Conformal Mapping

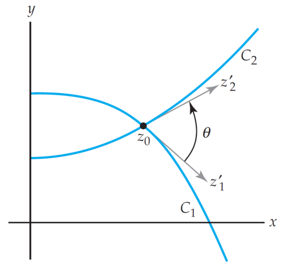
Introduction to Conformal Mapping

- We saw that a nonconstant linear mapping acts by rotating, magnifying, and translating points in the complex plane.

  As a result, the angle between any two intersecting arcs in the z-plane is equal to the angle between the images of the arcs in the w-plane under a linear mapping.
- Complex mappings that have this angle-preserving property are called conformal mappings.
- We will formally define conformal mappings and show that any analytic complex function is conformal at points where the derivative is nonzero.
- Consequently, all of the elementary functions we studied previously are conformal in some domain D.

The Angle Between Two Smooth Curves at a Point

- Suppose that $w = f(z)$ is a complex mapping defined in a domain D.
- Assume that C_1 and C_2 are smooth curves in D that intersect at z_0 and have a fixed orientation.
- Let $z_1(t)$ and $z_2(t)$ be parametrizations of C_1 and C_2 such that $z_1(t_0) = z_2(t_0) = z_0$, and such that the orientations on C_1 and C_2 correspond to the increasing values of the parameter t.
- Because C_1 and C_2 are smooth, the tangent vectors $z_1' = z_1'(t_0)$ and $z_2' = z_2'(t_0)$ are both nonzero.
- We define the **angle** between C_1 and C_2 to be the angle θ in the interval $[0, \pi]$ between the tangent vectors z_1' and z_2'.

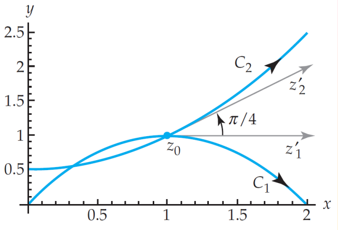


Equality of Angles in Magnitude and in Sense

- Suppose that under the complex mapping $w = f(z)$ the curves C_1 and C_2 in the z-plane are mapped onto the curves C_1' and C_2' in the w-plane, respectively.
- Because C_1 and C_2 intersect at z_0, we must have that C_1' and C_2' intersect at $f(z_0)$.
- If C_1' and C_2' are smooth, then the angle between C_1' and C_2' at $f(z_0)$ is the angle ϕ in $[0, \pi]$ between the tangent vectors w_1' and w_2'.
- We say that the angles θ and ϕ are **equal in magnitude** if $\theta = \phi$.
- In the z-plane, the vector z_1', whose initial point is z_0, can be rotated through the angle θ onto the vector z_2'. This rotation in the z-plane can be in either direction.
- In the w-plane, the vector w_1', whose initial point is $f(z_0)$, can be rotated in one direction through an angle of ϕ onto the vector w_2'.
- If the rotation in the z-plane is the same direction as the rotation in the w-plane, we say that the angles θ and ϕ are **equal in sense**.

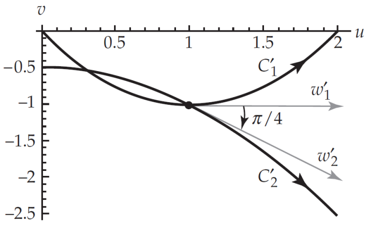
Magnitude and Sense of Angles

- The smooth curves C_1 and C_2 shown are given by $z_1(t) = t + (2t - t^2)i$ and $z_2(t) = t + \frac{1}{2}(t^2 + 1)i$, $0 \le t \le 2$, respectively. These curves intersect at $z_0 = z_1(1) = z_2(1) = 1 + i$. The tangent vectors at z_0 are $z_1' = z_1'(1) = 1$ and $z_2' = z_2'(1) = 1 + i$.

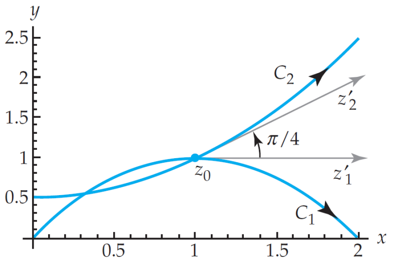


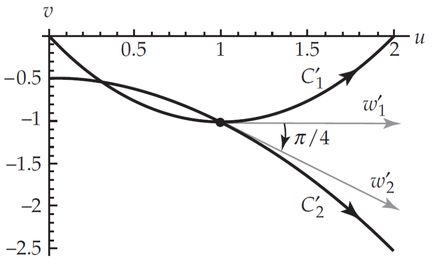
The angle between C_1 and C_2 at z_0 is $\theta = \frac{\pi}{4}$.

Under the complex mapping $w = \overline{z}$, the images of C_1 and C_2 are the curves C_1' and C_2'. They are parametrized by $w_1(t) = t - (2t - t^2)i$ and $w_2(t) = t - \frac{1}{2}(t^2 + 1)i$, $0 \le t \le 2$, and intersect at the point $w_0 = f(z_0) = 1 - i$.



Magnitude and Sense of Angles (Cont'd)





- At w_0, the tangent vectors to C_1' and C_2' are $w_1' = w_1'(1) = 1$ and $w_2' = w_2'(1) = 1 - i$.
    - The angle between C_1' and C_2' at w_0 is $\phi = \frac{\pi}{4}$. Therefore, the angles θ and ϕ are equal in magnitude.
    - The rotation through $\frac{\pi}{4}$ of the vector z_1' onto z_2' must be counterclockwise, whereas the rotation through $\frac{\pi}{4}$ of w_1' onto w_2' must be clockwise. Thus, ϕ and θ are not equal in sense.

Conformal Mapping

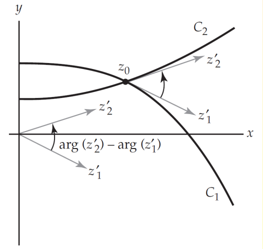
Definition (Conformal Mapping)

Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. We call $w = f(z)$ **conformal at** z_0 if, for every pair of smooth oriented curves C_1 and C_2 in D intersecting at z_0, the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C_1' and C_2' at $f(z_0)$ in both magnitude and sense.

