NUM–METH MAT–410: HW 08.09.20 FIXED 2

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1. Problem

Solve

$$\epsilon u'' + au' = f(x)$$

for the parameters

$$a = 1 \qquad \text{and}$$
$$f(x) = x^3.$$

This gives

$$u'' + \frac{u'}{\epsilon} = \frac{x^3}{\epsilon}. \tag{1.1}$$

Consider the boundary conditions

$$\begin{cases} \phi_0 &= \zeta_0 u(0) - \eta_0 \epsilon u'(0) \qquad \text{and} \\ \phi_1 &= \zeta_1 u(1) + \eta_1 \epsilon u'(1) \end{cases}$$

for the cases

$$\begin{cases} \zeta_0 = \zeta_1 = 1 \qquad ; \quad \eta_0 = \eta_1 = 0 \qquad \text{and} \\ \zeta_0 = \eta_0 = 1 \qquad ; \quad \zeta_1 = 1,\ \eta_1 = 0. \end{cases}$$

This yields two cases. Case 1 is

$$\begin{cases} \phi_0 &= u(0) \\ \phi_1 &= u(1) \end{cases} \tag{1.2}$$

and case 2 is

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$$\begin{cases} \phi_0 & = u(0) - \epsilon u'(0) \\ \phi_1 & = u(1) \end{cases} \tag{1.3}$$

2. General Solution to Equation (1.1)

2.1. Homogeneous Solution. Solve (1.1) as a homogeneous equation in the form

$$u_H = c_1 e^{\lambda_1} + c_2 e^{\lambda_2}. \tag{2.1}$$

Find the λs

$$\epsilon u'' + u' = 0$$
$$u'' + \frac{1}{\epsilon} u' = 0$$
$$\lambda^2 + \frac{1}{\epsilon}\lambda = 0$$
$$\lambda_1 = 0 \qquad \lambda_2 = -\frac{1}{\epsilon}.$$

Now substitute the λs into (2.1) and find the homogeneous solution

$$\begin{aligned} u_H &= c_1 + c_2 e^{-1/\epsilon \cdot x} \\ u_1 = 1 &\qquad u_2 = e^{-1/\epsilon \cdot x}. \end{aligned} \tag{2.2}$$

2.2. Particular Solution. Using the variation of parameters method we can find the particular solution with the help of (2.2)

$$u_P = -u_1 \int \frac{u_2 f(x)}{W(u_1, u_2)} dx + u_2 \int \frac{u_1 f(x)}{W(u_1, u_2)} dx$$
$$W(u_1, u_2) = u_1 u_2' - u_2 u_1' = -\frac{1}{\epsilon} e^{-1/\epsilon \cdot x}.$$

Plugging in W and cancelling, we get

$$u_P = -\int \frac{e^{-1/\epsilon \cdot x} \frac{1}{\epsilon} x^3}{-\frac{1}{\epsilon} e^{-1/\epsilon \cdot x}} dx + e^{-1/\epsilon \cdot x} \int \frac{\frac{1}{\epsilon} x^3}{-\frac{1}{\epsilon} e^{-1/\epsilon \cdot x}} dx$$
$$u_P = \int x^3 dx - e^{-1/\epsilon \cdot x} \int x^3 \cdot e^{1/\epsilon \cdot x} dx.$$

From here 3 rounds of integration by parts results in

$$u_P = \frac{1}{4} x^4 - e^{-1/\epsilon \cdot x} \left[\epsilon x^3 e^{1/\epsilon \cdot x} - 3\epsilon \left[\epsilon x^2 e^{1/\epsilon \cdot x} - 2\epsilon \left[\epsilon x e^{1/\epsilon \cdot x} - \epsilon \left(\epsilon e^{1/\epsilon \cdot x} \right) \right] \right] \right]$$

which can be reduced to

$$u_P = \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x + 6\epsilon^4. \tag{2.3}$$

Now we can find the complete solution by adding the homogeneous solution (2.2) and the particular solution (2.3) together

$$\begin{aligned} u &= u_H + u_P \\ u &= c_1 + c_2 \cdot e^{-1/\epsilon \cdot x} + \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x + 6\epsilon^4 \\ u' &= -\frac{1}{\epsilon}c_2 e^{-1/\epsilon \cdot x} + x^3 - 3\epsilon x^2 + 6\epsilon^2 x - 6\epsilon^3 \end{aligned} \tag{2.4}$$

3. Solution to BVP 1

Combining (1.2) with (2.4) yields

$$\begin{aligned} \phi_0 &= u(0) = c_1 + c_2 + 6\epsilon^4 \\ \phi_1 &= u(1) = c_1 + c_2 e^{-1/\epsilon} + \frac{1}{4} - \epsilon + 3\epsilon^2 - 6\epsilon^3 + 6\epsilon^4. \end{aligned}$$

Rearranging the equations we get

$$\begin{aligned} c_1 &= \phi_0 - 6\epsilon^4 - \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3}{1 - e^{-1/\epsilon}} \\ c_2 &= \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3}{1 - e^{-1/\epsilon}}. \end{aligned}$$

Plugging c_1 and c_2 into (2.4) yields

$$\begin{aligned} u = {} & \phi_0 - 6\epsilon^4 - \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3}{1 - e^{-1/\epsilon}} \\ & + \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3}{1 - e^{-1/\epsilon}} \cdot e^{-1/\epsilon \cdot x} \\ & + \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x + 6\epsilon^4 \\ u = {} & (e^{-1/\epsilon \cdot x} - 1)\frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3}{1 - e^{-1/\epsilon}} \\ & + \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x + \phi_0 \end{aligned}$$

4. Solution to BVP 2

Combining (1.3) with (2.4) yields

$$\begin{aligned} \phi_0 &= u(0) - \epsilon u'(0) = c_1 + 2c_2 + 12\epsilon^4 \\ \phi_1 &= u(1) = c_1 + c_2 e^{-1/\epsilon} + \frac{1}{4} - \epsilon + 3\epsilon^2 - 6\epsilon^3 + 6\epsilon^4. \end{aligned}$$

Rearranging the equations we get

$$
\begin{aligned}
c_1 &= \phi_0 - 12\epsilon^4 - 2\left(\frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3 - 6\epsilon^4}{2 - e^{-1/\epsilon}}\right) \\
c_2 &= \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3 - 6\epsilon^4}{2 - e^{-1/\epsilon}}.
\end{aligned}
$$

Plugging c_1 and c_2 into (2.4) yields

$$
\begin{aligned}
u &= \phi_0 - 12\epsilon^4 - 2\left(\frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3 - 6\epsilon^4}{2 - e^{-1/\epsilon}}\right) \\
&\quad + \frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3 - 6\epsilon^4}{2 - e^{-1/\epsilon}} \cdot e^{-1/\epsilon \cdot x} \\
&\quad + \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x + 6\epsilon^4 \\
u &= \phi_0 + (e^{-1/\epsilon \cdot x} - 2)\frac{\phi_0 - \phi_1 + 1/4 - \epsilon + 3\epsilon^2 - 6\epsilon^3 - 6\epsilon^4}{2 - e^{-1/\epsilon}} \\
&\quad + \frac{1}{4}x^4 - \epsilon x^3 + 3\epsilon^2 x^2 - 6\epsilon^3 x - 6\epsilon^4
\end{aligned}
$$