The $3x+1$ Problem: An Annotated Bibliography, II (2000-2009)

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ABSTRACT. The $3x+1$ problem concerns iteration of the map $T : \mathbb{Z} \to \mathbb{Z}$ given by

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \ . \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ . \end{cases}$$

The $3x+1$ Conjecture asserts that each $m \geq 1$ has some iterate $T^{(k)}(m) = 1$. This is the second installment of an annotated bibliography of work done on the $3x+1$ problem and related problems, mainly covering the period 2000 through 2009, with some related later papers (which were preprints by 2009). At present the $3x+1$ Conjecture remains unsolved.

1. Introduction

The $3x+1$ problem is most simply stated in terms of the *Collatz function* $C(x)$ defined on integers as "multiply by three and add one" for odd integers and "divide by two" for even integers. That is,

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod{2} \ , \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ , \end{cases}$$

The $3x+1$ *problem* (or *Collatz problem*) is to prove that starting from any positive integer, some iterate of this function takes the value 1. The problem other names: it has also been called Kakutani's problem, the Syracuse problem, and Ulam's problem.

Much work on the problem is stated in terms of the $3x+1$ *function*

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ . \end{cases}$$

The $3x+1$ *Conjecture* states that every $m \geq 1$ has some iterate $T^{(k)}(m) = 1$.

The $3x+1$ Conjecture has now been verified up to $17 \times 2^{58} > 4.899 \times 10^{18}$ (as of Feb. 21, 2008) by an ongoing computation run by T. Oliveira e Silva (2004+). An independent computation of Roosendaal(2004+) verifies it to $612 \times 2^{50} > 6.89 \times 10^{17}$.

At present the $3x+1$ conjecture remains unsolved. The proofs claimed in Yamada (1981), Cadogan (2006) and Bruckman (2008) are incomplete.

Surveys on results on the $3x+1$ problem can be found in Lagarias (1985), Müller (1991), and the first chapter of Wirsching (1998a), described in the first installment of the annotated bibliography, Lagarias (2003+). A more recent survey appears in Chamberland (2003).

2. Terminology

We use the following definitions. The *trajectory* or *foward orbit* of an integer m is the set

$$O^+(m) := \{m,\ T(m)\ ,\ \ T^{(2)}(m), \ldots\}\ .$$

The *stopping time* $\sigma(m)$ of m is the least k such that $T^{(k)}(m) < m$, and is ∞ if no such k exists. The *total stopping time* $\sigma_\infty(m)$ is the least k such that m iterates to 1 under k applications of the function T i.e.

$$\sigma_\infty(m) := \inf\ \{k\ :\ T^{(k)}(m) = 1\}.$$

The *scaled total stopping time* or *gamma value* $\gamma(m)$ is given by

$$\gamma(m) := \frac{\sigma_\infty(m)}{log\ m}.$$

The *height* $h(m)$ is the least k for which the Collatz function $C(x)$ has $C^{(k)}(m) = 1$. It is also given by

$$h(m) := \sigma_\infty(m) + d(m),$$

where $d(m)$ counts the number of iterates $T^{(k)}(m) \equiv 1 \pmod 2$ for $0 \le k < \sigma_\infty(m)$. Finally, the function $\pi_a(x)$ counts the number of n with $|n| \le x$ that contain a in their forward orbit under T.

3. Bibliography

This bibliography covers research articles, survey articles and PhD theses on the $3x+1$ problem and related problems from 2000 to the present. The first installment of the annotated bibliography is Lagarias(2003+), which covers the period 1963–1999. Articles in Chinese have the authors surname listed first.

1. Ethan Akin (2004), *Why is the* $3x+1$ *Problem Hard?,* In: *Chapel Hill Ergodic Theory Workshops* (I. Assani, Ed.), Contemp. Math. vol 356, Amer. Math. Soc. 2004, pp. 1–20. (MR 2005f:37031).

   This paper analyzes the $3x+1$ problem by viewing the map T as acting on the domain $\mathbb{Z}_2$ of 2-adic integers. The map T is topologically conjugate over $\mathbb{Z}_2$ to the 2-adic shift map

$$S(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod 2\ , \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2\ , \end{cases}$$

by a conjugacy map $Q_3 : \mathbb{Z}_2 \to \mathbb{Z}_2$, i.e. $Q_3 \circ T = S \circ Q_3$. (The map Q_3 equals the map denoted Q_∞ in Lagarias (1985), and is the inverse of the map Φ in Bernstein (1994).) The $3x+1$ Conjecture can be reformulated in terms of the behavior of Q_3 acting on integers, namely that Q_3 maps $\mathbb{Z}^+$ into $\frac{1}{3}\mathbb{Z}$. Consider more generally for any odd rational a the map $T_a(x)$ which sends $x \mapsto \frac{ax+1}{2}$ or $\frac{x}{2}$, according as x is an odd or even 2-adic integer. The author observes there is an associated conjugacy map $Q_a : \mathbb{Z}_2 \to \mathbb{Z}_2$ with the same property as above, and formulates the Rationality Conjecture for Q_a, which asserts that Q_a maps the rationals with odd denominators to rationals. He shows that the Rationality conjecture is true for $a = \pm 1$ and is false for any odd rational a that is not an integer. For the remaining cases of odd integer a, where the Rationality Conjecture remains unsolved, he presents a heuristic argument suggesting that it should be true for $a = \pm 3$ and false for all odd integers $|a| \geq 5$.

2. João F. Alves, Mário M. Graca, M. E. Sousa Dias, and José Sousa Ramos (2005), *A linear algebra approach to the conjecture of Collatz,* Lin. Alg. Appl. **394** (2005), 277–289. (MR2100588)

   This paper studies the conjecture that the only periodic orbit of the Collatz map on the positive integers goes through $n = 1$. They form an $n \times n$ zero-one matrix A_n whose entries are
$$A_{i,j} = 1 \quad \text{if} \quad T(i) = j, \quad 1 \leq i, j \leq n.$$
   where $T(n)$ is the $3x+1$ function, and $A_{i,j} = 0$ otherwise. The assertion that $\{1, 2\}$ is the only periodic orbit of T on the positive integers is shown to be equivalent to $\det(I - xA_n) = 1 - x^2$ for all $n \geq 1$. They prove that $\det(I - xA_n) = \det(I - xA_{n-1})$ for all $n \not\equiv 8 \pmod{18}$. They deduce that if there is another periodic orbit on the positive integers then there exists $m \equiv 8 \pmod{18}$ such that $n = \frac{m}{2}$ is in a periodic orbit. Various further conditions are deduced in the case $n \equiv 8 (\mathrm{mod}\ 18)$, e.g. $\det(I - xA_n) = \det(I - xA_{n-1})$ if $n \equiv 8 \pmod{54}$.

3. Tewodrus Amdeberhan, Dante Manna and Victor H. Moll (2008), *The 2-adic valuation of a sequence arising from a rational integral,* J. Combinatorial Theory, Series A, **115** (2008), no. 8, 1474–1486. (MR 2009k:11194)

   This paper studies certain integer sequences $\{A_{k,m} : k \geq 0\}$ arising from evaluation of the integral
$$N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 4ax^2 + 1)^{m+1}}$$
   expanded in Taylor series in the parameter a, as
$$N_{0,4}(a; m) = \frac{\pi}{\sqrt{2} m! (4(2a+1))^{\frac{m+1}{2}}} \sum_{k=0}^\infty A_{k,m} \frac{a^k}{k!}.$$
   These sequences are given by
$$A_{k,m} = \frac{k! m!}{2^{m-k}} \sum_{j=k}^m 2^j \binom{2m-2j}{m-j} \binom{m+j}{m} \binom{j}{k}.$$
   In section 6 a relation is shown between divisibility of $A_{1,m}$ by powers of 2 and the $3x+1$ problem. Namely $a_m := ord_2(A_{1,m}) - 1$ gives the number of iterations of the $3x+1$ map

$T(x)$ starting from $x_0 = m$, in which the parity of the iterates does not change, i.e

$$m \equiv T(m) \equiv \cdots \equiv T^{a_m - 1}(m) \not\equiv T^{a_m}(m) \pmod{2}.$$

This is given as Theorem 6.1 of the paper.

4. Paul Andaloro (2000), *On total stopping times under* $3X + 1$ *iteration,* Fibonacci Quarterly **38** (2000), 73–78. (MR 2000m:11024).

   This paper shows various results on the minimal elements having a given stopping time, where the "stopping time" is defined to be the number of odd elements in the trajectory up to and including 1. It also obtains a new congruential "sufficient set" criterion to verify the $3x+1$ Conjecture. It shows that knowing that the $3x+1$ Conjecture is true for all $n \equiv 1 \pmod{16}$ implies that it is true in general.

5. Paul Andaloro (2002), *The* $3X + 1$ *problem and directed graphs,* Fibonacci Quarterly **40** (2002), 43–54. (MR 2003a:11018).

   This paper considers various "compressed" versions of the $3x + 1$ graph, in which only a subset of the vertices are retained with certain directed paths in original $3x + 1$ graph iterates of $T(\cdot)$ replaced by single directed edges. The initial "compressed" graph corresponds to odd integers, and the paper introduces two further "compressed" graphs with fewer allowed vertices. In each case, the $3x+1$ Conjecture is equivalent to the graph being weakly connected, i.e. being connected when viewed as an undirected graph. The paper shows that certain kinds of vertex pairs in such graphs are weakly connected, typically for allowed vertices in certain congruence classes (mod 2^k) for small k.

6. Stefan Andrei, Manfred Kudlek, Radu Stefan Niculescu (2000), *Some results on the Collatz problem*, Acta Informatica **37** (2000), 145–160. (MR 2002c:11022).

   This paper studies the internal structure of $3x + 1$-trees. Among other things, it observes the that integers of the form $2^m k - 1$ eventually iterate to the integers $3^m k - 1$, respectively. and that integers of the form $2^{3m}k - 5$ iterate to the integers $3^{2m}k - 5$, and integers of the form $2^{11m}k - 17$ iterate to the integers $3^{7m}k - 17$. These facts are related to the cycles associated to -1, -5 and -17, respectively.

7. Stefan Andrei, Wei-Ngan Chin, and Huu Hai Nguyen (2007+), *A Functional View over the Collatz's Problem,* preprint.

   This paper describes chains in the $3x + 1$ tree using a phrase structure grammar.

8. David Applegate and Jeffrey C. Lagarias (2003), *Lower bounds for the total stopping time of* $3x + 1$ *iterates,* Math. Comp. **72** (2003), 1035–1049. (MR 2004a:11016).

   This paper proves there are infinitely many positive n which have a finite total stopping time $\sigma_\infty(n) > 6.14316 \log n$. It also shows that there is a positive c such that at least $cx^{1/60}$ of all integers $1 < n \le x$ have a finite total stopping time $\sigma_\infty(n) > 5.9 \log n$. The proofs are computer-intensive, and produce a "certificate" encoding a proof, which is based on a search of $3x+1$ trees to depth 60. The "certificates" are quite large, involving about 350 million trees for the lower bound $6.14316 \log n$, which corresponds to a density of odd integers in a trajectory (the" ones-ratio") of $\frac{14}{29} \approx 0.483$.

This rigorous bound is below the bound $\sigma_\infty(n) \approx 6.95212 \log n$ that one expects to hold for almost all integers, which corresponds to a ones-ratio of $\frac{1}{2}$. The paper gives heuristic arguments suggesting that the method of this paper might prove $\sigma_\infty(n) \approx 6.95212 \log n$ holds for infinitely many n, but that it would likely require a search of $3x + 1$ trees to depth at least 76. This would require a very large computation.

9. David Applegate and Jeffrey C. Lagarias (2006), *The $3x+1$ semigroup*, J. Number Theory **177** (2006), 146–159. (MR 2006k:11037).

   This paper considers a weak version of the $3x + 1$ problem proposed by Farkas (2005). It considers the multiplicative semigroup $\mathcal{R}$ of positive rational numbers generated by $\{\frac{2n+1}{3n+2} : n \geq 0\}$ together with $\{2\}$. The *weak $3x + 1$ conjecture* asserts that this semigroup contains all positive integers. The relation to the $3x + 1$ problem is that the semigroup contains 1 and its generators encode the action of the inverse $3x+1$ function. It follows that the truth of the $3x + 1$ conjecture implies the truth of the weak $3x + 1$ conjecture. This paper proves the conjecture. Its main result shows that the semigroup $\mathcal{R}$ consists of all positive rational numbers $\frac{a}{b}$ such that 3 does not divide b. The proof is an induction motivated by certain results established in Lagarias (2006).

10. Edward Belaga and Maurice Mignotte (2000), *Cyclic Structure of Dynamical Systems Associated with $3x + d$ Extensions of Collatz Problem,* U. Strasbourg report 2000-18, 57 pages. (`http://hal.archives-ouvertes.fr/IRMA-ACF`, file hal-00129656)

    This paper is a theoretical and experimental study of the distribution of number and lengths of finite cycles to the $3x+d$ map, for $d \equiv \pm 1 \pmod 6$. It contains much data and a wealth of interesting results and conjectures.

11. Edward Belaga (2003), *Effective polynomial upper bounds to perigees and numbers of $(3x + d)$-cycles of a given oddlength,* Acta Arithmetica **106**, No. 2, (2003), 197–206. (MR 2003m:11120).

    Let d be a positive odd integer, and consider the $3x + d$ map $T_d(x) = \frac{3x+d}{2}$ if x is odd; $T_d(x) = \frac{x}{2}$ if x is even, acting on the domain of positive integers. This paper shows that for any cycle C of the $3x + d$ map of length l containing k odd elements, the smallest element $prg(C)$ in the cycle satisfies

    $$prg(C) \leq \frac{d}{2^{l/k} - 3}.$$

    From this follows

    $$\log_2 3 < \frac{\text{length}(C)}{\text{oddlength}(C)} \leq \log_2(d + 3)$$

    The author shows that the upper bound is sharp, and gives evidence that the lower bound is probably asymptotically approachable. Using bounds from transcendence theory (linear forms in logarithms) the author gives an upper bound for the total number of cycles $U_{d,k}$ of odd-length k. This upper bound is dk^{c_0}, for a constant c_0, and states that one may take $c_0 = 32$. He also shows that the largest element S of any such cycle is bounded above by

    $$S < dk^{c_0}(\frac{3}{2})^k.$$

12. Edward Belaga and Maurice Mignotte (2006a), *Walking Cautiously into the Collatz Wilderness: Algorithmically, Number Theoretically, and Randomly,* Fourth Colloquium on Mathematics and Computer Science, DMTS (Discrete Mathematics and Computer Science) Proceedings **AG**, 2006, 249–260.

This paper discusses many open questions about the $3x + 1$ map and related maps, and recent new numerical evidence supporting them. It formulates some new conjectures.

13. Edward Belaga and Maurice Mignotte (2006b), *The Collatz problem and its generalizations: Experimental Data. Table 1. Primitive cycles of* $3x + d$ *mappings,* Univ. of Strasbourg preprint 2006-015, 9 pages+400+ page table .

This paper gives detailed tables of primitive cycles on the positive integers for $3x+d$ maps for $1 < d < 20000$. The authors conjecture they obtain the complete list of such cycles, for these values of d. The text prior to the table formulates some interesting new conjectures.

14. Edward Belaga and Maurice Mignotte (2006c), *The Collatz problem and its generalizations: Experimental Data. Table 2. Factorization of Collatz numbers* $2^l - 3^k$*,* Univ. of Strasbourg preprint 2006-018, 6 pages text+156 page table.

These tables give known divisors of numbers of form $D = 2^l - 3^k$ for $l \leq 114$. Numbers D of this form give $3x + D$ problems having many primitive cycles.

15. Vitaly Bergelson, Michal Misiurewicz and Samuel Senti (2006), *Affine Actions of a Free Semigroup on the Real Line,* Ergodic Theory and Dynamical Systems **26** (2006), 1285–1305. MR2266362 (2008f:37019).

This extends the analysis of Misiurewicz and Rodrigues (2005) to semigroups generalizing those associated to the $3x + 1$ map. The current paper considers the orbits of a semigroup generated by $T_0(x) = ax$, $T_1(x) = bx + 1$, in which $0 < a < 1 < b$, viewed as acting on the positive real numbers $\mathbb{R}_{\geq 0}$. It defines various notions of "uniform distribution" on the positive real axis, and derives various results concerning uniform distribution of orbits of such semigroups. In particular, if the iterates of the transformation are decreasing on average, and if one assumes the symbolic dynamics of T_0, T_1 are drawn from an ergodic invariant measure ν on the shift space, then it proves the existence and uniqueness of an invariant measure μ on $\mathbb{R}_{\geq 0}$, which depends on ν. See Theorem F of the paper for a precise statement.

16. Konstantin Borovkov and Dietmar Pfeifer (2000), *Estimates for the Syracuse problem via a probabilistic model,* Theory of Probability and its Applications **45**, No. 2 (2000), 300–310. (MR 1 967 765).

