NumMeth MAT-410 Lab 1 Report

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Abstract

This report deals with numerical solutions to second order differential equations using finite difference methods. The problems solved using finite difference methods were chosen according to my `readme` file. They are solved numerically using the Central Difference Method and Ilin's Scheme. The function's and numerical solution's behavior for various parameter values are discussed.

Contents

1 Introduction

This report deals with three different differential based on one general system of equations. These equations are solved both analytically and numerically using finite difference methods. The resulting graphs of the analytic solutions and the numerical solutions are then graphed for various parameter values and their behavior is investigated according to the requirements in our task PDF.

1.1 Problem Functions

All problem functions in this report are versions of the following general equation

$$\begin{cases} \varepsilon \cdot u''(x) + a(x) \cdot u'(x) - b(x) \cdot u(x) & = f(x), \quad x \in (0,1) \\ \zeta_0 \cdot u(0) - \eta_0 \cdot \varepsilon \cdot u'(0) & = \phi_0 \\ \zeta_1 \cdot u(1) + \eta_1 \cdot \varepsilon \cdot u'(1) & = \phi_1. \end{cases} \tag{1}$$

Here $\varepsilon > 0$, $a(x), b(x), f(x)$ are functions on the interval $[0;1]$ where $b(x) \geq 0$. The three problems discussed here are versions of this general problem (1). The first equation from my readme file is the following:

$$\begin{cases} \varepsilon \cdot u''(x) + u'(x) & = x^3, \quad x \in (0,1) \\ u(0) & = \phi_0 \\ u(1) & = \phi_1. \end{cases} \tag{2}$$

Here ϕ_0 and ϕ_1 can be freely chosen. The analytic solution to this equation that I found for our first homework assignment is the following:

$$u = (e^{-1/\varepsilon \cdot x} - 1)\frac{\phi_0 - \phi_1 + 1/4 - \varepsilon + 3\varepsilon^2 - 6\varepsilon^3}{1 - e^{-1/\varepsilon}}$$
$$+ \frac{1}{4}x^4 - \varepsilon x^3 + 3\varepsilon^2 x^2 - 6\varepsilon^3 x + \phi_0.$$

The boundary conditions of this problem make it a Dirichlet problem. I will refer to this problem as PROBLEM 1 from now on. The second equation from my readme file is:

$$\begin{cases} \varepsilon \cdot u''(x) + u'(x) & = x^3, \quad x \in (0,1) \\ u(0) - u'(0) & = \phi_0 \\ u(1) & = \phi_1. \end{cases} \tag{3}$$

This equation's analytic solution is:

$$u = \phi_0 + (e^{-1/\varepsilon \cdot x} - 2)\frac{\phi_0 - \phi_1 + 1/4 - \varepsilon + 3\varepsilon^2 - 6\varepsilon^3 - 6\varepsilon^4}{2 - e^{-1/\varepsilon}}$$
$$+ \frac{1}{4}x^4 - \varepsilon x^3 + 3\varepsilon^2 x^2 - 6\varepsilon^3 x - 6\varepsilon^4.$$

Again ϕ_0 and ϕ_1 can be freely chosen. This problem is a Robin problem because it contains a derivate in the boundary conditions. This problem will be referred to as PROBLEM 2. The third problem this report covers is number 3 from our PDF. It has the form:

$$\begin{cases} \varepsilon \cdot u''(x) + \left(3 \cdot (1+x)^2 - \frac{2 \cdot \varepsilon}{1+x}\right) \cdot u'(x) & = \frac{3 \cdot \varepsilon}{2 \cdot (1+x)^2} - \frac{3\dot{(}1+x)}{2}, \quad x \in (0,1) \\ u(0) - \frac{1}{3} \cdot \varepsilon \cdot u'(0) & = \frac{\varepsilon}{6} - \frac{1}{1 - e^{-7/\varepsilon}} \\ u(1) & = 1 - \frac{\ln(2)}{2} \end{cases} \tag{4}$$

and it's analytic solution is

$$u(x) = \frac{1 - e^{\frac{1 - (1+x)^3}{\varepsilon}}}{1 - e^{-7/\varepsilon}} - \frac{\ln(1+x)}{2}.$$

This problem is also a Robin problem and will be referred to as PROBLEM 3.

1.2 Finite Difference Schemes

There are two finite difference schemes that my program is capable of and this report covers. The first scheme is the Central Difference Scheme for which

$$\gamma_i = 1 \quad (i = 1, 2, \ldots, n)$$

and Ilin's Scheme where

$$\gamma_i = R_i \cdot \text{cotanh}(R_i) \quad (i = 1, 2, \ldots, n)$$
$$R_i \equiv \frac{a(x_i) \cdot h}{2 \cdot \varepsilon}$$
$$h = \frac{1}{n-1}.$$

This is the only difference between Ilin's Scheme and the Central Difference Scheme. They both use the Thomas Algorithm with the values described in our tasks to solve the equations (2), (3), and (4) numerically.

