles is the region where both series converge, i.e., at a specified point z^* in this region, both series converge to same value $f(z^*) = \frac{1}{1-z^*}$. Outside the colored region at least one of the two series must diverge.

Laurent Series

Isolated Singularities

- Suppose that $z = z_0$ is a singularity of a complex function f, i.e., a point at which f fails to be analytic.
- The point $z = z_0$ is said to be an **isolated singularity** of the function f if there exists some deleted neighborhood, or punctured open disk, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic.
  Example: The points $z = 2i$ and $z = -2i$ are singularities of $f(z) = \frac{z}{z^2+4}$. Both $2i$ and $-2i$ are isolated singularities since f is analytic at every point in the neighborhood defined by $|z - 2i| < 1$, except at $z = 2i$, and at every point in the neighborhood defined by $|z - (-2i)| < 1$, except at $z = -2i$. In other words, f is analytic in the deleted neighborhoods $0 < |z - 2i| < 1$ and $0 < |z + 2i| < 1$.
- A singular point $z = z_0$ of a function f is **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0.
  Example: The branch point $z = 0$ is a nonisolated singularity of $\mathrm{Ln} z$ since every neighborhood of $z = 0$ contains points on the negative real axis.

A New Kind of Series

- If $z = z_0$ is a singularity of a function f, then certainly f cannot be expanded in a power series with z_0 as its center.
- About an isolated singularity $z = z_0$, it is still possible to represent f by a series involving both negative and nonnegative integer powers of $z - z_0$, i.e.,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots .$$

- Example: Consider the function $f(z) = \frac{1}{z-1}$. The point $z = 1$ is an isolated singularity of f and, consequently, the function cannot be expanded in a Taylor series centered at that point. Nevertheless, f can expanded in a series of the previous form that is valid for all z near 1: $f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{z-1} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots$. This series representation is valid for $0 < |z-1| < \infty$.

Principal Part and Analytic Part

- Using summation notation, we can rewrite

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

- The part with negative powers $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$ is called the **principal part** of the series. It converges for $\left|\frac{1}{z-z_0}\right| < r^*$ or, equivalently, for $|z-z_0| > \frac{1}{r^*} = r$.
- The part consisting of the nonnegative powers $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, is called the **analytic part** of the series. It converges for $|z-z_0| < R$.
- Thus, the sum converges when z satisfies both $|z-z_0| > r$ and $|z-z_0| < R$, i.e., when z is a point in an annular domain defined by $r < |z-z_0| < R$.
- By summing over negative and nonnegative integers, we can rewrite $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$.

An Example

- The function $f(z) = \frac{\sin z}{z^4}$ is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series.
- However, $\sin z$ is an entire function having Maclaurin series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots,$$

  which converges for $|z| < \infty$.
- By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z:

$$f(z) = \frac{\sin z}{z^4} = \overbrace{\frac{1}{z^3} - \frac{1}{3!z}}^{\text{principal part}} \overbrace{+\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}^{\text{analytic part}}.$$

- The analytic part converges for $|z| < \infty$.
- The principal part is valid for $|z| > 0$.
- The series converges for all z, but $z = 0$, i.e., is valid for $0 < |z| < \infty$.

Laurent Series and Laurent's Theorem

- A series representation of a function f consisting of both negative and nonnegative powers of $z - z_0$ is called a **Laurent series** or a **Laurent expansion** of f about z_0 on the annulus $r < |z - z_0| < R$.

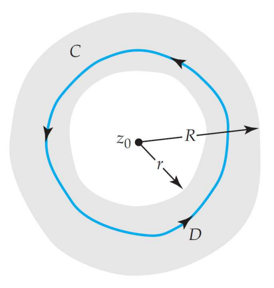
Theorem (Laurent's Theorem)

Let f be analytic within the annulus D defined by $r < |z - z_0| < R$. Then f has the series representation $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ valid for $r < |z - z_0| < R$.

