# ACTUARIAL MATHEMATICS NOTES

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## Contents



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## 1. Class 01.09.2020

### 1.1. General Introduction.

- introduction of professor
- this is course 1 of 2
- quite popular elective course – he is not sure why
- we'll dig deep inside the workings of insurance and the like
- it's a very well paid profession
- who is an actuary:
  - completely independent professional
  - actuarial association assigns audits to actuaries
  - basically a type of auditor
  - actuarial expertise is needed to make investments
- this course will teach us the basics
- to become an actuary, you will have to pass 6 exams, we'll learn stuff for the first 2
- re-insurance: some of the richest companies in the world
- Parmenter is the main text book
- **course content**: chapters 1, 2, 3, and 4

### 1.2. Accumulation Function.

- the simplest financial transaction is an investment
- **principal**: initial investment
- **accumulated value**: total amount the money grows to

- **Amount Function**: amount of money at time $t$ from investment of the principal – $A(t)$, $t$ is measured in years, $A(0)$ is the principal
- **Accumulation Function**: how much money increases as a percent value, where $a(0) = 1$ (as there has been no change)
$$a(t) = \frac{A(t)}{A(0)}$$
- accumulation functions can be any function where $a(0) = 1$, additionally one would hope that it is increasing
- continuity is not required, depends on how interest is paid – if fractional values of $t$ make sense it may be continuous, but if interest is paid discretely, it may be stepwise
- three types of accumulation functions
  1. amount of interest earned each year is constant – linear graph, simple interest
  2. the amount of interest increases over the years – exponential graph, compound interest
  3. if interest is paid out at fixed periods of time a piecewise function is used – the amount of interest might be constant or increasing
- **Interest = Accumulated Value - Principal**
- to make this more practical, the *effective rate of interest* $i$ is used
- $i$ is the interest earned on a principal of 1 over the period of 1 year – amount of interest earned over 1 year divided by the value at the beginning of the year
$$i = a(1) - 1$$
- $i$ can also be calculated with the amount function
$$i = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)}$$
- $i$ can be calculated for the $n$th year by
$$i = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{A(n) - A(n-1)}{A(n-1)}$$

1.3. **Simple Interest.**

- primarily used between integer periods of time
- $a(t)$ is a straight line here – the increase is linear
- general form of the equation is
$$a(t) = 1 + it$$
- interest earned each year is constant – interest does not earn interest
- if the principal is $k$ at $t = 0$
$$A(t) = k(1 + it)$$
- the effective rate of interest is not constant, it decreases over time
$$i_n = \frac{i}{1 + i(n-1)}$$

- **exact simple interest**: count the last day, not the first

$$t = \frac{\text{number of days}}{365}$$

- **ordinary simple interest (Banker's Rule)**: count the last day, not the first

$$t = \frac{\text{number of days}}{360}$$

- international markets use ordinary simple interest

## 2. Class 03.09.2020

### 2.1. Compound Interest.

- most important special case
- effective interest rate is fixed
- interest earns interest itself
- because the interest affects itself, the function is exponential

$$a(t) = (1+i)^t, \quad t \geq 0$$

- amount function for compound interest is

$$A(t) = k(1+i)^t$$

- the effective interest rate for compound interest is constant
- what values to choose for $t$ is done like with simple interest, either *exact* or *ordinary*
- if we want to find some value between integers, we linearly interpolate it

$$A(t) = A(\lfloor t \rfloor) + (t - \lfloor t \rfloor) \cdot (A(\lceil t \rceil) - A(\lfloor t \rfloor))$$

- to find the time it takes a principal to accumulate to a certain value, use logs

$$t = \frac{\log\left(\frac{\text{future value}}{\text{principal}}\right)}{\log(1+i)}$$

- compound and simple interest graphs only intersect at $(0,1)$ and at $(1, 1+i)$, this furthermore gives two cases

$$\begin{cases} \text{simple i.} > \text{compound i.} & \text{for} \quad 0 < t < 1 \\ \text{compound i.} > \text{simple i.} & \text{for} \quad t > 1 \end{cases}$$

