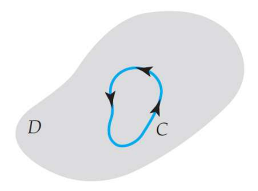
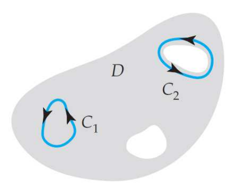
Cauchy-Goursat Theorem

Simply and Multiply Connected Domains

- A **domain** is an open connected set in the complex plane.
- A domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.





Example: The entire complex plane is a simply connected domain. The annulus defined by $1 < |z| < 2$ is not simply connected.

- A domain that is not simply connected is called a **multiply connected domain**.
  - A domain with one "hole" is **doubly connected**;
  - A domain with two "holes" **triply connected**, and so on.

Example: The open disk $|z| < 2$ is a simply connected domain. The open circular annulus $1 < |z| < 2$ is doubly connected.

Cauchy's Theorem

Cauchy's Theorem (1825)

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then, for every simple closed contour C in D,

$$\oint_C f(z)dz = 0.$$

- We apply Green's theorem and the Cauchy-Riemann equations. Recall from calculus that, if C is a positively oriented, piecewise smooth, simple closed curve forming the boundary of a region R within D, and if the real-valued functions $P(x,y)$ and $Q(x,y)$ along with their first-order partial derivatives are continuous on a domain that contains C and R, then $\oint_C Pdx + Qdy = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$. Since f' is continuous throughout D, the real and imaginary parts of $f(z) = u + iv$ and their first partial derivatives are continuous throughout D.

Proof of Cauchy's Theorem

- We have by Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

By continuity of u, v and their first partial derivatives, $\oint_C f(z)dz = \oint_C u(x,y)dx - v(x,y)dy + i\oint_C v(x,y)dx + u(x,y)dy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dA + i\iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dA$. f being analytic in D, u and v satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore,

$$\begin{aligned} \oint_C f(z)dz &= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right)dA + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)dA \\ &= 0. \end{aligned}$$

The Cauchy-Goursat Theorem

- Edouard Goursat proved in 1883 that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem:

Cauchy-Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then, for every simple closed contour C in D,

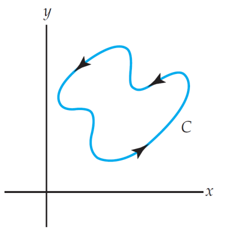
$$\oint_C f(z)dz = 0.$$

- Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can also be stated as:

  If f is analytic at all points within and on a simple closed contour C, then $\oint_C f(z)dz = 0$.

Applying the Cauchy-Goursat Theorem I

- Evaluate $\oint_C e^z dz$, where the contour C is shown below.



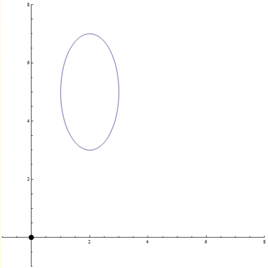
$f(z) = e^z$ is entire. Thus, it is analytic at all points within and on the simple closed contour C. It follows from the Cauchy-Goursat theorem that $\oint_C e^z dz = 0$.

- We have $\oint_C e^z dz = 0$, for any simple closed contour in the complex plane.
- Moreover, for any simple closed contour C and any entire function f, such as $f(z) = \sin z$, $f(z) = \cos z$, and $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n = 0, 1, 2, \ldots$, we also have

$$\oint_C \sin z dz = 0, \ \oint_C \cos z dz = 0, \ \oint_C p(z) dz = 0, \ \text{etc.}$$

Applying the Cauchy-Goursat Theorem II

- Evaluate $\oint_C \frac{1}{z^2} dz$, where C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$. The rational function $f(z) = \frac{1}{z^2}$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the simple closed elliptical contour C.