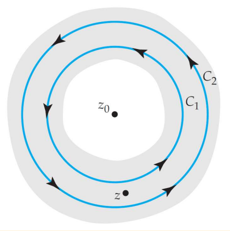
The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds,$$

$k = 0, \pm 1, \pm 2, \ldots,$ where C is a simple closed curve that lies entirely within D and has z_0 in its interior.



Proof of Laurent's Theorem I



- Let C_1 and C_2 be concentric circles with center z_0 and radii r_1 and R_2, where $r < r_1 < R_2 < R$. Let z be a fixed point in D that satisfies $r_1 < |z - z_0| < R_2$. By introducing a crosscut between C_2 and C_1, Cauchy's formula gives $f(z) = \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}ds$.

  We can write $\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}ds = \sum_{k=0}^{\infty} a_k(z-z_0)^k$, where $a_k = \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{(s-z_0)^{k+1}}ds$, $k = 0, 1, 2, \ldots$. We have $-\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{(z-z_0)-(s-z_0)}ds = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{z-z_0}\left(\frac{1}{1-\frac{s-z_0}{z-z_0}}\right)ds = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{z-z_0}\left(1+\frac{s-z_0}{z-z_0}+\cdots+\left(\frac{s-z_0}{z-z_0}\right)^{n-1}+\frac{(s-z_0)^n}{(z-s)(z-z_0)^{n-1}}\right)ds = \sum_{k=1}^{n}\frac{a_{-k}}{(z-z_0)^k} + R_n(z)$, $a_{-k} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{(s-z_0)^{-k+1}}ds$, $R_n(z) = \frac{1}{2\pi i(z-z_0)^n}\oint_{C_1}\frac{f(s)(s-z_0)^n}{z-s}ds$.

Proof of Laurent's Theorem II

- Now let $d = |z - z_0|$ and let M denote the maximum value of $|f(z)|$ on C_1. Using $|s - z_0| = r_1$ and $|z - s| = |z - z_0 - (s - z_0)|$ $\geq |z - z_0| - |s - z_0| = d - r_1$, the ML-inequality gives: $|R_n(z)| = \left|\frac{1}{2\pi i(z-z_0)^n} \oint_{C_1} \frac{f(s)(s-z_0)^n}{z-s} ds\right| \leq \frac{1}{2\pi d^n} \frac{Mr_1^n}{d-r_1} 2\pi r_1 = \frac{Mr_1}{d-r_1} \left(\frac{r_1}{d}\right)^n$. Because $r_1 < d$, $\left(\frac{r_1}{d}\right)^n \to 0$ as $n \to \infty$, and so $|R_n(z)| \to 0$ as $n \to \infty$. Thus we have shown that $-\frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{s-z} ds = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^k$.
- Therefore, overall we have

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^k + \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$

By summing over all integer powers,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k,\ a_k = \oint_C \frac{f(z)}{(z - z_0)^{k+1}} dz,\ k = 0, \pm 1, \pm 2, \ldots.$$

Remarks

- In the case when $a_{-k} = 0$ for $k = 1, 2, \ldots$, the principal part is zero and the Laurent series reduces to a Taylor series.
- The annular domain $r < |z - z_0| < R$ need not have a "ring" shape. Some other possible annular domains are:
  - (i) $r = 0$, R finite; In this case, the series converges in $0 < |z - z_0| < R$, i.e., the domain is a punctured open disk.
  - (ii) $r \neq 0$, $R = \infty$; In this case, the annular domain is $r < |z - z_0|$ and consists of all points exterior to the circle $|z - z_0| = r$.
  - (iii) $r = 0$, $R = \infty$; In this case, the domain is defined by $0 < |z - z_0|$. This represents the entire complex plane except the point z_0.
- Finding the Laurent series of a function in a specified annular domain is generally difficult, but in many instances we can obtain a desired Laurent series by either
  - employing a known power series expansion of a function; or by
  - creative manipulation of a suitably chosen geometric series.

Finding Laurent Expansions I

- Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains.

  (a) $0 < |z| < 1$ (b) $1 < |z|$ (c) $0 < |z-1| < 1$ (d) $1 < |z-1|$.

