When Machine Learning Meets Complex Systems - Exercise 1 Part (d) and (e)

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Introduction

The following report outlines the methods used for solving exercises (d) and (e) of task number 1.5 from exercise sheet one.
The general task 1.5 from exericse sheet one was about analyzing the Henon map, which is defined in equation 1.

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned} \tag{1}$$

Evaluation of Fixed Point Stability - Part (d)

0.1 Task Description

A fixed point of a map is linearly stable if and only if all eigenvalues $\lambda_\pm$ of the Jacobian satisfy $|\lambda_\pm| < 1$. Determine the stability of the fixed points of the Henon map, as a function of a and b. Show that one fixed point is always unstable, while the other is stable for a slightly larger than a_0. Show that this fixed point loses stability in a flip bifurcation, i.e. that $\lambda = -1$ at $a_1 = \frac{4}{3}(1-b)^2$.

0.2 Stability analysis of fixed points

In part (c) of exercise 1.5 the eigenvalues of the Jacobian matrix of the Henon map were determined as shown in equation 2.

$$\lambda_{\pm}(x_{\pm}) = -a x_{\pm} \pm \sqrt{(a x_{\pm})^2 + b} \tag{2}$$

The fixed points of the Henon map $x_{\pm}$, which were derived in part (b) of exercise 1.5 are determined by the parameters a and b according to equation 3.

$$x_{\pm} \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a} \tag{3}$$

Therefore, the stability of both fixed points $x_{\pm}$ can be evaluated from equations 2 and 3, by assessing wether both eigenvales λ_+ and λ_- fulfill the stability criterion $|\lambda_{\pm}| < 1$ for each of the fixed points.

Furthermore, for the stability evaluation of the fixed points equations 4, 5 and 6 reveal that $|b| < 1$ needs to be satisfied for each of the fix points to be stable. Therefore, parameter b is restricted to $-1 < b < 1$.

$$\begin{aligned}\det(J-\lambda) &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - \operatorname{tr}(J)\lambda + \det(J) \end{aligned} \tag{4}$$

$$\lambda_{\pm} = \frac{\operatorname{tr}(J)}{2} \pm \frac{1}{2}\sqrt{\operatorname{tr}(J)^2 - 4\cdot\det(J)}$$

In part (c) of this exercise the Jacobian was determined as in equation 5.

$$J(x_n) = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix}, \quad \operatorname{tr}(J) = -2ax_n, \quad \det(J) = -b \tag{5}$$

Using equations 4 and 5 the relation in equation 6 can be derived, which forces the condition $|b| < 1$ for stable fixed points.

$$\begin{aligned}\lambda_+ \cdot \lambda_- = \left(\frac{\operatorname{tr}(J)}{2}\right)^2 - \left(\left(\frac{\operatorname{tr}(J)}{2}\right)^2 - \det(J)\right) = \det(J) \\ |b| = |\det(J)| = |\lambda_+ \cdot \lambda_-| = |\lambda_+| \cdot |\lambda_-| < 1\end{aligned} \tag{6}$$

Additionally, in part (b) of this exercise a lower boundary $a > a_0(b) = -\frac{(1-b)^2}{4}$ was derived for a that needs to be satisfied for $x_{\pm} \in \mathbb{R}$ to be real fixed points.

Having specified the necessary conditions for parameters $a > a_0(b)$ and $|b| < 1$ to obtain real stable fixed points, the stability analysis can be conducted numerically for both fixed points $x_+(a, b)$ and $x_-(a, b)$, whereby each of these fixed points is only stable when $|\lambda_+(a, b)| < 1$ and $|\lambda_-(a, b)| < 1$ both satisfy the stability criterion.

The results of the stability analyses for the fixed points $x_\pm$, which are depicted in Figure 1 and Figure 2, show three different regions that classify the fixed point and its stability. The black region covers the value pairs (a, b) that result in complex fixed points and are therefore excluded by the criterion $a > a_0(b)$. Furthermore, the red and green regions represent all value pairs (a, b) that result in real fixed points $x_\pm$, where the green area classifies stabil fixed points and the red area unstable fixed points. Once more it needs to be recognized that both eigenvalues $|\lambda_\pm| < 1$ need to fulfill the stability criterion simoultaneoulsy for the fixed point to be stable.