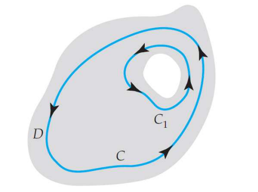


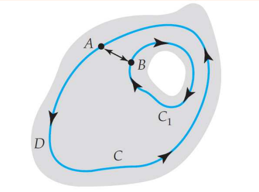
Thus, again by the Cauchy-Goursat Theorem, we get

$$\oint_C \frac{1}{z^2} dz = 0.$$

Cauchy-Goursat Theorem for Multiply Connected Domains

- If f is analytic in a multiply connected domain D, then we cannot conclude that $\oint_C f(z)dz = 0$, for every simple closed contour C in D.
- Suppose that D is a doubly connected domain and C and C_1 are simple closed contours placed as follows:





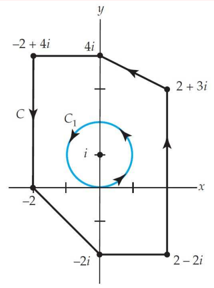
Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1.

By introducing the crosscut AB, the region bounded between the curves is now simply connected. So: $\oint_C f(z)dz + \int_{AB} f(z)dz + \oint_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$ or $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

- This is sometimes called the **principle of deformation of contours**.
- It allows evaluation of an integral over a complicated simple closed contour C by replacing C with a more convenient contour C_1.

Applying Deformation of Contours

- Evaluate $\oint_C \frac{1}{z-i} dz$, where C is the black contour:



We choose the more convenient circular contour C_1 drawn in blue. By taking the radius of the circle to be $r = 1$, we are guaranteed that C_1 lies within C. C_1 is the circle $|z - i| = 1$.
It can be parametrized by

$$z = i + e^{it}, \ 0 \le t \le 2\pi.$$

From $z - i = e^{it}$ and $dz = ie^{it} dt$, we get:

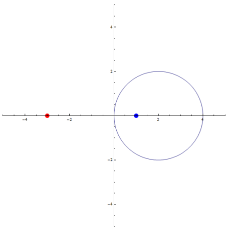
$$\begin{aligned} \oint_C \frac{1}{z-i} dz &= \oint_{C_1} \frac{1}{z-i} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt \\ &= i \int_0^{2\pi} dt = 2\pi i. \end{aligned}$$

A Generalization

- This result can be generalized: If z_0 is any constant complex number interior to any simple closed contour C, and n an integer, we have
$$\oint_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1 \end{cases}.$$
- That the integral is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem.
  - When $n = 0$ or negative, $\frac{1}{(z-z_0)^n}$ is a polynomial and therefore entire. Then, clearly, $\oint_C \frac{1}{(z-z_0)^n} dz = 0$.
  - It is not very difficult to see that the integral is still zero when n is a positive integer different from 1.
- Analyticity of the function f at all points within and on a simple closed contour C is sufficient to guarantee that $\oint_C f(z) dz = 0$.
- This result emphasizes that analyticity is not necessary, i.e., it can happen that $\oint_C f(z) dz = 0$ without f being analytic within C.
Example: If C is the circle $|z| = 1$, then $\oint_C \frac{1}{z^2} dz = 0$, but $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$ within C.

Applying the Formula for the Integral of $1/(z-z_0)^n$

- Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is circle $|z-2|=2$.
  The denominator factors as $z^2+2z-3=(z-1)(z+3)$. Thus, the integrand fails to be analytic at $z=1$ and $z=-3$.



Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. By partial fractions

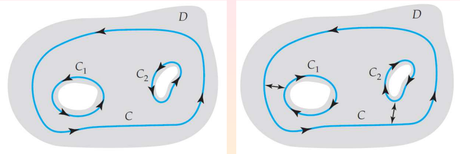
$$\frac{5z+7}{z^2+2z-3}=\frac{3}{z-1}+\frac{2}{z+3}.$$

Hence, $\oint_C \frac{5z+7}{z^2+2z-3}dz=3\oint_C \frac{1}{z-1}dz+2\oint_C \frac{1}{z+3}dz$. The first integral has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem. Hence,

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=3(2\pi i)+2(0)=6\pi i.$$

Cauchy-Goursat Theorem: Multiply Connnected Domains

- If C, C_1, and C_2 are simple closed contours as shown below



and f is analytic on each of the three contours as well as at each point interior to C but exterior to both C_1 and C_2, then by introducing crosscuts between C_1 and C and between C_2 and C, we get $\oint_C f(z)dz + \oint_{-C_1} f(z)dz + \oint_{-C_2} f(z)dz = 0$, whence $\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$.