## 3. Class 08.09.2020

### 3.1. Present Value and Discount.

3.1.1. *Present Value.*

- we define the *present value $t$ years in the past* as the amount of money that will accumulate to the principal in $t$ years
- this is the reverse of what we have been calculating thus far

| | | |
|---|---|---|
| Present Value | Principal | Accumulated Value |

- $v$ is the amount of money needed to accumulate to 1 within 1 year

$$v = \frac{1}{1+i}$$

- how $v$ works can be seen in this timeline that shows the evolution of $a(t)$

| | | |
|---|---|---|
| $\frac{1}{1+i}$ | $1$ | $1+i$ |
| $-1$ | $0$ | $1$ |

- for compound interest, $v$ is

$$v^t = \frac{1}{(1+i)^t} \tag{3.1}$$

- this is simply an inverted formula of $a(t)$ for compound interest
- for simple interest the present value is called $x$

$$x = \frac{1}{1+it}$$

3.1.2. *Discount.*

- imagine \$100 was invested and accumulated to \$112 in 1 year
- \$100 was the starting figure and interest (\$12) was added to it
- we could look at it the other way around and say \$112 is the starting value and at the start of the year \$12 was subtracted from it
- \$12 here is an amount of *discount*
- it's the same as interest, only the point of view is different
- discount focuses on the end of the year, so it is defined as

$$d = \frac{a(1) - 1}{a(1)}$$

- this only differs from the definition of $i$ in the denominator, which is $a(0)$ for $i$ because the beginning of the year is the focus
- effective rate of discount in the $n$th year is

$$d_n = \frac{a(n) - a(n-1)}{a(n)}$$

- some identities relating to $i$ are

$$
\begin{aligned}
d &< i \\
d &= \frac{i}{1+i} \\
1-d &= v \\
i &= \frac{d}{1-d}
\end{aligned}
$$

- now the rules for finding the present and accumulated values are reversed

$$
\begin{aligned}
\text{present value}: &\quad (1-d)^t \\
\text{accumulated value}: &\quad \frac{1}{(1-d)^t}
\end{aligned}
$$

## 4. Class 10.09.2020

### 4.1. **Nominal Rate of Interest.**

- $a(t) = (1+i)^t$ will be assumed in this section
- effective rates of interest can be given for any length of time
- to apply our previous formulae, we need to make sure that $t$ is the number of *effective interest periods*
- generally, these periods are not years, but shorter periods
- a yearly rate or 12% "convertible semiannually" actually means that you pay 6% twice a year – in this case it would actually be 12.36%
- the effective interest rate increases the shorter the intervals between payments are
- the 12% is a **nominal rate of interest**, meaning it is convertible over a period other than 1 year
- $i^{(m)}$ denotes the nominal rate of interest convertible $m$ times a year

$$1+i = \left[1 + \frac{i^{(m)}}{m}\right]^m$$

- we can also define a nominal rate of discount $d^{(m)}$

$$1-d = \left[1 - \frac{d^{(m)}}{m}\right]^m$$

- we also see that

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = \left[1 - \frac{d^{(n)}}{n}\right]^{-n}$$

### 4.2. **Force of Interest.**

- our goal is to find nominal rates of interest that are equivalent to a certain effective annual rate of interest
- for example $i = 0.12$ with the functions above gives the values

| $m$ | 1 | 2 | 5 | 10 | 50 |
|---|---|---|---|---|---|
| $i^{(m)}$ | 0.12 | 0.1166 | 0.1146 | 0.1140 | 0.1135 |