- The term **conformal mapping** will also be used to refer to a complex mapping $w = f(z)$ that is conformal at z_0.
- If $w = f(z)$ maps a domain D onto a domain D' and if $w = f(z)$ is conformal at every point in D, then we call $w = f(z)$ a **conformal mapping of D onto D'**.
- Example: If $f(z) = az + b$ is a linear function with $a \neq 0$, then $w = f(z)$ is conformal at every point in the complex plane.
- Example: We just saw that $w = \overline{z}$ is not a conformal mapping at the point $z_0 = 1 + i$ since θ and ϕ are not equal in sense.

Angles between Curves

- Consider smooth curves C_1 and C_2, parametrized by $z_1(t)$ and $z_2(t)$, respectively, which intersect at $z_1(t_0) = z_2(t_0) = z_0$.
- The requirement that C_1 is smooth ensures that the tangent vector to C_1 at z_0, given by $z_1' = z_1'(t_0)$, is nonzero, and, so, $\arg(z_1')$ is defined and represents an angle between z_1' and the positive x-axis.
- The tangent vector to C_2 at z_0, given by $z_2' = z_2'(t_0)$, is nonzero, and $\arg(z_2')$ represents an angle between z_2' and the positive x-axis.
- The angle θ between C_1 and C_2 at z_0 is the value $\arg(z_2') - \arg(z_1')$ in $[0, \pi]$, provided that we can rotate z_1' counterclockwise about 0 through the angle θ onto z_2'. In the case that a clockwise rotation is needed, then $-\theta$ is the value in the interval $(-\pi, 0)$. In either case, we get both the magnitude and sense of the angle between C_1 and C_2 at z_0.



Example of Angles between Curves

- Consider again the smooth curves C_1 and C_2 given by $z_1(t) = t + (2t - t^2)i$ and $z_2(t) = t + \frac{1}{2}(t^2 + 1)i$, $0 \leq t \leq 2$, respectively, that intersect at the point $z_0 = z_1(1) = z_2(1) = 1 + i$.

  Their images under $w = \overline{z}$ are $w_1(t) = t - (2t - t^2)i$ and $w_2(t) = t - \frac{1}{2}(t^2 + 1)i$, $0 \leq t \leq 2$, and intersect at the point $w_0 = f(z_0) = 1 - i$.

  The unique value of $\arg(z_2') - \arg(z_1') = \arg(1 + i) - \arg(1) = \frac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, that lies in the interval $[0, \pi]$ is $\frac{\pi}{4}$. Therefore, the angle between C_1 and C_2 is $\theta = \frac{\pi}{4}$, and the rotation of z_1' onto z_2' is counterclockwise.

  The expression $\arg(w_2') - \arg(w_2') = \arg(1 - i) - \arg(1) = -\frac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, has no value in $[0, \pi]$, but has the unique value $-\frac{\pi}{4}$ in the interval $(-\pi, 0)$. Thus, the angle between C_1' and C_2' is $\phi = \frac{\pi}{4}$, and the rotation of w_1' onto w_2' is clockwise.

Analytic Functions

Theorem (Conformal Mapping)

If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.

- Suppose that f is analytic in a domain D containing z_0, and that $f'(z_0) \neq 0$. Let C_1 and C_2 be two smooth curves in D parametrized by $z_1(t)$ and $z_2(t)$, respectively, with $z_1(t_0) = z_2(t_0) = z_0$. Assume that $w = f(z)$ maps the curves C_1 and C_2 onto the curves C_1' and C_2'. We wish to show that the angle θ between C_1 and C_2 at z_0 is equal to the angle ϕ between C_1' and C_2' at $f(z_0)$ in both magnitude and sense. We may assume, by renumbering C_1 and C_2, if necessary, that $z_1' = z_1'(t_0)$ can be rotated counterclockwise about 0 through the angle θ onto $z_2' = z_2'(t_0)$. The angle θ is the unique value of $\arg(z_2') - \arg(z_1')$ in the interval $[0, \pi]$. C_1' and C_2' are parametrized by $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$.

Proof of the Conformal Mapping Theorem

- C_1' and C_2' are parametrized by $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$. Using the chain rule $w_1' = w_1'(t_0) = f'(z_1(t_0)) \cdot z_1'(t_0) = f'(z_0) \cdot z_1'$, and $w_2' = w_2'(t_0) = f'(z_2(t_0)) \cdot z_2'(t_0) = f'(z_0) \cdot z_2'$. Since C_1 and C_2 are smooth, both z_1' and z_2' are nonzero. Furthermore, by hypothesis, $f'(z_0) \neq 0$. Therefore, both w_1' and w_2' are nonzero, and the angle ϕ between C_1' and C_2' at $f(z_0)$ is a value of $\arg(w_2') - \arg(w_1') = \arg(f'(z_0) \cdot z_2') - \arg(f'(z_0) \cdot z_1')$. Now we obtain:

$$\begin{aligned} &\arg(f'(z_0) \cdot z_2') - \arg(f'(z_0) \cdot z_1') \\ &= \arg(f'(z_0)) + \arg(z_2') - [\arg(f'(z_0)) + \arg(z_1')] \\ &= \arg(z_2') - \arg(z_1'). \end{aligned}$$

  The unique value in $[0, \pi]$ is θ. Therefore, $\theta = \phi$ in both magnitude and sense, and consequently $w = f(z)$ is a conformal mapping at z_0.
- Example: (a) The entire function $f(z) = e^z$ is conformal at every point in the complex plane since $f'(z) = e^z \neq 0$, for all z in $\mathbb{C}$.
  (b) The entire $g(z) = z^2$ is conformal at all points z, $z \neq 0$.

Critical Points

- The function $g(z) = z^2$ is not a conformal mapping at $z_0 = 0$ because $g'(0) = 0$.
- In general, if a complex function f is analytic at a point z_0 and if $f'(z_0) = 0$, then z_0 is called a **critical point** of f.
- Although it does not follow from the Conformal Mapping Theorem, it is true that analytic functions are not conformal at critical points.
- More specifically, the following magnification of angles occurs at a critical point:

Theorem (Angle Magnification at a Critical Point)

Let f be analytic at the critical point z_0. If $n > 1$ is an integer such that $f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$, then the angle between any two smooth curves intersecting at z_0 is increased by a factor of n by the complex mapping $w = f(z)$. In particular, $w = f(z)$ is not a conformal mapping at z_0.

