# Laplace's Equation

- Separation of variables – two examples
- Laplace's Equation in Polar Coordinates
  - Derivation of the explicit form
  - An example from electrostatics
- A surprising application of Laplace's eqn
  - Image analysis
  - This bit is NOT examined

# Laplace's Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

In the vector calculus course, this appears as $\nabla^2 \phi = 0$ where $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$

Note that the equation has **no** dependence on time, just on the spatial variables x,y. This means that Laplace's Equation describes **steady state** situations such as:

• steady state temperature distributions

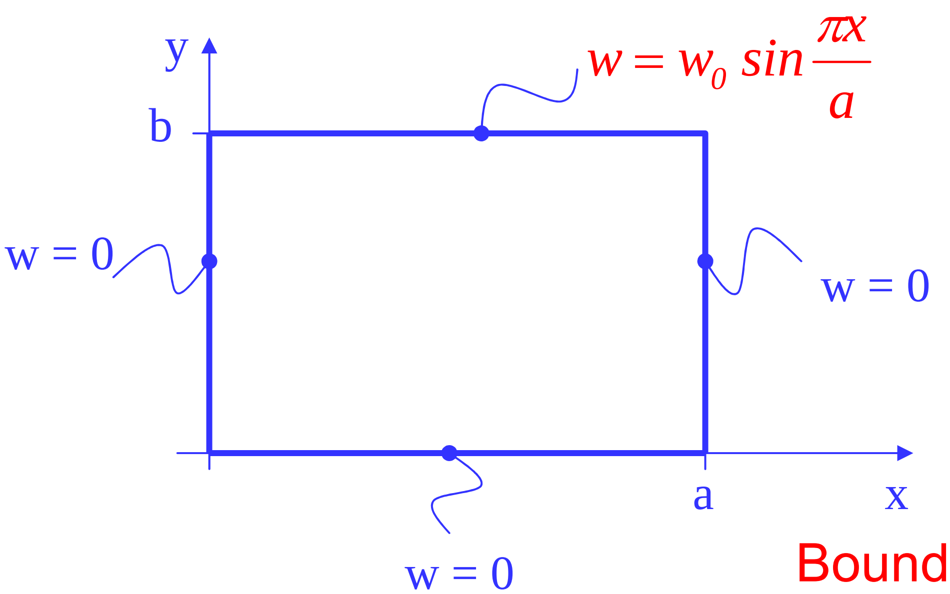
• steady state stress distributions

• steady state potential distributions (it is also called the *potential equation*

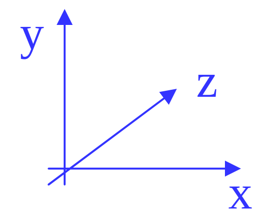
• steady state flows, for example in a cylinder, around a corner, …

# Stress analysis example: Dirichlet conditions

Steady state stress analysis problem, which satisfies Laplace's equation; that is, a stretched elastic membrane on a rectangular former that has prescribed out-of-plane displacements along the boundaries




w(x,y) is the displacement in z-direction




To solve:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

## Boundary conditions

$w(0, y) = 0,$ for $0 \le y \le b$

$w(x, 0) = 0,$ for $0 \le x \le a$

$w(a, y) = 0,$ for $0 \le y \le b$

$w(x, b) = w_0 \sin\dfrac{\pi}{a} x,$ for $0 \le x \le a$

# Solution by separation of variables

$$w(x, y) = X(x)Y(y)$$

from which $$X''Y + XY'' = 0$$

and so $$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

as usual … $$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

where *k* is a constant that is either equal to, >, or < 0.

# Case $k=0$

$$X(x) = (Ax + B), Y(y) = (Cy + D)$$

$w(0, y) = 0 \Rightarrow B = 0$ or $C = D = 0$

if $C = D = 0$, then $Y(y) \equiv 0$, so $w(x, y) \equiv 0$

Continue with $B = 0$ :$w(x, y) = Ax(Cy + D)$

$w(x,0) = 0 \Rightarrow ADx = 0$

Either $A = 0$ (so $w \equiv 0$) or $D = 0$

Continue with $w(x, y) = ACxy$

$w(a, y) = 0 \Rightarrow ACay = 0 \Rightarrow A = 0$ or $C = 0 \Rightarrow w(x, y) \equiv 0$

That is, the case $k=0$ is not possible

# Case *k>0*

Suppose that $k = \alpha^2$, so that

$$w(x, y) = (A\cosh\alpha x + B\sinh\alpha x)(C\cos\alpha y + D\sin\alpha y)$$

Recall that $\cosh 0 = 1, \sinh 0 = 0$

$w(0, y) = 0 \Rightarrow A(C\cos\alpha y + D\sin\alpha y) = 0$

$C = D = 0 \Rightarrow w(x, y) \equiv 0$

Continue with $A = 0 \Rightarrow w(x, y) = B\sinh\alpha x(C\cos\alpha y + D\sin\alpha y)$

$w(x,0) = 0 \Rightarrow BC\sinh\alpha x = 0$

$B = 0 \Rightarrow w(x, y) \equiv 0$

Continue with $C = 0 \Rightarrow w(x, y) = BD\sinh\alpha x\sin\alpha y$

$w(a, y) = 0 \Rightarrow BD\sinh\alpha a\sin\alpha y = 0$

so either $B = 0$ or $D = 0 \Rightarrow w(x, y) \equiv 0$

Again, we find that the case *k>0* is not possible

# Final case *k<0*

Suppose that $k = -\alpha^2$

$$w(x, y) = (A\cos\alpha x + B\sin\alpha x)(C\cosh\alpha y + D\sinh\alpha y)$$

$$w(0, y) = 0 \Rightarrow A(C\cosh\alpha y + D\sinh\alpha y) = 0$$

as usual, $C = D = 0 \Rightarrow w \equiv 0$

continue with $A = 0 \Rightarrow w(x, y) = B\sin\alpha x(C\cosh\alpha y + D\sinh\alpha y)$

$$w(x,0) = 0 \Rightarrow BC\sin\alpha x = 0$$

$B = 0 \Rightarrow w \equiv 0$

continue with $C = 0 \Rightarrow w(x, y) = BD\sin\alpha x\sinh\alpha y$

$$w(a, y) = 0 \Rightarrow BD\sin\alpha a\sinh\alpha y = 0$$

$B = 0$ or $D = 0 \Rightarrow w \equiv 0$

$$\sin\alpha a = 0 \Rightarrow \alpha = n\frac{\pi}{a} \Rightarrow w_n(x, y) = BD\sin n\frac{\pi}{a}x\sinh n\frac{\pi}{a}y$$

