Logarithmic Functions

Complex Logarithm

- In real analysis, the natural logarithm function $\ln x$ is often defined as an inverse function of the real exponential function e^x. We use $\log_e x$ to represent the real logarithmic function.
- The situation is different in complex analysis because the complex exponential function e^z is not a one-to-one function on its domain $\mathbb{C}$.
- Given a fixed nonzero complex number z, the equation $e^w = z$ has infinitely many solutions e.g., $\frac{1}{2}\pi i$, $\frac{5}{2}\pi i$, and $-\frac{3}{2}\pi i$ are all solutions to $e^w = i$.
- In general, if $w = u + iv$ is a solution of $e^w = z$, then $|e^w| = |z|$ and $\arg(e^w) = \arg(z)$. Thus, $e^u = |z|$ and $v = \arg(z)$, or, equivalently, $u = \log_e |z|$ and $v = \arg(z)$. Therefore, given a nonzero complex number z we have shown that: if $e^w = z$, then $w = \log_e |z| + i\arg(z)$.
- This set of values defines a multiple-valued function $w = G(z)$, called the **complex logarithm** of z and denoted by $\ln z$.

Definition of the Complex Logarithmic Function

Definition (Complex Logarithm)

The multiple-valued function $\ln z$ defined by:

$$\ln z = \log_e |z| + i\arg(z)$$

is called the **complex logarithm**.

- The notation $\ln z$ will always be used to denote the multiple valued complex logarithm.
- By switching to exponential notation $z = re^{i\theta}$, we obtain the following alternative description of the complex logarithm:

$$\ln z = \log_e r + i(\theta + 2n\pi),\ \ n = 0, \pm 1, \pm 2, \ldots .$$

- The complex logarithm can be used to find all solutions to the exponential equation $e^w = z$, when z is a nonzero complex number.

Solving Exponential Equations I

- Find all complex solutions to the equation $e^w = i$.

  For each equation $e^w = z$, the set of solutions is given by $w = \ln z$, where $w = \log_e |z| + i\arg(z)$. For $z = i$, we have $|z| = 1$ and $\arg(z) = \frac{\pi}{2} + 2n\pi$. Thus, we get $w = \ln i = \log_e 1 + i(\frac{\pi}{2} + 2n\pi)$, whence

  $$w = \frac{(4n+1)\pi}{2} i, \ n = 0, \pm 1, \pm 2, \ldots.$$

  Therefore, each of the values: $w = \ldots, -\frac{7\pi}{2} i, -\frac{3\pi}{2} i, \frac{\pi}{2} i, \frac{5\pi}{2} i, \ldots$ satisfies the equation $e^w = i$.

Solving Exponential Equations II

- Find all complex solutions to the equation $e^w = 1 + i$ and to the equation $e^w = -2$.
- For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4} + 2n\pi$. Thus, we get

$$\begin{aligned} w &= \ln(1+i) = \log_e \sqrt{2} + i(\tfrac{\pi}{4} + 2n\pi) \\ &= \tfrac{1}{2}\log_e 2 + \tfrac{(8n+1)\pi}{4} i, \; n = 0, \pm 1, \pm 2, \ldots . \end{aligned}$$

- Since $z = -2$, we have $|z| = 2$ and $\arg(z) = \pi + 2n\pi$. Thus, $w = \ln(-2) = \log_e 2 + i(\pi + 2n\pi)$. That is,

$$w = \log_e 2 + (2n+1)\pi i, \; n = 0, \pm 1, \pm 2, \ldots .$$

Logarithmic Identities

- Complex logarithm satisfies the following identities, which are analogous to identities for the real logarithm:

Theorem (Algebraic Properties of $\ln z$)

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$;

(ii) $\ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2$;

(iii) $\ln z_1^n = n \ln z_1$.

- $\ln z_1 + \ln z_2 = \log_e |z_1| + i\arg(z_1) + \log_e |z_2| + i\arg(z_2) = \log_e |z_1| + \log_e |z_2| + i(\arg(z_1) + \arg(z_2))$.
  The real logarithm satisfies $\log_e a + \log_e b = \log_e(ab)$, for $a > 0$ and $b > 0$, so $\log_e |z_1 z_2| = \log_e |z_1| + \log_e |z_2|$. Also, $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$. Therefore, $\ln z_1 + \ln z_2 = \log_e |z_1 z_2| + i\arg(z_1 z_2) = \ln(z_1 z_2)$.
- Parts (ii) and (iii) are similar.