This paper studies a multiplicative random walk imitating the $3x + 1$ iteration. Let X_0 be given and set $Y_j = X_0 X_1 \cdots X_j$ where each X_i for $i \geq 1$ are i.i.d. random variables assuming the value $\frac{1}{2}$ or $\frac{3}{2}$ with probability $\frac{1}{2}$ each. Let $\sigma_\infty(X_0, \omega)$ be a random variable equal to the smallest J such that $Y_j < 1$. Then $E[\sigma_\infty(X_0, \omega)] = (\frac{1}{2} \log \frac{4}{3})^{-1} \log X_0$ and

the normalized variable

$$\hat{\sigma}_\infty(X_0, \omega) = \frac{\sigma_\infty(X_0, \omega) - c_1 \log X_0}{c_2 (\log X_0)^{1/2}}$$

with $c_1 = (\frac{1}{2}\log\frac{4}{3})^{-1} = 6.95212$ and $c_2 = c_1^{3/2}(\frac{1}{2}\log 3)$, has a distribution converging to a standard normal distribution as $X_0 \to \infty$ (Theorem 5). Various refinements of this result are given. Comparisons are made to empirical $3x+1$ data for values around $n \approx 10^6$, which show good agreement.

17. Barry Brent (2002+), $3X+1$ *dynamics on rationals with fixed denominator,* `eprint: arXiv math.DS/0204170`.

    This paper reports on computer experiments looking for cycles with greatest common divior 1 for the $3x+k$ problem, for various $k \equiv \pm 1 (\mathrm{mod}\ 6)$. It suggests that for $k = 7$, $k = 19$ and $k = 31$ there is only one such cycle on the positive integers. Various other statistics are reported on.

18. Thomas Brox (2000), *Collatz cycles with few descents,* Acta Arithmetica **92** (2000), 181–188. (MR 2001a:11032)

    This paper considers the variant of the $3x+1$ map, call it T_1, that divides out all powers of 2 at each step. A cycle is written $\{x_1, x_2, ... x_m\}$ where each x_i is an odd integer. A *descent* means $|T_1(x)| < |x|$. Let $|C|$ denote the number of (odd) elements in a $3x+1$ cycle C, and let $d(C)$ denote the number of descents in C. The author proves that the the number of $3x+1$ cycles C that have $d(C) < 2log|C|$ is finite. In particular the number of Collatz cycles with $d(C) < r$ is finite for any fixed r. The proof uses a bound of Baker and Feldman and is in principle effective. Thus for each r there exists an algorithm to determine all cycles with $d(C) < r$. This greatly strengthens the result of R. Steiner (1978) who detemined all cycles with one descent.

19. Paul S. Bruckman (2008), *A proof of the Collatz conjecture,* International Journal of Mathematical Education in Science and Technology, **39**, No. 3 (2008), 403–407. [Erratum: **39**, No. 4 (2008), 567.] (MR 2009d:11043b)

    This paper asserts a proof of the Collatz conjecture. However the argument given has a gap which leaves the proof incomplete. The erratum points out this gap and withdraws the proof.

The gap is, suppose N_0 is the starting value, and that N_k is the k-th odd iterate to occur. Let E_k denote the number of divisions by 2 that occur in reaching N_k, then $2^{E_k} N_k - 3^k N_0 = S_k$, where S_k is the positive integer $S_k = \sum_{j=0}^{k-1} 2^{E_j} 3^{k-1-j}$. Next determine (A_k, B_k) by requiring $2^{E_k} B_k - 3^k A_k = 1$, with $0 \le B_k < 3^k$. The author notes that there is an integer T_k such that

$$N_0 = A_k + T_k 2^{E_k}, \qquad N_k = B_k + T_k 3^k.$$

Here T_k depends on k and may be positive or negative. The author then argues by contradiction, asserting in Section 2 the claim that the minimal counterexample N_0

must have $2^{E_k} < 3^k$ for all $k \geq 1$, which would imply that the sequence of iterates of N_0 diverges. The argument justifying this claim has a gap, it supposes $2^{E_k} > 3^k$, and asserts the contradiction that $N_k < N_0$. But $N_k \geq N_0$ may hold if T_k is sufficiently negative, and in fact for every starting value $n_0 \geq 1$ the values $T_k \to -\infty$ as $k \to \infty$. This can be seen from the equation (11) saying that $n_0 = A_k S_k + T_k 2^{N_k}$, noting that n_0 is fixed, $S_k \to \infty$ by its recurrence $S_{k+1} = 3S_k + 2^{E_k}$ and $A_k \geq 1$. (The author's argument as given would prove the cycle starting at 1 does not exist.)

20. Charles C. Cadogan (2000), *The* $3x+1$ *problem: towards a solution*, Caribbean J. Math. Comput. Sci. **10** (2000), paper 2, 11pp. (MR 2005g:11032)

    The paper studies trajectories of the $3x+1$ problem. calling two integers n_1 and n_2 equivalent, written $n_1 \sim n_2$, if their trajectories eventually coalesce. Various results are obtained giving sufficient conditions for equivalence. The author conjectures that for each positive odd integer n, $9n+4 \sim 3n+1$. His main result is that this conjecture implies the truth of the $3x+1$ Conjecture.

21. Charles C. Cadogan (2003), *Trajectories in the* $3x+1$ *problem,* J. of Combinatorial Mathematics and Combinatorial Computing, **44** (2003), 177–187. (MR 2004a:11017)

    This paper describes various pairs of trajectories that coalesce under the $3x+1$ iteration. For example the trajectories of $3n+1$ and $4n+1$ coalesce, and the trajectory of $16k+13$ coalesces with that of $3k+4$. The main result (Theorem 3.9) gives a certain infinite family of coalescences.

22. Charles C. Cadogan (2006), *A Solution to the* $3x+1$ *Problem,* Caribbean J. Math. Comp. Sci. **13** (2006), 1–11.

    This paper asserts a proof of the $3x+1$ conjecture. However the argument given has a gap which leaves the proof incomplete. Namely, on the line just before equation (2.6) the expression $1+2t_{i,j} \sim 1+3n_{i+1,j}$ should instead read $1+2t_{i,j} \sim 1+3n_{i,j}$, as given by equation (2.5). Hence instead of obtaining equation (2.6) in the form $t_{i,j} \sim t_{i+1,j} \sim t_{i+2,j}$, one only obtains $t_{i,j} \sim t_{i+1,j}$. This renders the proof of Theorem 2.15 incomplete, as it depends on equation (2.6). Then the induction step in Lemma 3.1 cannot be completed, as it depends on Theorem 2.15. Finally the main result Theorem 3.3 has a gap since it depends on Lemma 3.1.

23. Mónica del Pilar Canales Chacón and Michael Vielhaber (2004), *Structural and Computational Complexity of Isometries and their Shift Commutators,* Electronic Colloquium on Computational Complexity, Report No. 57 (2004), 24 pp. (electronic).

    The paper considers functions on $f : \{0,1\}^\infty \to \{0,1\}^\infty$ computable by invertible transducers. They give several formulations for computing such maps and consider several measures of computational complexity of such functions, including tree complexity $T(f,h)$, which measures the local branching of a tree computation, where h is the tree height of a vertex. They also study the bit complexity $B(f,n)$ which is the complexity of computing the first n input/ output symbols. Tree complexity is introduced in H. Niederreiter and M. Vielhaber, J. Complexity **12** (1996), 187–198 (MR 97g:94025).

The $3x+1$ function is considered as an example showing that some of the general complexity bounds they obtain are sharp. Interpreting the domain $\{0,1\}^{\infty}$ as the 2-adic integers, the map Q_{∞} associated to the $3x+1$ map given in Lagarias (1985) [Theorem L] is a function of this kind. It is invertible and the inverse map Q_{∞}^{-1} is studied in Bernstein (1994) and Bernstein and Lagarias (1996). In Theorem 33 the authors give a 5-state shift automaton that computes the "shift commutator" of the $3x+1$ function, which they show takes a 2-adic integer a to a if a is even, and to $3a+2$ if a is odd. In Theorem 34 they deduce that the tree complexity of Q_{∞} is bounded by a constant. Here Q_{∞} corresponds to their function $T(\mathbf{c},\cdot)$.

24. Mark Chamberland (2003), *Una actualizachio del problema* $3x+1$*,* Butletti de la Societat Catalana, **22** (2003) 19–45. (MR 2004i:11019).

This is a survey paper (in Catalan) describing recent results on the $3x+1$ problem, classified by area.

Note. An English version of this paper: "An Update on the $3x+1$ Problem" is posted on the author's webpage: `http://www.math.grin.edu/~chamberl/`

25. Dean Clark, Periodic solutions of arbitrary length in a simple integer iteration, Advances in Difference Equations, **2006** Article 35847, pp. 1–9.

This paper studies the second order nonlinear recurrence

$$y_{n+1} = \lceil a y_n \rceil - y_{n-1}$$

in which the initial conditions (y_0, y_1) are integers, and a is a constant with $\{a \in \mathbb{R} : |a| < 2, \}$. These generate an infinite sequence. For $a = \frac{3}{2}$, presuming integer initial conditions, the recurrence can be rewritten as

$$y_{n+1} := \begin{cases} \dfrac{3y_n+1}{2} - y_{n-1} & \text{if} \;\; y_n \equiv 1 \pmod{2} \\ \\ \dfrac{3y_y}{2} - y_{n-1} & \text{if} \;\; y_n \equiv 0 \pmod{2}\,. \end{cases}$$

which the author views as a second-order analogue of the $3x+1$ iteration. The main result of the paper shows that for fixed $|a| < 2$, all orbits of the recurrence are purely periodic.

In addition, the author shows when $a = \frac{p}{q}$ is a non-integer rational with $|a| < 2$, then there are orbits of arbitrarily large period.

The author presents some computer plots of values $T(y_{n-1}, y_n) = (y_n, y_{n+1})$ in the plane, For $a = \frac{1+\sqrt{5}}{2}$ and other values, the periodic orbits plotted this way can appear complex, approximating fractal-like shapes.

26. Lisbeth De Mol (2008), *Tag Systems and Collatz-like functions*, Theoretical Computer Science **390** (2008), 92–101.

A Post tag stystem $\mathcal{T}$ with alphabet size μ and shift ν consists of an alphabet $\mathcal{A} = \{a_1, a_2, \cdots, a_\mu\}$ and rules $a_i \mapsto E_i$, where E_i is a finite word with symbols drawn from this alphabet. Given an input word w with leftmost symbol a_i, the tag system iteration produces an output word w' which chops off the ν leftmost symbols and tags on to the right end of w the word E_i, thus the new word w' has length $|w'| = |w| + |E_i| - \nu$. Given an initial word w_0, the tag system produces subsequent words $w_1, w_2, ...$ It is said to halt at step n if w_n is the empty word. The *halting problem* for a tag system $\mathcal{T}$ is to determine for each possible input word w_0 whether or not the tag system eventually halts on that input. The *reachability problem* for $\mathcal{T}$ is, given a word x, to determine for each input w_0 whether or not the word x is eventually reached under repeated iteration. Let $TS(\mu, \nu)$ denote the collection of all Tag systems with parameters (μ, ν). In 1921 Emil Post showed the halting and reachability problems are decidable for all tag systems in $TS(2, 2)$; his proofs were not published. He could not resolve the case of $TS(2, 3)$. In 1961 Minsky showed that the halting problem, and hence the reachability problem are undecidable for tag systems in general. It was later shown that a universal Turing machine is encodable as a tag system in $TS(576, 2)$, so both the halting and reachability problems are unsolvable in this class.

In this paper the author shows the $3x+1$ problem is encodable as a reachability problem in $TS(3, 2)$. The tag system takes $\mathcal{A} = \{a, b, c\}$ with rules $a \mapsto bc,\ b \mapsto a,\ c \mapsto aaa$. The reachability question concerns reaching $x = a$, Starting from $w_0 = a^n$ (word repeating the letter a n times) the next word that is a power of a that the tag system reaches is $a^{T(n)}$, where $T(n)$ is the $3x + 1$ function. To answer this reachability problem for all iterates requires solving the $3x+1$ problem. It follows that in should be a difficult problem to find a decision procedure for the reachability problem on the class $TS(3, 2)$. The author also shows that the iteration of a generalized Collatz function which is affine on congruence classes (mod d) is encodable in a tag system in some $TS(\mu, d)$ with $\mu \leq 2d + 3$.

27. Diego Domenici (2009) *A few observations on the Collatz problem,* Inter. J. Appl. Math. Stat. **14** (2009), No. J09, 97–107. (MR 2524878)

    The paper proves results encoding the $3x+1$ conjecture as a problem of representing integers in particular forms. It shows a necessary and sufficient condition for the $3x + 1$ conjecture to hold is that each positive integer n has a representation

$$n = \frac{2^m}{3^l} - \sum_{k=1}^{1} \frac{2^{b_k}}{3^k}$$

with $1 \leq l \leq m - 3$ and $0 \leq b_1 < b_2 < \cdots < b_l \leq m - 4$.

28. Jean-Guillaume Dumas (2008), *Caractérisation des quenines et leur représentation spirale* Mathématiques et Sciences Humaines [Mathematics and Social Science], **184** No. 4, 9–23.

    This paper solves the problem raised by Queneau (1963) on allowable spiral rhyme patterns generalizing the 'sestina" rhyme pattern of Arnaut Daniel, a 12-th century troubadour. One considers iterations of the $(3x + 1)$-like function

$$\sigma_n(x) := \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{2n+1-x}{2} & \text{if } x \text{ is odd} \end{cases}$$

on the domain $\{1, 2, ..., n\}$. This function gives a permutation in the symmetric group S_n of this domain, which is called by the author a *quenine*. We also recall the inverse permutation $\delta_{2,n}(x) = \sigma_n^{-1}(x)$ given by

$$\delta_{2,n}(x) := \begin{cases} 2x & \text{if } 1 \le x \le \frac{n}{2} \\ 2n+1-2x & \text{if } \frac{n}{2} < x \le n. \end{cases}$$

The Raymond Queneau numbers n (or 2-admissible numbers) are those numbers for which this permutation is a cyclic permutation.

The author first recalls (Theorem 1) the results of Berger (1969), which used the inverse permutation $\delta_{2,n}(x)$ above to put restrictions on Queneau numbers. The author then proves (Theorem 2) that an integer n is a Queneau number if and only if either $p = 2n+1$ is prime, and either (i) $p \equiv 3$ or $5 \pmod 8$ and 2 is a primitive root of p, i.e. $ord_p(2) = 2n$, or (ii) $p \equiv 7 \pmod 8$ and $ord_p(2) = n$. This establishes a corrected form of a Conjecture of Roubaud (1993).

The author goes on to consider generalizations of quenines introduced by Roubaud (1993). He gives pictures showing the various spiral patterns associated with such permutations. To start, he sets

$$\delta_{3,n}(x) := \begin{cases} 3x & \text{if } 1 \le x \le \frac{n}{3} \\ 2n+1-3x & \text{if } \frac{n}{3} < x \le \frac{2n}{3} \\ 3x-(2n+1) & \text{if } \frac{2n}{3} < x \le n. \end{cases}$$

More generally, one can replace 3 with any $g \le n$, to get a g-spiral permutation, whose definition can be worked out from a picture of the spiral. Call such a permutation *g-admissible* if it is a cyclic permutation. Theorem 3 gives a necessary and sufficient condition for n to be g-admissible, which is that $p = 2n+1$ is prime and either (i) g is a primitive root (mod p), or (ii) n is odd, and $ord_p(g) = n$. The paper includes a table giving for $n \le 1000$ such that $p = 2n+1$ is prime the minimal g for which n is g-admissible.

Various other types of permutations are considered: a *pérecquine* is one associated to the permutation

$$\pi_n(x) := \begin{cases} 2x & \text{if } 1 \le x \le \frac{n}{2} \\ 2x-(n+1) & \text{if } \frac{n}{2} < x \le n. \end{cases}$$

This pattern is named after Georges Perec, an Oulipo member, cf. Queneau (1963). Theorem 5 shows that this permutation is cycle if and only if $p = n+1$ is prime and 2 is a primitive root (mod p).

29. Jeffrey P. Dumont and Clifford A. Reiter (2001), *Visualizing Generalized 3x+1 Function Dynamics,* Computers and Graphics **25** (2001), 883–898.

This paper describes numerical and graphical experiments iterating generalizations of the $3x+1$ function. It plots basins of attraction and false color pictures of escape times for various generalizations of the $3x+1$ function to the real line, as in Chamberland (1996), and to the complex plane, as in Letherman, Schleicher and Wood (1999). It introduces a new generalization to the complex plane , the *winding 3x+1 function*,

$$W(z) := \frac{1}{2}\left(3^{mod_2(z)}z + mod_2(z)\right),$$

in which

$$mod_2(z) := \frac{1}{2}(1 - e^{\pi i z}) = (\sin\frac{\pi z}{2})^2 - \frac{i}{2}\sin \pi z,$$

Plots of complex basins of attraction for Chamberland's function appear to have a structure resembling the Mandlebrot set, while the basins of attraction of the winding $3x+1$ function seems to have a rather different structure. The programs were written in the computer language J.