Angle Magnification at Critical Points

- Example: Find all points where the mapping $f(z) = \sin z$ is conformal. The function $f(z) = \sin z$ is entire and we have that $f'(z) = \cos z$. Moreover, $\cos z = 0$ if and only if $z = \frac{(2n+1)\pi}{2},\ n = 0, \pm 1, \pm 2, \ldots,$ and so each of these points is a critical point of f.
  - Therefore, by the Conformal Mapping Theorem, $w = \sin z$ is a conformal mapping at z, for all $z \neq \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$.
  - Furthermore, by the Angle Magnification Theorem, $w = \sin z$ is not a conformal mapping at z if $z = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$. Because $f''(z) = -\sin z = \pm 1$ at the critical points of f, the theorem indicates that angles at these points are increased by a factor of 2.

Linear Fractional Transformations

Linear Fractional Transformations

- We studied complex linear mappings $w = az + b$ where a and b are complex constants and $a \neq 0$. Such mappings act by rotating, magnifying, and translating points in the complex plane.
- We also looked at the complex reciprocal mapping $w = \frac{1}{z}$. An important property, when defined on the extended complex plane, is that it maps certain lines to circles and certain circles to lines.
- A more general type of mapping that has similar properties is a linear fractional transformation:

Definition (Linear Fractional Transformation)

If a, b, c and d are complex constants with $ad - bc \neq 0$, then the complex function defined by:

$$T(z) = \frac{az + b}{cz + d}$$

is called a **linear fractional transformation**.

- These are also called **Möbius** or **bilinear transformations**.
- If $c = 0$, then T is a linear mapping.

Properties of Linear Fractional Transformations

- If $c \neq 0$, then we can write $T(z) = \frac{az+b}{cz+d} = \frac{bc-ad}{c}\frac{1}{cz+d} + \frac{a}{c}$. Setting $A = \frac{bc-ad}{c}$ and $B = \frac{a}{c}$, we see that the transformation T is written as the composition $T(z) = f \circ g \circ h(z)$, where

$$f(z) = Az + B, \quad h(z) = cz + d, \quad g(z) = \frac{1}{z}.$$

- The domain of T is the set of all z, such that $z \neq -\frac{d}{c}$.
- Since $T'(z) = \frac{ad-bc}{(cz+d)^2}$ and $ad - bc \neq 0$, linear fractional transformations are conformal on their domains.
- The condition $ad - bc \neq 0$ also ensures that T is one-to-one.
- If $c \neq 0$, then $T(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(z+\frac{b}{a})}{z+\frac{d}{c}} = \frac{\phi(z)}{z-(-\frac{d}{c})}$, where $\phi(z) = \frac{a}{c}(z + \frac{b}{a})$. Because $ad - bc \neq 0$, we have that $\phi(-\frac{d}{c}) \neq 0$, and, hence, the point $z = -\frac{d}{c}$ is a simple pole of T.

Linear Fractional Transformation on the Extended Plane

- Since T is defined for all points in the extended plane except the pole $z = -\frac{d}{c}$ and the ideal point ∞, we need only extend the definition of T to include these points.
  - Because $\lim_{z \to -\frac{d}{c}} \frac{cz+d}{az+b} = \frac{0}{a(-\frac{d}{c})+b} = \frac{0}{-ad+bc} = 0,$ it follows that $\lim_{z \to -d/c} \frac{az+b}{cz+d} = \infty$.
  - Moreover, $\lim_{z \to \infty} \frac{az+b}{cz+d} = \lim_{z \to 0} \frac{a/z+b}{c/z+d} = \lim_{z \to 0} \frac{a+zb}{c+zd} = \frac{a}{c}$.
- Thus, if $c \neq 0$, we regard T as a one-to-one mapping of the extended complex plane defined by: $T(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \neq -\frac{d}{c}, \infty \\ \infty, & \text{if } z = -\frac{d}{c} \\ \frac{a}{c}, & \text{if } z = \infty \end{cases}$

A Linear Fractional Transformation

- Find the images of the points $0, 1+i, i$ and ∞ under the linear fractional transformation

$$T(z) = \frac{2z+1}{z-i}.$$

  - For $z = 0$, $T(0) = \dfrac{2(0)+1}{0-i} = \dfrac{1}{-i} = i$.
  - For $z = 1+i$, $T(1+i) = \dfrac{2(1+i)+1}{(1+i)-i} = \dfrac{3+2i}{1} = 3+2i$.
  - For $z = i$, $T(i) = \infty$.
  - Finally, for $z = \infty$, $T(\infty) = \dfrac{2}{1} = 2$.

Circle-Preserving Property

- The reciprocal mapping $w = \frac{1}{z}$ has two important properties:
  - The image of a circle centered at $z = 0$ is a circle;
  - The image of a circle with center on the x- or y-axis and containing the pole $z = 0$ is a vertical or horizontal line.
- Linear fractional transformations have a similar mapping property:

Theorem (Circle-Preserving Property)

If C is a circle in the z-plane and if T is a linear fractional transformation, then the image of C under T is either a circle or a line in the extended w-plane. The image is a line if and only if $c \neq 0$ and the pole $z = -\frac{d}{c}$ is on the circle C.

- When $c = 0$, T is a linear function, and we saw that linear functions map circles onto circles.
- Assume that $c \neq 0$. Then $T(z) = f \circ g \circ h(z)$, where $f(z) = Az + B$ and $h(z) = cz + d$ are linear functions and $g(z) = \frac{1}{z}$ is the reciprocal function. Since h is a linear mapping, the image C' of the circle C under h is a circle.

Proof of the Circle-Preserving Property

- $z \xrightarrow{h(z)=cz+d} w \xrightarrow{g(w)=1/w} \xi \xrightarrow{f(\xi)=A\xi+B}$
  We examine two cases:
  - Case 1: Assume that the origin $w=0$ is on the circle C'. This occurs if and only if the pole $z=-\frac{d}{c}$ is on the circle C. If $w=0$ is on C', then the image of C' under $g(z)=\frac{1}{z}$ is either a horizontal or vertical line L. Since f is linear, the image of L under f is also a line. Thus, if the pole $z=-\frac{d}{c}$ is on C, then the image of C under T is a line.
  - Case 2: Assume that the point $w=0$ is not on C', i.e., the pole $z=-\frac{d}{c}$ is not on the circle C. Let C' be the circle $|w-w_0|=\rho$. If we set $\xi=g(w)=\frac{1}{w}$ and $\xi_0=g(w_0)=\frac{1}{w_0}$, then for any point w on C' we have $|\xi-\xi_0|=\left|\frac{1}{w}-\frac{1}{w_0}\right|=\frac{|w-w_0|}{|w|\cdot|w_0|}=\rho|\xi_0||\xi|$. It can be shown that the ξ satisfying $|\xi-a|=\lambda|\xi-b|$ form a line if $\lambda=1$ and a circle if $0<\lambda\neq 1$. A comparison with $a=\xi_0$, $b=0$, and $\lambda=\rho|\xi_0|$, taking into account that $w=0$ is not on C', yields $|w_0|\neq\rho$, or, equivalently, $\lambda=\rho|\xi_0|\neq 1$. This implies that the set of points ξ is a circle. Finally, since f is a linear function, the image of this circle under f is again a circle. We conclude that the image of C under T is a circle.