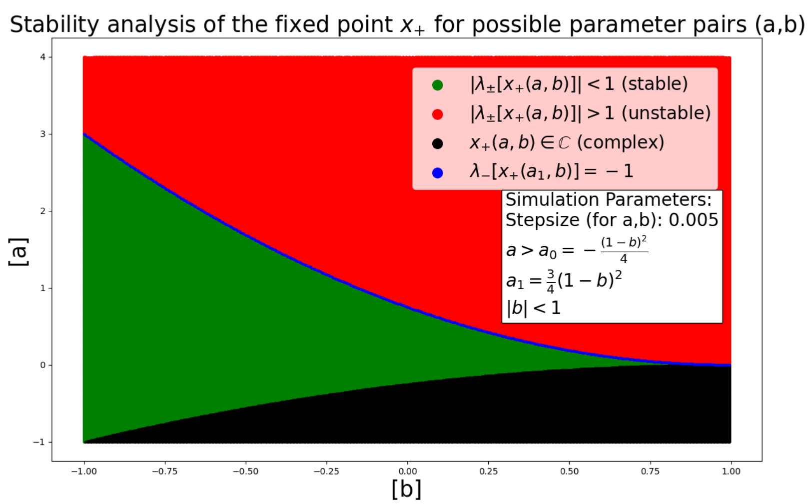

Figure 1: Stability analysis for the fixed points $x_+(a, b)$

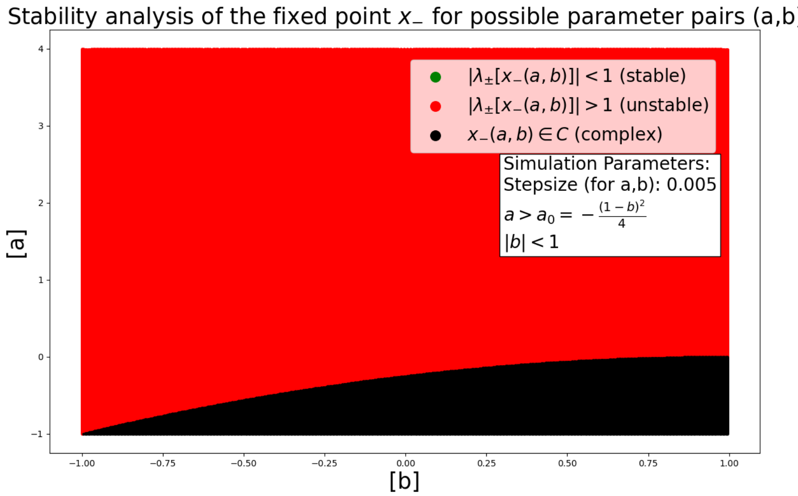

Figure 2: Stability analysis for the fixed points $x_-(a,b)$

Figures 1 and 2 reveal that only the fixed point x_+ can be stable for legitimate parameter pairs (a,b), whereas the fixed point x_- is always unstable. Furthermore, Figure 1 shows that the fixed point x_+ looses its stability when $a \geq a_1 = \frac{3}{4}(1-b)^2$, because $|\lambda_-(x_+(a_1,b))| \geq 1$ does not fulfill the stability criterion anymore, which can also be derived analytically as shown in equations 7 - 10.

$$\begin{aligned} a_1 \cdot x_+(a_1,b) &= \frac{-(1-b)+\sqrt{(1-b)^2+4a_1}}{2} \\ &= \frac{-(1-b)+\sqrt{4(1-b)^2}}{2} \\ &= \frac{-(1-b)}{2} + |(1-b)| = \frac{1-b}{2} \\ &\rightarrow \; ax_+(a > a_1, b) > \frac{1-b}{2} \end{aligned} \tag{7}$$

$$\text{using } \; (1+b),\; (1-b) \; > \; 0 \,\forall\, b \; \in \; (-1,1) \tag{8}$$

