ACTUARIAL MATHEMATICS HOMEWORK 4

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1. IAA Problems

Ex 2–1.

$$
\begin{aligned}
1+i &= \left[1+\frac{i^{(4)}}{4}\right]^4 \\
1+i &= \left[1+\frac{0.05}{4}\right]^4 \\
1+i &= (1.0125)^4 \\
i &= (1.0125)^4 - 1 \\
i &= 0.050945
\end{aligned}
$$

Ex 2–2.

$$
\begin{aligned}
\left(1+\frac{0.05}{12}\right)^{12t} &= \left(1+\frac{i}{365}\right)^{365t} \\
12t\ln\left(1+\frac{0.05}{12}\right) &= 365t\ln\left(1+\frac{i}{365}\right) \\
\frac{12t}{365t}\ln\left(1+\frac{0.05}{12}\right) &= \ln\left(1+\frac{i}{365}\right) \\
\frac{12}{365}\ln\left(1+\frac{0.05}{12}\right) &= \ln\left(1+\frac{i}{365}\right) \\
\exp\left(\frac{12}{365}\ln\left(1+\frac{0.05}{12}\right)\right) &= 1+\frac{i}{365} \\
\left(1+\frac{0.05}{12}\right)^{12/365} - 1 &= \frac{i}{365} \\
i &= 365\cdot\left(1+\frac{0.05}{12}\right)^{12/365} - 365 \\
i^{(365)} &= 0.0498995 \\
i &= 0.05116
\end{aligned}
$$

Ex 2–3.

$$(0.05)^{12} = 2.44141\cdot 10^{-16}$$

Date: September 21, 2020.

Ex 2–4.

$$\begin{aligned}
1+i &= \left[1+\frac{i^{(m)}}{m}\right]^m \\
i^{(m)} &= m\left(\sqrt[m]{1+i}-1\right) \\
\lim_{m\to\infty} i^{(m)} &= \lim_{m\to\infty} m\left(\sqrt[m]{1+i}-1\right) \\
\lim_{m\to\infty} i^{(m)} &= \lim_{m\to\infty} \frac{\sqrt[m]{1+i}-1}{\frac{1}{m}} \\
&\text{we apply L'Hopital's rule and cancel} \\
&= \lim_{m\to\infty}\left[(1+i)^{1/m}\cdot\ln(1+i)\right] \\
&= \ln(1+i)
\end{aligned}$$

Ex 2–5.

$$\begin{aligned}
\delta &= \ln(1+i) \\
i &= \left[1+\frac{0.05}{365}\right]^{365}-1 \\
\delta &= \ln\left[1+\frac{0.05}{365}\right]^{365} \\
\delta &= 365\cdot\ln\left(1+\frac{0.05}{365}\right) \\
\delta &= 0.0499966
\end{aligned}$$

Ex 2–7.

$$1+i = \left[1+\frac{i^{(m)}}{m}\right]^m$$

$$1-d = \left[1-\frac{d^{(m)}}{m}\right]^m$$

$$\left[1+\frac{i^{(m)}}{m}\right]^m = \left[1-\frac{d^{(n)}}{n}\right]^{-n}$$

Ex 2–8.

$$\begin{aligned}
i &= \frac{d}{1-d} \\
i &= \frac{0.05}{1-0.05} \\
i &= 0.05263
\end{aligned}$$

P 2–3.

$$\begin{aligned}
\left[1+\frac{i^{(m)}}{m}\right]^{m} &= \left[1-\frac{d^{(n)}}{n}\right]^{-n} \\
\left[1+\frac{i^{(1/4)}}{1/4}\right]^{1/4} &= \left[1-\frac{d^{(4)}}{4}\right]^{-4} \\
1+\frac{i^{(1/4)}}{1/4} &= \left[1-\frac{d^{(4)}}{4}\right]^{-16} \\
\frac{i^{(1/4)}}{1/4} &= \left[1-\frac{d^{(4)}}{4}\right]^{-16} - 1 \\
i^{(1/4)} &= 4\left[1-\frac{d^{(4)}}{4}\right]^{-16} - 4
\end{aligned}$$

P 2–4.