# Solution

Applying the first three boundary conditions, we have

$$\boxed{w(x,y) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}}$$

The final boundary condition is: $w(x,b) = w_0 \sin\frac{\pi x}{a}$

which gives: $w_0 \sin\frac{\pi x}{a} = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$

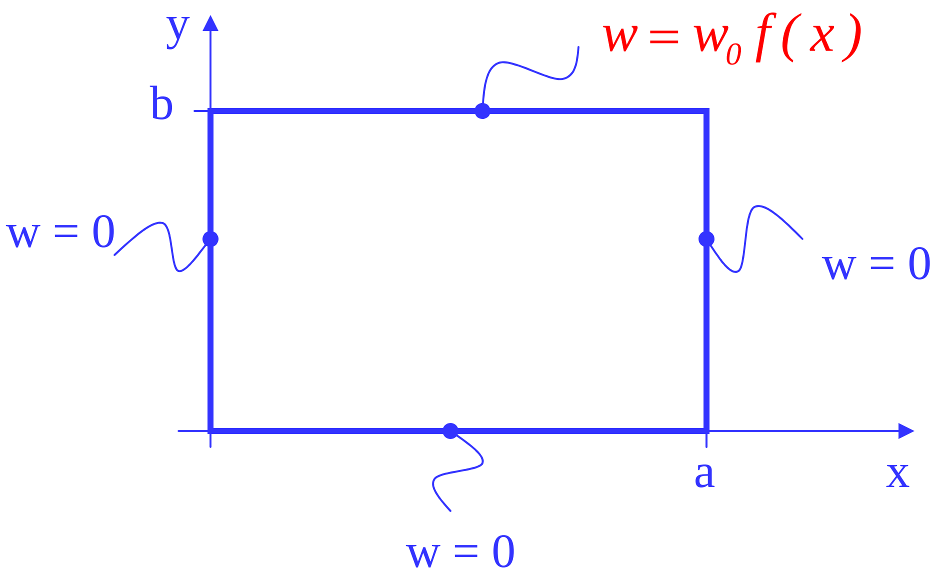
We can see from this that *n* must take only one value, namely 1, so that $K_1 = \dfrac{w_0}{\sinh\frac{\pi b}{a}}$

and the final solution to the stress distribution is

$$\boxed{w(x,y) = \frac{w_0}{\sinh\frac{\pi b}{a}} \sin\frac{\pi x}{a} \sinh\frac{\pi y}{a}}$$

Check out: http://mathworld.wolfram.com/HyperbolicSine.html

# More general boundary condition




Then

$$w_0 f(x) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

and as usual we use orthogonality formulae/HLT to find the $K_n$

# Types of boundary condition

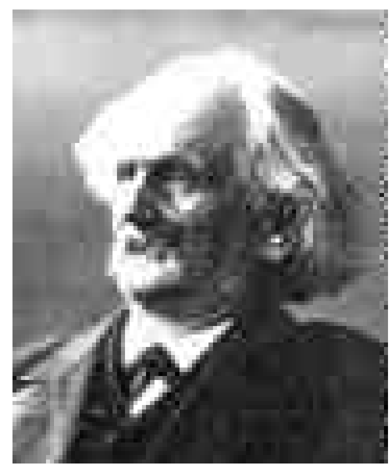
1. The *value* $\phi(x, y)$ is specified at each point on the boundary: "Dirichlet conditions"
2. The *derivative normal to the boundary* $\frac{\partial \phi}{\partial \mathbf{n}}(x, y)$ is specified at each point of the boundary: "Neumann conditions"
3. A mixture of type 1 and 2 conditions is specified

Johann Dirichlet (1805-1859)

http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Dirichlet.html

Carl Gottfried Neumann (1832 -1925)

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Neumann_Carl.html

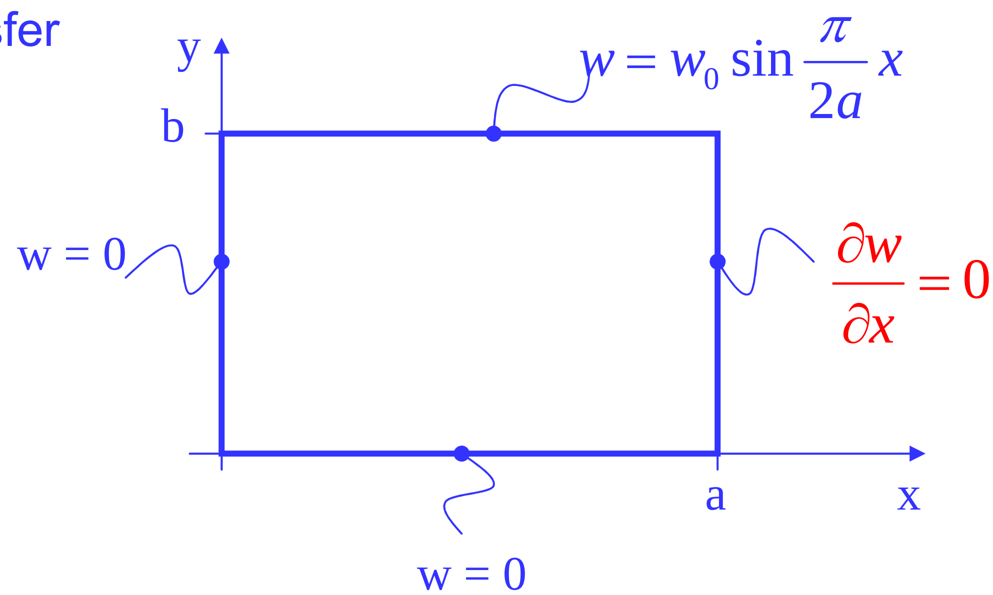


# A mixed condition problem

A steady state heat transfer problem

To solve:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$




Boundary conditions

$w(0, y) = 0,$ for $0 \le y \le b$

$w(x, 0) = 0,$ for $0 \le x \le a$

$\left.\frac{\partial w}{\partial x}\right|_{x=a} = 0,$ for $0 \le y \le b$

There is no flow of heat across this boundary; but it does not necessarily have a constant temperature along the edge

$w(x, b) = w_0 \sin \frac{\pi}{2a} x,$ for $0 \le x \le a$

# Solution by separation of variables

$$w(x, y) = X(x)Y(y)$$

from which $$X''Y + XY'' = 0$$

and so $$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

as usual … $$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

where *k* is a constant that is either equal to, >, or < 0.