- In parts (a) and (b) we want only powers of z, whereas in parts (c) and (d) we want powers of $z-1$.

(a) $f(z) = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\left(1 + z + z^2 + z^3 + \cdots\right)$. The infinite series in the brackets converges for $|z| < 1$, but after we multiply this expression by $\frac{1}{z}$, the resulting series $f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$ converges for $0 < |z| < 1$.

(b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1$. We write the given function $f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$. The series in the brackets converges for $|\frac{1}{z}| < 1$ or equivalently for $1 < |z|$. Thus, the required Laurent series is $f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$.

Finding Laurent Expansions I

(c) We add and subtract 1 in the denominator: $f(z) = \frac{1}{(1-1+z)(z-1)} = \frac{1}{z-1}\frac{1}{1+(z-1)} = \frac{1}{z-1}\left(1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$. The requirement that $z \neq 1$ is equivalent to $0 < |z-1|$, and the geometric series in brackets converges for $|z-1| < 1$. Thus, the last series converges for z satisfying $0 < |z-1| < 1$.

(d) As in part (b), $f(z) = \frac{1}{z-1}\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\frac{1}{1+\frac{1}{z-1}} = \frac{1}{(z-1)^2}\left(1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \cdots\right) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \cdots$. Because the series within the brackets converges for $|\frac{1}{z-1}| < 1$, the final series converges for $1 < |z-1|$.

More Laurent Series Expansions I

- Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for

$$\text{(a) } 0 < |z-1| < 2 \qquad \text{(b) } 0 < |z-3| < 2.$$

(a) We need to express $z-3$ in terms of $z-1$. This can be done by writing $f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\frac{1}{1-\frac{z-1}{2}} =$
$\frac{-1}{2(z-1)^2}\left(1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \cdots\right) =$
$-\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$.

(b) To obtain powers of $z-3$, we write $z-1 = 2+(z-3)$ and $f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3}[2+(z-3)]^{-2} = \frac{1}{4(z-3)}[1+\frac{z-3}{2}]^{-2} =$

$$\frac{1}{4(z-3)}\left(1 + \frac{(-2)}{1!}\left(\frac{z-3}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{z-3}{2}\right)^3 + \cdots\right).$$

The series in the brackets is valid for $|\frac{z-3}{2}| < 1$ or $|z-3| < 2$. Multiplying by $\frac{1}{4(z-3)}$ gives a series that is valid for $0 < |z-3| < 2$:
$f(z) = \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \cdots$.

More Laurent Series Expansions II

- Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

  By partial fractions we can rewrite f as $f(z) = \frac{8z+1}{z(1-z)} = \frac{1}{z} + \frac{9}{1-z}$. Now we have

$$\frac{9}{1-z} = 9 + 9z + 9z^2 + \cdots .$$

  The foregoing geometric series converges for $|z| < 1$, but after we add the term $\frac{1}{z}$ to it, the resulting Laurent series

$$f(z) = \frac{1}{z} + 9 + 9z + 9z^2 + \cdots$$

  is valid for $0 < |z| < 1$.

More Laurent Series Expansions III

- Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

  The center $z = 2$ is a point of analyticity of the function f. Our goal now is to find two series involving integer powers of $z-2$, one converging for $1 < |z-2|$ and the other converging for $|z-2| < 2$. Decompose f into partial fractions: $f(z) = \frac{-1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$.
  - $f_1(z) = \frac{-1}{z} = \frac{-1}{2+z-2} = \frac{-1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{-1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \cdots\right) = \frac{-1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$. This series converges for $|\frac{z-2}{2}| < 1$ or $|z-2| < 2$.
  - $f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2}\left(1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \cdots\right) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$. It converges for $|\frac{1}{z-2}| < 1$ or $1 < |z-2|$.

  Thus, we get $f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2}$ $-\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$. This representation is valid for z satisfying $1 < |z-2| < 2$.

More Laurent Series Expansions IV

- Expand $f(z) = \frac{e^3}{z}$ in a Laurent series valid for $0 < |z| < \infty$.