Mapping Lines to Circles with $T(z)$

- The key observation in the foregoing proof was that a linear fractional transformation can be written as a composition of the reciprocal function and two linear functions.
- The image of any line L under the reciprocal mapping $w = \frac{1}{z}$ is a line or a circle.
- Therefore, using similar reasoning, we can show:

Proposition (Mapping Lines to Circles with $T(z)$)

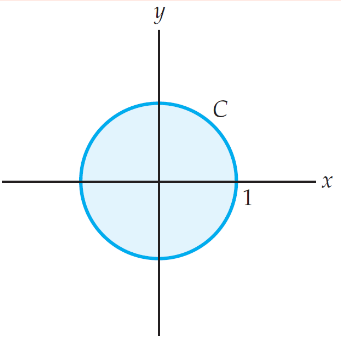
If T is a linear fractional transformation, then the image of a line L under T is either a line or a circle. The image is a circle if and only if $c \neq 0$ and the pole $z = -\frac{d}{c}$ is not on the line L.

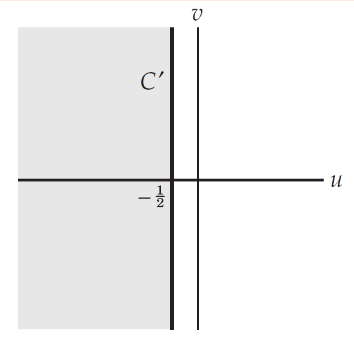
Image of a Circle I

- Find the image of the unit circle $|z| = 1$ under the linear fractional transformation $T(z) = \frac{z+2}{z-1}$. What is the image of the interior $|z| < 1$ of this circle?
  - The pole of T is $z = 1$ and this point is on the unit circle $|z| = 1$. Thus, by the Circle-Preserving Theorem, the image of the unit circle is a line. Since the image is a line, it is determined by any two points. Because $T(-1) = -\frac{1}{2}$ and $T(i) = -\frac{1}{2} - \frac{3}{2}i$, we see that the image is the line $u = -\frac{1}{2}$.
  - For the second question, note that a linear fractional transformation is a rational function, and so it is continuous on its domain. As a consequence, the image of the interior $|z| < 1$ of the unit circle is either the half-plane $u < -\frac{1}{2}$ or the half-plane $u > -\frac{1}{2}$. Using $z = 0$ as a test point, we find that $T(0) = -2$, which is to the left of the line $u = -\frac{1}{2}$, and so the image is the half-plane $u < -\frac{1}{2}$.

Illustration of Example I

- The unit circle $|z| = 1$ is mapped by $T = \frac{z+2}{z-1}$ onto the line $u = -\frac{1}{2}$:





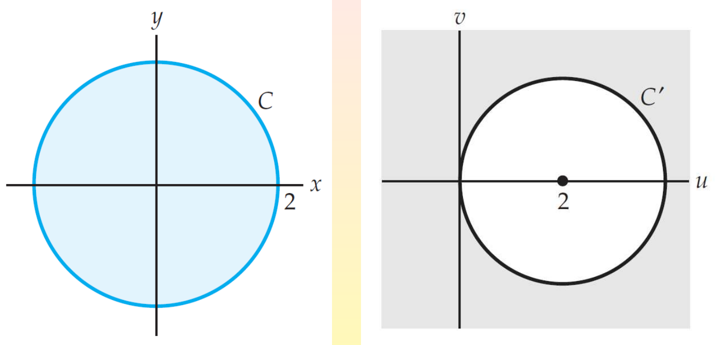
- Moreover, the interior $|z| < 1$ is mapped onto the half-plane $u < -\frac{1}{2}$.

Image of a Circle II

- Find the image of the unit circle $|z| = 2$ under the linear fractional transformation $T(z) = \frac{z+2}{z-1}$. What is the image of the disk $|z| \leq 2$ under T?
  - The pole $z = 1$ does not lie on the circle $|z| = 2$. The Circle Mapping Theorem indicates that the image of $|z| = 2$ is a circle C'. The circle $|z| = 2$ is symmetric with respect to the x-axis. So, if z is on the circle $|z| = 2$, then so is $\overline{z}$. Moreover, for all z, $T(\overline{z}) = \frac{\overline{z}+2}{\overline{z}-1} = \frac{\overline{z+2}}{\overline{z-1}} = \overline{\left(\frac{z+2}{z-1}\right)} = \overline{T(z)}$. Hence, if z and $\overline{z}$ are on $|z| = 2$, then we must have that both $w = T(z)$ and $\overline{w} = \overline{T(z)} = T(\overline{z})$ are on the circle C'. It follows that C' is symmetric with respect to the u-axis. Since $z = 2$ and -2 are on the circle $|z| = 2$, the two points $T(2) = 4$ and $T(-2) = 0$ are on C'. The symmetry of C' implies that 0 and 4 are endpoints of a diameter, and so C' is the circle $|w - 2| = 2$.
  - Using $z = 0$ as a test point, we find that $w = T(0) = -2$, which is outside the circle $|w - 2| = 2$. Therefore, the image of the interior of the circle $|z| = 2$ is the exterior of the circle $|w - 2| = 2$.

Illustration of Example II

- The circle $|z| = 2$ is mapped by $T = \frac{z+2}{z-1}$ onto the circle $|w - 2| = 2$:



- Moreover, the interior $|z| < 2$ is mapped onto the exterior $|w - 2| > 2$.

