Predator-Prey Equations: Modeling Food Chains

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Phase Planes
Two-Species Food Chain
General Behavior
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Outline

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Background

- predator-prey equations are differential equations describing populations of predators and prey
- the most famous ones are Lotka-Volterra equations
- independently derived by Alfred J. Lotka (1880–1949) and Vito Volterra (1860–1940) in the 1920s
- Volterra observed fish, Lotka chemical reactions and got the same equations – both are the same system [2]

Lotka-Volterra Equations

- simplest predator-prey equations
- describe interactions of one predator and one prey

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy & \text{Prey} \\ \dfrac{dy}{dt} = -cy + dxy & \text{Predator} \end{cases} \tag{1}$$

- parameters $a, b, c, d > 0$, following [1]

Expanded Lotka-Volterra Equations

- expansion of (1) to include another predator
- interactions: y eats x and z eats y

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy & \text{Prey} \\ \dfrac{dy}{dt} = -cy + dxy - eyz & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = -fz + gyz & \text{Apex Predator} \end{cases} \tag{2}$$

- parameters $a, b, c, d, e, f, g > 0$, following [1]

Phase Planes

- space where all points are states of a system, e.g. system (1)
- moving points form trajectories (lines), closed trajectories are periodic solutions
- stationary points are equilibria, don't change over time
- give qualitative insights into equation without solving [5]

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Phase Planes - Example

$x' = y$ (angle) and $y' = -\sin(x)$ (angular velocity)

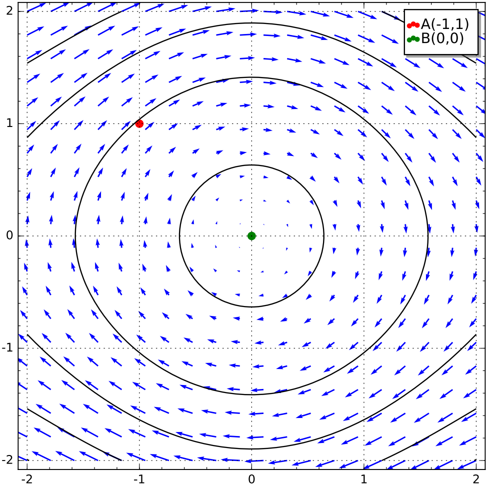

Figure 1: Phase plane of a pendulum with contour lines

Two-Species Food Chain

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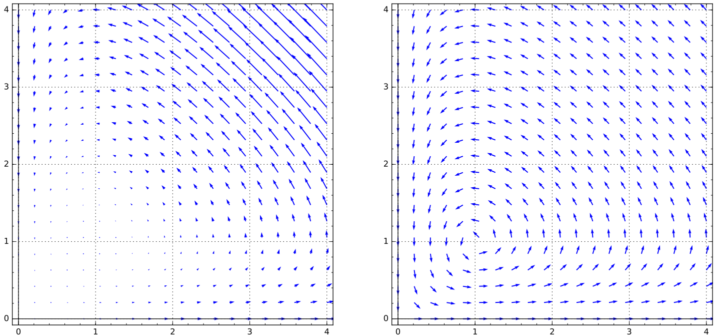
Equations

- standard Lotka-Volterra equations
- equivalent to system (2) with $z = 0$

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey} \\ \dfrac{dy}{dt} = y(x-1) & \text{Predator} \end{cases} \tag{3}$$

- from (1), parameters $a = b = c = d = 1$ chosen for simplicity

Phase Plane



(a) Phase plane (b) Normalized phase plane

Figure 2: Two-species system phase planes with $a = b = c = d = 1$

Phase Plane with Contours

- solutions to (1) have the form (4)

$$C = a \ln y - by + c \ln x - dx \tag{4}$$

- for $a = b = c = d = 1$ in (3) we get

$$C = \ln y - y + \ln x - x$$

- this can be used to graph solutions of (3) in a phase plane

Phase Plane with Contours

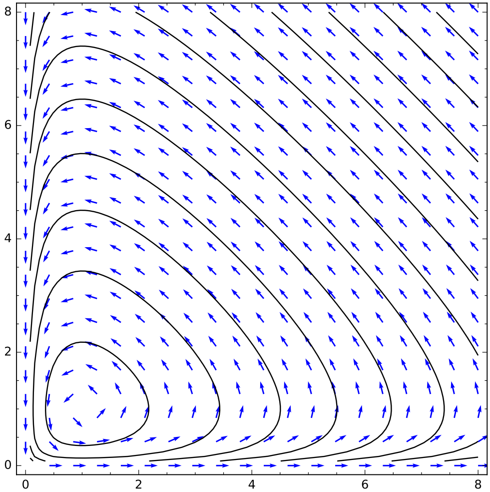


Figure 3: Two-species system phase plane with normalized vectors, contour lines, and $a = b = c = d = 1$

