Iso-Geometric Physics-Informed Neural Networks

September 10, 2023

B-Splines and NURBs

B-Splines or 'Basis-Splines' are a special type of spline which can be used for function approximation or as a basis functions in iso-geometric analysis (IGA). A B-Spline of order n is a piecewise polynomial of degree $n-1$ in a variable x. To define the B-Spline knot vectors are necessary, which are simply given in the form

$$\Xi = (\xi_0, \xi_1, ..., \xi_N), \tag{1}$$

where $\xi_i \in \mathbb{R}$ and $\xi_i \leq \xi_j$ for $i \leq j$ i.e. the entries of the knot vectors are non-decreasing. The first B-Spline is defined using (1) as

$$B_i^1 := \begin{cases} 1 & \text{for } \xi_i \leq x < \xi_{i+1} \\ 0 & \text{else} \end{cases} . \tag{2}$$

Higher-order B-Splines can be derived using the Cox-de-Boor recursion formula given as

$$B_i^{n+1} = \omega_i^k(x) B_i^k(x) + \left(1 - \omega_{i+1}^k(x)\right) B_{i+1}^k(x), \tag{3}$$

where

$$\omega_i^k(x) := \begin{cases} \frac{x-\xi_i}{\xi_{i+k}-\xi_i} & \xi_{i+k} \neq \xi_i \\ 0 & \text{else} \end{cases} . \tag{4}$$

To parametrize a curve with B-Splines, one has to define control points $\{\mathbf{c}_i\}_{i=1}^{N_c} \subset \mathbb{R}^2$ which warp the path of the curve. The curve $C : \mathbb{R} \to \mathbb{R}^2$ is then given as an embedding

$$C(x) = \sum_{i=1}^{N_c} \mathbf{c}_i \, B_i^n(x). \tag{5}$$

Similarly, parametrizations of surfaces can be constructed using tensor products of B-Spline curves. In this case however, the B-Spline surface dependent on both spatial variables x and y and consequently $S : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$S(x,y) = \sum_{j=1}^{N_c} \sum_{i=1}^{N_c} \mathbf{c}_{ij} \, B_i^n(x) B_j^n(y), \tag{6}$$

where $\mathbf{c}_{ij} \in \mathbb{R}^2$ are points in a 2D control grid.

Differentiating B-Splines

Here how to calculate the derivative and that derivatives of splines are again splines

Integrating over B-Splines

To integrate over a B-Spline, one has to consider path integrals and transformation laws. In the case we are considering a path integral along a B-Spline curve in $\mathbb{R}^2$, is given by

$$I(f) = \int_{[0,1]} f\left(C(x)\right) \left|J_C\left(x\right)\right| \mathrm{d}x, \tag{7}$$

where J_C is the jacobian of (5). The norm of the jacobian is given as

$$|J_C(x)| = \sqrt{\sum_{i=1}^{N_c} \left(\frac{\mathrm{d}B_i^n}{\mathrm{d}x}\left(x\right)\right)^2 \left(\mathbf{c}_{i,1}^2 + \mathbf{c}_{i,2}^2\right)}, \tag{8}$$

where $\mathbf{c}_i = (\mathbf{c}_{i,1}, \mathbf{c}_{i,2})$. Generally, we evaluate (7) using numerical quadrature. An example of this would be Gauss-Legendre qaudrature, where we use quadrature points $X_q = \{x_1, ..., x_{N_q}\}$ and weights $\Omega_q = \{\omega_1, ..., \omega_{N_q}\}$ to approximate an integral via

$$I(f) = \int_{[-1,1]} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{N_q} \omega_i\, f(x_i) \tag{9}$$

For integrals over arbitrary domains $[a, b]$, one has to rescale the quadrature points and weights using the transformation

$$\overline{\omega}_i = \frac{b-a}{2}\omega_i \quad \text{and} \quad \overline{x}_i = \frac{b-a}{2}x_i + \frac{a+b}{2}. \tag{10}$$

