Newton's Second Law ($F = ma$) states that

$$F = (\rho\Delta x)\,\frac{\partial^2 u}{\partial t^2} \tag{2}$$

where ρ is the linear density of the string (ML^{-1}) and Δx is the length of the segment. The force comes from the tension in the string only - we ignore any external forces such as gravity. The horizontal tension is constant, and hence it is the vertical tension that moves the string vertically (obvious).

Balancing the forces in the horizontal direction gives

$$T\,(x + \Delta x, t)\cos\theta\,(x + \Delta x, t) = T\,(x, t)\cos\theta\,(x, t) = \tau = const \tag{3}$$

where τ is the constant horizontal tension. Balancing the forces in the vertical direction yields

$$\begin{aligned} F &= T\,(x + \Delta x, t)\sin\theta\,(x + \Delta x, t) - T\,(x, t)\sin\theta\,(x, t) \\ &= T\,(x + \Delta x, t)\cos\theta\,(x + \Delta x, t)\tan\theta\,(x + \Delta x, t) - T\,(x, t)\cos\theta\,(x, t)\tan\theta\,(x, t) \end{aligned}$$

Substituting (3) and (1) yields

$$\begin{aligned} F &= \tau\,(\tan\theta\,(x + \Delta x, t) - \tan\theta\,(x, t)) \\ &= \tau\left(\frac{\partial u}{\partial x}\,(x + \Delta x, t) - \frac{\partial u}{\partial x}\,(x, t)\right). \end{aligned} \tag{4}$$

Substituting F from (2) into Eq. (4) and dividing by Δx gives

$$\rho\frac{\partial^2 u}{\partial t^2}\,(\xi, t) = \tau\frac{\frac{\partial u}{\partial x}\,(x + \Delta x, t) - \frac{\partial u}{\partial x}\,(x, t)}{\Delta x}$$

for $\xi \in [x, x + \Delta x]$. Letting $\Delta x \to 0$ gives the <u>1-D Wave Equation</u>

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\tau}{\rho} > 0. \tag{5}$$

Note that c has units $[c] = \left[\frac{\text{Force}}{\text{Density}}\right]^{1/2} = LT^{-1}$ of speed.

1.1 Boundary conditions

Ref: Guenther & Lee §4.2 (p. 94), Myint-U & Debnath §4.4

In order to guarantee that Eq. (5) has a unique solution, we need initial and boundary conditions on the displacement $u\,(x, t)$. There are now 2 initial conditions and 2 boundary conditions.