Predator-Prey Equations: Modeling Food Chains

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May 7, 2020

Outline

Introduction

Background

- differential equations describing populations of predators and prey
- most famous are Lotka-Volterra equations
- derived by Alfred James Lotka (1880–1949) and Vito Volterra (1860–1940) [2] in the 1920s
- Volterra observed fish, Lotka chemical reactions – both are the same system [2]

Lotka-Volterra Equations

- simplest predator-prey equations
- parameters $a, b, c, d > 0$, following [1]

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy & \text{Prey} \\ \dfrac{dy}{dt} = -cy + dxy & \text{Predator} \end{cases} \tag{1}$$

Expanded Lotka-Volterra Equations

- expansion of (1)
- parameters $a, b, c, d, e, f, g > 0$, following [1]

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy & \text{Prey} \\ \dfrac{dy}{dt} = -cy + dxy - eyz & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = -fz + gyz & \text{Apex Predator} \end{cases} \tag{2}$$

Phase Planes

- space where all points are solutions of a system like (1)
- moving points form trajectories
- stationary points are equilibria [5]
- example of pendulum in FIGURE 1
- give insights into equation without solving

Phase Planes - Example

$x' = y$ and $y' = -\sin(x)$

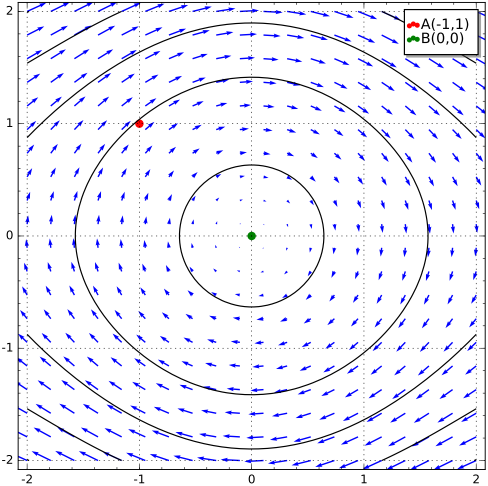


Figure 1: Phase plane of a pendulum with contour lines

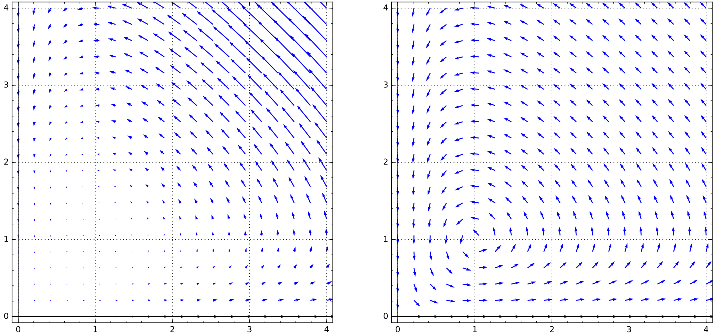
Two-Species Food Chain

Two-Species Food Chain

- standard Lotka-Volterra equations
- equivalent to (2) with $z = 0$
- parameters $a = b = c = d = 1$ chosen for simplicity

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey,} \\ \dfrac{dy}{dt} = y(x-1) & \text{Predator.} \end{cases} \tag{3}$$

Phase Plane



(a) Phase plane (b) Normalized phase plane

Figure 2: Two-species system phase planes with $a = b = c = d = 1$

Phase Plane with Contours

$$C = a \ln y - by + c \ln x - dx \tag{4}$$

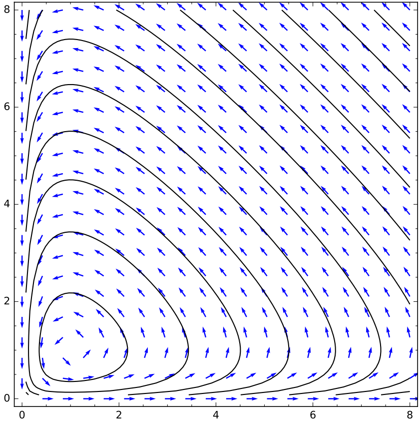


Figure 3: Two-species system phase plane with normalized vectors, contour lines, and $a = b = c = d = 1$

