Conditional Probability and General Multiplication Rule

Objectives:

- Identify Independent and dependent events
- Find Probability of independent events
- Find Probability of dependent events
- Find Conditional probability
- General multiplication rule

Consider the following two problems:

(1) Select 2 cards from a standard deck of 52 cards with replacement. What is the probability of obtaining two kings?
(2) Select 2 cards from a standard deck of 52 cards without replacement. What is the probability of obtaining two kings?

Both problems seem similar as we are trying to find the probability of selecting two cards. But there is a difference in these two problems. In the first problem, we are selecting two cards ***with replacement*** and in the second problem we are selecting two cards ***without replacement***. In the first problem, probability of obtaining a king on the first selection does not affect the probability of obtaining a king on the second selection. While in the second problem, once you obtain a king on the first selection, you only have 51 cards to select your second king from. So the probability of obtaining a king on the second selection is not same as selecting a king.

Two events in the first problem are called independent events. Two events in the second problem are called dependent events.

*** Independent Events:** Two events are ***independent*** if the occurrence of one of the events does not affect the probability of the occurrence of the other event.

*** Dependent Events:** Two events are ***dependent*** if the occurrence of one of the events affects the probability of the occurrence of the other event. In other words, events that are not independent are dependent.

If two events A and B are independent events, then probability of event A and B is given by the following rule:

$$\mathbf{P}(A \textbf{ and } B) = \mathbf{P}(A) * \mathbf{P}(B).$$

EXAMPLE 1: Select two cards from the standard deck of 52 cards *with replacement*. Find the probability of selecting two kings.

Solution: Let A be the event that first card selected is king and B be the event that second card selected is king.

Then P(A) = $\frac{4}{52}$ as there are 4 kings in a deck. Now we replace the first card back into the deck.

So now P(B) = $\frac{4}{52}$ as well. Hence, the probability of selecting two kings *with replacement* is

$$P(A \text{ and } B) = P(A) * P(B) = \frac{4}{52} * \frac{4}{52} = \frac{1}{13} * \frac{1}{13} = \frac{1}{169} \approx 0.0059.$$

EXAMPLE 2: Suppose you toss a coin and then roll a dice. What is the probability of obtaining a tail and then rolling a 5.

Solution: Let A be the event that tail appears when you toss a coin and B be the event that 5 appears when you roll a dice.

Then P(A) = $\frac{1}{2}$ and P(B) = $\frac{1}{6}$. Hence,

$$P(A \text{ and } B) = P(A) * P(B) = \frac{1}{2} * \frac{1}{6} = \frac{1}{12} \approx 0.0833.$$

Now let's go back to the second problem we discussed at the beginning. We know that the two events are dependent in the second problem. So how do we find the probability in that problem? Let's think about it. Let A be the event that the first card selected is king and B be the event that the second card selected is a king. Then we know that probability of selecting a king from a standard deck is $\frac{1}{13}$. Since we do not put the first king we selected back in the deck, the probability of selecting a second king is affected by the first event A as now we only have 51 cards left in the deck of which only 3 are kings. So now we need another rule to find this probability.

General Multiplication Rule

For any two events A and,

$$\mathbf{P(A \text{ and } B) = P(A) * P(B \mid A)}$$

$$\textbf{or } \mathbf{P(A \text{ and } B) = P(B) * P(A \mid B)}$$

where P(B | A) and P(A | B) are the conditional probabilities.

* **<u>Conditional Probability:</u>** A conditional probability is the probability of an event occurring, given that another event has already occurred. The conditional probability of event B occurring, given that event A has already occurred, is denoted by P(B | A) and is read as "probability of B, given A." Note, from the general multiplication rule, we have the following conditional probability formula.

Conditional Probability (when P(B) ≠ 0)

$$\mathbf{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \frac{\mathbf{P}(\boldsymbol{A}\ \mathbf{and}\ \boldsymbol{B})}{\mathbf{P}(\boldsymbol{B})}$$

EXAMPLE 3: Select two cards from the standard deck of 52 cards *without replacement*. Find the probability of selecting two kings.

Solution: Let A be the event that first card selected is king and B be the event that second card selected is king.