# Case *k=0*

$$X(x) = (Ax + B), Y(y) = (Cy + D)$$

$w(0, y) = 0 \Rightarrow B = 0$ or $C = D = 0$

if $C = D = 0$, then $Y(y) \equiv 0$, so $w(x, y) \equiv 0$

Continue with $B = 0$ :$w(x, y) = Ax(Cy + D)$

$w(x,0) = 0 \Rightarrow ADx = 0$

Either $A = 0$ (so $w \equiv 0$) or $D = 0$

Continue with $w(x, y) = ACxy$

$$\left.\frac{\partial w}{\partial x}\right|_{x=a} = 0 \Rightarrow ACy = 0 \Rightarrow A = 0 \text{ or } C = 0 \Rightarrow w(x, y) \equiv 0$$

That is, the case *k=0* is not possible

# Case *k>0*

Suppose that $k = \alpha^2$, so that

$$w(x,y) = (A\cosh \alpha x + B\sinh \alpha x)(C\cos \alpha y + D\sin \alpha y)$$

Recall that $\cosh 0 = 1, \sinh 0 = 0$

$$w(0,y) = 0 \Rightarrow A(C\cos \alpha y + D\sin \alpha y) = 0$$

$$C = D = 0 \Rightarrow w(x,y) \equiv 0$$

Continue with $A = 0 \Rightarrow w(x,y) = B\sinh \alpha x(C\cos \alpha y + D\sin \alpha y)$

$$w(x,0) = 0 \Rightarrow BC\sinh \alpha x = 0$$

$$B = 0 \Rightarrow w(x,y) \equiv 0$$

Continue with $C = 0 \Rightarrow w(x,y) = BD\sinh \alpha x \sin \alpha y$

$$\left.\frac{\partial w}{\partial x}\right|_{x=a} = 0 \Rightarrow \alpha BD\cosh \alpha a \sin \alpha y = 0$$

so either $B = 0$ or $D = 0 \Rightarrow w(x,y) \equiv 0$

Again, we find that the case *k>0* is not possible

# Final case *k*<0

Suppose that $k = -\alpha^2$

$$w(x, y) = (A\cos\alpha x + B\sin\alpha x)(C\cosh\alpha y + D\sinh\alpha y)$$

$w(0, y) = 0 \Rightarrow A(C\cosh\alpha y + D\sinh\alpha y) = 0$

as usual, $C = D = 0 \Rightarrow w \equiv 0$

continue with $A = 0 \Rightarrow w(x, y) = B\sin\alpha x(C\cosh\alpha y + D\sinh\alpha y)$

$w(x,0) = 0 \Rightarrow BC\sin\alpha x = 0$

$B = 0 \Rightarrow w \equiv 0$

continue with $C = 0 \Rightarrow w(x, y) = BD\sin\alpha x\sinh\alpha y$

$\left.\dfrac{\partial w}{\partial x}\right|_{x=a} = 0 \Rightarrow \alpha BD\cos\alpha a\sinh\alpha y = 0$

$B = 0$ or $D = 0 \Rightarrow w \equiv 0$

$\cos\alpha a = 0 \Rightarrow \alpha = \dfrac{(2n-1)\pi}{2a} \Rightarrow w_n(x, y) = BD\sin\dfrac{(2n-1)\pi}{2a}x\sinh\dfrac{(2n-1)\pi}{2a}y$

# Solution

Applying the first three boundary conditions, we have

$$\boxed{w(x,y)=\sum_{n=1}^{\infty}K_n\sin\frac{(2n-1)\pi}{2a}x\sinh\frac{(2n-1)\pi}{2a}y}$$

The final boundary condition is: $w(x,b)=w_0\sin\frac{\pi x}{2a}$

which gives: $w_0\sin\frac{\pi x}{2a}=\sum_{n=1}^{\infty}K_n\sin\frac{(2n-1)\pi}{2a}x\sinh\frac{(2n-1)\pi}{2a}b$

We can see from this that *n* must take only one value, namely 1, so that $K_1=\dfrac{w_0}{\sinh\frac{\pi}{2a}b}$

and the final solution to the stress distribution is

$$\boxed{w(x,y)=\frac{w_0}{\sinh\frac{\pi b}{a}}\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}}$$