30. Jeffrey P. Dumont and Clifford A. Reiter (2003), *Real dynamics of a 3-power extension of the $3x+1$ function,* Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis **10** (2003), 875–893. (MR2005e:37099).

This paper studies the real dynamics of the function

$$T(x) := \frac{1}{2}(3^{mod_2(x)}x + mod_2(x)),$$

in which the function

$$mod_2(x) := (\sin\frac{1}{2}\pi x)^2.$$

This function agrees with the $3x+1$-function on the integers. The authors show that this function has negative Schwartzian derivative on the region $x > 0$. They study its periodic orbits and critical points, and show that any cycle of positive integers is attractive. They define an extension of the notion of total stopping time to all real numbers x that are attracted to the periodic orbit $\{1, 2\}$, representing the number of steps till the orbit enters the immediate basin of attraction of this attracting periodic orbit. They formulate the *odd critical point conjecture*, which asserts for an odd positive integer $n \geq 3$ with associated nearby critical point c_n, that the critical point c_n is attracted to the periodic orbit $\{1, 2\}$, and that c_n and n have the same total stopping time.

31. Hershel M. Farkas (2005), *Variants of the $3N+1$ problem and multiplicative semigroups*, In: Geometry, Spectral Theory, Groups and Dynamics: Proceedings in Memory of Robert Brooks, Contemporary Math., Volume 387, Amer. Math. Soc., Providence, 2005, pp. 121–127. (MR 2006g:11052)

This paper formulates some weakenings of the $3x+1$ problem where stronger results can be proved. It first shows that iteration of the map

$$F(n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv \quad 0 \pmod{3}); \\ \frac{3n+1}{2} & \text{if } n \equiv 7 \text{ or } 11 \pmod{12}; \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 5 \pmod{12}; \end{cases}$$

on the positive integers has all trajectories get to 1. The trajectories of the iterates of these functions can have arbitrarily long subsequences on which the iterates increase. The author then asks questions of the type: "Which integers can be represented in a multiplicative semigroup whose generators are a specified infinite set of rational numbers?"

He proves that the integers represented by the multiplicative semigroup generated by $\{\frac{d(n)}{n} : n \geq 1\}$, where $d(n)$ is the divisor function, represents exactly the set of positive odd integers. The analysis involves the function $F(n)$ above. Finally the author proposes as an open problem a weakened version of the $3x+1$ problem, which asks: "Which integers are represented by the multiplicative semigroup generated by $\{\frac{2n+1}{3n+2} : n \geq 1\}$ together with $\{2\}$?" The truth of the $3x+1$ Conjecture implies that all positive integers are so represented.

Note. Applegate and Lagarias (2006) prove that all positive integers are represented in the semigroup above.

Of the author's three results, Theorem 1 holds for $\tilde{T}(x)$. The arguments of Theorems 2 and 3 presented appear to apply to $\tilde{T}(x)$ as well; line 4 of Theorem 3 needs to be modified to $\tilde{T}^{(L+1)}(n) \equiv \tilde{T}^{(L)}(n) + 1 \pmod{2}$. The "proof" of Theorem 2 seems not to adequately relate the mathematics and the metamathematics.

32. David Gluck and Brian D. Taylor (2002), *A new statistic for the* $3x+1$ *problem,* Proc. Amer. Math. Soc. **130** (2002), 1293–1301. (MR 2002k:11031).

    This paper considers iterations of the Collatz function $C(x)$. If $\mathbf{a} = (a_1, a_2, ..., a_n)$ is a finite Collatz trajectory starting from a_1, with $a_n = 1$ being the first time 1 is reached, they assign the statistic

    $$C(\mathbf{a}) = \frac{a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1}{a_1^2 + a_2^2 + ... + a_n^2}.$$

    They prove that $\frac{9}{13} < C(\mathbf{a}) < \frac{5}{7}$. They find sequences of starting values that approach the upper and lower bounds, given that the starting values terminate.

33. Jeffrey R. Goodwin (2003), *Results on the Collatz conjecture,* Annalele Stiintifice ale Universitatii "Al. I. Cuza" din lasi serie noua. Informatica (Romanian), **XIII** (2003) pp. 1–16. MR2067520 (2005b:11025). [Scientific Annals of the "Al. I. Cuza" University of Iasi, Computer Science Section, Tome XIII, 2003, 1–16]

    This paper partitions the inverse iterates of 1 under the $3x+1$ map into various subsets, and studies their internal recursive structure.

34. He, Sheng Wang (2003), $3n+1$ *problem's simplifying and structure property of* $\{T(n)\}(n \in \mathbb{N})$ (Chinese), Acta Scieniarum Naturalium Universitatis Neimonggol [Nei Menggu da xue xue bao. Zi ran ke xue] (2003), No. 2.

    [I have not seen this paper.]

35. Kenneth Hicks, Gary L. Mullen, Joseph L. Yucas and Ryan Zavislak (2008), *A Polynomial Analogue of the* $3N+1$ *Problem?*, American Math. Monthly **115** (2008), No. 7, 615–622.

    This paper studies an iteration on polynomials resembling the $3x+1$ problem. It considers the function on the polynomial ring $GF_2[x]$, where GF_2 is the finite field with 2 elements, given by

    $$C_1(f(x)) = \begin{cases} (x+1)f(x)+1 & \text{if } f(0) \neq 0 \\ \frac{f(x)}{x} & \text{if } f(0) = 0. \end{cases} \quad (1)$$

They show that the iteration always converges to a constant; for an n-th degree starting polynomial in at most $n^2 + 2n$ steps. They also show there exist starting polynomials of degree n which take at least $3n$ steps to get to a constant. They observe that their result applies more generally to the map C_F acting on monic polynomials $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$, where F is an arbitrary field, by:

$$C_F(f(x)) = \begin{cases} (x - \frac{a_0}{a_j})f(x) + \frac{a_0^2}{a_j} & \text{if } f(0) \neq 0 \\ \frac{f(x)}{x} & \text{if } f(0) = 0, \end{cases} \tag{2}$$

where a_j is the smallest value $1 \leq j \leq n$ such that $a_j \neq 0$. Here a degree n polynomial takes at most $n^2 + 2n$ steps to get to a constant polynomial.

Note. Iterations of a polynomial ring over a finite field were first introduced in Matthews and Leigh (1987). They exhibited a mapping having a provably divergent trajectory.

36. Wernt Hotzel (2003), *Beitäge zum* $3n + 1$-*Problem,* Dissertation: Univsität Hamburg 2003, 62pp. [Zbl 1066.11501]

This thesis consists of 8 short chapters.

Chapter 2 describes symbolic dynamics of the $3x+1$ map T, calling it the Collatzfunction d, and noting it extends to the domain of 2-adic integers $\mathbb{Z}_2$. He encodes a symbolic dynamics of forward iterates, using binary labels (called I and O, described here as 1 and 0). He introduces the $3x + 1$ conjugacy map, denoting it T.

Chapter 3 studies the set of rational periodic points, letting $\mathbb{Q}_2$ denote the set of rationals with odd denominators. He observes that there are none in the open interval $(-1, 0)$.

Chapter 4 studies periodic cycles using Farey sequences, following Halbeisen and Hungerbühler (1997).

Chapter 5 studies unbounded orbits, observing that for rational inputs $r \in \mathbb{Q}_2$ has an unbounded orbit if and only if $r \notin \mathbb{Q}_2$.

Chapter 6 studies rational cycles with a fixed denominator N,

Chapter 7 studies periodic points of the $3x + 1$ conjugacy map. The fixed point $\frac{1}{3}$ is known. An algorithm which searches for periodic points is described.

Chapter 8 describes an encryption algorithms based on the $3x + 1$ function.

37. Huang, Guo Lin and Wu, Jia Bang (2000), *One-to-one correspondence between the natural numbers and the parity vectors in the Collatz problem* (Chinese), J. of South-Central University for the Nationalities, Natural Science Ed. [Zhong nan min zu da xue xue bao. Zi ran ke xue ban] **19** (2000), No. 3, 59–61.

English summary: "It is shown in this paper that $M_N^\circ = \{0, 1, 2, \cdots, 2^N - 1\}$ and V_N denotes the set of all v_n of truncations up to the N-th term, viz. $\{x_0, x_1, ..., x_{N-1}\}$ of the parity vector $v = \{x_0, x_1, \cdots\}$, if $m \in M_N^\circ$ and $m \to v_N(m)$ then the mapping $\sigma : M_N^\circ \to V_N$ is one-to-one. The following lemmas are from this. Lemma 1. Let $M_N = \{1, 2, \cdots, 2^N\}$, if $m \in M_N$ and $m \to v_N(m)$ then the correspondence is one-to-one. Lemma 2. Let $\mathbf{v}$ denote the set of all parity vectors $\mathbf{v} = \{x_0, x_1, \cdots\}$, if $m \in \mathbb{N}$ and $m \to \mathbf{v}(m)$ then the correspondence is one-to-one. Thus the investigation on any natural number can be converted to the investigation of its parity vector."

Note. This result is implicit in Terras (1976), and properties of the parity vector are described in Lagarias (1985).

38. Yasuaki Ito and Koji Nakano (2009), *A Hardware-Software Cooperative Approach to the Exhaustive verification of the Collatz conjecture,* International Symp. on Parallel and Distributed Processing with Applications, 2009, pp. 63–70.

    [I have not seen this paper.]

39. Ke, Wei (2001), *Generalization Concerning the* $3x + 1$ *Problem* (Chinese), Journal of Jinan University (Science and Technology) [Jinan da xue xue bao. Zi ran ke xue ban] (2001), No. 4.

    [I have not seen this paper.]

40. Ke, Yong-Sheng (2005), *The proof of the hypothesis about* "$3x + 1$" (Chinese), Journal of Tianjin Vocational Institute [Teacher's College] [Tianjin zhi ye ji shu shi fax xue yuan xue bao] (2005), No. 2.

    [I have not seen this paper.]

41. Immo O. Kerner (2000), *Die Collatz-Ulam-Kombination CUK*, Rostocker Informatik-Berichte No. 24 (2000), preprint.

    This paper attibutes the Collatz problem to Collatz in 1937. Let $c(n)$ be a function counting the number of iterations of the Collatz function $C(n)$ to get to 1, with $c(1) = 0$ and $c(3) = 7$, and so on. It reviews some basic results on the Collatz function, and introduces an auxiliary two-variable function $B(n, t)$ to describe the iteration, and plots some graphs of its behavior.

42. Stefan Kohl (2005), *Restklassenweise affine Gruppen*, Universität Stuttgart, Ph. D. Dissertation, 2005. (eprint: `http://deposit.ddb.de/dokserv?idn=977164071`)

    In this thesis, the author studies the semigroup $Rcwa(\mathbb{Z})$consisting of all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which there exists a modulus $m = m(f)$ such that the restriction of f to each residue class $(\bmod\ m)$ is an affine map. It also considers the group $RCWA(\mathbb{Z})$ consisting of the set of invertible elements of $Rcwa(\mathbb{Z})$. The group $RCWA(\mathbb{Z})$ is a subgroup of the infinite permutation group of the the integers. Both the $3x + 1$ function and the Collatz function belong to the semigroup $Rcwa(\mathbb{Z})$. The original Collatz map (see Klamkin (1963)) given by $f(3n) = 2n$, $f(3n - 1) = 4n - 1$, $f(3n - 2) = 4n - 3$ is a permutation belonging to $RCWA(\mathbb{Z})$.

    Some of the results of the thesis are as follows. The group $RCWA(\mathbb{Z})$ is not finitely generated (Theorem 2.1.1). It has finite subgroups of any isomorphism type (Theorem 2.1.2). It has a trivial center (Theorem 2.1.3). It acts highly transitively on $\mathbb{Z}$ (Theorem 2.1.5). All nontrivial normal subgroups also act highly transitively on $\mathbb{Z}$, so that it has no notrivial solvable normal subgroup (Corollary 2.1.6). It has an epimorphism *sgn* onto $\{1, -1\}$, so has a normal subgroup of index 2 (Theorem 2.12.8). Given any two subgroups, it has another subgroup isomorphic to their direct product(Corollary 2.3.3).

It has only finitely many conjugacy classes of elements having a given odd order, but it has infinitely many conjugacy classes having any given even order (Conclusion 2.7.2).

The author notes that the $3x+1$ function can be embedded as a permutation in $RCWA(\mathbb{Z}\times\mathbb{Z})$, as $(x,y) \mapsto (\frac{3x+1}{2}, 2y)$ if $x \equiv 1 (\bmod\ 2)$; $\mapsto (\frac{x}{2}, y)$ if $x \equiv 0, 2 (\bmod\ 6)$; $\mapsto (\frac{x}{2}, 2y+1)$if $x \equiv 4 (\bmod\ 6)$, where it represents the iteration projected onto the x-coordinate.

The author develops an algorithm for efficiently computing periodically linear functions, whether they are permutations or not. Periodically linear functions are functions which are defined as affine functions on each congruence class $j \pmod M$ for a fixed modulus M, as in Lagarias (1985). The $3x+1$ function is an example of such a function. The author has written a corresponding package RCWA (Residue Class-Wise Affine Groups) for the computational algebra and group theory system GAP (groups, Algorithms, Programming). This package is available for download at:
`http://www.gap-system.org/Packages/rcwa.html`.
A manual for RCWA can be found at:
`http://www.gap-system.org/Manuals/pkg/rcwa/doc/manual.pdf`

43. Stefan Kohl (2007), *Wildness of iteration of certain residue class-wise affine mappings*, Advances in Applied Math. **39** (2007), 322–328. (MR 2008g:11041)

A mapping $f : \mathbb{Z} \to \mathbb{Z}$ is called *residue class-wise affine* (abbreviated RCWA) if it is affine on residue classes $(\bmod\ m)$ for some fixed $m \geq 1$. (This class of functions was termed *periodically linear* in Lagarias (1985).) The smallest such m is called the *modulus* of f. This class of functions is closed under pointwise addition and under composition. A function f is called *tame* if the modulus of its k-th iterate remains bounded as $k \to \infty$; it is *wild* otherwise. The author shows that if $f : \mathbb{Z} \to \mathbb{Z}$ is an RCWA -function, which is surjective, but not injective, then f is necessarily wild. The paper also presents counterexamples showing that each of the three other possible combinations of hypotheses of (non-)surjectivitiy or of (non)-injectivity of f permits no conclusion whether it is tame or wild.

44. Stefan Kohl (2008a), *On conjugates of Collatz-type mappings,* Int. J. Number Theory **4**, No. 1 (2008), 117–120. (MR 2008m: 11055)

A map $f : \mathbb{Z} \to \mathbb{Z}$ is said to be *almost contracting* if there is a finite set S such that every trajectory of f visits this finite set. This property holds if and only if there is a permutation σ of the integers such that $g = \sigma^{-1} \circ f \circ \sigma$ decreases absolute value off a finite set, a property that is called
em monotonizable. Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is a surjective, but not injective, RCWA mapping (see Kohl(2006+) for a definition) having the property that the preimage set of any integer under f is finite. The main result asserts that if f is almost contracting and k such that the k-th iterate $f^{(k)}$ decreases almost all integers, then any permutation σ that establishes the almost contracting property of f cannot itself be an RCWA mapping.

The $3x+1$ function $T(x)$ is believed to be a function satisfying the hypotheses of the author's main result: It is surjective but not injective, and is believed to be almost contracting. The almost contacting property for the $3x+1$ map is equivalent to establishing that its iteration on $\mathbb{Z}$ has only finitely many cycles and no divergent trajectories.

45. Stefan Kohl (2008b), *Algorithms for a class of infinite permutation groups,* J. of Symbolic

Computation **43**, No. 8 (2008), 545–581. (MR 2009e:20003)

A mapping $f : \mathbb{Z} \to \mathbb{Z}$ is called *residue-class-wise afffine* (RCWA) if there is a positive integer m such that it is an affine mapping when restricted to each residue class (mod m). The $3x+1$ mapping T is of this kind, as is a permutation constructed by Collatz. This paper describes a collection of algorithms and methods for computing in permutation groups and monoids whose members are all RCWA mappings.

46. Stefan Kohl (2010), *A simple group generated by involutions interchanging residue classes of the integers*, Math. Z. **264** (2010), no. 4, 927–938.

The author defines a simple group of permutations of the integers, generated by compositions of certain RCWA functions.

47. Pavlos B. Konstadinidis (2006), *The real* $3x+1$ *problem*, Acta Arithmetica **122** (2006), 35–44. (MR 2007c:11029)

The author extends the $3x+1$ function to the real line as:

$$U(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } \lfloor x \rfloor \equiv 1 \pmod{2}\ . \\ \dfrac{x}{2} & \text{if } \lfloor x \rfloor \equiv 0 \pmod{2}\ . \end{cases}$$

The paper shows that the only periodic orbits of the function $U(x)$ on the positive real numbers are those on the positive integers. The paper also considers some related functions.