- $A(t)$ at t is the amount of money in the account
- $A'(t)$ is the rate of change of $A(t)$, how it changes over time – how the amount of money changes over time
- $A'(t) = \delta(t)A(t)$ because $A'(t)$ involves some type of logarithm of a time dependent function of i and so does $\delta(t)$

P 2–5. Initial investment of \$1,000 accumulating with $\delta = \frac{1}{1+t}$. We can find the accumulation function by integration.

$$\begin{aligned}
a(t) &= e^{\int_0^t \delta_r dr} \\
\ln(a(t)) &= \int_0^t \frac{1}{1+r} dr \\
&= [\ln(1+r)]_0^t \\
&= \ln(1+t) + \ln(1) \\
\ln(a(t)) &= \ln(1+t) \\
a(t) &= 1+t \\
a(5) &= 1+5 = 6
\end{aligned}$$

This means that out \$1,000 accumulate to \$6,000 over the course of 5 years.

P 2–6. For simple interest $a(t) = 1 + it$, to find δ_t that provides an equivalent return we have to

$$\begin{aligned}
\delta_t &= \frac{a'(t)}{a(t)} \\
&= \frac{(1+it)'}{1+it} \\
\delta_t &= \frac{i}{1+it}
\end{aligned}$$

P 2–7.

$$\begin{aligned} d &= 1 - v \\ &= 1 - \frac{1}{1+i} \\ &= \frac{1+i}{1+i} - \frac{1}{1+i} \\ d &= \frac{i}{1+i} \quad \square \end{aligned}$$

There is a similar equation

$$\begin{aligned} d &= 1 - v \\ v &= 1 - d \\ 1 - d &= \left[1 - \frac{d^{(m)}}{m}\right] \\ v &= 1 - d = \left[1 - \frac{d^{(m)}}{m}\right] \end{aligned}$$

P 2–8.

$$\begin{aligned} d &= iv \\ d &= i \cdot \frac{1}{1+i} \\ d &= \frac{i}{1+i} \quad \square \end{aligned}$$

Are there similar equations with $i^{(m)}$ and $d^{(m)}$? Not really, only in the case where $d = i = 0$ and $v = 1$

$$\begin{aligned} d^{(m)} &= i^{(m)} v \\ 1 - d &= (1+i)v \\ 1 - d &= (1+i)\frac{1}{1+i} \\ 1 - d &= 1 \\ d &= 0 \end{aligned}$$

P 2–9. Both compound interest and simple interest are strictly increasing functions, meaning their slopes are always positive (for all t).

Simple interest:

$$\begin{aligned} a_s(t) &= 1 + it \\ a_s'(t) &= i > 0 \end{aligned}$$

Compound interest:

$$\begin{aligned} a_c(t) &= (1+i)^t = e^{t \ln(1+i)} \\ a_c'(t) &= \ln(1+i)(1+i)^t > 0 \end{aligned}$$

The functions only intersect at $t = 0$ and $t = 1$.

Intersect at $t = 0$

$$\begin{aligned} a_c(0) &= a_s(0) \\ 1 &= 1 \end{aligned}$$

Intersect at $t = 1$

$$\begin{aligned} a_c(1) &= a_s(1) \\ 1 + i &= 1 + i \end{aligned}$$

Simple interest forms a straight line while compound interest is concave upwards. Because the slope for simple interest is constant, compound interest will at some point grow quicker than simple interest.

The concavity combined with the slope tell us that for $0 < t < 1$ $a_c(t) < a_s(t)$ and for $1 < t$ $a_c(t) > a_s(t)$

P 2–10.

(a)

$$\begin{aligned} \frac{d}{di} d &= \frac{d}{di} \frac{i}{1+i} \\ \left[i(1+i)^{-1}\right]' &= i'(1+i)^{-1} + i\left[(1+i)^{-1}\right]' \\ &= (1+i)^{-1} + i \cdot -(1+i)^{-2} \\ &= \frac{1}{(1+i)^{-2}} \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dv} \delta &= \frac{d}{dv} \ln(1+i) \\ 1 + i &= (1-d)^{-1} \\ &= \frac{d}{dv} \ln\left((1-d)^{-1}\right) \\ v &= 1 - d \\ &= \frac{d}{dv} \ln\left(v^{-1}\right) \\ &= -v^{-2} \cdot v \\ &= -\frac{1}{v} \end{aligned}$$

2. Parmenter p. 36

2–1. Draw a timeline:

$+7,000$						$+4,000$				X
0	1	2	3	4	5	6	7	8	9	10
			$-5,000$		$-3,000$					

$i = 0.06$, how much is in the account after 10 years?

$$X = 7,000(1.06)^{10} - 5,000(1.06)^7 - 3,000(1.06)^5 + 4,000(1.06)^4$$
$$X = 6,053.01369$$

2–2. Draw a timeline:

$+2,000$	$+3,000$			$-4,000$				X
'97	'98	'99	'00	'01	'02	'03	'04	'05
0	1	2	3	4	5	6	7	8

$i = 0.13$, how much is owed in 2005?

$$X = 2,000(1.13)^8 + 3,000(1.13)^7 - 4,000(1.13)^4$$
$$X = 5,852.81039$$

2–3. Draw a timeline:

X	-200	-500	-500	-700
0	1	2	3	4

$i = 0.12$, at these payments, how much money can be borrowed?

$$X(1.12)^4 - 200(1.12)^3 - 500(1.12)^2 - 500(1.12) - 700 = 0$$
$$X = \frac{200}{1.12} + \frac{500}{(1.12)^2} + \frac{500}{(1.12)^3} + \frac{700}{(1.12)^4}$$
$$X = 1,377.92115$$

2–4. Draw a timeline:

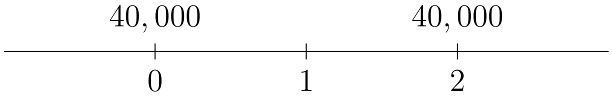
X		800	500			700
0	1	2	3	4	5	6

$i = 0.13$, at what time would a single payment of 2,100 be equivalent to the ones above?

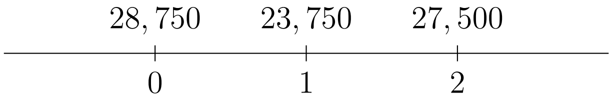
$$2,100(1.13)^t = 800(1.13)^4 + 500(1.13)^3 + 700$$
$$t = \frac{\ln\left(800(1.13)^4 + 500(1.13)^3 + 700\right) - \ln(2,100)}{\ln(1.13)}$$
$$t = 2.13418$$

This result is the time from $t = 6$ backwards, so to find the time from the origin, we subtract our t from 6, getting $t = 3.86582$.

2–5. Draw timelines:



OR

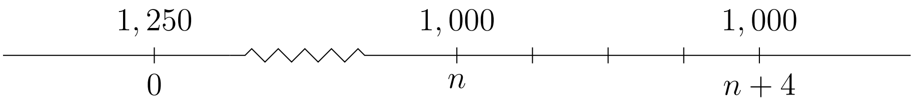


For which i are these two payment plans equivalent?

$$\begin{aligned}
40,000 + 40,000(1+i)^2 &= 28,750(1+i)^2 + 23,750(1+i) + 27,500 \\
1 + i &= c \\
0 &= 11,250c^2 + 23,750c + 12,500 \\
0 &= 9c^2 - 19c + 10 \\
c_1 = 1 \qquad c_2 &= \frac{10}{9} \\
i_1 = 0 \qquad i_2 &= \frac{1}{9}
\end{aligned}$$

2–6.

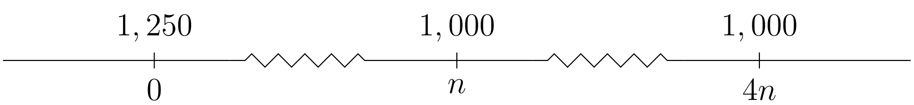
(a) draw a timeline



$i = 0.08$, find n if 1,250 is the present value

$$\begin{aligned}
1,000 + 1,000(1,08)^4 &= 1,250(1,08)^{n+4} \\
1,000(1 + 1.08^4) &= 1,250(1,08)^{n+4} \\
1.08^{n+4} &= \frac{1,000(1 + 1.08^4)}{1,250} \\
n + 4 &= \frac{\ln(1,000(1 + 1.08^4)) - \ln(1,250)}{\ln(1.08)} \\
n &= 8.26035 - 4 \\
n &= 4.26035
\end{aligned}$$

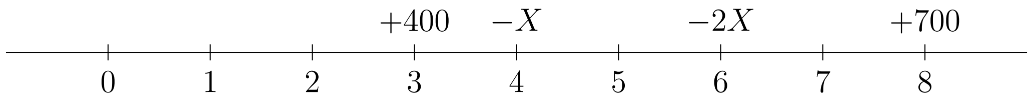
(b) draw a timeline



if $i = 0.08$ and 1,250 the present value, find n

$$
\begin{aligned}
1,000 + 1,000(1.08)^{3n} &= 1,250(1.08)^{4n} \\
0 &= 1.25(1.08)^{4n} - (1.08)^{3n} - 1 \\
0 &= 1.25\left((1.08)^n\right)^4 - \left((1.08)^n\right)^3 - 1 \\
u &= (1.08)^n \\
0 &= 1.25u^4 - u^3 - 1 \\
u_1 < 0, \quad u_2, u_3 \text{ are complex} \quad u_4 &= 1.22995467 \\
n &= \frac{\ln(u_4)}{\ln(1.08)} \\
n &= \frac{\ln(1.22995467)}{\ln(1.08)} \\
n &= 2.6893
\end{aligned}
$$