  We know that for $|z| < \infty$,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

We obtain the Laurent series for f by simply replacing z by $\frac{3}{z}$, when $z \neq 0$:

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots .$$

This series is valid for $z \neq 0$, that is, for $0 < |z| < \infty$.

Remarks

(i) Replacing the complex variable s with the usual symbol z, we see that when $k = -1$, the formula for the Laurent series coefficients yields

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz,$$

or more important,

$$\oint_C f(z) dz = 2\pi i a_{-1}.$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; i.e., the series we obtain is unique.

Zeros and Poles

Review of Laurent Series

- Suppose $z = z_0$ is an isolated singularity of a complex function f, and that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

  is the Laurent series representation of f valid for the punctured open disk $0 < |z - z_0| < R$.

- The part of the series with the negative powers of $z - z_0$, i.e.,

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}$$

  is the **principal part** of the series.

- We will classify the isolated singularity $z = z_0$ according to the number of terms in the principal part.

Classification of Isolated Singular Points

- An isolated singular point $z = z_0$ of a complex function f is given a classification depending on whether the principal part of its Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

contains zero, a finite number, or an infinite number of terms:

(i) If the principal part is zero, that is, all the coefficients a_{-k} are zero, then $z = z_0$ is called a **removable singularity**.

(ii) If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**. If, in this case, the last nonzero coefficient in $\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$ is a_{-n}, $n \geq 1$, then $z = z_0$ is called a **pole of order** n. If $z = z_0$ is a pole of order 1, then the principal part contains exactly one term with coefficient a_{-1} and the pole is called a **simple pole**.

(iii) If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

Form of Laurent Series Based on Classification

- The form of a Laurent series for a function f, when $z = z_0$ is one of the various types of isolated singularities is summarized below:

$z = z_0$	Laurent Series for $0 < \lvert z - z_0 \rvert < R$
Removable Singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of Order n	$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$ $+ a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Simple Pole	$\frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential Singularity	$\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}$ $+ a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

A Removable Singularity

- Recall the Maclaurin series for $\sin z$: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$. Divide by z to get
$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$
 Thus, all the coefficients in the principal part of the Laurent series are zero. Hence, $z = 0$ is a removable singularity of the function $f(z) = \frac{\sin z}{z}$.
- If a function f has a removable singularity at $z = z_0$, then we can supply an appropriate definition for the value of $f(z_0)$ so that f becomes analytic at $z = z_0$.

  Example: Since the right-hand side of the series above is 1 when we set $z = 0$, it makes sense to define $f(0) = 1$. Hence the function $f(z) = \frac{\sin z}{z}$ is now defined and continuous at every complex number z. Indeed, f is also analytic at $z = 0$ because it is represented by the Taylor series $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ centered at 0 (a Maclaurin series).

Poles and Essential Singularities

(a) Dividing $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ by z^2 shows that, for $0 < |z| < \infty$,

$$\frac{\sin z}{z^2} = \overbrace{\frac{1}{z}}^{\text{principal part}} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots .$$

Since $a_{-1} \neq 0$, $z = 0$ is a simple pole of the function $f(z) = \frac{\sin z}{z^2}$. Similarly, $z = 0$ is a pole of order 3 of the function $f(z) = \frac{\sin z}{z^4}$.

(b) The Laurent series of $f(z) = \frac{1}{(z-1)^2(z-3)}$ for $0 < |z-1| < 2$:

$$f(z) = \overbrace{-\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)}}^{\text{principal part}} - \frac{1}{8} - \frac{z-1}{16} - \cdots .$$

Since $a_{-2} = -\frac{1}{2} \neq 0$, we conclude that $z = 1$ is a pole of order 2.

(c) The principal part of the Laurent expansion of $f(z) = e^{3/z}$ valid for $0 < |z| < \infty$ contains an infinite number of nonzero terms. This shows that $z = 0$ is an essential singularity of f.