Cauchy-Goursat Theorem for Multiply Connnected Domains

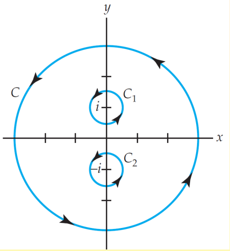
Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation, such that $C_1, C_2, \ldots, C_n$ are interior to C, but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then $\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$.

Integrals in Multiply Connected Domains

- Evaluate $\oint_C \frac{1}{z^2+1} dz$, where C is the circle $|z| = 4$.

  The denominator of the integrand factors as $z^2 + 1 = (z - i)(z + i)$. So, the integrand $\frac{1}{z^2+1}$ is not analytic at $z = i$ and at $z = -i$. Both points lie within C. Using partial fractions, $\frac{1}{z^2+1} = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}$. whence $\oint_C \frac{1}{z^2+1} dz = \frac{1}{2i} \oint_C (\frac{1}{z-i} - \frac{1}{z+i}) dz$.

  Surround $z = i$ and $z = -i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. The choice $|z - i| = \frac{1}{2}$ for C_1 and $|z + i| = \frac{1}{2}$ for C_2 will suffice.



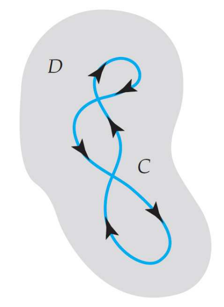
  We have $\oint_C \frac{1}{z^2+1} dz =$

  $\frac{1}{2i} \oint_{C_1} (\frac{1}{z-i} - \frac{1}{z+i}) dz + \frac{1}{2i} \oint_{C_2} (\frac{1}{z-i} - \frac{1}{z+i}) dz = \frac{1}{2i} \oint_{C_1} \frac{1}{z-i} dz - \frac{1}{2i} \oint_{C_1} \frac{1}{z+i} dz + \frac{1}{2i} \oint_{C_2} \frac{1}{z-i} dz - \frac{1}{2i} \oint_{C_2} \frac{1}{z+i} dz = \frac{1}{2i} 2\pi i - 0 + 0 - \frac{1}{2i} 2\pi i = 0.$

Non-Simple Closed Contours

- Throughout the foregoing discussion we assumed that C was a simple closed contour, in other words, C did not intersect itself.
- It can be shown that the Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D.
- For a contour C that is closed but not simple, if f is analytic in D, then

$$\oint_C f(z)dz = 0.$$



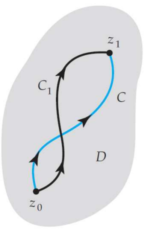
Independence of Path

Path Independence

Definition (Independence of the Path)

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be **independent of the path** if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.

- The Cauchy-Goursat theorem holds for closed contours, not just simple closed contours, in a simply connected domain D.
- Suppose that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1. C joined with $-C_1$ forms a closed contour. Thus, if f is analytic in D, $\int_C f(z)dz + \int_{-C_1} f(z)dz = 0$. Therefore, $\int_C f(z)dz = \int_{C_1} f(z)dz$.

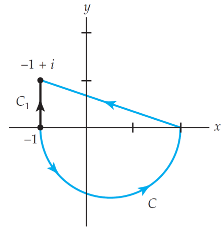


Theorem (Analyticity Implies Path Independence)

Suppose that a function f is analytic in a simply connected domain D and C is any contour in D. Then $\int_C f(z)dz$ is independent of the path C.

Choosing a Different Path

- Evaluate $\int_C 2z dz$, where C is the contour shown in blue.