Case $x = 0$

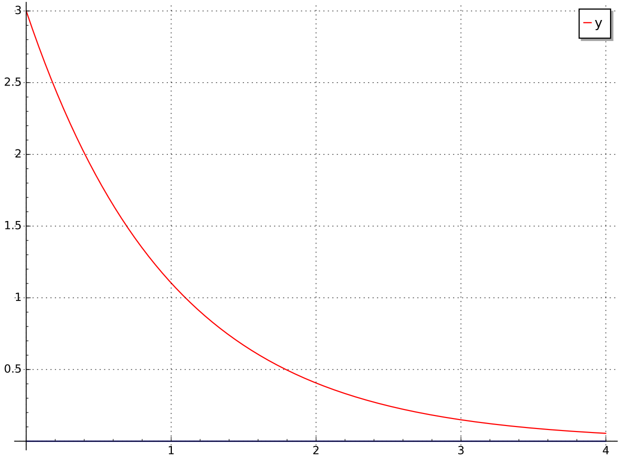


Figure 4: Two-species system graph for $x = 0$, $y = 3$, and $a = b = c = d = 1$

Case $y = 0$

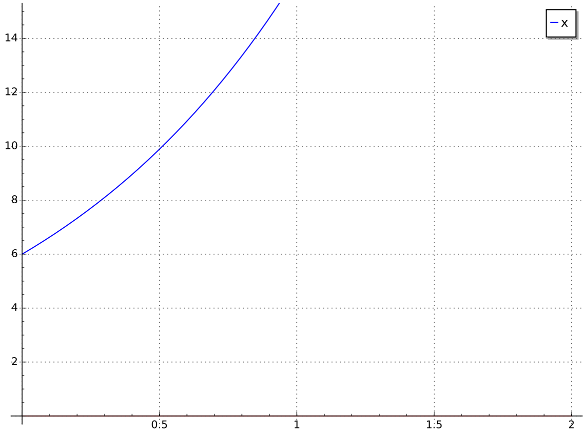


Figure 5: Two-species system graph for $x = 6$, $y = 0$, and $a = b = c = d = 1$

Case $x = 6$, $y = 3$ Contour

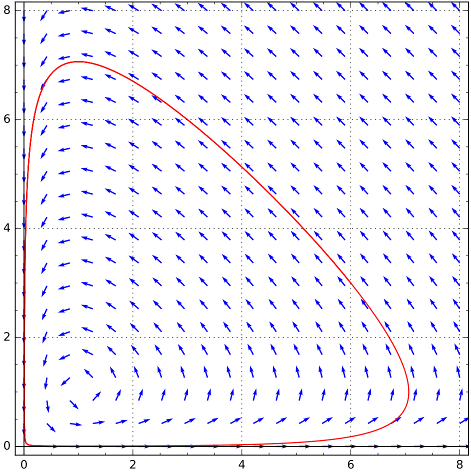


Figure 6: Two-species system contour for $x = 6$, $y = 3$, and $a = b = c = d = 1$

Case $x = 6$, $y = 3$ Graph

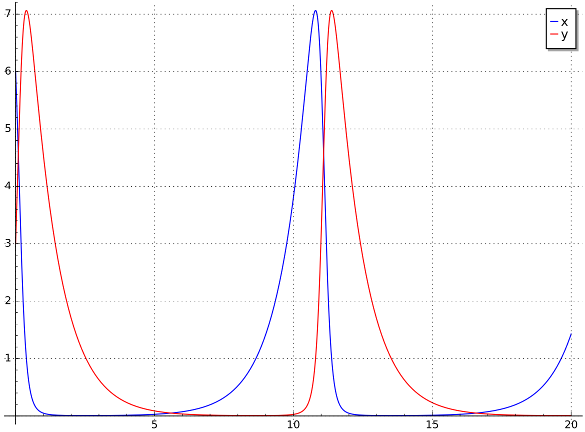


Figure 7: Two-species system graph for $x = 6$, $y = 3$, and $a = b = c = d = 1$

Three-Species Food Chain

Three-Species Food Chain

- equation (2) with simple parameters
- $a = b = c = d = e = f = g = 1$ makes the system

