NumMeth MAT-410 Lab 2 Report

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1 Introduction

In this report I will detail my solutions to the tasks from our lab 2 PDF regarding a weighted scheme for a one-dimensional heat equation. I was assigned problem 4 and schemes 1, 2, and 3. For these schemes, I will talk about the conditions for the Courant number and how to guarantee the monotony of each of the schemes. Then the influence of monotony on the correspondence of the numerical and analytic solutions will be discussed. The possible deterioration in case monotony is not guaranteed will also be covered. Finally, the three schemes will be compared with respect to their graphs and errors for the same tasks.

1.1 The Task

The heat equation this report covers is the following:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + f(x,t); \qquad 0 < x < 1; \qquad t > 0.$$

The parameter ε lies on the interval $(0,1]$ and can be specified using the parameter k as $\varepsilon = 2^{-k}$. In this report, $f(x,t) = 0$. The initial condition for this equation is

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1$$

and the boundary conditions are

$$\begin{aligned} u(0,t) &= \psi_0(t), \\ u(1,t) &= \psi_1(t), \qquad t \ge 0. \end{aligned}$$

This equation has both x and t as parameters, meaning that it changes in x and in time t.

The problem I am working with is problem 4 from our lab 2 PDF. It has two additional parameters, $T1$ and $T2$, which are the initial temperatures on the intervals $[0,0.5)$ and $(0.5,1]$ respectively. At the point $x = 0.5$, the temperature equals $\frac{T1+T2}{2}$.

1.2 The Schemes

This report covers 3 schemes, schemes 1, 2, and 3 from the lab 2 PDF. The schemes are different ways of solving the above equation by first discretizing the domain of x and t into grid points. The step size in x is called h and the step size in t is called τ. The equation is solved by iteration through t from 0 to t_{max} and for each t_m by solving a system of equations from $x = 0$ to $x = 1$. Then the weighted scheme (V.2.4) from page 2 of lab 2 PDF gives this system of equations. (V.2.4) can be transformed into a form (V.2.8) that can easily be solved using the tridiagonal matrix algorithm (progonka). (V.2.8) is the formula for this algorithm

$$-Au_{i-1}^{m} + Bu_i^m - Cu_{i+1}^m = A^0 u_{i-1}^{m-1} + B^0 u_i^{m-1} + C^0 u_{i+1}^{m-1}.$$

The parameters for this algorithm are defined in (V.2.9)–(V.2.11). (V.2.11) contains the definition of the parameters in the equation above. They are defined as

$$\begin{cases} A = C = \theta \cdot K, \\ B = 1 + A + C, \\ A^0 = C^0 = (1-\theta) \cdot K, \\ B^0 = 1 - A^0 - c^0, \\ K = \frac{\varepsilon \tau}{h^2}. \end{cases}$$

Here K is the Courant number and θ (the "weight", $0 \le \theta \le 1$) is the parameter that differentiates the schemes from each other. The following three schemes are covered in this report:

1. Explicit scheme (ES), where $\theta = 0$,
2. Crank–Nicolson scheme (CNS), where $\theta = 0.5$,
3. Implicit scheme (IS), where $\theta = 1$.

The parameter θ dramatically changes the way these three schemes behave, as will be demonstrated in the following sections.

2 Monotonicity

All three schemes are considered monotonous if the following conditions holds (V.2.13)

$$\max\left(0, 1 - \frac{1}{2K}\right) \leq \theta \leq 1.$$

In effect this means that for these schemes K can only have values such that $K \leq 0.5$ or $K \leq \frac{1}{2(1-\theta)}$.

2.1 Explicit Scheme

For this scheme $\theta = 0$, so

$$K \leq \frac{1}{2(1-\theta)}; \quad K \leq \frac{1}{2(1)}; \quad K \leq 0.5.$$

If $K \leq 0.5$, ES is monotonous.

2.2 Crank–Nicolson Scheme

For this scheme $\theta = 0.5$, so

$$K \leq \frac{1}{2(1-\theta)}; \quad K \leq \frac{1}{2(0.5)}; \quad K \leq 1.$$

If $K \leq 1$, CNS is monotonous.

2.3 Implicit Scheme

For this scheme $\theta = 1$, so

$$K \leq \frac{1}{2(1-1)}; \quad K \leq \frac{1}{2(0)}.$$

Here, the limit of K cannot be found, so I would say that IS is monotonous for all K.

3 Monotonicity and Accuracy

Monotonicity has a positive influence on the accuracy of an approximation because it tends to prohibit oscillations in the numerical solution. Nonetheless, it is not a guarantee for accuracy. As an example of this, consider the case where $T1 = -2$, $T2 = 2$, $N = 65$ (number of nodes in space), $M = 257$ (number of nodes in time), $\varepsilon = 0.125$, and $t_{max} = 7$.

Using the formula laid out above, we can find K to be equal to 14. This means that both ES and CNS are not monotonous. For ES the numerical solution break down completely, starts to oscillate, and the error becomes so large that the variable holding it overflows – this would be the expected outcome. For CNS however, which is also not monotonous, this is not the case. CNS only slightly oscillates but converges to the analytic solution. This behavior can be seen in Figure 1.

