

Laplace's Equation

- Separation of variables – two examples
- Laplace's Equation in Polar Coordinates
 - Derivation of the explicit form
 - An example from electrostatics
- A surprising application of Laplace's eqn
 - Image analysis
 - This bit is NOT examined

Laplace's Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

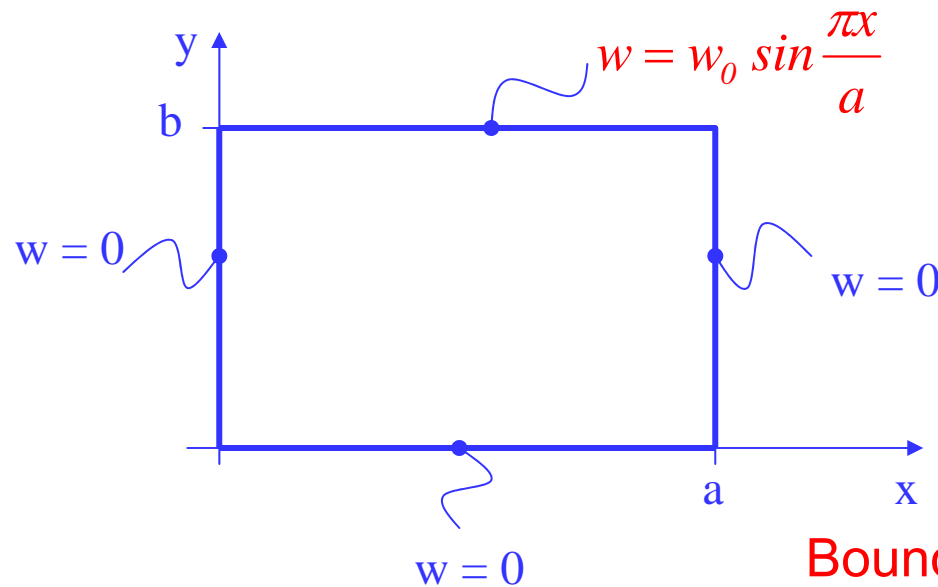
In the vector calculus course, this appears as $\nabla^2 \phi = 0$ where $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$

Note that the equation has **no** dependence on time, just on the spatial variables x, y . This means that Laplace's Equation describes **steady state** situations such as:

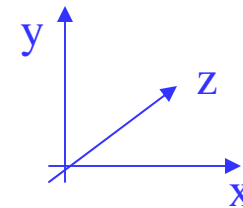
- steady state temperature distributions
- steady state stress distributions
- steady state potential distributions (it is also called the *potential equation*)
- steady state flows, for example in a cylinder, around a corner, ...

Stress analysis example: Dirichlet conditions

Steady state stress analysis problem, which satisfies Laplace's equation; that is, a stretched elastic membrane on a rectangular former that has prescribed out-of-plane displacements along the boundaries



$w(x,y)$ is the displacement in z -direction



To solve:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

Boundary conditions

$$w(0, y) = 0, \quad \text{for } 0 \leq y \leq b$$

$$w(x, 0) = 0, \quad \text{for } 0 \leq x \leq a$$

$$w(a, y) = 0, \quad \text{for } 0 \leq y \leq b$$

$$w(x, b) = w_0 \sin \frac{\pi}{a} x, \quad \text{for } 0 \leq x \leq a$$

Solution by separation of variables

$$w(x, y) = X(x)Y(y)$$

from which $X''Y + XY'' = 0$

and so $\frac{X''}{X} + \frac{Y''}{Y} = 0$

as usual ... $\frac{X''}{X} = -\frac{Y''}{Y} = k$

where k is a constant that is either equal to, $>$, or < 0 .

Case $k=0$

$$X(x) = (Ax + B), Y(y) = (Cy + D)$$

$$w(0, y) = 0 \Rightarrow B = 0 \text{ or } C = D = 0$$

if $C = D = 0$, then $Y(y) \equiv 0$, so $w(x, y) \equiv 0$

Continue with $B = 0$: $w(x, y) = Ax(Cy + D)$

$$w(x, 0) = 0 \Rightarrow ADx = 0$$

Either $A = 0$ (so $w \equiv 0$) or $D = 0$

Continue with $w(x, y) = ACxy$

$$w(a, y) = 0 \Rightarrow ACay = 0 \Rightarrow A = 0 \text{ or } C = 0 \Rightarrow w(x, y) \equiv 0$$

That is, the case $k=0$ is not possible

Case $k > 0$

Suppose that $k = \alpha^2$, so that

$$w(x, y) = (A \cosh \alpha x + B \sinh \alpha x)(C \cos \alpha y + D \sin \alpha y)$$

Recall that $\cosh 0 = 1, \sinh 0 = 0$

$$w(0, y) = 0 \Rightarrow A(C \cos \alpha y + D \sin \alpha y) = 0$$

$$C = D = 0 \Rightarrow w(x, y) \equiv 0$$

$$\text{Continue with } A = 0 \Rightarrow w(x, y) = B \sinh \alpha x (C \cos \alpha y + D \sin \alpha y)$$

$$w(x, 0) = 0 \Rightarrow BC \sinh \alpha x = 0$$

$$B = 0 \Rightarrow w(x, y) \equiv 0$$

$$\text{Continue with } C = 0 \Rightarrow w(x, y) = BD \sinh \alpha x \sin \alpha y$$

$$w(a, y) = 0 \Rightarrow BD \sinh \alpha a \sin \alpha y = 0$$

$$\text{so either } B = 0 \text{ or } D = 0 \Rightarrow w(x, y) \equiv 0$$

Again, we find that the case $k > 0$ is not possible

Final case $k < 0$

Suppose that $k = -\alpha^2$

$$w(x, y) = (A \cos \alpha x + B \sin \alpha x)(C \cosh \alpha y + D \sinh \alpha y)$$

$$w(0, y) = 0 \Rightarrow A(C \cosh \alpha y + D \sinh \alpha y) = 0$$

$$\text{as usual, } C = D = 0 \Rightarrow w \equiv 0$$

$$\text{continue with } A = 0 \Rightarrow w(x, y) = B \sin \alpha x (C \cosh \alpha y + D \sinh \alpha y)$$

$$w(x, 0) = 0 \Rightarrow BC \sin \alpha x = 0$$

$$B = 0 \Rightarrow w \equiv 0$$

$$\text{continue with } C = 0 \Rightarrow w(x, y) = BD \sin \alpha x \sinh \alpha y$$

$$w(a, y) = 0 \Rightarrow BD \sin \alpha a \sinh \alpha y = 0$$

$$B = 0 \text{ or } D = 0 \Rightarrow w \equiv 0$$

$$\sin \alpha a = 0 \Rightarrow \alpha = n \frac{\pi}{a} \Rightarrow w_n(x, y) = BD \sin n \frac{\pi}{a} x \sinh n \frac{\pi}{a} y$$