Linear Fractional Transformations as Matrices

- With the linear fractional transformation $T(z) = \frac{az+b}{cz+d}$ we associate the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- The assignment is not unique because, if e is a nonzero complex number, then $T(z) = \frac{az+b}{cz+d} = \frac{eaz+eb}{ecz+ed}$. But, if $e \neq 1$, then the two matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} = eA$ are not equal.
- It is easy to verify that the composition $T_2 \circ T_1$ of $T_1(z) = \frac{a_1 z+b_1}{c_1 z+d_1}$ and $T_2(z) = \frac{a_2 z+b_2}{c_2 z+d_2}$ is represented by the product of matrices $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}$.

Inverse Linear Fractional Transformations and Matrices

- The formula for $T^{-1}(z)$ can be computed by solving the equation $w = T(z)$ for z. This formula is represented by the inverse of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:
  $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. By identifying $e = \frac{1}{ad-bc}$ in the multiplicative relation between matrices corresponding to the same linear fractional transformation, we can also represent $T^{-1}(z)$ by the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Using Matrices

- Suppose $S(z) = \frac{z-i}{iz-1}$ and $T(z) = \frac{2z-1}{z+2}$. Use matrices to find $S^{-1}(T(z))$.

  We represent the linear fractional transformations S and T by the matrices $\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. The transformation S^{-1} is given by $\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}$. So, the composition $S^{-1} \circ T$ is given by $\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2+i & 1+2i \\ 1-2i & 2+i \end{pmatrix}$. Therefore, $S^{-1}(T(z)) = \dfrac{(-2+i)z+1+2i}{(1-2i)z+2+i}$.

Cross-Ratio

- The cross-ratio is a method to construct a linear fractional transformation $w = T(z)$, which maps three given distinct points z_1, z_2 and z_3 on the boundary of D to three given distinct points w_1, w_2 and w_3 on the boundary of D'.

Definition (Cross-Ratio)

The cross-ratio of the complex numbers z, z_1, z_2 and z_3 is the complex number $\dfrac{z - z_1}{z - z_3}\dfrac{z_2 - z_3}{z_2 - z_1}$.

- When computing a cross-ratio, we must be careful with the order of the complex numbers. E.g., the cross-ratio of $0, 1, i$ and 2 is $\frac{3}{4} + \frac{1}{4}i$, whereas the cross-ratio of $0, i, 1$ and 2 is $\frac{1}{4} - \frac{1}{4}i$.
- The cross-ratio can be extended to include points in the extended complex plane by using the limit formula. E.g., the cross-ratio of, say, ∞, z_1, z_2 and z_3 is given by $\lim_{z\to\infty} \dfrac{z - z_1}{z - z_3}\dfrac{z_2 - z_3}{z_2 - z_1}$.

Cross-Ratios and Linear Fractional Transformations

Theorem (Cross-Ratios and Linear Fractional Transformations)

If $w = T(z)$ is a linear fractional transformation that maps the distinct points z_1, z_2 and z_3 onto the distinct points w_1, w_2 and w_3, respectively, then, for all z,

$$\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}.$$

- Let $R(z) = \frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}$. Note that $R(z_1) = 0$, $R(z_2) = 1$, $R(z_3) = \infty$. Let $S(z) = \frac{z-w_1}{z-w_3}\frac{w_2-w_3}{w_2-w_1}$. For S, $S(w_1) = 0$, $S(w_2) = 1$, $S(w_3) = \infty$. Therefore, the points z_1, z_2 and z_3 are mapped onto the points w_1, w_2 and w_3, respectively, by the linear fractional transformation $S^{-1}(R(z))$. Hence, $0, 1$ and ∞ are mapped onto $0, 1$ and ∞, respectively, by the composition $T^{-1}(S^{-1}(R(z)))$. The only linear fractional transformation that maps 0, 1 and ∞ onto 0, 1, and ∞ is the identity. Thus, $T^{-1}(S^{-1}(R(z))) = z$, or $R(z) = S(T(z))$. With $w = T(z)$, we get $R(z) = S(w)$, i.e., $\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}$.

Constructing a Linear Fractional Transformation I

- Construct a linear fractional transformation that maps the points $1, i$ and -1 on the unit circle $|z| = 1$ onto the points $-1, 0, 1$ on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.
  - Identifying
$$z_1 = 1,\ z_2 = i,\ z_3 = -1, \quad w_1 = -1,\ w_2 = 0,\ w_3 = 1,$$
the desired mapping $w = T(z)$ must satisfy
$$\frac{z-1}{z-(-1)} \frac{i-(-1)}{i-1} = \frac{w-(-1)}{w-1} \frac{0-1}{0-(-1)}.$$
We get $i(w-1)(z-1) = (w+1)(z+1)$, whence $w(z-1)i - w(z+1) = (z+1) + (z-1)$, giving
$$w = \frac{(z+1)+(z-1)i}{-(z+1)+(z-1)i} = \frac{(z-i)(i+1)}{(iz-1)(i+1)} = \frac{z-i}{iz-1}.$$
  - Using the test point $z = 0$, we obtain $T(0) = i$. Therefore, the image of the interior $|z| < 1$ is the upper half-plane $v > 0$.

Constructing a Linear Fractional Transformation II

- Construct a linear fractional transformation that maps the points $-i, 1$ and ∞ on the line $y = x - 1$ onto the points $1, i$ and -1 on the unit circle $|w| = 1$.

  The cross-ratio of z, $z_1 = -i$, $z_2 = 1$, and $z_3 = \infty$ is $\lim_{z_3 \to \infty} \frac{z+i}{z-z_3} \frac{1-z_3}{1+i} = \lim_{z_3 \to 0} \frac{z+i}{z-1/z_3} \frac{1-1/z_3}{1+i} = \lim_{z_3 \to 0} \frac{z+i}{zz_3-1} \frac{z_3-1}{1+i} = \frac{z+i}{1+i}$. By the theorem, with $w_1 = 1$, $w_2 = i$ and $w_3 = -1$, the desired mapping $w = T(z)$ must satisfy

$$\frac{z+i}{1+i} = \frac{w-1}{w+1} \frac{i+1}{i-1}.$$

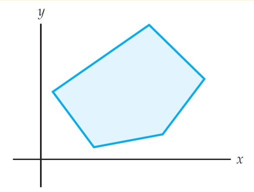
  After solving for w and simplifying we obtain

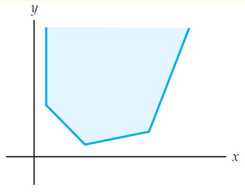
$$w = T(z) = \frac{z+1}{-z+1-2i}.$$

Schwarz-Christoffel Transformations

Polygonal Regions

- A **polygonal region** in the complex plane is a region that is bounded by a simple, connected, piecewise smooth curve consisting of a finite number of line segments.
- The boundary curve of a polygonal region is called a **polygon** and the endpoints of the line segments in the polygon are called **vertices**.
- If a polygon is a closed curve, then the region enclosed by the polygon is called a **bounded polygonal region**.