$$
\begin{aligned}
\lambda_{\pm}(x_+(a_1,b),a_1,b) &= -a_1x_+ \pm \sqrt{(a_1x_+)^2+b} \\
&= \frac{-(1-b)}{2} \pm \sqrt{\frac{(1-b)^2}{4}+b} \\
&= \frac{-(1-b)}{2} \pm \frac{1}{2}|(1+b)| \\
&= \frac{1}{2}(-(1-b) \pm (1+b)) = \frac{1}{2}(-1 \pm 1 + b \pm b) \\
&\rightarrow\ \lambda_+(x_+(a_1,b),a_1,b) = b, \\
&\rightarrow\ \lambda_-(x_+(a_1,b),a_1,b) = -1
\end{aligned} \tag{9}
$$

For $a \geq a_1$:

$$
\begin{aligned}
& ax_+(a \geq a_1,b) \geq a_1x_+ = \frac{1-b}{2} > 0 \\
& \sqrt{(ax_+)^2+b} \ \geq \ \sqrt{(a_1x_+)^2+b} \\
& -ax_+ - \sqrt{(ax_+)^2+b} \ \leq -a_1x_+ - \sqrt{(a_1x_+)^2+b} \\
& \rightarrow\ \lambda_-(x_+(a \geq a_1,b), a \geq a_1, b) \leq -1 \\
& \Rightarrow x_+(a \geq a_1,b) \text{ is unstable}
\end{aligned} \tag{10}
$$

Evaluation of 2-Cycles - Part (e)

The 2-Cycle of a mapping $f(x,y)$ is generally defined by two points (x_1,y_1) and (x_2,y_2), whereby the first point always maps into the second point when applying the respective mapping and the second point always maps again into the first point.

0.3 Task Description

Consider the Henon map with $-1 < b < 1$. Show that the map has a 2-Cycle for $a > a1 = \frac{4}{3}(1-b)^2$. For which values of a is the 2-Cycle stable?

0.4 Derive 2-Cycles of the Henon map

A 2-Cycle is defined by (x_1,y_1), (x_2,y_2) that satisfy:

$$
\begin{aligned}
&f(x_1, y_1) \rightarrow (x_2, y_2) \\
&f(x_2, y_2) \rightarrow (x_1, y_1) \\
&f(f(x_1, y_1)) \rightarrow (x_1, y_1)
\end{aligned}
\tag{11}
$$

Evaluating the necessary conditions for a 2-Cycle, which are shown in equation 11, for the Henon map defined by eaquation 1, yields the characteristic equations 12 and 13 for (x_1, y_1) and (x_2, y_2) to build a 2-Cycle:

$$
\begin{aligned}
x_1 &= y_2 + 1 - a x_2^2 \\
y_1 &= b x_2 \\
x_2 &= y_1 + 1 - a x_1^2 \\
y_2 &= b x_1
\end{aligned}
\tag{12}
$$

Using the equations for y_1 and y_2 to derive equations for x_1 and x_2 yields:

$$
\begin{aligned}
x_2 &= b x_2 + 1 - a x_1^2 \\
x_1 &= b x_1 + 1 - a x_2^2 \\
\rightarrow x_2 &= \frac{1 - a x_1^2}{1 - b} \\
\rightarrow x_1 &= \frac{1 - a x_2^2}{1 - b}
\end{aligned}
\tag{13}
$$

Evaluating this fourth order polynomial with WolframAlpha, four solutions are obtained for x_1 and x_2. Comparing the first two solutions for x_1 and x_2 from equation 14 with equation 3 from part (d), shows that these solutions are the fixed points of the Henon map. Fixed points always also satisfy the necessary 2-Cycle conditions from equation 11, as a fixed points always maps again into itself and therefore also maps into itself after applying the mapping $f(x, y)$ two consecutive times.

$$
x_1 = x_2 = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a}
\tag{14}
$$

