Two–dimensional Poisson Equations

Theory and basic formulas

Let

$$\Omega = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\},$$

a *unit–square* in the plain (x, y) and the borders

$$\begin{array}{c} \partial\Omega = \{x = 0, 0 \leq y \leq 1\} \cup \{x = 1, 0 \leq y \leq 1\} \\ \cup\{0 \leq x \leq 1, y = 0\} \cup \{0 \leq x \leq 1, y = 1\} \end{array}.$$

In the domain Ω consider the following boundary value problem, usually called the *Dirichlet problem* or *Poisson Problem*

$$-\Delta u(x, y) \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y), \quad (x, y) \in \overset{0}{\Omega} \equiv \Omega \backslash \partial\Omega$$

$$u(x, y) = 0, \quad (x, y) \in \partial\Omega$$

These equations describe a stationary thermal field whose source is $f(x, y)$. Operator Δ is the *Laplace Operator.* Consider the uniform grid in Ω

$$\Omega^h \equiv \left\{(x_i, y_j)|x_i = (i - 1)h; \quad y_j = (j - 1)h; \quad i, j = 1, ..., n; \quad h = \frac{1}{n-1}\right\}.$$

Let's introduce the notation: inner part of the mesh area:

$$\Omega^{h,0} \equiv \Omega^h \cap \overset{0}{\Omega}$$

and the boundary of the grid domain

$$\partial\Omega^h \equiv \Omega^h \cap \partial\Omega$$

and the grid function defined in the domain Ω^h

$$u^h \equiv \left\{u^h_{i,j}\right\}^n_{i,j=1}$$

Consider the following grid problem for u^h

$$-\Delta^h u^h_{i,j} \equiv -\left(\frac{u^h_{i+1,j} - 2u^h_{i,j} + u^h_{i-1,j}}{h^2} + \frac{u^h_{i,j+1} - 2u^h_{i,j} + u^h_{i,j-1}}{h^2}\right) = f^h_{i,j}, \quad (x_i, y_j) \in \Omega^{h,0}$$

$$u^h_{i,j} = 0, \quad (x_i, y_j) \in \partial\Omega^h$$

Here $f^h_{i,j} \equiv f(x_i, y_j)$. It follows that (V.1.3)

1. exists
2. unique
3. continuous

and depends on f^h. The approximate solution shall be denoted by $\bar{u}^h_{i,j}$.

To solve the equation, consider the *two–parameter family of iterative methods*

$$\Delta^h_0 u^h_{i,j} \equiv \frac{4}{h^2} u^h_{i,j}$$

$$\Delta^h_1 u^h_{i,j} \equiv -\frac{1}{h^2}\left(u^h_{i-1,j} + u^h_{i,j-1}\right)$$

$$\Delta^h_2 u^h_{i,j} \equiv -\frac{1}{h^2}\left(u^h_{i+1,j} + u^h_{i,j+1}\right), \quad (x_i, y_j) \in \Omega^{h,0}$$

We see that

$$-\Delta^h = \Delta^h_1 + \Delta^h_0 + \Delta^h_2$$

The equations can be solved on the grid Ω^h by going left–right and bottom–up, i.e. in a double loop we'd have

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for (int j = 1; j <= n; ++j)
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for the outer loop and

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for (int i = 1; i <= n; ++i)
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for the inner loop. The proposed method is the following:

$$\left(\Delta^h_0 + \theta\Delta^h_1\right) \cdot \frac{u^{(k)} - u^{(k-1)}}{\tau} - \Delta^h u^{(k-1)} = f^h, \quad \text{in the area } \Omega^{h,0}$$

$$u^{(k)} = 0, \text{ on } \partial\Omega^h, \quad k = 1, 2, ...$$

$$u^{(0)} \quad \text{– given in } \Omega^h.$$

Here, k is an iteration parameter, and

$$u^{(k)} \equiv \left\{u^{(k)}_{i,j}\right\}^n_{i,j=1}$$

is the grid function – the solution to the kth iteration step. In a point wise form, our equations are

$$u^{(k)} = \frac{\theta}{4}\left(u^{(k)}_{i-1,j} + u^{(k)}_{i,j-1}\right) + \frac{\tau-\theta}{4}\left(u^{(k-1)}_{i-1,j} + u^{(k-1)}_{i,j-1}\right)$$
$$+\frac{\tau}{4}\left(u^{(k-1)}_{i+1,j} + u^{(k-1)}_{i,j+1}\right) + (1-\tau)u^{(k-1)}_{i,j} + \frac{\tau h^2}{4}\cdot f^h; \quad (i,j)\in\Omega^{h,0}$$

$$u^{(k)}_{i,j} = 0, \quad (i,j)\in\partial\Omega^h$$

Properties of the iterative algorithm

Denote

$$\mu \equiv 4\cdot\sin^2\left(\frac{\pi h}{2}\right), \quad v \equiv 4\cdot\cos^2\left(\frac{\pi h}{2}\right)$$