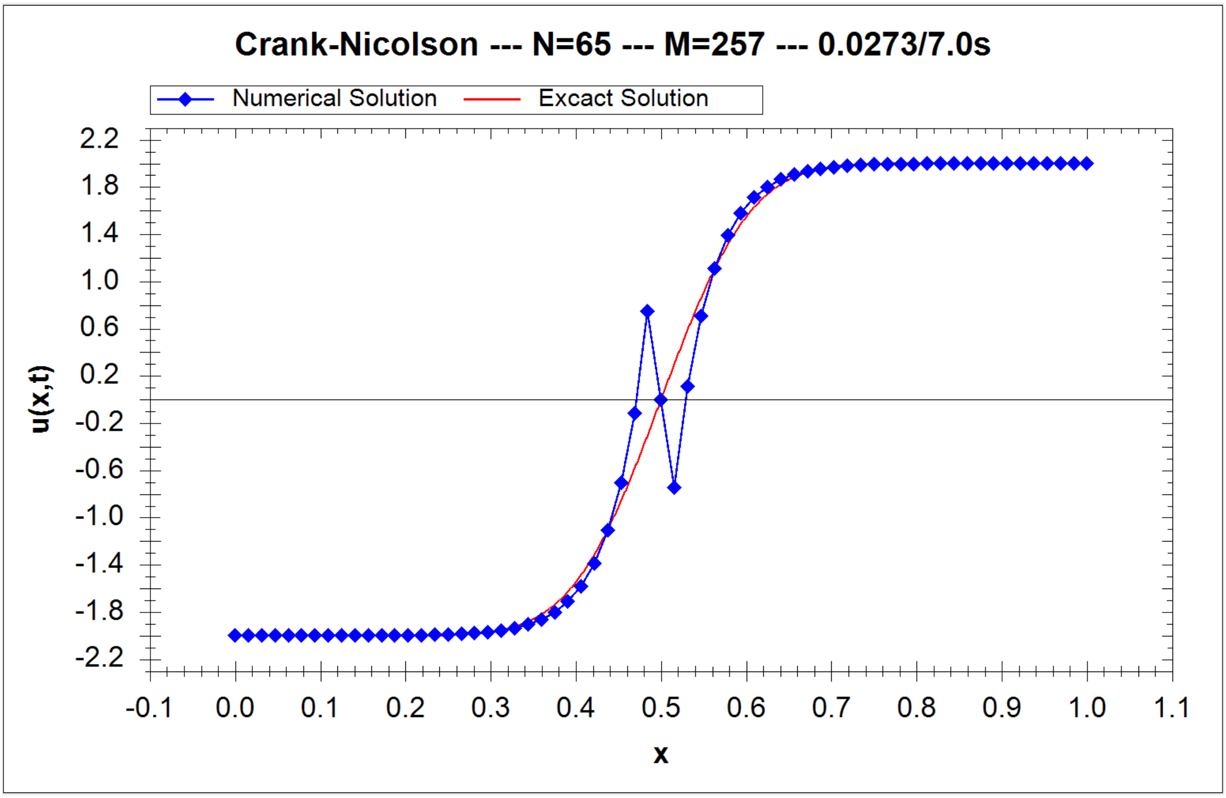

Figure 1: CNS for the values stated above at $t = 0.0273$ while not monotonous

As t increases and the sharp transition between the temperature levels becomes more smooth, the oscillations become smaller and finally stop.

IS does not have trouble with this example because it is monotonous for any value of K. Figure 2 compares the error of CNS and IS for this example as t progresses.

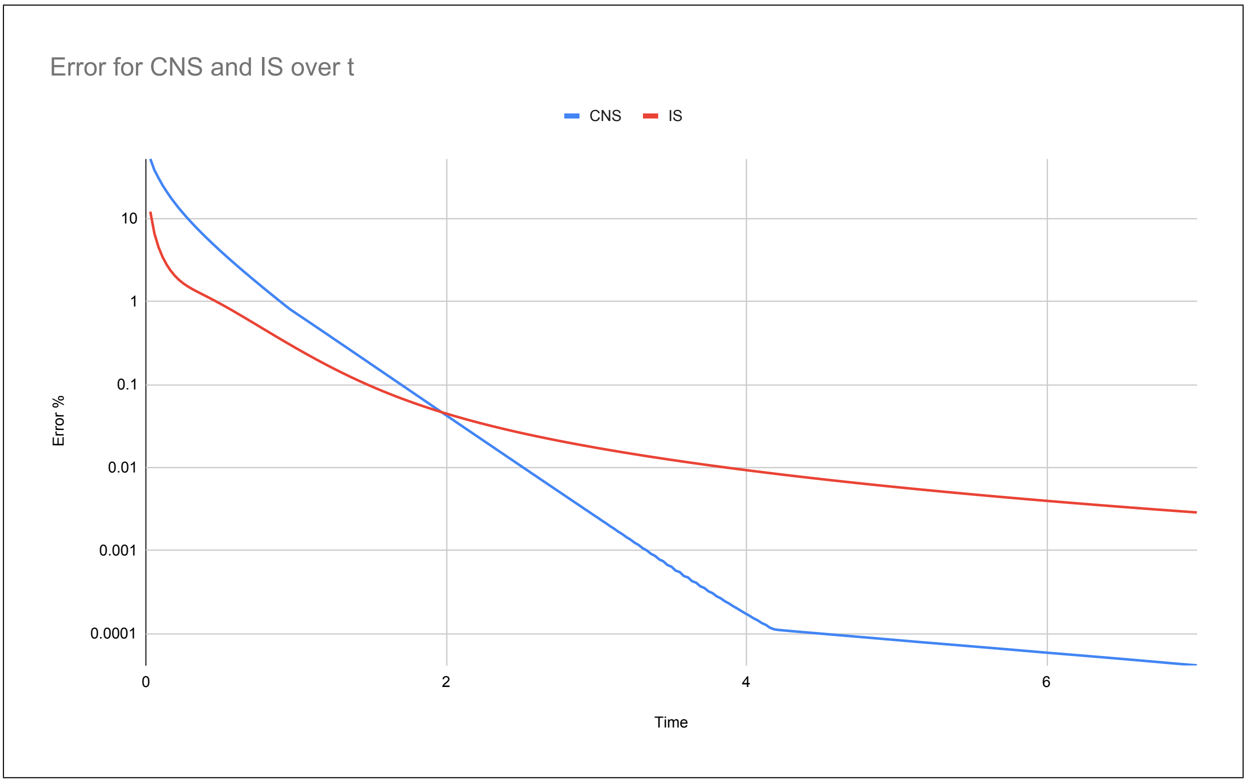

Figure 2: CNS and IS errors over time with a $\log_{10}$ y axis

We can see that IS starts out with a lower error but soon CNS becomes more accurate because it is a generally more accurate method. The fact that it is not monotonous makes it less accurate at the start but as soon as the function becomes less extreme and easier to approximate CNS becomes more accurate. This example illustrates that even if a scheme is not monotonous it can still be accurate, maybe even more accurate than a monotonous scheme.

Monotonicity is a beneficial condition for a scheme and has a positive effect on its accuracy. Nonetheless, the differences between schemes may have a larger influence and a more accurate scheme might be able to overcome the disadvantages of not being monotonous.

4 Deterioration of Solutions

A related question is whether a scheme that is not monotonous will deteriorate over time, becoming less accurate as time goes on. If a scheme is not monotonous, it might start to oscillate and deteriorate. To illustrate this point, I chose $T1 = -2$, $T2 = 2$, $t_{max} = 7$, $\varepsilon = 2^{-12}$, $N = 65$, and $M = 5$. This gives a Courant number of $K = 1.75$. Again, ES and CNS are not monotone and thus might deteriorate.

Starting at $t = 0$, all three schemes have the exact same shape, as see in Figure 3.

