ACTUARIAL MATHEMATICS HOMEWORK 5

MORITZ M. KONARSKI

1. Parmenter Exercises 2–11 to 2–19

2–11. Plan A:

150	200	250
0	1	2

Plan B:

87	425	50
0	1	2

Find the i for which plan A is better than plan B for the consumer.

$$150(1+i)^2 + 200(1+i) + 250 < 87(1+i)^2 + 425(1+i) + 50$$

Find the intersections and then check which plan is better in the interval

$$150(1+i)^2 + 200(1+i) + 250 < 87(1+i)^2 + 425(1+i) + 50$$
$$1 + i = c$$
$$0 = 63c^2 - 225c + 200$$
$$c_1 = \frac{40}{21} \qquad c_2 = \frac{5}{3}$$
$$i_1 = \frac{19}{21} \qquad i_2 = \frac{2}{3}$$

For $\frac{2}{3} < i < \frac{19}{21}$ plan A is better than plan B.

2–12. Draw a timeline:

300	200	100	$= 800$
0	1	2	3

For which i do the payments equal 800?

$$800 = 300(1+i)^3 + 200(1+i)^2 + 100(1+i)$$
$$1 + i = 1.12926$$
$$i = 0.12926$$

Date: September 27, 2020.

2–13. Draw a timeline:

$5,000$		$-3,000$	$5,000$		$-10,000$
$'95$	$'96$	$'97$	$'98$	$'99$	$'00$

For which i do the payments equal 0?

$$
\begin{aligned}
0 &= 5,000(1+i)^5 - 3,000(1+i)^3 + 5,000(1+i)^2 - 10,000 \\
1+i &= 1.097 \\
i &= 0.097
\end{aligned}
$$

2–14. Draw a timeline:

50	100	100	100	X
0 mo	4 mo	8 mo	12 mo	16 mo

(a) What is X if $i = 0.12$?

$$
\begin{aligned}
i^{(4)} &= 4\left(\sqrt[4]{1+0.12} - 1\right) = 0.11495 \\
600 &= 50\left(1+i^{(4)}\right)^4 + 100\left(1+i^{(4)}\right)^3 + 100\left(1+i^{(4)}\right)^2 + 100\left(1+i^{(4)}\right) + X \\
X &= 148.32615
\end{aligned}
$$

(b) What is i if $X = 350$?

$$
\begin{gathered}
600 = 50\left(1+i^{(4)}\right)^4 + 100\left(1+i^{(4)}\right)^3 + 100\left(1+i^{(4)}\right)^2 + 100\left(1+i^{(4)}\right) + 350 \\
1 + i^{(4)} = c \\
c_1 = 0.85852 \qquad c_2 = -2.06683 \\
i_1^{(4)} = -0.13415 \qquad i_2^{(4)} = -0.99703
\end{gathered}
$$

2–15. 7% effective interest at the end of each year, every 4 years 5% bonus on existing money. Find the effective rate for

(a) 3 years

$$(1+0.07)^3 - 1 = 0.22504$$

(b) 4 years

$$(1+0.07)^4 \cdot 1.05 - 1 = 0.37634$$

(c) 5 years

$$(1+0.07)^5 \cdot 1.05 \cdot 1.07 - 1 = 0.47268$$

2–16. The sum of all payments of 1 every 3 years from 1 to n while n approaches infinity is

$$\lim_{n\to\infty}\left(1+1\cdot\left(1+i^{(m)}\right)+1\cdot\left(1+i^{(m)}\right)^3+\cdots+1\cdot\left(1+i^{(m)}\right)^n\right)$$

And we know that the present value times the interest to the power of n approaching infinity is this

$$\lim_{n\to\infty}\left(\frac{125}{91}\cdot\left(1+i^{(m)}\right)^n\right)$$

We can set these two equal

$$\lim_{n\to\infty}\left(1+1\cdot\left(1+i^{(m)}\right)+1\cdot\left(1+i^{(m)}\right)^3+\cdots+1\cdot\left(1+i^{(m)}\right)^n\right)$$
$$=\lim_{n\to\infty}\left(\frac{125}{91}\cdot\left(1+i^{(m)}\right)^n\right)$$

and divide both sides by $\left(1+i^{(m)}\right)^{n-1}$

$$\lim_{n\to\infty}\left(1\cdot\left(1+i^{(m)}\right)^{n-1}+1\cdot\left(1+i^{(m)}\right)^{n-2}+\cdots+1+1\cdot\left(1+i^{(m)}\right)\right)$$
$$=\lim_{n\to\infty}\left(\frac{125}{91}\cdot\left(1+i^{(m)}\right)\right)$$

Taking the limit we get

$$1+\left(1+i^{(m)}\right)=\frac{125}{91}\cdot\left(1+i^{(m)}\right)$$

We find that

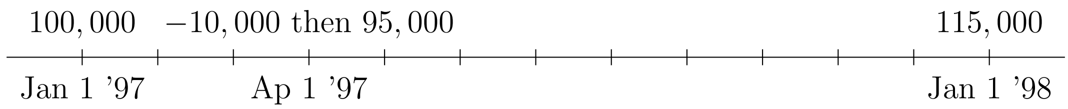
$$i^{(m)}=\frac{1}{\frac{125}{91}-1}-1$$

and thus

$$i=\left[1+\frac{\frac{1}{\frac{125}{91}-1}-1}{1/3}\right]^{1/3}-1=0.820085$$

Testing the result for n from 0 to 50,000, the results match relatively well. The sum of individual payments is 8.506E+13004 and the present value function gives 5.264E+13004.

