ACTUARIAL MATHEMATICS NOTES

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Contents

Date: November 2, 2020.

1. Class 01.09.2020

1.1. General Introduction.

- introduction of professor
- this is course 1 of 2
- quite popular elective course – he is not sure why
- we'll dig deep inside the workings of insurance and the like
- it's a very well paid profession
- who is an actuary:
  - completely independent professional
  - actuarial association assigns audits to actuaries
  - basically a type of auditor
  - actuarial expertise is needed to make investments
- this course will teach us the basics
- to become an actuary, you will have to pass 6 exams, we'll learn stuff for the first 2
- re-insurance: some of the richest companies in the world
- Parmenter is the main text book
- **course content**: chapters 1, 2, 3, and 4

1.2. Accumulation Function.

- the simplest financial transaction is an investment

- **principal**: initial investment
- **accumulated value**: total amount the money grows to
- **Amount Function**: amount of money at time t from investment of the principal – $A(t)$, t is measured in years, $A(0)$ is the principal
- **Accumulation Function**: how much money increases as a percent value, where $a(0) = 1$ (as there has been no change)

$$a(t) = \frac{A(t)}{A(0)}$$

- accumulation functions can be any function where $a(0) = 1$, additionally one would hope that it is increasing
- continuity is not required, depends on how interest is paid – if fractional values of t make sense it may be continuous, but if interest is paid discretely, it may be stepwise
- three types of accumulation functions
  (1) amount of interest earned each year is constant – linear graph, simple interest
  (2) the amount of interest increases over the years – exponential graph, compound interest
  (3) if interest is paid out at fixed periods of time a piecewise function is used – the amount of interest might be constant or increasing
- **Interest = Accumulated Value - Principal**
- to make this more practical, the *effective rate of interest* i is used
- i is the interest earned on a principal of 1 over the period of 1 year – amount of interest earned over 1 year divided by the value at the beginning of the year

$$i = a(1) - 1$$

- i can also be calculated with the amount function
$$i = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)}$$
- i can be calculated for the nth year by
$$i = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{A(n) - A(n-1)}{A(n-1)}$$

1.3. **Simple Interest.**

- primarily used between integer periods of time
- $a(t)$ is a straight line here – the increase is linear
- general form of the equation is
$$a(t) = 1 + it$$
- interest earned each year is constant – interest does not earn interest
- if the principal is k at $t = 0$
$$A(t) = k(1 + it)$$
- the effective rate of interest is not constant, it decreases over time
$$i_n = \frac{i}{1 + i(n-1)}$$
- **exact simple interest**: count the last day, not the first
$$t = \frac{\text{number of days}}{365}$$
- **ordinary simple interest (Banker's Rule)**: count the last day, not the first
$$t = \frac{\text{number of days}}{360}$$
- international markets use ordinary simple interest

2. Class 03.09.2020

2.1. Compound Interest.

- most important special case
- effective interest rate is fixed
- interest earns interest itself
- because the interest affects itself, the function is exponential

$$a(t) = (1+i)^t, \quad t \geq 0$$

- amount function for compound interest is

$$A(t) = k(1+i)^t$$

- the effective interest rate for compound interest is constant
- what values to choose for t is done like with simple interest, either *exact* or *ordinary*
- if we want to find some value between integers, we linearly interpolate it

$$A(t) = A(\lfloor t \rfloor) + (t - \lfloor t \rfloor) \cdot (A(\lceil t \rceil) - A(\lfloor t \rfloor))$$

- to find the time it takes a principal to accumulate to a certain value, use logs

$$t = \frac{\log\left(\frac{\text{future value}}{\text{principal}}\right)}{\log(1+i)}$$

- compound and simple interest graphs only intersect at $(0,1)$ and at $(1, 1+i)$, this furthermore gives two cases

$$\begin{cases} \text{simple i.} > \text{compound i.} \quad \text{for} \quad 0 < t < 1 \\ \text{compound i.} > \text{simple i.} \quad \text{for} \quad t > 1 \end{cases}$$