A polygonal region that is not bounded is called an **unbounded polygonal region**.

In the case of an unbounded polygonal region, the ideal point ∞ is also called a vertex of the polygon.

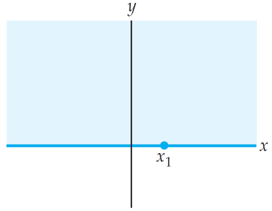
Special Cases I

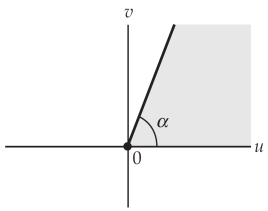
- Before providing a general formula for a conformal mapping of the upper half-plane $y \geq 0$ onto a polygonal region, we examine the complex mapping
$$w = f(z) = (z - x_1)^{\alpha/\pi},$$
where x_1 and α are real numbers and $0 < \alpha < 2\pi$.
- This mapping is the composition of a translation $T(z) = z - x_1$ followed by the real power function $F(z) = z^{\alpha/\pi}$.
  - T translates in a direction parallel to the real axis. The x-axis is mapped onto the u-axis with $z = x_1$ mapping onto $w = 0$.
  - For F, we replace z by $re^{i\theta}$ to obtain: $F(z) = (re^{i\theta})^{\alpha/\pi} = r^{\alpha/\pi}e^{i(\alpha\theta/\pi)}$. Thus, the complex mapping $w = z^{\alpha/\pi}$:
    - magnifies or contracts the modulus r of z to the modulus $r^{\alpha/\pi}$ of w;
    - rotates z through $\frac{\alpha}{\pi}$ radians about the origin to increase or decrease an argument θ of z to an argument $\frac{\alpha\theta}{\pi}$ of w.

Thus, $w = F(T(z)) = (z - x_1)^{\alpha/\pi}$ maps a ray emanating from x_1 and making an angle of ϕ radians with the real axis onto a ray emanating from 0 making an angle of $\frac{\alpha\phi}{\pi}$ radians with the real axis.

Mapping of the Upper Half-Plane

- Consider again $w = f(z) = (z - x_1)^{\alpha/\pi}$ on the half-plane $y \geq 0$. This set consists of the point $z = x_1$ together with the set of rays $\arg(z - x_1) = \phi$, $0 \leq \phi \leq \pi$. The image under $w = (z - x_1)^{\alpha/\pi}$ consists of the point $w = 0$ together with the set of rays $\arg(w) = \frac{\alpha\phi}{\pi}$, $0 \leq \frac{\alpha\phi}{\pi} \leq \alpha$.





We conclude that the image of the half-plane $y \geq 0$ is the point $w = 0$ together with the wedge $0 \leq \arg(w) \leq \alpha$. The function f has derivative: $f'(z) = \frac{\alpha}{\pi}(z - x_1)^{(\alpha/\pi)-1}$. Since $f'(z) \neq 0$ if $z = x + iy$ and $y > 0$, it follows that $w = f(z)$ is a conformal mapping at any point z with $y > 0$.

Mapping f, with $f'(z) = A(z - x_1)^{(\alpha_1/\pi)-1}(z - x_2)^{(\alpha_2/\pi)-1}$

- Consider a new function f, analytic in $y > 0$ and whose derivative is:
$$f'(z) = A(z - x_1)^{(\alpha_1/\pi)-1}(z - x_2)^{(\alpha_2/\pi)-1},$$
where x_1, x_2, α_1 and α_2 are real, $x_1 < x_2$, and A is a complex constant.
- Note a parametrization $w(t)$, $a < t < b$, gives a line segment if and only if there is a constant value of $\arg(w'(t))$ for all $a < t < b$.
- We determine the images of the intervals $(-\infty, x_1), (x_1, x_2)$ and (x_2, ∞) on the real axis under $w = f(z)$.
  - If we parametrize $(-\infty, x_1)$ by $z(t) = t$, $-\infty < t < x_1$, then $w(t) = f(z(t)) = f(t)$, $-\infty < t < x_1$. Thus, $w'(t) = f'(t) = A(t - x_1)^{(\alpha_1/\pi)-1}(t - x_2)^{(\alpha_2/\pi)-1}$. An argument of $w'(t)$ is then given by: $\text{Arg}(A) + (\frac{\alpha_1}{\pi} - 1)\text{Arg}(t - x_1) + (\frac{\alpha_2}{\pi} - 1)\text{Arg}(t - x_2)$. Since $-\infty < t < x_1$, $t - x_1 < 0$, and, so $\text{Arg}(t - x_1) = \pi$. Since $x_1 < x_2$, $t - x_2 < 0$, whence $\text{Arg}(t - x_2) = \pi$. Hence, $\text{Arg}(A) + \alpha_1 + \alpha_2 - 2\pi$ is a constant value of $\arg(w'(t))$ for all t in $(-\infty, x_1)$. We conclude that the interval $(-\infty, x_1)$ is mapped onto a line segment by $w = f(z)$.

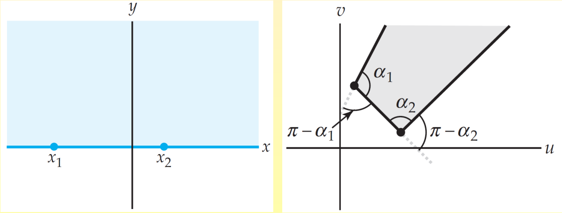
Mapping f (Cont'd)

- By similar reasoning we determine that both (x_1, x_2) and (x_2, ∞) also map onto line segments:

Interval	An Argument of w'	Change in Argument
$(-\infty, x_1)$	$\text{Arg}(A) + \alpha_1 + \alpha_2 - 2\pi$	0
(x_1, x_2)	$\text{Arg}(A) + \alpha_2 - \pi$	$\pi - \alpha_1$
(x_2, ∞)	$\text{Arg}(A)$	$\pi - \alpha_2$

Since f is an analytic (and, hence, continuous) mapping, the image of the half-plane $y \geq 0$ is an unbounded polygonal region.

