An Overview of Financial Mathematics

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Abstract

This document is meant to be a quick introduction to interest theory. It is written specifically for actuarial students preparing to take exam FM/2 jointly offered by the Society of Actuaries and Casualty Actuarial Society. As such, this guide will follow the course objectives for the August 2012 sitting. That being said, I hope it will be of some use to anyone studying financial mathematics. It is best used in conjunction with another more rigorous text. I recommend *Mathematical Interest Theory* by Vaaler & Daniel. Both the Actex and ASM manuals are very helpful in preparing for exam FM thanks to their thousands of practice problems. This guide covers the time value of money, annuities, loans, bonds, stocks, interest rates, and an introduction to derivatives, forwards and futures.

1 Introduction to the Time Value of Money

1.1 Interest Explained

The central idea of interest theory is that the same amount of money is worth different amounts at different times. In other words, a rational person would rather have five thousand dollars right now than the promise of five thousand dollars in three years – the rational person has the option to save the money for three years *or* to do something else with it. This *or* adds value to having the money now.

Because of this difference in values a lender who loans money to a borrower can reasonably charge him for it. The money that the borrower pays to the lender is called interest. Most of us, much to Polonius' dismay, are both borrowers and lenders. We've borrowed money to pay for our cars and therefore have to pay some amount of interest on that loan. At the same time we've lent money to the bank (by depositing our money there) and they pay us interest.

There is a reason that the interest rate, the rate that determines the amount of interest owed, is different in our arrangement with the car dealership and the bank's arrangement with us: we are riskier. The dealership has fair reason to worry that I might split without paying for all of the car or find myself bankrupt and unable to pay. The dealership is trying to put a price on the risk they are accepting. Therefore, if I am able to convince the dealer I'm a sure thing to pay it all back, I might be able to get a lower rate. On the other hand, I'm not at all concerned that the bank is going to run away with my money so I can't reasonably charge them a high rate for borrowing my money. However, I am still assuming *some* risk, the bank might go under, so I do charge some interest.

These are just two examples of hundreds of different ways that people and organizations can lend money to each other. This review guide is more concerned with valuing the interest and determining the value of money at different times than working out what a fair interest rate is or why interest exists. Those topics will be explored in exams MFE/3F and MLC/3L. For now, just assume a non-negotiable interest rate and don't worry too much about why exactly it exists.

1.2 Simple and Compound Interest

Let's assume that you deposit some money into a savings account at a bank and plan to leave it there for a while without depositing into it or withdrawing from it. The amount of money in the account will therefore increase over time as the bank adds the interest payments to the savings account. This phenomenon is called the growth of money or time value of money. The money can grow in one of two main ways.

The first is that each year the bank might add some percent of the original amount deposited. For example, you might deposit \$100 into the account. Then one year later the bank adds an interest payment of amount \$5 so that the savings account contains \$105. Then one year after that, two years after the original deposit was made, the bank adds another \$5 into the account so that the account contains \$110. This model of growth is called simple interest and can be represented by the following equation:

$$A(t) = K(1 + it) \tag{1}$$

where A, the amount function, is the value of the account at time t; K, the principal, is the amount originally deposited; i is the interest rate governing the account; and t is the time since the original deposit. In this example, $A(0) = \$100$, $A(1) = \$105$, $A(2) = \$110$, $K = \$100$, and $i = .05$ (remember, the interest rate is just that, a rate). Sometimes it is more convenient to deal with the case in which just one dollar was originally invested.

$$a(t) = 1 + it \tag{2}$$

Here a is called the accumulation function. Note that $A(t) = Ka(t)$. Simple interest is not used very much in the real world for an obvious reason. Look back over the example and see if you can figure out why. A savvy investor, instead of doing what the investor in the example did, would take his \$105 and open a brand new account. That way at the end of the second year the bank would add $105 * .05 = 5.25$ to his account. This investor was able to take advantage of the system and get an $A(2)$ equal to \$110.25. The banks forseeing this problem usually offer a savings account that grows according to compound interest.

$$A(t) = Ka(t) = K(1 + i)^t \tag{3}$$

To explore simple and compound interest in more depth it is necessary that I introduce some new vocabulary. The effective interest rate, $i_{[t_1, t_2]}$, tells us the rate that corresponds to the amount of interest paid in the interval $[t_1, t_2]$.

$$i_{[t_1, t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)} = \frac{A(t_2) - A(t_1)}{A(t_1)} \tag{4}$$

Note that the interval $[n - 1, n]$ is called the nth period. With these definitions in hand, it is clear that for simple interest the effective interest rate is constantly decreasing while for compound interest it is the same regardless of the period (look back to the example and verify this for yourself).

