4.4: LAPLACE'S EQUATION IN CIRCULAR REGIONS

KIAM HEONG KWA

1. The Laplacian in Polar Coordinates

The differential operator

$$\Delta = \nabla^2 \tag{1.1}$$

is usually called the **Laplacian**. It carries a function u to the divergence of the gradient of u :

$$u \xrightarrow{\nabla} \nabla u \xrightarrow{\nabla \cdot} \nabla \cdot \nabla u = \nabla^2 u = \Delta u. \tag{1.2}$$

A solution to the **Laplace's equation**

$$\Delta u = 0 \tag{1.3}$$

is simply a function whose image vanishes under the action of the Laplacian Δ.

In two dimensions and in rectangular coordinates x and y,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.4}$$

In polar coordinates r and θ,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{1.5}$$

where

$$\begin{aligned} x &= r\cos\theta, & r^2 &= x^2 + y^2, \\ y &= r\sin\theta, & \tan\theta &= \frac{y}{x}. \end{aligned} \tag{1.6}$$

The equivalence of (1.4) and (1.5) can be derived using the chain rule for differentiation and (1.6). Interested readers are referred to section 4.1 of the text.

Date: May 9, 2011.

2. Laplace's Equation in Circular Regions

Recall that the steady-state (or time independent) solution u in a two dimension heat conduction problem is defined by the condition that $\Delta u = 0$ together with some given boundary conditions. Over a circular region centered at the origin, this can be made explicit using the polar coordinates:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ 0 < r < a,\ 0 < \theta < 2\pi, \tag{2.1}$$

with

$$u(a,\theta) = f(\theta),\ 0 < \theta < 2\pi. \tag{2.2}$$

These equations describe a **Dirichlet problem** over a circular disk of radius a centered at the origin. Note that *f is necessarily 2π-periodic in θ and so is any solution u of* (2.1) *and* (2.2)*.*

This boundary value problem can be solved using the method of separation of variables also.

Step 1: Separating Variables in (2.1)**.** We begin by searching for nonzero product solutions

$$u(r,\theta) = R(r)\Theta(\theta), \tag{2.3}$$

where R is a function of r alone and Θ is a function of θ alone. Recall that *u is necessarily 2π-periodic in θ and hence so is Θ.*

Plugging (2.3) into (2.1) yields

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

Then multiplying this equation with $\dfrac{r^2}{R\Theta}$ gives

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

Rearranging this equation separates the variables:

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda. \tag{2.4}$$

Here λ is a separation constant. Equivalently, we get

$$r^2R'' + rR' - \lambda R = 0, \tag{2.5a}$$

$$\Theta'' + \lambda\Theta = 0. \tag{2.5b}$$

Step 2: Solving the Separated Equations (2.5a) **and** (2.5b)**.** We first solve (2.5b) under the stipulation of the **periodic boundary condition**; that is, Θ is 2π-periodic.

Suppose $\lambda < 0$, so that $\lambda = -\mu^2$, where $\mu = \sqrt{-\lambda} > 0$. Then $\Theta'' - \mu^2\Theta = 0$, from which it follows that $\Theta(\theta) = c_\mu e^{\mu\theta} + d_\mu e^{-\mu\theta}$, where c_μ and d_μ are integration constants. The periodic boundary condition implies that $c_\mu + d_\mu = \Theta(0) = \Theta(2k\pi) = c_\mu e^{2k\mu\pi} + d_\mu e^{-2k\mu\pi}$ for all $k \in \mathbb{Z}$. It follows that $c_\mu + d_\mu = \lim_{k\to\infty}\left(c_\mu e^{2k\mu\pi} + d_\mu e^{-2k\mu\pi}\right) = \begin{cases} \infty & \text{if } c_\mu > 0, \\ 0 & \text{if } c_\mu = 0, \\ -\infty & \text{if } c_\mu < 0. \end{cases}$ This would have contradicted the fact that $c_\mu + d_\mu$ is bounded unless $c_\mu = 0$, from which it also follows that $c_\mu + d_\mu = 0$, so that $d_\mu = 0$. Hence if $\lambda < 0$, the only solution of (2.5b) that satisfies the periodic boundary condition is the trivial solution $\Theta \equiv 0$.