Solution

Applying the first three boundary conditions, we have

$$w(x, y) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

The final boundary condition is: $w(x, b) = w_0 \sin \frac{\pi x}{a}$

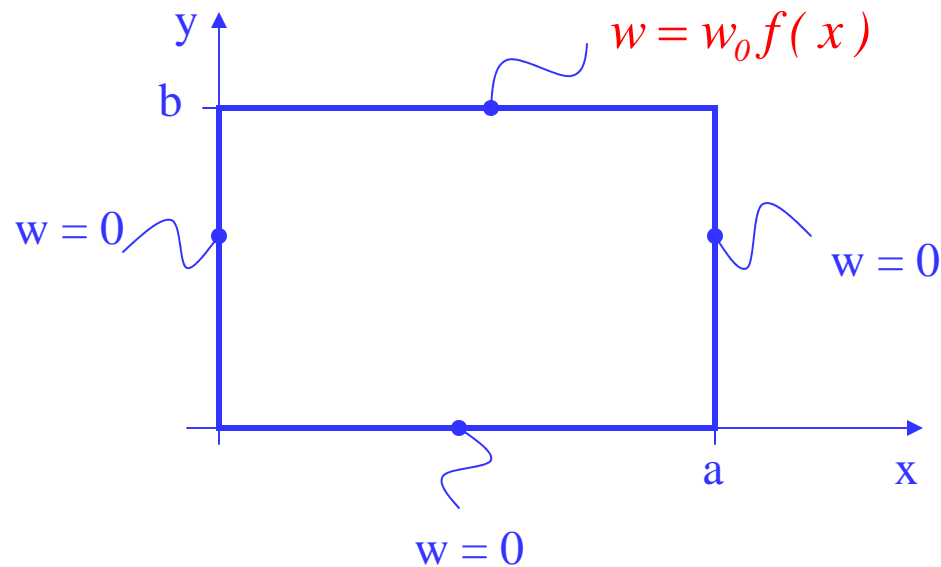
which gives:
$$w_0 \sin \frac{\pi x}{a} = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

We can see from this that n must take only one value, namely 1, so that $K_1 = \frac{w_0}{\sinh \frac{\pi b}{a}}$

and the final solution to the stress distribution is

$$w(x, y) = \frac{w_0}{\sinh \frac{\pi b}{a}} \sin \frac{\pi x}{a} \sinh \frac{\pi y}{a}$$

More general boundary condition



Then

$$w_0 f(x) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

and as usual we use orthogonality formulae/HLT to find the K_n

Types of boundary condition

1. The *value* $\phi(x, y)$ is specified at each point on the boundary: “**Dirichlet conditions**”
2. The *derivative normal to the boundary* $\frac{\partial \phi}{\partial \mathbf{n}}(x, y)$ is specified at each point of the boundary: “**Neumann conditions**”
3. A mixture of type 1 and 2 conditions is specified



Johann Dirichlet (1805-1859)

<http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Dirichlet.html>

Carl Gottfried Neumann (1832 -1925)

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Neumann_Carl.html

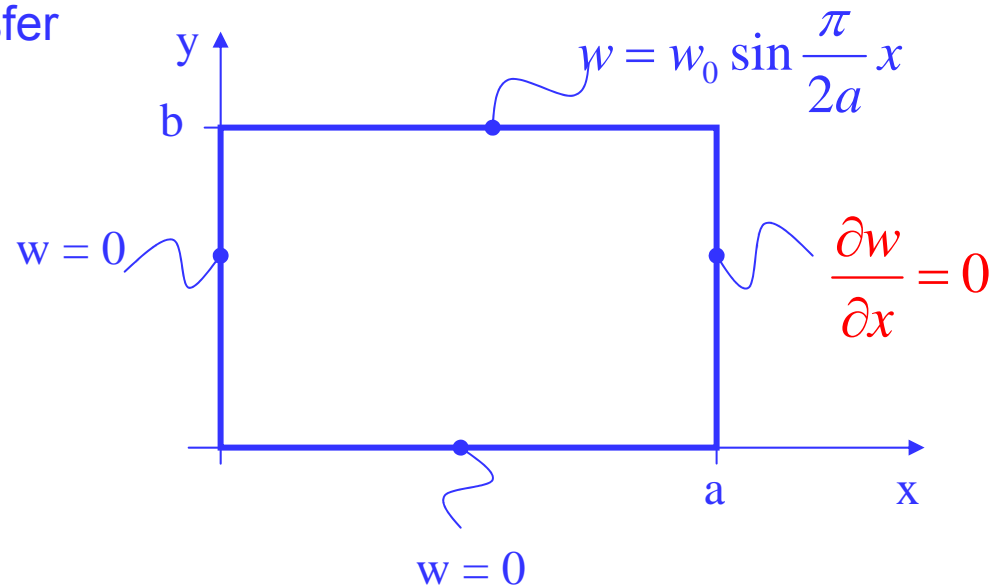


A mixed condition problem

A steady state heat transfer problem

To solve:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$



Boundary conditions

$$w(0, y) = 0, \quad \text{for } 0 \leq y \leq b$$

$$w(x, 0) = 0, \quad \text{for } 0 \leq x \leq a$$

$$\left. \frac{\partial w}{\partial x} \right|_{x=a} = 0, \quad \text{for } 0 \leq y \leq b$$

$$w(x, b) = w_0 \sin \frac{\pi}{2a} x, \quad \text{for } 0 \leq x \leq a$$

There is no flow of heat across this boundary; but it does not necessarily have a constant temperature along the edge

Solution by separation of variables

$$w(x, y) = X(x)Y(y)$$

from which $X''Y + XY'' = 0$

and so $\frac{X''}{X} + \frac{Y''}{Y} = 0$

as usual ... $\frac{X''}{X} = -\frac{Y''}{Y} = k$

where k is a constant that is either equal to, $>$, or < 0 .