1.3   Nominal Interest Rates

In our example the interest was paid at the end of each year. But there is no reason that the interest must be paid yearly. Indeed, it is very often paid monthly, quarterly, or semi-annually. Ideally, we would always be given the effective rate for the interval and the length of the interval. For example, an

investment might pay 1% per month (if it doesn't say interest is simple or compound, it is assumed to be compound), and we would know that at the end of each month our investment would be worth 1.01 times what it was worth at the beginning of the month. Unfortunately, this is not always the case.

The nominal interest rate is the interest rate per period times the number of those periods in a year. In our example we would say the investment pays interest at a nominal rate of 12% convertible/credited/payable monthly. This nominal rate is written $i^{(12)} = 12\%$ (note the paranthesis around the 12, it's not an exponent) and we say "i upper twelve". Note that if we deposited $100 in this account we would not have $112 at the end of the year but rather $(100)(1.01)(1.01)...(1.01) = (100)(1.01)^{12} = \$112.68$. We can use formula (4) to determine that this account pays interest at an annual EIR of $\frac{112.68-100}{100} = 12.68\%$. There is a specific formula for converting a nominal interest rate to an effective interest rate, but I prefer to think about what's actually happening to the money.

$$a(t) = (1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{tm} \tag{5}$$

You can then rearrange whatever you want to solve for what the problem requires.

1.4 Discount Rates

So far, I've talked about what happens when an amount of money changes hands at the beginning of a time period and interest is paid on that money. But sometimes, two parties make an arrangement where the money changes hands at the end of a time period instead. Since the money is worth more now than in the future, the person getting the money at the end of the year ought to get a cheap price on it. In other words, he gets it at a discounted price.

A simple example best demonstrates the difference between discount and interest rates. The interest rate is the amount the money grows. The investor lends $100 to the bank, the bank pays him 5% interest, and at the end of the year the investor gets $105. The discount rate is the rate at which the money is discounted. The investor lends the bank $95 and at the end of the year the bank pays him $100. In this case the discount rate was 5%.

We calculate the effective discount rate in a similar fashion to how we calculated the effective interest rate.

$$d_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_2)} = \frac{A(t_2) - A(t_1)}{A(t_2)} \tag{6}$$

In our introductory example we had just one discount time period, but there can, of course, be multiple periods. Both simple and compound discounting

exist.

$$a(t) = \frac{1}{1 - dt} \tag{7}$$

$$a(t) = (1 - d)^{-t} \tag{8}$$

Also note that discount rates can be be written as nominal rates. The concepts are all equivalent to the interest rate concepts. The following equality will be useful.

$$a(t) = (1 - d)^{-t} = \left(1 - \frac{d^{(p)}}{p}\right)^{-pt} \tag{9}$$

The previous example also shows how interest and discount rates are related. A different investor at a different bank could have lent the bank \$95 and watched that money grow, according to some interest rate, to \$100. Be aware that the \$95 did not grow at an interest rate of 5% but $\frac{100-95}{95} = .0526$ or 5.26%. A little algebraic manipulation of equations (4) and (6) reveals that

$$(1 + i)(1 - d) = 1 \tag{10}$$

With the concept of discounting we are now well prepared to truly understand the notion of the time value of money.

1.5   Present and Future Values

After reading the previous example comparing interest and discount rates a natural thing to wonder is how much money did the investor have to invest at the first bank where the interest rate was 5% to end the period with \$100. To figure this out, simply use equation (3) and solve for K. In fact, this sort of calculation is done so often that a new function, called the discount function, is used. The discount function is, of course, the inverse of the accumulation function.

$$v(t) = \frac{1}{a(t)} \tag{11}$$

To make very clear what values we are referring to, we use the terms present value and future value. The present value is the value of the money right now. The future value is the value of that same money plus the accumulated interest.

