NumMeth MAT-410 Lab 1 Report

Moritz M. Konarski

October 13, 2020

Abstract

This report deals with numerical solutions to second order differential equations using finite difference methods. The problems were chosen according to my readme file. The finite difference methods in this report are the Central Difference Scheme and the Ilin Scheme. With their help the test problems' analytical and numerical solutions for various parameter values are considered.

Contents

1 Introduction

This report deals with three differential equations based on one general system of equations. These equations are solved both analytically and numerically. The numerical solutions are found using finite difference methods. The resulting graphs of the solutions for various parameters are shown. Their behavior is investigated according to the requirements in our task PDF.

1.1 Problem Functions

All problem functions in this report are versions of the following general equation

$$\begin{cases} \varepsilon \cdot u''(x) + a(x) \cdot u'(x) - b(x) \cdot u(x) &= f(x), \quad x \in (0,1) \\ \zeta_0 \cdot u(0) - \eta_0 \cdot \varepsilon \cdot u'(0) &= \phi_0, \\ \zeta_1 \cdot u(1) + \eta_1 \cdot \varepsilon \cdot u'(1) &= \phi_1. \end{cases} \tag{1}$$

Here $\varepsilon > 0$, $a(x), b(x), f(x)$ are functions on the interval $[0;1]$ where $b(x) \geq 0$. The following problems are versions of the general problem (1). The first equation from my readme file is the following:

$$\begin{cases} \varepsilon \cdot u''(x) + u'(x) &= x^3, \quad x \in (0,1) \\ u(0) &= \phi_0, \\ u(1) &= \phi_1. \end{cases} \tag{2}$$

Here ϕ_0 and ϕ_1 can be freely chosen. I found the analytic solution to this equation for our first homework assignment.

$$\begin{aligned} u = (e^{-1/\varepsilon \cdot x} - 1)&\frac{\phi_0 - \phi_1 + 1/4 - \varepsilon + 3\varepsilon^2 - 6\varepsilon^3}{1 - e^{-1/\varepsilon}} \\ &+ \frac{1}{4}x^4 - \varepsilon x^3 + 3\varepsilon^2 x^2 - 6\varepsilon^3 x + \phi_0 \end{aligned}$$

The boundary conditions of this problem make it a Dirichlet problem. I will refer to this problem as PROBLEM 1 from now on. The second equation from my readme file is:

$$\begin{cases} \varepsilon \cdot u''(x) + u'(x) &= x^3, \quad x \in (0,1) \\ u(0) - u'(0) &= \phi_0, \\ u(1) &= \phi_1. \end{cases} \tag{3}$$

This equation's analytic solution is:

$$\begin{aligned} u = \phi_0 + (e^{-1/\varepsilon \cdot x} - 2)&\frac{\phi_0 - \phi_1 + 1/4 - \varepsilon + 3\varepsilon^2 - 6\varepsilon^3 - 6\varepsilon^4}{2 - e^{-1/\varepsilon}} \\ &+ \frac{1}{4}x^4 - \varepsilon x^3 + 3\varepsilon^2 x^2 - 6\varepsilon^3 x - 6\varepsilon^4. \end{aligned}$$

Again ϕ_0 and ϕ_1 can be freely chosen. This problem is a Robin problem because it contains a derivative in its boundary condition. This problem will be referred to as PROBLEM 2. The third problem this report covers is number 3 from our PDF. It has the form:

$$\begin{cases} \varepsilon \cdot u''(x) + \left(3 \cdot (1+x)^2 - \frac{2 \cdot \varepsilon}{1+x}\right) \cdot u'(x) &= \frac{3 \cdot \varepsilon}{2 \cdot (1+x)^2} - \frac{3(1+x)}{2}, \quad x \in (0,1) \\ u(0) - \frac{1}{3} \cdot \varepsilon \cdot u'(0) &= \frac{\varepsilon}{6} - \frac{1}{1 - e^{-7/\varepsilon}}, \\ u(1) &= 1 - \frac{\ln(2)}{2}, \end{cases} \tag{4}$$

and it's analytic solution is

$$u(x) = \frac{1 - e^{\frac{1-(1+x)^3}{\varepsilon}}}{1 - e^{-7/\varepsilon}} - \frac{\ln(1+x)}{2}.$$

This problem is also a Robin problem and will be referred to as PROBLEM 3.

1.2 Finite Difference Schemes

There are two finite difference schemes that my program is capable of and this report covers. The first scheme is the Central Difference Scheme (CDS) for which

$$\gamma_i = 1,$$
$$\theta_i = 0,$$

and the Ilin Scheme (IS) where

$$\gamma_i = R_i \cdot \operatorname{cotanh}(R_i),$$
$$\theta_i = \operatorname{cotanh}(R_i) - \frac{1}{R_i}.$$

This is the only difference between IS and the CDS. They both use the Thomas Algorithm with the values described in our task PDF to solve the equations (2), (3), and (4) numerically. For the Thomas Algorithm the factors A_i, B_i, C_i from our advection-diffusion PDF p. 20 are used. My boundary conditions are based on the precise boundary conditions that are outlined in lab 1 PDF on page 3 (IV.1.3). These precise conditions allow precise approximations of the boundaries and by extension the whole function.