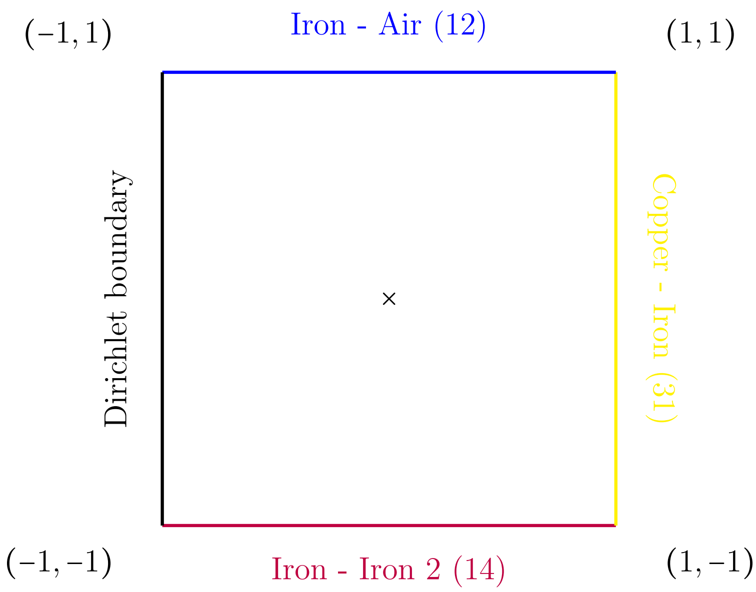
As in the IGA framework, we typically parametrize the B-Splines over the unit cube and typically choose $a = 0$ and $b = 1$. Similarly, when integrating over a B-Spline surface, we evaluate

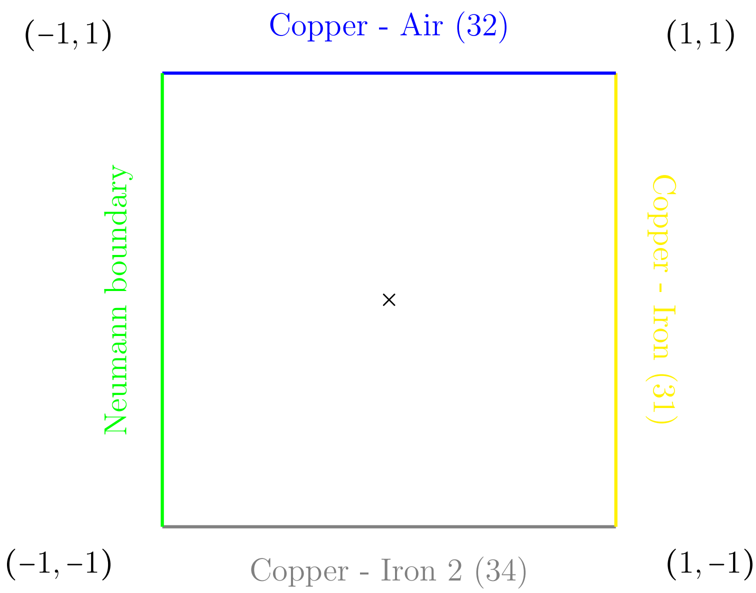
$$\int_{[0,1]^2} f\left(S(x,y)\right) |J_S(x,y)|\,\mathrm{d}x\mathrm{d}y \approx \sum_{i=1}^{N_q}\sum_{j=1}^{N_q} \omega_i\,\omega_j\, f\left(S(x_i,y_i)\right) |J_S(x_i,y_i)|, \tag{11}$$

where $|J_S(x_i, y_i)|$ denotes the jacobian determinant given by the formula

$$|J_S(x,y)| = \left(\sum_{i=1}^{N_q}\sum_{j=1}^{N_q} \frac{\partial B_i^n}{\partial x}(x) B_i^n(y)\mathbf{c}_{i,1}\right)\left(\sum_{i=1}^{N_q}\sum_{j=1}^{N_q} B_i^n(x)\frac{\partial B_i^n}{\partial y}(y)\mathbf{c}_{i,2}\right) \tag{12}$$