$$FV = PV * a(t) \tag{12}$$

It should be becoming clear how the discount rate plays a role in this. Let's say an investor wants his investment to grow to \$1 over the course of the year. The present value he has to invest is $\frac{1}{1+i}$. Recall that the discount rate is the rate that tells us how much the future value is on sale or marked down. Therefore,

$$d = 1 - \frac{1}{1 + i} = \frac{i}{1 + i} \tag{13}$$

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This is simply another way of writing what we already algebraically showed in equation (7). Finally, we see that

$$d = 1 - v \tag{14}$$

1.6 The Force of Interest

While generally accounts pay interest in discrete intervals, it is theoretically possible that an account could pay interest continuously. To calculate what the accumulation function is under continuous interest you can do a couple things. One, you can take a limit of a nominal rate. First, we will define the force of interest, δ, as follows.

$$\lim_{m \to \infty} i^{(m)} = ln(1 + i) = \delta \tag{15}$$

It's not necessary that you be able to prove it for exam FM, but it's a good proof to know. With that definition and a recollection of the definition of e in hand we can finally calculate the accumulation function.

$$\lim_{m \to \infty} \left(1 + \frac{i^{(m)}}{m}\right)^m = \lim_{m \to \infty} e^{i^{(m)}} = e^{\delta} \tag{16}$$

A completely different approach is to start with what continuous interest actually means. It means that at every point in time your instantaneous change in your account is δ times the amount of money currently in the account, since that's the interest amount payable. Mathematically we can write:

$$\frac{d}{dt} a(t) = \delta a(t) \tag{17}$$

This is just an elementary differential equation with solution:

$$a(t) = e^{\delta t} \tag{18}$$

I'll reassure you now that you don't need to be able to derive the accumulation function under continuously compounded interest. Simply memorizing the accumulation function is sufficient.

The last explanation introduced the idea of the instantaneous change in the account value. While solving the differential equation we see that $\delta = \frac{a'(t)}{a(t)}$. In that case, the force of interest, δ, was constant. However, in accounts earning discrete interest there is also a force of interest at all times. And, as luck would have it, the formula is the same. The only difference will be that δ will depend on time, i.e., it's not a constant. In general, given a δ function in terms of t we can determine what $a(t)$ is.

$$a(t) = e^{\int_0^t \delta(u) du} \tag{19}$$

1.7 Big Equation

The previous sections have introduced the most common ways that money can grow. All of the interest theory presented so far can be represented by one big equality.

$$a(t) = (1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = (1-d)^{-t} = \left(1 - \frac{d^{(p)}}{p}\right)^{-pt} = e^{\delta t} \quad (20)$$

And if, instead of finding the future value of some amount of money, you need to find a present value use $v(t) = \frac{1}{a(t)}$.

1.8 Inflation

Inflation is defined as the natural increase in the cost of goods. Inflation as a concept is complicated and beyond the scope of the overview, but its impact on the value of money is important. As a simple example, in 1980 \$100 could buy 278 widgets while today it can only buy 100 widgets. It is reasonable to assume that in the future \$100 will buy even fewer widgets. In other words, inflation eats away at the value of money. Because of this, there is an important rate, the real interest rate, that describes how the purchasing power of money grows over time.

$$1 + j = \frac{1+i}{1+r} \quad (21)$$

where j is the real interest rate, i is the interest rate, and r is the rate of inflation. Be forewarned that the notation will change in the later exams. It is standard exam FM notation, though.

1.9 Equations of Value

Up to this point I have only presented situations that involve a single present value and a single future value that can be related by using a single discount or accumulation factor. However, the real meat of financial mathematics is putting these concepts to use in more complicated cash flows.

A classic example might ask us to find the present value of three payments assuming a compound interest rate of 6%. A payment of \$1000 made one year from now, a payment of \$5000 made two years from now, and a payment of \$3000 made three years from now. The present value can simply be thought of as the sums of the present values of four different future values.

$$PV = (1000)(1.06)^{-1} + (5000)(1.06)^{-2} + (3000)(1.06)^{-3} = \$7912.24 \quad (22)$$

Another problem might assume that the interest rate is 4% in the first year and 7% in the second year. Find the present value of \$2000 paid at the end of the

first year and $3000 paid at the end of the second year. For problems like this it is helpful to draw a timeline.

$$PV = (2000)(1.04)^{-1} + (3000)(1.07)^{-1}(1.04)^{-1} = \$4618.98 \qquad (23)$$

To find the future value of this set of cash flows at the end of year 2, you can either bring forward the present value amount we solved for or the individual cash flows. Draw a timeline.

$$FV = (4618.98)(1.04)(1.07) = \$5140.00 \qquad (24)$$

$$FV = (2000)(1.07) + 3000 = \$5140.00 \qquad (25)$$

There are several different ways to solve these sorts of problems. It helps to always break it down into little bits – money moves from one period to the next period at some rate. It's always possible to solve equations of value this way. There are many, many examples of them and I encourage you to do enough of them until you feel comfortable moving money around according to the theory laid down in this first section.