Check out: http://mathworld.wolfram.com/HyperbolicSine.html

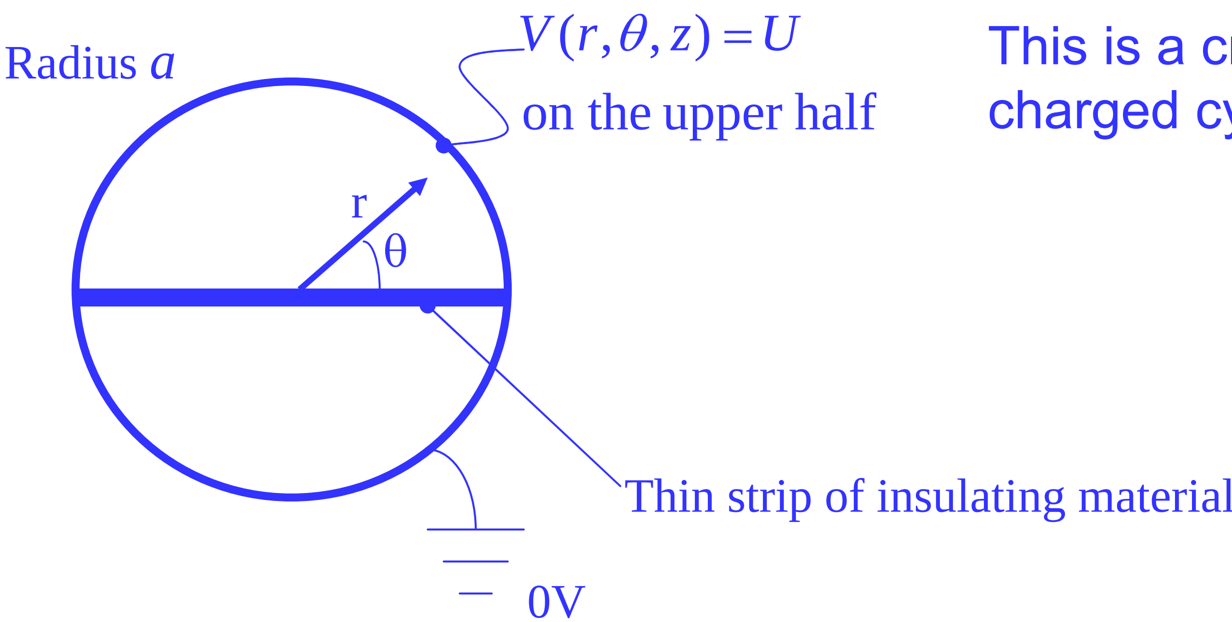
# PDEs in other coordinates…

- In the vector algebra course, we find that it is often easier to express problems in coordinates other than (x,y), for example in polar coordinates (r,Θ)
- Recall that in practice, for example for finite element techniques, it is usual to use curvilinear coordinates … but we won't go that far

We illustrate the solution of Laplace's Equation using polar coordinates*

*Kreysig, Section 11.11, page 636

# A problem in electrostatics



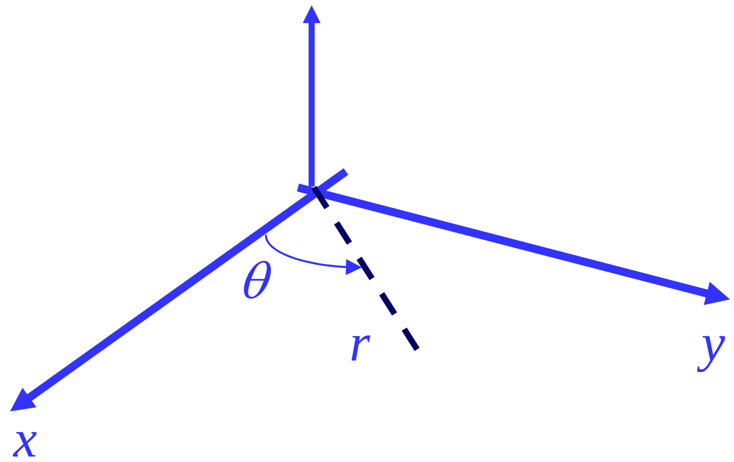

This is a cross section of a charged cylindrical rod.

I could simply TELL you that Laplace's Equation in cylindrical polars is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

… brief time out while I DERIVE this

# 2D Laplace's Equation in Polar Coordinates




$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ where $$x = x(r,\theta),\ \ y = y(r,\theta)$$

$$u(x,y) = u(r,\theta)$$

So, Laplace's Equation is $\nabla^2 u(r,\theta) = 0$

We next derive the ***explicit*** polar form of Laplace's Equation in 2D

Recall the chain rule: $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Use the product rule to differentiate again

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} \qquad (*)$$

and the chain rule again to get these derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x}=\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x}+\frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x}=\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x}+\frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}$$

# The required partial derivatives

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}, \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

in like manner ….

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}, \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{r^4}$$

# Back to Laplace's Equation in polar coordinates

Plugging in the formula for the partials on the previous page to the formulae on the one before that we get:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial u}{\partial r}\frac{y^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{-2xy}{r^3} + \frac{\partial u}{\partial \theta}\frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{y^2}{r^4}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} + \frac{\partial u}{\partial r}\frac{x^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{2xy}{r^3} - \frac{\partial u}{\partial \theta}\frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2}{r^4}$$

So Laplace's Equation in polars is

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

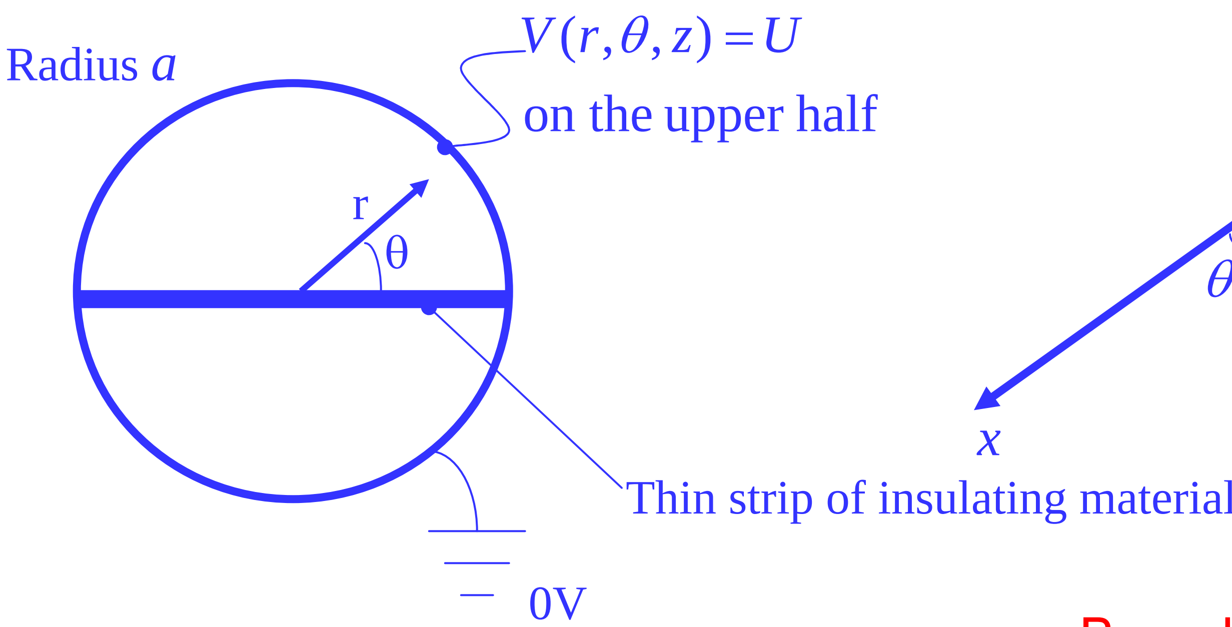
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