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey} \\ \dfrac{dy}{dt} = y(x-z-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases} \tag{5}$$

Case $z = 0$

Equation becomes the same as two-species system

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey} \\ \dfrac{dy}{dt} = y(x-0-1) = y(x-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = 0(y-1) = 0 & \text{Apex Predator} \end{cases}$$

Case $z = 0$ Contour

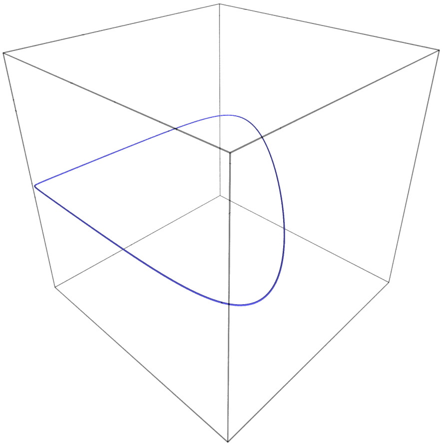


Figure 8: Three-species system contour for $x = 6$, $y = 3$, $z = 0$, $a = b = c = d = e = f = g = 1$

Case $z = 0$ Graph

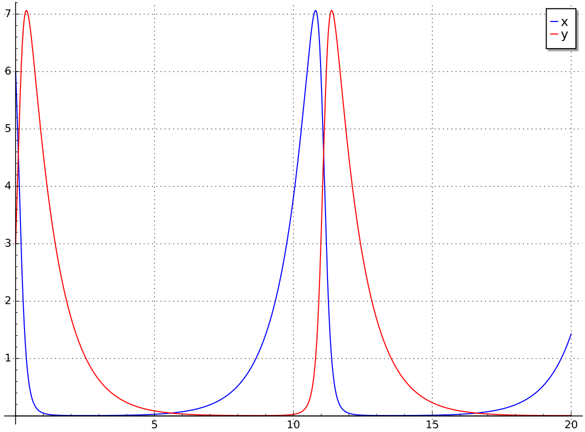


Figure 9: Three-species system graph for $x = 6$, $y = 3$, $z = 0$, $a = b = c = d = e = f = g = 1$

Case $x = 0$

If $x = 0$ we get the following equations

$$\begin{cases} \dfrac{dx}{dt} = 0(1-y) = 0 & \text{Prey} \\ \dfrac{dy}{dt} = y(0-z-1) = y(-z-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases}$$

Case $x = 0$ Graph

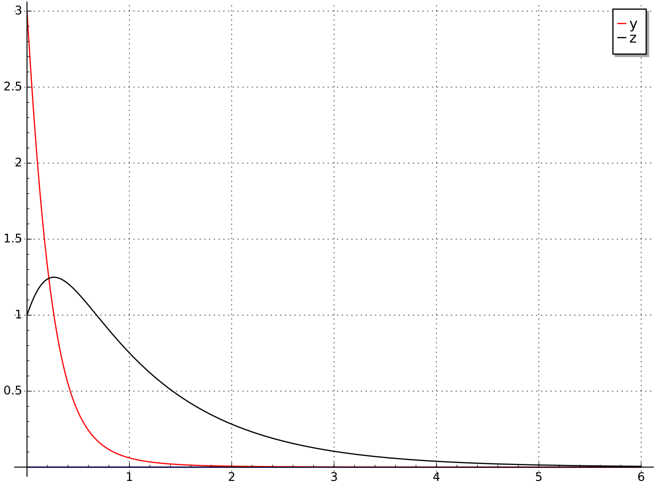


Figure 10: Three-species system graph for $x = 0$, $y = 3$, $z = 1$, $a = b = c = d = e = f = g = 1$