48. Alex V. Kontorovich and Steven J. Miller (2005), *Benford's law, values of L-functions, and the* $3x+1$ *problem*, Acta Arithmetica **120** (2005), 269–297. (MR 2007c:11085).

Benford's law says that the leading digit of decimal expansions of certain sequences are not uniformly distributed, but have the probability of digit j being $\log_{10}(1+\frac{1}{j})$. This paper gives a general method for verifying Benford's law for certain sequences. These include special values of L-functions and ratios of certain $3x+1$ iterates, the latter case being covered in Theorem 5.3. It considers the $3x+1$ iteration in the form of Sinai(2003a) and Kontorovich and Sinai (2002), which for an odd integer x has $m(x)$ being the next odd integer occurring in the $3x+1$ iteration. For a given real base $B > 1$ it looks at the distribution of the quantities $\log_B(x_m/(\frac{3}{4})^m x_0) (\mathrm{mod}\ 1)$ as x_0 varies over odd integers in $[1, X]$ and $X \to \infty$. It then takes a second limit as $m \to \infty$ and concludes that the uniform distribution is approached, provided B is such that $\log_2 B$ is an irrational number of finite Diophantine type. The case $B = 10$ corresponds to Benford's law. The theorem applies when $B = 10$ because $\log_2 10$ is known to be of finite Diophantine type by A. Baker's results on linear forms in logarithms. A main result used in the proof is the Structure Theorem in Kontorovich and Sinai (2002). Note that the assertion of Theorem 5.3 concerns a double limit: first $X \to \infty$ and then $m \to \infty$. See Lagarias and Soundararajan (2006) for related results.

49. Alex V. Kontorovich and Yakov G. Sinai (2002), *Structure Theorem for* (d, g, h)*-maps,* Bull. Braz. Math. Soc. (N.S.) **33** (2002), 213–224. (MR 2003k:11034).

This paper studies (d, g, h)-maps, in which $g > d \geq 2$, with g relatively prime to d, and $h(n)$ is a periodic integer-valued function with period d, with $h(n) \equiv -n (\text{mod } d)$ and $0 < |h(n)| < g$. The (d, g, h)-map is defined on $\mathbb{Z} \backslash d\mathbb{Z}$ by

$$T(x) := \frac{gx + h(gx)}{d^k}, \quad \text{with} \quad d^k || gx + h(gx).$$

A path of m iterates can be specified by the values $(k_1, k_2, ..., k_m)$ and a residue class ϵ (mod dg), and set $k = k_1 + k_2 + ... + k_m$. The structure theorem states that exactly $(d-1)^m$ triples (q, r, δ) with $0 \leq q < d^k$, $0 < r < g^m$ and $\delta \in E = \{j : 1 \leq j < dg, d \nmid j, g \nmid j\}$ produce a given path, and for such a triple (q, r, δ) and all $x \in \mathbb{Z}$,

$$T^{(m)}(gd(d^k x + q) + \epsilon) = gd(g^m x + r) + \delta.$$

They deduce that, as $m \to \infty$, a properly logarithmically scaled version of iterates converges to a Brownian motion with drift $\log g - \frac{d}{d-1} \log d$. More precisely, fix m, and choose points $0 = t_0 < t_1 < ... < t_r = 1$ and set $m_i = \lfloor t_i m \rfloor$ and $y_i = \log T^{(m_i)}(x)$. Then the values $y_i - y_{i-1}$ converge to a Brownian path. These results imply that when the drift is negative, almost all trajectories have a finite stopping time with $|T^{(m)}(x)| < |x|$.

50. Benjamin Kraft and Kennan Monks (2010), *On conjugacies for the $3x + 1$ map induced by continuous endomorphisms of the shift dynamical system,* Discrete Math. **310** (2010), no. 13-14, 1875–1883. (MR 2011e:37027)

The continuous endomorphisms of the shift dynamical system were classified by Maria Monks (2009). A conjugacy Φ between the $3x + 1$ map T extended to the 2-adic integers to the one-sided shift S was shown in Lagarias (1985), i.e. $\Phi \circ S \circ \Phi^{-1} = T$. This paper studies maps $H_f = \Phi \circ f \circ \Phi^{-1}$ where f is a continuous endomorphism of the 2-adic integers $\mathbb{Z}_2$ (a.k. a. binary shift dynamical system). It generalizes results of Maria Monks (2009).

51. Ilia Krasikov and J. C. Lagarias (2003), *Bounds for the $3x + 1$ problem using difference inequalities*, Acta Arithmetica **109** (2003), no. 3, 237–258. (MR 2004i:11020)

This paper deals with the problem of obtaining lower bounds for $\pi_a(x)$, the counting function for the number of integers $n \leq x$ that have some $3x + 1$ iterate $T^{(k)}(n) = a$. It improves the nonlinear programming method given in Applegate and Lagarias (1995b) for extracting lower bounds from the inequalities of Krasikov (1989). It derives a nonlinear program family directly from the Krasikov inequalities (mod 3^k) whose associated lower bounds are expected to be the best possible derivable by this approach. The nonlinear program for $k = 11$ gives the improved lower bound: If $a \not\equiv 0 \pmod 3$, then $\pi_a(x) > x^{.841}$ for all sufficiently large x. The interest of the new nonlinear program family is the (not yet realized) hope of proving $\pi_a(x) > x^{1-\epsilon}$ by this approach, taking a sufficiently large k.

52. Stuart A. Kurtz and Janos Simon (2007), *The undecidability of the generalized Collatz problem,* In: J-Y. Cai et al, Ed., *Theory and Applications of Models of Computation, 4-th International Conference, TAMC 2007, Shanghai, China, May 22-25, 2007.* Lecture Notes in Computer Science No. 4484, Springer-Verlag: New York (2007), pp. 542–553.

(MR 2374341)

A generalized Collatz function g is a function mapping the nonnegative integers to itself such that there is a modulus m such that for each congruence class $x \equiv i \pmod{m}$ it is given by $g(x) = a_i x + b_i$ when where (a_i, b_i) are nonnegative rational numbers such that $(i + km)a_i + b_i$ is an integer for all integer k. Generalized Collatz functions can easily be listed with a Gödel numbering k_e.

The authors build on the work of Conway (1972), who considered the subclass of such functions with all $b_i = 0$, which we term multiplicative generalized Collatz functions. Conway proved that the following problem is undecidable: Given as input (g, n) where g is a multiplicative generalized Collatz function and n a nonnegative integer, can it be decided whether there is some $i \geq 1$ such that the i-th iterate $g^{(i)}(2^n)$ is a power of 2? Conway obtained the result by showing that the partially defined function $M(n)$ determined by such a function g (with $2^{M(n)}$ being the first power of 2 encountered as an iterate $g^{(i)}(2^n)$ if one exists, and $M(n)$ is undefined otherwise) could be any unary partial recursive function. Here the authors formulate:

Generalized Collatz Problem (GCP). Given a representation of a generalized Collatz function g, can it be decided for all $x \geq 0$ whether there exists some $i \geq 1$ such that $g^{(i)}(x) = 1$?

They first formulate (Theorem 1): Let M_e be an acceptable Gödel numbering of partial recursive functions. Then from the index e one can compute a representation of a generalized Collatz function g_e such that M_e is total if and only if for every $x \geq 1$ there exist a value $ige1$ such that $g_e^{(i)}(x) = 1$. From this result they formulate (Theorem 2): The problem GCP is Π_0^2-complete. This locates precisely in the degrees of unsolvability hierarchy the difficulty of this undecidable problem.

This is a conference paper and does not supply detailed proofs.

53. Jeffrey C. Lagarias (2003+), *The $3x + 1$ Problem: An Annotated Bibliography (1963–1999)*,
eprint: `arxiv:math.NT/0309224 Sept. 13, 2003, v11`.

This is the initial installment of the annotated bibliography. It contains over 180 items.

54. Jeffrey C. Lagarias (2006), *Wild and Wooley Numbers*, American Mathematical Monthly, **113** (2006), 97–108. (MR 2007g:11029).

This paper considers some problems about multiplicative semigroups of positive rationals motivated by work of Farkas (2005) on variants of the $3x+1$ problem. The *wild semigroup* $\mathcal{S}$ is the semigroup of positive rational numbers generated by $\{\frac{3n+1}{2n+1} : n \geq 0\}$ together with $\frac{1}{2}$, and the *Wooley semigroup* is the sub-semigroup generated by $\{\frac{3n+1}{2n+1} : n \geq 0\}$ without $\frac{1}{2}$, This paper considers the question of which integers occur in these semigroups. The wild integer semigroup is the set of all integers in $\mathcal{S}$, and generators of the wild integer semigroup are termed *wild numbers*. The paper develops evidence in favor of the conjecture that the wild numbers consist of the set of all primes, excluding

3. It shows that 3 is not a wild number, that all other primes below 50 are wild numbers, and that there are infinitely many wild numbers. The term “wild numbers” was suggested by the novel “The Wild Numbers” by Philibert Schogt. The conjecture above was proved subsequently in Applegate and Lagarias (2006).

55. Jeffrey C. Lagarias and Neil J. A. Sloane (2004), *Approximate squaring*, Experimental Math. **13** (2004), 113–128. (MR 2005c:11098).

This paper studies iteration of the “approximate squaring” map $f(x) = x\lceil x \rceil$, and asks the question whether for a rational starting value $x_0 = r > 1$ some iterate is an integer. It conjectures that the answer is always “yes”, and proves it for rationals r with denominator 2. It shows that this holds for most rationals having a fixed denominator $d \geq 3$ with an exceptional set of integers below x of size at most $O(x^{\alpha_d})$ for certain constants $0 < \alpha_d < 1$. It then considers a variant of this problem on the p-adic numbers, where an exceptional set exists and is shown to have Hausdorff dimension equal to α_p.

The paper also studies the iteration of “approximate multiplication” maps $f_r(x) = r\lceil x \rceil$, where r is a fixed rational number. It conjectures that for $r > 1$ all but a finite number of integer starting values have some subsequent iterate that is an integer, and proves this for rationals r with denominator 2. It shows for rationals r with denominator d that the size of the exceptional set of integers below x that have no integer in their forward orbit under f_r has cardinality at most $O(x^{\beta_d})$ with $\beta_d = \frac{\log(d-1)}{\log d}$. It suggests that this conjecture is likely to be hard in the general case, by noting an analogy with iteration of the map appearing in Mahler’s Z-number problem, see Mahler (1968).

56. Jeffrey C. Lagarias and K. Soundararajan (2006), *Benford’s Law for the* $3x+1$ *Function,* J. London Math. Soc. **74** (2006), 289–303. (MR 2007h:37007)

Kontorovich and Miller (2005) proved results concerning Benford’s law for initial $3x + 1$ iterates, in a double limit as the number of iterates $N \to \infty$. This paper proves a quantitative version of Benford’s law valid for finite N. Benford’s law (to base B) for an infinite sequence $\{x_k : k \geq 1\}$ of positive quantities x_k is the assertion that $\{\log_B x_k : k \geq 1\}$ is uniformly distributed (mod 1). This paper studies the initial iterates $x_k = T^{(k)}(x_0)$ for $1 \leq k \leq N$ of the $3x + 1$ function, where N is fixed. It shows that for most initial values x_0, such sequences approximately satisfy Benford’s law, in the sense that the discrepancy of the finite sequence $\{\log_B x_k : 1 \leq k \leq N\}$ is small. The precise result treats the uniform distribution of initial values $1 \leq x_0 \leq X$, with $x \geq 2^N$, and shows that for any (real) base $B > 1$ the discrepancy is smaller than $2N^{-\frac{1}{36}}$ for all but an exceptional set $|\mathcal{E}(X, B)|$ of cardinality $|\mathcal{E}(X, B)| \leq c(B)N^{-\frac{1}{36}}X$, where $c(B)$ is independent of N and X.

57. Eero Lehtonen (2008), *Two undecidable versions of Collatz’s problems*, Theor. Comp. Sci. **407** (2008), 596–600. (MR 2009k:68092)

The author gives two constructions. He first constructs a function

$$f(x) = \begin{cases} 3x + t & \text{if } n \in A_t, \text{ for } t = -9, -8, ..., 8, 9, \\ \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \end{cases}$$

in which $\{A_t : \; -9 \le t \le 9\}$ are recursive sets of nonnegative integers that partition the class of odd positive integers, for which the problem of deciding if a given integer iterates to 1 is undecidable. It encodes the halting problem for Turing machines, when halting occurs if and only if the function iterations eventually arrive at 1. His second construction is a variant of Collatz's original function $f(3n) = 2n, f(3n+1) = 4n+1, f(3n+2) = 4n+3$, which is a permutation of the positive integers $\mathbb{N}$. He constructs a recursive function $f : \mathbb{N} \to \mathbb{N}$ which is a bijection of $\mathbb{N}$, such that the question whether a given input n is in a finite cycle under iteration of f is undecidable.

58. Dan Levy (2004), *Injectivity and Surjectivity of Collatz Functions*, Discrete Math. **285** (2004), 190–199. (MR 2005f:11036).

    This paper gives necessary and sufficient conditions on members of a class of generalized Collatz maps of the form $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod{d}$ to be injective maps, resp. surjective maps, on the integers. These give as a corollary a criterion of Venturini (1997) for such a map to be a permutation of the integers.

    The author frames some of his results in terms of concepts involving integer matrices. He introduces a notion of *gcd matrix* if its elements can be written $M_{ij} = gcd(m_i, m_j)$ and a *difference matrix* if its elements can be written $M_{ij} = m_i - m_j$. Then he considers a relation that M is a *total non-divisor* of N if $M_{ij} \nmid N_{ij}$ for all i, j. Then the author's condition for injectivity of a generalized Collatz map above is that the $d \times d$ gcd matrix $M_{ij} = gcd(m_i, m_j)$ is a total non-divisor of the $d \times d$ difference matrix $N_{ij} = q_i - q_j$, with $q_j = \frac{r_j - j m_j}{d}$.

    A very interesting result of the author is an explicit example of an injective function $T(\cdot)$ in the class above which has a (provably) divergent trajectory, and which has iterates both increasing and decreasing in size. This particular map T is not surjective.

59. Li, Xiao Chun (2002), *Compressive Iteration for the $3N + 1$ conjecture* (Chinese), J. Huazhong Univ. of Science and Technology (Natural Science) [Hua zhong gong xue yuan] **30** (2002), no. 2.

    [I have not seen this paper.]

60. Li, Xiao Chun (2003), *Contractible iteration for the $3n + 1$ conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **31** (2003), no. 7, 115–116. (MR 2025640).

    English summary: "The concept of contractible iteration for the $3N + 1$ conjecture was presented. The results of contractible iteration were gien as follows: some equivalence propositions for the $3N + 1$ conjecture; sequence of Syracuse order; proof of term formula of n; some theories about $t_a(n) = t_c(n)$."

    Note. The adequate stopping time $t_a(n)$ and coefficient stopping time $t_c(n)$ are those given in Wu and Hao (2003). Terras (1976) made a conjecture equivalent to asserting the equality $t_a(n) = t_c(n)$ always holds.

61. Li, Xiao Chun (2004), *Same-flowing numbers and super contraction iteration in $3N + 1$ conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **32** (2004) no. 10, 30–32.

English summary: "The concept of same-flowing numbers in $3N + 1$ conjecture was given. Some definitions and theorems were set up. Research into contraction iteration of odd numbers $2a + 1 (a \in \mathbb{N}_d)$ could be converted to research into Syracuse order of odd number a and contraction iteration simplified. Infinite sequence same-flowing to $3a (a \in \mathbb{N}_d)$ was constructed. The concept of super contraction iterations and term formula of x was given."

Note. The super-contraction iteration keeps track of the successive odd integers in the Collatz iteration.

62. Li, Xiao Chun (2005), *Necessary condition for periodic numbers in the* $3N+1$ *conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **33** (2005), no. 11, 102–103. (MR 2209315).

    English summary: "After the introduction of the conception of periodic numbers in $3N + 1$ conjecture, numerical theory function was defined, the definition of sum-line number in binary bit described and the Gauss function given. At the same time, some properties of number theory function and calculation formula of were discussed. By the use of number theory function, a necessary condition of periodic numbers in $3N + 1$ conjecture was proposed. The presentation of this necessary condition could play a certain role in further solution to periodic numbers question in $3N + 1$ conjecture."

63. Li, Xiao Chun (2006), *Some properties of super contraction iteration in the* $3N + 1$ *conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **34** (2006), no. 8, 115–117. (MR 2287431).