2 Analysis

This section will investigate the behavior of the analytical and numerical solutions to (2), (3), and (4). The parameters n (number of nodes) and ε will be changed to investigate their influence on the problem functions and solutions. Both the Central Difference Scheme and Ilin Scheme will be compared regarding their accuracy.

2.1 Behavior as ε Tends to Zero

As ε tends to zero the shape of all three functions becomes more pronounced. What I mean is that when ε is large ($\varepsilon \geq 0.5$), the functions have more smoothly changing slopes. When ε approaches zero, the functions have more abrupt changes in their slope. These abrupt changes in slope take place at the boundary layers of the functions which are at $x = 0$ for all three problems. Figure 1 shows this behavior for (2). It shall also represent (3) because even though they are different functions their graphs are similar enough for this illustration. Figure 2 shows this behavior for (4), showing that this problem also develops a clear boundary layer at $x = 0$. For (2) and (3) we have $\phi_0 = 0.25$ and $\phi_1 = 0.75$ as the boundary values.

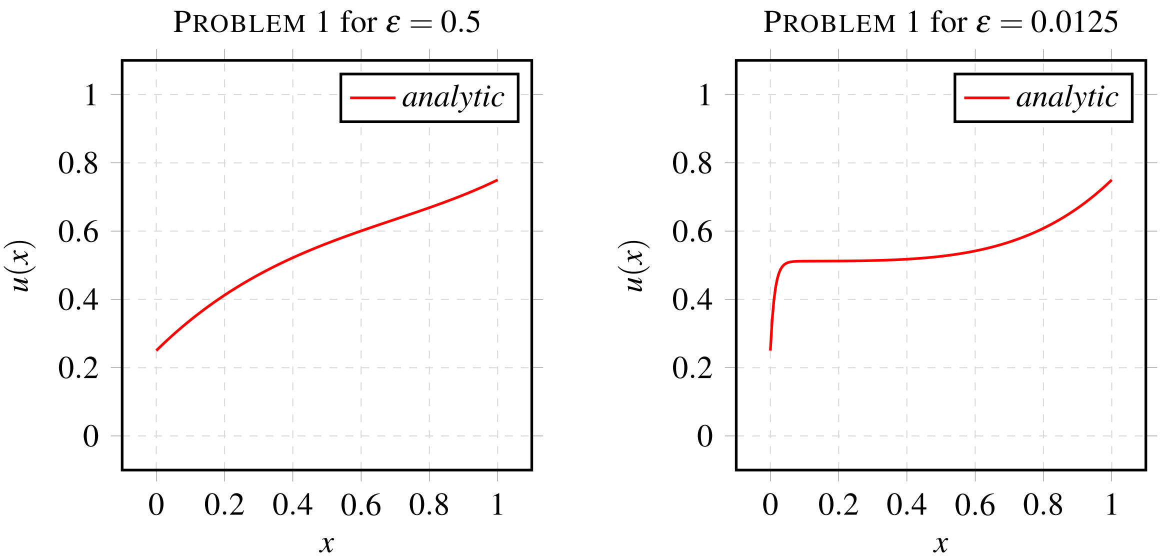

Figure 1: Behavior of (2) for $\phi_0 = 0.25$, $\phi_1 = 0.75$, different ε

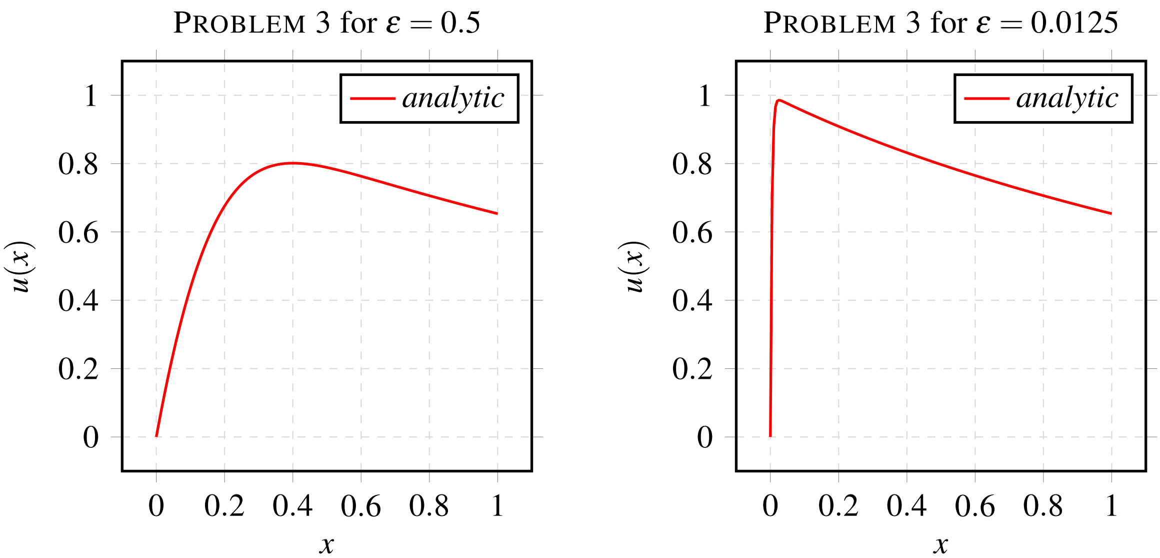

Figure 2: Behavior of (4) for different ε

Figure 1 and Figure 2 allow us to conclude that all three functions have a boundary layer at $x = 0$ and that a smaller ε leads to a more pronounced boundary layer with a more drastic change in slope.