Zeros and Multiplicities

- A number z_0 is a **zero** of a function f if $f(z_0) = 0$.
- We say that an analytic function f has a **zero of order** n at $z = z_0$ if z_0 is a zero of f and of its first $n-1$ derivatives, but not of its n-th derivative, i.e., $f(z_0) = 0$, $f'(z_0) = 0$, $f''(z_0) = 0$, $\ldots$, $f^{(n-1)}(z_0) = 0$, but $f^{(n)}(z_0) \neq 0$.
- A zero of order n is also referred to as a **zero of multiplicity** n.

  Example: Consider $f(z) = (z-5)^3$.

  $$f(5) = 0,\ f'(5) = 0,\ f''(5) = 0, \text{ but } f'''(5) = 6 \neq 0.$$

  Thus, f has a zero of order (or multiplicity) 3 at $z_0 = 5$.
- A zero of order 1 is called a **simple zero**.

Order of Zeros

Theorem (Zero of Order n)

A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- Partial Proof ("only if" Part): Given that f is analytic at z_0, it can be expanded in a Taylor series that is centered at z_0 and is convergent for $|z - z_0| < R$. Since, in a Taylor series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$, $a_k = \frac{f^{(k)}(z_0)}{k!}$, $k = 0, 1, \ldots$, it follows that the first n terms are zero. So $f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + a_{n+2}(z - z_0)^{n+2} + \cdots = (z - z_0)^n \left(a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots \right)$. Letting $\phi(z) = a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots$, we conclude $f(z) = (z - z_0)^n \phi(z)$, where ϕ is an analytic function, such that $\phi(z_0) = a_n \neq 0$ because $a_n = \frac{f^{(n)}(z_0)}{n!} \neq 0$.

Computing the Order of a Zero Using a Power Series

- The analytic function $f(z) = z \sin z^2$ has a zero at $z = 0$.
  If we replace z by z^2 in the Maclaurin series for $\sin z$, we obtain

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots .$$

  Then, by factoring z^2 out, we can rewrite f as

$$f(z) = z \sin z^2 = z^3 \phi(z),$$

  where $\phi(z) = 1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots$ and $\phi(0) = 1$.
  This shows that $z = 0$ is a zero of order 3 of f.

Poles of Order n

- A pole of order n may be characterized analogously to the characterization of zeros:

Theorem (Pole of Order n)

A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written $f(z) = \frac{\phi(z)}{(z-z_0)^n}$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- Partial Proof ("only if" Part): Since f is assumed to have a pole of order n at z_0, it can be expanded in a Laurent series $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \cdots$, valid in some punctured disk $0 < |z - z_0| < R$. By factoring out $\frac{1}{(z-z_0)^n}$, $f(z) = \frac{\phi(z)}{(z-z_0)^n}$, where $\phi(z) = a_{-n} + \cdots + a_{-2}(z - z_0)^{n-2} + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \cdots$. This is a power series valid for the open disk $|z - z_0| < R$. Since $z = z_0$ is a pole of order n of f, $a_{-n} \neq 0$.

Zeros and Poles

- A zero $z = z_0$ of an analytic function f is *isolated* in the sense that there exists some neighborhood of z_0 for which $f(z) \neq 0$ at every point z in that neighborhood except at $z = z_0$.
- As a consequence, if z_0 is a zero of a nontrivial analytic function f, then the function $\frac{1}{f(z)}$ has an isolated singularity at the point $z = z_0$.

Theorem (Pole of Order n)

If the functions g and h are analytic at $z = z_0$ and h has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $f(z) = \frac{g(z)}{h(z)}$ has a pole of order n at $z = z_0$.

- Because h has a zero of order n, $h(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$. Thus, f can be written $f(z) = \frac{g(z)/\phi(z)}{(z-z_0)^n}$. Since g and ϕ are analytic at $z = z_0$ and $\phi(z_0) \neq 0$, it follows that the function g/ϕ is analytic at z_0 and $g(z_0)/\phi(z_0) \neq 0$. We conclude that the function f has a pole of order n at z_0.