    English summary: "In researching of $3N+1$ conjecture, by using operator of dividing even factor, this paper provided the concept of super contraction and series of iteration super contraction iteration, which greatly increased iteration velocity in comparison with former contraction iteration. At the same time, this paper first provided the concept of omission numbers and obtained the cyclic relation between contraction iteration and super contraction iteration, and obtained no unique property about precursor numbers of monadic-order to odd number in super contraction iteration, and to the odd number y of $4k + 3$, obtained no unique property about precursor numbers of monadic-order to odd number in super contraction iteration. At last, this paper first provided necessary condition of existence of cyclic numbers in super contraction iteration, which can play active role in cyclic numbers researching of $3N + 1$ conjecture. All presented definitions and theories can simplify in researching world well-known problem $3N + 1$ conjecture in number theory. This paper also provides new method in $3N + 1$ conjecture continuous researching."

    Note. As in his earlier paper Li (2004), the author studies odd numbers appearing in the $3x + 1$ iteration.

64. Li, Xiao Chun and Liang, You Min (2002) *Research on the Syracuse Operator and its properties* (Chinese), Journal of Air Force Radar Academy (2002), No. 3.

    [I have not seen this paper.]

65. Li, Xiao Chun and Liu, Jun (2006), *Equivalence of the* $3N+1$ *and* $3N+3k$ *conjecture and some related properties* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **34** (2006), no. 7, 120-121. (MR 2287654)

    English summary: "$3N+1$ conjecture in number theory was extended to $3N+3^k$ one. It was pointed out that $3N+3$ conjecture is equivalent to $3N+3^k$ one. Some properties related to $3N+3$ were obtained. The generalization of $3N+1$ conjecture and these newly obtained properties not only simplify the operation about $4K+3$-odd numbers in $3N+3$ conjecture but also provide new way in researching of $3N+1$ conjecture."

66. Li, Xiao Chun and Wu, Jia Bang (2004), *Study of periodic numbers in the* $3N+1$ *conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **32** (2004), no. 10, 100–101. (MR 2121229)

    English summary: "A necessary condition about the existence of periodic numbers in the $3N+1$ conjecture: $S^l(mx_i) = nx_i$, and its generalized necessary condition: $S^l(\sigma_b x_i) = n^{b+1}x_i$ were presented. A formula about periodic numbers was given, where $x_1 = \frac{r_1}{1-3^l/2^k}$. It was proved that the length of periodic numbers being 2 or 3 could not exist."

    Note. The numbers x_i are the successive odd numbers appearing in the $3x+1$ iteration. The authors exclude periodic orbits on the odd itntegers, where there are exactly 2 or 3 odd numbers.

67. Josefine López and Peter Stoll (2009), *The* $3x+1$ *conjugacy map over a Sturmian word,* Integers **9** (2009), A13, 141–162. (MR 2506145).

    This paper studies the $3x+1$ conjugacy map $\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ on the 2-adic integers, which conjugates the $3x+1$ map $T(x)$ to the 2-adic shift map, in the sense $\Phi^{-1} \circ T \circ \Phi = S$, and which was studied by Bernstein and Lagarias (1996). It is unknown whether there is any aperiodic $x \in \mathbb{Z}_2$ such that $\Phi(x)$ is periodic; this is conjectured not to happen. This paper studies this function for x whose 2-adic expansion is a Sturmian word, corresponding to a line with irrational slope. This paper finds a generalized continued fraction expansion for $\frac{-1}{\Phi(x)}$ in this case convergent in the metric on the 2-adic integers $\mathbb{Z}_2$. It explicitly computes a number of examples, suggesting that the images $\Phi(x)$ then have 2-adic expansions of full complexity.

68. Florian Luca (2005), *On the nontrivial cycles in Collatz's problem*, SUT Journal of Mathematics **41** (2005), no. 1, 31–41 (MR 2006e:11034).

    This paper establishes conditions on non-trivial cycles on the positive integers of the $3x+1$ function. Let n denote the cycle length, and let $\{x_1, ..., x_k\}$ denote the set of odd integers that appear in such a cycle, so that $1 \le k < n$, and let l_j denote the number of iterates between x_i and x_{i+1}, so that $x_{i+1} = \frac{3x_i+1}{2^{l_i}}$, and $n = \sum_{i=1}^{k} l_i$. Let $1 \le J \le k$ denote the number of blocks of consecutive l_j taking the same value, say L_j, with the block length N_j for $1 \le j \le J$, so that $l_i = l_{i+1} = ... = l_{i+N_j-1} = L_j$ with $l_{i-1} \ne L_j, l_{i+N_j} \ne L_j$. Then $L_1N_1 + ... + L_jN_j = n$. Call an *ascent* a value $L_j = 1$, i.e. a consecutive string of increasing values $x_i < x_{i+1} < ... < x_{i+N_j-1}$ with $x_{i-1} > x_i$ and $x_{i+N_j} < x_{i+N_j-1}$. The author's main result (Theorem 1) states that there is an absolute constant C_1 such that there are at least $C_1 \log n$ ascents in any cycle. In particular for any fixed c, there are only finitely many nontrivial cycles of positive integers having at most c ascents. This result improves on that of Mimuro (2001). The proof uses transcendence results coming from linear forms in logarithms.

This result complements a result of Brox(2000), who showed that there are only finitely many integer cycles having at most $2 \log k$ *descents*, where a descent is a value i such that $x_{i+1} < x_i$.

Note. A misprint occurs in the the definition of l_i on page 32, in condition (ii) "largest" should read "smallest".

69. Maurice Margenstern (2000), *Frontier between decidability and undecidability: a survey*, Theor. Comput. Sci. **231** (2000), 217–251. (MR 2001g:03079).

This paper surveys results concerning the gap between decidability and undecidability, as measured by the number of states or symbols used in a Turing machine, Diophantine equations, the word problem, molecular computations. Post systems, register machines, neural networks and cellular automata. The $3x + 1$ function (and related functions) have been studied as a test for possible undecidability of behavior for small state Turing machines, which are so small they are not known to simulate a universal Turing machine. The author summarizes known results in Figure 12, summarizing how large a Turing machine has to be (number of states, number of symbols) to encode the $3x+1$ problem. For related results, see Margenstern and Matiyasevich (1999) and Michel (2004).

Note. This paper is the journal version of a conference paper in: M. Margenstern, Ed., International Colloquium on Universal Machines and Computations, MCU '98 , Metz, France, March 23-27,1998, Proceedings, Volume I, IUT: Metz 1998.

70. Keith R. Matthews (2005+), *The generalized* $3x+1$ *mapping,* preprint, 23pp., dated Oct. 31, 2005, downloadable as pdf file from: `http://www.maths.uq.edu.au/~krm/interests.html`

This paper discusses the behavior of $3x + 1$-like mappings and surveys many example functions as considered in Matthews and Watts (1984, 1985), Leigh (1986), Leigh and Matthews (1987) and Matthews (1992), see also Venturiri (1992). A particularly tantalizing example is

$$U(x) = \begin{cases} 7x+3 & \text{if } x \equiv 0 \pmod{3} \\[2ex] \dfrac{7x+2}{3} & \text{if } x \equiv 1 \pmod{3} \\[2ex] \dfrac{x-2}{3} & \text{if } x \equiv 2 \pmod{3}\ . \end{cases}$$

Almost all trajectories contain an element $n \equiv 0 \pmod{3}$ and once a trajectory enters the set $\{n : n \equiv 0 \pmod{3}\}$ it stays there. Matthews offers \$100 to show that if a trajectory has all iterates $U^{(k)}(x) \equiv \pm 1 \pmod{3}$ then it must eventually enter one of the cycles $\{1, -1\}$ or $\{-2, -4, -2\}$. The paper also considers some maps on the rings of integers of an algebraic number field, for example $U : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ given by

$$U(\alpha) = \begin{cases} \dfrac{(1-\sqrt{2})\alpha}{\sqrt{2}} & \text{if } \alpha \equiv 0 \pmod{(\sqrt{2})} , \\ \dfrac{3\alpha+1}{\sqrt{2}} & \text{if } \alpha \equiv 1 \pmod{(\sqrt{2})} . \end{cases}$$

The author conjectures that if $U^{(k)}(\alpha) = x_k + y_k\sqrt{2}$ is a divergent trajectory, then $x_k/y_k \to -\sqrt{2}$ as $k \to \infty$.

71. Karl Heinz Metzger (2000), *Untersuchungen zum* $(3n+1)$*-Algorithmus. Teil II: Die Konstruktion des Zahlenbaums,* PM (Praxis der Mathematik in der Schule) **42** (2000), 27–32.

The author treats the Collatz function $C(x)$, and studies the directed graph ("number-forest") on the positive integers formed with edges iterating the Collatz function. This graph might have many connected components. The $3x+1$ Conjecture asserts this graph forms a single tree, with one extra directed edge added making the trivial cycle at the bottom. The author constructs the graph as follows. To each odd number the author forms a directed path containing all numbers $\{2^k u : k \geq 0\}$ (a "Spross"), with directed edges from $2^{k+1}u \to 2^k u$. He then arranges an infinite table of all positive integers, in which the odd numbers u are placed successively in a bottom horizontal row and the "Sprossen" are then put in vertical columns over them. To the "Sprossen" edges he now adds directed edges starting from the odd nodes u at the bottom row taking $u \to 3u+1$. If $u \equiv 1 \pmod 4$ these edges move horizontally to the left and upward at least two rows, ("leftmovers") while if $u \equiv 3 \pmod 4$ they move horizontally to the right and upward exactly one row ("rightmovers"). He then considers the successive odd integers reached during an iteration. He shows that any chain of successive rightmoving odd numbers reached under iteration must be finite (Satz 9), i.e. a number of form $4n+1$ is eventually reached in the iteration. Satz 10 then asserts that positive integers can all be accounted for in a rearranged table having only odd numbers of form $4n+1$ along its bottom row. The results through Satz 9 are rigorous; maybe Satz 10 too.

The final Satz 11 asserts the truth of the $3x+1$ Conjecture. However its proof on page 32 is incomplete. The proof seems to implicitly assume the decrease in size during leftmoving steps overcomes the increase in size of iterates during rightmoving steps, but this is not rigorously shown.

72. Karl Heinz Metzger (2003), *Untersuchungen zum* $(3n+1)$*-Algorithmus, Teil III: Gesetzmässigkeiten der Ablauffolgen,* PM (Praxis der Mathematik in der Schule) **45** (2003), No. 1, 25–32.

This paper is independent of parts I and II, which had gaps in some proofs, and does not reference any of the papers Metzger (1995) (1999) (2000).

It studies iterates of the Collatz function $C(n)$, viewing the iterates $\pmod 6$. It observes a kind of self-similar structure in the tree of inverse iterates, and introduces some symbolic dynamics to describe it. It gives formulas for elements on certain branches of the tree. If one writes $n = 6(k-1)+i$ with $0 \leq i \leq 5$ and $C(n) = 6(\bar{k}-1)+j$ then it describes how to update k to $\bar{k}$, over a sequence of iterates.

73. Pascal Michel (2004), *Small Turing machines and generalized busy beaver competition,* Theor. Comp. Sci. **326** (2004), 45–56. [MR 2005e:68049, Zbl 1071.68025]

Let $TM(k,l)$ denote the set of one-tape Turing machines with k states and $l$symbols; the tape is two-way infinite. Much work has been done on how large k and l must be to encode an undecidable halting problem. It is known that the halting problem is decidable if $k = 1$ or $l = 1$ and also for classes $T(2,3), T(3,2)$ (hence $T(2,2)$. It is known that universal Turing machines exist in the classes $T(2,18), T(3,9), T(4,6), T(5,5), T(7,4), T(10,3)$ and $T(19,2)$, hence the halting problem is undecidable in these classes. (See Y.Rogozhin, Small universal Turing machines, Theor. Comp. Sci. **168** (1996), 215–240.) It is also known that the $3x+1$ problem can be encoded in a Turing machine in classes $T(2,8), T(3,5)$, $T(4,4)$, $T(5,3)$ and $T(2,10)$. It follows that there is no method currently known for solving the halting problem for Turing machines in these classes. (See Margenstein (2000) for a survey, and also Michel (1993).) Here the author shows there are Turing machines computing $3x+1$-like functions for which halting is not known in the small Turing classes $T(2,4), T(3,3), T(5,2)$. Thus the only remaining class for which decidability of the halting problem is likely to be possible is $T(4,2)$.

For the class $T(2,4)$ the author constructs a machine that can encode iteration of the function $g : \mathbb{Z}_{\geq 0} \times \{0,1\} \rightarrow \mathbb{Z}_{\geq 0} \times \{0,1\}$ given by

$$\begin{array}{rcc} g(3k, \quad 0) & = & (5k+1,1) \\ g(3k+1,0) & = & \text{halt} \\ g(3k+2,0) & = & (5k+4,0) \\ g(3k, \quad 1) & = & \text{halt} \\ g(3k+1,1) & = & (5k+5,0) \\ g(3k+2,1) & = & (5k+7,1). \end{array}$$

It is not known whether or not every input value $\mathbb{Z}_{\geq 0} \times \{0,1\}$ iteration of the function g eventually halts.

The paper also constructs machines in small Turing classes that achieve new records on the maximal number of steps before halting on empty input (busy beaver function $S(k,l)$) and for the maximal number of symbols printed before halting on empty input ($\Sigma(k,l)$. These show $S(2,3) \geq 38$, $S(2,4) \geq 7195$, $S(3,3) \geq 40737$ and $\Sigma(2,3) \geq 9, \Sigma(2,4) \geq 90$ and $\Sigma(3,3) \geq 208$. The author conjectures the values for $(2,3)$ and $(2,4)$ are best possible.

74. Tomoaki Mimuro (2001), *On certain simple cycles of the Collatz conjecture,* SUT Journal of Mathematics, **37**, No. 2 (2001), 79–89. (MR 2002j:11018).

The paper shows there are only finitely many positive integer cycles of the $3x+1$ function whose symbol sequence has the form $1^i\ (10^j)^k$, where i, j, k vary over nonnegative integers. (The symbol seqence is read left to right.) This result includes the trivial cycle starting from $n = 1$, whose symbol sequence is (10), where $(i,j,k) = (0,1,1)$. Suppose that the periodic orbit has period $p = i + k(j+1)$ terms, of which $d = i + k$ are odd. The author shows by elementary arguments that there are no integer orbits of the above type with $\frac{3}{4} > \frac{3^d}{2^p}$, and the trivial cycle is the unique solution with $\frac{3}{4} = \frac{3^d}{2^p}$. Using bounds from transcendence theory (linear forms in logarithms) he shows that there are finitely many values of (i,j,k) giving an integer orbit with $1 > \frac{3^d}{2^p} > \frac{3}{4}$, with an effective bound on their size. Any orbit on the positive integers necessarily has $1 > \frac{3^d}{2^p}$, so the result follows.

For other papers using transcendence theory to classify some types of periodic orbits, see Steiner (1978), Belaga and Mignotte (1999), Brox (2000), Simons (2005), Simons and de Weger (2005). A further improvement on this result is given in Luca (2005).

Note. There are two known integer orbits on the negative integers of the author's form. They are $n = -1$ with symbol seqence (1), where $(i, j, k) = (1, *, 0)$, and $\frac{3^d}{2^p} = \frac{3}{2}$, and $n = -5$ with symbol sequence (110) where $(i, j, k) = (1, 1, 1)$, and $\frac{3^d}{2^p} = \frac{9}{8}$. The author's finiteness result might conceivably be extended to the range $\frac{3}{2} \geq \frac{3^d}{2^p} \geq 1$ and so cover them.

75. Michal Misiurewicz and Ana Rodriguez (2005), *Real* $3X + 1$, Proc. Amer. Math. Soc., **133** (2005), 1109–1118. (MR 2005j:37011).

The authors consider the semigroup generated by the two maps $T_1(x) = \frac{x}{2}$ and $T_2(x) = \frac{3x+1}{2}$. They show this semigroup is a free semigroup on two generators. The forward orbit of a positive input x_0 under this semigroup is

$$O^+(x_0) := \{T_{i_1} \circ \ldots \circ T_{i_n}(x_0) : n \geq 1, \text{ each } i_k \in \{0, 1\}\}.$$

They show that each orbit $O^+(x_0)$ is dense on $(0, \infty)$. Furthermore they show that starting from x_0 one can get an iterate $T^{(n_0)}(x_0)$ within a given error ϵ of a given value y while remaining in the bounded region

$$\min(x, y - \epsilon \leq T_{i_1} \circ \cdots T_{i_j}(x_0) \leq \max(11x + 4, 4y - x). \;\; 1 \leq j \leq n_0.$$

They show that orbits having a periodic point are dense in $(0, \infty)$. Finally they show that the group of homeomorphisms of the line generated by T_1, T_2 consists of all maps $x \mapsto 2^k 3^l x + \frac{m}{2^i 3^j}$, in which k, l, m are integers and i, j are nonnegative. It is not a free group. These results concern topological dynamics; for results concerning measurable dynamics see Bergelson, Misiurewicz and Senti (2006).

76. Kenneth G. Monks (2002), $3X + 1$ *Minus the* $+$, Discrete Math. Theor. Comput. Sci. **5** (2002), 47–53. (MR 2203f:11030).