Principal Value of Complex Logarithm

- The complex logarithm of a positive real has infinitely many values.
- Example: $\ln 5$ is the set of values $\log_e 5 + 2n\pi i$, where n is any integer, whereas $\log_e 5$ has a single value $\log_e 5 = 1.6094$. The unique value of $\ln 5$ corresponding to $n = 0$ is the same as $\log_e 5$.
- In general, this value of the complex logarithm is called the **principal value of the complex logarithm** since it is found by using the principal argument $\text{Arg}(z)$ in place of the argument $\arg(z)$.
- We denote the principal value of the logarithm by the symbol $\text{Ln}z$, which, thus, defines a function, whereas $\ln z$ is multi-valued.

Definition (Principal Value of the Complex Logarithm)

The complex function $\text{Ln}z$ defined by:

$$\text{Ln}z = \log_e |z| + i\text{Arg}(z)$$

is called the **principal value of the complex logarithm**.

Computing the Principal Value of the Complex Logarithm

- Compute the principal value of the complex logarithm $\mathrm{Ln} z$ for

$$\text{(a) } z = i \quad \text{(b) } z = 1 + i \quad \text{(c) } z = -2$$

(a) For $z = i$, we have $|z| = 1$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$. So we get

$$\mathrm{Ln} i = \log_e 1 + \frac{\pi}{2} i = \frac{\pi}{2} i.$$

(b) For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\mathrm{Arg}(z) = \frac{\pi}{4}$. Thus,

$$\mathrm{Ln}(1+i) = \log_e \sqrt{2} + \frac{\pi}{4} i = \frac{1}{2} \log_e 2 + \frac{\pi}{4} i.$$

(c) For $z = -2$, we have $|z| = 2$ and $\mathrm{Arg}(z) = \pi$, whence

$$\mathrm{Ln}(-2) = \log_e 2 + \pi i.$$

- Warning! The algebraic identities for the complex logarithm are not necessarily satisfied by the principal value of the complex logarithm.

Lnz as an Inverse Function

- Because Lnz is one of the values of the complex logarithm $\ln z$, it follows that: $e^{\text{Ln}z} = z$, for all $z \neq 0$.
- This suggests that the logarithmic function Lnz is an inverse function of e^z.
- Because the complex exponential function is not one-to-one on its domain, this statement is not accurate.
- The relationship between these functions is similar to the relationship between the squaring function z^2 and the principal square root function $z^{1/2} = \sqrt{|z|}e^{i\text{Arg}(z)/2}$.
- The exponential function must first be restricted to a domain on which it is one-to-one in order to have a well-defined inverse function.
- In fact, e^z is a one-to-one function on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$.

Lnz as an Inverse Function (Cont'd)

- We show that if the domain of e^z is restricted to the fundamental region, then Lnz is its inverse function.

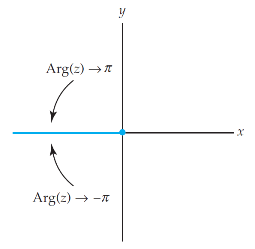
  Consider a point $z = x + iy$, $-\infty < x < \infty$, $-\pi < y \leq \pi$. We have $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$, n an integer. Thus, y is an argument of e^z. Since z is in the fundamental region, we also have $-\pi < y \leq \pi$, whence y is the principal argument of e^z, i.e., $\text{Arg}(e^z) = y$. In addition, for the real logarithm we have $\log_e e^x = x$, and so $\text{Ln}e^z = \log_e |e^z| + i\text{Arg}(e^z) = \log_e e^x + iy = x + iy$. Thus, we have shown that $\text{Ln}e^z = z$, if $-\infty < x < \infty$ and $-\pi < y \leq \pi$.

Lnz as an Inverse Function of e^z

If the complex exponential $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, then f is one-to-one and the inverse function of f is the principal value of the complex logarithm $f^{-1}(z) = \text{Ln}z$.

Discontinuities of Lnz

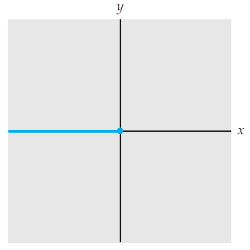
- The principal value of the complex logarithm Lnz is discontinuous at $z = 0$ since this function is not defined there.
- Lnz turns out to also be discontinuous at every point on the negative real axis.
- This may be intuitively clear since the value of Lnz for a point z near the negative x-axis in the second quadrant has imaginary part close to π, whereas the value of a nearby point in the third quadrant has imaginary part close to $-\pi$.



- The function Lnz is, however, continuous on the set consisting of the complex plane excluding the non-positive real axis.