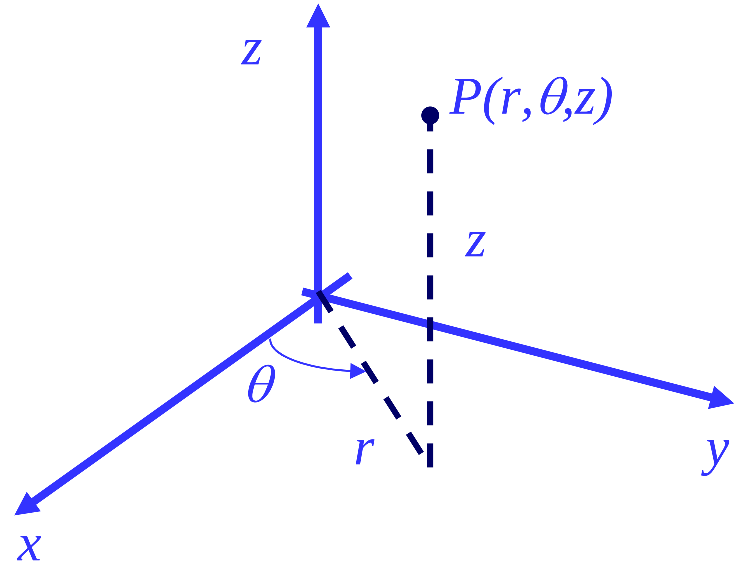
is equivalent to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

## Example of Laplace in Cylindrical Polar Coordinates (r, Θ, z)

Consider a cylindrical capacitor




Laplace's Equation in cylindrical polars is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Boundary conditions

$$V(a,\theta) = U \quad \forall \theta : 0 \le \theta \le \pi$$

$$V(a,\theta) = 0 \quad \forall \theta : \pi \le \theta \le 2\pi$$

In the polar system, note that the solution must repeat itself every θ = 2π

*V* should remain finite at *r=0*

There is no variation in $V$ in the z-direction, so $$\frac{\partial V}{\partial z} = 0$$

This means we can treat it as a 2D problem $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Using separation of variables

$$V = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\ R\Theta'' = 0$$

$$-\frac{\Theta''}{\Theta} = \frac{R'' + R'/r}{R/r^2}$$

As before, this means

$$-\frac{\Theta''}{\Theta} = \frac{R'' + R'/r}{R/r^2} = k, \quad \text{a constant}$$

# The case *k=0*

$$\Theta'' = 0 \Rightarrow \Theta(\theta) = a\theta + b$$

$$R'' + \frac{R'}{r} = 0 \Rightarrow R(r) = \left(C \ln r + D\right)$$

and so $V(r,\theta) = (a\theta + b)\left(C \ln r + D\right)$

The solution has to be periodic in *2π*: *a=0*

The solution has to remain finite as *r→0: c=0*

$$\boxed{V(r,\theta) = bd = g, \text{a constant}}$$

# The case *k<0*

Suppose that $k = -m^2$

$$\Theta'' + m^2\Theta = 0 \Rightarrow \Theta(\theta) = \left(A_m \cosh m\theta + B_m \sinh m\theta\right)$$

$$R'' + \frac{R'}{r} + \frac{m^2}{r^2}R = 0 \Rightarrow R(r) = \left(C_m r^{-m} + D_m r^{m}\right)$$

The solution has to be periodic in *θ*, with period *2π*. This implies that $A_m = B_m = 0 \Rightarrow V(r,\theta) \equiv 0$

# The case *k>0*

Suppose that $k = n^2$

$$\Theta'' + n^2\Theta = 0 \Rightarrow \Theta(\theta) = (A_n \cos n\theta + B_n \sin n\theta)$$

$$R'' + \frac{R'}{r} - \frac{n^2}{r^2}R = 0 \Rightarrow R(r) = \left(C_n r^n + D_n r^{-n}\right)$$

$$V(r,\theta) = (A_n \cos n\theta + B_n \sin n\theta)\left(C_n r^n + D_n r^{-n}\right)$$

Evidently, this is periodic with period *2π*

To remain finite as *r→0* $\quad D_n = 0$

# The solution

$$V(r,\theta) = g + \sum_n r^n \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

Notice that we have not yet applied the voltage boundary condition!!
Now is the time to do so

$$g + \sum_n a^n \left(A_n \cos n\theta + B_n \sin n\theta\right) = V(a,\theta) = \begin{cases} U & 0 \le \theta \le \pi \\ 0 & \pi < \theta \le 2\pi \end{cases}$$

Integrating *V* from *0* to *2π:* $\int_0^{2\pi} V(a,\theta)\,d\theta = \pi U$

Left hand side: $\int_0^{2\pi} g + \sum_n a^n \left(A_n \cos n\theta + B_n \sin n\theta\right) d\theta = 2\pi g$

and so $g = \dfrac{U}{2}$

# Solving for $A_m$ and $B_m$

So far, the solution is $$V(a,\theta)=\frac{U}{2}+\sum_n a^n\left(A_n\cos n\theta+B_n\sin n\theta\right)$$

We apply the orthogonality relationships:

$$\int_0^{2\pi} V(a,\theta)\cos m\theta d\theta=\frac{U}{2}\int_0^{2\pi}\cos m\theta d\theta+\int_0^{2\pi}\sum_n a^n(A_n\cos n\theta+B_n\sin n\theta)\cos m\theta d\theta$$

$$U\int_0^{\pi}\cos m\theta d\theta=0+A_m a^m\pi$$

$0=A_m a^m\pi$, and so $A_m=0$, for all $m$

$$\int_0^{2\pi} V(a,\theta)\sin m\theta d\theta=\frac{U}{2}\int_0^{2\pi}\sin m\theta d\theta+\int_0^{2\pi}\sum_n a^n(A_n\cos n\theta+B_n\sin n\theta)\sin m\theta d\theta$$

$$U\int_0^{\pi}\sin m\theta d\theta=\frac{U}{2}\left[\frac{\cos m\theta}{m}\right]_0^{2\pi}+B_m a^m\pi$$