Then $P(A) = \frac{4}{52}$ as there are 4 kings in a deck. Now we do not replace the first card back into the deck. So now selecting a second king, given that the first card selected is king is $P(B \mid A) = \frac{3}{51}$ as now there are 3 kings left among 51 cards. Hence, the probability of selecting two kings *without replacement* is

$$P(A \text{ and } B) = P(A) * P(B \mid A) = \frac{4}{52} * \frac{3}{51} = \frac{1}{13} * \frac{1}{17} = \frac{1}{221} \approx 0.0045.$$

EXAMPLE 4: A committee consists of four women and three men. The committee will randomly select two people to attend a conference in Hawaii. Find the probability that both are women.

Solution: Let A be the event that first person selected is woman and B be the event that second person selected is woman.

Then $P(A) = P(B) = \frac{4}{7}$ as there are 4 women in the committee of 7 people. Now we selected a woman as the first person to attend the conference, we cannot select her as a second person to attend the conference. So now there are 6 people left to select from and only 3 of them are women. So to find the probability of selecting both women is

$$P(A \text{ and } B) = P(A) * P(B \mid A) = \frac{4}{7} * \frac{3}{6} = \frac{4}{7} * \frac{1}{2} = \frac{2}{7} \approx 0.2857.$$

Exercises 3.2

(1) **Flipping a Coin:** What is the probability of obtaining five heads in a row when flipping a coin?

(2) **Traffic Fatalities:** The following data represent the number of traffic fatalities in the United States in 2005 by person type for male and female drivers.

Person Type	**Male**	**Female**	**Total**
Driver	20,795	6,598	**27,393**
Passenger	5,190	4,896	**10,086**
Total	**25,985**	**11,494**	**37,479**

(a) What is the probability that a randomly selected traffic fatality who was female was a passenger?
(b) What is the probability that a randomly selected passenger fatality was female?
(c) Suppose you are a police officer called to the scene of a traffic accident with a fatality. The dispatcher states that the victim was driving, but the gender is not known. Is the victim more likely to be male or female? Why?

(3) **Income by Region:** According to the U.S. Census Bureau, 19.1% of U.S. households are in the Northeast. In addition, 4.4% of U.S. households earn $75,000 per year or more and are located in the Northeast. Determine the probability that a randomly selected U.S. household earns more than $75,000 per year, given that the household is located in the Northeast.

(4) **Drawing a Card:** Suppose that a single card is selected from a standard 52-card deck. What is the probability that the card drawn is a king? Now suppose that a single card is drawn from a standard 52-card deck, but we are told that the card drawn is a king? Did the knowledge that the card is a heart change the probability that the card was a king? What term is used to describe this result?

(5) **Cigar Smoking:** The data in the following table show the results of a national study of 137,243 U.S. men that investigated the association between cigar smoking and death from cancer. **Note:** "Current cigar smoker" means cigar smoker at time of death.

	Died from Cancer	**Did Not Die from Cancer**
Never Smoked Cigars	782	120,747
Former Cigar Smoker	91	7,757
Current Cigar Smoker	141	7,725

(a) What is the probability that a randomly selected individual from the study who died from cancer was a former cigar smoker?
(b) What is the probability that a randomly selected individual from the study who was a former cigar smoker died from cancer?

(6) **High School Dropouts:** According to the U.S. Census Bureau, 8.4% of high school dropouts are 16- to 17-year-olds. In addition, 6.2% of high school dropouts are blonde 16- to 17-year-olds. What is the probability that a randomly selected dropout is blonde, given that he or she is 16 to 17 years old?

(7) **Marital Status:** The following data, in thousands, represent the marital status of Americans 25 years old or older and their levels of education in 2006.

	Did not Graduate from High School	**High School Graduate**	**Some College**	**College Graduate**	**Total**
Never Married	4,803	9,575	5,593	11,632	**31,603**
Married, spouse present	13,880	35,627	19,201	46,965	**115,673**
Married, spouse absent	1,049	1,049	503	944	**3,545**
Separated	1,162	1,636	790	1,060	**4,648**
Widowed	4,070	5,278	1,819	2,725	**13,892**
Divorced	2,927	7,725	4,639	7,232	**22,523**
Total	**27,891**	**60,890**	**32,545**	**70,558**	**191,884**

(a) What is the probability that a randomly selected individual who has never married is a high school graduate?
(b) What is the probability that a randomly selected individual who is a high school graduate has never married?

