Math 241: Laplace equation in polar coordinates; consequences and properties

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Laplace on a disk

- Next up is to solve the Laplace equation on a disk with boundary values prescribed on the circle that bounds the disk.
- We'll use polar coordinates for this, so a typical problem might be:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the disk of radius $R = 3$ centered at the origin, with boundary condition

$$u(3,\theta) = \begin{cases} 1 & 0 \le \theta \le \pi \\ \sin^2\theta & \pi < \theta < 2\pi \end{cases}$$

Separation of variables

- We search for separated solutions: $u(r,\theta) = R(r)\Theta(\theta)$. So

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

or

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\lambda.$$

- We need Θ to be periodic with period 2π (so that u will be well-defined as a function of x and y) — so $\lambda = n^2$ $(n = 0, 1, 2, \ldots)$ and

$$\Theta = \begin{cases} a_0 & n = 0 \\ a_n \cos(n\theta) + b_n \sin(n\theta) & n = 1, 2, \ldots \end{cases}$$

Separated solutions

- The R equation becomes $r^2R'' + rR' + n^2R = 0$, for $n = 0, 1, 2, \ldots$.
- This is a Cauchy-Euler equation (look in your Math 240 book) and the solution is

$$R = \begin{cases} c_1 + c_2 \ln r & n = 0 \\ c_1 r^n + c_2 r^{-n} & n = 1, 2, \ldots \end{cases}$$

- Because we don't want the solution to go to infinity at the center of the disk (where $r = 0$), we set $c_2 = 0$ in both cases. So our separated solutions are:

$$u(r, \theta) = R(r)\Theta(\theta) = \begin{cases} a_0 & n = 0 \\ a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) & n = 1, 2, \ldots \end{cases}$$

Fourier series

- As usual, we'll make a series out of our separated solutions and try to match the boundary condition;

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

- Each term in the series satisfies $\nabla^2 u = 0$ and is well-defined on the disk. Now we need to match the boundary condition for $r = 3$ (from the first slide):

$$\begin{aligned} u(3,\theta) &= a_0 + \sum_{n=1}^{\infty} 3^n (a_n \cos(n\theta) + b_n \sin(n\theta)) \\ &= \begin{cases} 1 & 0 \le \theta \le \pi \\ \sin^2\theta & \pi < \theta < 2\pi \end{cases} \end{aligned}$$

Fourier coefficients

- We calculate the Fourier coefficients as usual:

$$a_0 = \frac{1}{2\pi}\left(\int_0^{\pi} 1\,d\theta + \int_{\pi}^{2\pi} \sin^2\theta\,d\theta\right) = \frac{3}{4}$$

and

$$\begin{aligned} a_n &= \frac{1}{3^n\pi}\left(\int_0^{\pi} \cos(n\theta)\,d\theta + \int_{\pi}^{2\pi} \sin^2\theta\,\cos(n\theta)\,d\theta\right) \\ &= \begin{cases} -\frac{1}{36} & n = 2 \\ 0 & n \neq 2,\; n > 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{3^n\pi}\left(\int_0^{\pi} \sin(n\theta)\,d\theta + \int_{\pi}^{2\pi} \sin^2\theta\,\sin(n\theta)\,d\theta\right) \\ &= \begin{cases} 0 & n \text{ even} \\ \dfrac{1}{3^n\pi}\left(\dfrac{2}{n} + \dfrac{4}{n(n^2-4)}\right) & n \text{ odd} \end{cases} \end{aligned}$$

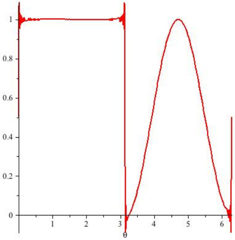
The solution

- So the solution is

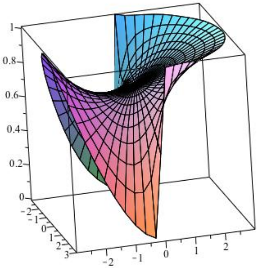
$$u(r,\theta) = \frac{3}{4} - \frac{r^2}{36}\cos(2\theta) + \sum_{n=0}^{\infty} \frac{r^{2n+1}}{3^{2n+1}\pi} \frac{2(4n^2+4n-1)}{(2n+3)(4n^2-1)} \sin(2n+1)\theta.$$