2–17.



(a) Find the time weighted interest

$$i=\frac{105,000}{100,000}\cdot\frac{115,000}{95,000}-1$$
$$i=0.27105$$

(b) Find the simple interest rate for the year

$$1 + i = c$$

$$115,000 = 100,000c^{12} - 10,000c^{9}$$

$$c_1 = 1.02009 \qquad c_2 < 0$$

$$i^{(12)} = 0.02009$$

$$i = 0.26959$$

(c) Find the simple interest rate for the year assuming equal spacing of payments

100,000 | −10,000 then 95,000 | 115,000

Jan 1 '97 | Jul 1 '97 | Jan 1 '98

$$1 + i = c$$

$$115,000 = 100,000c^{2} - 10,000c$$

$$c_1 = 1.12354 \qquad c_2 < 0$$

$$i^{(2)} = 0.12354$$

$$i = 0.26234$$

2–18.

100,000 | −10,000 | +5,000 | 115,000

Jan 1 '97 | Ap 1 '97 | Jul 1 '97 | Jan 1 '98

(a) Find the simple interest rate for the year

$$1 + i = c$$

$$115,000 = 100,000c^{4} - 10,000c^{3} + 5,000c^{2}$$

$$c_1 = 1.04889 \qquad c_2, c_3, c_4 \text{ are negative or complex}$$

$$i^{(4)} = 0.04889$$

$$i = 0.21037$$

(b) Find the simple interest rate for the year assuming equal spacing of payments

100,000 | −10,000 | +5,000 | 115,000

Jan 1 '97 | Ap 1 '97 | Jul 1 '97 | Jan 1 '98

$$1 + i = c$$

$$115,000 = 100,000c^{3} - 10,000c^{2} + 5,000c$$

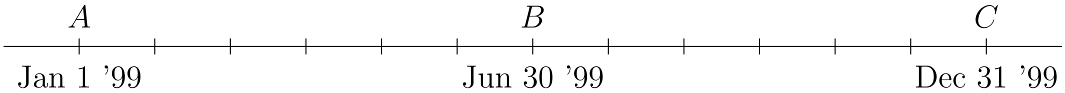
$$c_1 = 1.06569 \qquad c_2, c_3 < 0$$

$$i^{(3)} = 0.0.6569$$

$$i = 0.210299$$

(c) You cannot find the time weighted interest because we don't know the value immediately before or after the deposit of 5,000.

2–19.



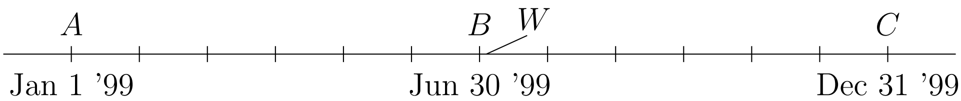
(a) Find the time weighted interest

$$i = \frac{B-A}{A} \cdot \frac{C-B}{B}$$

and the simple interest

$$i = \frac{C}{A}$$

(b) Find the simple and time weighted interest rate for a deposit W immediately after B



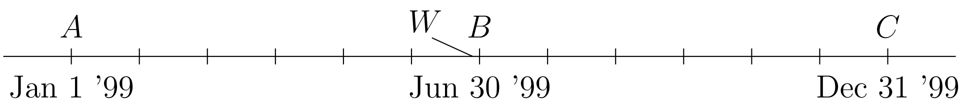
Time weighted interest

$$i = \frac{B-A}{A} \cdot \frac{C-B+W}{B-W} = \left(\frac{B}{A} - 1\right)\left(\frac{C}{B-W} + 1\right)$$

Simple interest

$$i_{1,2}^{(2)} = \frac{-W \pm \sqrt{W^2 + 4AC}}{2A} - 1$$

(c) Find the simple and time weighted interest rate for a deposit W immediately before B



Time weighted interest

$$i = \frac{B-A-W}{A} \cdot \frac{C-B}{B} = \left(\frac{B-W}{A} - 1\right)\left(\frac{C}{B} - 1\right)$$

Simple interest

$$i_{1,2}^{(2)} = \frac{-W \pm \sqrt{W^2 + 4AC}}{2A} - 1$$

(d) The simple rates of interest in (b) and (c) are identical because for simple interest it does not matter when a deposit happens relative to a known account balance as long as the deposit still happens at the same time (which it does).

(e) Show that if W is a deposit ($W > 0$), the time weighted interest in (b) is greater than the one in (c)

$$\left(\frac{B-W}{A} - 1\right)\left(\frac{C}{B} - 1\right) < \left(\frac{B}{A} - 1\right)\left(\frac{C}{B-W} + 1\right)$$

Comparing one factor from each side we see that

$$\frac{B-W}{A} - 1 < \frac{B}{A} - 1$$

because $B - W < B$. Examining the second pair of factors we see

$$\frac{C}{B} - 1 < \frac{C}{B - W} + 1$$

because again $B - W < B$. Thus we may conclude that

$$\left(\frac{B - W}{A} - 1\right)\left(\frac{C}{B} - 1\right) < \left(\frac{B}{A} - 1\right)\left(\frac{C}{B - W} + 1\right)$$