Since we have already evaluated the fixed points and are only interested in the "real" 2-Cycles of the Henon map at this point, the two solutions from equation 14 are discarded and no longer considered for further analysis.
The two remaining solutions x_{12} from equation 15 define the "real" 2-Cycles of

the Henon map where $x_1 \neq x_2$. When reading equation 15 it should be observed that the $(+)$ and the $(-)$ solution are both solutions for x_1 and x_2. The color coding is only used at this point to indicate that if x_1 is identified with the $(-)$ solution, x_2 needs to be identified with the $(+)$ solution.

$$
\begin{aligned}
x_{12} &= \frac{(1-b) \pm \sqrt{4a - 3(b-1)^2}}{2a} \\
y_{12} &= b x_{21}
\end{aligned}
\tag{15}
$$

From equation 15 it can be obtained that for real 2-Cycles the condition from equation 16 neets to be satisfied:

$$
\begin{aligned}
4a - 3(b-1)^2 &\geq 0 \\
\rightarrow a &\geq \frac{3}{4}(1-b)^2
\end{aligned}
\tag{16}
$$

As $a = \frac{3}{4}(1-b)^2$ results in a fixed point again, because $x_1 = x_2$, a is restricted to $a > \frac{3}{4}(1-b)^2$ to get a real 2-Cycle from equation 15.

0.5 Stability analysis of 2-Cycles

In part (d) of this task, the Jacobian matrix of the Henon map was used to analyze the stability of fixed points. To evaluate the stability of the 2-Cycles, an analogous procedure can be conducted that requires the Jacobians $J(x_1)$ and $J(x_2)$ of the 2-Cycle. For this procedure we can define an additional mapping $g(x, y) := f(f(x, y))$, where $f(x, y)$ represents the mapping defined by the Henon map. According to the definition of $g(x, y)$ fixed points of $g(x, y)$ are 2-Cycles of $f(x, y)$ and therefore 2-Cycles of the Henon map.
As the procedure to analyze fixed points is known from part (d), the definition of $g(x, y)$ yields the relation in equation 17 for 2-Cycles (x_1, y_1), (x_2, y_2) when applying the chain rule for derivatives.

$$
\begin{aligned}
g'(x_1, y_1) &= f'(f(x_1, y_1)) \cdot f'(x_1, y_1) \\
&= f'(x_2, y_2) \cdot f'(x_1, y_1)
\end{aligned}
\tag{17}
$$

For this multidimensional mapping the evaluation of $f'(x, y)$ leads to the Jacobian of the corresponding point $J(x, y)$. Equation 17 yields the relation from equation 19 analogous to the procedure in part (d). Therefore, the 2-Cycles of the Henon map are only stable, if the Eigenvalues of M for the corresponding 2-Cycle satisfy the stability criterion $|\lambda_\pm| < 1$.

$$J(x_n, y_n) = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix}. \tag{18}$$

$$\begin{aligned} M(x_1, y_1, x_2, y_2) &:= J(x_1, y_1) \cdot J(x_2, y_2) \\ M &= \begin{pmatrix} 4a^2 x_1 x_2 + b & -2ax_1 \\ -2abx_2 & b \end{pmatrix}. \end{aligned} \tag{19}$$

According to equation 4 the eigenvalues of M are determined by equation 20:

$$\begin{aligned} \lambda_\pm &= \frac{\mathrm{tr}(M)}{2} \pm \frac{1}{2}\sqrt{\mathrm{tr}(M)^2 - 4 \cdot \det(M)} \\ \mathrm{tr}(M) &= 4a^2 x_1 x_2 + 2b \\ \det(M) &= b^2 \end{aligned} \tag{20}$$

The results of the stability analysis for the 2-Cycles are shown in Figure 3, where analagous to part (d) the red area represents unstable 2-Cycles and the green area stable 2-Cycles. Addtionally, the lower boundary for $a > a_1(b) = \frac{3}{4}(1-b)^2$ is drawn in the graphic that further restricts the possible parameter pairs (a, b), which is necessary to obtain real 2-Cycles. When comparing Figure 3 with Figure 1 at this point, it can be verified again that the real two cycles start where the fixed point x_+ looses stability for $a > a_1(b)$.
Furthermore, it makes no difference for the stability of the 2-Cycles wether $(x_1, y_1) \to (x_2, y_2)$ or $(x_2, y_2) \to (x_1, y_1)$, because the matrix M from equation 19 would only be transposed in the latter case and transposed matrices have the same eigenvalues $\lambda_\pm$. So the stability of the 2-Cycle does not depend on the choice of the first point that starts the 2-Cycle.