Case $k=0$

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if $C = D = 0$, then $Y(y) \equiv 0$, so $w(x, y) \equiv 0$

Continue with $B = 0$: $w(x, y) = Ax(Cy + D)$

$$w(x, 0) = 0 \Rightarrow ADx = 0$$

Either $A = 0$ (so $w \equiv 0$) or $D = 0$

Continue with $w(x, y) = ACxy$

$$\left. \frac{\partial w}{\partial x} \right|_{x=a} = 0 \Rightarrow ACy = 0 \Rightarrow A = 0 \text{ or } C = 0 \Rightarrow w(x, y) \equiv 0$$

That is, the case $k=0$ is not possible

Case $k > 0$

Suppose that $k = \alpha^2$, so that

$$w(x, y) = (A \cosh \alpha x + B \sinh \alpha x)(C \cos \alpha y + D \sin \alpha y)$$

Recall that $\cosh 0 = 1, \sinh 0 = 0$

$$w(0, y) = 0 \Rightarrow A(C \cos \alpha y + D \sin \alpha y) = 0$$

$$C = D = 0 \Rightarrow w(x, y) \equiv 0$$

$$\text{Continue with } A = 0 \Rightarrow w(x, y) = B \sinh \alpha x (C \cos \alpha y + D \sin \alpha y)$$

$$w(x, 0) = 0 \Rightarrow BC \sinh \alpha x = 0$$

$$B = 0 \Rightarrow w(x, y) \equiv 0$$

$$\text{Continue with } C = 0 \Rightarrow w(x, y) = BD \sinh \alpha x \sin \alpha y$$

$$\left. \frac{\partial w}{\partial x} \right|_{x=a} = 0 \Rightarrow \alpha BD \cosh \alpha a \sin \alpha y = 0$$

$$\text{so either } B = 0 \text{ or } D = 0 \Rightarrow w(x, y) \equiv 0$$

Again, we find that the case $k > 0$ is not possible

Final case $k < 0$

Suppose that $k = -\alpha^2$

$$w(x, y) = (A \cos \alpha x + B \sin \alpha x)(C \cosh \alpha y + D \sinh \alpha y)$$

$$w(0, y) = 0 \Rightarrow A(C \cosh \alpha y + D \sinh \alpha y) = 0$$

$$\text{as usual, } C = D = 0 \Rightarrow w \equiv 0$$

$$\text{continue with } A = 0 \Rightarrow w(x, y) = B \sin \alpha x (C \cosh \alpha y + D \sinh \alpha y)$$

$$w(x, 0) = 0 \Rightarrow BC \sin \alpha x = 0$$

$$B = 0 \Rightarrow w \equiv 0$$

$$\text{continue with } C = 0 \Rightarrow w(x, y) = BD \sin \alpha x \sinh \alpha y$$

$$\left. \frac{\partial w}{\partial x} \right|_{x=a} = 0 \Rightarrow \alpha BD \cos \alpha a \sinh \alpha y = 0$$

$$B = 0 \text{ or } D = 0 \Rightarrow w \equiv 0$$

$$\cos \alpha a = 0 \Rightarrow \alpha = \frac{(2n-1)\pi}{2a} \Rightarrow w_n(x, y) = BD \sin \frac{(2n-1)\pi}{2a} x \sinh \frac{(2n-1)\pi}{2a} y$$

Solution

Applying the first three boundary conditions, we have

$$w(x, y) = \sum_{n=1}^{\infty} K_n \sin \frac{(2n-1)\pi}{2a} x \sinh \frac{(2n-1)\pi}{2a} y$$

The final boundary condition is: $w(x, b) = w_0 \sin \frac{\pi x}{2a}$

$$\text{which gives: } w_0 \sin \frac{\pi x}{2a} = \sum_{n=1}^{\infty} K_n \sin \frac{(2n-1)\pi}{2a} x \sinh \frac{(2n-1)\pi}{2a} b$$

We can see from this that n must take only one value, namely 1, so that $K_1 = \frac{w_0}{\sinh \frac{\pi}{2a} b}$

and the final solution to the stress distribution is

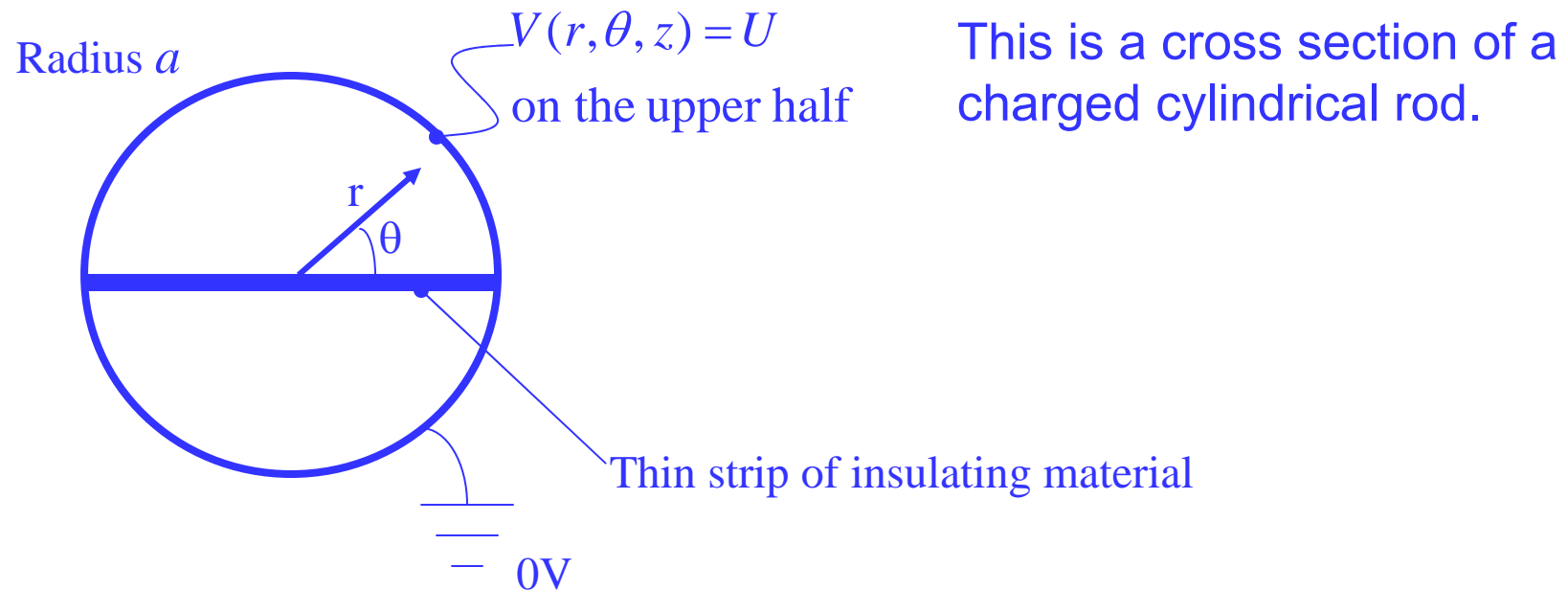
$$w(x, y) = \frac{w_0}{\sinh \frac{\pi b}{a}} \sin \frac{\pi x}{a} \sinh \frac{\pi y}{a}$$

PDEs in other coordinates...