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Case $x = 0$ – No Prey

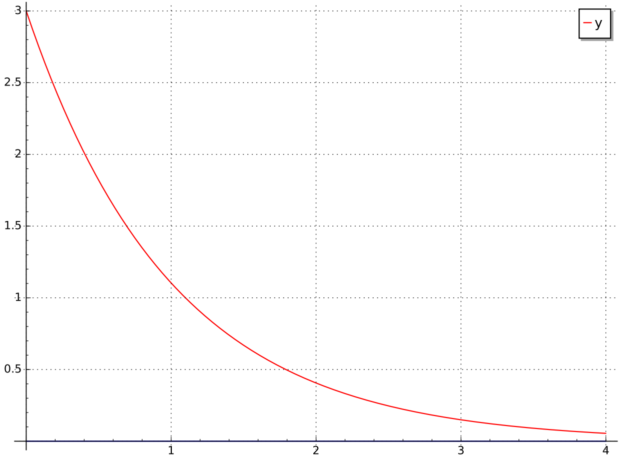

Figure 4: Two-species system graph for $x = 0$, $y = 3$, and $a = b = c = d = 1$

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Case $y = 0$ – No Predators

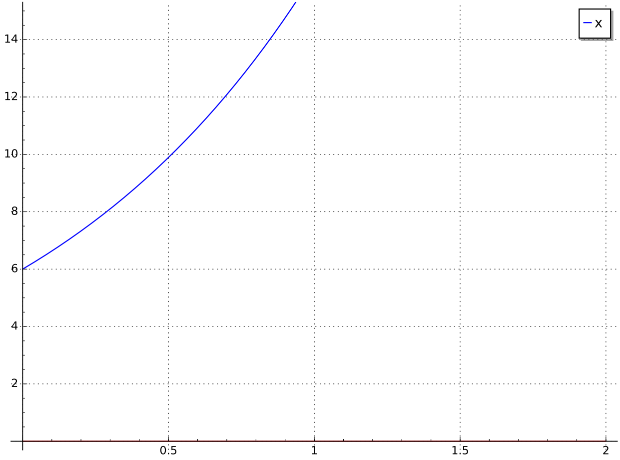

Figure 5: Two-species system graph for $x = 6$, $y = 0$, and $a = b = c = d = 1$

Case $x = 6$, $y = 3$ – Contour

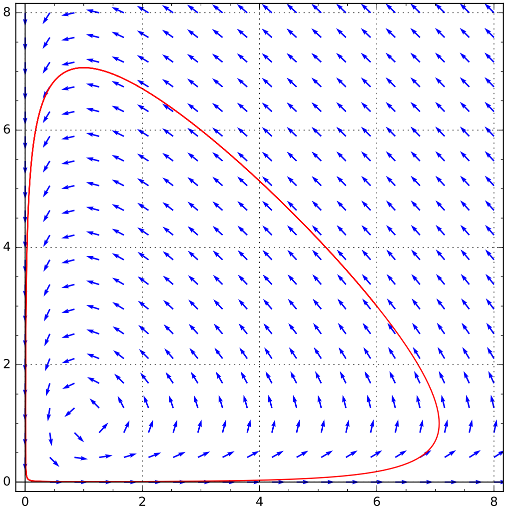

Figure 6: Two-species system contour for $x = 6$, $y = 3$, and $a = b = c = d = 1$

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Case $x = 6$, $y = 3$ – Graph

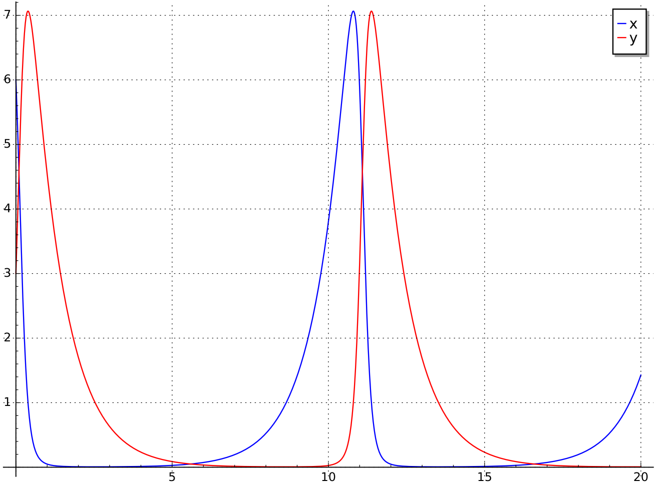

Figure 7: Two-species system graph for $x = 6$, $y = 3$, and $a = b = c = d = 1$

Three-Species Food Chain

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Equations

- equation (2) with parameters chosen for simplicity

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey} \\ \dfrac{dy}{dt} = y(x-z-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases} \tag{5}$$