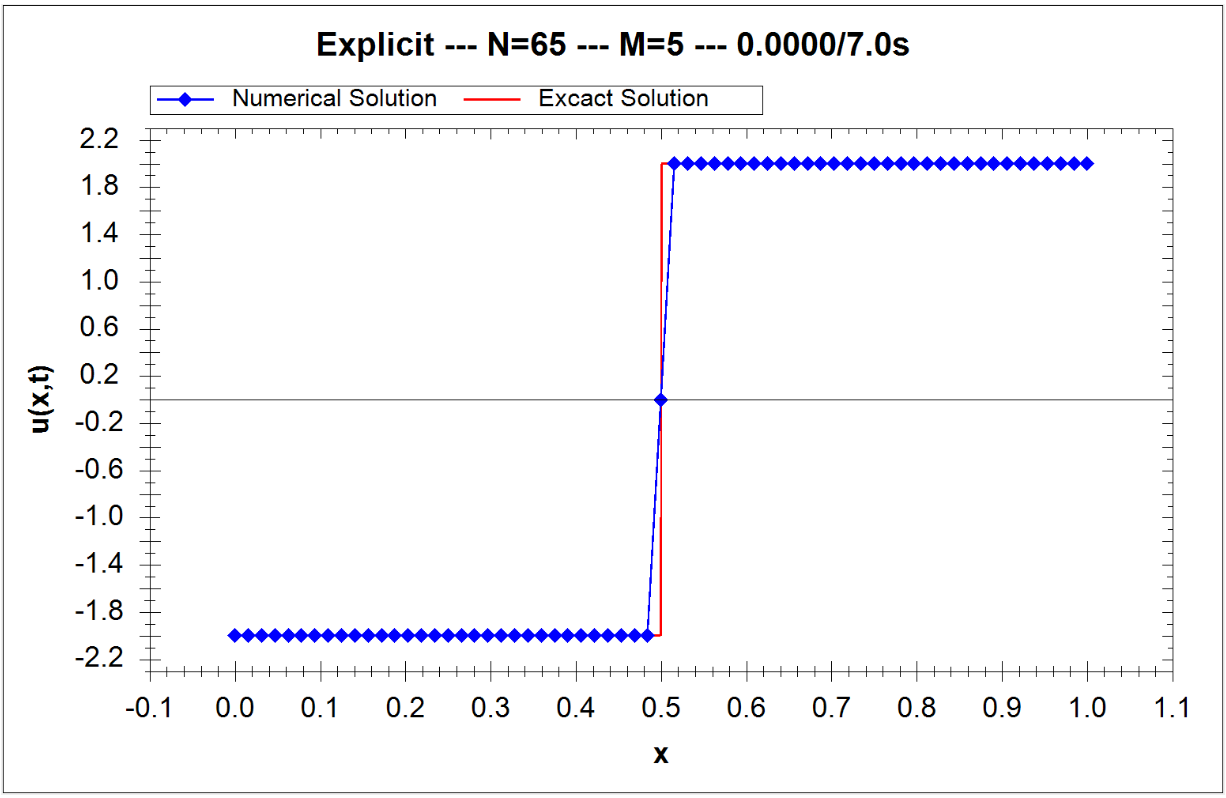

Figure 3: ES at $t = 0$

One time step further, ES has already started to strongly oscillate, see Figure 4.

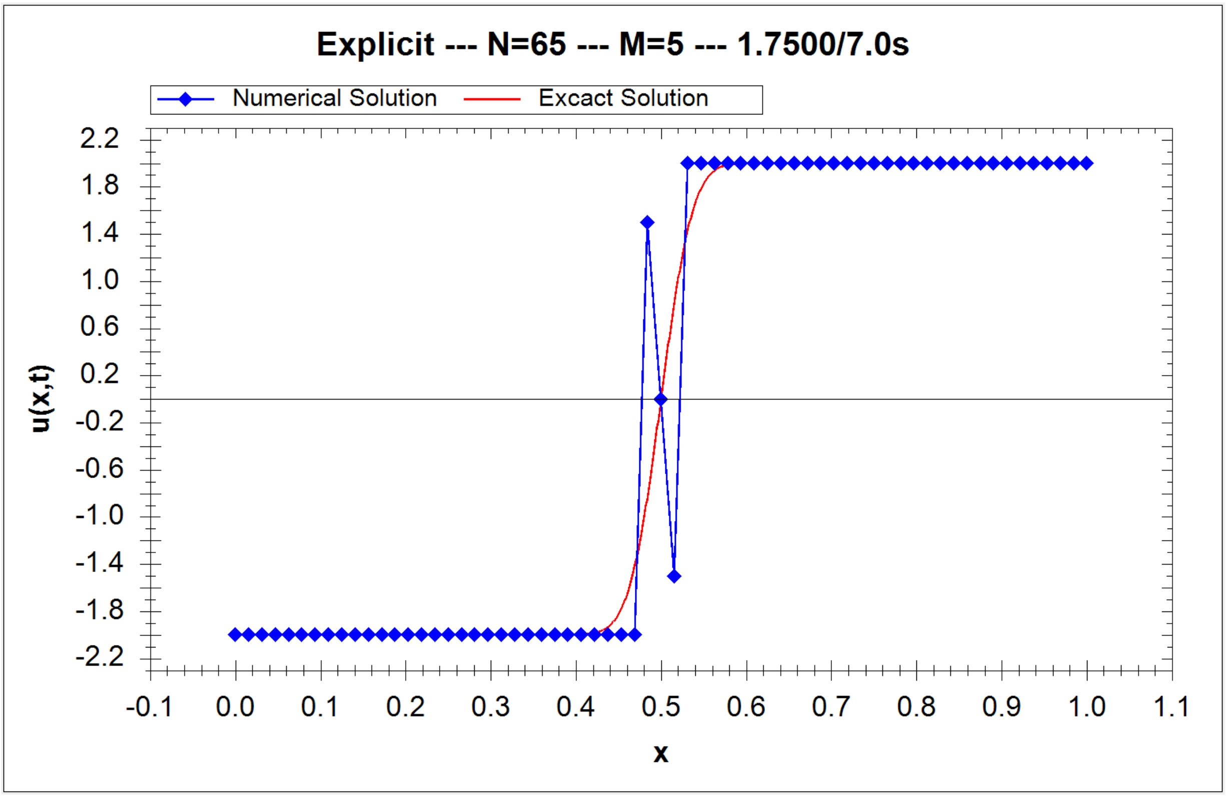

Figure 4: ES at $t = 1.75$

In contrast to this, the other scheme that is not monotone, CNS, does not oscillate, see Figure 5.