Suppose $\lambda = 0$. Then $\Theta'' = 0$, from which it follows that $\Theta(\theta) = a_0 + b_0\theta$, where a_0 and b_0 are integration constants. The periodic boundary condition implies that $a_0 = \Theta(0) = \Theta(2k\pi) = a_0 + 2kb_0\pi$ for all $k \in \mathbb{Z}$. This holds if and only if $b_0 = 0$, while a_0 being arbitrary. Hence if $\lambda = 0$, then $\Theta(\theta) = a_0$ is a solution of (2.5b) that satisfies the periodic boundary condition for arbitrary value of a_0.

Finally, suppose $\lambda > 0$, so that $\lambda = \mu^2$, where $\mu = \sqrt{\lambda} > 0$. Then $\Theta'' + \mu^2\Theta = 0$, from which it follows that $\Theta(\theta) = a_\mu \cos\mu\theta + b_\mu \sin\mu\theta$, where a_μ and b_μ are integration constants. This shows that the least nonnegative period of Θ is $\dfrac{2\pi}{\mu}$ unless $\Theta \equiv 0$. Thus the periodic boundary condition implies that if $\Theta \neq 0$, then $\dfrac{2\pi}{\mu}n = 2\pi$ or $\mu = n$ for some $n \in \mathbb{N}$.

To conclude, the only values of λ for which (2.5b) admits nontrivial 2π-periodic solutions are

$$\lambda = \lambda_n = n^2,\ n = 0, 1, 2, \cdots, \tag{2.6}$$

and for each such λ-value, the nontrivial 2π-periodic solutions are constant multiples of

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \tag{2.7}$$

where a_n and b_n are integration constants.

Next, we solve (2.5a) for nontrivial solutions with $\lambda = \lambda_n = n^2$, $n = 0, 1, 2, \cdots$:

$$r^2 R'' + rR' - n^2 R = 0. \tag{2.8}$$

This is in fact an **Euler equation**. The change of variables $r = ae^z$ transforms it to the equation

$$\frac{d^2 R}{dz^2} - n^2 R = 0, \tag{2.9}$$

from which it follows that

$$R(z) = \begin{cases} c_1 + c_2 z & \text{if } n = 0, \\ c_1 e^{nz} + c_2 e^{-nz} & \text{if } n = 1, 2, \cdots, \end{cases} \tag{2.10}$$

where c_1 and c_2 are integration constants.

Exercise 1. *Show that the change of variables $r = ae^z$ transforms* (2.8) *into* (2.9)*.*

In terms of r, we have

$$R(r) = \begin{cases} c_1 + c_2 \ln \dfrac{r}{a} & \text{if } n = 0, \\ c_1 \left(\dfrac{r}{a}\right)^n + c_2 \left(\dfrac{r}{a}\right)^{-n} & \text{if } n = 1, 2, \cdots. \end{cases} \tag{2.11}$$

Note that $\lim_{r \to 0+} |R(r)| = \infty$ unless $c_2 = 0$. For boundedness, we set $c_2 = 0$. By setting $c_1 = 1$ and $c_2 = 0$, we get

$$R_n(r) = \begin{cases} 1 & \text{if } n = 0, \\ \left(\dfrac{r}{a}\right)^n & \text{if } n = 1, 2, \cdots, \end{cases} \tag{2.12}$$

as the bounded solutions of (2.5a) when $\lambda = \lambda_n = n^2$, $n = 0, 1, 2, \cdots$.