- In the vector algebra course, we find that it is often easier to express problems in coordinates other than (x,y) , for example in polar coordinates (r,Θ)
- Recall that in practice, for example for finite element techniques, it is usual to use curvilinear coordinates ... but we won't go that far

We illustrate the solution of Laplace's Equation using polar coordinates*

A problem in electrostatics

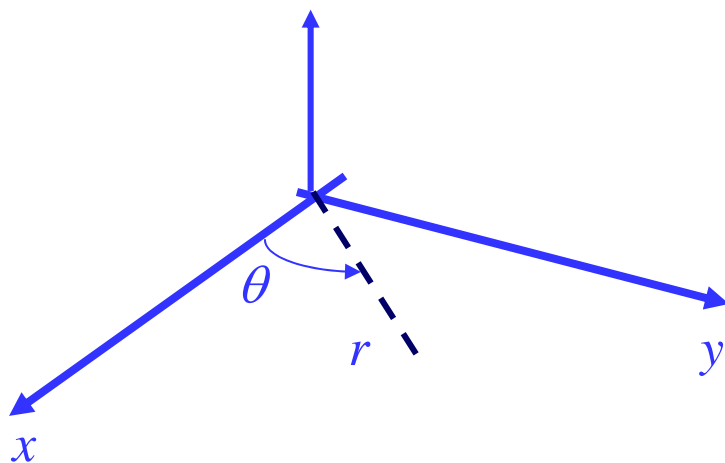


I could simply TELL you that Laplace's Equation in cylindrical polars is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

... brief time out while
I DERIVE this

2D Laplace's Equation in Polar Coordinates



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{where} \quad x = x(r, \theta), \quad y = y(r, \theta)$$

$$u(x, y) = u(r, \theta)$$

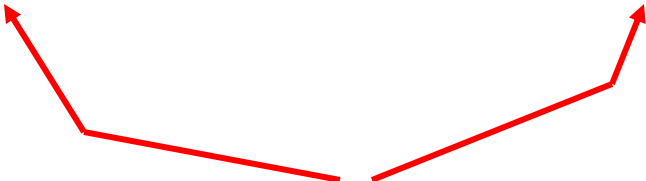
So, Laplace's Equation is $\nabla^2 u(r, \theta) = 0$

We next derive the **explicit** polar form of Laplace's Equation in 2D

Recall the chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}$$

Use the product rule to differentiate again

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial r} \right) \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \theta} \right) \frac{\partial \theta}{\partial x} \quad (*)$$


and the chain rule again to get these derivatives

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial r} \right) = \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \right) \frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial r^2} \frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r} \frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \theta} \right) = \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial \theta} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial \theta} \right) \frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial r \partial \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2} \frac{\partial \theta}{\partial x}$$

The required partial derivatives

$$x = r \cos \theta \quad y = r \sin \theta \quad r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2 \Rightarrow 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

in like manner

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}, \quad \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{r^4}$$

Back to Laplace's Equation in polar coordinates

Plugging in the formula for the partials on the previous page to the formulae on the one before that we get:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \frac{x^2}{r^2} + \frac{\partial u}{\partial r} \frac{y^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta} \frac{-2xy}{r^3} + \frac{\partial u}{\partial \theta} \frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2} \frac{y^2}{r^4}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} \frac{y^2}{r^2} + \frac{\partial u}{\partial r} \frac{x^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta} \frac{2xy}{r^3} - \frac{\partial u}{\partial \theta} \frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2} \frac{x^2}{r^4}$$

So Laplace's Equation in polars is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

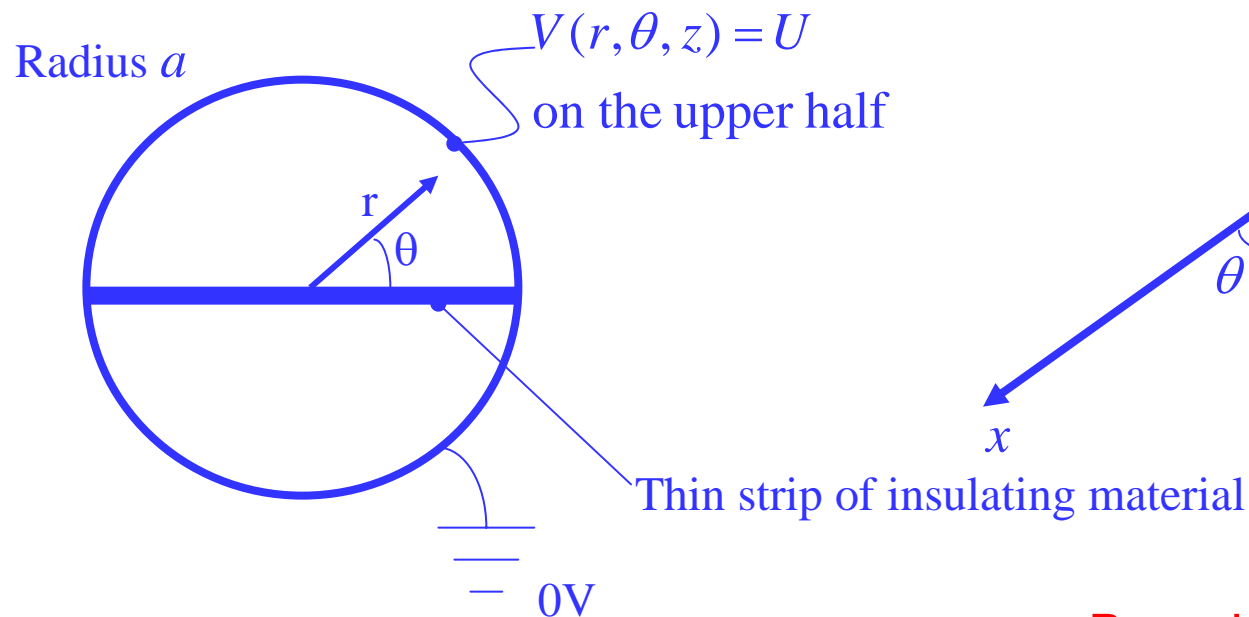
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is equivalent to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

