Predator-Prey Equations: Modeling Food Chains

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 differential equations describing populations of predators and prey

- ▶ most famous are Lotka-Volterra equations
- ▶ derived by Alfred James Lotka (1880–1949) and Vito Volterra (1860–1940) [2] in the 1920s
- ▶ Volterra observed fish, Lotka chemical reactions both are the same system [2]

Lotka-Volterra Equations

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simplest predator-prey equations

 \blacktriangleright parameters a, b, c, d > 0, following [1]

$$\begin{cases} \frac{dx}{dt} = ax - bxy & \text{Prey} \\ \frac{dy}{dt} = -cy + dxy & \text{Predator} \end{cases}$$
 (1)

Expanded Lotka-Volterra Equations

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- ▶ expansion of (1)
- \triangleright parameters a, b, c, d, e, f, g > 0, following [1]

$$\begin{cases} \frac{dx}{dt} = ax - bxy & \text{Prey} \\ \frac{dy}{dt} = -cy + dxy - eyz & \text{Intermediate Predator} \\ \frac{dz}{dt} = -fz + gyz & \text{Apex Predator} \end{cases}$$

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➤ space where all points are solutions of a system like (1)

- moving points form trajectories
- ▶ stationary points are equilibria [5]
- ► example of pendulum in Figure 1
- ▶ give insights into equation without solving

Phase Planes - Example

$$x' = y$$
 and $y' = -\sin(x)$

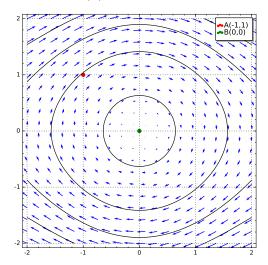


Figure 1: Phase plane of a pendulum with contour lines

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- ▶ standard Lotka-Volterra equations
- equivalent to (2) with z = 0
- ightharpoonup parameters a=b=c=d=1 chosen for simplicity

$$\begin{cases} \frac{dx}{dt} = x(1-y) & \text{Prey,} \\ \frac{dy}{dt} = y(x-1) & \text{Predator.} \end{cases}$$
 (3)

Phase Plane

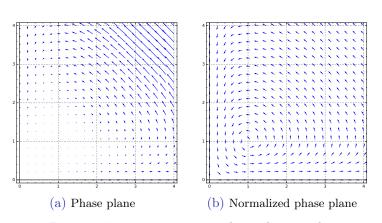


Figure 2: Two-species system phase planes with a=b=c=d=1

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Phase Plane with Contours

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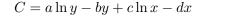
(4)

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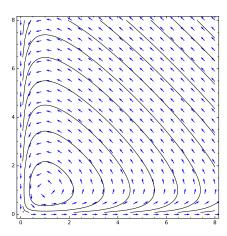


Figure 3: Two-species system phase plane with normalized vectors, contour lines, and a = b = c = d = 1

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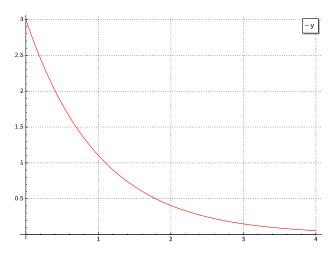


Figure 4: Two-species system graph for $x=0,\,y=3,$ and a=b=c=d=1

Case y = 0

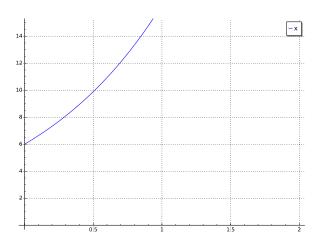


Figure 5: Two-species system graph for $x=6,\ y=0,$ and a=b=c=d=1

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Case x = 6, y = 3 Contour

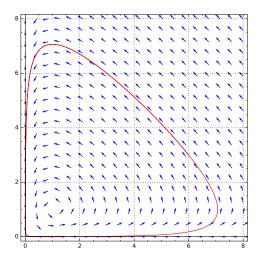


Figure 6: Two-species system contour for $x=6,\,y=3,$ and a=b=c=d=1

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Case x = 6, y = 3 Graph

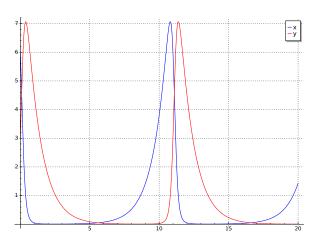


Figure 7: Two-species system graph for $x=6,\ y=3,$ and a=b=c=d=1

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• equation (2) with simple parameters

$$ightharpoonup a=b=c=d=e=f=g=1$$
 makes the system

$$\begin{cases} \frac{dx}{dt} = x(1-y) & \text{Prey} \\ \frac{dy}{dt} = y(x-z-1) & \text{Intermediate Predator} \\ \frac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases}$$
(5)

Case z=0

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Equation becomes the same as two-species system

$$\begin{cases} \frac{dx}{dt} = x(1-y) & \text{Prey} \\ \frac{dy}{dt} = y(x-0-1) = y(x-1) & \text{Intermediate Predator} \\ \frac{dz}{dt} = 0(y-1) = 0 & \text{Apex Predator} \end{cases}$$