Examples

(a) Inspection of the rational function

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

shows that the denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at any of these points, it follows from the theorem that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.

(b) $z = 0$ is a zero of order 3 of $z \sin z^2$. The reciprocal function

$$f(z) = \frac{1}{z \sin z^2}$$

has a pole of order 3 at $z = 0$.

Remarks

(i) If a function f has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ from any direction. Thus, we can write $\lim_{z \to z_0} f(z) = \infty$.

(ii) A function f is **meromorphic** if it is analytic throughout a domain D, except possibly for poles in D. It can be proved that a meromorphic function can have at most a finite number of poles in D.

E.g., the rational function

$$f(z) = \frac{1}{z^2 + 1}$$

is meromorphic in the complex plane.

Residues and Residue Theorem

Residue

- If a complex function f has an isolated singularity at a point z_0, then f has a Laurent series representation $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$, which converges for all z in some deleted neighborhood $0 < |z - z_0| < R$ of z_0.
- We now focus on the coefficient a_{-1} and its importance in the evaluation of contour integrals.
- The coefficient a_{-1} is called the **residue** of the function f at the isolated singularity z_0 and denoted

$$a_{-1} = \text{Res}(f(z), z_0).$$

- Recall, if the principal part of the series valid for $0 < |z - z_0| < R$ contains a finite number of terms with a_{-n} the last nonzero coefficient, then z_0 is a pole of order n; if the principal part contains an infinite number of terms with nonzero coefficients, then z_0 is an essential singularity.

Examples of Residues

(a) We have seen that $z = 1$ is a pole of order two of the function $f(z) = \frac{1}{(z-1)^2(z-3)}$. The Laurent series valid for the deleted neighborhood $0 < |z-1| < 2$ is

$$f(z) = -\frac{1/2}{(z-1)^2} + \frac{-1/4}{z-1} - \frac{1}{8} - \frac{z-1}{16} - \cdots .$$

Thus, the coefficient of $\frac{1}{z-1}$ is $a_{-1} = \operatorname{Res}(f(z), 1) = -\frac{1}{4}$.

(b) We also saw that $z = 0$ is an essential singularity of $f(z) = e^{3/z}$. Its Laurent series is

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots ,\ 0 < |z| < \infty.$$

Hence, the coefficient of $\frac{1}{z}$ is $a_{-1} = \operatorname{Res}(f(z), 0) = 3$.

Residue at a Simple Pole

- We examine ways of obtaining a_{-1} when z_0 is a pole of a function f, without the necessity of expanding f in a Laurent series at z_0.
- We begin with the residue at a simple pole:

Theorem (Residue at a Simple Pole)

If f has a simple pole at $z = z_0$, then

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

- Since f has a simple pole at $z = z_0$, its Laurent expansion convergent on a punctured disk $0 < |z - z_0| < R$ has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

where $a_{-1} \neq 0$. By multiplying both sides of this series by $z - z_0$ and then taking the limit as $z \to z_0$ we obtain $\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} [a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots] = a_{-1} = \text{Res}(f(z), z_0)$.

Residue at a Pole of Order n

Theorem (Residue at a Pole of Order n)

If f has a pole of order n at $z = z_0$, then

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

- Since f has a pole of order n at $z = z_0$, its Laurent expansion, convergent on a punctured disk $0 < |z - z_0| < R$, has the form $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \cdots$, where $a_{-n} \neq 0$. We multiply by $(z - z_0)^n$, $(z - z_0)^n f(z) = a_{-n} + \cdots + a_{-2}(z - z_0)^{n-2} + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \cdots$ and then differentiate $n - 1$ times:

$$\frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! a_{-1} + n! a_0 (z - z_0) + \cdots .$$

Therefore, as $z \to z_0$, $\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! a_{-1}$.

Finding Residue at a Pole

- The function $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$. Use the theorems to find the residues.