The exterior angles between successive sides of the boundary are the changes in argument of w'. Thus, the interior angles of the polygon are α_1 and α_2.



Schwarz-Christoffel Formula

- The foregoing discussion can be generalized to produce a formula for the derivative f' of a function f that maps the half-plane $y \geq 0$ onto a polygonal region with any number of sides.

Theorem (Schwarz-Christoffel Formula)

Let f be a function that is analytic in the domain $y > 0$ and has the derivative

$$f'(z) = A(z - x_1)^{(\alpha_1/\pi)-1}(z - x_2)^{(\alpha_2/\pi)-1} \cdots (z - x_n)^{(\alpha_n/\pi)-1},$$

where $x_1 < x_2 < \cdots < x_n$, $0 < \alpha_i < 2\pi$, for $1 \leq i \leq n$, and A is a complex constant. Then the upper half-plane $y \geq 0$ is mapped by $w = f(z)$ onto an unbounded polygonal region with interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$.

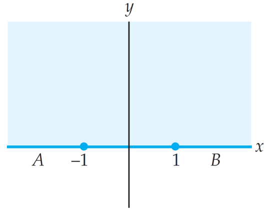
- By the Conformal Mapping Theorem, the function given by the Schwarz-Christoffel formula is a conformal mapping in $y > 0$.
- Even though the mapping from the upper half-plane onto a polygonal region is defined for $y \geq 0$, it is only conformal in $y > 0$.

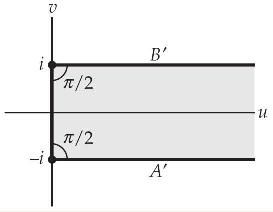
Remarks on the Schwarz-Christoffel Formula

- In practice we usually have some freedom in the selection of the points x_k on the x-axis. A judicious choice can simplify the computation of $f(z)$.
- The Schwarz-Christoffel Theorem provides a formula only for the derivative of f. A general formula for f is given by an integral $f(z) = A\int (z-x_1)^{(\alpha_1/\pi)-1}(z-x_2)^{(\alpha_2/\pi)-1}\cdots(z-x_n)^{(\alpha_n/\pi)-1}dz + B$, where A and B are complex constants. Thus, f is the composition of $g(z) = \int (z-x_1)^{(\alpha_1/\pi)-1}(z-x_2)^{(\alpha_2/\pi)-1}\cdots(z-x_n)^{(\alpha_n/\pi)-1}dz$ and the linear mapping $h(z) = Az + B$. The linear mapping h allows us to rotate, magnify (or contract), and translate the polygonal region produced by g.
- The Schwarz-Christoffel Formula can also be used to construct a mapping of the upper half-plane $y \geq 0$ onto a bounded polygonal region. To do so, we apply the formula using only $n-1$ of the n interior angles of the bounded polygonal region.

Using the Schwarz-Christoffel Formula I

- Use the formula to construct a conformal mapping from the upper half-plane onto the polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$. The polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$, is the semi-infinite strip:





The interior angles are $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, and the vertices are $w_1 = -i$ and $w_2 = i$. To find the desired mapping, we set $x_1 = -1$ and $x_2 = 1$. Then $f'(z) = A(z+1)^{-1/2}(z-1)^{-1/2}$. By the Theorem, $w = f(z)$ is a conformal mapping from the half-plane $y \geq 0$ onto the polygonal region $u \geq 0$, $-1 \leq v \leq 1$.

Using the Schwarz-Christoffel Formula I (Cont'd)

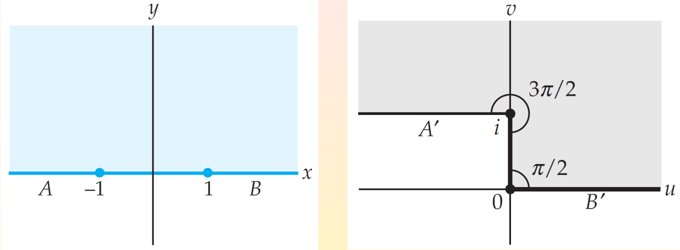
- For f(z), $f'(z) = A(z+1)^{-1/2}(z-1)^{-1/2}$ is integrated.

  Since z is in the upper half-plane $y \geq 0$, we first use the principal square root to write $f'(z) = \frac{A}{(z^2-1)^{1/2}}$. Since $(-1)^{1/2} = i$, we have $f'(z) = \frac{A}{(z^2-1)^{1/2}} = \frac{A}{[-1(1-z^2)]^{1/2}} = \frac{A}{i}\frac{1}{(1-z^2)^{1/2}} = -Ai\frac{1}{(1-z^2)^{1/2}}$. An antiderivative is given by $f(z) = -Ai\sin^{-1} z + B$, where $\sin^{-1} z$ is the single-valued function obtained by using the principal square root and principal value of the logarithm and where A and B are complex constants.

  If we choose $f(-1) = -i$ and $f(1) = i$, then the constants A and B must satisfy $\left\{\begin{array}{rcr} -Ai\sin^{-1}(-1) + B = Ai\frac{\pi}{2} + B & = & -i \\ -Ai\sin^{-1}(1) + B = -Ai\frac{\pi}{2} + B & = & i. \end{array}\right\}$. By adding these two equations we see that $2B = 0$, or, $B = 0$. By substituting $B = 0$ into either equation we obtain $A = -\frac{2}{\pi}$. Therefore, $f(z) = i\frac{2}{\pi}\sin^{-1} z$.