Concluding the analysis of fixed points and 2-Cycles, Figure 4 shows the bifurcation diagrams of the x-coordinate of the Henon map for some selected b-values, where the transition from stable fixed points to stable 2-Cycles can be found at $a = a_1(b)$. With a further increase in parameter a the 2-Cycles lead into period doubling at some point and result in chaotic behaviour eventually.

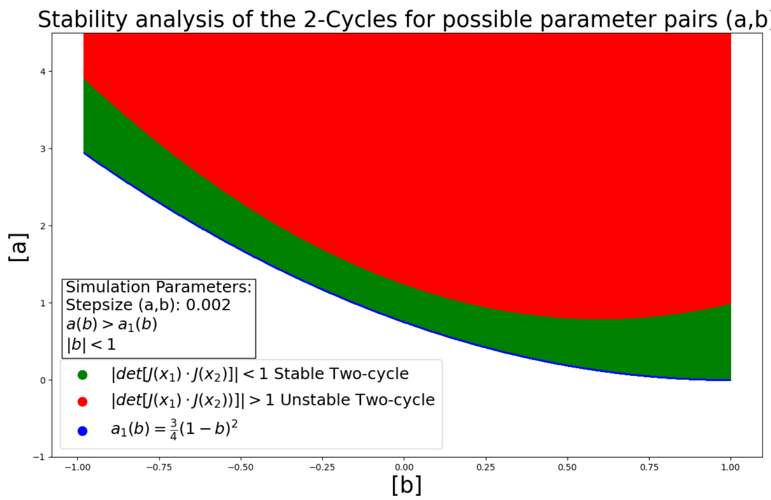

Figure 3: Stability analysis of the 2-Cycles.

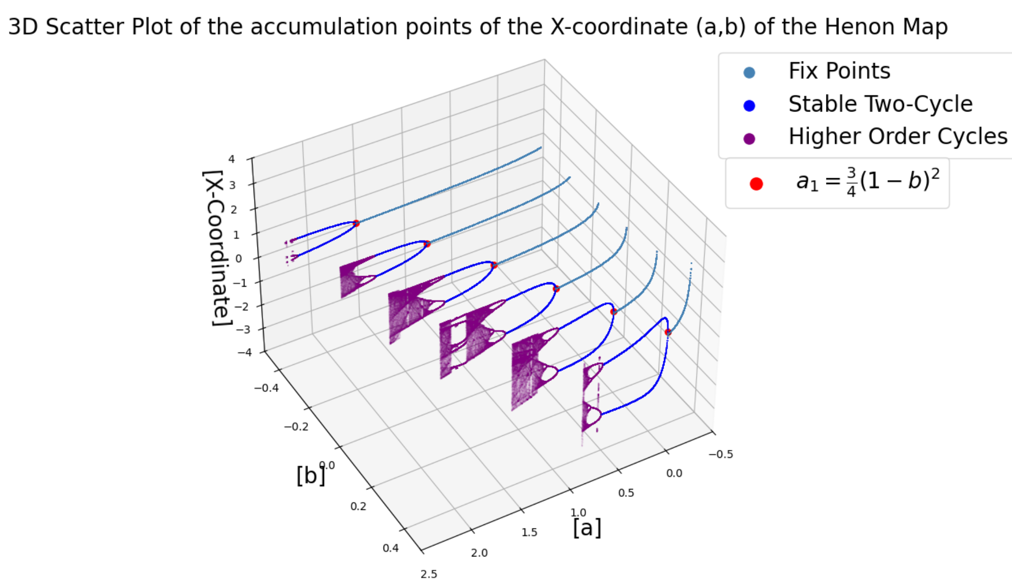

Figure 4: Bifurcation diagrams of the x-coordinate of the Henon map for selected b-values: [-0.45, -0.25, -0.05, 0.15, 0.35, 0.55]