$\frac{2U}{m}=B_m a^m\pi$, for odd $m=(2n-1)$

$$\boxed{V(r,\theta)=\frac{U}{2}+\sum_n\frac{2U}{(2n-1)\pi a^{(2n-1)}}r^{(2n-1)}\sin(2n-1)\theta}$$

$$V(r,\theta)=\frac{U}{2}+\frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{r^{(2n-1)}}{(2n-1)a^{(2n-1)}}\sin(2n-1)\theta$$

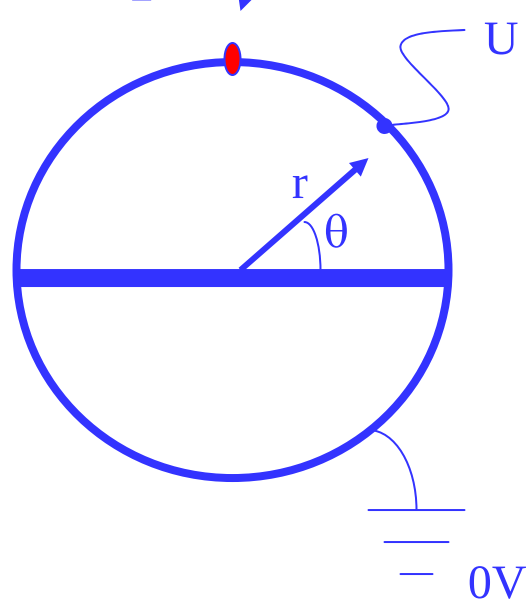
Check for $r = a$, $\theta = \pi/2$:

$$V(r,\theta)=\frac{U}{2}+\frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{a^{(2n-1)}}{(2n-1)a^{(2n-1)}}\sin(2n-1)\frac{\pi}{2}$$

$$=\frac{U}{2}+\frac{2U}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\ldots\ldots\right]$$

$$=\frac{U}{2}+\frac{2U}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right]$$

$$=\frac{U}{2}+\frac{2U}{\pi}\left[\frac{\pi}{4}\right]=U$$



# An application in image analysis

- We saw that the Gaussian is a solution to the heat/diffusion equation
- We have studied Laplace's equation
- The next few slides hint at the application of what we have done so far in image analysis
- This is aimed at engaging your interest in PDEs … it is not examined

# Laplace’s Equation in image analysis

How do we compute the edges?

Image fragment

Edge map

- Remove the noise by smoothing
- Find places where the second derivative of the image is zero