- we see that $i^{(m)}$ decreases as $m$ increases
- $m$ is approaching a limit, using L'Hopital's rule we can find it

$$\delta = \ln(1+i)$$
$$e^{\delta} = 1+i$$

- $\delta$ is called the **force of interest**
- it represents the nominal rate of interest that is convertible *continuously* – serving as a good approximation of $i^{(m)}$ for large $m$, like dayly conversions
- the second form of $\delta$ is useful because it makes conversions easier
- the derivative of $(1+i)^t$ by $t$ $(D)$ can be rewritten to be

$$\delta = \ln(1+i) = \frac{D[(1+i)^t]}{(1+i)^t} = \frac{D[a(t)]}{a(t)}$$

- for compound interest $\delta = \ln(1+i)$, but for arbitrary accumulation functions it is

$$\delta_t = \frac{D[a(t)]}{a(t)}$$
$$\delta_t = D[\ln(a(t))]$$

- if $\delta_r$ is given and we want to find $a(t)$ we use

$$a(t) = e^{\int_0^t \delta_r dr}$$

- we note that $i > \delta$
- the force of discount is the same as the force of interest

## 5. Class 15.09.2020

### 5.1. Equation of Value.

- interest problems only involve 4 quantities:
  (1) principal value
  (2) accumulated value
  (3) period of investment
  (4) rate of interest
- each one of them can be calculated if the other 3 are known
- when multiple investments are made, the time diagram is the most important tool
- then an *equation of value* is set up to find the value
- again, be careful with interpolation between integral durations with compound interest
- finding an appropriate rate of interest such that money increases generally involves logarithms involves

5.1.1. *Example 1.*

- Alice borrows 5,000 at 18% convertible semiannually
- after 2 years, she pays back 3,000
- 3 years after that she pays 2,000
- how much does she owe 7 years after taking out the loan?
- time diagram:

$$\begin{array}{cccccccc} 5,000 & & -3,000 & & & -2,000 & & X \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

- because the interest rate is convertible semiannually, our nominal rate is $i = 0.09$
- using the diagram we see

$$X = 5,000(1.09)^{14} - 3,000(1.09)^{10} - 2,000(1.09)^{4} = 6,783.38$$

- in the same way payments here are negative loans, withdrawals can be seen as negative deposits

5.1.2. *Example 2.*

- John borrows 3,000
- 2 years later he borrows another 4,000
- 2 years after that he borrows 5,000
- $i = 0.18$
- at what time would a single loan of 12,000 be equivalent? – at what time would the amount owed be the same as a loan of 12,000?
- draw a timeline:

$$\begin{array}{cccccc} 3,000 & & 4,000 & & & 5,000 \\ \hline 0 & 1 & 2 & t & 3 & 4 \\ & & & 12,000 & & \end{array}$$

- solution:

$$\begin{aligned} 12,000v^{t} &= 3,000 + 4,000v^{2} + 5,000v^{4} \\ v &= \frac{1}{1.18} \\ v^{t} &= \frac{3 + 4v^{2} + 5v^{4}}{12} \\ t &= \frac{\ln(3 + 4v^{2} + 5v^{4}) - \ln(12)}{\ln(v)} \\ t &= 2.11789 \end{aligned}$$

## 6. Class 24.09.2020

### 6.1. **Arithmetic and geometric sequences.**

6.1.1. *Arithmetic sequences.*

- $a, a+d, a+2d, a+3d$
- $n$th term: $a_n = a + (n-1)d$
- sum of first $n$ terms: $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

6.1.2. *Geometric sequences.*

- $a, ar, ar^2, ar^3$
- $n$th term: $a_n = ar^{n-1}$
- sum of first $n$ terms: $S_n = \frac{a(1-r^n)}{a-r}$

6.2. **Basic Results.**

- *annuity* – payments made of regular intervals
- generally, all the payments are of the same magnitude
- annuity is generally a payment of 1 over $n$ periods
- we do have to find the equivalent rate of interest for the payment periods
- a payment plan for a general annuity

$$\begin{array}{cccccc} a_{\overline{n}|} & 1 & 1 & 1 & & s_{\overline{n}|} \\ \hline 0 & 1 & 2 & 3 & \cdots & n \end{array}$$