2.2 Accuracy as n Increases

When the number of nodes n increases, the accuracy of the approximation increases, too. In other words, the numerical solution converges to the analytic solution as n increases. The error of the approximation is calculated using the formula

$$\text{Err} \equiv \frac{\max\limits_{1\le i\le n} |u(x_i) - u_i^h|}{\max\limits_{1\le i\le n} |u(x_i)|} \cdot 100\%.$$

A small error indicates that the numeric approximations are converging to the analytic solutions. In Figure 3 and Figure 4 the error of CDS and IS approximations dependent on n is shown. Figure 3 illustrates that CDS is not very accurate for low n, with PROBLEM 3 even having an error of 402.26% for $n = 3$. But as n increases, the error decreases quickly. For the first two problems it is below 10% at only 17 nodes. For PROBLEM 3 is takes 64 nodes to reach the same accuracy which suggests that it is harder to approximate this problem than the first two problems.

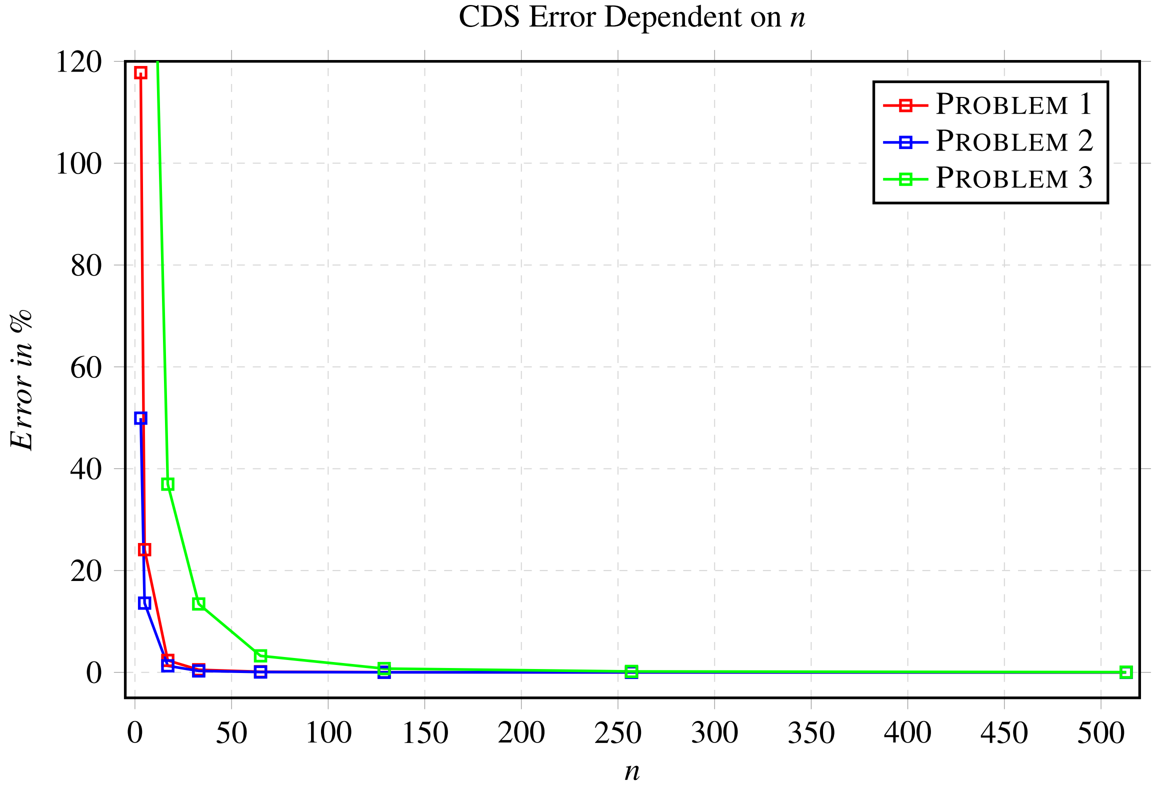

Figure 3: Accuracy of CDS dependent on n. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $\varepsilon = 0.05$

Figure 4 shows that IS is more accurate for low n than CDS, where the largest error is again PROBLEM 3 with $n = 5$ at 38%. As n increases, this error decreases quickly, too. Again the first two problems converge faster than the third problem.