$$-\left(\sum_{i=1}^{N_q}\sum_{j=1}^{N_q} \frac{\partial B_i^n}{\partial x}(x) B_i^n(y)\mathbf{c}_{i,2}\right)\left(\sum_{i=1}^{N_q}\sum_{j=1}^{N_q} B_i^n(x)\frac{\partial B_i^n}{\partial y}(y)\mathbf{c}_{i,1}\right) \tag{13}$$

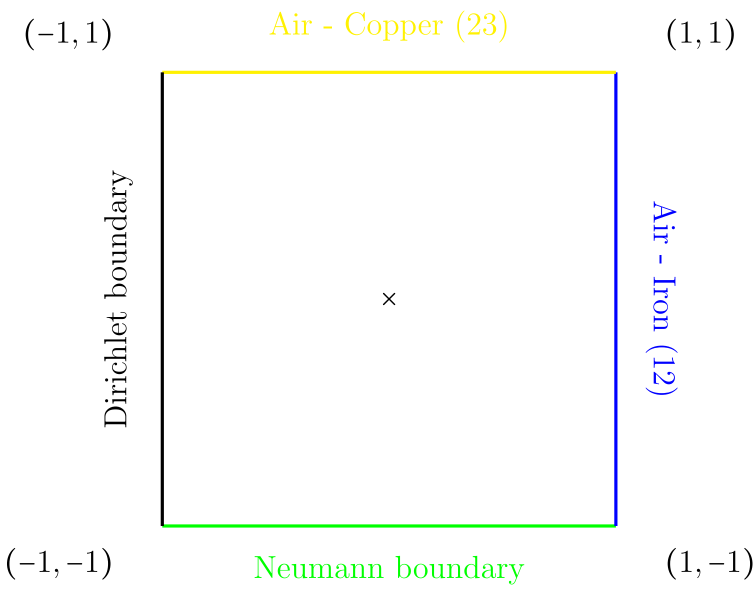




Quadrupole Magnet

Iron domain

- Blue Interface: Iron - Air, Label 12: $(y = 1)$ $N(-1, 1)$ is at the Dirichlet boundary condition. $N(1, 1)$ is at the intersection of the domains $1, 3, 4$.
- Purple Interface: Iron - Iron 2, Label 14: $(y = 1)$ $N(1, -1)$ is at the intersection of the domains $1, 2, 3$. $N(-1, -1)$ is at the Dirichlet boundary
- Black Dirichlet boundary condition at $N(-1, y)$
- Yellow Iron - Copper, Label 13: $(x = -1)$ $N(-1, y)$ is at the interface of iron yoke and the copper coil. $N(1, -1)$ is at the intersection of the domains $1, 2, 3$. $N(1, 1)$ is at the intersection of the domains $1, 3, 4$.