Continuity

- Recall that a complex function $f(z) = u(x, y) + iv(x, y)$ is continuous at a point $z = x + iy$ if and only if both u and v are continuous real functions at (x, y).
- The real and imaginary parts of $\text{Ln}z$ are $u(x, y) = \log_e |z| = \log_e \sqrt{x^2 + y^2}$ and $v(x, y) = \text{Arg}(z)$, respectively.
- From calculus, we know that the function $u(x, y) = \log_e \sqrt{x^2 + y^2}$ is continuous at all points in the plane except $(0, 0)$ and the function $v(x, y) = \text{Arg}(z)$ is continuous on $|z| > 0$, $-\pi < \arg(z) < \pi$.
- Therefore, it follows that $\text{Ln}z$ is a continuous function on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, i.e., f_1 defined by: $f_1(z) = \log_e r + i\theta$ is continuous on the domain where $r = |z| > 0$ and $-\pi < \theta = \arg(z) < \pi$.



Analyticity

- Since the function f_1 agrees with the principal value of the complex logarithm $\text{Ln} z$ where they are both defined, it follows that f_1 assigns to the input z one of the values of the multiple-valued function $F(z) = \ln z$.
- I.e., we have shown that the function f_1 is a branch of the multiple-valued function $F(z) = \ln z$.
- This branch is called the **principal branch of the complex logarithm**. The nonpositive real axis is a branch cut for f_1 and the point $z = 0$ is a branch point.
- The branch f_1 is an analytic function on its domain:

Theorem (Analyticity of the Principal Branch of $\ln z$)

The principal branch f_1 of the complex logarithm is an analytic function and its derivative is given by: $f_1'(z) = \dfrac{1}{z}$.

- We prove that f_1 is analytic by using polar coordinates.

Analyticity (Proof)

- Because f_1 is defined on the domain $r > 0$ and $-\pi < \theta < \pi$, if z is a point in this domain, then we can write $z = re^{i\theta}$, with $-\pi < \theta < \pi$. Since the real and imaginary parts of f_1 are $u(r,\theta) = \log_e r$ and $v(r,\theta) = \theta$, respectively, we find that: $\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial v}{\partial \theta} = 1$, $\frac{\partial v}{\partial r} = 0$, and $\frac{\partial u}{\partial \theta} = 0$. Thus, u and v satisfy the Cauchy-Riemann equations in polar coordinates $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Because u, v, and the first partial derivatives of u and v are continuous at all points in the domain, it follows that f_1 is analytic in this domain. In addition, the derivative of f_1 is given by: $f_1'(z) = e^{-i\theta}(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}) = \frac{1}{re^{i\theta}} = \frac{1}{z}$.
- Because $f_1(z) = \mathrm{Ln} z$, for each point z in the domain, it follows that $\mathrm{Ln} z$ is differentiable in this domain, and that its derivative is given by f_1'. That is, if $|z| > 0$ and $-\pi < \arg(z) < \pi$ then:

$$\frac{d}{dz}\mathrm{Ln} z = \frac{1}{z}.$$

Derivatives of Logarithmic Functions I

- Find the derivatives of the function $z\mathrm{Ln}z$ in an appropriate domain.

  The function $z\mathrm{Ln}z$ is differentiable at all points where both of the functions z and $\mathrm{Ln}z$ are differentiable. Because z is entire and $\mathrm{Ln}z$ is differentiable on the domain $|z| > 0, -\pi < \arg(z) < \pi$, $z\mathrm{Ln}z$ is differentiable on the domain defined by $|z| > 0, -\pi < \arg(z) < \pi$:

$$\frac{d}{dz}[z\mathrm{Ln}z] = \frac{d}{dz}z \cdot \mathrm{Ln}z + z\frac{d}{dz}\mathrm{Ln}z = \mathrm{Ln}z + z\frac{1}{z} = \mathrm{Ln}z + 1.$$

Derivatives of Logarithmic Functions II

- Find the derivatives of the function $\mathrm{Ln}(z+1)$ in an appropriate domain.

  The function $\mathrm{Ln}(z+1)$ is a composition of the functions $\mathrm{Ln}z$ and $z+1$. Because the function $z+1$ is entire, it follows from the chain rule that $\mathrm{Ln}(z+1)$ is differentiable at all points $w=z+1$ such that $|w|>0$ and $-\pi<\arg(w)<\pi$. To determine the corresponding values of z for which $\mathrm{Ln}(z+1)$ is not differentiable, we first solve for z in terms of w to obtain $z=w-1$. The equation $z=w-1$ defines a linear mapping of the w-plane onto the z-plane given by translation by -1. Under this mapping the non-positive real axis is mapped onto the ray emanating from $z=-1$ and containing the point $z=-2$. Thus, $\mathrm{Ln}(z+1)$ is differentiable at all points z that are not on this ray.

$$\frac{d}{dz}\mathrm{Ln}(z+1)=\frac{1}{z+1}\cdot 1=\frac{1}{z+1}.$$