2 Annuities

An annuity is a kind of cash flow that involves paying the owner a fixed sum of money each year for some number of years. The key thing to remember is that the amount of money paid is fixed. This will allow us to reduce PV and FV calculations for annuities to short formulas instead of strings of values summed together as in section 1.9.

2.1 Annuities-Immediate

An immediate annuity is an annuity that makes payments at the end of specified intervals. The payments are always the same amount, the time intervals are always the same length, and the interest rate is constant. Score! Assuming an end-of-interval payment amount 1, an effective interest rate i per period, and n intervals the present value of the annuity-immediate is as follows.

$$PV = (1+i)^{-1} + (1+i)^{-2} + ... + (1+i)^{-n} \qquad (26)$$

Recall this very important equation for the sum of the terms of a geometric sequence.

$$\text{Sum} = \frac{\text{first term} - \text{next after last term}}{1 - \text{ratio}} \qquad (27)$$

From equation (26), we can see that the first term is $(1+i)^{-1}$, the next term after the last is $(1+i)^{-(n+1)}$, and the ratio is $(1+i)^{-1}$. We let the present value be represented by the symbol $a_{\overline{n}|}$ pronounced "a angle n".

$$PV = a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i} = \frac{1 - v^n}{i} \tag{28}$$

The future value, $s_{\overline{n}|}$, "s angle n", can be determined using either the method we used in (24) or the method used in (25). As a reminder the present value, $a_{\overline{n}|}$, is the value one full interval before the first payment. The future value is the value the instant after the nth payment.

$$FV = s_{\overline{n}|} = (a_{\overline{n}|})(1+i)^n = \frac{(1+i)^n - 1}{i} \tag{29}$$

Another useful relationship exists between $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i \tag{30}$$

We have been assuming that the level payment is of amount 1. If it's some other amount, say P (remember that the payment amount must be constant), then the present value is simply $P(a_{\overline{n}|})$ and the future value is $P(s_{\overline{n}|})$

2.2 Annuities-Due

This is exactly the same idea as an immediate annuity, except instead of the payments being made at the end of each interval they're made at the beginning of each interval. We write the present value as $(\ddot{a}_{\overline{n}|})$ and say "a double dot angle n". Notice that the present value is the value of the cash flows the instant before the first payment.

$$\ddot{a}_{\overline{n}|} = (1+i)(a_{\overline{n}|}) = \frac{1 - v^n}{d} \tag{31}$$

It's worth taking a few minutes to make sure you understand why it's true. Remember that the payments of $a_{\overline{n}|}$ are farther away so their present value must be lower. The future value formulas are analogous.

$$FV = \ddot{s}_{\overline{n}|} = (1+i)(s_{\overline{n}|}) = \frac{(1+i)^n - 1}{d} \tag{32}$$

There are a number of relationships between present and future values and between annuities due and immediate. In reality they're all kind of silly. But in terms of learning it's good to think through them and in terms of passing exam FM it's good to know them as they can really speed up a solution.

$$a_{\overline{n}|} + 1 = \ddot{a}_{\overline{n+1}|} \tag{33}$$

$$\ddot{s}_{\overline{n}|} + 1 = s_{\overline{n+1}|} \tag{34}$$

$$\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d \tag{35}$$

$$(a_{\overline{n}|})(1+i)^n(1+i) = \ddot{s}_{\overline{n}|} \tag{36}$$

$$\ddot{s}_{\overline{n}|}(1+i)^{-n}(1+i)^{-1} = a_{\overline{n}|} \tag{37}$$

As before, for payment amounts other than 1, multiply the annuity symbol by the amount of the payment.