2–7. Draw timeline:



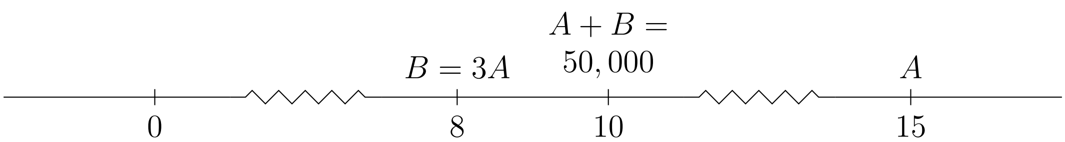
if $i = 0.14$, find X

$$
\begin{aligned}
400(1.14)^5 + 700 &= X(1.14)^4 + 2X(1.14)^2 \\
X &= \frac{400(1.14)^5 + 700}{(1.14)^4 + 2(1.14)^2} \\
X &= 342.843
\end{aligned}
$$

2–8. At what point are 1,000 at $i = 0.12$ equal to twice 1,000 at $i = 0.09$?

$$
\begin{aligned}
1,000(1.12)^t &= 2 \cdot 1000(1.08)^t \\
t\ln(1.12) &= \ln(2) + t\ln(1.08) \\
t &= \frac{\ln(2)}{\ln(1.12) - \ln(1.08)} \\
t &= 19.05945
\end{aligned}
$$

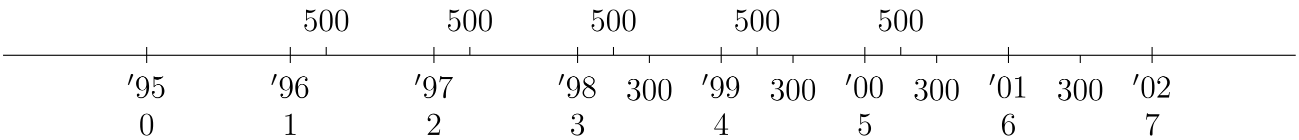
2–9. Draw a timeline:



$i_B = 0.08$, $i_A = 0.09$
At $t = 8$ $B = 3A$ and thus $B(1.08)^8 = 3A(1.09)^8$ and $B_8 = 3A(1.09)^8(1.08)^{-8}$.
This means that at $t = 10$, $B_{10} = B_8(1.08)^2$ or $B_{10} = 3A(1.09)^8(1.08)^{-6}$.

$$\begin{aligned}
A_{10} + B_{10} &= 52,000 \\
A(1.09)^{10} + 3A(1.09)^{8}(1.08)^{-6} &= 52,000 \\
A_8 &= \frac{52,000}{(1.09)^2 + 3(1.08)^{-6}} \\
A_{15} &= A_8 \cdot (1.09)^7 \\
A_{15} &= \frac{52,000}{(1.09)^2 + 3(1.08)^{-6}} * (1.09)^7 \\
A_{15} &= 30,876.94409
\end{aligned}$$

2–10. Timeline for this problem:



Find the total value of the payments on March 15, 2002

$$V = \sum_{j=2}^{6} 500\left(1 + \frac{17}{400}\right)^{4j} + \sum_{j=1}^{4} 300\left(1 + \frac{17}{400}\right)^{4j-1}$$

(a) Value on March 15, 2005

$$\begin{aligned}
X &= V \cdot \left(1 + \frac{17}{400}\right)^{3} \\
X &= 7,678.792345
\end{aligned}$$

(b) Value on March 15, 1999

$$\begin{aligned}
X &= V \cdot \left(1 + \frac{17}{400}\right)^{-3} \\
X &= 5,981.864088
\end{aligned}$$

(c) Value on March 15, 1995

$$\begin{aligned}
X &= V \cdot \left(1 + \frac{17}{400}\right)^{-7} \\
X &= 5,064.449988
\end{aligned}$$