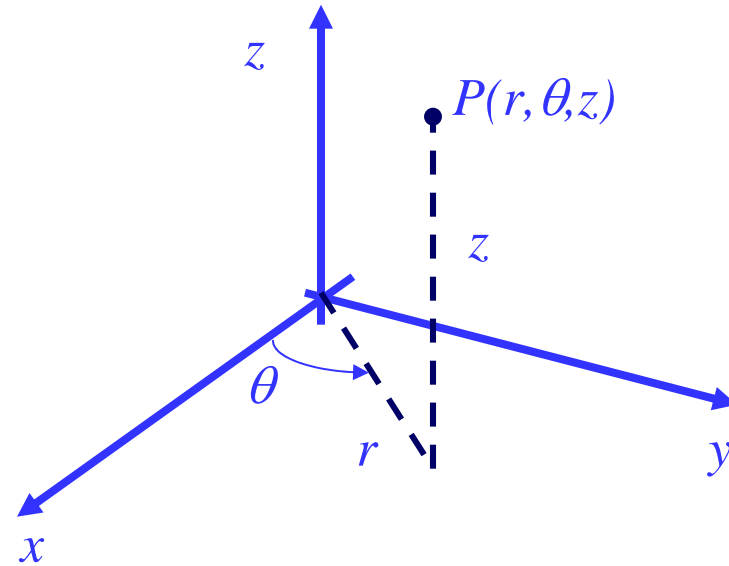
Example of Laplace in Cylindrical Polar Coordinates (r, θ, z)

Consider a cylindrical capacitor



Laplace's Equation in cylindrical polars is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



Boundary conditions

$$V(a, \theta) = U \quad \forall \theta : 0 \leq \theta \leq \pi$$

$$V(a, \theta) = 0 \quad \forall \theta : \pi \leq \theta \leq 2\pi$$

In the polar system, note that the solution must repeat itself every $\theta = 2\pi$

V should remain finite at $r=0$

There is no variation in V in the z -direction, so $\frac{\partial V}{\partial z} = 0$

This means we can treat it as a 2D problem $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0$

Using separation of variables

$$V = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$-\frac{\Theta''}{\Theta} = \frac{R'' + R'/r}{R/r^2}$$

As before, this means

$$-\frac{\Theta''}{\Theta} = \frac{R'' + R'/r}{R/r^2} = k, \text{ a constant}$$

The case $k=0$

$$\Theta'' = 0 \Rightarrow \Theta(\theta) = a\theta + b$$

$$R'' + \frac{R'}{r} = 0 \Rightarrow R(r) = (C \ln r + D)$$

$$\text{and so } V(r, \theta) = (a\theta + b)(C \ln r + D)$$

The solution has to be periodic in 2π : $a=0$

The solution has to remain finite as $r \rightarrow 0$: $c=0$

$$V(r, \theta) = bd = g, \text{ a constant}$$

The case $k < 0$

Suppose that $k = -m^2$

$$\Theta'' + m^2 \Theta = 0 \Rightarrow \Theta(\theta) = (A_m \cosh m\theta + B_m \sinh m\theta)$$

$$R'' + \frac{R'}{r} + \frac{m^2}{r^2} R = 0 \Rightarrow R(r) = (C_m r^{-m} + D_m r^m)$$

The solution has to be periodic in θ , with period 2π . This implies that $A_m = B_m = 0 \Rightarrow V(r, \theta) \equiv 0$

The case $k > 0$

Suppose that $k = n^2$

$$\Theta'' + n^2 \Theta = 0 \Rightarrow \Theta(\theta) = (A_n \cos n\theta + B_n \sin n\theta)$$

$$R'' + \frac{R'}{r} - \frac{n^2}{r^2} R = 0 \Rightarrow R(r) = (C_n r^n + D_n r^{-n})$$

$$V(r, \theta) = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n})$$

Evidently, this is periodic with period 2π

To remain finite as $r \rightarrow 0$ $D_n = 0$

The solution

$$V(r, \theta) = g + \sum_n r^n (A_n \cos n\theta + B_n \sin n\theta)$$

Notice that we have not yet applied the voltage boundary condition!!
Now is the time to do so

$$g + \sum_n a^n (A_n \cos n\theta + B_n \sin n\theta) = V(a, \theta) = \begin{cases} U & 0 \leq \theta \leq \pi \\ 0 & \pi < \theta \leq 2\pi \end{cases}$$

Integrating V from 0 to 2π : $\int_0^{2\pi} V(a, \theta) d\theta = \pi U$

Left hand side: $\int_0^{2\pi} g + \sum_n a^n (A_n \cos n\theta + B_n \sin n\theta) d\theta = 2\pi g$

$$\text{and so } g = \frac{U}{2}$$

Solving for A_m and B_m

So far, the solution is $V(a, \theta) = \frac{U}{2} + \sum_n a^n (A_n \cos n\theta + B_n \sin n\theta)$

We apply the orthogonality relationships:

$$\int_0^{2\pi} V(a, \theta) \cos m\theta d\theta = \frac{U}{2} \int_0^{2\pi} \cos m\theta d\theta + \int_0^{2\pi} \sum_n a^n (A_n \cos n\theta + B_n \sin n\theta) \cos m\theta d\theta$$

$$U \int_0^{\pi} \cos m\theta d\theta = 0 + A_m a^m \pi$$

$0 = A_m a^m \pi$, and so $A_m = 0$, for all m

$$\int_0^{2\pi} V(a, \theta) \sin m\theta d\theta = \frac{U}{2} \int_0^{2\pi} \sin m\theta d\theta + \int_0^{2\pi} \sum_n a^n (A_n \cos n\theta + B_n \sin n\theta) \sin m\theta d\theta$$

$$U \int_0^{\pi} \sin m\theta d\theta = \frac{U}{2} \left[\frac{\cos m\theta}{m} \right]_0^{2\pi} + B_m a^m \pi$$

$$\frac{2U}{m} = B_m a^m \pi, \text{ for odd } m = (2n-1)$$

$$V(r, \theta) = \frac{U}{2} + \sum_n \frac{2U}{(2n-1)\pi a^{(2n-1)}} r^{(2n-1)} \sin(2n-1)\theta$$

$$V(r, \theta) = \frac{U}{2} + \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{r^{(2n-1)}}{(2n-1)a^{(2n-1)}} \sin(2n-1)\theta$$