This paper formulates a FRACTRAN program (see Conway(1987)) of the form $R_i(n) \equiv r_i n$ if $n \equiv i (\text{mod } d)$, such that the $3x + 1$ Conjecture is true if and only if the R-orbit of 2^m contains 2, for all positive integers m. The author determines information on the behavior under iteration of the function $R(n)$ for all positive integers n, not just powers of 2. He then deduces information on the possible structure of an integer $3x + 1$ cycle (for the function $T(\cdot)$), namely that the sum of its even elements must equal the sum of its odd elements added to the number of its odd elements. Finally he notes that this fact can be deduced directly without using the FRACTRAN encoding.

77. Kenneth G. Monks and Jonathan Yazinski (2004), *The Autoconjugacy of the* $3x+1$ *function,* Discrete Mathematics **275** (2004), No. 1, 219–236. MR2026287 (2004m:11030).

This paper studies the iteration of the $3x + 1$ map $T(x)$ on the 2-adic integers $\mathbb{Z}_2$. It shows that the set of $\text{Aut}(T) = \{U \in \text{Aut}(\mathbb{Z}_2) : UTU^{-1} = T\}$ consists of the identity map and a map $\Omega = \Phi \circ V \circ \Phi^{-1}$ where $V(x) = -1 - x$ is the map reversing the bits in a 2-adic integer and Φ is the $3x+1$ Conjugacy map studied in Bernstein and Lagarias (1996). It formulates the Autoconjugacy Conjecture that $\Omega(\mathbb{Q}_{odd}) \subseteq \mathbb{Q}_{odd}$, and proves this conjecture is equivalent to no rational number with odd denominator having a divergent T-orbit. It defines a notion of self-conjugate cycle under the $3x+1$ map, which is a periodic orbit C such that $\Omega(C) = C$. It proves that $\{1, 2\}$ is the only self-conjugate cycle of integers. It shows that all self-conjugate cycles consist of positive rational numbers.

78. Kenneth M. Monks (2006), *The sufficiency of arithmetic progressions for the* $3x+1$ *conjecture,* Proc. Amer. Math. Soc. **134** (2006), No. 10, 2861–2872. MR2231609 (2007c:11030).

    This paper shows that the $3x+1$ conjecture is true if it is true for all the integers in any arithmetic progression $\{A + Bn : n \geq 0\}$, provided $A \geq 0, B \geq 1$. It gives analogous reductions for the divergent orbits conjecture and the nontrivial cycles conjecture.

79. Maria Monks (2009), *Endomorphisms of the shift dynamical systems, discrete derivatives, and applications,* Discrete Math. **309** (2009), 5196–5205. (MR 2010i:37217)

    The continuous endomorphisms of the one-sided shift dynamical system S on the 2-adic integers are induced by block maps from binary sequences of length n to those of length 1. Let D denote the endomorphism associated to $00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 1, 11 \mapsto 0$.

    Also let $V(x) = 1 - x$. A map $h : \mathbb{Z}_2 \to \mathbb{Z}_2$ is *solenoidal* if for each $k \geq 1$ the first k digits of the input vector x uniquely determine the first k digits of the output vector $h(x)$. The main result is the only continuous endomorphisms of S whose parity vector is solenoidal are $D, V \circ D, S$, and $V \circ S$.

    The map D is interpreted as a "discrete derivative," and its dynamics under iteration are studied. The 2-adic integers which under iteration reach a fixed point are characterized. It is shown that each integer $N \in \mathbb{Z}$ is eventually periodic under iteration of D, with minimal period a power of 2.

    Finally, a necessary and sufficient condition for the truth of the $3x+1$ conjecture is given in terms of these concepts. There is a unique conjugacy R taking D to the $3x+1$ map T, i.e $R \circ D \circ R^{-1} = T$. In Theorem 5.2 the $3x+1$ conjecture is shown equivalent to the action of R^{-1} on positive integers m being eventually periodic with preperiod of length 2 and period of length a power of 2.

80. Helmut Müller (2009), Uber Periodenlängen and die Vermutungen von Collatz und Crandall, [On period lengths and the conjectures of Collatz and Crandall], Mitt. Math. Ges. Hamburg, **28** (2009), 121–130.

    [I have not seen this paper.]

81. Tomas Oliveira e Silva (2004+), *Computational verification of* $3x+1$ *conjecture,* web document at `http://www.ieeta.pt/ ˜tos/`; email: `tos@ieeta.pt`.

    In Oliveira e Silva (1999) the author reported on computations verifying the $3x+1$ conjecture for $n < 3 \cdot 2^{53} = 2.702 \times 10^{16}$. In 2004 he implemented an improved version of this algorithm. As of February 2008 his computation verified the 3x+1 conjecture up to $17 \cdot 2^{58} > 4.899 \times 10^{18}$. This is the current record value for verifying the $3x+1$ conjecture. Compare Roosendaal (2004+).

82. Reiko Ohira and Michinori Yamashita (2004), *A Generalization of the Collatz problem* (Japanese), PC Literacy [Pasocon Literacy] (Personal Computer Users Application Technology Association] **31** (2006), No. 4, pp. 16–21.

    The paper presents the $3x = 1$ function and then describes some speed-ups of the iteration, by only stopping at steps that are odd numbers, with the previous step dividing by a power 2^e with $e > 1$. It then defines the p-Collatz function for an odd p by

$$f_p(x) = \begin{cases} \dfrac{px + (p-2)}{2} & \text{if } x \equiv 1 \pmod 2 \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 . \end{cases}$$

    The authors study cycles of these functions. They note the identity $f_p(p-2) = \frac{(p-2)(p+1)}{2}$. This yields for $p = 2^m - 1$ that $f_p(p-2) = (p-2)2^{m-1}$, so that $p-2$ is in a periodic orbit; the case $p = 3$ gives the trivial cycle of the $3x = 1$ function. It notes that $f_p(p^2-4) = \frac{(p-2)(p-1)^2}{2}$. This implies for $p = 2^m - 1$ then $f_p(p^2-4) = (p-2)2^{2m-1}$, so this enters the periodic orbit given by $p-2$. The paper gives a table of periodic orbits found for various small$p$ including $p = 5, 7, 9, 15, 17, 25, 27, 29$. (Note that for $p \geq 5$ most trajectories of the map f_p are expected to be divergent.)

83. Reiko Ohira and Michinori Yamashita(2004), *On the p-Collatz problem* (Japanese), PC Literacy [Pasocon Literacy] (Personal Computer Users Application Technology Association] **31** (2006), No. 4, pp. 61–64.

    [I have not seen this paper.]

84. Pan, Hong-liang (2000), *Notes on the* $3x+1$ *problem* (Chinese), Journal of Suzhou University, Natural Science [Suzhou da xue xue bao. Zi ran ke xue ban] **16** (2000), No. 4, 13–16.

    The author considers the $3x+1$ function $T(n)$, and first shows the following statement is equivalent to the $3x+1$ Conjecture: The smallest element P_n reached in the trajectory of $n \geq 1$ always satisfies $P_n \equiv 1 \pmod 4$.

    Next, given $n \geq 1$, let $y_k(n) = 1$ (resp. -1) according as the k-th iterate $T^{(k)}(n)$ is odd (resp. even), and define

$$D_k(n) := \frac{T^{(k)}(n)}{2^{y_0(n)+y_1(n)+\cdots+y_{k-1}(n)}}.$$

    The author proves that $D_{k+1}(n) \leq D_k(n)$ for all $k \geq 1$. Since $D_k(n)$ is nonnegative it follows that the limit $D_n := \lim_{k\to\infty} D_k(n)$ exists. For initial values that enter the trivial cycle, one has $D_n > 0$, since eventually the $y_k(n)$ alternate between $+1$ and -1. The author shows that $D_n = 0$ for all initial values $n \geq 1$ that either enter a non-trivial cycle or else have a divergent trajectory. It follows that the $3x+1$ conjecture is equivalent to the assertion that $D_n > 0$ for all $n \geq 1$.

85. Joseph L. Pe (2004), *The* $3x+1$ *Fractal,* Computers & Graphics **28** (2004), 431–435.

This paper considers iteration of the following extension of the Collatz function to complex numbers z, which he terms the complex Collatz function. Define $C(z) = \frac{z}{2}$ if $\lceil |z| \rceil$ is an even integer, and $C(z) = 3z + 1$ otherwise. A complex number has the *tri-convergence property* if its iterates contain three subsequences which converge to 1, 4 anbd 2, respectively. The $3x+1$ conjecture now asserts that all positive integers have the tri-convergence property. He gives a sufficient condition for a complex number to have this property, and uses it to show that $z = 1 + i$ has the tri-convergence property. He states that it is unlikely that $z = 3+5i$ has this property. The $3x+1$ problem now asserts that all positive integers have the tri-convergence property. He gives some density plots of iterates exhibiting where they are large or small; self-similarity patterns are evident in some of them. The author makes conjectures about some of these patterns, close to the negative real axis.

86. Yuval Peres, Károly Simon and Boris Solomak (2006), *Absolute continuity for random iterated function systems with overlaps,* J. London Math. Soc. **74**, No. 2 (2006), 739–756. MR2286443 (2007m:37053).

This paper contains results which apply to a question of Ya. G. Sinal which was motivated by the $3x+1$ iteration. It is stated in Section 3, as follows. Let $0 < a < 1$ be constant let Z_i $(i \geq 1)$ be independent identically distributed discrete random variables taking values $1+a$ or $1-a$, each with probability $\frac{1}{2}$. Form the random variable

$$X := 1 + Z_1 + Z_1 Z_2 + ... + Z_1 Z_2 \cdots Z_n + ...$$

With probability one this sum converges, and the distribution of X is given by a measure ν^a supported on the interval $I_a = [\frac{1}{a}, +\infty)$. Sinai asked for which values of a is the measure ν^a absolutely continuous with respect to Lebesgue measure. It is known that when $\frac{\sqrt{3}}{2} < a < 1$, the measure ν^a is singular with respect to Lebesgue measure. Conjecturally ν^a is absolutely continuous for almost all a in the interval $(0, \frac{\sqrt{3}}{2})$, and this is unsolved.

The results of this paper imply the truth of a "randomly perturbed" version of this conjecture. In Corollary 3.1 the authors consider

$$X = 1 + Z_1^{'} + Z_1^{'} Z_2^{'} + ... + Z_1^{'} Z_2^{'} \cdots Z_n^{'} + ...$$

in which $Z_i^{'} := Z_i Y_i$, in which each Y_i is drawn from a fixed absolutely continuous distribution Y supported on $(1-\epsilon_1, 1+\epsilon_2)$ for small positive ϵ_1, ϵ_2, having a bounded density and expectation $\mathbb{E}[\log Y] = 0$. The Y_i and Z_i are drawn independently of all other Y_i and all other Z_i. Let $\nu_{\mathbf{y}}^a$ be the conditional distribution of $Z_i^{'}$ given a fixed sequence of draws $\mathbf{y} = (y_1, y_2, y_3, ...)$ for all the Y_i. The conclusions are:

(a) If $0 < a < \frac{\sqrt{3}}{2}$, then almost every choice of draws $\mathbf{y} = (y_1, y_2, y_3, ...)$ yields an absolutely continuous conditional distribution $\nu_{\mathbf{y}}^a$.

(b) If $\frac{\sqrt{3}}{2} \leq a < 1$, then $\nu_{\mathbf{y}}^a$ is always singular with respect to Lebesgue measure. For almost all $\mathbf{y}$ the Hausdorff dimension of the support of $\nu_{\mathbf{y}}^a$ is $(2\log 2)/\log(\frac{1}{1-a^2})$.

The paper contains many other results.

87. Qu, Jing-hua (2002), *The $3x+1$ problem and its necessary and sufficient condition* (Chinese), Journal of Sangqiu Teacher's College (2002), No. 5.

[I have not seen this paper.]

88. Eric Roosendaal (2004+), *On the $3x+1$ problem*, web document, available at: `http://www.ericr.nl/wondrous/index.html`

The author maintains an ongoing distributed search program for verifying the $3x+1$ Conjecture to new records and for searching for extremal values for various quantities associated to the $3x+1$ function. These include quantities termed the glide, delay, residue, completeness, and gamma. Many people are contributing time on their computers to this project.

As of February 2008 the $3x+1$ Conjecture is verified up to $612 \times 2^{50} \approx 6.89 \times 10^{17}$. The largest value $\gamma(n)$ found so far is 36.716918 at $n = 7,219,136,416,377,236,271,195 \approx 7.2 \times 10^{21}$.

[The current record for verification of the $3x+1$ conjecture published in archival literature is that of Oliveira e Silva (1999). Note that Oliveira e Silva has extended his computations to 4.899×10^{18}, the current record.]

89. Jean-Louis Rouet and Marc R. Feix (2002), *A generalization of the Collatz problem. Building cycles and a stochastic approach,* J. Stat. Phys. **107**, No. 5/6 (2002), 1283–1298.
(MR 2003i:11035).

The paper studies the class of functions $U(x) = (l_i x + m_i)/n$ if $x \equiv i \pmod{n}$, with $il_i + m_i \equiv 0 \pmod{n}$. These functions include the $3x+1$ function as a special case. They show that there is a bijection between the symbolic dynamics of the first k iterations and the last k digits of the input x written in base n if and only if n is relatively prime to the product of the l_i. They show that for fixed n and any given $\{m_i : 1 \le j \le k\}$ and can find a set of coefficients $\{l_i : 0 \le i \le n-1\}$ and $\{m_i : 0 \le i \le n-1\}$ with n relatively prime to the product of the l_i which give these values as a k-cycle, $U(m_i) = m_{i+1}$ and $U(m_k) = m_0$. They give numerical experiments indicating that for maps of this kind on k digit inputs (written in base n) "stochasticity" persists beyond the first k iterations. For the $3x+1$ problem itself (with base $n = 2$), it is believed that "stochasticity" persists for about $c_0\ k$ iterations, with $c_0 = \frac{2}{\log_2(4/3)} = 4.8187$, as described in Borovkov and Pfeifer (2000), who also present supporting numerical data.

90. Giuseppe Scollo (2005), *ω-rewriting the Collatz problem,* Fundamenta Informaticae **64** (2005), 401–412. [MR 2008g:68060, Zbl 1102.68051]

This paper reformulates the Collatz iteration dynamics as a term rewriting system. ω-rewriting allows infinite input sequences and infinite rewriting. In effect the dynamics is extended to a larger domain, allowing infinitary inputs. The author shows the inputs extend to the the $3x+1$ problem on rationals with odd denominator. The infinitary extension seems analogous to extending the $3x+1$ map to the 2-adic integers.

91. Douglas J. Shaw (2006), *The pure numbers generated by the Collatz sequence*, Fibonacci Quarterly, **44** , No. 3, (2006), 194-201.

A positive integer is called *pure* if its entire tree of preimages under the Collatz map $C(x)$ contains no integer that is smaller than it is; otherwise it is called *impure*. Equivalently, an integer n is *impure* if there is some $r < n$ with $C^{(k)}(r) = n$ for some $k \geq 1$. Thus $n = 4$ is impure since $C^{(5)}(3) = 4$. This paper develops congruence conditions characterizing pure and impure numbers, e.g. all $n \equiv 0 \pmod{18}$ are impure, while all $n \equiv 9 \pmod{18}$ are pure. It proves that the set of pure numbers and impure numbers each have a natural density. The density of impure numbers satisfies $\frac{91}{162} < d < \frac{2}{3}$. The subtlety in the structure of the set of pure numbers concerns which numbers $n \not\equiv 0 \pmod{3}$ are pure.

92. Shi, Qian Li (2005a), *Properties of cycle sets* (Chinese), J. Yangzte Univ. Natural Science. [Changjiang daxue xuebao. Zi ke ban] **2** (2005), No. 7, 191–192. (MR 2241924)

    [I have not seen this paper.]

93. Shi, Qian Li (2005b), *Properties of the* $3x{+}1$ *problem* (Chinese), J. Yangzte Univ. Natural Science. [Changjiang daxue xuebao. Zi ke ban] **2** (2005), No. 10, 287–289. (MR 2241263)

    [I have not seen this paper.]

94. John L. Simons (2005), *On the non-existence of* 2*-cycles for the* $3x + 1$ *problem*, Math. Comp. **74** (2005), 1565–1572. MR2137019(2005k:11050).

    The author defines an *m-cycle* of the $3x + 1$ problem to be an periodic orbit of the $3x + 1$ function that contains m local minima, i.e. contains m blocks of consecutive odd integers. This is equal to the number of *descents* in the terminology of Brox (2000), which correspond to local maxima. The result of Steiner (1978) shows there are no non-trivial 1-cycles of the $3x + 1$ problem on the positive integers. The author proves there are no non-trivial 2-cycles for the $3x + 1$ function. See Simons and de Weger (2005) for an impressive generalization of this result.