Figure 3 and Figure 4 illustrate that PROBLEM 3 is harder to approximate than PROBLEM 1 or PROBLEM 2. This is most likely caused by PROBLEM 3's more pronounced boundary layer which is harder to approximate. Furthermore we see that if n is large enough, the numeric solution converges to the analytic solution. As an example of this see Figure 5 which shows that for large n, the numeric solution using IS converges to the analytical one, even though the problem's Robin boundary condition is not handled well for low n.

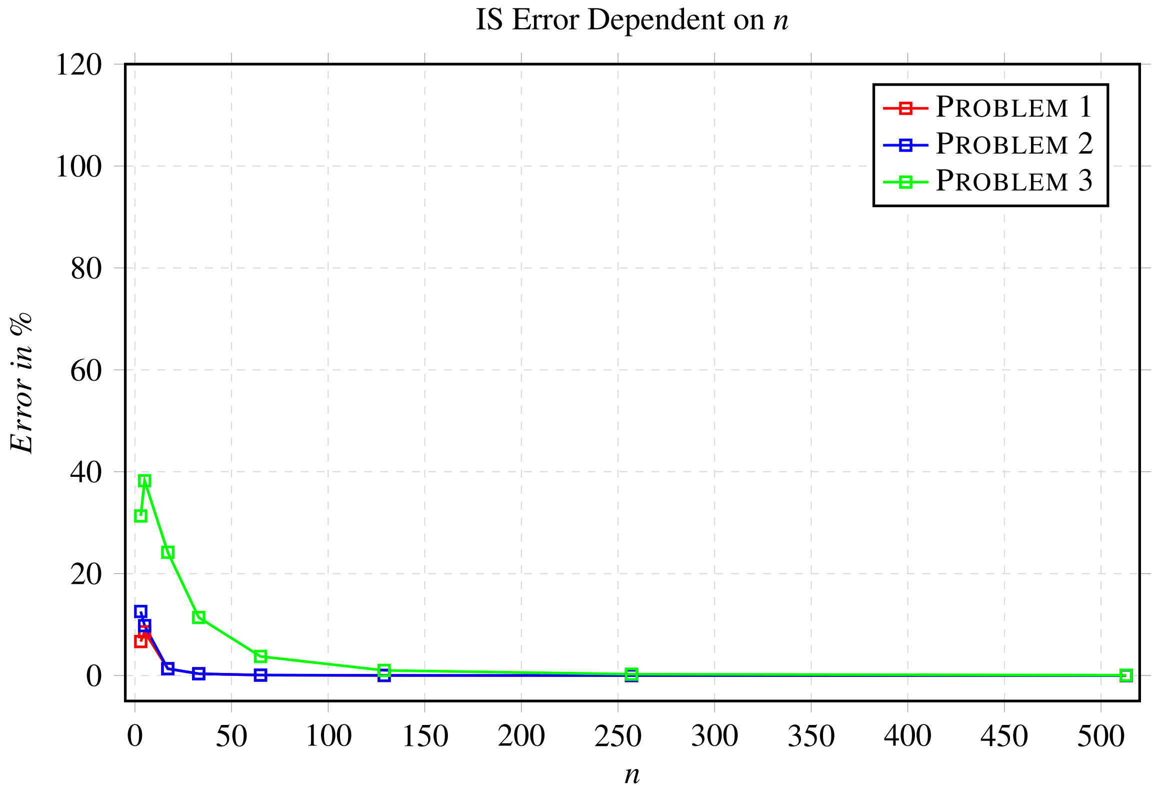

Figure 4: Accuracy of IS dependent on n. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $\varepsilon = 0.05$