The function $f(z) = 2z$ is entire. By the theorem, we can replace the piecewise smooth path C by any convenient contour C_1 joining $z_0 = -1$ and $z_1 = -1 + i$. We choose the contour C_1 to be the vertical line segment $x = -1, 0 \leq y \leq 1$.

Since $z = -1 + iy$, $dz = idy$. Therefore,

$$\begin{aligned} \int_C 2z dz &= \int_{C_1} 2z dz \\ &= \int_0^1 2(-1 + iy) i dy \\ &= \int_0^1 (-2i - 2y) dy \\ &= (-2iy - y^2)\Big|_0^1 \\ &= -1 - 2i. \end{aligned}$$

Antiderivatives

- A contour integral $\int_C f(z)dz$ that is independent of the path C is usually written $\int_{z_0}^{z_1} f(z)dz$, where z_0 and z_1 are the initial and terminal points of C.

Definition (Antiderivative)

Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$, for each z in D, then F is called an **antiderivative** of f.

Example: The function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$ since $F'(z) = \sin z$.

- The most general antiderivative, or **indefinite integral**, of a function $f(z)$ is written $\int f(z)dz = F(z) + C$, where $F'(z) = f(z)$ and C is some complex constant.
- Differentiability implies continuity, whence, since an antiderivative F of a function f has a derivative at each point in a domain D, it is necessarily analytic and hence continuous at each point in D.

Fundamental Theorem for Contour Integrals

Fundamental Theorem for Contour Integrals

Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then, for any contour C in D with initial point z_0 and terminal point z_1,

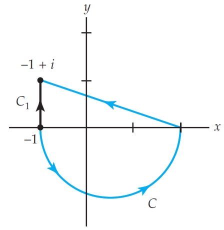
$$\int_C f(z)dz = F(z_1) - F(z_0).$$

- We prove the FTCI in the case when C is a smooth curve parametrized by $z = z(t)$, $a \leq t \leq b$. The initial and terminal points on C are $z(a) = z_0$ and $z(b) = z_1$. Since $F'(z) = f(z)$, for all z in D,

$$\begin{array}{rcl} \int_C f(z)dz & = & \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt \\ & = & \int_a^b \frac{d}{dt}F(z(t))dt = F(z(t))|_a^b \\ & = & F(z(b)) - F(z(a)) \\ & = & F(z_1) - F(z_0). \end{array}$$

Applying the Fundamental Theorem I

- The integral $\int_C 2zdz$, where C is shown



is independent of the path. Since $f(z) = 2z$ is an entire function, it is continuous. Moreover, $F(z) = z^2$ is an antiderivative of f since $F'(z) = 2z = f(z)$. Hence, by the Fundamental Theorem, we have

$$\begin{aligned} \int_{-1}^{-1+i} 2zdz &= z^2\Big|_{-1}^{-1+i} \\ &= (-1+i)^2 - (-1)^2 \\ &= -1-2i. \end{aligned}$$

Applying the Fundamental Theorem II

- Evaluate $\int_C \cos z dz$, where C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.

  $F(z) = \sin z$ is an antiderivative of $f(z) = \cos z$, since $F'(z) = \cos z = f(z)$. Therefore, by the Fundamental Theorem, we have

$$\begin{aligned} \int_C \cos z dz &= \int_0^{2+i} \cos z dz \\ &= \sin z\big|_0^{2+i} \\ &= \sin(2+i) - \sin 0 \\ &= \sin(2+i). \end{aligned}$$

Some Conclusions

- Observe that if the contour C is closed, then $z_0 = z_1$ and, consequently, $\oint_C f(z)dz = F(z_1) - F(z_0) = 0$.
- Since the value of $\int_C f(z)dz$ depends only on the points z_0 and z_1, this value is the same for any contour C in D connecting these points:

  If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.

- Moreover, we have a sufficient condition:

  If f is continuous and $\int_C f(z)dz$ is independent of the path C in a domain D, then f has an antiderivative everywhere in D.