Combining (2.7) and (2.12) yields the nontrivial product solutions

$$u_0(r, \theta) = a_0 \text{ and } u_n(r, \theta) = \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \ n = 1, 2, \cdots, \tag{2.13}$$

of (2.1).

Step 3: Fourier Series Solution of the Entire Problem. The series solution

$$u(r,\theta)=\sum_{n=0}^{\infty}u_n(r,\theta)=a_0+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(a_n\cos n\theta+b_n\sin n\theta)\,. \tag{2.14}$$

of (2.1) and (2.2) is obtained by superposing the product solutions (2.13) and requiring that

$$f(\theta)=u(a,\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\,.$$

Recognizing this as the Fourier series of f, we set

$$\begin{aligned} a_0&=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta,\\ a_n&=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta,\\ b_n&=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta\,d\theta \text{ for } n=1,2,\cdots\,. \end{aligned} \tag{2.15}$$

Example 1 (Exercise 4.4.2 in the text)**.** *We solve* (2.1) *and* (2.2) *with* $a=1$ *(the radius of the unit disk) and* $f(\theta)=\sin 2\theta$*. By* (2.14) *and* (2.15)*, we have*

$$u(r,\theta)=a_0+\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)\,,$$

where

$$\begin{aligned} a_0&=\frac{1}{2\pi}\int_0^{2\pi}\sin 2\theta\,d\theta=0,\\ a_n&=\frac{1}{\pi}\int_0^{2\pi}\sin 2\theta\cos n\theta\,d\theta=0,\\ b_n&=\frac{1}{\pi}\int_0^{2\pi}\sin 2\theta\sin n\theta\,d\theta\\ &=\begin{cases}1 & \text{if } n=2,\\ 0 & \text{if } n\neq 2\end{cases} \quad \text{for } n=1,2,\cdots\,. \end{aligned}$$

Note that we have used the orthogonality of the cosine and sine functions in the above calculations. Alternatively, recall that we only need *to make sure that*

$$\sin 2\theta = f(\theta) = u(1,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Hence

$$u(r,\theta) = r^2 \sin 2\theta = 2r\cos\theta \cdot r\sin\theta = 2xy.$$

Example 2 (Exercise 4.4.4 in the text)**.** *We solve* (2.1) *and* (2.2) *with* $a = 1$ *(the radius of the unit disk) and* $f(\theta) = \begin{cases} \pi - \theta & \text{if } 0 \le \theta \le \pi, \\ 0 & \text{if } \pi \le \theta < 2\pi. \end{cases}$ *By* (2.14) *and* (2.15)*, we have*

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = 0 \\
&= \frac{1}{2\pi}\int_0^{\pi} (\pi-\theta)\,d\theta + \frac{1}{2\pi}\int_\pi^{2\pi} 0\,d\theta \\
&= \frac{\pi}{4}, \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = 0, \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi-\theta)\cos n\theta\,d\theta + \frac{1}{\pi}\int_\pi^{2\pi} 0\cdot\cos n\theta\,d\theta \\
&= \frac{1}{\pi}\left[(\pi-\theta)\frac{\sin n\theta}{n} - \frac{\cos n\theta}{n^2}\right]_0^{\pi} \\
&= \frac{1-(-1)^n}{\pi n^2}, \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi-\theta)\sin n\theta\,d\theta + \frac{1}{\pi}\int_\pi^{2\pi} 0\cdot\sin n\theta\,d\theta \\
&= \frac{1}{\pi}\left[(\pi-\theta)\cdot -\frac{\cos n\theta}{n} - \frac{\sin n\theta}{n^2}\right]_0^{\pi} \\
&= \frac{1}{n} \textit{ for } n = 1, 2, \cdots.
\end{aligned}$$

Hence

$$u(r,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} r^n \left[\frac{1-(-1)^n}{\pi n^2} \cos n\theta + \frac{1}{n} \sin n\theta \right].$$

3. Laplace's Equation: Varying the Region and the Boundary Conditions

Other than circular disks, the methods in the previous section can be applied to solve Laplace's equation over planar regions conveniently described in polar coordinates. In addition, other than Dirichlet boundary conditions, Neumann or Robin boundary conditions may also be present. We shall illustrate this point of view with various examples.