Pictures

- Boundary values, 100 terms



- Graph of $u(r, \theta)$, 200 terms:



Generally

The solution of the Dirichlet problem $\nabla^2 u = 0$ on a disk of radius R with boundary condition $u = f(\theta)$ on the boundary of the disk, is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta)\, d\theta$, and for $n > 0$,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta.$$

From

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

we see that when $r = 0$, we get that

$$u(0,\theta) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta.$$

$u(0,\theta)$ is the value of u at the center of the circle, and the last expression is the average value of f (that is to say, the average value of u) on the circle of radius R.
This will be true for any disk contained in the domain where u is harmonic (i.e., where $\nabla^2 u = 0$.

Mean-value property

The reasoning on the last slide proves:

Theorem (Mean-value property for harmonic functions)

Let u be a harmonic function ($\nabla^2 u = 0$ in a region that contains a disk D whose center is the point p, whose radius is r and whose circumference is the circle C. Then

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p + re^{i\theta})\,d\theta = \frac{1}{2\pi r}\int_C u\,ds.$$

In other words, the value of u at the center of the disk D is equal to the average of the values of u on the circumference of the disk.

Maximum principle

Since the average of a set of numbers cannot exceed the maximum of the numbers, we have

Theorem (Maximum principle for harmonic functions)

If u is harmonic on a region R, then the maximum value of u must occur at a boundary point of R. Moreover, if the maximum point also occurs at an interior point of R, then u must be a constant function.

Clearly, there is also a minimum principle for harmonic functions. So if we know that the maximum and minimum of a harmonic function on a domain cannot be exceed the maximum on the boundary or be less than the minimum on the boundary.

Solid MVP

If we multiply the equality in the mean-value property by $r\,dr$ and integrate from 0 to R, we get

$$\int_0^R u(p) r\,dr = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(r + e^{i\theta}) r\,dr\,d\theta$$

Since $u(p)$ is a constant the left side equals $u(p)R^2/2$.
Recognizing the double integral on the right as the integral of $u\,dA$ over the entire disk D of radius R centered at p gives:

Theorem (Solid mean value property for harmonic functions)

If u is harmonic on a disk D with center p and radius R, then the value of u at the center of the disk is equal to the average of u over the disk:

$$u(p) = \frac{1}{\pi R^2} \iint_D u\,dA.$$

Geometry

We can use the solid MVP to prove a remarkable fact about harmonic functions. But first we need a little geometric fact:

Lemma

Let x and y be points in the plane, let $D_x(R)$ be the disk of radius R centered at x, and let $D_y(R)$ be the disk of radius R centered at y. Then the area of the set of points contained in **one but not both** *of $D_x(R)$ or $D_y(R)$ is less than $2\pi R|x-y|$.*

To see this, note that if q is in $D_x(R)$ but not $D_y(R)$, then $|q-x| < R$ and $|q-y| > R$, so $R - |x-y| < |q-x| < R$ (for the left inequality, note that $|q-x| = |(q-y)-(x-y)| > |q-y| - |x-y| > R - |x-y|$. And the ring of points at distance between $R-|x-y|$ and R from x has area less than $2\pi R|x-y|$.

Liouville's theorem

Now for our remarkable fact:

Theorem (Liouville's theorem for harmonic functions)

Suppose u is a harmonic function defined on the entire plane. If u is bounded (that is, if there is a number M so that $-M < u < M$ everywhere in the plane, then u is a constant function.

To prove this, let x and y be any points in the plane. Then

$$u(x) - u(y) = \frac{1}{\pi R^2}\left(\iint_{D_x(R)} u\,dA - \iint_{D_y(R)} u\,dA\right)$$

For large R, a lot of the two integrals cancel because the disks overlap. And because u is bounded we can conclude

$$|u(x) - u(y)| \leq \frac{1}{\pi R^2} M(area(S))$$

where S is the set of points contained in one but not both of $D_x(R)$ or $D_y(R)$.

End of proof of Liouville's theorem

From the geometric lemma, we conclude that

$$|u(x) - u(y)| \leq \frac{1}{\pi R^2}(2\pi R|x - y|)$$

Since this last quantity goes to zero as $R \to \infty$, we conclude that $u(x) = u(y)$. And since x and y were arbitrary, u must be constant. This concludes the proof of Liouville's theorem.