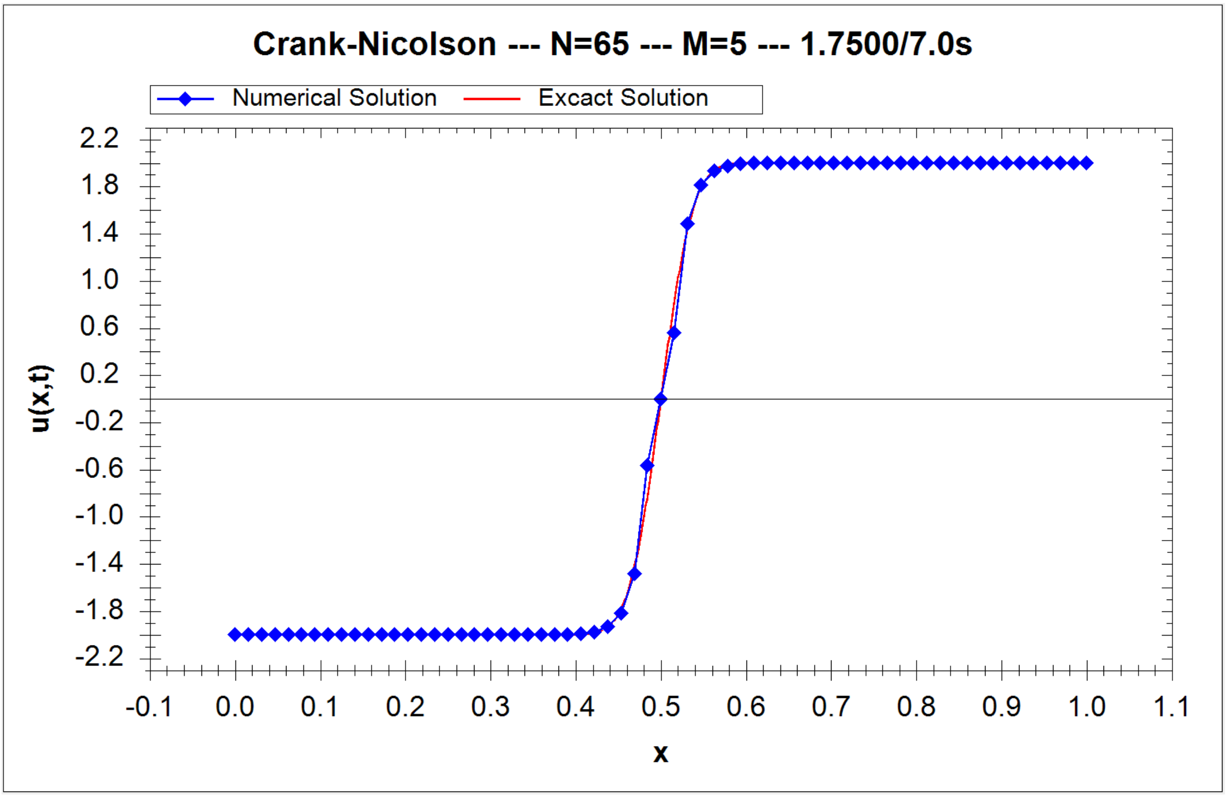

Figure 5: CNS at $t = 1.75$

The behavior seen here continues for the whole time span of the analysis. Figure 6 shows the errors of all three schemes for this problem.

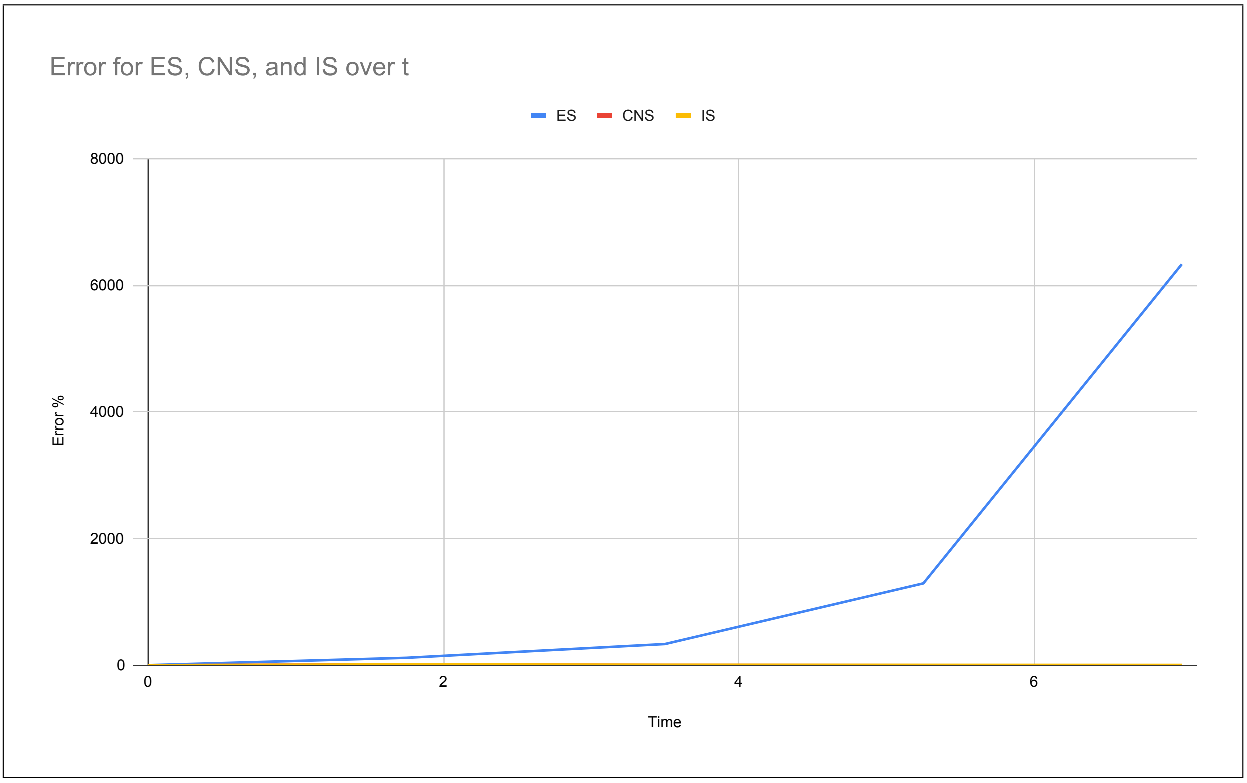

Figure 6: ES, CNS, and IS errors over time

It is obvious from Figure 6 that ES deteriorates and the error becomes very large. CNS on the other hand remains accurate and does not deteriorate. In Figure 7 the axis has been adjusted to show CNS and IS more clearly. The same pattern as in the previous section emerges, as IS is more accurate for the initial values of t when the shape of the function is more extreme but then CNS becomes more accurate as the functions smoothes out.

