Numerical Methods for Equations of Mathematical Physics

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Contents

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In-Class Notes

General Organization

- discussion of classes to take
- we will take two electives in spring and not this semester
- *Mathematical Models in Economics* and *Mathematical Models in Geophysics* in spring

Introduction

- we took *Equations of Mathematical Physics*
- our resources are: *Google Classroom*, *Google Drive*
- we will use *Visual Studio* with *C#* and *Zed Graph* to write our programs

Grading

- 3 different projects totaling 80 point
  1. different methods for solving advection-diffusion transport problems
  2. different methods to solve heat equation
  3. methods to solve elliptic equation through iteration
- **see syllabus**
- final exam is 20 point

Evolutionary Advection-Diffusion Equation

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + a_1 \cdot \frac{\partial u}{\partial x} + a_2 \cdot \frac{\partial u}{\partial y} + a_3 \cdot \frac{\partial u}{\partial z} - b \cdot u$$

- this could be simplified to

$$\frac{\partial u}{\partial t} = k \cdot \nabla^2 u + \vec{a} \nabla u - b \cdot u$$

- if we take the one-dimensional form of this equation, convection and advection are the same
- we get a stationary equation that is simple because it is one-dimensional and does not depend on time
- we have several forms that are called different names – three types of boundary problems

- we can re-write the equation in operator form
- thus we can define $F(x)$ in operator form – this will make is simpler
- theoretical results show that unique solutions belong to a class of functions that have continuous derivatives
- problem number 1 has one classic equation
- we will consider different approaches to solving this problem

Approximation of Derivatives

- see *Theorem 1* and (4) and (5)
- we will consider a uniform mesh with a step
- it has n greet points
- we'll consider an approximation of derivatives on this grid
- we will use *Taylor's formula* iteratively in (4)
- last point is remainder where we consider the unknown point θ
- in the future we will suppose that our solution is smooth enough – there are at least 4 continuous derivatives
- we denote the order of the remainder with respect to h meaning it tends to 0 if h tends to 0
- look at other slides to look at what slides and symbols we are using
- *approximation for the first derivative by forward difference*
- we can also obtain a backward difference and a central difference

After Class Notes

Syllabus

- classical and modern numerical methods for partial differential equations
- we'll consider boundary-value and initial-boundary-value problems
- topics:
    1. advection-diffusion
    2. poisson equations
    3. non-stationary heat equations
- 100 points in total, 80 points for 3 lab projects, 20 points for final exam

Advection-Diffusion Equation

- a combination of the diffusion (advection) and convection equations
- describes physical systems where these two things are happening
- the general equation is

$$\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c) - \nabla \cdot (\vec{v} c) + R$$

- c is the variable we want to study (could be x)
- D is the diffusivity, it describes how quickly two things diffuse into each other
- $\vec{v}$ is the velocity field that stuff is moving through – it describes time and space
- R describes sources or sinks of material – whether or not material increases or decreases
- ∇ is the gradient – a vector of partial derivatives of the function by each of the variables

- $\nabla\cdot$ is the divergence – whether or not material moves away from a point or not

Common Simplification

$$\frac{\partial c}{\partial t} = D\nabla^2 c - \vec{v} \cdot \nabla c$$

- often no sources or sinks are assumed
- if the flow is incompressible, the divergence is 0
- the diffusion coefficient is assumed constant

Equation From Class

$$\frac{\partial u}{\partial t} = k \cdot \nabla^2 u + \vec{a}\nabla u - b \cdot u$$

- k is the diffusivity
- u is the variable in question
- $\vec{a}$ is the velocity field
- $-b \cdot u$ is a negative sink in the process