Copper domain

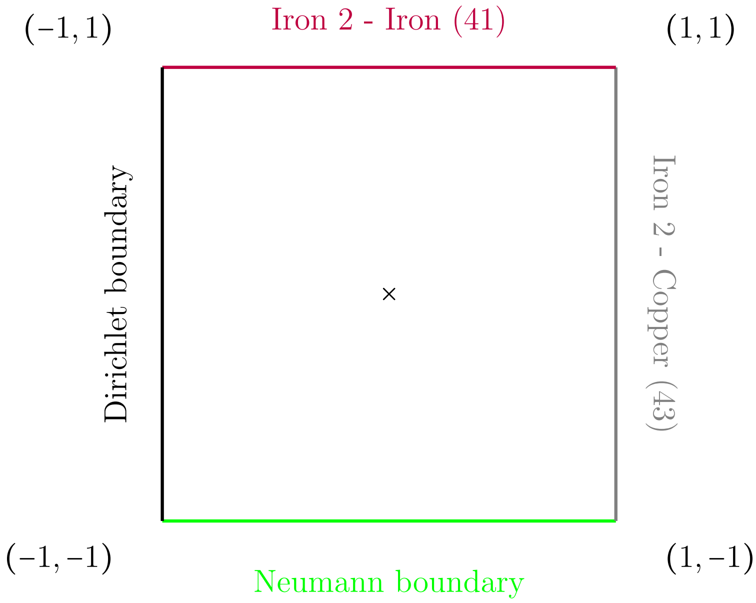
- Blue Interface: Iron - Air, Label 12: ($y = 1$) $N(-1,1)$ is at the Dirichlet boundary condition. $N(1,1)$ is at the intersection of the domains $1, 3, 4$.
- Yellow Interface: Copper - Iron, Label 41: ($x = 1$) $N(1,-1)$ is at the intersection of the domains $1, 2, 3$. $N(1,1)$ is at the intersection of the domains $1, 3, 4$
- Green Neumann boundary condition at $N(-1, y)$
- Gray: Copper - Iron 2, Label 24: ($y = -1$) $N(x,-1)$ is at the interface of copper coil and the lower right iron yoke. $N(1,-1)$ is at the intersection of the domains $1, 3, 4$. $N(1,1)$ is at the lower right Dirichlet boundary condition.

Ansatz function for domain solution: $v(x, y) = (1-x)(1-y)(y+1)$
Interface functions:

- $f_{31}(y) = \frac{1}{2}(y+1)(1-y)(1+y)$ and $f_{13}(x) = \frac{1}{2}(y+1)(1-y)(1+y)$
- $f_{32}(y) = \frac{1}{2}(x+1)(1-x)$ and $f_{34}(y) = -\frac{1}{2}(1-x)(x-1)$
- $f_{34}(x, y) = (1+x)(1+y)$ and

Air domain

- Yellow Interface: Air - Copper, Label 23: ($y = 1$) $N(-1,1)$ is at the Dirichlet boundary condition. $N(1,1)$ is at the intersection of the domains $1, 2, 3$.
- Green Neumann boundary condition ($y = -1$) $N(1,-1)$ is at the Air - iron interface. $N(-1,-1)$ is at the origin.
- Black Dirichlet boundary condition at $N(-1, y)$
- Blue Air - Iron, Label 13: ($x = 1$) $N(1, y)$ is at the interface of iron yoke and the air coil. $N(1,-1)$ is at the interface between the Iron pole and the air domain. $N(1,1)$ is at the intersection of the domains $1, 2, 3$.



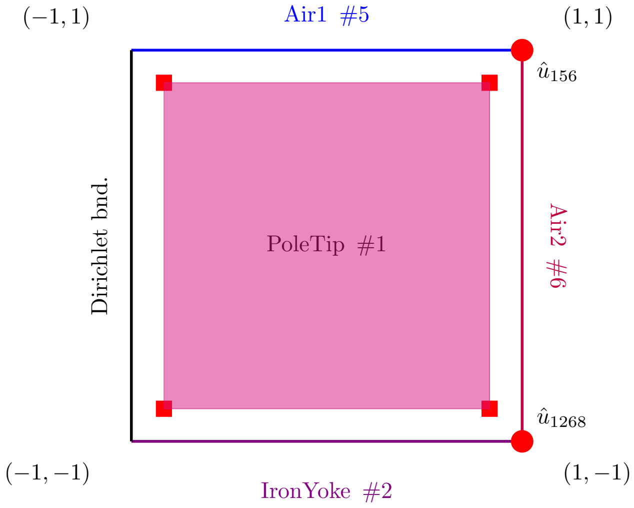
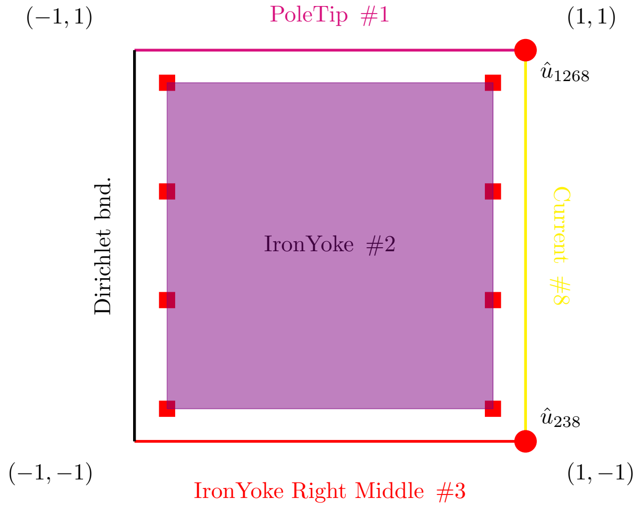
Iron 2 domain