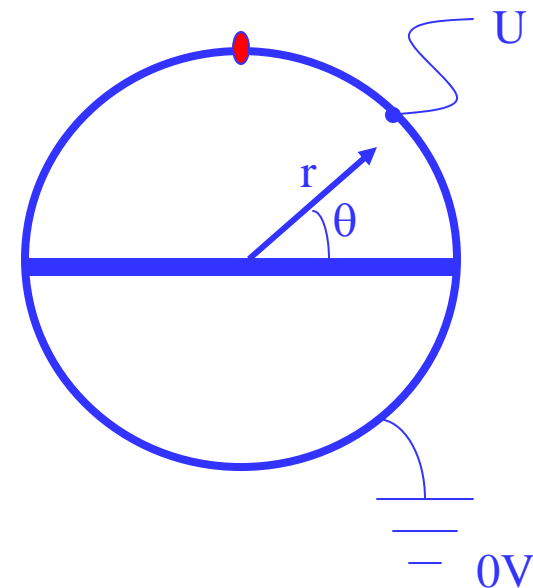
Check for $r = a$, $\theta = \pi/2$:

$$V(r, \theta) = \frac{U}{2} + \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{a^{(2n-1)}}{(2n-1)a^{(2n-1)}} \sin(2n-1)\frac{\pi}{2}$$

$$= \frac{U}{2} + \frac{2U}{\pi} \left[\sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \dots \right]$$

$$= \frac{U}{2} + \frac{2U}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$$

$$= \frac{U}{2} + \frac{2U}{\pi} \left[\frac{\pi}{4} \right] = U$$



An application in image analysis

- We saw that the Gaussian is a solution to the heat/diffusion equation
- We have studied Laplace's equation
- The next few slides hint at the application of what we have done so far in image analysis
- This is aimed at engaging your interest in PDEs ... it is **not** examined

Laplace's Equation in image analysis

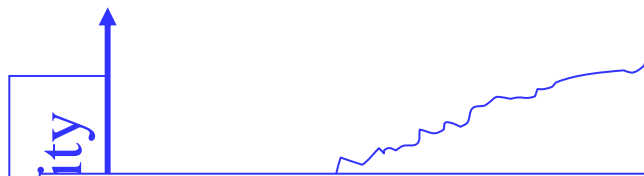
How do we compute the edges?



Image fragment



Edge map



- Remove the noise by smoothing
- Find places where the second derivative of the image is zero

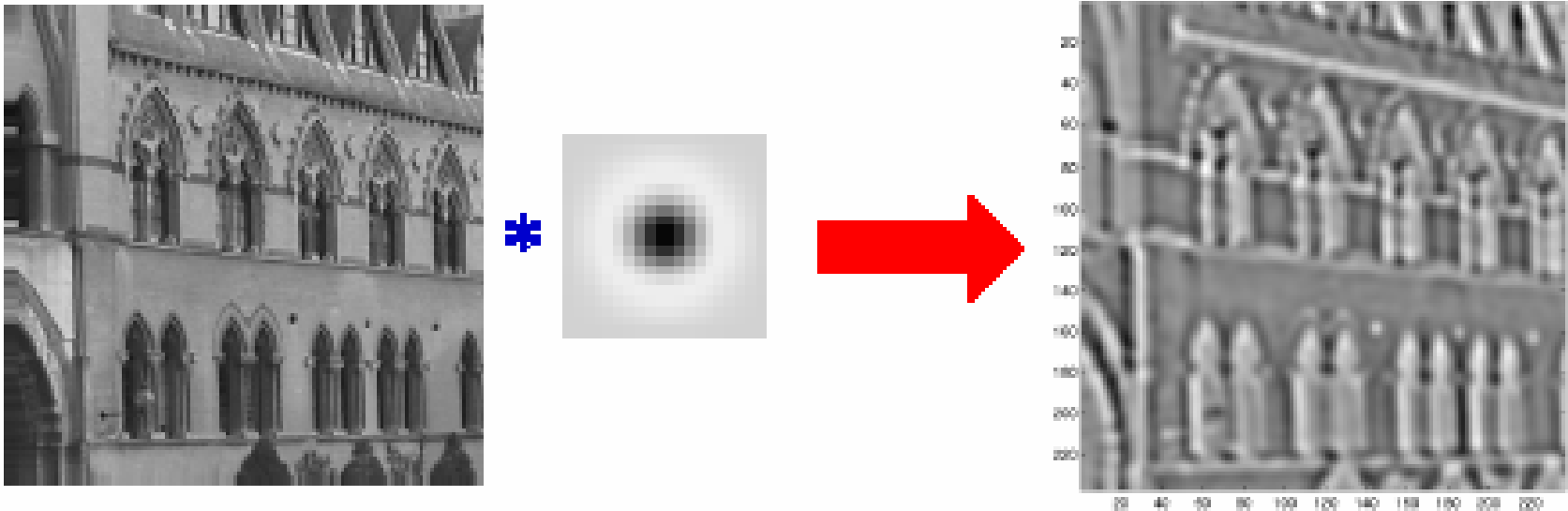
Signal position, x

Zero crossing at step. But
Constant gradient - max at the step.
Note amplified noise