- Assume f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D and that F is a function defined by $F(z) = \int_{z_0}^z f(s)ds$, where s denotes a complex variable, z_0 is a fixed point in D, and z represents any point in D. We wish to show that $F'(z) = f(z)$, i.e., that $F(z) = \int_{z_0}^z f(s)ds$ is an antiderivative of f in D.

$F(z) = \int_{z_0}^{z} f(s)ds$ is an Antiderivative of f in D

- We have
  $F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds = \int_{z}^{z+\Delta z} f(s)ds.$
  Because D is a domain, we can choose Δz so that $z + \Delta z$ is in D. Moreover, z and $z + \Delta z$ can be joined by a straight segment. With z fixed, we can write $f(z)\Delta z = f(z) \int_{z}^{z+\Delta z} ds = \int_{z}^{z+\Delta z} f(z)ds$ or
  $f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)ds$. Therefore, we have
  $\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds$. Since f is continuous at the point z, for any $\varepsilon > 0$, there exists a $\delta > 0$, so that $|f(s) - f(z)| < \epsilon$ whenever $|s - z| < \delta$. Consequently, if we choose Δz so that $|\Delta z| < \delta$, it follows from the ML-inequality, that
  $\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \left|\frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds\right| =$
  $\left|\frac{1}{\Delta z}\right| \left|\int_{z}^{z+\Delta z} [f(s) - f(z)]ds\right| \leq \left|\frac{1}{\Delta z}\right| \varepsilon |\Delta z| = \varepsilon$. Hence,
  $\lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$ or $F'(z) = f(z)$.

Existence of Antiderivative

- If f is an analytic function in a simply connected domain D, it is continuous throughout D. This implies, by the Path Independence Theorem, that path independence holds for f in D. Therefore,

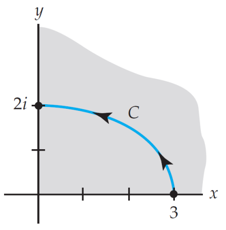
Theorem (Existence of Antiderivative)

Suppose that a function f is analytic in a simply connected domain D. Then f has an antiderivative in D, i.e., there exists a function F such that $F'(z) = f(z)$, for all z in D.

- We have seen that, for $|z| > 0$, $-\pi < \arg(z) < \pi$, $\frac{1}{z}$ is the derivative of $\text{Ln}z$. Thus, under some circumstances $\text{Ln}z$ is an antiderivative of $\frac{1}{z}$, but one must be careful!
  If D is the entire complex plane without the origin, $\frac{1}{z}$ is analytic in this multiply connected domain. If C is any simple closed contour containing the origin, it does not follow that $\oint_C \frac{1}{z} dz = 0$. In this case, $\text{Ln}z$ is not an antiderivative of $\frac{1}{z}$ in D since $\text{Ln}z$ is not analytic in D ($\text{Ln}z$ fails to be analytic on the non-positive real axis).

Using the Logarithmic Function

- Evaluate $\int_C \frac{1}{z} dz$, where C is the contour shown:



Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$, i.e., the first quadrant. In this case, $\text{Ln}z$ is an antiderivative of $\frac{1}{z}$ since both these functions are analytic in D.

Therefore,

$$\int_C \frac{1}{z} dz = \int_3^{2i} \frac{1}{z} dz = \text{Ln}z\big|_3^{2i} = \text{Ln}(2i) - \text{Ln}3.$$

Recall $\text{Ln}(2i) = \log_e 2 + \frac{\pi}{2} i$ and $\text{Ln}3 = \log_e 3$. Hence,
$\int_C \frac{1}{z} dz = \log_e 2 + \frac{\pi}{2} i - \log_e 3 = \log_e \frac{2}{3} + \frac{\pi}{2} i.$

Using an Antiderivative of $z^{-1/2}$

- Evaluate $\int_C \frac{1}{z^{1/2}}dz$, where C is the line segment between $z_0 = i$ and $z_1 = 9$.