2.3 Annuities with Arithmetic Progression

The actuarial exam will require that you be able to find the present value of some special non-level-payment annuities. Annuities whose payments are geometrically increasing can be dealt with using equation (27). There also exist annuities whose payments increase by some fixed amount each interval. Consider an annuity that pays P at the end of the first period, $P + Q$ at the end of the second period, and $P + (n-1)Q$ and the end of the nth period. Then the present value of the increasing annuity at time 0 can be represented as follows.

$$(I_{P,Q}a)_{\overline{n}|} = Pa_{\overline{n}|} + Q\frac{(a_{\overline{n}|} - n(1+i)^{-n})}{i} \tag{38}$$

This equation makes a lot of sense and you should derive it yourself. This will help you understand what all the parts mean which will in turn help you adapt the equation to many situations. And using the same logic as before:

$$(I_{P,Q}s)_{\overline{n}|} = (I_{P,Q}a)_{\overline{n}|}(1+i)^n \tag{39}$$

$$(I_{P,Q}\ddot{a})_{\overline{n}|} = (I_{P,Q}a)_{\overline{n}|}(1+i) \tag{40}$$

$$(I_{P,Q}\ddot{s})_{\overline{n}|} = (I_{P,Q}a)_{\overline{n}|}(1+i)^n(1+i) \tag{41}$$

For completeness I'll include the rest of this section, but I really urge you to understand what's going on. If you can't derive the following equations it doesn't mean you haven't memorized enough to be ready for the test, it means you don't understand enough to be ready for the test.

$$(I_{1,1}a)_{\overline{n}|} = (Ia)_{\overline{n}|} = \frac{(1+i)(a_{\overline{n}|}) - n(1+i)^{-n}}{i} \tag{42}$$

In the event that $P = n$ and $Q = -1$ we have what's called a decreasing annuity.

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \tag{43}$$

$$(Ds)_{\overline{n}|} = (Da)_{\overline{n}|}(1+i)^n \tag{44}$$

2.4 Perpetuities

A perpetuity is exactly what it sounds like - an annuity that goes on perpetually. It should be clear right away that a perpetuity can't have a future value since we've been defining the future value as the value of the annuity after the last payment, which in this case doesn't exist. To calculate the present value of the perpetuity, $a_{\overline{\infty}|}$, write out (26) for this stream of cash flows and then realize that for (27) the next term after the last must be $(1+i)^{-(\infty+1)} = 0 \ \forall i$.

$$a_{\overline{\infty}|} = \frac{1}{i} \tag{45}$$

$$\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} + 1 = \frac{1+i}{i} = \frac{1}{d} \tag{46}$$

$$(I_{P,Q}a)_{\overline{\infty}|} = \frac{P}{i} + \frac{Q}{i^2} \tag{47}$$

Remember that in all three of these cases – annuity immediate, annuity due, and perpetuity – i is the interest rate per period. So if you're given $i^{(4)}$ make sure to convert to $i = \frac{i^{(4)}}{4}$ and to use the number of years times 4 as the number of payments.

2.5 The Weird Annuity

Almost all of the annuities and perpetuities that the actuarial exam will throw at you can be dealt with using a timeline, the formulas from the previous five sections, and a little common sense. However, it is worth your time to memorize the formula for the weird annuity. It pays an amount $1/m$ at the end of each mth of the first year, amount $2/m$ at the end of each mth of the second year, up to amount n/m at the end of each mth of the nth year. The symbol for this kind of annuity is $(Ia)_{\overline{n}|}^{(m)}$.

$$(Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|} - n(1+i)^{-n}}{i^{(m)}} \tag{48}$$

2.6 Continuously Paying Annuities

The idea of these is that instead of getting paid at the end of every year, or every month, the annuitant gets paid at the end of every instant. The present value of this continuously paying annuity is symbolized by $\bar{a}_{\overline{n}|}$. You can derive the following formula by either taking the limit of equation (48) as m goes to infinity or integrating over the appropriate time span.

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - (1+i)^{-n}}{\delta} = \frac{1 - e^{-n\delta}}{\delta} \tag{49}$$

The general formula for a continuously paying annuity is really all you need to memorize.

$$PV = \int_0^n f(t)v^t dt \tag{50}$$

Where $f(t)$ is the rate of payment. Very often it is 1 per year in which case the formula for the present value is what we already saw in equation (49). Other times, as in an increasing continuously paying annuity that pays an amount t at time t, the present value is $PV = \int_0^n tv^t dt$. And by now you should know how

to move a lump sum from the present to the future, simply multiply by $(1+i)^n$.
The perpetuity $(n \to \infty)$ formulas pop right out of equations (49) and (50).

$$\bar{a}_{\overline{\infty}|} = \frac{1}{\delta} \tag{51}$$

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2} \tag{52}$$

where $f(t) = 1$ for (51) and $f(t) = t$ for (52).

3   Loans

Patience!

References

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