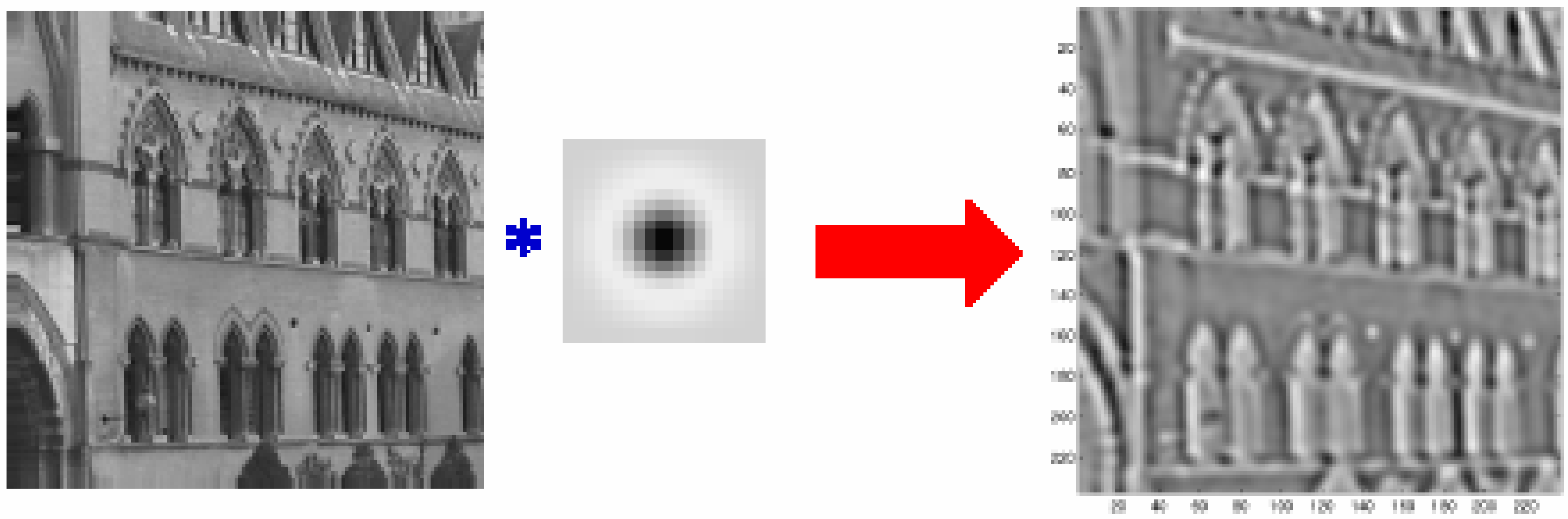
Signal position, x

Constant gradient + max at the step. Note amplified noise

Zero crossing at step. But doubly amplified noise

# Gaussian smoothing

Blurring with a Gaussian filter is one way to tame noise

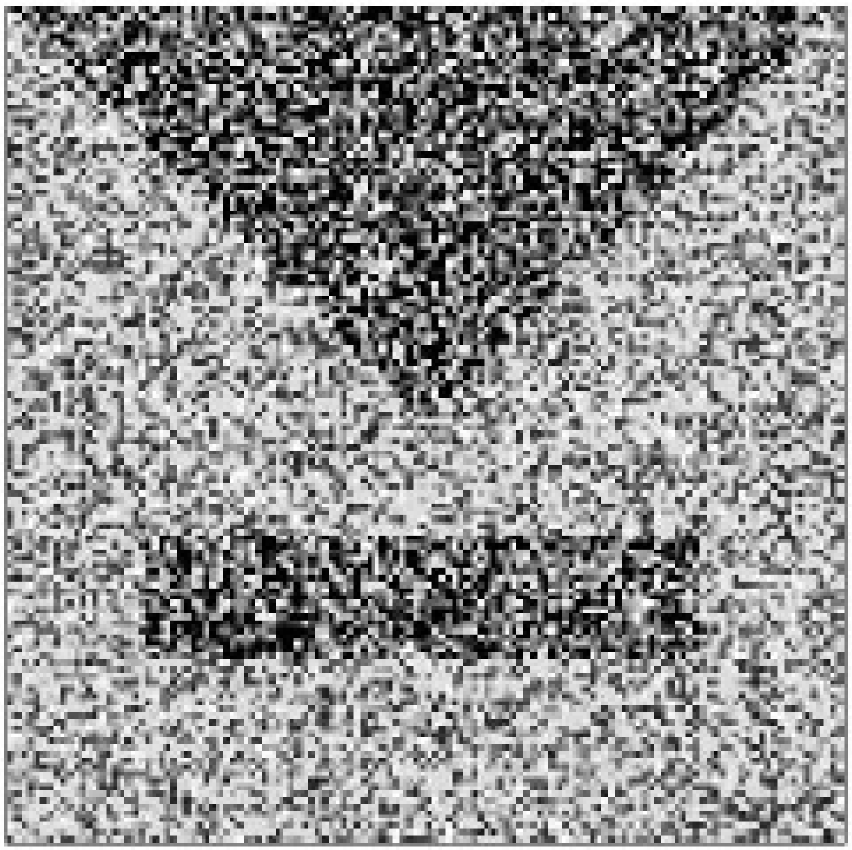


Zero crossings of a second derivative, isotropic operator, after Gaussian smoothing

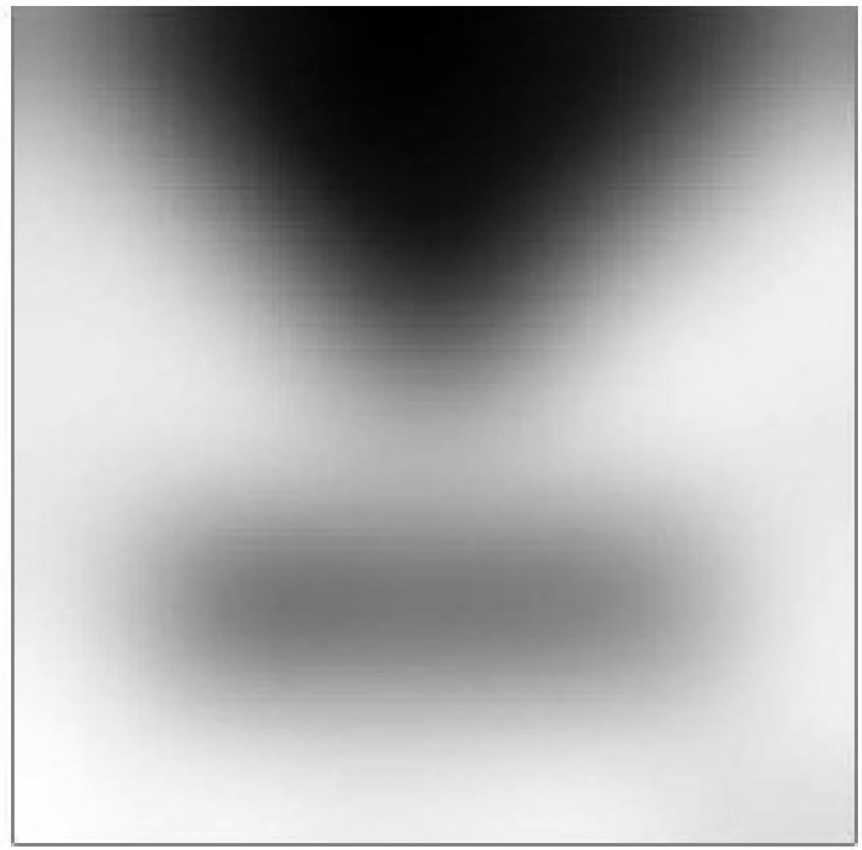
$$\nabla^2 I_{smooth} = 0$$

An application of Laplace's Equation!