- present value of the annuity is $a_{\overline{n}|}$

$$a_{\overline{n}|} = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{i}$$

- accumulated value of the annuity is $s_{\overline{n}|}$

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n = \frac{(1+i)^n - 1}{i}$$

- to find actual value, we can multiply the present value with the actual value
- other symbols and values for annuities

$$\begin{array}{cccccc} a_{\overline{n}|} & \ddot{a}_{\overline{n}|} & & & s_{\overline{n}|} & \ddot{s}_{\overline{n}|} \\ \hline 0 & 1 & 2 & \cdots & n & n+1 \end{array}$$

- present value of the annuity described on the first payment $\ddot{a}_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

- accumulated value one period after the last payment $\ddot{s}_{\overline{n}|}$

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

- we note two more identities

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n$$

$$1 = d \cdot \ddot{a}_{\overline{n}|} + v^n$$

## 7. Class 06.10.2020

### 7.1. Annuities.

- annuities can be viewed from many different angles with the same result
- *annuity-immediate* are payments at the end of periods
- *annuity-due* are payments made at the beginning of periods

### 7.2. Perpetuities.

- annuity whose payments continue forever

$$\begin{aligned} a_{\overline{\infty}|} &= \lim_{n\to\infty} a_{\overline{n}|} \\ &= \frac{1}{i} \end{aligned}$$

- we also have the perpetuity at the time of the first payment

$$\begin{aligned} \ddot{a}_{\overline{\infty}|} &= a_{\overline{\infty}|}(1+i) \\ &= \frac{1}{d} \end{aligned}$$

### 7.3. Unknown time and unknown rate of interest.

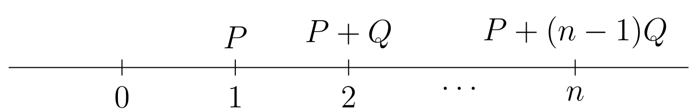
- a fund of 5,000 will be used to award scholarships of 500 for as long as possible. If $i = 0.09$, how many scholarships can be awarded?

$$\begin{aligned} 500 \cdot a_{\overline{n}|} \leq 5,000 &< 500 \cdot a_{\overline{n+1}|} \\ a_{\overline{n}|} \leq 10 &< a_{\overline{n+1}|} \\ a_{\overline{n}|} &= 10 \\ \frac{1-v^n}{i} &= 10 \\ n &= \frac{\ln(1-10i)}{\ln(v)} \end{aligned}$$

## 8. Class 13.10.2020

### 8.1. Varying Annuities.

- general type of a varying annuity




- we can find the value 1 year before the first payment with

$$A = Pa_{\overline{n}|} + Q\left[\frac{a_{\overline{n}|} - nv^n}{i}\right]$$

- the accumulated value of these payments is

$$A(1+i)^n = Ps_{\overline{n}|} + Q\left[\frac{s_{\overline{n}|} - n}{i}\right]$$

8.1.1. *Increasing Annuity.*

- here $P = Q = 1$
- the present value for this annuity is

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

- the accumulated value for this annuity is

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

8.1.2. *Decreasing Annuity.*

- here $P = 1$ and $Q = -1$
- the present value for this annuity is

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

- the accumulated value for this annuity is

$$(Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

8.1.3. *Geometric Annuity.*

- here $Q$ changes in a geometric way, by $c$
- then the sum of this annuity can be found with

$$r = \frac{1+i}{c}$$
$$S_n = Q(c)^{n-1}\left[\frac{1-r^n}{1-r}\right]$$

- to find the present value

$$r = \frac{1+i}{c}$$
$$P_n = Q(c)\left[\frac{1-r^n}{1-r}\right]$$

## 9. Class 15.10.2020

### 9.1. **Amortization.**

- repay a loan by the *amortization method* – installment payments at periodic intervals
- knowing the outstanding principal is important because you need to know how much you owe
- *prospective method*: outstanding principal is the present value of the outstanding payments at that time
- *retrospective method*: original principal accumulated until then minus the accumulated value of all the payments made until then
- this means that we either need to find $a_{\overline{n}|}$ or "original principal $s_{\overline{n}|}$ – payments $s_{\overline{n}|}$"