  Since $z = 3$ is a simple pole,

  $$\operatorname{Res}(f(z), 3) = \lim_{z \to 3} (z-3) f(z) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}.$$

  At the pole of order 2,

  $$\begin{aligned} \operatorname{Res}(f(z), 1) &= \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} (z-1)^2 f(z) = \lim_{z \to 1} \frac{d}{dz} \frac{1}{z-3} \\ &= \lim_{z \to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}. \end{aligned}$$

Second Method for Computing a Residue at a Simple Pole

- Suppose a function f can be written as a quotient $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are analytic at $z = z_0$. If $g(z_0) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

- Since h has a zero of order 1 at z_0, we must have $h(z_0) = 0$ and $h'(z_0) \neq 0$. By definition of the derivative, $h'(z_0) = \lim_{z \to z_0} \dfrac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \dfrac{h(z)}{z - z_0}$. Therefore, $\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0)\dfrac{g(z)}{h(z)} = \lim_{z \to z_0} \dfrac{g(z)}{\frac{h(z)}{z - z_0}} = \dfrac{g(z_0)}{h'(z_0)}$.

Applying the Second Method

- The polynomial $z^4 + 1$ can be factored as

$$(z - z_1)(z - z_2)(z - z_3)(z - z_4),$$

where z_1, z_2, z_3, and z_4 are the four distinct roots of the equation $z^4 + 1 = 0$ (or, equivalently, the four fourth roots of -1). It follows that the function $f(z) = \frac{1}{z^4+1}$ has four simple poles. By the root formula $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{5\pi i/4}$, and $z_4 = e^{7\pi i/4}$. We compute the residues:

$$\begin{aligned}
\mathrm{Res}(f(z), z_1) &= \tfrac{1}{4z_1^3} = \tfrac{1}{4}e^{-3\pi i/4} = -\tfrac{1}{4\sqrt{2}} - \tfrac{1}{4\sqrt{2}}i \\
\mathrm{Res}(f(z), z_2) &= \tfrac{1}{4z_2^3} = \tfrac{1}{4}e^{-9\pi i/4} = \tfrac{1}{4\sqrt{2}} - \tfrac{1}{4\sqrt{2}}i \\
\mathrm{Res}(f(z), z_3) &= \tfrac{1}{4z_3^3} = \tfrac{1}{4}e^{-15\pi i/4} = \tfrac{1}{4\sqrt{2}} + \tfrac{1}{4\sqrt{2}}i \\
\mathrm{Res}(f(z), z_4) &= \tfrac{1}{4z_4^3} = \tfrac{1}{4}e^{-21\pi i/4} = -\tfrac{1}{4\sqrt{2}} + \tfrac{1}{4\sqrt{2}}i.
\end{aligned}$$

Using the Original Formula

- We could have calculated each of the residues of $f(z) = \frac{1}{z^4+1}$ using $\mathrm{Res}(f(z), z_i) = \lim_{z \to z_i} (z - z_i) f(z)$.
- E.g., at z_1,

$$\begin{aligned} \mathrm{Res}(f(z), z_1) &= \lim_{z \to z_1} (z - z_1) \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\ &= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \\ &= \frac{1}{(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{5\pi i/4})(e^{\pi i/4}-e^{7\pi i/4})}. \end{aligned}$$

In simplifying the denominator of the last expression considerably more algebra is involved than using the second method.

Cauchy's Residue Theorem

- Complex integrals $\oint_C f(z)dz$ can sometimes be evaluated by summing the residues at the isolated singularities of f within C:

Theorem (Cauchy's Residue Theorem)

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k).$$

- Suppose $C_1, C_2, \ldots, C_n$ are circles centered at $z_1, z_2, \ldots, z_n$, respectively, such that C_k has a radius r_k small enough so that $C_1, C_2, \ldots, C_n$ are mutually disjoint and are interior to the simple closed curve C. We saw that $\oint_{C_k} f(z)dz = 2\pi i \text{Res}(f(z), z_k)$, whence, we have $\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$.

Evaluation by the Residue Theorem I

- Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$, where
  - (a) the contour C is the rectangle defined by $x=0, x=4, y=-1, y=1$;
  - (b) the contour C is the circle $|z|=2$.