  We take $f_1(z) = z^{1/2}$ to be the principal branch of the square root function. In the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the function $\frac{1}{f_1(z)} = \frac{1}{z^{1/2}} = z^{-1/2}$ is analytic and possesses the antiderivative $F(z) = 2z^{1/2}$. Hence,

$$\begin{aligned} \int_C \frac{1}{z^{1/2}}dz &= \int_i^9 \frac{1}{z^{1/2}}dz \\ &= 2z^{1/2}\Big|_i^9 \\ &= 2[3 - (\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2})] \\ &= (6 - \sqrt{2}) - i\sqrt{2}. \end{aligned}$$

Integration-By-Parts

- In calculus indefinite integrals of certain kinds can be evaluated by integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

More compactly, $\int udv = uv - \int vdu$.

- Suppose f and g are analytic in a simply connected domain D. Then

$$\int f(z)g'(z)dz = f(z)g(z) - \int g(z)f'(z)dz.$$

- In addition, if z_0 and z_1 are the initial and terminal points of a contour C lying entirely in D, then

$$\int_{z_0}^{z_1} f(z)g'(z)dz = f(z)g(z)\Big|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz.$$

The Mean Value Theorem for Definite Integrals

- The **Mean Value Theorem for Definite Integrals**: If f is a real function continuous on the closed interval $[a, b]$, then there exists a number c in the open interval (a, b), such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

- Let f be a complex function analytic in a simply connected domain D. Then, f is continuous at every point on a contour C in D with initial point z_0 and terminal point z_1.

  Unfortunately, no analog of the Mean Value Theorem exists for the contour integral $\int_{z_0}^{z_1} f(z)dz$.

Cauchy's Integral Formulas

Cauchy's First Formula

- If f is analytic in a simply connected domain D and z_0 is a point in D, the quotient $\frac{f(z)}{z-z_0}$ is not defined at z_0 and, hence, is not analytic in D.
- Therefore, we cannot conclude that the integral of $\frac{f(z)}{z-z_0}$ around a simple closed contour C that contains z_0 is zero.
- Indeed,the integral of $\frac{f(z)}{z-z_0}$ around C has the value $2\pi i f(z_0)$.

Theorem (Cauchy's Integral Formula)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

- Let D be a simply connected domain, C a simple closed contour in D, and z_0 an interior point of C. In addition, let C_1 be a circle centered at z_0 with radius small enough so that C_1 lies within the interior of C. By the principle of deformation of contours, $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$.

Proof of Cauchy's Integral Formula

- From $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$, we get by adding and subtracting $f(z_0)$ in the numerator: $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z_0)-f(z_0)+f(z)}{z-z_0} dz = f(z_0) \oint_{C_1} \frac{1}{z-z_0} dz + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz$. We know that $\oint_{C_1} \frac{1}{z-z_0} dz = 2\pi i$, whence $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz$.
  Since f is continuous at z_0, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(z) - f(z_0)| < \varepsilon$, whenever $|z - z_0| < \delta$. In particular, if we choose C_1 to be $|z - z_0| = \frac{1}{2}\delta < \delta$, then by the *ML*-inequality, $\left|\oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz\right| \leq \frac{\varepsilon}{\delta/2} 2\pi \frac{\delta}{2} = 2\pi\varepsilon$. Thus, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle C_1 to be sufficiently small. This implies that the integral is 0. We conclude that $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$.

Using Cauchy's Integral Formula

- Cauchy's integral formula shows that the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.
- Since we often work problems without a simply connected domain explicitly defined, a more practical restatement is:

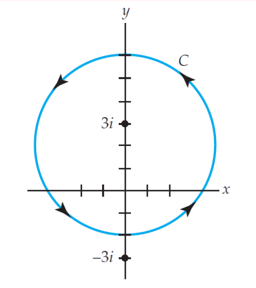
> If f is analytic at all points within and on a simple closed contour C, and z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$.

- Example: Evaluate $\oint_C \frac{z^2-4z+4}{z+i} dz$, where C is the circle $|z| = 2$.