- $a = b = c = d = e = f = g = 1$ as parameters

Case $z = 0$

No apex predators – becomes the same system as the two-species system (3)

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) & \text{Prey} \\ \dfrac{dy}{dt} = y(x-0-1) = y(x-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = 0(y-1) = 0 & \text{Apex Predator} \end{cases}$$

Case $z = 0$ Contour

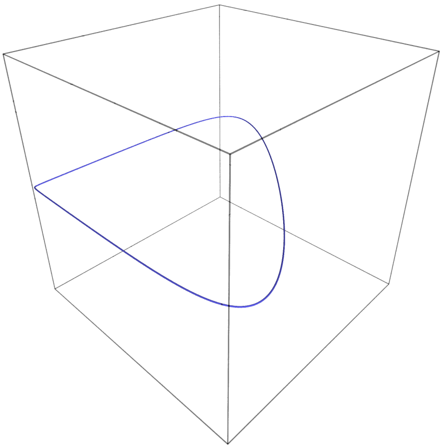

Figure 8: Three-species system contour for $x = 6$, $y = 3$, $z = 0$, $a = b = c = d = e = f = g = 1$

Case $z = 0$ Graph

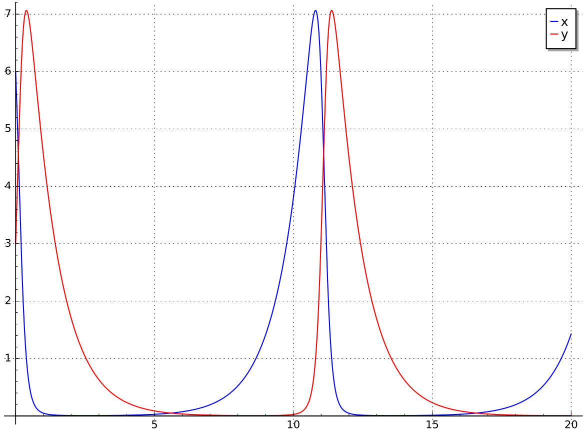

Figure 9: Three-species system graph for $x = 6$, $y = 3$, $z = 0$, $a = b = c = d = e = f = g = 1$

Case $x = 0$

If $x = 0$ we get the following equations

$$\begin{cases} \dfrac{dx}{dt} = 0(1-y) = 0 & \text{Prey} \\ \dfrac{dy}{dt} = y(0-z-1) = y(-z-1) & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(y-1) & \text{Apex Predator} \end{cases}$$

Case $x = 0$ Graph

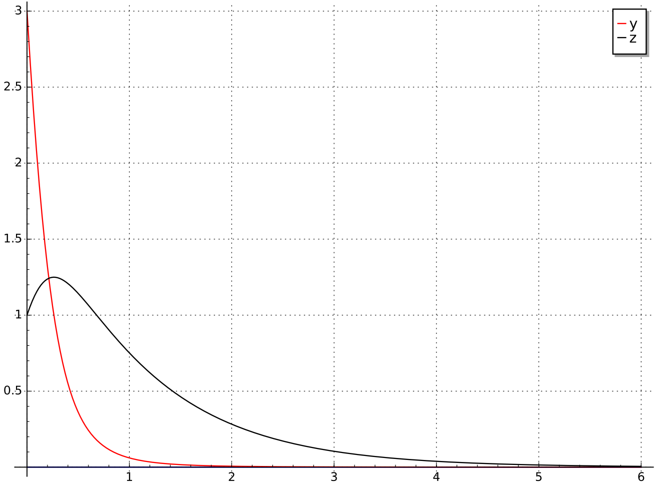

Figure 10: Three-species system graph for $x = 0$, $y = 3$, $z = 1$, $a = b = c = d = e = f = g = 1$