2 Analysis

This section will investigate the behavior of the three problem functions (analytical and numerical) for different parameters and how they are approximated by different schemes.

2.1 Behavior as ε Tends to Zero

As ε tends to zero the shape of all three functions becomes more pronounced. What I mean is that when ε is large ($\varepsilon \geq 0.5$), the functions have slowly changing slopes. When ε approaches zero, the functions, have sharp "turns" where their slope drastically changes. FIGURE **??** shows this behavior for (2), FIGURE **??** shows this behavior for (3), and FIGURE **??** shows this behavior for (4). For (2) and (3) we have $\phi_0 = 0.25$ and $\phi_1 = 0.75$.

Here we have a figure showing the general behavior of the error as n increases. The first figure shows the central difference scheme, the second one the Ilin Scheme.

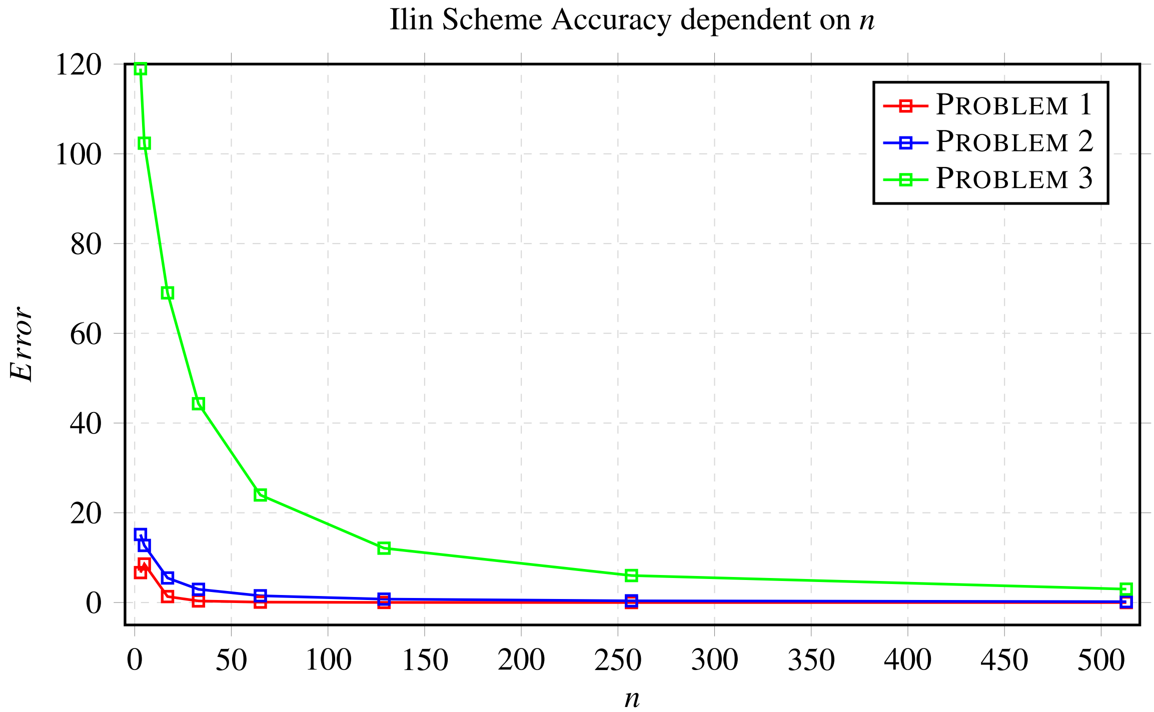

Figure 1: Accuracy of the Ilin Scheme dependent on n. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $\varepsilon = 0.05$