Example 3 (Exercise 4.4.14)**.** *Consider the boundary value problem in the wedge* $0 < r < 1$, $0 < \theta < \dfrac{\pi}{2}$*:*

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{3.1}$$

with the boundary conditions

(3.2)

$$u(r,0) = u\left(r, \frac{\pi}{2}\right) = 0 \text{ for } 0 < r < 1 \text{ and } \frac{\partial u}{\partial r}(\theta, 1) = \theta \text{ for } 0 < \theta < \frac{\pi}{2}.$$

A boundary condition such as $\dfrac{\partial u}{\partial r}(\theta,1) = \theta$ *that specifies the normal derivative of* u *on the boundary is called a* **Neumann condition***.*

Step 1: Separating the Variables in (3.1)**.** *As in the previous section, by searching for nonzero product solutions* $u(r,\theta) = R(r)\Theta(\theta)$*, we obtain the separated equations*

$$r^2 R'' + rR' - \lambda R = 0, \tag{3.3a}$$

$$\Theta'' + \lambda\Theta = 0, \tag{3.3b}$$

where λ *is a separation constant.*

Step 2: Solving the Separated Equations (3.3a) **and** (3.3b)**.** *The homogeneous boundary conditions* $u(r,0) = u\left(r, \dfrac{\pi}{2}\right) = 0$ *in* (3.2) *together with the nontriviality of product solutions imply that* $\Theta(0) = \Theta\left(\dfrac{\pi}{2}\right) = 0$*. These boundary conditions of* Θ *imply that the only* λ*-values for which* (3.3b) *admits nontrivial solutions are*

$$\lambda = \lambda_n = 4n^2, \; n = 1, 2, \cdots . \tag{3.4}$$

For each of these λ*-values, say* $\lambda_n = 4n^2$*, every nontrivial solution of* (3.3b) *is of the form*

$$\Theta_n(\theta) = b_n \sin 2n\theta, \tag{3.5}$$

where b_n *is an integration constant.*

Next, we search for nontrivial solutions of (3.3a) *with* $\lambda = \lambda_n = 4n^2$, $n = 1, 2, \cdots$*:*

$$r^2 R'' + rR' - 4n^2 R = 0. \tag{3.6}$$

Recall that this is an Euler equation and the change of variables $r = e^z$ *converts it to the equation*

$$\frac{d^2 R}{dz^2} - 4n^2 R = 0.$$

The general solution of (3.6) *is then*

$$R(r) = c_1 r^{2n} + c_2 r^{-2n},$$

where c_1 *and* c_2 *are integration constants. Note that* $\lim_{r\to 0+} |R(r)| = \infty$ *unless* $c_2 = 0$*. For boundedness, we set* $c_2 = 0$*. By setting* $c_1 = 1$ *and* $c_2 = 0$*, we get*

$$R_n(r) = r^{2n} \tag{3.7}$$

as the bounded solutions of (3.3a) *when* $\lambda = \lambda_n = 4n^2$*,* $n = 1, 2, \cdots$*.*

Combining (3.5) *and* (3.7) *yields the nontrivial product solutions*

$$u_n(r,\theta) = b_n r^{2n} \sin 2n\theta, \; n = 1, 2, \cdots , \tag{3.8}$$

of (3.1) *that satisfy the homogeneous boundary conditions* $u(r,0) = u\left(r, \dfrac{\pi}{2}\right) = 0$ *in* (3.2)*.*

Step 3: Fourier Series Solution of the Entire Problem. *The series solution of* (3.1) *and* (3.2) *is obtained by superposing the product solutions* (3.8)*:*