Case $y = 0$

If $y = 0$ we get the following system of equations

$$\begin{cases} \dfrac{dx}{dt} = x(1-0) = x & \text{Prey} \\ \dfrac{dy}{dt} = 0(x-z-1) = 0 & \text{Intermediate Predator} \\ \dfrac{dz}{dt} = z(0-1) = -z & \text{Apex Predator} \end{cases}$$

Case $y = 0$ Graph

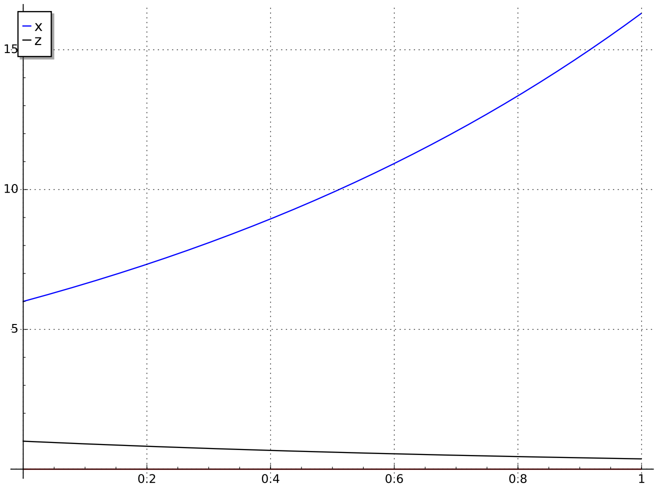

Figure 11: Three-species system graph for $x = 6$, $y = 0$, $z = 1$, $a = b = c = d = e = f = g = 1$

Further Analysis; Case $ga > fb$

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- in [1] the authors use further criteria to classify (2)

$$ga > fb, \qquad ga < gb, \qquad ga = fb.$$

- For example $a = g = 1.1$ and all other constants 1

$$\begin{cases} \dfrac{dx}{dt} = 1.1x - xy & \text{Prey,} \\ \dfrac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \dfrac{dz}{dt} = -z + 1.1yz & \text{Apex Predator.} \end{cases}$$

Case $ga > fb$ Graph

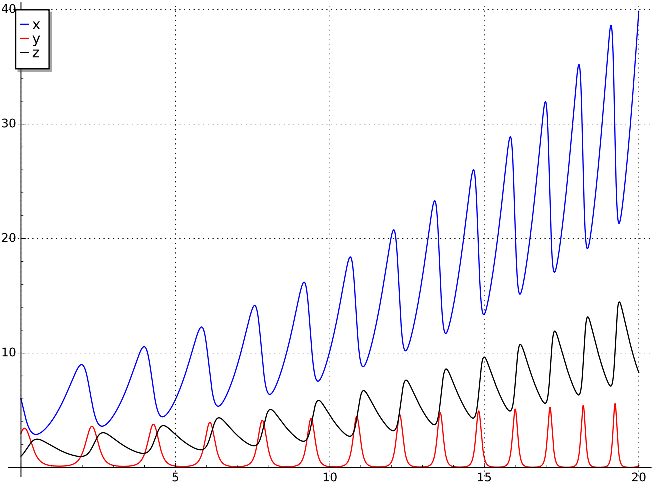

Figure 12: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, $b = c = d = e = f = 1$, and $a = g = 1.1$

Case $ga < fb$

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For example $b = f = 1.1$ and all other constants equal 1

$$\begin{cases} \dfrac{dx}{dt} = x - 1.1xy & \text{Prey,} \\ \dfrac{dy}{dt} = -y + xy - yz & \text{Intermediate Predator,} \\ \dfrac{dz}{dt} = -1.1z + yz & \text{Apex Predator.} \end{cases}$$

Case $ga < fb$ Graph

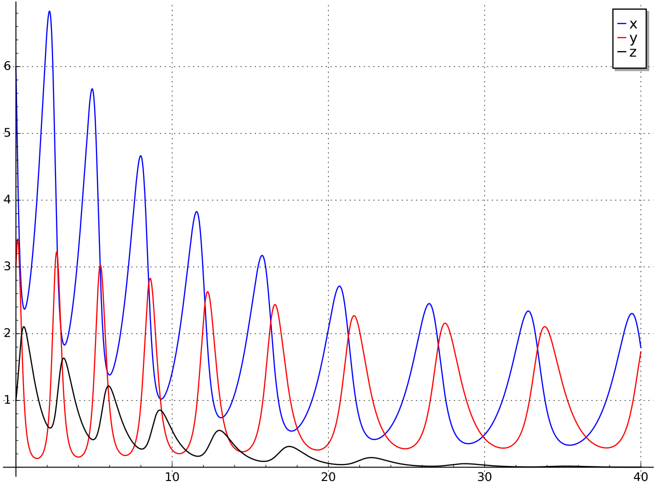

Figure 13: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, $a = c = d = e = g = 1$, and $b = f = 1.1$