Case $y = 0$

If $y = 0$ we get the following system of equations

$$\begin{cases} \dfrac{dx}{dt} = x(1-0) = x & \text{Prey} \\ \dfrac{dy}{dt} = 0(x-z-1) = 0 & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(0-1) = -z & \text{Apex Predator} \end{cases}$$

Case $y = 0$ Graph

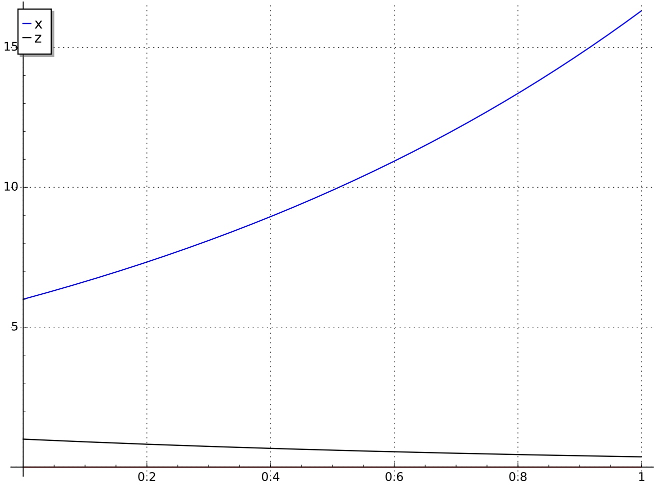


Figure 11: Three-species system graph for $x = 6$, $y = 0$, $z = 1$, $a = b = c = d = e = f = g = 1$

Further Analysis; Case $ga > fb$

- in [1] the authors use further criteria to classify (2)

$$ga > fb, \qquad ga < gb, \qquad ga = fb.$$

- For example $a = g = 1.1$ and all other constants 1

$$\begin{cases} \dfrac{dx}{dt} = 1.1x - xy & \text{Prey,} \\ \dfrac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \dfrac{dz}{dt} = -z + 1.1yz & \text{Apex Predator.} \end{cases}$$

Case $ga > fb$ Graph

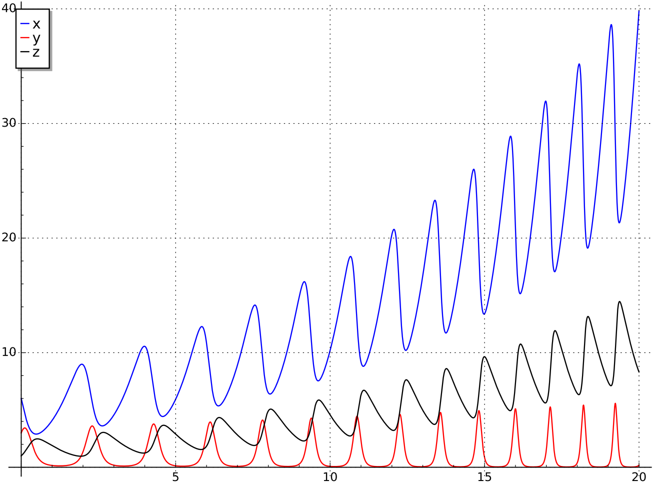


Figure 12: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, $b = c = d = e = f = 1$, and $a = g = 1.1$

Case $ga < fb$

For example $b = f = 1.1$ and all other constants equal 1

$$\begin{cases} \dfrac{dx}{dt} = x - 1.1xy & \text{Prey,} \\ \dfrac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \dfrac{dz}{dt} = -1.1z + yz & \text{Apex Predator.} \end{cases}$$