3. Class 08.09.2020

3.1. Present Value and Discount.

3.1.1. *Present Value.*

- we define the *present value t years in the past* as the amount of money that will accumulate to the principal in t years
- this is the reverse of what we have been calculating thus far

Present Value	Principal	Accumulated Value

- v is the amount of money needed to accumulate to 1 within 1 year

$$v = \frac{1}{1+i}$$

- how v works can be seen in this timeline that shows the evolution of $a(t)$

$\frac{1}{1+i}$	1	$1+i$
-1	0	1

- for compound interest, v is

$$v^t = \frac{1}{(1+i)^t} \tag{3.1}$$

- this is simply an inverted formula of $a(t)$ for compound interest
- for simple interest the present value is called x

$$x = \frac{1}{1+it}$$

3.1.2. *Discount.*

- imagine \$100 was invested and accumulated to \$112 in 1 year
- \$100 was the starting figure and interest (\$12) was added to it

- we could look at it the other way around and say \$112 is the starting value and at the start of the year \$12 was subtracted from it
- \$12 here is an amount of *discount*
- it's the same as interest, only the point of view is different
- discount focuses on the end of the year, so it is defined as

$$d = \frac{a(1) - 1}{a(1)}$$

- this only differs from the definition of i in the denominator, which is $a(0)$ for i because the beginning of the year is the focus
- effective rate of discount in the nth year is

$$d_n = \frac{a(n) - a(n-1)}{a(n)}$$

- some identities relating to i are

$$\begin{aligned} d &< i \\ d &= \frac{i}{1+i} \\ 1 - d &= v \\ i &= \frac{d}{1-d} \end{aligned}$$

- now the rules for finding the present and accumulated values are reversed

$$\begin{aligned} \text{present value :} &\qquad (1-d)^t \\ \text{accumulated value :} &\qquad \frac{1}{(1-d)^t} \end{aligned}$$

4. Class 10.09.2020

4.1. Nominal Rate of Interest.

- $a(t) = (1+i)^t$ will be assumed in this section
- effective rates of interest can be given for any length of time
- to apply our previous formulae, we need to make sure that t is the number of *effective interest periods*
- generally, these periods are not years, but shorter periods
- a yearly rate or 12% "convertible semiannually" actually means that you pay 6% twice a year – in this case it would actually be 12.36%
- the effective interest rate increases the shorter the intervals between payments are
- the 12% is a **nominal rate of interest**, meaning it is convertible over a period other than 1 year
- $i^{(m)}$ denotes the nominal rate of interest convertible m times a year

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m$$

- we can also define a nominal rate of discount $d^{(m)}$

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m$$

- we also see that

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = \left[1 - \frac{d^{(n)}}{n}\right]^{-n}$$

4.2. Force of Interest.

- our goal is to find nominal rates of interest that are equivalent to a certain effective annual rate of interest
- for example $i = 0.12$ with the functions above gives the values

m	1	2	5	10	50
$i^{(m)}$	0.12	0.1166	0.1146	0.1140	0.1135

- we see that $i^{(m)}$ decreases as m increases
- m is approaching a limit, using L'Hopital's rule we can find it

$$\delta = \ln(1+i)$$

$$e^{\delta} = 1+i$$

- δ is called the **force of interest**
- it represents the nominal rate of interest that is convertible *continuously* – serving as a good approximation of $i^{(m)}$ for large m, like dayly conversions
- the second form of δ is useful because it makes conversions easier
- the derivative of $(1+i)^t$ by t (D) can be rewritten to be

$$\delta = \ln(1+i) = \frac{D[(1+i)^t]}{(1+i)^t} = \frac{D[a(t)]}{a(t)}$$

- for compound interest $\delta = \ln(1+i)$, but for arbitrary accumulation functions it is

$$\delta_t = \frac{D[a(t)]}{a(t)}$$

$$\delta_t = D[\ln(a(t))]$$

- if δ_r is given and we want to find $a(t)$ we use

$$a(t) = e^{\int_0^t \delta_r dr}$$

- we note that $i > \delta$
- the force of discount is the same as the force of interest