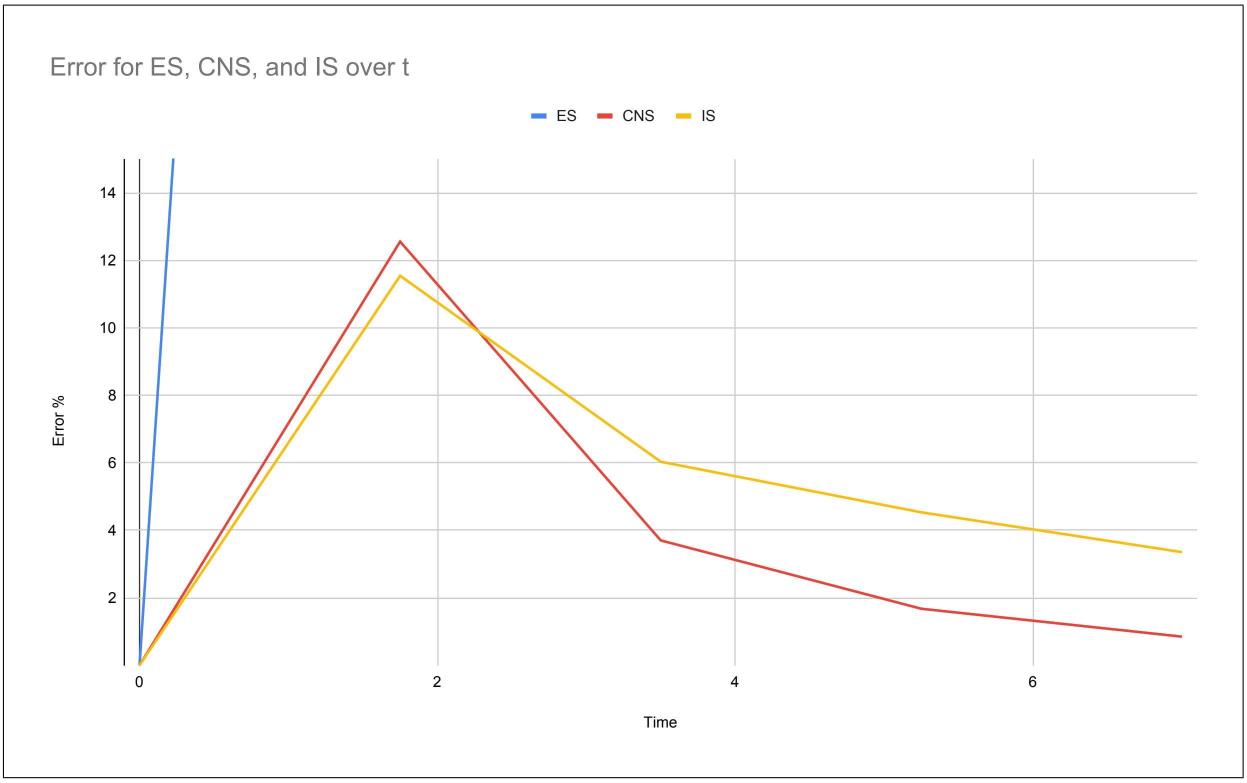

Figure 7: ES, CNS, and IS errors over time; adjusted axis

A non-monotonous scheme might deteriorate if it starts to oscillate strongly. ES has shows that if it is not monotonous it will strongly oscillate and deteriorate, while CNS, despite not being monotonous, will not oscillate and thus it will not deteriorate. IS is again not influenced by the monotonicity as it is always monotonous.

5 Comparison of the Schemes

5.1 Comparison 1

To compare all 3 schemes in addition to the previous sections, I chose 2 more illustrative examples. The first example has $N = 5$, $M = 5$, $T1 = -2$, $T2 = 2$, $\varepsilon = 1$, $t_{max} = 1$, and a Courant number of $K = 4$. Again, ES and CNS are not monotonous. At $t = 0$ all three schemes look the same as shown here by IS in Figure 8.

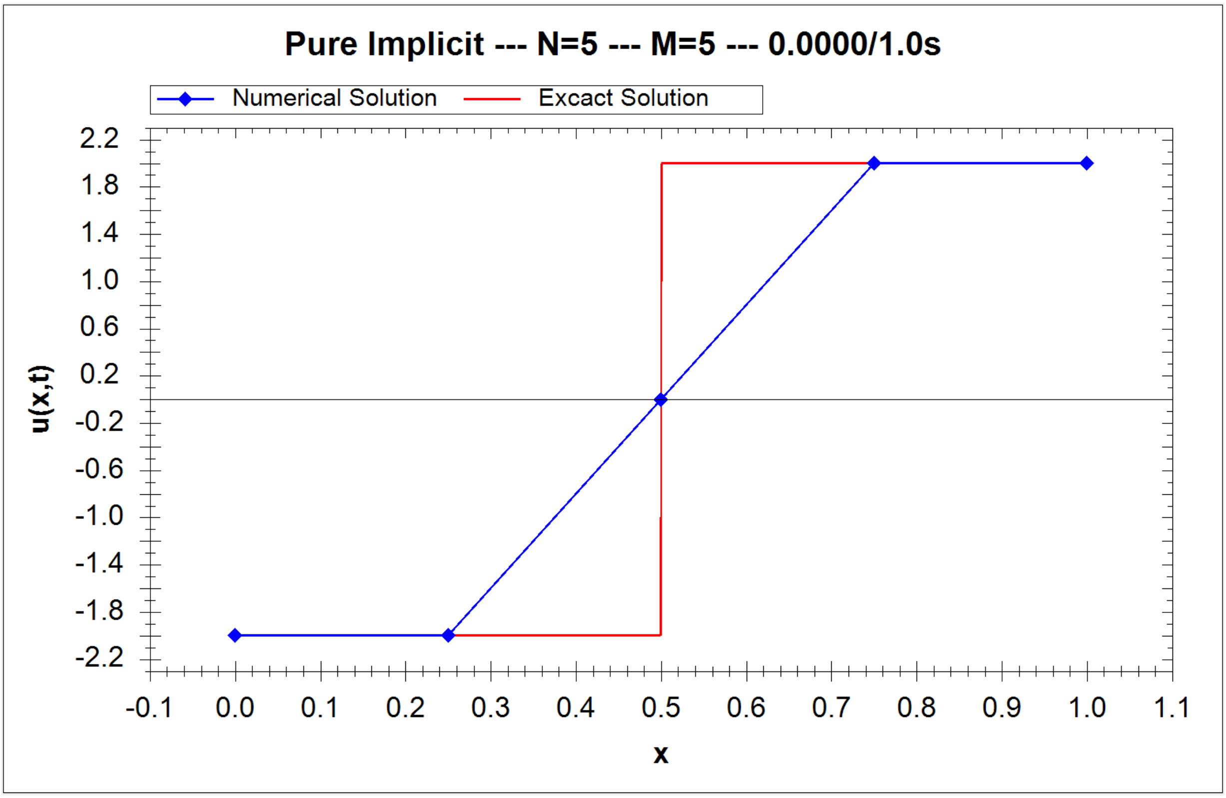

Figure 8: IS at $t = 0$

In the next step, the differences between the schemes become obvious. Figure 9 shows ES at $t = 0.25$. We can see the oscillation and should take note of the changed axis minimum and maximum in this graph.