### 9.2. Amortization Schedules.

- a payment $X$ can be divided into its principal and interest parts like so:
  1. know or find the outstanding principal 1 time interval before $X$, let's call it $P$
  2. the interest portion of $X$ is $iP$
  3. the principal portion of $X$ is $X - iP$
- if a loan in paid back in equal payments of $X$ for $n$ years, the interest part of the $k$th payment is

$$X(1 - v^{n-k+1})$$

- the principal part of the $k$th payment is

$$Xv^{n-k+1}$$

- an amortization schedule is simply a table showing the payments and how they are made up

| Duration | Payment | Interest | Principal Repaid | Outstanding Principal |
|---|---|---|---|---|
| 0 | | | | 1,000.00 |
| 1 | 150 | 110.00 | 40.00 | 960.00 |
| 2 | 150 | 105.60 | 44.40 | 915.60 |
| 3 | 150 | 100.72 | 49.28 | 866.32 |
| $\vdots$ | 150 | $\vdots$ | $\vdots$ | $\vdots$ |
| 12 | 150 | 23.93 | 126.07 | 91.51 |
| 13 | 101.58 | 10.07 | 91.51 | 0.00 |

## 10. Class 22.10.2020

### 10.1. Sinking Funds.

- you pay interest each month but nothing more
- at the end you simply pay the full loan amount back
- generally you invest the money into a **sinking fund** in the meantime – if you get a higher interest rate than you pay, you could even make money

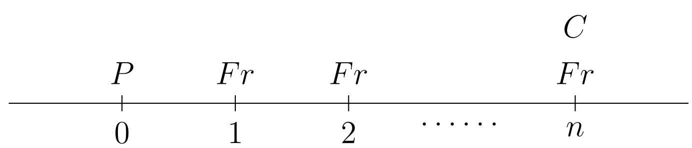
### 10.2. Yield Rates.

- only the payments made directly by or to the person(s) should be considered for that person(s)'s yield rate
- simply solve the problem you are given using a calculator

## 11. Class 11.19.2020

**11.1. Price of a bond.** A bond is a certificate in which, in return for receiving an initial sum of money from the investor, the borrower agrees to pay interest at a specified rate (the coupon rate) until a specified date (the maturity date), and, at that time, to pay a fixed sum (the redemption value). The coupon rate is customarily quoted as a nominal rate convertible semiannually, and is applied to the face (or par) value, which is stated on the front of the bond. Usually the face and redemption values are equal, but this is not always the case.

- basically a different way to handle loans
- e.g. face amount of 500, redeemable at par in 10 years, coupon rate 11% convertible semi-annually $\rightarrow$ investor gets 20 payments of (0.055)(500) and a lump sum payment of 500 at the end
- if one can buy the bond above for less than 500, their yield rate will be higher, if they pay more, their yield rate will be lower
- $F$ – face value or par of the bond
- $r$ – coupon rate per interest period, the quoted rate is generally $2r$; each interest payment is $Fr$
- $C$ – redemption value of the bond, often $D = F$
- $i$ – yield rate per interest period
- $n$ – number of interest periods until redemption date
- $P$ – purchase price of the bond to get yield rate $i$
- a timeline of a standard bond




- formula for $P$

$$P = (Fr)a_{\overline{n}|i} + C(1+i)^{-n}$$

- formula for $i$ as an iterative method

$$i = \frac{Fr(1-v^n)}{P - Cv^n}$$

- formula for $n$

$$n = \frac{\ln\left(\frac{P - \frac{Fr}{i}}{C - \frac{Fr}{i}}\right)}{\ln v}$$

- if an investor buys a bond for less than the redemption value, he buys it at a discount – the other way around he buys it at a premium $(P - C)$