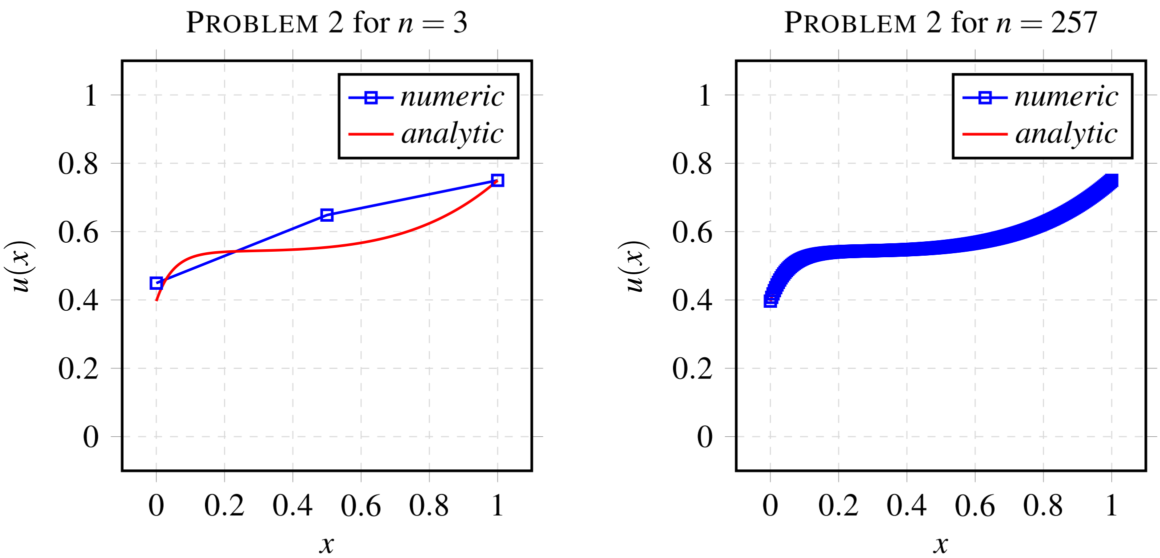

Figure 5: Convergence of (3) with IS for $\phi_0 = 0.25$, $\phi_1 = 0.75$, $\varepsilon = 0.05$

Considering the effect of n on the accuracy of the numerical approximation we can say that an increase in n leads to a decrease in the error and a better convergence of the numerical solution to analytical solution.

2.3 Effect of ε on Accuracy

Epsilon has a negative effect on the accuracy of approximations when it gets smaller. This is caused by the more extreme shape of the functions around the boundary layer that makes is harder to successfully approximate them. Especially the CDS has difficulties here as it starts to oscillate around the drastic change in slope near the boundary layer. These oscillations then continue for the length of the function, making the whole approximation imprecise. IS can successfully avoid these oscillations as the graphs below illustrate. The exponential functions involved in the hyperbolic cotangent in R_i of IS are the reason for this.

Figure 6 shows that when $\varepsilon \to 0$ the error of CDS increases dramatically ($n = 33$ here). For PROBLEM 1 and PROBLEM 2 the maximum error for $\varepsilon = 0.0001$ is 164.2% and 41.4% respectively. For PROBLEM 3 the maximum error for this value of ε is 367.7% and even for $\varepsilon = 0.001$ the error is 199%. Neither of these values fit on the graph.