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin 2n\theta. \tag{3.9}$$

The coefficients b_n *are chosen to satisfy the boundary conditions in* (3.2)*:*

$$\theta = \frac{\partial u}{\partial r}(\theta, 1) = \sum_{n=1}^{\infty} 2nb_n \sin 2n\theta.$$

This can be achieved by taking the series on the right side as the Fourier sine series of θ*, so that*

$$\begin{aligned} 2nb_n &= \frac{2}{\pi/2}\int_0^{\pi/2} \theta \sin 2n\theta \, d\theta \\ &= \frac{4}{\pi}\left[\theta \cdot -\frac{\cos 2n\theta}{2n} + \frac{\sin 2n\theta}{4n^2}\right]_0^{\pi/2} \\ &= \frac{2(-1)^{n+1}}{n}, \end{aligned}$$

so that

$$b_n = \frac{(-1)^{n+1}}{n^2} \text{ for } n = 1, 2, \cdots .$$

Hence

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} r^{2n} \sin 2n\theta. \tag{3.10}$$

Example 4 (Exercise 4.4.18 in the text)**.** *We will solve* (2.1) *on the unit disk (*$a = 1$*) with the* **Robin boundary condition**

$$\frac{\partial u}{\partial r}(1,\theta) + 2u(1,\theta) = 100 - 2\cos 2\theta \text{ for } 0 < \theta < 2\pi. \tag{3.11}$$

Many parts of the computation is similar to what we have done in the previous section.

Steps 1 and 2: Separating the Variables in (2.1) **and Solving the Separated Equations.** *As in the previous section, by searching for nonzero product solutions* $u(r,\theta) = R(r)\Theta(\theta)$*, we obtain the separated equations* (2.5a) *and* (2.5b)*. It is also natural to impose the periodic boundary condition that* u *is* 2π*-periodic in* θ *and thus so is* Θ*. Then a verbatim line of argument leads to the nontrivial product solutions* (2.13)*.*

Step 3: Fourier Series Solution of the Entire Problem. *The series solution of* (2.1) *and* (3.11) *is obtained by superposing the product solutions* (2.13)*, so that*

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right) \tag{3.12}$$

and

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} \left(a_n \cos n\theta + b_n \sin n\theta\right). \tag{3.13}$$

The coefficients a_n *and* b_n *are chosen such that*

$$\begin{aligned} 100 - 2\cos 2\theta &= \frac{\partial u}{\partial r}(1,\theta) + 2u(1,\theta) \\ &= \sum_{n=1}^{\infty} n \left(a_n \cos n\theta + b_n \sin n\theta\right) \\ &\qquad + 2a_0 + \sum_{n=1}^{\infty} 2 \left(a_n \cos n\theta + b_n \sin n\theta\right) \\ &= 2a_0 + \sum_{n=1}^{\infty} (n+2) \left(a_n \cos n\theta + b_n \sin n\theta\right). \end{aligned} \tag{3.14}$$

By the orthogonality of the cosine and sine functions, it is readily seen that

$$2a_0 = 100,\ (n+2)a_n = \begin{cases} -2 & \text{if } n = 2, \\ 0 & \text{if } n \neq 2, \end{cases} \text{ and } (n+2)b_n = 0 \text{ for } n = 1, 2, \cdots.$$

Hence

$$a_n = \begin{cases} 50 & \text{if } n = 0, \\ -\dfrac{1}{2} & \text{if } n = 2, \\ 0 & \text{if } n \neq 0, 2, \end{cases} \qquad \text{and } b_n = 0 \text{ for } n = 1, 2, \cdots.$$

Hence

$$u(r,\theta) = 50 - \frac{r^2}{2}\cos 2\theta.$$

As a remark, it is worth noting that the coefficients a_n *and* b_n *can be calculated by taking the series at the end of* (3.14) *as the Fourier series of* $100 - 2\cos 2\theta$.