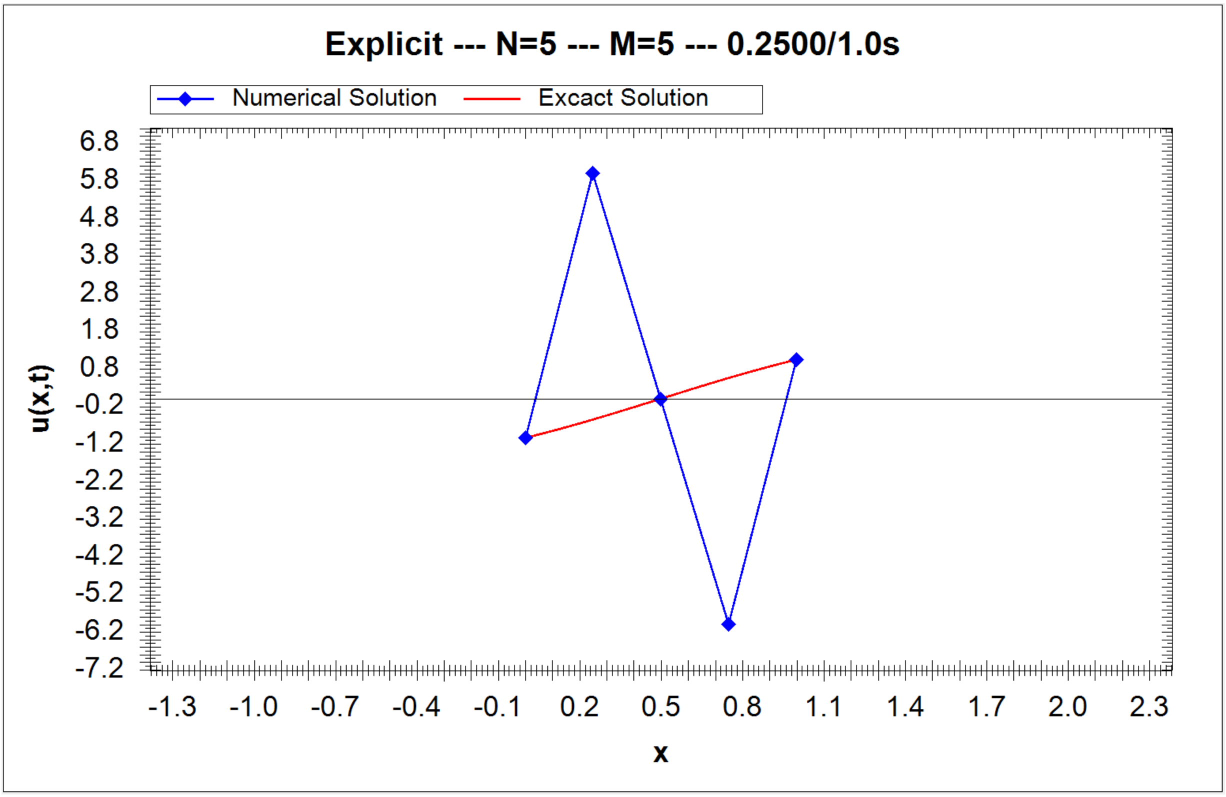

Figure 9: ES at $t = 0.25$

The behavior of CNS at the point $t = 0.25$ is shown in Figure 10.

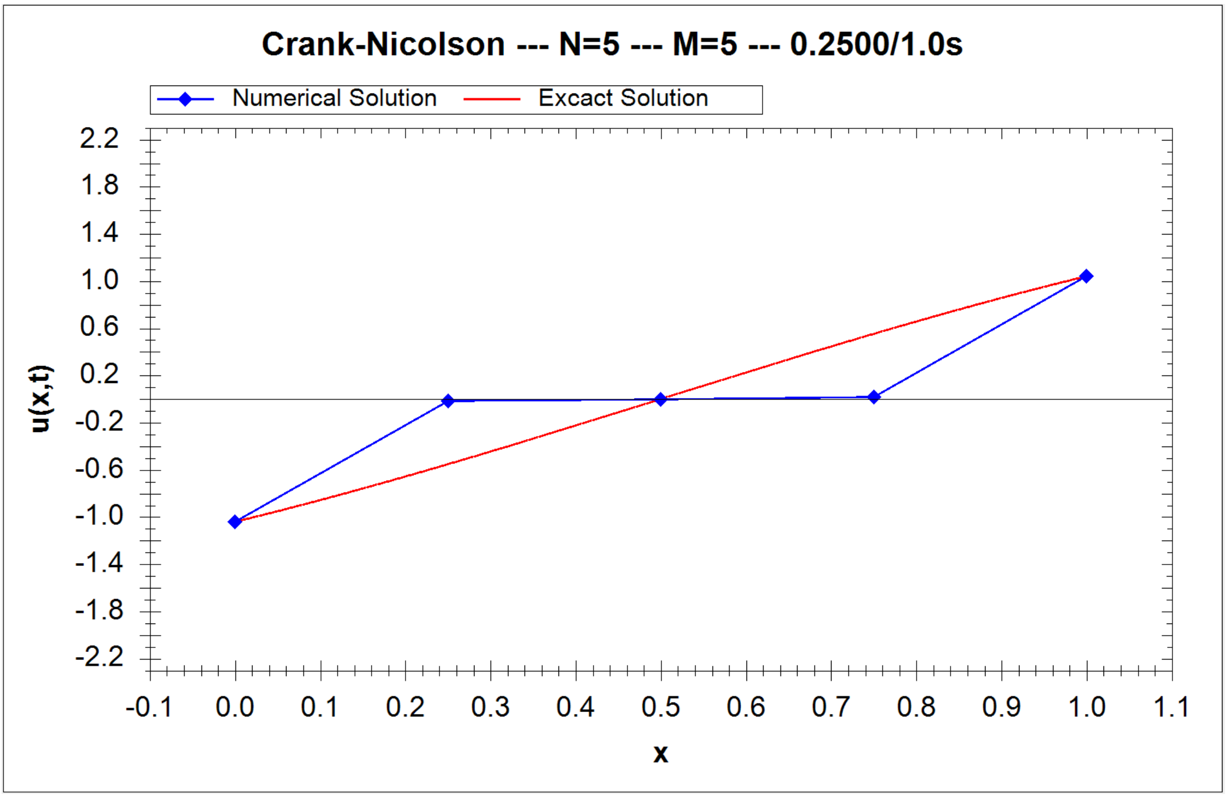

Figure 10: CNS at $t = 0.25$

Finally, IS at $t = 0.25$ can be seen in Figure 11.