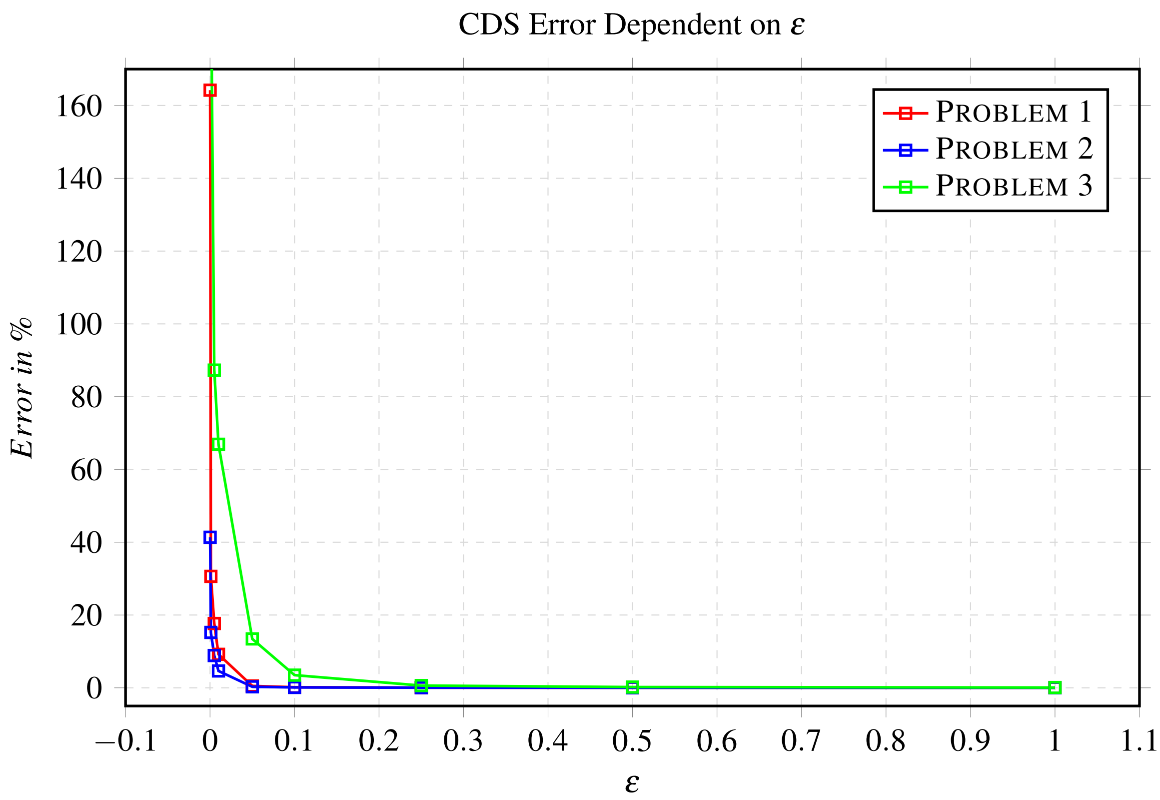

Figure 6: Accuracy of CDS dependent on ε. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $n = 33$

The large error in PROBLEM 3 can be explained by the aforementioned oscillations that are caused by the pronounced boundary layer in that problem. To illustrate this, Figure 7 shows the analytic and numerical solutions for $n = 33$ and $\varepsilon = 0.1$ and $\varepsilon = 0.001$ respectively. The graph on the right shows the oscillations that explain the high error.

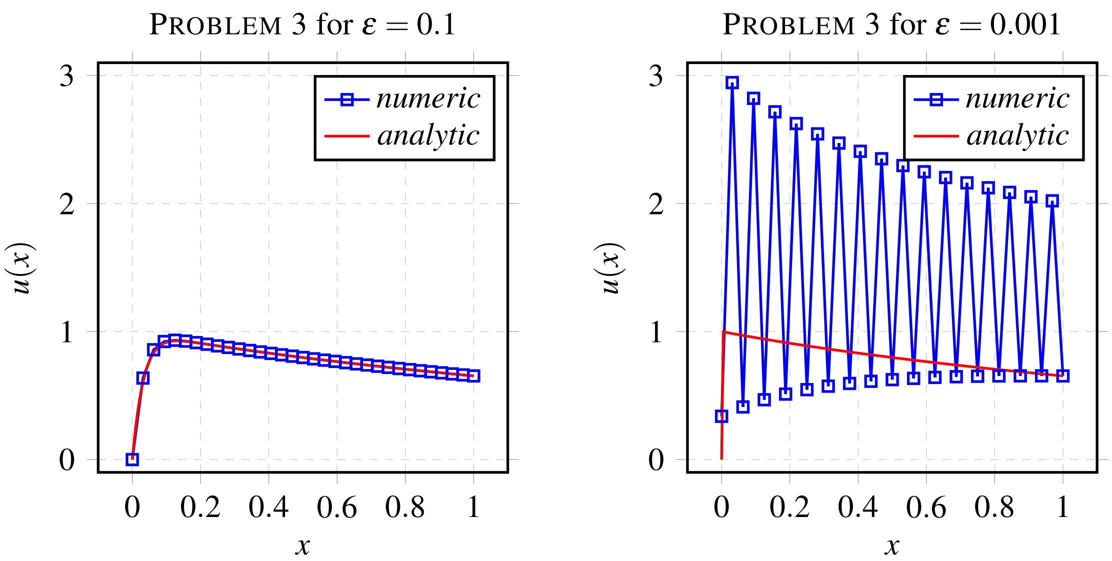

Figure 7: Behavior of (4) with CDS for $n = 33$

While CDS is fairly accurate for most values of ε, when it becomes too small, the numerical solution begins to oscillate and becomes very inaccurate. IS does not face that problem as it's R_i is not prone to oscillations. Figure 8 shows the errors for $\varepsilon \to 0$ for IS. We see that the error of IS also increases, but much less than for CDS ($n = 33$ also here). For PROBLEM 1 and PROBLEM 2 the maximum error for $\varepsilon = 0.0001$ is 8.5% and 3.8% respectively. For PROBLEM 3 the maximum error for this value can't be found with my program. It returns `NaN` (not a number). I think the reason for this is that PROBLEM 3 is not monotone for any values of ε that I tested. This causes the numeric solution to break down for small ε. Furthermore, the conditions for stability are also not met. For $\varepsilon = 0.001$ the error is 39.4% for this problem.

The explanation for the great increase in accuracy compared to CDS is that IS does not start to oscillate for small values of ε and thus keeps its accuracy. Figure 8 shows the much smaller errors for small ε of IS compared to CDS (note that the axes are the same as in Figure 6).

Figure 9 shows the same problem and parameters as Figure 7 and demonstrates that IS does not oscillate in a situation where CDS did.