Consider the following areas on the parametric plane

$$\Pi_1 \equiv \left\{(\theta,\tau)|0\le\theta, 0<\tau<\theta+\min\left\{\frac{2}{\mu}(2-\theta), \frac{(2-\theta)^3}{\theta\mu v + 2(2-\theta)^2}\right\}\right\}$$

$$\Pi_2 \equiv \left\{(\theta,\tau)|0\le\theta\le 2; \theta + \frac{(2-\theta)^3}{\theta\mu v + 2(2-\theta)^2} \le \tau < \theta + \frac{2}{v}(2-\theta)\right\}$$

If $(\theta,\tau)\in\Pi_1\cup\Pi_2$, then the iterative process converges and the following estimate holds

$$||u^h - u^{(k)}||_\infty \le q^k\cdot||u^h - u^{(0)}||_\infty$$

where q is

$$q^2 = 1 - \tau\mu\cdot\frac{\mu(\theta-\tau)+2(2-\theta)}{2\mu\theta + (2-\theta)^2}, \text{ for } (\theta,\tau)\in\Pi_1$$

or

$$q^2 = 1 - \tau v\cdot\frac{v(\theta-\tau)+2(2-\theta)}{2v\theta + (2-\theta)^2}, \text{ if } (\theta,\tau)\in\Pi_2.$$

Because the process is iterative, we need to know how many times we need to iterate to reduce the initial error m times. k satisfies the estimate

$$k \ge \frac{\ln(m)}{\ln\left(\frac{1}{q}\right)}$$

The value $\ln\left(\frac{1}{q}\right)$ is called the *rate of convergence of the iterative method* – the larger the value, the faster it converges.

Schemes

Jacobi's Method

$\theta = 0, \tau = 1$. In this case, we are in the region Π_2, and we find

$$q = \cos(\pi h).$$

The rate of convergence is

$$\frac{1}{q} \approx 1 + \frac{\pi^2 h^2}{2} \to \ln\left(\frac{1}{q}\right) = O\left(\frac{\pi^2 h^2}{2}\right), \text{ for } h \to 0.$$

Seidel's Method

$\theta = \tau = 1$, we are in the region Π_1. We find q as

$$q^2 = \frac{1}{1 + 8\sin^2\left(\frac{\pi h}{2}\right)}$$

and

$$\frac{1}{q} = O\left(1 + \pi^2 h^2\right)$$

thus the speed of convergence is

$$\ln\left(\frac{1}{q}\right) = O\left(\pi^2 h^2\right), \text{ for } h \to 0.$$

Test Tasks

Eigenfunctions of the grid Laplace operator Δ^h

The parameters $l = m = 2$ are selected. This pair determines the solution

$$f^h_{i,j} = \lambda_{l,m} - \sin(\pi l x_i) \cdot \sin(\pi m y_j); \quad i, j = 1, 2, ..., n$$

where

$$\lambda_{l,m} = \frac{4}{h^2} \cdot \left[\sin^2\left(\frac{\pi l h}{2}\right) + \sin^2\left(\frac{\pi m h}{2}\right)\right]; \quad l = m = 2.$$

The exact solution depends on l, m

$$u^h_{i,j} = \sin(\pi l x_i) \cdot \sin(\pi m y_j), \quad i, j = 1, 2, ..., n$$

Program requirements

1. iterative process with choice of θ and τ
2. test problems
3. choice of grid nodes n
$$n = 1 + d^k, k = 2, 3, ...$$
4. finding the error
$$\overset{\max}{(x_i, y_j) \in \Omega^h} \; |u(x_i, y_j) - \bar{u}^h_{i,j}$$
5. entering σ, the condition for stopping the iterative process, else
$$\sigma = (N-1)^{-3}$$
6. show number of iterations required for the method to converge
7. simultaneously render $\sin(\pi l x) \cdot \sin(\pi m y)$ and piecewise linear interpolation of $\bar{u}^h_{i,j}$
   2D graphs are sufficient, with the slices
$$\left\{(x, y) | x = \left[\frac{n}{2}\right], y \in [0, 1]\right\}$$
$$\left\{(x, y) | x \in [0, 1], y = \left[\frac{n}{2}\right]\right\}$$
8. Additional capabilities
   1. render the areas Π_1 and Π_2 dependent on the parameters
   2. 3D graphs of the exact solution and a linear interpolation of the grid function

Report requirements

Use the function $u^{(0)} \equiv 0$ everywhere

1. How does the change in the number of grid nodes (parameter n) affect the speed of convergence of methods?
2. Does the "complexity of the solution" (parameters l and m in Problem 1) affect the convergence of methods?
3. Compare the effectiveness of the methods.
   1. number of required iterations for convergence
   2. errors
   3. visual proximity of the two graphs
4. *Additional tasks*: explore the domains Π_1 and Π_2 as well as q^2, find the parameters (θ, τ) that make the method converge the fastest