Gaussian smoothing



Blurring with a Gaussian filter is one way to tame noise

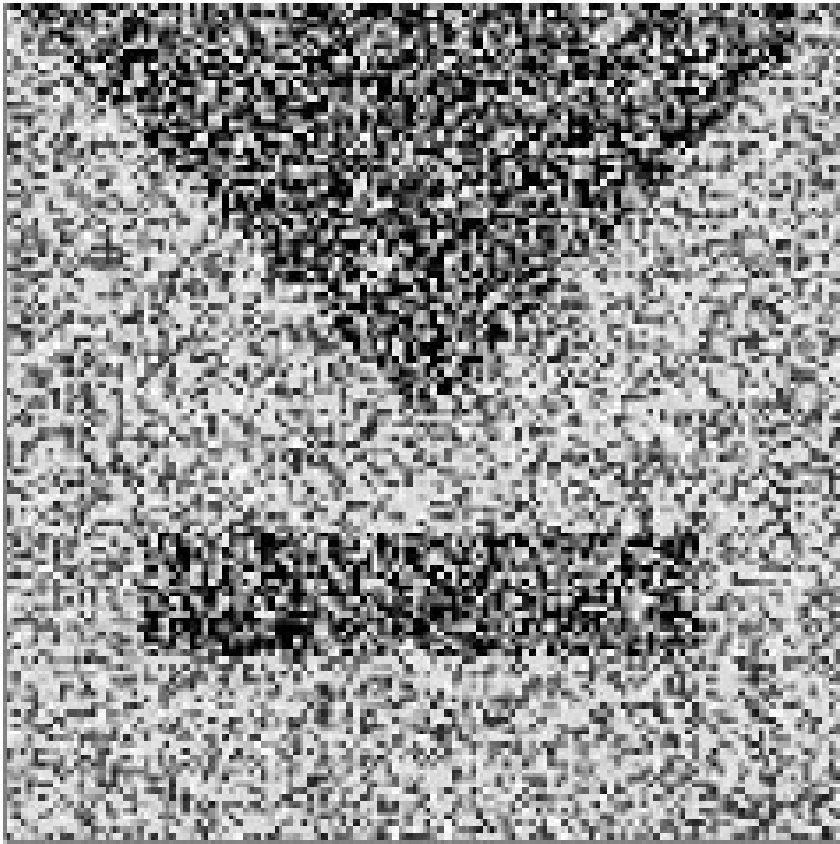


Zero crossings of a second derivative, isotropic operator,
after Gaussian smoothing

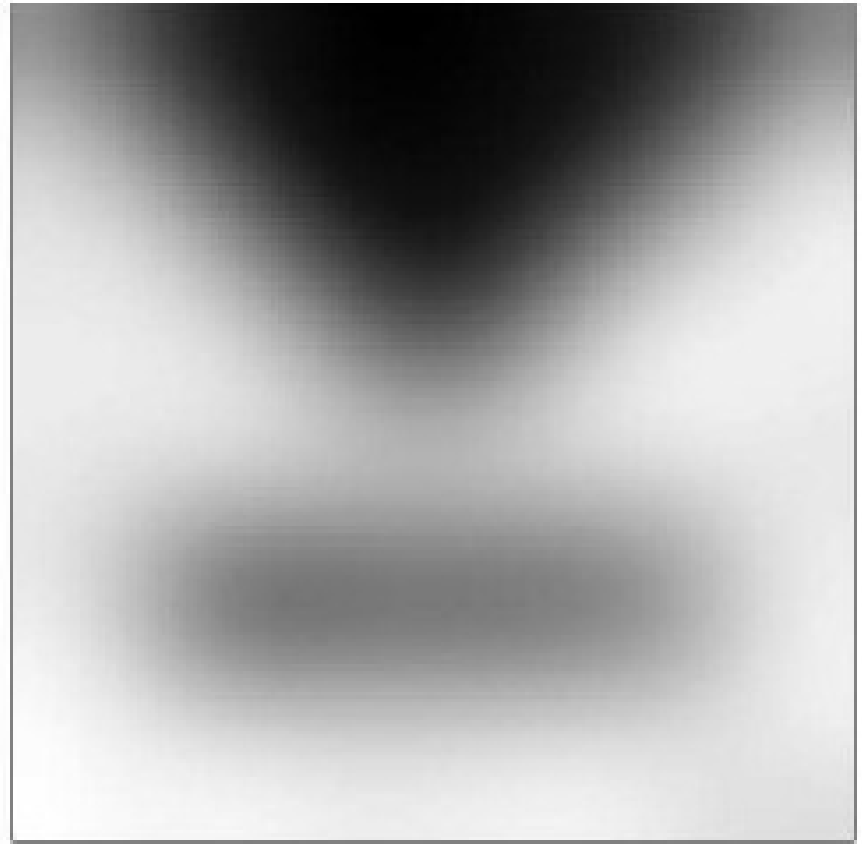
$$\nabla^2 I_{smooth} = 0$$

An application of Laplace's Equation!

Limits of isotropic Gaussian blurring



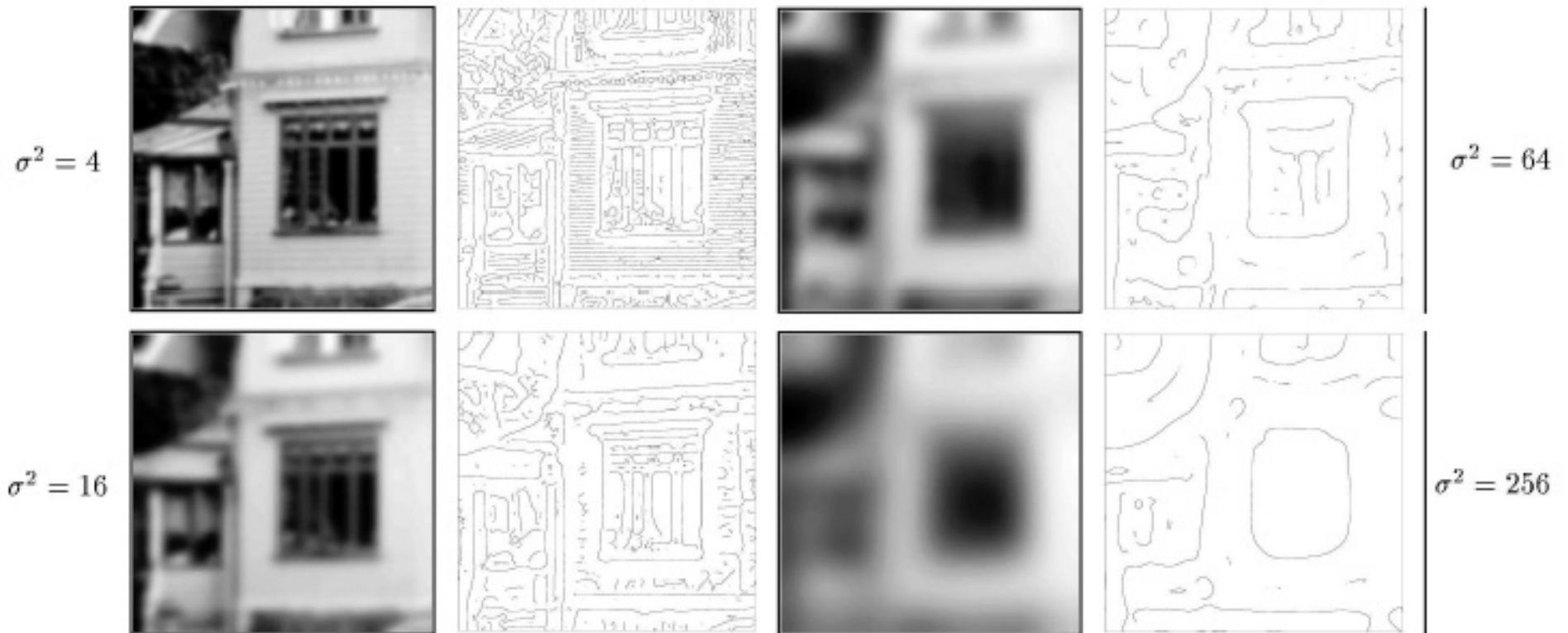
A noisy image



Gaussian blurring

Gaussian is isotropic – takes no account of orientation of image features – so it gives crap edge features

$$\nabla^2 (G_\sigma * I) = 0$$



As the blurring is increased, by increasing the standard deviation of the Gaussian, the structure of the image is quickly lost.