### 11.2. **Book value.**

- the outstanding value of the bond at time $t$
- it is the present value of all future payments
- book value at time $t$ as the $t$th coupon has been paid

$$B_t = (Fr)a_{\overline{n-t}|} + Cv^{n-t}$$

- the book value generally changes from $P$ at $t = 0$ to $C$ at $t = n$
- also, for successive book values

$$B_{t+1} = B_t(1+i) - Fr$$

- between book values we assume simple interest at rate $i$
- flat price of the bond is what we generally calculate – the one that goes up and down
- market price (amortized value) is the interpolation between successive book values – we don't have the payment spikes in there

### 11.3. Bond amortization schedules.

- basically the same idea as with loan amortization schedules
- book value is in the last column of the table
- this will show how book value changes from $P$ to $C$
- book value at $t$ is $B_t$
- amount of coupon at $t+1$ is $Fr$
- amount of interest is $B_t i$
- change in book value is $Fr - B_t i$
- *Example 1*: amount of interest and change in book value for the 15th coupon:
  - interest: $B_t i = 950.83 \cdot 0.06 = 57.05$
  - it's greater than the coupon value
  - we modify the book value by the difference of new − old, so we actually increase it
- for a complete schedule, we start with finding the price using the main formula for it
- then we need to do the normal stuff

### 11.4. Other topics.

- *Different interest periods*: we must convert one to the other in a sensible way – i.e. coupons are the defining part
- *Changing coupons*: apply the changing annuities stuff to the price formula to find the price at a certain yield rate
- *Changing interest rates*: calculate the price using a piecewise function to find the present value of all the coupon payments at different rates and also the present value of the redemption value for the different rates
- *Callable bonds*: borrower can redeem the bond anytime between the call date and the usual maturity date, no coupons are paid once the bond is redeemed. Sometimes different redemption values are paid for different redemption dates
- **important**: generally it is best to take the earliest value if $i < r$, else use the last value. If there is doubt, find all possible values and compare
- *Duration of a Bond*: $R_n$ are payments made at times $t_n$

$$\bar{d} = \frac{\sum_{j=1}^{n} t_j v^{t_j} R_j}{\sum_{j=1}^{n} v^{t_j} R_j}$$

- the modified duration or volatility is

$$\bar{v} = \frac{\bar{d}}{1+i}$$

## 12. Notes 16.12.2020 – Bonds

### 12.1. Introduction.

- bonds are low-risk growth opportunities
- bonds promise payments at certain dates (coupon payments)
- at the redemption date the last of the coupon payments occurs
- issue date is the date the bond is bought
- the term is the time from the issue date until the maturity date
- a bond is noncallable if the maturity date is fixed
- if a bond only has a payment at the maturity date it is simply a loan (called zero-coupon bonds or pure discount bonds)
- coupon bonds have payments prior to the redemption date
- coupon payments tend to be level and evenly spaced, the redemption payment tends to be larger

### 12.2. Alphabet soup and basic price formula.

- $F$ — face/par value of the bond only used for coupon payments (is often equal to the redemption value)
- $Fr = \frac{F\alpha}{m}$ — coupon rate, paid $m$ times per year based on convertible rate $\alpha$, $r$ is the effective rate
- $n$ — number of coupon periods in the bond term, $N$ is the number of years, so $n = Nm$
- $C$ — redemption amount paid at the end of $n$ periods
- if $F = C$, the bond is redeemable-at-par or a par-value bond; if $C$ is not given, $F = C$ should be assumed
- $g = \frac{Fr}{C}$ — modified coupon rate, expresses the coupon rate in terms of $C$, $Fr = Cg$
- $i$ — annual effective yield rate, $j$ — investors effective yield rate per period