Using the Schwarz-Christoffel Formula II

- Use the formula to construct a conformal mapping from the upper half-plane onto the polygonal region shown:



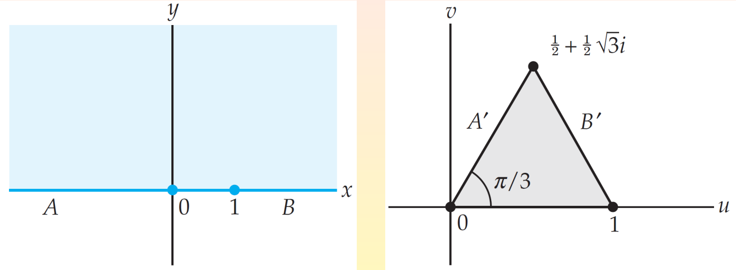
This is an unbounded polygonal region with interior angles $\alpha_1 = \frac{3\pi}{2}$ and $\alpha_2 = \frac{\pi}{2}$ at the vertices $w_1 = i$ and $w_2 = 0$, respectively. If we select $x_1 = -1$ and $x_2 = 1$ to map onto w_1 and w_2, respectively, then $f'(z) = A(z+1)^{1/2}(z-1)^{-1/2}$. Note $(z+1)^{1/2}(z-1)^{-1/2} = \left(\frac{z+1}{z-1}\right)^{1/2} = \frac{z+1}{(z^2-1)^{1/2}}$. Therefore, $f'(z) = A\left[\frac{z}{(z^2-1)^{1/2}} + \frac{1}{(z^2-1)^{1/2}}\right]$.

Using the Schwarz-Christoffel Formula II (Cont'd)

- An antiderivative of $f'(z) = A\left[\frac{z}{(z^2-1)^{1/2}} + \frac{1}{(z^2-1)^{1/2}}\right]$ is given by $f(z) = A[(z^2-1)^{1/2} + \cosh^{-1} z] + B,$ where A and B are complex constants, and where $(z^2-1)^{1/2}$ and $\cosh^{-1} z$ represent branches of the square root and inverse hyperbolic cosine functions defined on the domain $y > 0$. Because $f(-1) = i$ and $f(1) = 0$, the constants A and B must satisfy the system of equations
$$\left\{\begin{array}{rcl} A(0+\cosh^{-1}(-1)) + B = A\pi i + B & = & i \\ A(0+\cosh^{-1} 1) + B = B & = & 0 \end{array}\right\}.$$ Therefore,
$A = \frac{1}{\pi},\ B = 0,$ and the desired mapping is
$f(z) = \frac{1}{\pi}(z^2-1)^{1/2} + \cosh^{-1} z.$

Using the Schwarz-Christoffel Formula III

- Use the formula to construct a conformal mapping from the upper half-plane onto the polygonal region bounded by the equilateral triangle with vertices $w_1 = 0$, $w_2 = 1$, and $w_3 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$.



The region has interior angles $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}$. Since the region is bounded, we can find a desired mapping by using the formula with $n - 1 = 2$ of the interior angles. After selecting $x_1 = 0$ and $x_2 = 1$, $f'(z) = Az^{-2/3}(z-1)^{-2/3}$.

Using the Schwarz-Christoffel Formula III (Cont'd)

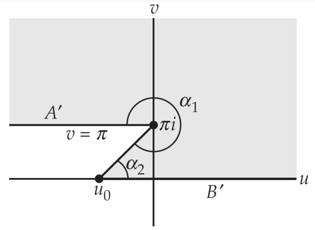
- There is no antiderivative of $f'(z) = Az^{-2/3}(z-1)^{-2/3}$ that can be expressed in terms of elementary functions. Since f' is analytic in the simply connected domain $y > 0$, we know that an antiderivative f does exist in this domain. It is given by the integral formula

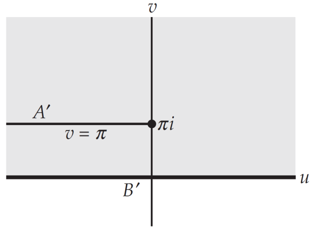
$$f(z) = A\int_0^z \frac{1}{s^{2/3}(s-1)^{2/3}}\,ds + B,$$

where A and B are complex constants. Requiring that $f(0) = 0$ allows us to solve for the constant B. We have $f(0) = A\int_0^0 \frac{1}{s^{2/3}(s-1)^{2/3}}\,ds + B = 0 + B = B$, and, so, $B = 0$. If we also require that $f(1) = 1$, then $f(1) = A\int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}}\,ds = 1$. If Γ denotes value of the integral $\Gamma = \int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}}\,ds$, then $A = \frac{1}{\Gamma}$.

Using the Schwarz-Christoffel Formula IV

- Use the formula to construct a conformal mapping from the upper half-plane onto the non-polygonal region defined by $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u \leq 0$, deleted.





Let u_0 be a point on the non-positive u-axis in the w-plane. We can approximate the non-polygonal region by a polygonal region: The vertices of this polygonal region are $w_1 = \pi i$ and $w_2 = u_0$, with corresponding interior angles α_1 and α_2. If we choose the points $z_1 = -1$ and $z_2 = 0$ to map onto the vertices $w_1 = \pi i$ and $w_2 = u_0$, respectively, then $f'(z) = A(z+1)^{(\alpha_1/\pi)-1} z^{(\alpha_2/\pi)-1}$.

Using the Schwarz-Christoffel Formula IV (Cont'd)

- As u_0 approaches $-\infty$ along the u-axis, the interior angle α_1 approaches 2π and the interior angle α_2 approaches 0. With these limiting values, $f'(z) = A(z+1)^{(\alpha_1/\pi)-1} z^{(\alpha_2/\pi)-1}$ suggests that our desired mapping f has derivative $f'(z) = A(z+1)^1 z^{-1} = A(1+\frac{1}{z})$. Thus, $f(z) = A(z + \mathrm{Ln} z) + B$, with A and B complex constants.
  - Consider $g(z) = z + \mathrm{Ln} z$ on the upper half-plane $y \geq 0$.
    - For the half-line $y = 0, -\infty < x < 0$, if $z = x + 0i$, then $\mathrm{Arg}(z) = \pi$, and so $g(z) = x + \log_e |x| + i\pi$. When $x < 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to -1. Thus, the image of the negative x-axis under g is the horizontal half-line $v = \pi$, $-\infty < u < -1$.
    - For the half-line $y = 0$, $0 < x < \infty$, if $z = x + 0i$, then $\mathrm{Arg}(z) = 0$, and so $g(z) = x + \log_e |x|$. When $x > 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to ∞. Therefore, the image of the positive x-axis under g is the u-axis.

  The image of the half-plane $y \geq 0$ under $g(z) = z + \mathrm{Ln} z$ is the region $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u < -1$ deleted.

  In order to obtain the region we want, we should compose g with a translation by 1. Hence, the desired mapping is $f(z) = z + \mathrm{Ln}(z) + 1$.