Can we do better? Can we make blurring respect edges?

Anisotropic diffusion

$$\partial_t I = \nabla^T (g(x; t) \nabla I)$$

$$g(x; t) = e^{-\frac{|\nabla I_\sigma|^2}{k^2}}, \text{ for some constant } k, \text{ or}$$

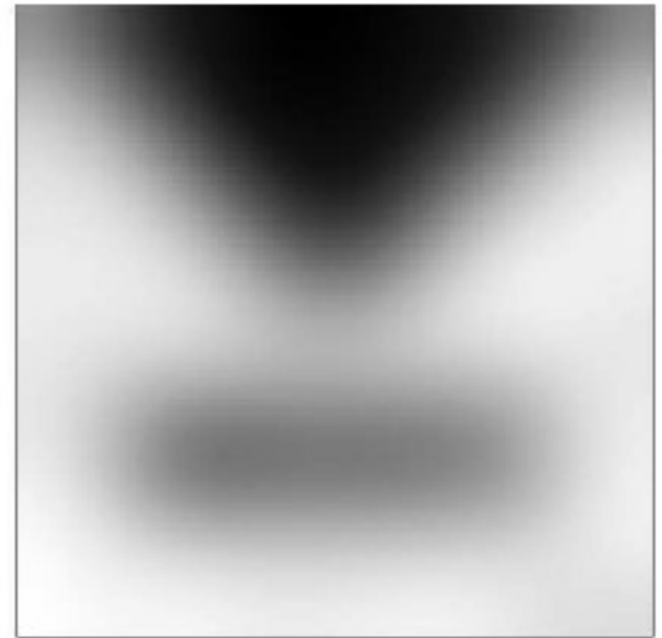
$$g(x; t) = \frac{1}{1 + |\nabla I_\sigma|^2 / k^2}$$

This is a non-linear version of Laplace's Equation, in which the blurring is small across an edge feature (low gradient) and large along an edge.

Anisotropic blurring
of the noisy image

Top right: Gaussian

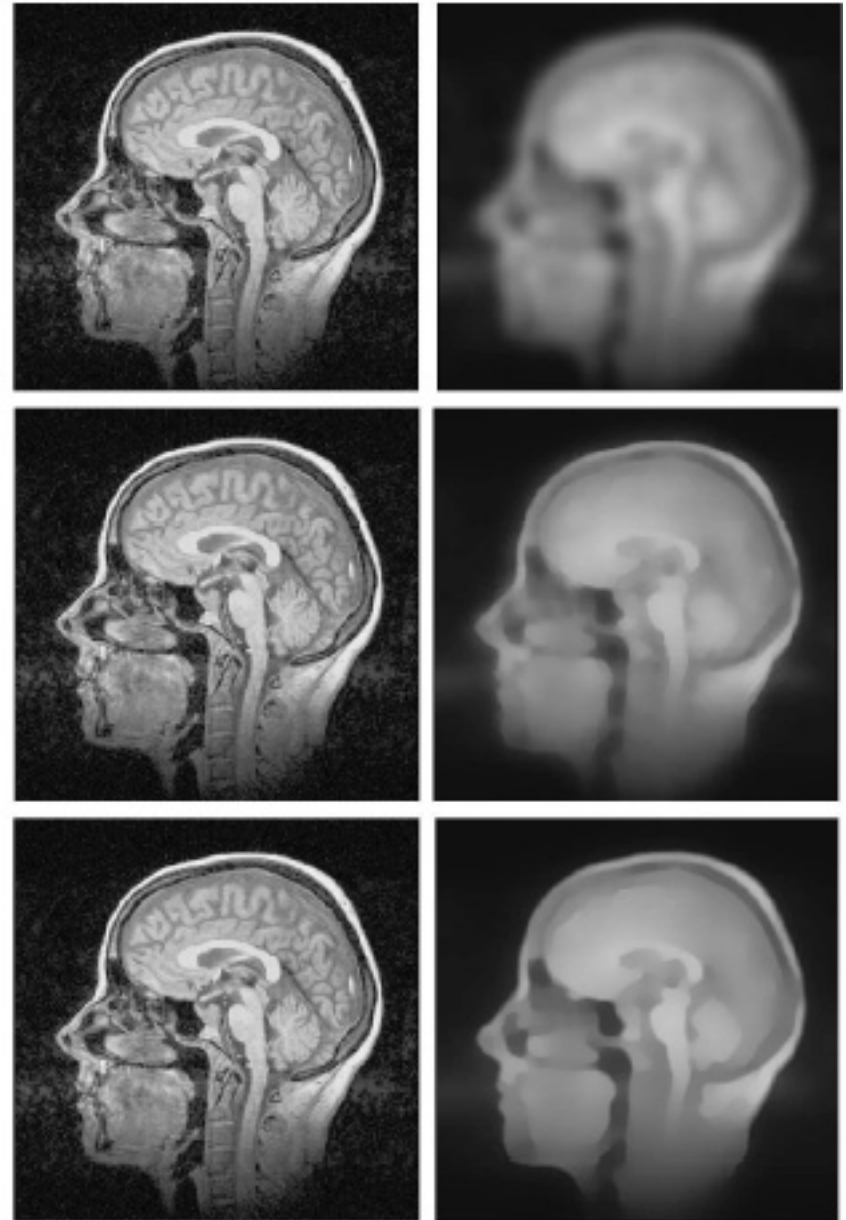
Bottom left and right:
different anisotropic
blurrings



Example of
anisotropic diffusion –
brain MRI images,
which are very noisy.

Top: Gaussian blur

Middle and bottom:
anisotropic blur



Anisotropic
blurring of the
house image
retaining important
structures at
different degrees
of non-linear
blurring



Two final examples:
Left – original image
Right – anisotropic blurring