Case z = 0 Contour

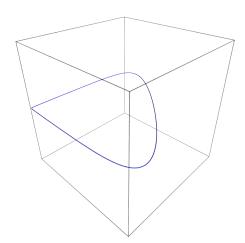


Figure 8: Three-species system contour for x=6, y=3, z=0, a=b=c=d=e=f=g=1

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Case z = 0 Graph

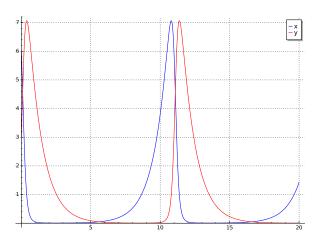


Figure 9: Three-species system graph for $x=6,\ y=3,\ z=0,$ a=b=c=d=e=f=g=1

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Case x = 0

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If x = 0 we get the following equations

$$\begin{cases} \frac{dx}{dt} = 0(1-y) = 0 & \text{Prey} \\ \frac{dy}{dt} = y(0-z-1) = y(-z-1) & \text{Intermediate Predator} \\ \frac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases}$$

Case x = 0 Graph

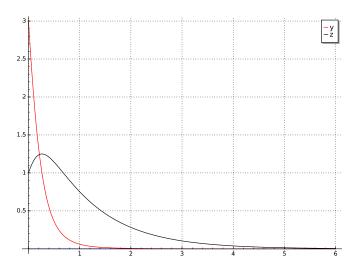


Figure 10: Three-species system graph for $x=0,\,y=3,\,z=1,$ a=b=c=d=e=f=g=1

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Case y = 0

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If y = 0 we get the following system of equations

$$\begin{cases} \frac{dx}{dt} = x(1-0) = x & \text{Prey} \\ \frac{dy}{dt} = 0(x-z-1) = 0 & \text{Intermediate Predator} \\ \frac{dz}{dt} = z(0-1) = -z & \text{Apex Predator} \end{cases}$$

Case y = 0 Graph

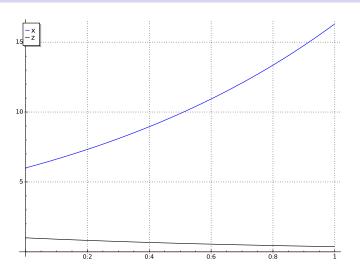


Figure 11: Three-species system graph for x=6, y=0, z=1, a=b=c=d=e=f=g=1

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Further Analysis; Case ga > fb

▶ in [1] the authors use further criteria to classify (2)

$$ga > fb, \qquad ga < gb, \qquad ga = fb.$$

▶ For example a = g = 1.1 and all other constants 1

$$\begin{cases} \frac{dx}{dt} = 1.1x - xy & \text{Prey,} \\ \frac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \frac{dz}{dt} = -z + 1.1yz & \text{Apex Predator.} \end{cases}$$

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Case ga > fb Graph

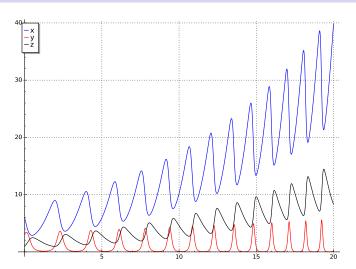


Figure 12: Three-species system graph for $x=6,\,y=3,\,z=1,$ b=c=d=e=f=1, and a=g=1.1

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For example b=f=1.1 and all other constants equal 1

$$\begin{cases} \frac{dx}{dt} = x - 1.1xy & \text{Prey,} \\ \frac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \frac{dz}{dt} = -1.1z + yz & \text{Apex Predator.} \end{cases}$$

Case ga < fb Graph

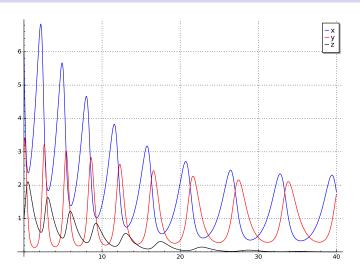


Figure 13: Three-species system graph for x=6, y=3, z=1, a=c=d=e=g=1, and b=f=1.1

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Case ga < fb Contour

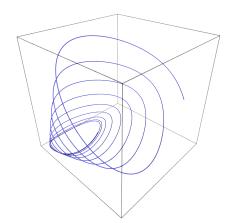


Figure 14: Three-species system contour for $x=6,\,y=3,$ $z=1,\,a=c=d=e=g=1,$ and b=f=1.1

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Case ga = fb

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- original equation (5) does not change
- \blacktriangleright all constants equal 1, a=b=c=d=e=f=g=1
- ▶ system could be periodic

Case ga = fb Graph

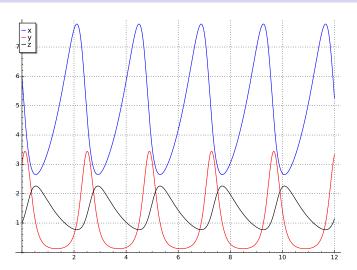


Figure 15: Three-species system graph for x=6, y=3, z=1, and a=b=c=d=e=f=g=1

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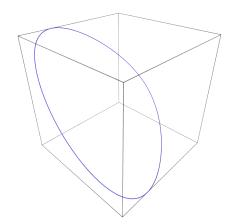


Figure 16: Three-species system contour for $x=6,\ y=3,$ z=1, and a=b=c=d=e=f=g=1

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► models fit general intuition

- inaccuracies: unlimited growth, only one species as prey
- equation had great impact on ecology [6]
- ▶ Lotka-Volterra equations (1) are among the most famous differential equations

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