  We identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula,
  $\oint_C \frac{z^2-4z+4}{z+i} dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = \pi(-8 + 6i)$.

Another Application of Cauchys Integral Formula

- Evaluate $\oint_C \frac{z}{z^2+9} dz$, where C is the circle $|z-2i| = 4$.



By factoring the denominator as $z^2+9 = (z-3i)(z+3i)$, we see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. By rewriting the integrand as $\dfrac{z}{z^2+9} = \dfrac{\frac{z}{z+3i}}{z-3i}$, we identify $f(z) = \frac{z}{z+3i}$

The function f is analytic at all points within and on the contour C. Hence, by Cauchy's integral formula

$$\oint_C \frac{z}{z^2+9} dz = \oint_C \frac{\frac{z}{z+3i}}{z-3i} dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i.$$

Cauchy's Second Formula

- We prove that the values of the derivatives $f^{(n)}(z_0)$, $n = 1, 2, 3, \ldots$ of an analytic function are also given by an integral formula.

Theorem (Cauchy's Integral Formula for Derivatives)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

- Partial Proof (for $n = 1$): By the definition of the derivative and Cauchy's Integral Formula, $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} =$
$$\lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz - \oint_C \frac{f(z)}{z - z_0} dz \right] =$$
$$\lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz.$$

Prof of Cauchy's Second Formula for $n = 1$

- We work out some preliminaries:
  - Continuity of f on the contour C guarantees that f is bounded, i.e., there exists real number M, such that $|f(z)| \leq M$, for all points z on C.
  - In addition, let L be the length of C and let δ denote the shortest distance between points on C and the point z_0. Thus, for all points z on C, we have $|z - z_0| \geq \delta$, or $\frac{1}{|z-z_0|^2} \leq \frac{1}{\delta^2}$.
  - Furthermore, if we choose $|\Delta z| \leq \frac{1}{2}\delta$, then $|z - z_0 - \Delta z| \geq ||z - z_0| - |\Delta z|| \geq \delta - |\Delta z| \geq \frac{1}{2}\delta$, whence $\frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{\delta}$.

Now, $\left|\oint_C \frac{f(z)}{(z-z_0)^2}dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz\right| = \left|\oint_C \frac{-\Delta z\ f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz\right| \leq \frac{2ML|\Delta z|}{\delta^3}$. The last expression approaches zero as $\Delta z \to 0$, whence

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz.$$

Using Cauchy's Integral Formula for Derivatives

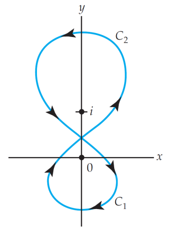
- Evaluate $\oint_C \frac{z+1}{z^4+2iz^3} dz$, where C is the circle $|z|=1$.

  Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integrand as $\dfrac{z+1}{z^4+2iz^3} = \dfrac{\frac{z+1}{z+2i}}{z^3}$ we can identify, $z_0=0,\ n=2$, and $f(z) = \frac{z+1}{z+2i}$. The quotient rule gives $f'(z) = \frac{-1+2i}{(z+2i)^2}$ and $f''(z) = \frac{2-4i}{(z+2i)^3}$, whence $f''(0) = \frac{2i-1}{4i}$. Therefore, we get

$$
\begin{aligned}
\oint_C \frac{z+1}{z^4+4z^3} dz &= \frac{2\pi i}{2!} f''(0) \\
&= \frac{2\pi i}{2!} \frac{2i-1}{4i} \\
&= -\frac{\pi}{4} + \frac{\pi}{2} i.
\end{aligned}
$$

Another Application of the Integral Formula for Derivatives

- Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}dz$, where C is the figure-eight contour shown below:



Although C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2. We write $\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz = -\oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = -I_1 + I_2$.