(8) **Defense System:** Suppose that a satellite defense system is established in which four satellites acting independently have a 0.9 probability of detecting an incoming ballistic missile. What is the probability that at least one the four satellites detects an incoming ballistic missile? Would you feel safe with such a system?

(9) **Casino Visits:** According to a December 2007 Gallup poll, 24% of American adults have visited a casino in the past 12 months.

(a) What is the probability that 4 randomly selected adult Americans have visited a casino in past 12 months? Is this result unusual?
(b) What is the probability that 4 randomly selected adult Americans have *not* visited a casino in the past 12 months? Is this result unusual?

(10) **Health Insurance Coverage:** The following data represent in thousands, the type of health insurance coverage of people by age in the year 2006.

	Age				**Total**
	< 18	**18-44**	**45-64**	**> 64**	
Private health insurance	47,906	74,375	57,505	21,904	**201,690**
Government health insurance	22,109	12,375	11,304	33,982	**80,270**
No health insurance	8,661	27,054	10,737	541	**46,993**
Total	**78,676**	**114,304**	**79,546**	**56,427**	**328,953**

(a) What is the probability that a randomly selected individual who is less than 18 years old has no health insurance?
(b) What is the probability that a randomly selected individual who has no health insurance is less than 18 years old?

(11) **Left-Handed People:** In a sample of 1000 people, 120 are left-handed. Two unrelated people are selected at random without replacement.

(a) Find the probability that both people are left-handed.
(b) Find the probability that neither person is left-handed.
(c) Find the probability that at least one of the two people is left-handed.

(12) **Light Bulbs:** Twelve light bulbs are tested to see if they last as long as the manufacturer claims they do. Three light bulbs fail the test. Two light bulbs are selected at random without replacement.

(a) Find the probability that both light bulbs failed the test.
(b) Find the probability that both light bulbs passed the test.
(c) Find the probability that at least one light bulb failed the test.

(13) **Guessing**: A multiple-choice quiz has three questions. Each with five answer choices. Only one of the choices is correct. You have no idea what the answer is to any question and have to guess each answer.

(a) Find the probability of answering the first question correctly.
(b) Find the probability of answering the first two questions correctly.
(c) Find the probability of answering all three questions correctly.
(d) Find the probability of answering none of the questions correctly.
(e) Find the probability of answering at least one of the questions correctly.

(14) **Christmas Lights:** Christmas lights are often designed with a series circuit. This means that when one light burns out the entire string of light goes black. Suppose that the lights are designed so that the probability a bulb will last 2 years is 0.995. The success or failure of a bulb is independent of the success or failure of other bulbs.

(a) What is the probability that in a string of 100 lights all 100 will last 2 years?
(b) What is the probability that at least one bulb will burn out in 2 years?

(15) **Birthdays**: Three people are selected at random. Find the probability that (a) all three share the same birthday and (b) none of the three shares the same birthday. Assume 365 days in a year.

Answers

(1) 0.03125

(2) (a) 0.42 ; (b) 0.485 ; (c) Male

(3) 0.230

(4) 1/13 ; 1/13 ; no ; independent

(5) (a) 0.090 ; (b) 0.012

(6) 0.738

(7) (a) 0.303 ; (b) 0.157

(8) 0.9999

(9) (a) 0.0033 ; yes ; (b) 0.3336 ; no

(10) (a) 0.110 ; (b) 0.184

(11) (a) 0.014 ; (b) 0.774 ; (c) 0.226

(12) (a) 0.045 ; (b) 0.545 ; (c) 0.455

(13) (a) 0.2 ; (b) 0.04 ; (c) 0.008 ; (d) 0.512 ; (e) 0.488

(14) (a) 0.6058 ; (b) 0.3942

(15) (a) 0.00000751 ; (b) 0.992)