Poisson's and Laplace's Equations

Poisson equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\rho(x, y)$$

Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Discretization of Laplace equation: set $u_{ij} = u(x_i, y_j)$ and $\Delta x = \Delta y = h$

$$\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right)/h^2 = 0$$

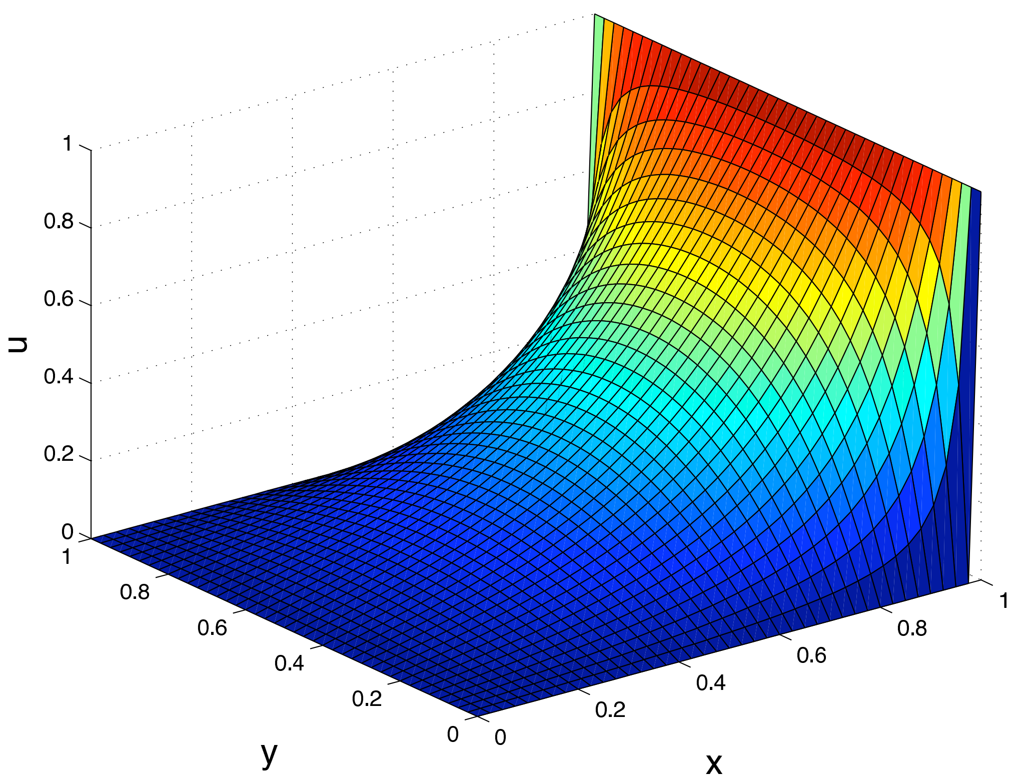
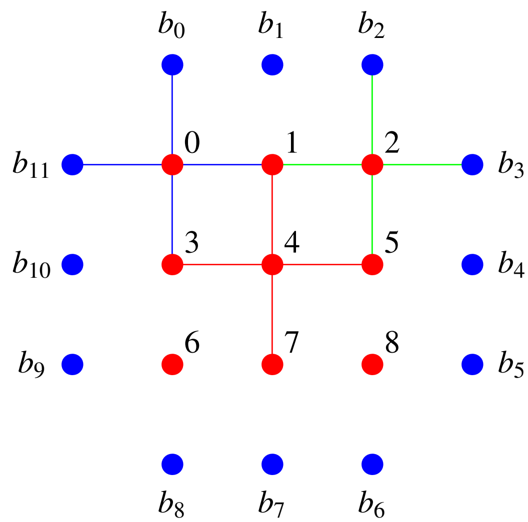


Figure 1: Numerical solution to the model Laplace problem on a 40×40 grid.

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0 \Longrightarrow Au = b$$

$$\textit{iterative method} \ \ u_{ij} \leftarrow \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$



$$Au = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = b = \begin{bmatrix} -b_0 - b_{11} \\ -b_1 \\ -b_2 - b_3 \\ -b_{10} \\ 0 \\ -b_4 \\ -b_8 - b_9 \\ -b_7 \\ -b_6 - b_5 \end{bmatrix}$$

Note: For an $N\Delta x \times N\Delta y$ grid with Dirichlet boundary conditions, A is $(N-1)^2 \times (N-1)^2$ and there are $N-3$ zeros between the diagonals and the fringes. The bandwidth $w = N - 1$.