5. Class 15.09.2020

5.1. **Equation of Value.**

- interest problems only involve 4 quantities:
  1. principal value
  2. accumulated value
  3. period of investment
  4. rate of interest
- each one of them can be calculated if the other 3 are known
- when multiple investments are made, the time diagram is the most important tool
- then an *equation of value* is set up to find the value
- again, be careful with interpolation between integral durations with compound interest
- finding an appropriate rate of interest such that money increases generally involves logarithms involves

5.1.1. *Example 1.*

- Alice borrows 5,000 at 18% convertible semiannually
- after 2 years, she pays back 3,000
- 3 years after that she pays 2,000
- how much does she owe 7 years after taking out the loan?
- time diagram:

$5,000$		$-3,000$			$-2,000$		X
0	1	2	3	4	5	6	7

- because the interest rate is convertible semiannually, our nominal rate is $i = 0.09$
- using the diagram we see

$$X = 5,000(1.09)^{14} - 3,000(1.09)^{10} - 2,000(1.09)^{4} = 6,783.38$$

- in the same way payments here are negative loans, withdrawals can be seen as negative deposits

5.1.2. *Example 2.*

- John borrows 3,000
- 2 years later he borrows another 4,000
- 2 years after that he borrows 5,000
- $i = 0.18$
- at what time would a single loan of 12,000 be equivalent? – at what time would the amount owed be the same as a loan of 12,000?
- draw a timeline:

$3,000$		$4,000$			$5,000$
0	1	2	t	3	4
			$12,000$		

- solution:

$$12,000v^t = 3,000 + 4,000v^2 + 5,000v^4$$

$$v = \frac{1}{1.18}$$

$$v^t = \frac{3 + 4v^2 + 5v^4}{12}$$

$$t = \frac{\ln(3 + 4v^2 + 5v^4) - \ln(12)}{\ln(v)}$$

$$t = 2.11789$$

6. Class 24.09.2020

6.1. Arithmetic and geometric sequences.

6.1.1. *Arithmetic sequences.*

- $a, a+d, a+2d, a+3d$
- nth term: $a_n = a + (n-1)d$
- sum of first n terms: $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

6.1.2. *Geometric sequences.*

- a, ar, ar^2, ar^3
- nth term: $a_n = ar^{n-1}$
- sum of first n terms: $S_n = \frac{a(1-r^n)}{a-r}$

6.2. Basic Results.

- *annuity* – payments made of regular intervals
- generally, all the payments are of the same magnitude
- annuity is generally a payment of 1 over n periods
- we do have to find the equivalent rate of interest for the payment periods
- a payment plan for a general annuity

$a_{\overline{n}\vert}$	1	1	1		$s_{\overline{n}\vert}$
0	1	2	3	$\cdots$	n

- present value of the annuity is $a_{\overline{n}|}$

$$a_{\overline{n}|} = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{i}$$

- accumulated value of the annuity is $s_{\overline{n}|}$

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n = \frac{(1+i)^n - 1}{i}$$

- to find actual value, we can multiply the present value with the actual value

- other symbols and values for annuities

$a_{\overline{n}|}$	$\ddot{a}_{\overline{n}|}$			$s_{\overline{n}|}$	$\ddot{s}_{\overline{n}|}$
0	1	2	$\cdots$	n	$n+1$

- present value of the annuity described on the first payment $\ddot{a}_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}$$

- accumulated value one period after the last payment $\ddot{s}_{\overline{n}|}$

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

- we note two more identities

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n$$

$$1 = d \cdot \ddot{a}_{\overline{n}|} + v^n$$

7. Class 06.10.2020

7.1. **Annuities.**

- annuities can be viewed from many different angles with the same result
- *annuity-immediate* are payments at the end of periods
- *annuity-due* are payments made at the beginning of periods

7.2. **Perpetuities.**

- annuity whose payments continue forever

$$\begin{aligned} a_{\overline{\infty}|} &= \lim_{n\to\infty} a_{\overline{n}|} \\ &= \frac{1}{i} \end{aligned}$$