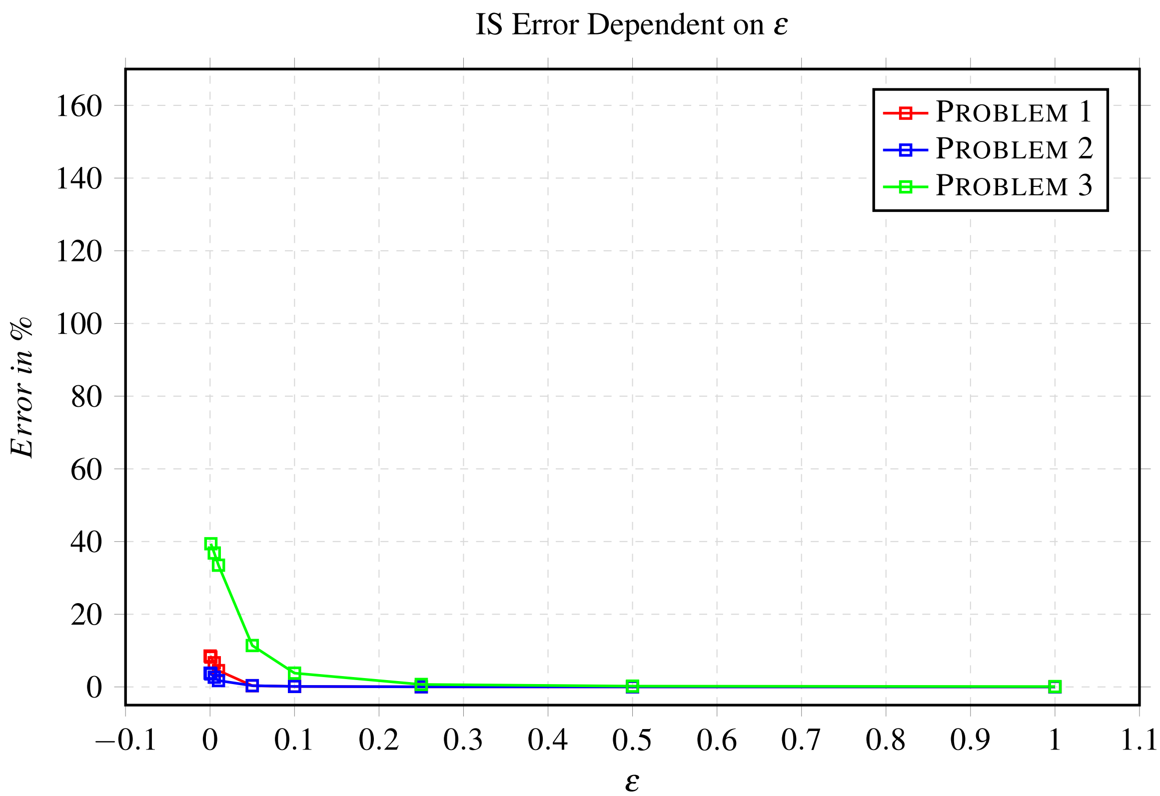

Figure 8: Accuracy of IS dependent on ε. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $n = 33$