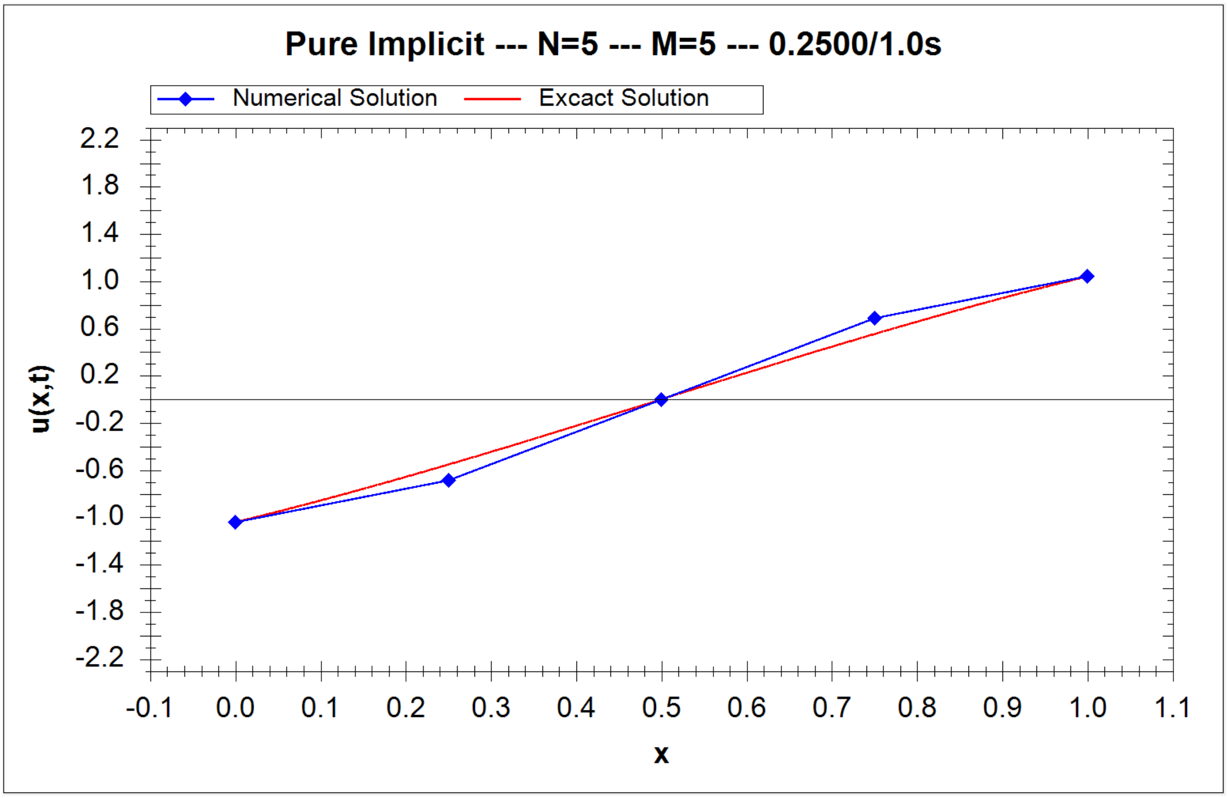

Figure 11: IS at $t = 0.25$

These three figures support the assumption we have made previously that ES oscillates and deteriorates when it is not monotonous and that CNS is more accurate even when not monotonous. Furthermore, IS is again the most accurate scheme when the shape of the function is more extreme. Figure 12 shows the graphs of the errors and summarizes these observations. A difference here is that IS remains the most accurate scheme even as t increases.

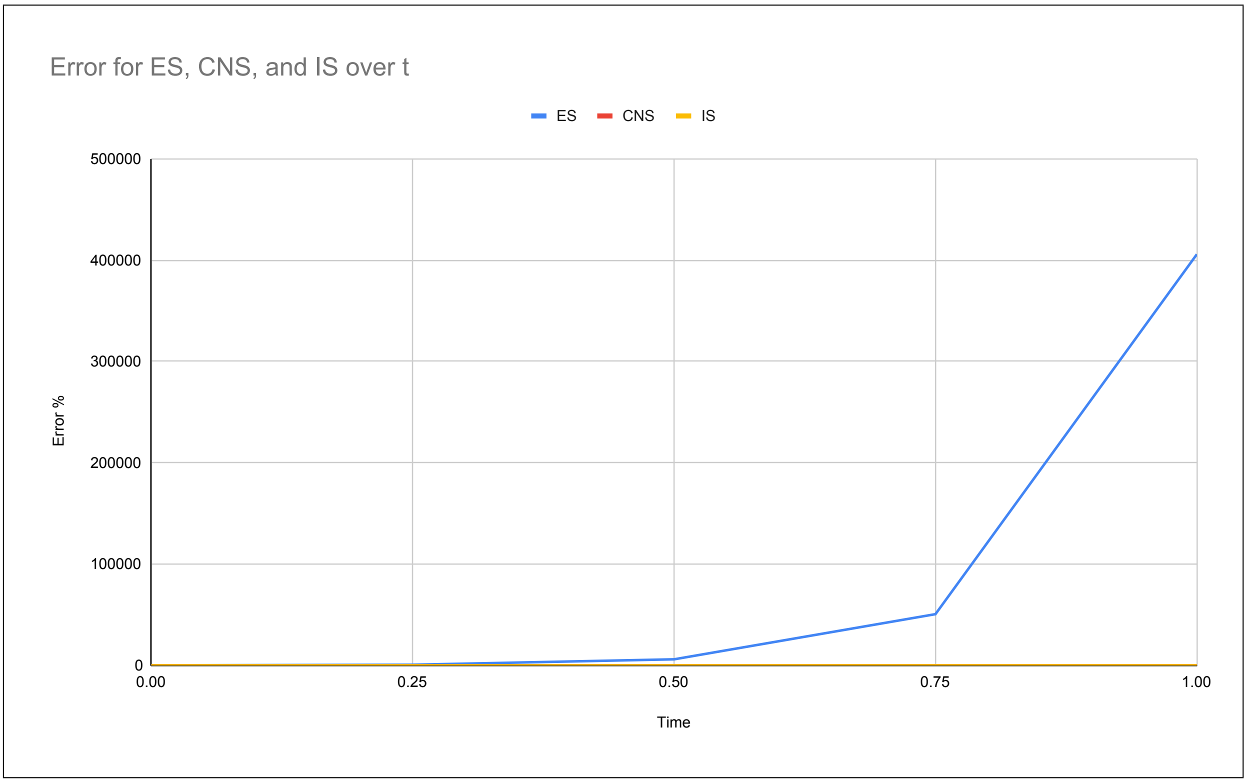

Figure 12: ES, CNS, and IS errors over time

To see the behavior, Figure 13 has an adjusted y axis.

