Progressions

$$a_n = 3 \cdot a_{n-1} + 4 \cdot 7^{n-1}$$

When we write down the first couple elements of the series we get

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 3 \cdot 0 + 4 \cdot 7^0 = 4 \\
a_2 &= 3 \cdot 4 + 4 \cdot 7^1 = 40 \\
a_3 &= 3 \cdot 40 + 4 \cdot 7^2 = 316 \\
a_4 &= 3 \cdot 316 + 4 \cdot 7^3 = 2320
\end{aligned}$$

When we look at these first 5 elements, we can see that all of them contain $3 \cdot a_{n-1}$ and $4 \cdot 7^{n-1}$. Now we can make a guess that our progression depends in some way on 7^n and 3^n (because we multiply by 3 every time, which is analogous to 3^n). With this in mind we can try to split our results (the a_ns) into 3^n and 7^n

$$\begin{gathered}
a_0 = 0 = 1 - 1 = 7^0 - 3^0 \\
a_1 = 4 = 7 - 3 = 7^1 - 3^1 \\
a_2 = 40 = 49 - 9 = 7^2 - 3^2 \\
a_3 = 316 = 343 - 27 = 7^3 - 3^3 \\
a_4 = 2320 = 2401 - 81 = 7^4 - 3^4
\end{gathered}$$

From this it is obvious that our progression can also be written as

$$a_n = 7^n - 3^n.$$

Taylor Series

$$f(x) = \frac{1 - e^{2x}}{x} \quad ; \qquad a = 0$$

To find the series at $a = 0$ we replace e^{2x} with its Taylor Series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The first couple derivatives of $g(x) = e^{2x}$ are

$$g'(x) = 2 \cdot e^{2x}, \qquad g''(x) = 2 \cdot 2 \cdot e^{2x}, g'''(x) = 2 \cdot 2 \cdot 2 \cdot 2e^{2x}, \quad ...$$

Plugging in plugging in $g(x)$ into the formula for the Taylor Series with $a = 0$ into we get

$$\frac{e^0}{0!}x^0, \quad \frac{2 \cdot e^0}{1!}x^1, \quad \frac{2 \cdot 2 \cdot e^0}{2!}x^2, \quad \frac{2 \cdot 2 \cdot 2 \cdot e^0}{3!}x^3, ...$$

We see that this can be written as

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n.$$

We now put this expression into $f(x)$ and get

$$f(x) = \frac{1 - \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n}{x}$$

We extract the first element of the series

$$f(x) = \frac{1 - \left(\frac{2^0}{0!}x^0 + \sum_{n=1}^{\infty} \frac{2^n}{n!}x^n\right)}{x}$$

which becomes

$$f(x) = \frac{-\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n}{x}.$$

Dividing by x gives

$$f(x) = -\sum_{n=1}^{\infty} \frac{2^n}{n!} x^{n-1}.$$

Multivariable Integrals

$$\int_0^1 \int_{\sqrt{y}}^1 y \frac{e^{x^2}}{x^3} dx dy$$

The limits of the inner integral $\int_{\sqrt{y}}^1$ can be changed to $\int_0^{x^2}$ by changing the order of integration. This yields

$$\int_0^1 \int_0^{x^2} y \frac{e^{x^2}}{x^3} dy dx.$$

This is possible because $\sqrt{y} \to 1$ for x is equivalent to $0 \to x^2$ for y. The other limit stays $0 \to 1$. We can now easily integrate the inner integral

$$\int_0^1 \int_0^{x^2} y \frac{e^{x^2}}{x^3} dy dx = \int_0^1 \left[\frac{1}{2} y^2 \frac{e^{x^2}}{x^3} \right]_{y=0}^{y=x^2} dx = \int_0^1 \frac{1}{2} x^4 \frac{e^{x^2}}{x^3} dx = \frac{1}{2} \int_0^1 x e^{x^2} dx.$$

We can solve the resulting integral by substituting $u = x^2$ and $du = 2xdx$ or $dx = \frac{du}{2x}$ (the limits don't change in this particular case), giving

$$\frac{1}{2} \int_0^1 x e^u \frac{du}{2x} = \frac{1}{2} \int_0^1 \frac{e^u}{2} du = \frac{1}{4} \left[e^u \right]_0^1 = \frac{1}{4} (e^1 - e^0) = \frac{e-1}{4}.$$