

B. PDE on rectangular domains, separation of variables.

6.8. Laplace equation in a rectangle, Fourier series.

We want to solve the Dirichlet boundary value problem for the Laplace equation in a rectangle

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < H, \\ u(0, y) = g_1(y), \\ u(L, y) = g_2(y), \\ u(x, 0) = g_3(x), \\ u(x, H) = g_4(x). \end{array} \right. \quad (1)$$

By superposition, we know that we can split problem (1) into four similar problems, each of which satisfies one of the four boundary functions and the zero boundary condition on the other three sides. For example, let us find u_1 :

$$\left\{ \begin{array}{l} \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \\ u_1(0, y) = g_1(y), \\ u_1(L, y) = 0, \\ u_1(x, 0) = 0, \\ u_1(x, H) = 0. \end{array} \right. \quad (2)$$

We use the method of **separation of variables**. Let $u_1(x, y) = X(x)Y(y)$. Then the Laplace equation can be written as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \quad (3)$$

Since the two sides of (3) are functions of different variables, we conclude that they must be constant, which we set to be λ . Thus (3) becomes

$$X'' = \lambda X, \quad (4)$$

$$Y'' = -\lambda Y. \quad (5)$$

We see that we can let $Y(y)$ satisfy the boundary condition

$$Y(0) = Y(H) = 0. \quad (6)$$

Then, the Y equation (5) and the boundary condition (6) have solution

$$Y = c \sin\left(\frac{n\pi y}{H}\right) \quad (7)$$

for

$$\lambda = \left(\frac{n\pi}{H}\right)^2, \quad (8)$$

where $n = 1, 2, \dots$. For the λ in (8), we find that the general solution to (4) is

$$X(x) = Ee^{\frac{n\pi}{H}x} + Fe^{-\frac{n\pi}{H}x}.$$

Or

$$X(x) = a_1 \cosh\left[\frac{n\pi}{H}(x - L)\right] + a_2 \sinh\left[\frac{n\pi}{H}(x - L)\right].$$

The shift in x by L is selected to satisfy the boundary condition at $x = L$ conveniently. We impose that $X(L) = 0$, which implies $a_1 = 0$. Thus

$$X(x) = a_2 \sinh\left[\frac{n\pi}{H}(x - L)\right].$$

In summary, solutions u of the product form $X(x)Y(y)$ satisfying the zero condition on the three corresponding sides of (2) are

$$u(x, y) = a_2 \sinh\left[\frac{n\pi}{H}(x - L)\right] \sin\left(\frac{n\pi y}{H}\right)$$

for any constant a_2 and all $n = 1, 2, 3, \dots$. By superposition, we find that

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi}{H}(x - L)\right] \sin\left(\frac{n\pi y}{H}\right) \quad (9)$$

is also a solution to the Laplace equation with zero value on the three zero-value sides of the rectangle in (2) for any real numbers a_n . We want to choose a_n such that u in (9) satisfies the fourth nonzero boundary condition:

$$u(0, y) = \sum_{n=1}^{\infty} a_n \sinh\left[-\frac{n\pi L}{H}\right] \sin\left(\frac{n\pi y}{H}\right) = g_1(y).$$

It is known that any smooth function $g_1(y)$ with $g_1(0) = g_1(H) = 0$ can be expressed as a **Fourier sine series**

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right), \quad 0 \leq y \leq H, \quad (10)$$

where

$$A_n = \frac{2}{H} \int_0^H g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy. \quad (11)$$

Thus, we can take

$$a_n = \frac{A_n}{\sinh\left[-\frac{n\pi L}{H}\right]}$$

so that (9) satisfies all four boundary conditions. Hence we find a solution to (2):

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{H \sinh\left[-\frac{n\pi L}{H}\right]} \left(\int_0^H g_1(\eta) \frac{\sin(n\pi\eta)}{H} d\eta \right) \sinh\left[\frac{n\pi}{H}(x - L)\right] \sin\left(\frac{n\pi y}{H}\right). \quad (12)$$

Notes.

1. The cosh and sinh functions are $\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$.
2. Problem (5) and (6) is called an **eigenvalue problem**, where we need find both a number λ and a nonzero solution $Y(y)$.
3. The Fourier sine series (10)(11) is a special case of the general Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq L,$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx, \\ a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

For a proof, see Dirichlet-Jordan Convergence Theorem, p.164, Sect. 4.5.1, Keener.)

4. Other boundary conditions, such as Neumann boundary condition can be solved similarly (See homework).
5. Nonhomogeneous equation (Poisson equation) can be solved also (See next lecture).
6. Laplace equation in a disk can be solved by separation of variables in addition to the complex variables method.

Excerpt: Story of Fourier

Joseph Fourier's father was a tailor in Auxerre. After the death of his first wife, with whom he had three children, he remarried and Joseph was the ninth of the twelve children of this second marriage. Joseph's mother died when he was nine years old and his father died the following year.

It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies*. The memoir was read to the Paris Institute on 21 December 1807 and a committee consisting of Lagrange, Laplace, and others was set up to report on the work. Now this memoir is very highly regarded but at the time it caused controversy.

There were two reasons for the committee to feel unhappy with the work. The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series, what we now call Fourier series. Further clarification by Fourier still failed to convince them.

The second objection was made by Biot against Fourier's derivation of the equations of transfer of heat. Fourier had not made reference to Biot's 1804 paper on this topic but Biot's paper is certainly incorrect. Laplace, and later Poisson, had similar objections.

The Institute set as a prize competition subject the propagation of heat in solid bodies for the 1811 mathematics prize. Fourier submitted his 1807 memoir together with additional work on the cooling of infinite solids and terrestrial and radiant heat. Only one other entry was received and the committee set up to decide on the award of the prize, Lagrange, Laplace, Malus, Haüy, and Legendre awarded Fourier the prize. The report was not however completely favourable and states:

... the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

With this rather mixed report there was no move in Paris to publish Fourier's work. His work was later published in 1822.