95. John L. Simons (2007), *A simple (inductive) proof for the non-existence of* 2*-cycles for the* $3x + 1$ *problem*, J. Number Theory **123** (2007), 10-17.

    This paper gives complements Simons (2005) by giving another proof of the non-existence of 2-cycles (periodic orbits containing exactly two blocks of consecutive odd elements) for the $3x + 1$ function on the positive integers. The last section sketches a proof that the $3x - 1$ function on the positive integers has a single 2-cycle with minimal element $n_0 = 17$; this is equivalent to showing that the $3x + 1$ function on the negative integers has a single 2-cycle, starting from $n_0 = -17$. The author's method applies to the $3x \pm q$ problem, with fixed q with $gcd(6, q) = 1$ to find a finite list of 2-cycles; however it does not extend to classify m-cycles with $m \geq 3$. For results on m-cycles, see Simons and de Weger(2005).

96. John L. Simons (2008), *On the (non)-existence of* m*-cycles for generalized Syracuse sequences,* Acta Arithmetica **131** (2008), No. 3, 217–254. (MR 2008m:11056)

    The author defines an *m-cycle* of the $3x + 1$ problem to be a periodic orbit of the $3x + 1$ function that contains m local minima, i.e. contains m blocks of consecutive

odd integers. This paper genralizes a proof of Simons and de Weger (2005) to give criteria for the non-existence of m-cycles for the $3x+1$ problem of generalized Collatz sequences such as the $3x+q$ problem, the $px+1$ problem and the inverse Collatz problem. (Note that such problems may have m-cycles for various small values of m.)

97. John L. Simons (2008), *Exotic Collatz Cycles,* Acta Arithmetica **134** (2008), 201–209. (MR 2438845)

The author considers the $px+q$ problem, with $\gcd(p.q) = 1$, $p \geq 5$ odd. He studies primitive m-cycles. He shows, modulo a conjecture, that for each p there exist a q with the number of cycles being arbitrarily large. He calls such examples "exotic" because empirically such problems appear to have most orbits diverging to infinity.

98. John L. Simons (2008a+), *Post-transcendence conditions for the existence of m-cycles for the $3x+1$ problem,* preprint.

The author defines an *m-cycle* of the $3x+1$ problem to be a periodic orbit of the $3x+1$ function that contains m local minima, i.e. contains m blocks of consecutive odd integers. This paper extends the bounds of Simons and de Weger for non-existence of m-cycles for the Collatz problem to larger values of m, including all $m \leq 73$.

99. John L. Simons (2008b+), *On isomorphism between Farkas sequences and Collatz sequences,* preprint.

The author considers generalizations of sequences studied by Farkas (2005). These are sequences taking odd integers to odd integers, of two types, $F_1(a,b,c,d(x)$ and $F_2(a,b,c,d)(x)$ where a,b,c,d are odd integers. They are given by

$$F_1(x) = \begin{cases} \frac{ax+b}{2} & \text{if } x \equiv 1 (\text{mod } 4); \\ \frac{cx+d}{2} & \text{if } x \equiv 3 (\text{mod } 4). \end{cases}$$

and

$$F_2(x) = \begin{cases} \frac{x}{3} & \text{if } x \equiv 3, 9 \ (\text{mod } 12); \\ \frac{ax+b}{2} & \text{if } x \equiv 1, 5 \ (\text{mod } 12); \\ \frac{cx+d}{2} & \text{if } x \equiv 7, 11 \ (\text{mod } 12). \end{cases}$$

He introduces a notion of *isomorphism* between orbits $\{x_n\}$ and $\{y_n\}$ of two recurrence sequences, namely that there are nonzero integers α, β and an integer γ such that

$$\alpha x_k + \beta y_n = \gamma, \quad \text{for all} \ \ n \geq n_0.$$

This notion takes place on the orbit level and is weaker than that of conjugacy of the maps. For example, he observes that each $3x+1$ orbit is isomorphic to some $3x+q$ orbit.

The author shows that the some sequences $F_1(a,b,c,d)(x)$ are isomorphic to some $px+q$ orbits. He shows that no $F_2(a,b,c,d(x)$ orbit is isomorphic to a $px+1$-map orbit.

100. John L. Simons and Benne M. M. de Weger (2004), *Mersenne en het Syracuseprobleem* [Mersenne and the Syracuse problem] (Dutch), Nieuw Arch. Wiskd. **5** (2004), no. 3, 218–220. (MR2090398).

This paper is a brief survey of the work of the authors on cycles for the $3x+1$ problem, given in Simons(2005) and Simons and de Weger (2005). It also considers cycles for the $3x+q$ problem, with $q>0$ an odd number. It defines the invariant $S(q)$ to be the sums of the lengths of the cycles of the $3x+1$ function on the positive integers. The invariant $S(q)$ is not known to be finite for even a single value of q, though the $3x+1$ conjecture implies that $S(1)=2$. The paper observes that $S(3^t)=S(1)$, for each $t\geq 1$. It also considers the case that $q=M_k:=2^k-1$ is a Mersenne number. It gives computational evidence suggesting that the minimum of $S(M_k^2)-S(M_k)$, taken over all $k\geq 3$, occurs for $k=3$, with $S(M_3)=6$ and $S(M_3^2)=44$.

101. John L. Simons and Benne M. M. de Weger (2005), *Theoretical and computational bounds for m-cycles of the $3n+1$ problem*, Acta Arithmetica, **117** (2005), 51–70. (MR 2005h:11049).

The authors define an *m-cycle* of the $3x+1$ problem to be an orbit of the $3x+1$ function that contains m local minima, i.e. contains m blocks of cosecutive odd integers. This is equal to the number of descents in the terminology of Brox (2000), which correspond to local maxima. These methods rule out infinite classes of possible symbol sequences for cycles. The author's main result is that for each fixed m there are only finitely many m-cycles, there are no non-trivial such cycles on the positive integers for $1\leq m\leq 68$, and strong constraints are put on any such cycle of length at most 72. The finiteness result on m-cycles was established earlier by Brox (2000). To obtain their sharp computational results they use a transcendence result of G. Rhin [Progress in Math., Vol 71 (1987), pp. 155-164] as well as other methods, and extensive computations. [Subsequent to publication, the authors showed there are also no notrivial such cycles for $69\leq m\leq 74$, see `http://www.win.tue.nl/~bdeweger/3n+1_v1.41.pdf`]

102. Yakov G. Sinai (2003a), *Statistical $(3X+1)$-Problem*, Dedicated to the memory of Jürgen K. Moser. Comm. Pure Appl. Math. **56** No. 7 (2003), 1016–1028. (MR 2004d:37007).

This paper analyzes iterations of the variant of the $3x+1$ map that removes all powers of 2 at each step, so takes odd integers to odd integers; the author restricts the iteration to the set Π of positive integers congruent to $\pm 1 (\bmod\ 6)$, which is closed under the iteration. It gives a Structure Theorem for the form of the iterates having a given symbolic dynamics. The discussion in section 5 can be roughly stated as asserting: There is an absolute constant $c>0$ such that "most" $3x+1$ trees (mod 3^m) contain at most $e^{c\sqrt{m\log m}}2^{2m}/3^m$ nodes whose path to the root node has length at most $2m$ and which has exactly m odd iterates. Here one puts a probability density on such trees which for a given tree counts the number of such nodes divided by the total number of such nodes summed over all trees, and "most" means that the set of such trees having the property contains $1-O(1/m)$ of the total probability, as $m\to\infty$. (The total number of such nodes is 2^{2m}, and the total number of trees is $2\cdot 3^{m-1}$.) Furthermore at least $\frac{1}{m}$ of the probability is distributed among trees having at most $M^c 2^{2m}/3^m$ such nodes. From this latter result follows an entropy inequality (Theorem 5.1) which is the author's main result: The entropy H_m of this probability distribution satisfies $H_m\geq m\log 3-O(\log\ m)$. For comparison the uniform distribution on $[1,3^m]$ has the maximal possible entropy $H=m\log 3$. He conjectures that the entropy satisfies $H_m\geq m\log 3-O(1)$.

See Kontorovich and Sinai (2002) for related results on the paths of iterates of $3x+1$-like

maps.

103. Yakov G. Sinai (2003b), *Uniform distribution in the* $(3x+1)$ *problem*, Moscow Math. Journal **3** (2003), No. 4, 1429–1440. (S. P. Novikov 65-th birthday issue). MR2058805 (2005a:11026).

Define the map $U(x)$ taking the set Π of postive integers congruent to $\pm 1 \pmod 6$ into itself, given by $U(x) = \frac{3x+1}{2^k}$ where $k = k(x)$ is the largest power of 2 dividing $3x+1$. Then consider all the preimages at depth m under $U(\cdot)$ of a given integer $y = 6r + \delta$ with integer $0 \le r < 3^m$ and $\delta = \pm 1$. This consists of the (infinite) set of all integers x such that $U^{(m)}(x) = y$. Let such a preimage x have associated data $(k_1, k_2, ..., k_m)$ with $k_j = k(U^{(j-1)}(x))$, and assign to x the weight $2^{-(k_1+...+k_m)}$ multiplied by $\frac{1}{3}$ if $\delta = 1$ and by $\frac{2}{3}$ if $\delta = -1$. Define the mass assigned to y to be the sum of the weights of all its preimages at depth m. The sum of these masses over $0 \le r < 3^m$ and $\delta = \pm 1$ adds up to 1 by the Structure Theorem proved in Sinai (2003). Let the scaled size of y be the rational number $\rho = \frac{y}{3^m}$. This now defines a probability distribution $P(m)$ on these values ρ viewed as a discrete set inside $[0, 1]$. The main theorem states that as $m \to \infty$ these probability distributions $P(m)$ weakly converge to the uniform distribution on $[0, 1]$.

104. Yakov G. Sinai (2004), *A theorem about uniform distribution,* Commun. Math. Phys. **252** (2004), 581–588. (F. Dyson birthday issue) MR2104890 (2005g:37009).

This paper presents a simplified and stronger version of the uniform distribution theorem given in Sinai (2003b).

105. Matti K. Sinisalo (2003+), *On the minimal cycle lengths of the Collatz sequences*, preprint., Univ. of Oulu, Finland.

This paper shows that the minimal length of a nontrivial cycle of the $(3x+1)$-function on the positive integers is at least 630,138,897. It uses a method similar to that of Eliahou (1993), and takes advantage of the verification of the $3x+1$ conjecture below the bound 2.70×10^{16} of Oliveira e Silva (1999). It also considers bounds for cycles of the $(3x-1)$-function.

106. Alain Slakmon and Luc Macot (2006), *On the almost convergence of Syracuse sequences*, Statistics and Probability Letters **76**, No. 15 (2006), 1625–1630.

The paper shows that the "random "Syracuse conjecture is true in the sense that random Syracuse sequences get smaller than some specified bound $B \ge 1$ almost surely. Consider identical independent $0-1$ random variables X_n having probability $P[X_n = 1] = p$, $P[X_n = 0] = q = 1-p$. The authors consider the random Syracuse model $S_{n+1} = \frac{3}{2}S_n + \frac{1}{2}$, if $X_{n+1} = 1$ and $S_{n+1} = \frac{1}{2}S_n$ if $X_{n+1} = 0$, starting from a given S_0. For the actual $3x+1$ problem one would take $p = q = \frac{1}{2}$. They then consider an auxiliary sequence Y_n with $Y_0 = S_0$, formed by the rule $Y_{n+1} = Y_n(1 + \sigma\gamma)$ if $X_{n+1} = 1$ and $Y_{n+1} = Y_n(1 - \sigma)$ if $X_n = 0$, where σ and γ are positive constants. They formulate results showing that if $p\gamma - q > 0$ then there is a positive threshold value c such that $Y_n \to \infty$ almost surely if $0 < \sigma < c$ and $Y_n \to 0$ almost surely if $\sigma > c$. Finally they show that the parameters (σ, γ) can be chosen so that $\sigma > c$ such that for all starting values $S_0 \ge B$, one has $T_n \ge S_n$ holding at every step, as long as $T_n \ge B$, while $T_n \to 0$ with probability one.

The conclude that some $S_n \le B$ with probability one.

107. Bart Snapp and Matt Tracy (2008), *The Collatz problem and Analogues,* J. Integer Sequences **11** (2008), Article 08.4.7, 10pp. (MR 2009i:11144)

The authors study a polynomial analogue of the Collatz problem, as considered in Hicks et al (2008), and then give an application to the original $3x+1$ problem on $\mathbb{Z}$. They consider for polynomials in $\mathbb{Z}/n\mathbb{Z}[X]$ the map

$$T_p(f) = \begin{cases} \dfrac{(x+1)f(x) - f(0)}{x} & \text{if } f(x) \text{ is not divisible by } x \\ \\ \dfrac{f(x)}{x} & \text{if } \ f(x) \text{ is divisible by } x. \end{cases}$$

They show that for $n = 2$ that all polynomials eventually iterate to 1 under this map, but for $n \ge 3$ there always exists some polynomial $f(x)$ that never iterates to 1 under this map. In Section 3 they apply this result for $n = 2$ to define an enlarged version of the Collatz graph on (all) integers which they call the *combined Collatz graph.* This graph glues together positive and negative integer iterates, adding solid and dotted edges, representing moves of the $3x+1$ and $3x-1$ iterations., respectively. They observe that the $3x+1$ conjecture is equivalent to the claim that all positive integers appear in this graph connected by undotted edges.

108. Tang, Ren Xian (2006), *About recursion relation of* $3x+1$ *Conjecture (Chinese),* J. of Hunan Univ. Science Engineering [Hunan ki ji da xue xue bao] (2006), No. 5

[I have not seen this paper.]

109. Toshio Urata (2002), *Some holomorphic functions connected with the Collatz problem,* Bulletin Aichi Univ. Education (Natural Science) [Aichi Kyoiku Daigaku Kenkyu hokoku. Shizen kagaku.] **51** (2002), 13–16.

The author constructs an entire function $F(z)$ which interpolates the values of the speeded up Collatz function $\phi(n)$ which takes odd integers to odd integers by dividing out all powers of 2, i.e. for an odd integer n, $\phi(n) = \frac{3n+1}{2^e(3n+1)}$, where $e(m) = ord_2(m)$. Furthermore the entire function $F(z)$ is constructed so that $F'(z) = 0$ at all positive odd integers. The author analyzes the holomorphic dynamics of iterating $F(z)$. He observes that $z = 1$ is a superattacting fixed point, that $z = 0$ is a repelling fixed point, and that there is an attracting fixed point z on the negative real axis with $-\frac{1}{20} < z < 0$. He proves that all positive odd integers are in the Fatou set of $F(z)$, and that all negative odd integers are in the Julia set of $F(z)$. He observes that the intersection of the Julia set with the negative real axis coincides with the inverse iterates of $z = 0$. Finally he notes that every component of the Fatou set is simply connected.

110. Toshio Urata (2003), *The Collatz problem over* 2*-adic integers,* Bulletin Aichi Univ. Education (Natural Science) [Aichi Kyoiku Daigaku Kenkyu hokoku. Shizen kagaku.] **52** (2003), 5–11.

This is an English version of Urata (1999). The author studies a 2-adic interpolation of the speeded-up Collatz function $\phi(n)$ defined on odd integers n by dividing out all powers of 2, i.e. for an odd integer n, $\phi(n) = \frac{3n+1}{2^{p(3n+1)}}$, where $p(m) = ord_2(m)$. Let $\mathbb{Z}_2^* = \{x \in \mathbb{Z}_2 : \ x \equiv 1 \pmod 2\}$ denote the 2-adic units. The author sets $OQ := \mathbb{Q} \cap \mathbb{Z}_2^*$, and one has $\mathbb{Z}_2^*$ is the closure $\overline{OQ}$ of $OQ \subset \mathbb{Z}_2$. The author shows that the map ϕ uniquely extends to a continuous function $\phi : \mathbb{Z}_2^* \backslash \{\frac{-1}{3}\} \to \mathbb{Z}_2^*$. He shows that if $f(x) = 2x + \frac{1}{3}$ then $f(x)$ leaves ϕ invariant, in the sense that $\phi(f(x)) = \phi(x)$ for all $x \in \mathbb{Z}_2^* \backslash \{\frac{-1}{3}\}$. It follows that $f(f(x)) = 4x + 1$ also leaves ϕ invariant.

To each $x \in \mathbb{Z}_2^* \backslash \{\frac{-1}{3}\}$ he associates the sequence of 2-exponents $(p_1, p_2, ...)$ produced by iterating ϕ. He proves that an element $x \in \mathbb{Z}_2^* \backslash \{\frac{-1}{3}\}$ uniquely determine x; and that every possible sequence corresponds to some value $x \in \mathbb{Z}_2^* \backslash \{\frac{-1}{3}\}$ He shows that all periodic points of ϕ on $\mathbb{Z}_2^*$ are rational numbers $x = \frac{p}{q} \in OQ$,and that there is a unique such periodic point for any finite sequence $(p_1, p_2, \cdots, p_m)$ of positive integers, representing 2-exponents, having period m. If $C(p_1, p_2, ..., p_m) = \sum_{j=0}^{m-1} 2^{p_1 + \cdots + p_j} 3^{m-1-j}$ then this periodic pont is

$$x = R(p_1, p_2 ..., p_m) := \frac{C(p_1, ..., p_m)}{2^{p_1 + \cdots + p_m} - 3^m}$$

He shows that an orbit is periodic if and only if its sequence of 2-exponents is periodic. Examples are given.