# Limits of isotropic Gaussian blurring



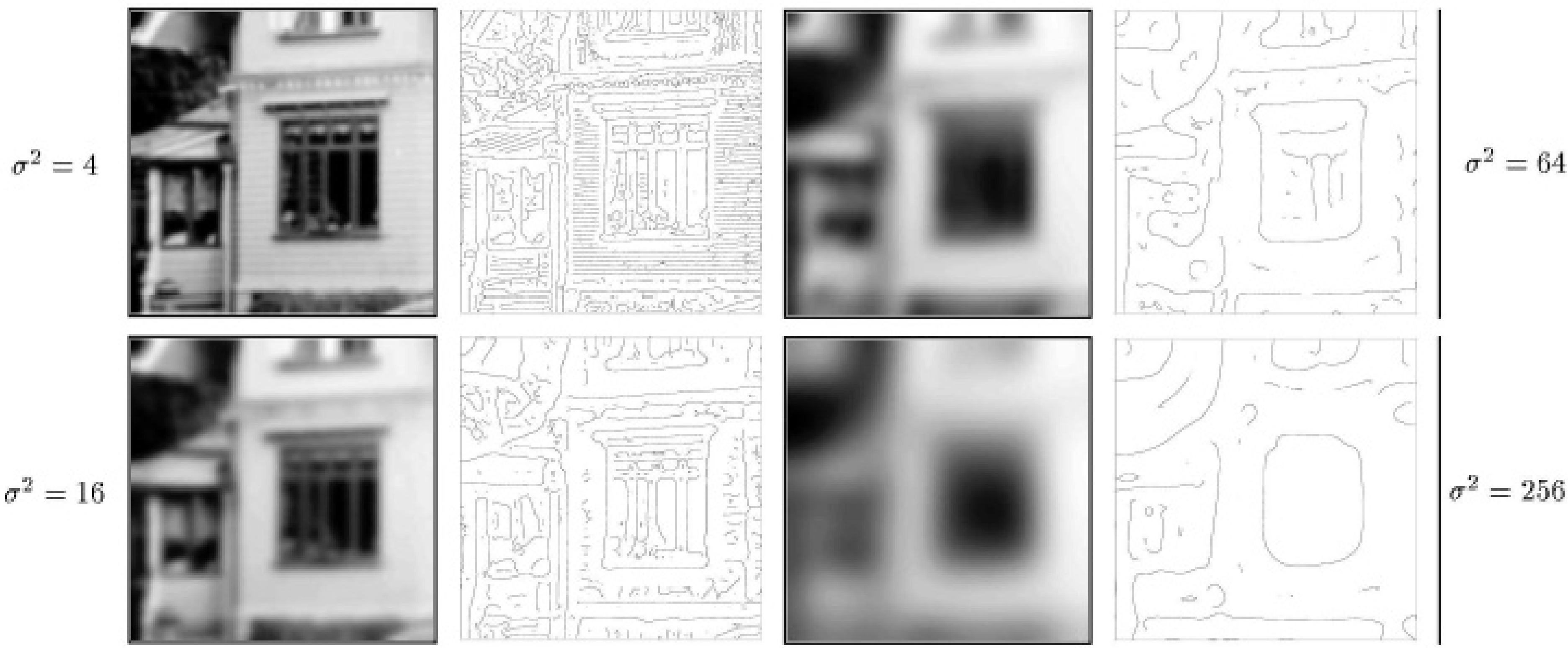
A noisy image



Gaussian blurring

Gaussian is isotropic – takes no account of orientation of image features – so it gives crap edge features

$$\nabla^2 (G_\sigma * I) = 0$$




As the blurring is increased, by increasing the standard deviation of the Gaussian, the structure of the image is quickly lost.

Can we do better? Can we make blurring respect edges?

# Anisotropic diffusion

$$\partial_t I = \nabla^T (g(x;t) \nabla I)$$

$$g(x;t) = e^{-\frac{|\nabla I_\sigma|^2}{k^2}}, \text{ for some constant k, or}$$

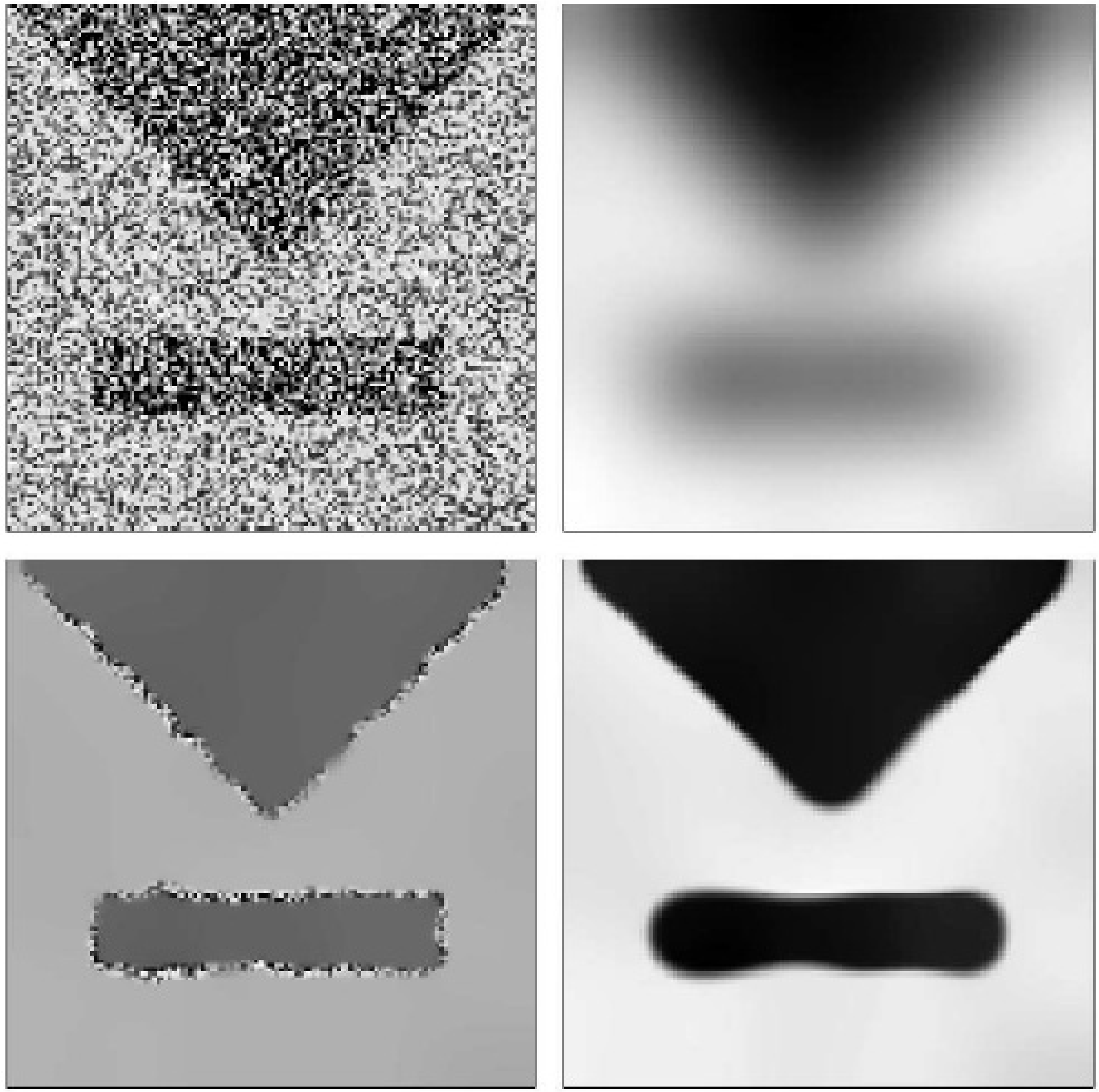
$$g(x;t) = \frac{1}{1 + |\nabla I_\sigma|^2 / k^2}$$

This is a non-linear version of Laplace's Equation, in which the blurring is small across an edge feature (low gradient) and large along an edge.

Anisotropic blurring of the noisy image

Top right: Gaussian

Bottom left and right: different anisotropic blurrings



Example of

anisotropic diffusion – brain MRI images, which are very noisy.

Top: Gaussian blur

Middle and bottom: anistropic blur

Anisotropic blurring of the house image retaining important structures at different degrees of non-linear blurring

Two final examples:

Left – original image

Right – anistropic blurring