Laplacian Matrix for $N = 6$

The 25×25 Laplacian matrix for $N = 6$ with zeros represented by white space. Note the 5 nonzero bands. Coincidentally the bandwidth $w = N-1 = 5$ for $N = 6$. Here $d = -4$.

$$
\begin{bmatrix}
d & 1 & & & & 1 \\
1 & d & 1 & & & & 1 \\
& 1 & d & 1 & & & & 1 \\
& & 1 & d & 1 & & & & 1 \\
& & & 1 & d & & & & & 1 \\
1 & & & & & d & 1 & & & & 1 \\
& 1 & & & & 1 & d & 1 & & & & 1 \\
& & 1 & & & & 1 & d & 1 & & & & 1 \\
& & & 1 & & & & 1 & d & 1 & & & & 1 \\
& & & & 1 & & & & 1 & d & & & & & 1 \\
& & & & & 1 & & & & & d & 1 & & & & 1 \\
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& & & & & & & & & & & & & & 1 & & & & 1 & d & & & & & 1 \\
& & & & & & & & & & & & & & & 1 & & & & & d & 1 \\
& & & & & & & & & & & & & & & & 1 & & & & 1 & d & 1 \\
& & & & & & & & & & & & & & & & & 1 & & & & 1 & d & 1 \\
& & & & & & & & & & & & & & & & & & 1 & & & & 1 & d & 1 \\
& & & & & & & & & & & & & & & & & & & 1 & & & & 1 & d
\end{bmatrix}
$$

1D, 2D, and 3D Laplacian Matrices

dimension	grid	n	bands	w	memory	complexity
1D	N	N	3	1	$2N$	$5N$
2D	$N \times N$	N^2	5	N	N^3	N^4
3D	$N \times N \times N$	N^3	7	N^2	N^5	N^7

Table 1: The Laplacian matrix is $n \times n$ in the large N limit, with bandwidth w. The memory required for Gaussian elimination due to fill-in is $\sim nw$. In 3D with $N = 100$, Gaussian elimination requires $\sim$ 80 GB of memory with 8-byte doubles, while for $N = 500$, Gaussian elimination requires $\sim$ 250 TB of memory, which is prohibitive. The complexity (operation count)—measured in flops—scales $\sim w^2 n$.

solver	100×100	200×200	400×400
full-matrix direct	1172	—	—
Jacobi	2.4	78	2005
Gauss-Seidel	2.0	32	540
SOR	0.07	0.45	3.5
banded-matrix direct	0.45	6.8	110

Table 2: Approximate CPU times in sec for the model Laplace problem solved in C (gcc −O) on three grids, using a single core of an Intel Core 2 Quad Processor at 2.66 GHz with 4 GB of memory.

The Iterative Idea

To solve $Au = b$, write

$$Mu^{(k+1)} = (M - A)u^{(k)} + b, \quad k = 0, 1, 2, \ldots$$

Then the error $e^{(k)} \equiv u^{(k)} - u$ satisfies

$$Me^{(k+1)} = (M - A)e^{(k)}, \qquad e^{(k+1)} = Be^{(k)}$$

where the iteration matrix $B = M^{-1}(M - A)$. Now

$$\left\|e^{(k)}\right\| = \left\|B^k e^{(0)}\right\| \sim \rho^k \to 0 \;\text{ iff }\; \rho < 1$$

where ρ = maximum of |eigenvalues of B| is the spectral radius of B. For the model Laplace problem on an $N \times N$ grid with $h = 1/N$,

$$\rho_J = \cos(\pi h) \approx 1 - \frac{\pi^2 h^2}{2}, \quad \rho_{GS} = \rho_J^2 = \cos^2(\pi h) \approx 1 - \pi^2 h^2$$

$$\rho_{SOR} = \frac{1 - \sin \pi h}{1 + \sin \pi h} \approx 1 - 2\pi h.$$

Jacobi Matrix

$$M_J = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

Gauss-Seidel Matrix

$$M_{GS} = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

SOR Matrix

$$M_{SOR} = \begin{bmatrix} -4/\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4/\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4/\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4/\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4/\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4/\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4/\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4/\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4/\omega \end{bmatrix}$$

Iterative Methods for Laplace's Equation

The best way to write the Jacobi, Gauss-Seidel, and SOR methods for Laplace's equation is in terms of the residual defined (at iteration k) by

$$r_{ij}^{(k)} = -4u_{ij}^{(k)} + u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}.$$

In matrix form, the residual (at iteration k) is

$$r^{(k)} = Au^{(k)} - b.$$

Then Jacobi's method is

$$u_{ij}^{(k+1)} = \frac{1}{4}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right) = u_{ij}^{(k)} + \frac{1}{4}r_{ij}^{(k)} \quad \text{(Jacobi)}$$

or in matrix form, with $A = L + D + U$ and $M = D$:

$$Du^{(k+1)} = -(L+U)u^{(k)} + b = Du^{(k)} - r^{(k)}$$

$$B_J = -D^{-1}(L+U)$$

$$u^{(k+1)} = u^{(k)} - D^{-1}r^{(k)} \quad \text{(Jacobi)}.$$

Note that $D^{-1} = -\frac{1}{4}I$. The Gauss-Seidel and SOR methods can be expressed most simply by using the *current* residual $\tilde{r}_{ij}$, where some nearest neighbor values of the solution have already been updated. For example, updating along rows from left to right and top to bottom:

$$\tilde{r}_{ij} = -4u_{ij}^{(k)} + u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k+1)} + u_{i,j-1}^{(k)}$$

or in matrix form:

$$\tilde{r} = Lu^{(k+1)} + (D+U)u^{(k)} - b.$$

Then Gauss-Seidel can be written as

$$u_{ij}^{(k+1)} = u_{ij}^{(k)} + \frac{1}{4}\tilde{r}_{ij} \quad \text{(Gauss-Seidel)}$$

or in matrix form with $M = D + L$:

$$(D+L)u^{(k+1)} = -Uu^{(k)} + b$$

$$B_{GS} = -(L+D)^{-1}U$$

$$Du^{(k+1)} = Du^{(k)} - \left(Lu^{(k+1)} + Du^{(k)} + Uu^{(k)} - b\right) = Du^{(k)} - \tilde{r}$$

$$u^{(k+1)} = u^{(k)} - D^{-1}\tilde{r} \quad \text{(Gauss-Seidel).}$$

SOR is derived by simply over-correcting Gauss-Seidel:

$$u_{ij}^{(k+1)} = u_{ij}^{(k)} + \frac{\omega}{4}\tilde{r}_{ij}, \quad 1 < \omega < 2 \quad \text{(SOR)}$$

or in matrix form with $M = \frac{D}{\omega} + L$ (and working backwards):

$$u^{(k+1)} = u^{(k)} - \omega D^{-1}\tilde{r}^{(k)} \quad \text{(SOR)}$$

$$\frac{D}{\omega}u^{(k+1)} = \frac{D}{\omega}u^{(k)} - \left(Lu^{(k+1)} + Du^{(k)} + Uu^{(k)} - b\right)$$

$$\left(\frac{D}{\omega} + L\right)u^{(k+1)} = \left(\frac{D}{\omega} - D - U\right)u^{(k)} + b.$$

$$B_{SOR} = \left(\frac{D}{\omega} + L\right)^{-1}\left(\frac{D}{\omega} - D - U\right) = (D+\omega L)^{-1}(1-\omega)D - \omega U)$$

solver	100×100	200×200	400×400
Jacobi	35061	145837	605755
Gauss-Seidel	18258	75778	314215
SOR	411	876	1858

Table 3: Number of iterative sweeps for the model Laplace problem on three $N \times N$ grids. For convergence of the iterative methods, $\epsilon = 10^{-5}h^2$. Note that the number of Gauss-Seidel iterations is approximately $\frac{1}{2}$ the number of Jacobi iterations, and that the number of SOR iterations is approximately $\frac{1}{N}$ times the number of Jacobi iterations, as predicted by theory.

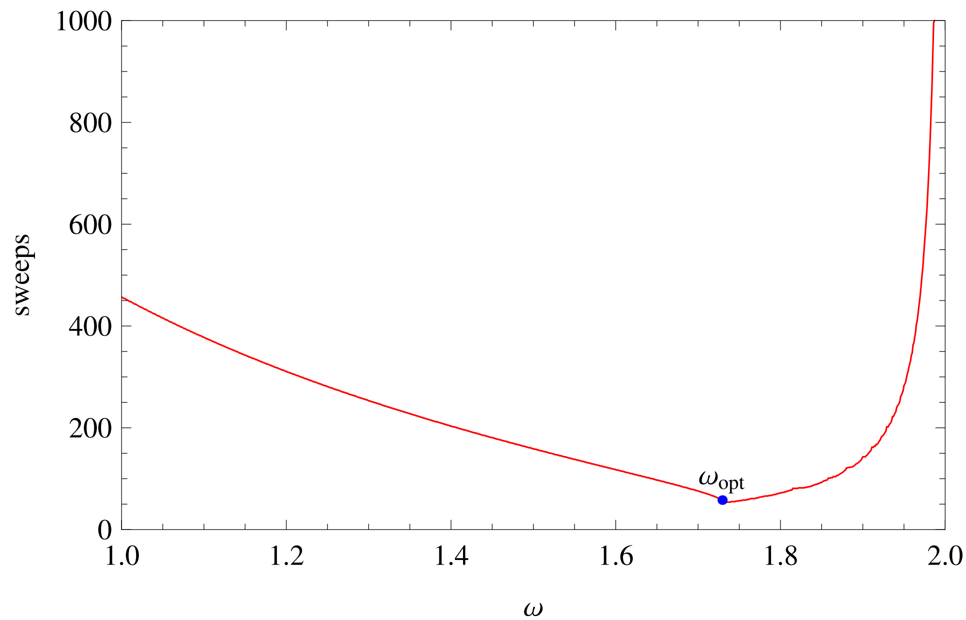

Figure 2: Number of sweeps required for convergence of SOR vs. ω on a 20×20 grid for the model Laplace problem.