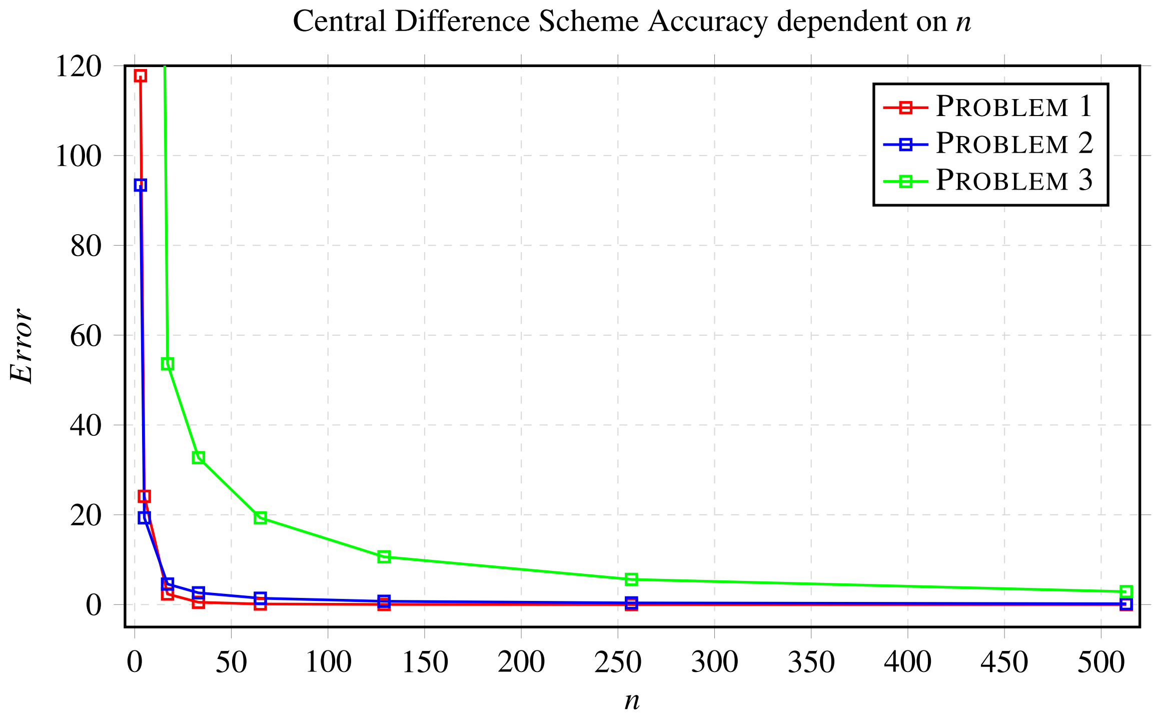

Figure 2: Accuracy of the Central Difference Scheme dependent on n. $\phi_0 = 0.25$, $\phi_1 = 0.75$, $\varepsilon = 0.05$

These graphs show that all three problems have their boundary layers at $x = 0$. This becomes evident when ε decreases because the slope around $x = 0$ increases sharply.

2.2 Accuracy as n Increases

When the number of nodes n increases, the accuracy of the approximation increases too. That means that all functions converge to the analytic solution when n increases. All of these graphs are using the Central Difference Scheme.

In FIGURE **??** the numerical solution behaves as expected and the accuracy increases as n increases and both functions become practically identical. FIGURE **??** and FIGURE **??** tell a different story. In these graphs the boundary at $x = 0$ poses a problem. Both (3) and (4) are Robin problems meaning that their boundary condition involves a derivate of u. Finite Difference Methods are not good at approximating these types of problems without modifications. Even the Ilin Scheme cannot approximate these problems. The general convergence of the numerical solution to the analytical solution holds even in these cases though.

2.3 Effect of ε on Accuracy

Epsilon in general has a negative effect on the accuracy of approximations when it gets smaller. This is caused by the more extreme slope of the functions that makes is harder to successfully approximate them. Especially the Central Difference Scheme (CDS) has difficulties here as it starts to oscillate around the drastic change in slope near the boundary layer. The Ilin Scheme (IS) can successfully avoid these oscillations as the graphs below illustrate.

In both FIGURE **??** and FIGURE **??** the oscillations are relatively small and confined to a small area around the boundary layer. In FIGURE **??** however the oscillations are much larger and, even though they dampen as x increases, they remain for the whole interval. The cause of these oscillations is ε and this can be seen in comparison with earlier graphs like FIGURE **??** where (4) is successfully approximated using CDS while here it does not work.

2.4 Accuracy of Difference Schemes

The Ilin Scheme is a lot more accurate than the Central Difference Scheme, for all numbers of nodes. They both converge for large numbers of n, but because the Ilin Scheme uses exponential functions (hyperbolic cotangent) instead of a simple central difference. This advantage the Ilin Scheme has has already been visible in the previous section, especially FIGURE **??**. The following graphs will further illustrate this point.

The Ilin Scheme successfully dampens the oscillations that the Central Difference Scheme experiences and thus increases the accuracy of the approximation for small numbers of nodes n.

3 Conclusion

To conclude, both the Central Difference Scheme and the Ilin Scheme approximate problems (2), (3), and (4) well. When the number of nodes is increased, the accuracy of the approximation increases, too. If the parameter ε is decreased, the function become harder to approximate because of a sharper change in slope and a very high slope at the boundary layer. A small ε can also cause oscillations in the Central Difference Scheme which decreases the accuracy of the approximation significantly. The Ilin Scheme is less prone to oscillations and thus more accurate. Neither scheme is capable of approximating boundary layers with Robin conditions because the finite difference scheme we are using is not made to approximate these problems.