- we also have the perpetuity at the time of the first payment

$$\begin{aligned} \ddot{a}_{\overline{\infty}|} &= a_{\overline{\infty}|}(1+i) \\ &= \frac{1}{d} \end{aligned}$$

7.3. **Unknown time and unknown rate of interest.**

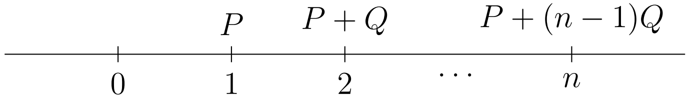
- a fund of 5,000 will be used to award scholarships of 500 for as long as possible. If $i = 0.09$, how many scholarships can be awarded?

$$\begin{aligned} 500 \cdot a_{\overline{n}|} \leq 5,000 &< 500 \cdot a_{\overline{n+1}|} \\ a_{\overline{n}|} \leq 10 &< a_{\overline{n+1}|} \\ a_{\overline{n}|} &= 10 \\ \frac{1-v^n}{i} &= 10 \\ n &= \frac{\ln(1-10i)}{\ln(v)} \end{aligned}$$

8. Class 13.10.2020

8.1. **Varying Annuities.**

- general type of a varying annuity



- we can find the value 1 year before the first payment with

$$A = Pa_{\overline{n}|} + Q\left[\frac{a_{\overline{n}|} - nv^n}{i}\right]$$

- the accumulated value of these payments is

$$A(1+i)^n = Ps_{\overline{n}|} + Q\left[\frac{s_{\overline{n}|} - n}{i}\right]$$

8.1.1. *Increasing Annuity.*

- here $P = Q = 1$
- the present value for this annuity is

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

- the accumulated value for this annuity is

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

8.1.2. *Decreasing Annuity.*

- here $P = 1$ and $Q = -1$
- the present value for this annuity is

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

- the accumulated value for this annuity is

$$(Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

8.1.3. *Geometric Annuity.*

- here Q changes in a geometric way, by c
- then the sum of this annuity can be found with

$$r = \frac{1+i}{c}$$
$$S_n = Q(c)^{n-1}\left[\frac{1-r^n}{1-r}\right]$$

- to find the present value

$$r = \frac{1+i}{c}$$
$$P_n = Q(c)\left[\frac{1-r^n}{1-r}\right]$$

9. Class 15.10.2020

9.1. Amortization.

- repay a loan by the *amortization method* – installment payments at periodic intervals
- knowing the outstanding principal is important because you need to know how much you owe
- *prospective method*: outstanding principal is the present value of the outstanding payments at that time
- *retrospective method*: original principal accumulated until then minus the accumulated value of all the payments made until then
- this means that we either need to find $a_{\overline{n}|}$ or "original principal $s_{\overline{n}|}-$ payments $s_{\overline{n}|}$"

9.2. Amortization Schedules.

- a payment X can be divided into its principal and interest parts like so:
  1. know or find the outstanding principal 1 time interval before X, let's call it P
  2. the interest portion of X is iP
  3. the principal portion of X is $X - iP$
- if a loan in paid back in equal payments of X for n years, the interest part of the kth payment is

$$X(1 - v^{n-k+1})$$

- the principal part of the kth payment is

$$Xv^{n-k+1}$$

- an amortization schedule is simply a table showing the payments and how they are made up

Duration	Payment	Interest	Principal Repaid	Outstanding Principal
0				1,000.00
1	150	110.00	40.00	960.00
2	150	105.60	44.40	915.60
3	150	100.72	49.28	866.32
⋮	150	⋮	⋮	⋮
12	150	23.93	126.07	91.51
13	101.58	10.07	91.51	0.00

10. Class 22.10.2020

10.1. Sinking Funds.

- you pay interest each month but nothing more
- at the end you simply pay the full loan amount back
- generally you invest the money into a **sinking fund** in the meantime – if you get a higher interest rate than you pay, you could even make money

10.2. Yield Rates.

- only the payments made directly by or to the person(s) should be considered for that person(s)'s yield rate
- simply solve the problem you are given using a calculator