ACTUARIAL MATHEMATICS HOMEWORK 3

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1–29.

(a) $i = 0.13 \to \delta = 0.122218$
(b) $d = 0.13 \to \delta = 0.139262$
(c) $i^{(4)} = 0.13 \to i = 0.136476 \to \delta = 0.127932$
(d) $d^{(5)} = 0.13 \to d = 0.123799 \to \delta = 0.132159$

1–30. Find derivatives

$$a(t) = 1 + it - (1+i)^t \qquad (1+i)^t = e^{t\cdot\ln(1+i)}$$
$$\delta = \ln(1+i)$$
$$a'(t) = i - \left[(1+i)^t\right]_t'$$
$$a'(t) = i - \left[e^{t\cdot\ln(1+i)}\right]'$$
$$a'(t) = i - \ln(1+i)e^{t\cdot\ln(1+i)}$$
$$a'(t) = i - \delta e^{t\cdot\delta}$$
$$a''(t) = \left[-\ln(1+i)e^{t\cdot\ln(1+i)}\right]'$$
$$a''(t) = -\left(\ln(1+i)\right)^2 \cdot e^{t\cdot\ln(1+i)}$$
$$a''(t) = -\delta^2 \cdot e^{t\cdot\delta}$$

Find $a'(t) = 0$

$$a'(t_0) = 0$$
$$0 = i - \delta e^{t_0\cdot\delta}$$
$$i = \delta e^{t_0\cdot\delta}$$
$$\ln i = \ln\delta + \ln e^{t_0\cdot\delta}$$
$$\ln i = \ln\delta + t_0\cdot\delta\cdot\ln e$$
$$t_0 = \frac{\ln i - \ln\delta}{\delta} = \frac{1}{\delta}[\ln i - \ln\delta]$$

We found the same extremum as in the exercise. Now we need to show that it is a maximum. For this we need $a''(t_0) < 0$

$$\begin{aligned} a''(t_0) &= -\delta^2 \cdot e^{t_0\cdot\delta} \\ &= -\delta^2 \cdot e^{\frac{1}{\delta}[\ln i - \ln\delta]\cdot\delta} \\ &= -\delta^2 \cdot e^{\ln i - \ln\delta} \\ &= -\delta^2 \cdot (i/\delta) \\ a''(t_0) &= -\delta\cdot i \end{aligned}$$
$$\delta > 0, \quad i > 0$$
$$a''(t_0) < 0 \quad \square$$

Date: September 13, 2020.

1–31.

(a)

$$
\begin{gathered}
\delta = \ln(1+i_0), \qquad e^{\delta} = i_0 + 1 \\
\delta_2 = 2 \cdot \delta \\
\delta_2 = \ln(1+i_1), \qquad e^{\delta_2} = i_1 + 1 \\
i_1 = e^{\delta_2} - 1, \qquad i_0 = e^{\delta} - 1 \\
i_1 > 2 \cdot i_0 \\
e^{\delta_2} - 1 > 2 \cdot \left(e^{\delta} - 1\right) \\
\left(e^{\delta}\right)^2 > 2e^{\delta} - 1 \\
\left(e^{\delta}\right)^2 - 2e^{\delta} + 1 > 0 \\
\left(e^{\delta} - 1\right)^2 > 0 \\
e^{\delta} > 1 \\
\delta > 0
\end{gathered}
$$

(b)

$$
\begin{gathered}
\delta = -\ln(1-d_0), \qquad e^{-\delta} = 1 - d_0 \\
\delta_2 = 2 \cdot \delta \\
\delta_2 = -\ln(1-d_1), \qquad e^{-\delta_2} = 1 - d_1 \\
d_1 = -e^{-\delta_2} + 1, \qquad d_0 = -e^{-\delta} + 1 \\
d_1 < 2 \cdot d_0 \\
-e^{-\delta_2} + 1 < 2 \cdot \left(-e^{-\delta} + 1\right) \\
\left(-e^{-\delta}\right)^2 + 1 < -2e^{-\delta} + 2 \\
0 < \left(e^{-\delta}\right)^2 - 2e^{-\delta} + 1 \\
0 < \left(e^{-\delta} - 1\right)^2 \\
e^{-\delta} < 1 \\
-\delta < 0
\end{gathered}
$$

1–32.

$$\lim_{i\to 0} \frac{i-\delta}{\delta^2} = 0.5$$

$$\lim_{i\to 0} \frac{i-\ln(1+i)}{(\ln(1+i))^2}, \quad \text{we get} \quad \frac{0}{0}$$

applying L'Hopital's Rule we get

$$\lim_{i\to 0} \frac{1-\frac{1}{1+i}}{2\ln(1+i)\frac{1}{1+i}}, \quad \text{we still get} \quad \frac{0}{0}$$

again applying L'Hopital's Rule we get

$$\lim_{i\to 0} \frac{(1+i)^{-2}}{(2\ln(1+i)(1+i)^{-1})'} =$$

$$\lim_{i\to 0} \frac{(1+i)^{-2}}{2((1+i)^{-2}-\ln(1+i)(1+i)^{-2})} =$$

$$\lim_{i\to 0} \frac{(1+i)^{-2}}{2(1+i)^{-2}(1-\ln(1+i))} =$$

$$\lim_{i\to 0} \frac{1}{2(1-\ln(1+i))} = \frac{1}{2-0} = 0.5$$

1–33.

$$\begin{aligned}
\delta_t &= 0.04(1+t)^{-1} \\
\delta_r &= 0.04(1+r)^{-1} \\
a(t) &= e^{\int_0^t \delta_r dr} \\
\ln(a(t)) &= \int_0^t \delta_r dr \\
&= \int_0^t 0.04(1+r)^{-1} dr \\
&= 0.04 \int_0^t (1+r)^{-1} dr \\
&= 0.04\left[\ln(1+r)\right]_0^t \\
&= 0.04\left[\ln(1+t) - \ln(1)\right] \\
\ln(a(t)) &= 0.04\ln(1+t) \\
a(t) &= (1+t)^{0.04}
\end{aligned}$$