111. Toshio Urata (2007), *Collatz's problem* (Japanese), Aichi University of Education Booklet, Math. Sciences Select 1, 60 pages.

This short book considers many aspects of the $3x + 1$ iteration, as discussed in his papers Urata (2002) (2003), (2005). In particular, it deals with holomorphic dynamics of iterating a function that is a speeded-up version of the Collatz function. It includes many pictures of parts of the Fatou and Julia set for these dynamics.

112. Toshio Urata and Kazuhiro Hamada (2005), *Positive values of a holomorphic function connected with the Collatz problem,* Bulletin Aichi Univ. Education (Natural Science), [Aichi Kyoiku Daigaku Kenkyu hokoku. Shizen kagaku.] **54** (2005), 1–10

The authors study the entire function $F(z)$ introduced in Urata (2002) which interpolates the speeded-up Collatz function $\phi(n)$ defined on the odd integers by $\phi(n) = \frac{3n+1}{2^{p(3n+1)}}$, where $p(m) := ord_2(m)$. They show that $F(x) > 0$ for all real $x > 0$. This holds even though $F(z)$ wildly oscillates on the positive real axis.

113. Tanguy Urvoy (2000), *Regularity of congruential graphs*, in: *Mathematical Foundations of Computer Science 2000 (Bratislava)*, Lecture Notes in Computer Science Vol. 1893, Springer: Berlin, 2000, pp. 680–689. (MR 2002d:68083).

The Collatz problem is viewed as a reachability problem on a directed graph with vertices labelled by positive integers n, and edges $2n \mapsto n$, $2n+1 \mapsto 3n+2$, This is an infinite, possibly disconneted graph, call it the Collatz graph. The $3x+1$ Conjecture for this graph can be formulated as a closed formula in monadic second order logic. The author considers more generally *congruential systems* which give infinite graphs with vertices labelled by positive integers and edges specified by a finite set of rules $pn + r \mapsto qn + s$, with $0 \le r < p, 0 \le s < q$.

In section 1.4 the author shows that a congruential system without remainders (all rules have $r = s = 0$) can be encoded as a Petri net with one free place. The reachability problem for a Petri net with initial position n given is known to be a decidable problem.

A large class of infinite graphs called regular graphs have a decidable monadic second order theory, see Courcelle [*Handbook of Theoretical Computer Science, Volume B,* Elsevier: New York 1990, pp. 193-242.] In section 3 the author shows that every regular graph with bounded vertex degrees can be obtained as a graph of some congruential system. In section 4 the author shows that the Collatz graph, first viewed as an undirected graph, and then directed with any choice of orientations of the edges, is never a regular graph.

114. Jean Paul Van Bendegem (2005), *The Collatz Conjecture: A Case Study in Mathematical Problem Solving*, Logic and Logical Philosophy **14**, No. 1 (2005), 7–23. (MR 2163301)

This philosophical essay concerns the issues of what mathematicans do beyond proving theorems. The work on the $3x + 1$ problem is discussed from this viewpoint. Such work includes: computer experiments, heuristic arguments concerning the truth of the conjecture, metamathemical heuristics concerning the likelihood of finding a proof, etc.

115. Stanislav Volkov (2006), *A probabilistic model for the $5k + 1$ problem and related problems,* Stochastic Processes and Applications **116** (2006), 662–674.

This paper presents a stochastic model for maps like the $5x + 1$ problem, in which most trajectories are expected to diverge. For the $5x + 1$ problem it is empirically observed that the number of values of $n \le x$ that have some iterate equal to 1 appears to grow like x^{α}, where $\alpha \approx 0.68$. The author develops stochastic models which supply a heuristic to estimate the value of α.

The stochastic model studied is a randomly labelled (rooted) binary tree model. At each vertex the left branching edge of the tree gets a label randomly drawn from a (discrete) real distribution X and the right branching edge gets a label randomly drawn from a (discrete) real distribution Y. Each vertex is labelled with the sum of the edge labels from the root; the root gets label 0. The rigorous results of the paper concern such stochastic models. The author assumes that both X and Y have positive expected values μ_x, μ_y, but that at least one random variable assumes some negative values. He also assumes that the moment generating functions of both variables are finite for all parameter values.

The author first considers for each real $\alpha > 0$ the total number of vertices $R_n(\alpha)$ at depth n in the tree having label $\le n\alpha$, and sets $R(\alpha) = \sum_{n=1}^{\infty} R_n(\alpha)$. He defines a large deviations rate function $\gamma(\alpha)$ associated to the random variable W that draws from $-X$ or $-Y$ with equal probability. He derives a large deviations criterion (Theorem 1) which states that if $\gamma(-\alpha) > \log 2$ then $R(\alpha)$ is finite almost surely, while if $\gamma(-\alpha) < \log 2$ then $R(\alpha)$ is infinite almost surely. The author next studies the quantity $Q(x)$ counting the number of vertices with labels smaller than x. This is a refinement of the case $\alpha = 0$ above. He supposes that $\gamma(0) > \log 2$ holds, and shows (Theorem 2) that this implies that $Q(x)$ is finite almost surely for each x. He then shows (Theorem 3) that

$$\beta := \lim_{x \to \infty} \frac{1}{x} \log Q(x)$$

exists almost surely and is given by

$$\beta := \max_{a \in (0, \frac{1}{2}(\mu_x + \mu_y)]} \frac{1}{a}(\log 2 - \gamma(-a)).$$

He constructs a particular stochastic model of this kind that approximates the $5x+1$ problem. For this model he shows that Theorem 3 applies and computes that $\beta \approx 0.678$.

The author observes that his stochastic model has similarities to the branching random walk stochastic model for the $3x+1$ problem studied in Lagarias and Weiss (1992), whose analysis also used the theory of large deviations.

Note. The exponential branching of the $5x+1$ tree above 1 allows one to prove that the number of such $n \le x$ that have some $5x+1$ iterate equal to 1 is at least x^{β} for some small positive β.

116. Wang, Nian-liang (2002), *The way to prove Collatz problem* (Chinese), Journal of Shangluo University (2002), No. 2.

    [I have not seen this paper.]

117. Wang, Xing-Yuan; Wang, Qiao Long; Fen, Yue Ping; and Xu, Zhi Wen (2003), *The distribution of the fixed points on the real axis of a generalized* $3x+1$ *function and some related fractal images* (Chinese), Journal of Image and Graphics [Zhongguo tu xiang tu xing xue bao] **8**(?) (2003), No. 1.

    [I have not seen this paper.]

118. Xing-yuan Wang and Xue-jing Yu (2007), *Visualizing generalized* $3x+1$ *function dynamics based on fractal,* Applied Mathematics and Computation **188** (2007), no. 1, 234–243. (MR2327110).

    This paper studies two complex-valued generalizations of the Collatz function. It replaces the function $mod_2(x)$ defined on integers x by the function $(\sin \frac{pix}{2})^2$, defined for complex x. These are then substituted in the definitions

$$C(x) = \frac{x}{2}(1 - mod_2(x)) + (3x+1)mod_2(x)$$

and

$$T(x) = \frac{3^{mod_2(x)} + mod_2(x)}{2^{1-mod_2(x)}}.$$

Both these functions agree with the Collatz function $C(n)$ on the positive integers. The first function simplifies to

$$C(x) = \frac{1}{4}(7x + 2 - (5x+2)\cos \pi x).$$

The paper studies iteration of the entire functions $C(x)$ and $T(x)$, from the viewpoint of escape time, stopping time and total stopping time. It presents graphics illustrating the results of the algorithms. The total stopping time plots exhibit vaguely Mandlebrot-like sets of various sizes located around the positive integers.

119. Wang, Xing-yuan and Yu, Xue-jing (2008), *Dynamics of the generalized $3x + 1$ function determined by its fractal images,* Progress Natural Science (English Edition) **18** (2008), 217–223. (MR 2009i:37020).

From the abstract: "Two different complex maps are obtained by generalizing the $3x + 1$ function to the complex plane, and fractal images for these two complex maps are constructed by using escape-time, stopping time and total-stopping-time arithmetic."

120. Günther J. Wirsching (2000), *Über das $3n+1$ Problem,* Elem. Math. **55** (2000), 142–155. (MR 2002h:11022).

This is a survey paper, which discusses the origin of the $3n + 1$ problem and results on the dynamics of the $3x + 1$ function.

121. Günther J. Wirsching (2001), *A functional differential equation and $3n + 1$ dynamics,* in: *Topics in Functional Differential and Functional Difference Equations (Lisbon 1999),* (T. Faria, E. Frietas, Eds.), Fields Institute Communications No. 29, Amer. Math. Soc. 2001, pp. 369–378. (MR 2002b:11035).

This paper explains how a functional differential equation arises in trying to understand $3n + 1$ dynamics. as given in Wirsching (1998a). It analyzes some properties of its solutions.

122. Günther J. Wirsching (2003) *On the problem of positive predecessor density in $3N + 1$ dynamics,* Disc. Cont. Dynam. Syst. **9** (2003), no. 3, 771–787. (MR 2004f:39028).

This paper discusses an approach to prove positive predecessor density, which formulates three conjectures which, if proved, would establish the result. This approach presents in more detail aspects of the approach taken in the author's Springer Lecture Notes volume, Wirsching (1998a).

123. Wu, Jia Bang and Huang, Guo Lin (2001), *On the traditional iteration and the elongate iteration of the $3N + 1$ conjecture* (Chinese), Journal of South-Central University for Nationalities. Natural Sciences Ed. [Zhong nan min zu da xue xue bao. Zi ran ke xue ban] **21** (2001), No. 4

[I have not seen this paper.]

124. Wu, Jia Bang and Hao, Shen Wang (2003), *On the equality of the adequate stopping time and the coefficient stopping time of n in the $3N + 1$ conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Technol. Nat. Sci. [Hua zhong gong xue yuan] **31** (2003), no. 5, 114–116. (MR 2000420)

English summary: "In the paper the equality of adequate stopping time $t_a(n)$ and the coefficient stopping time $t_c(n)$ in the $3N+1$ conjecture was discussed. It was proved that $t_a(n) = t_c(n)$ on condition that $d = \sum_{i=1}^{k-1} x_i(n)$ is not very large. Therefore it was conjectured that when the bound condition for d was unnecessary, or was automatically satisfied, $t_c(n) = t_a(n)$ in all cases."

Note: The notion of coefficient stopping time was introduced in Terras (1976) for the $3x+1$ function; here $t_c(n)$ is its analogue for the Collatz function. The adequate stopping time $t_a(n)$ is the analogue of the stopping time for the Collatz function. Terras (1976) made a conjecture equivalent to asserting the equality $t_a(n) = t_c(n)$ always holds.

125. Wu, Jia Bang and Huang, Guo Lin (2001a), *Family of consecutive integer pairs of the same height in the Collatz conjecture* (Chinese, English summary), Mathematica Applicata (Wuhan) [Ying yung shu hsüeh] **14** (2001), suppl. 21–25. (MR 1885838, Zbl 1002.11023)

English summary: "In the Collatz conjecture, certain runs of consecutive integers have the same height. In particularly in this paper, pairs of successive integers of the same height are investigated. It is found that families of consecutive integer pairs of the same height occur infinitely often and in different patterns."

Note. The authors show coalescence of iteration of the $3x + 1$ function for consecutive pairs in arithmetic progessions of form $\{(2^k m + r, 2^k m + r + 1) : m \geq 0\}$, for example $(32m+5, 32m+6), (64m+45, 64m+46), (128m+29, 128m+30)$. They exhibit infinitely many such arithmetic progressions, specified by the pattern of residues (modulo 2) of the successive iterates of the $3x + 1$ function, e.g. for $(128m + 29, 128m + 30)$ the patterns are $(1001101), (0111100)$, respectively.

126. Wu, Jia Bang and Huang, Guo Lin (2001b), *Elongate iteration for the* $3N + 1$ *Conjecture* (Chinese, English summary), J. Huazhong Univ. Sci. Tech. [Hua zhong gong xue yuan] **29** (2001), no.2 , 112–114. (MR 1887558)

English summary: "The concept of Elongate Iteration for the $3N + 1$ conjecture is presented and discussed. Some results of Elongate Iteration are given: **a.** The correspondence of numbers with parity vectors. **b.** Invariability of l-tuple. **c.** Proof of term formula of n. **d.** Some equivalence propositions for the $3N + 1$ conjecture. **e.** A property of coefficient stopping time $t_c(n)$, etc."

127. Wu, Wen Quan (2003), *An equivalent proposition of Collatz conjecture* (Chinese), Journal of Aba Teacher's College [Aba shi fan gao deng zhuan ke xue xiao xue bao] (2003), No. 3.

[I have not seen this paper.]

128. Xia, Ye (2003), *Some thoughts about* $3X + 1$ *Conjecture* (Chinese), [Zhongxue Shuxue Za Zhi (Chu zhong ban)] (2003), No. 6.

[I have not seen this paper.]

129. Xie, Jiu Guo (2006), *The interesting* $3x + 1$ *problem (Chinese),* J. of Hunan Univ. Science Technology [Hunan ki ji da xue xue bao] (2006), No. 11

[I have not seen this paper.]

130. Michinori Yamashita (2002), $3x + 1$ *problem from the* (e, k)*-perspective* (Japanese), PC Literacy [Pasocon Literacy] (Personal Computer Users Application Technology Association] **27** (2002), No. 10, 22–27.

[I have not seen this paper.]

131. Michinori Yamashita (2006), *Note on the* $3x \pm 1$ *Problem* (Japanese), PC Literacy [Pasocon Literacy] (Personal Computer Users Application Technology Association] (2006).

This paper studies the $3x + 1$ function $f(x) = T(x)$ and the $3x - 1$ function $g(x) = -T(-x)$. It lets $(e, k) = 2^e k - 1$, $[e, k] = 2^e k + 1$, where k is an odd integer. It proves a number of identities for these functions, such as $3(e, k) = (1, (e-1, 3k))$, given a table of what happens when one multiplies by a small number l. It studies patterns of successive iterations.

132. Roger E. Zarnowski (2001), *Generalized inverses and the total stopping time of Collatz sequences,* Linear and Multilinear Algebra **49** (2001), 115–130. (MR 2003b:15011).

The $3x + 1$ iteration is formulated in terms of a denumberable Markov chain with transition matrix P. The $3x + 1$ Conjecture is reformulated in terms of the limiting behavior of P^k. The group inverse $A^\sharp$ to an $n \times n$ matrix A is defined by the properties $AA^\sharp = A^\sharp A$, $AA^\sharp A = A$ and $A^\sharp AA^\sharp = A^\sharp$, and is unique when it exists. Now set $A = I - P$, an infinite matrix. Assuming there are no nontrivial cycles, the group inverse $A^\sharp$ exists, and satisfies $\lim_{k\to\infty} P^k = I - AA^\sharp$. An explicit formula is given for $A^\sharp$.

133. Roger E. Zarnowski (2008), *The congruence structure of the* $3x+1$ *map,* Fibonacci Quarterly **46/47** (2008/09), no. 2, 115–125. (MR 2010c:11037)

This paper studies the structure of forward iteration of the $3x + 1$ map taking congruence classes $(\bmod\ 2^n)$ to congruence classes $(\bmod\ 3^k)$ where k depends on the parity sequence of the iterates. He describes relations of image classes (which may overlap), using a concept termed "congruence class triangle."

This paper observes that the set A of nonnegative integers relatively prime to 3 is an invariant set under forward iteration of the $3x+1$ map. It then shows that under iteration the set of positive integers in any fixed congruence class (mod $2^a 3^b$) has forward orbits visiting every integer in A infinitely often. Thus to prove the $3x + 1$ conjecture it suffices to verify it on integers in any one of these congruence classes.

Note. There are quite a few prior results known on sufficiency of $3x + 1$ conjecture on a "thin" congruence implying it is true in general, e.g. Korec and Znam (1987).

134. Zhou, Yun Cai and Zhou, Bao Lan (2007), *On the Collatz Conjecture* (Chinese), J. Yangtze University Natural Science. [Changjiang daxue xuebao. Zi ke ban] (2007), No. 1, 24–26.

English summary: "The Collatz wave set (CWS) and the 3-1 mapping trim applied on it are defined, and the properties of CWS under trim mapping are discussed. Collatz's conjecture, namely, the $3x + 1$ problem, is also discussed from the viewpoint of CWS and 3-1 mapping, by which a new possibility is provided for proving Collatz's conjecture."

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