$$i = (1+j)^m - 1$$

- $I$ — nominal yield rate convertible $m$ times per year

$$j = \frac{I}{m} \qquad i = \left(1 + \frac{I}{m}\right)^m - 1$$

- $G = \frac{Fr}{j}$ — base amount, expresses the coupon amount in terms of yield rate $j$, then $Fr = Cg = Gj$
- $P$ — price the investor paid to get yield rate $j$
- $v_j = (1+j)^{-1}$ — find previous value
- $K = Cv_j^n$ — value of the redemption at issue date assuming compound interest
- the basic price formula is

$$P = (Fr)a_{\overline{n}|j} + Cv_j^n = (Fr)a_{\overline{n}|j} + K$$

- 

12.3. **The premium-discount formula.**

- because we have
$$a_{\overline{n}|j} = \frac{1 - v_j^n}{j}$$
and
$$v_j^n = 1 - j a_{\overline{n}|j}$$
we can write the price formula as
$$P = (Fr)a_{\overline{n}|j} + Cv_j^n = (Cg)a_{\overline{n}|j} + C(1 - j a_{\overline{n}|j})$$
and finally
$$P = C(g - j)a_{\overline{n}|j} + C$$
- this is called the premium-discount formula for a bond
- bonds sell at a premium if their price is higher than their redemption amount, and the premium is
$$P = C(g - j)a_{\overline{n}|j}$$
- if the price is lower than the redemption amount, the discount is
$$P = C(j - g)a_{\overline{n}|j}$$

12.4. **Other pricing formulas.**

- the base amount formula is
$$P = (C - G)v_j^n + G$$
- Makeham's formula is useful when K is known but the number of periods $n$ is not
$$P = \frac{g}{j}(C - K) + K$$
-

12.5. **Bond amortization.**

- because bond payments consist of interest and principal, we can make an amortization schedule
- we introduce $B_t$ as the balance of the debt at time $t$, depending on the yield rate $j$
- $B_0 = P$ and $B_n = C$, $B_t$ is called the book value at time $t$
- the interest due is $I_t = jB_{t-1}$
- the amount of principal adjustment is $P_t = B_{t-1} - B_t$
- to find $B_t$ the basic price formula or the premium-discount formula may be used
$$B_t = (Fr)a_{\overline{n-t}|j} + Cv_j^{n-t}$$
or for the premium-discount formula
$$B_t = C(g - j)a_{\overline{n-t}|j} + C$$

- using the premium-discount formula we can find

$$I_t = Cg - C(g-j)v_j^{n-t+1}$$

  and

$$P_t = C(g-j)v_j^{n-t+1}$$

  and we see that

$$P_t + I_t = Cg$$

- we also get a nice recursion formula

$$B_t = (1+j)B_{t-1} - Cg$$

12.6. **Valuing a bond after its date of issue.**

- the dirty value of a bond $D_T$ (called dirty because it is discontinuous at each coupon date)

$$D_T = (1+j)^f B_{\lfloor T \rfloor}, \quad f = T - \lfloor T \rfloor$$

- practical dirty values are more useful because they are continuous

$$D_T^{\text{prac}} = (1+fj)B_{\lfloor T \rfloor}, \quad f = T - \lfloor T \rfloor$$

- we also have practical clean values that are the best

$$C_T^{\text{prac}} = B_{\lfloor T \rfloor} + f(B_{\lfloor T \rfloor+1} - B_{\lfloor T \rfloor}), \quad f = T - \lfloor T \rfloor$$

- see page 287 (PDF) of the maths for actuaries book

12.7. **Callable bonds.**

- callable bonds are issued with call provision (agreement) that allow the issuer to repay it earlier
- the bond can be repaid at any one of the designated call dates
- the period before the first call date is called the lockout period
- for the investor callable bonds are worse because they are more difficult to judge and less safe
- to make up for this disadvantage, callable bonds generally have higher yield rates, and the redemption prices before maturity are often higher
- when considering such a bond, it is wise to look at all the possible redemption values to figure out which is the possible worst and focus on that
- if a bond is bought at a discount, the worst yield rate will be attained at the redemption date
- if a bond is bought at a premium, the worst yield rate is attained when the bond is called at the earliest possible time