Case $ga < fb$ Contour

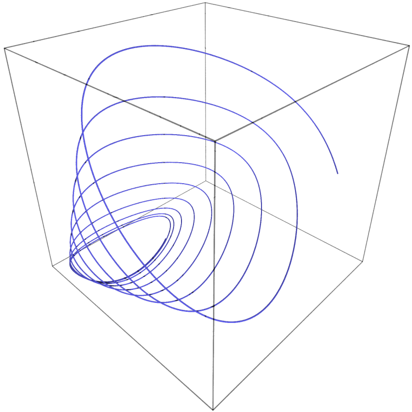

Figure 14: Three-species system contour for $x = 6$, $y = 3$, $z = 1$, $a = c = d = e = g = 1$, and $b = f = 1.1$

Case $ga = fb$

- original equation (5) does not change
- all constants equal 1, $a = b = c = d = e = f = g = 1$
- system could be periodic

Case $ga = fb$ Graph

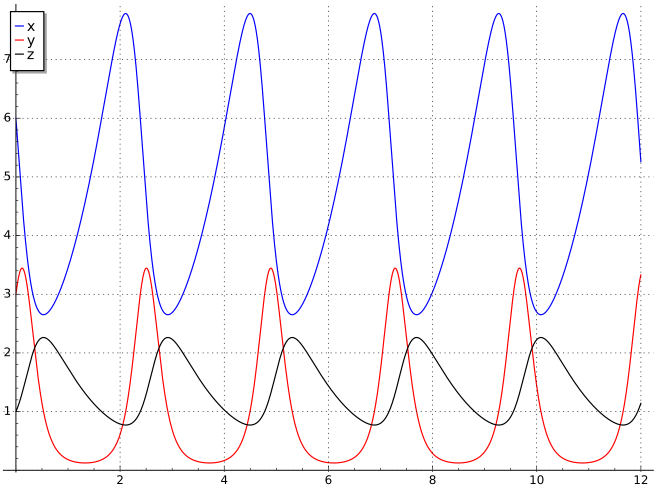

Figure 15: Three-species system graph for $x = 6$, $y = 3$, $z = 1$, and $a = b = c = d = e = f = g = 1$

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Case $ga = fb$ Contour

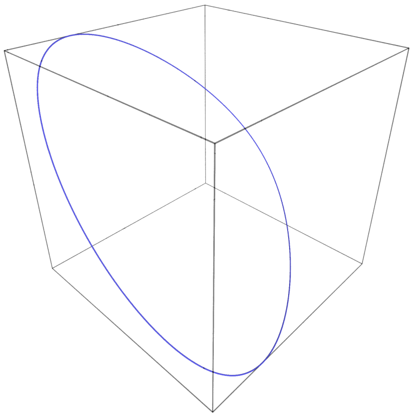

Figure 16: Three-species system contour for $x = 6$, $y = 3$, $z = 1$, and $a = b = c = d = e = f = g = 1$

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Conclusion

- models fit general intuition
- inaccuracies: unlimited growth, only one species as prey
- equation had great impact on ecology [6]
- Lotka-Volterra equations (1) are among the most famous differential equations

References

[1] E. Chauvet, J. E. Paullet, J. P. Previte, and Z. Walls, *Lotka-Volterra three-species food chain*, Mathematics Magazine, 75 (2002), pp. 243–255. Retrieved from: http://www.jstor.org/stable/3219158

[2] F. Hoppensteadt, *Predator-prey model*, Scholarpedia, 1 (2006), p. 1563. Retrieved from: http://www.scholarpedia.org/article/Predator-prey_model.

[3] G. Israel, *On the contribution of Volterra and Lotka to the development of modern biomathematics*, History and Philosophy of the Life Sciences, 10 (1988), pp. 37–49. Retrieved from: http://www.jstor.org/stable/23328998.

[4] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, Inc., 10 ed., 2011.

[5] D. H. Terman and E. M. Izhikevich, *State space*, Scholarpedia, 3 (2008), p. 1924. Retrieved from: http://www.scholarpedia.org/article/Phase_space.

[6] P. J. Wangersky, *Lotka-Volterra population models*, Annual Review of Ecology and Systematics, 9 (1978), pp. 189–218. Retrieved from: https://www.jstor.org/stable/2096748.

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