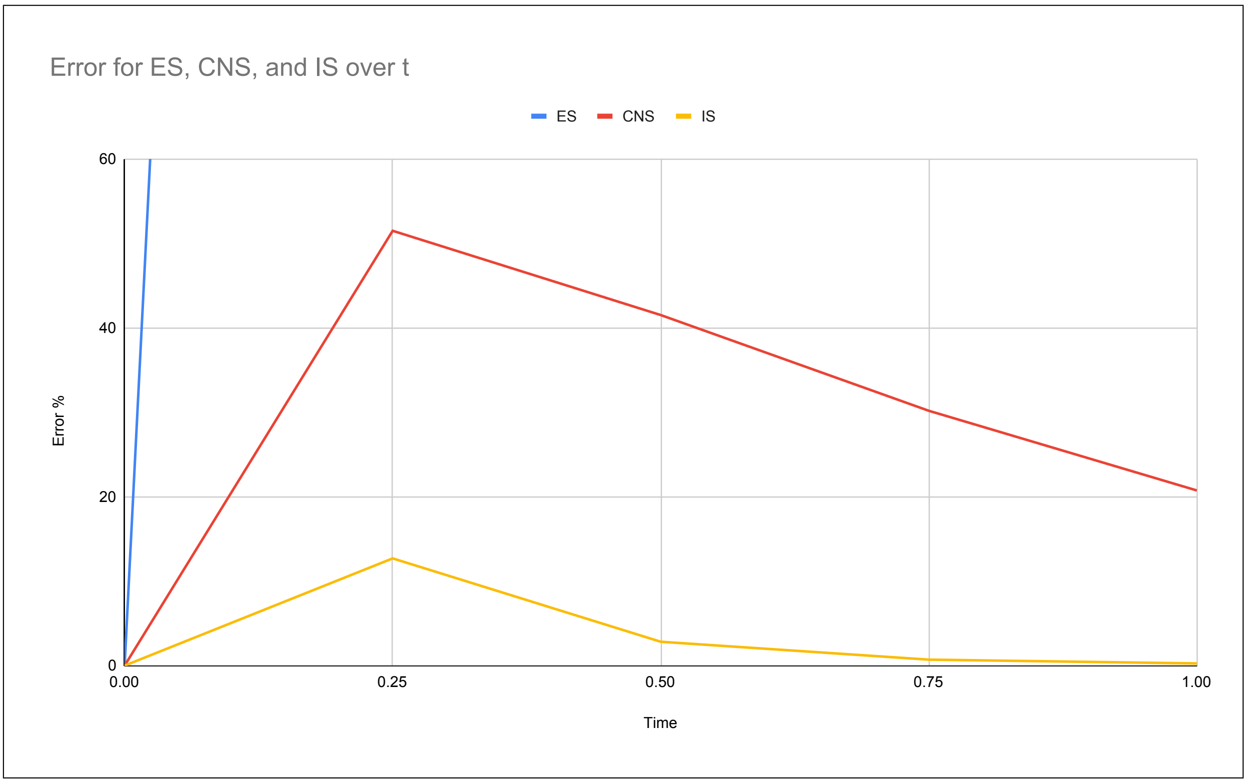

Figure 13: ES, CNS, and IS errors over time; y axis adjusted

5.2 Comparison 2

As a second comparison I finally chose parameters such that all 3 schemes are monotonous. Those parameters are $T1 = -2$, $T2 = 2$, $t_{max} = 8$, $\varepsilon = 2^{-9}$, $N = 129$, $M = 513$. The Courant number for this example is $K = 0.5$, which is just within the limits of ES. Figure 14 contains the errors of the 3 schemes plotted against the value of t for which they were found.

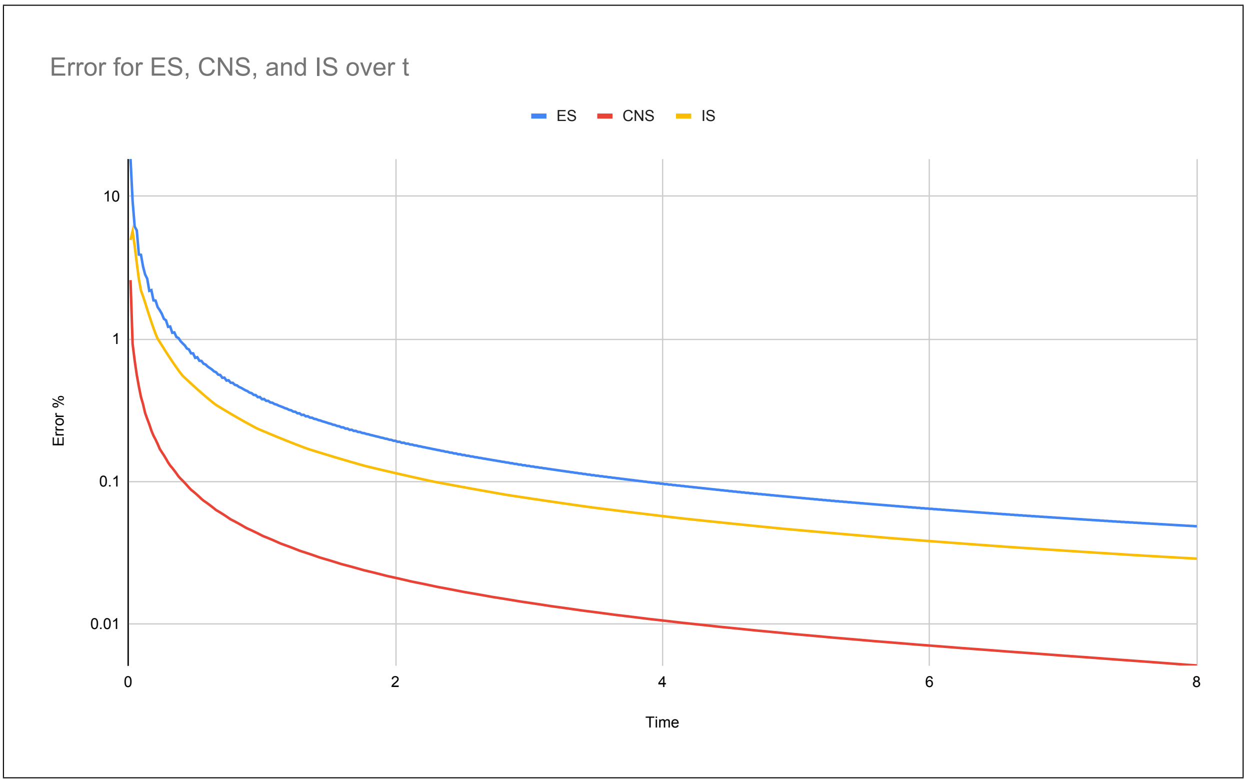

Figure 14: ES, CNS, and IS errors over time; $\log_{10}$ scale

We can see that ES is the least accurate for all values of t, followed by IS. The most accurate of the three schemes is CNS. This observation is compatible with our previous observation.

6 Conclusion

Taking into account all previous sections, I think that the implicit scheme is the most versatile of the three schemes that I examined. It is monotonous for all possible values which sets it apart from the other schemes. Being unconditionally monotonous is an advantage because it prevents oscillations from deteriorating the scheme's accuracy. The explicit scheme might be simple and easy to compute, but it is not accurate enough to take advantage of that (in my testing). While the implicit scheme is not quite as accurate as the Crank–Nicolson scheme, it suffers fewer problems with very abrupt changes in functions or monotonicity – in my opinion that gives it an advantage. Especially if the function (or its parameters) one wishes to numerically approximate is not completely known or investigated, the implicit scheme would make sure that the result would not be hurt by monotonicity or other problems.