(a) Since both $z=1$ and $z=3$ are poles within the rectangle, we have $\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i[\mathrm{Res}(f(z),1)+\mathrm{Res}(f(z),3)]$. We found these residues already: $\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i(-\frac{1}{4}+\frac{1}{4}) = 0$.

(b) Since only the pole $z=1$ lies within the circle $|z|=2$, we have $\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i \mathrm{Res}(f(z),1) = 2\pi i(-\frac{1}{4}) = -\frac{\pi}{2}i$.

Evaluation by the Residue Theorem II

- Evaluate $\oint_C \frac{2z+6}{z^2+4} dz$, where the contour C is the circle $|z-i|=2$.

  By factoring the denominator $z^2+4=(z-2i)(z+2i)$, we see that the integrand has simple poles at $-2i$ and $2i$. Only $2i$ lies within the contour C. Thus, $\oint_C \frac{2z+6}{z^2+4} dz = 2\pi i \mathrm{Res}(f(z), 2i)$. But $\mathrm{Res}(f(z), 2i) = \lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)} = \frac{6+4i}{4i} = \frac{3+2i}{2i}$. Hence, $\oint_C \frac{2z+6}{z^2+4} dz = 2\pi i \left(\frac{3+2i}{2i}\right) = \pi(3+2i)$.

Evaluation by the Residue Theorem III

- Evaluate $\oint_C \frac{e^z}{z^4+5z^3}dz$, where the contour C is the circle $|z|=2$.

  Writing the denominator as $z^4+5z^3 = z^3(z+5)$ reveals that the integrand $f(z)$ has a pole of order 3 at $z=0$ and a simple pole at $z=-5$. Only the pole $z=0$ lies within the given contour. Thus, we have

  $$\oint_C \frac{e^z}{z^4+5z^3}dz = 2\pi i \text{Res}(f(z),0) = 2\pi i \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}z^3\cdot\frac{e^z}{z^3(z+5)} =$$
  $$\pi i \lim_{z\to 0}\frac{d}{dz}\frac{e^z(z+4)}{(z+5)^2} = \pi i \lim_{z\to 0}\frac{(z^2+8z+17)e^z}{(z+5)^3} = \frac{17\pi}{125}i.$$

Evaluation by the Residue Theorem IV

- Evaluate $\oint_C \tan z dz$, where the contour C is the circle $|z| = 2$.

  The integrand $f(z) = \tan z = \frac{\sin z}{\cos z}$ has simple poles at the points where $\cos z = 0$. We saw that the only zeros of $\cos z$ are the real numbers $z = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$. Only $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are within the circle $|z| = 2$. Thus, we have

  $\oint_C \tan z dz = 2\pi i[\text{Res}(f(z), -\frac{\pi}{2}) + \text{Res}(f(z), \frac{\pi}{2})]$. With $f(z) = \frac{g(z)}{h(z)}$, where $g(z) = \sin z$, $h(z) = \cos z$, and $h'(z) = -\sin z$, we get

  $\text{Res}(f(z), -\frac{\pi}{2}) = \dfrac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1$. $\text{Res}(f(z), \frac{\pi}{2}) = \dfrac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$.

  Therefore, $\oint_C \tan z dz = 2\pi i[-1 - 1] = -4\pi i$.

Evaluation by the Residue Theorem V

- Evaluate $\oint_C e^{3/z}dz$, where the contour C is the circle $|z| = 1$.

  We saw that $z = 0$ is an essential singularity of the integrand $f(z) = e^{3/z}$. So we cannot use the formulas

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0)f(z)$$

or

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}(z - z_0)^n f(z)$$

to find the residue of f at that point. Nevertheless, the Laurent series of f at $z = 0$ gives

$$\mathrm{Res}(f(z), 0) = 3.$$

Hence, we have

$$\oint_C e^{3/z}dz = 2\pi i \mathrm{Res}(f(z), 0) = 2\pi i(3) = 6\pi i.$$