## 13. Interest Rate Sensitivity

### 13.1. Overview.

- interest rates are volatile and you can never be sure what you'll get
- thus you need to take care – immunizing your position means making sure that your position is good regardless of whether interest goes up or down
- for example, investing half in bonds that are good if interest is low and the other half in bonds that are good if interest is high can immunize one somewhat from changing rates
- duration of a cashflow is a measure of how sensitive the price is to changes in the interest rate
- Macaulay duration is a weighted average of the times of the cashflows

### 13.2. Overview.

- assuming a number of cash flows, the price of all of then to get a yield rare $i$ is
$$P(i) = \sum_{t\geq 0} C_t(1+i)^{-t}$$
- $i_0$ is the initial interest rate
- the Taylor polynomial for this sum is
$$\sum_{n=0}^{\infty} \frac{P^{(n)}(i_0)}{n!}(i-i_0)^n = P(i_0) + P'(i_0)(i-i_0) + \frac{P''}{2}(i_0)(i-i_0)^2 + \dots$$
- the first two terms give the tangent line approximation to $P(i)$ at rate $i_0$
$$P(i) \approx P(i_0) + P'(i_0)(i-i_0)$$
- the second-Taylor-polynomial approximation is
$$P(i) \approx P(i_0) + P'(i_0)(i-i_0) + \frac{P''}{2}(i_0)(i-i_0)^2$$
- for all these equations the derivates can be found with
$$P'(i) = -\sum_{t\geq 0} C_t(1+i)^{-t-1}$$
and
$$P''(i) = -\sum_{t\geq 0} C_t t(t+1)(1+i)^{-t-2}$$
- we have another approximation
$$\frac{P(i)-P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i-i_0)$$
- this allows us to estimate the fractional price increase
- thus we can talk about basis points, 100 of which correspond to $0.01 = 1\%$ in increased value ($i = q\% = 0.01q \to i = (q+i)\% = 0.01(q+1)$)
- If the yield is initially equal to $i_0$ and then it increases by one hundred basis points, the approximate relative price change is $\frac{P'(i_0)}{P(i_0)}$ percent.

- we can now define the modified duration
$$D(i,1) = -\frac{P'(i)}{P(i)} = \frac{\sum_{t\geq 0} C_t t(1+i)^{-t-1}}{\sum_{t\geq 0} C_t (1+i)^{-t}}$$
- general modified duration for $i$ convertible $m$ times is
$$D(i,m) = \left(\frac{1+i}{1+\frac{i^{(m)}}{m}}\right) D(i,1)$$
- the Macaulay duration is
$$D(i,\infty) = D(i,m)\left(1+\frac{i^{(m)}}{m}\right)$$
or alternatively
$$D(i,\infty) = \sum_{t\geq 0}\left(\frac{C_t(1+i)^{-t}}{P(i)}\right) t$$
- for a zero-coupon bond, $D(i,\infty) = N$ and $D(i,1) = \frac{N}{1+i}$

13.3. **Convexity.**

- while tangent approximation works well for small interest changes, larger changes can be modeled by the quadratic approximation
- thus we have the modified convexity as
$$C(i,1) = \frac{P''(i)}{P(i)}$$
- this allows us to rewrite our estimation as
$$\frac{P(i)-P(i_0)}{P(i_0)} \approx -D(i_0,1)(i-i_0) + C(i_0,1)\frac{(i-i_0)^2}{2}$$
- this indicates that if two investments have the same modified duration, the one with the larger absolute convexity is more susceptible to interest rate changes
- the Macaulay convexity is given by
$$C(i,\infty) = \sum_{t\geq 0}\left(\frac{C_t(1+i)^{-t}}{P(i)}\right) t^2$$
- from the Macaulay formulas we know that
$$C(i,m) = \frac{C(i,\infty) + \frac{1}{m}D(i,\infty)}{\left(1+\frac{i^{(m)}}{m}\right)^2}$$