  - $I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz = 2\pi i f(0) = 2\pi i(-3) = -6\pi i$.
  - For I_2, $f(z) = \frac{z^3+3}{z}$, whence $f'(z) = \frac{2z^3-3}{z^2}$, and $f'(i) = 3+2i$. Thus, $I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = \frac{2\pi i}{1!}f'(i) = 2\pi i(3+2i) = -4\pi + 6\pi i$.

Finally, $\oint_C \frac{z^3+3}{z(z-i)^2}dz = -I_1 + I_2 = 6\pi i + (-4\pi + 6\pi i) = -4\pi + 12\pi i$.

Consequences of the Integral Formulas

The Derivatives of an Analytic Function are Analytic

Theorem (Derivative of an Analytic Function Is Analytic)

Suppose that f is analytic in a simply connected domain D. Then f possesses derivatives of all orders at every point z in D. The derivatives $f', f'', f''', \ldots$ are analytic functions in D.

- If $f(z) = u(x, y) + iv(x, y)$ is analytic in a simply connected domain D, its derivatives of all orders exist at any point z in D. Thus, f', f'', f''', $\ldots$ are continuous. From

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \\ f''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \\ &\vdots \end{aligned}$$

we can also conclude that the real functions u and v have continuous partial derivatives of all orders at a point of analyticity.

Cauchy's Inequality

Theorem (Cauchy's Inequality)

Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \leq M$, for all points z on C, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

- From the hypothesis, $\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{M}{r^{n+1}}$. Thus, by Cauchy's Formula for Derivatives and the ML-inequality,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}.$$

- The number M depends on the circle $|z - z_0| = r$. But, if $n = 0$, then $M \geq |f(z_0)|$, for any circle C centered at z_0, as long as C lies within D. Thus, an upper bound M of $|f(z)|$ on C cannot be smaller than $|f(z_0)|$.

Liouville's Theorem

- Although the next result is known as "Liouville's Theorem", it was probably first proved by Cauchy.
- The gist of the theorem is that an entire function f, one that is analytic for all z, cannot be bounded unless f itself is a constant:

Theorem (Liouville's Theorem)

The only bounded entire functions are constants.

- Suppose f is an entire bounded function, i.e., $|f(z)| \leq M$, for all z. Then, for any point z_0, by Cauchy's Inequality, $|f'(z_0)| \leq \frac{M}{r}$. By making r arbitrarily large we can make $|f'(z_0)|$ as small as we wish. This means $f'(z_0) = 0$, for all points z_0 in the complex plane. Hence, by a preceding theorem, f must be a constant.

Fundamental Theorem of Algebra

- Liouville's Theorem enables us to establish the celebrated

Fundamental Theorem of Algebra

If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.

- Suppose that the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n > 0$, is not 0 for any complex number z. This implies that the reciprocal of p, $f(z) = \frac{1}{p(z)}$, is an entire function. Now

$$\begin{aligned} |f(z)| &= \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|} \\ &= \frac{1}{|z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|}. \end{aligned}$$

Thus, $|f(z)| \to 0$ as $|z| \to \infty$. So the function f must be bounded for finite z. By Liouville's Theorem, f is a constant. Hence, p is a constant. But this contradicts p not being a constant polynomial. Therefore, there must exist at least one z for which $p(z) = 0$.

Morera's Theorem

- Morera's theorem, which gives a sufficient condition for analyticity, is often taken to be the converse of the Cauchy-Goursat Theorem:

Theorem (Morera's Theorem)

If f is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$, for every closed contour C in D, then f is analytic in D.

- By the hypotheses of continuity of f and $\oint_C f(z)dz = 0$, for every closed contour C in D, we conclude that $\int_C f(z)dz$ is independent of the path. Then, the function F, defined by $F(z) = \int_{z_0}^{z} f(s)ds$ (where s denotes a complex variable, z_0 is a fixed point in D, and z any point in D) is an antiderivative of f, i.e., $F'(z) = f(z)$. Hence, F is analytic in D. In addition, $F'(z)$ is analytic in view of the analyticity of the derivative of any analytic function. Since $f(z) = F'(z)$, we see that f is analytic in D.