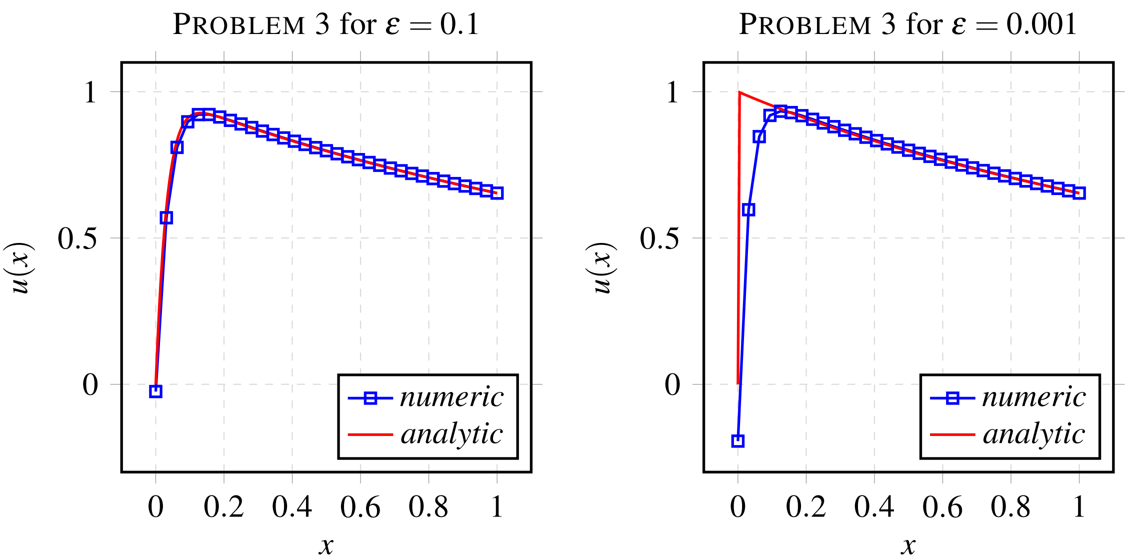

Figure 9: Behavior of (4) with IS for $n = 33$

We can see that as $\varepsilon \to 0$ approximating the analytical functions becomes more and more difficult. The parameter ε has negative influence on the convergence of the numerical solution to the analytical solution when it decreases. CDS begins to strongly oscillate around the boundary layer when ε is small and the approximation becomes pretty useless. IS does not have oscillations, but for PROBLEM 3 the numerical approximation is neither monotone nor stable which means that a solution might not exist for certain parameters. This is what happened when $\varepsilon = 0.0001$.

2.4 Accuracy of Difference Schemes

The two schemes discussed in this report both have advantages and disadvantages which will be discussed using PROBLEM 3 as the example because it was shown to be the most difficult problem to approximate.

When ε is decreased our previous analysis has shown that it becomes harder to approximate the functions because of the shape of the function near the boundary layer. Table 1 shows the error in % of the numerical solution compared to the analytical solution for PROBLEM 3 and both schemes. The last column of the table is the ratio of the error of CDS to the error of IS. This enables us to more easily compare the schemes. Table 1 shows that for large

ε	CDS Error in %	IS Error in %	$\mathrm{Err}_{\mathrm{CDS}}/\mathrm{Err}_{\mathrm{IS}}$
1	0.07529	0.10654	0.70668
0.5	0.20207	0.22170	0.91145
0.25	0.61485	0.68392	0.89900
0.1	3.48381	3.79717	0.91747
0.05	13.43621	11.39853	1.17876
0.01	66.91237	33.49386	1.99775
0.005	87.29208	36.84592	2.36911
0.001	199.07709	39.39860	5.05289
0.0001	367.72195	`NaN`	`NaN`

Table 1: Errors in % for (4) for CDS and IS, various ε, $n = 33$

ε CDS is the better approximation but as ε decreases their accuracies come closer. This trend continues and IS becomes more accurate than CDS. This is caused by the oscillations that CDS experiences for small ε making the approximation inaccurate. As IS does not experience oscillations, its error increases much slower. For PROBLEM 3 we see that IS can have limits, as it is neither monotone nor stable for this exercise and these parameters. This means that the solution is not stable and thus when ε gets too small my program cannot find a solution.

The other important parameter in this problem is n, the number of nodes. In Table 2 we see the error for the CDS and IS approximations of PROBLEM 3 and again their ratio. Table 2 demonstrates that for small n IS is more accurate because CDS tends to oscillate more for low n. Then, as n increases, CDS becomes more accurate. As n increases further the two methods begin to converge to an error of 0% and the difference between them continues to increase but the rate of change slows down. This means that when n increases, CDS converges a little bit faster than IS, but they both converge to the analytical solution.

n	CDS Error in %	IS Error in %	$\mathrm{Err}_{\mathrm{CDS}}/\mathrm{Err}_{\mathrm{IS}}$
3	402.26497	31.32132	12.84316
5	225.19784	38.23313	5.89012
17	36.95902	24.19590	1.52749
33	13.43621	11.39853	1.17876
65	3.24558	3.74921	0.86567
129	0.74625	1.00782	0.74045
257	0.18301	0.26429	0.69245
513	0.04564	0.06782	0.67295
1000	0.01198	0.01805	0.66371
10000	0.00012	0.00018	0.66666

Table 2: Errors in % for (4) for CDS and IS, various n, $\varepsilon = 0.05$

3 Conclusion

To conclude, both the Central Difference Scheme and the Ilin Scheme approximate problems (2), (3), and (4) well. When n is increased, the accuracy of the approximation increases, too. Both CDS and IS converge to the analytic function for large–enough n. For low n CDS tends to oscillate and thus IS is a better approximation for those cases. Even for Robin problems which are difficult to approximate at the boundary layer high n lead to convergence for both schemes.

The parameter ε has a negative influence on the accuracy of the numeric approximation when it becomes smaller. The shape of the function becomes more extreme and the finite difference methods become less accurate. IS can still approximate these cases well given the appropriate number of nodes. CDS has more trouble because it oscillates, especially around the boundary layer, making the approximation imprecise.

All in all I think that IS is the better method to approximate the solutions this report covers. It does not suffer from inaccuracy caused by oscillations if n or ε is small. It converges slower than CDS and is more computationally expensive because of the involved exponential functions, but in my opinion that is worth it. The only drawback that I found in my analysis is that for PROBLEM 3 with small ε IS is not monotonous and did not produce a result. On the other hand, the result obtained by CDS was 3.7 times off the mark which indicates that neither method can approximate very small ε well.