Case $ga < fb$ Graph

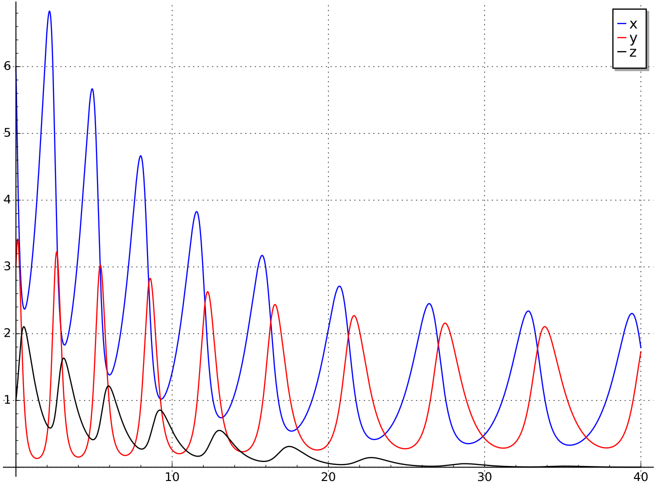


Figure 13: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, $a = c = d = e = g = 1$, and $b = f = 1.1$

Case $ga < fb$ Contour

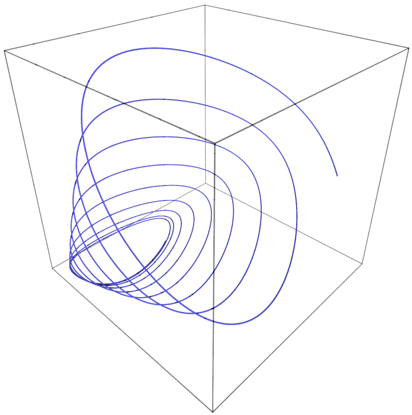


Figure 14: Three-species system contour for $x = 6$, $y = 3$, $z = 1$, $a = c = d = e = g = 1$, and $b = f = 1.1$

Case $ga = fb$

- original equation (5) does not change
- all constants equal 1, $a = b = c = d = e = f = g = 1$
- system could be periodic

Case $ga = fb$ Graph

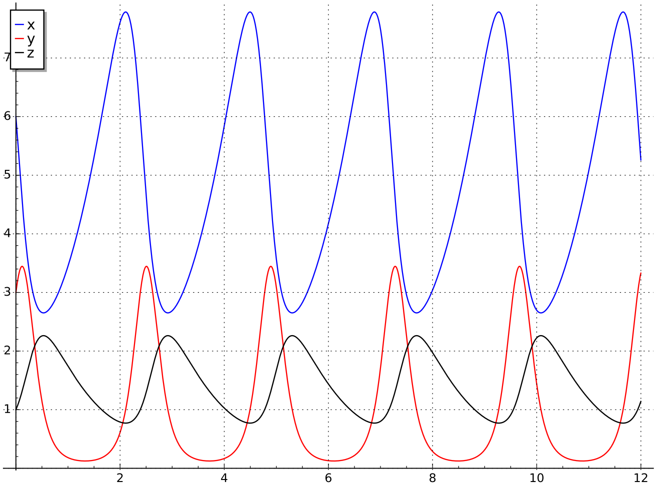


Figure 15: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, and $a = b = c = d = e = f = g = 1$

Case $ga = fb$ Contour

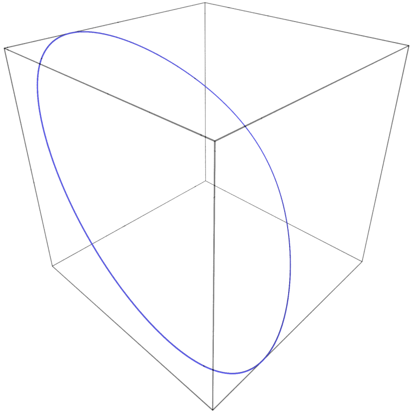


Figure 16: Three-species system contour for $x = 6$, $y = 3$, $z = 1$, and $a = b = c = d = e = f = g = 1$

Conclusion

Conclusion

- models fit general intuition
- inaccuracies: unlimited growth, only one species as prey
- equation had great impact on ecology [6]
- Lotka-Volterra equations (1) are among the most famous differential equations

Q and A

References

[1] E. Chauvet, J. E. Paullet, J. P. Previte, and Z. Walls, *Lotka-Volterra three-species food chain*, Mathematics Magazine, 75 (2002), pp. 243–255. Retrieved from: http://www.jstor.org/stable/3219158.

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