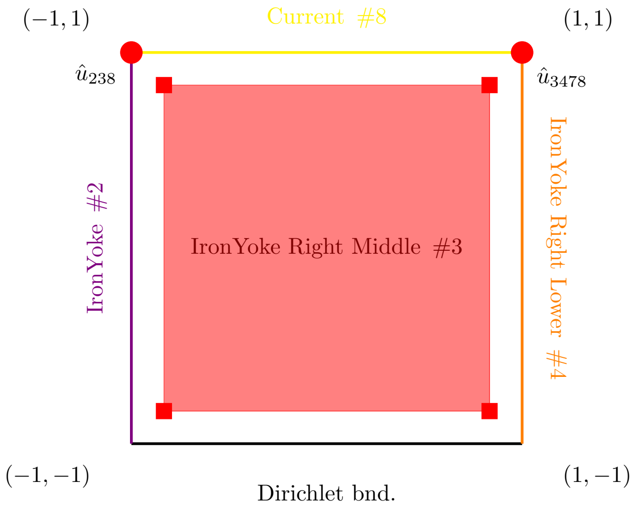
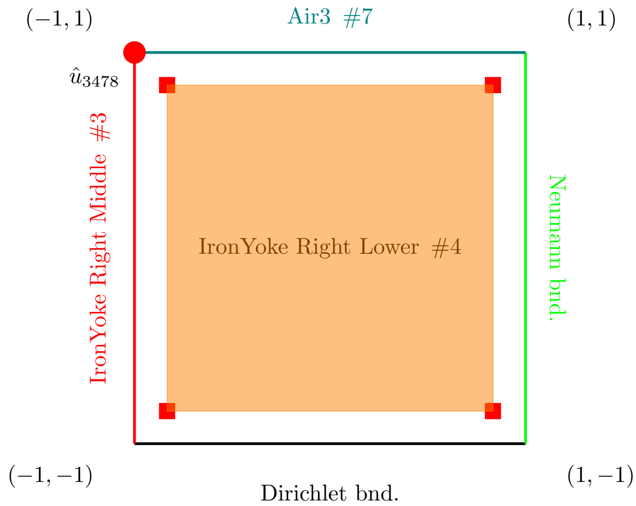
- Yellow Interface: Iron 2 - Copper, Label 43: $(x = 1)$ $N(1, -1)$ is at the Dirichlet boundary condition. $N(1, 1)$ is at the intersection of the domains $1, 3, 4$.
- Green Dirichlet boundary condition $(y = -1)$ $N(1, -1)$ is at the Air - iron interface. $N(-1, -1)$ is at the lower right corner.
- Black: Dirichlet boundary condition at $N(-1, y)$
- Blue Air - Iron, Label 13: $(x = 1)$ $N(1, y)$ is at the interface of iron yoke and the air coil. $N(1, -1)$ is at the interface between the Iron pole and the air domain. $N(1, 1)$ is at the intersection of the domains $1, 2, 3$.

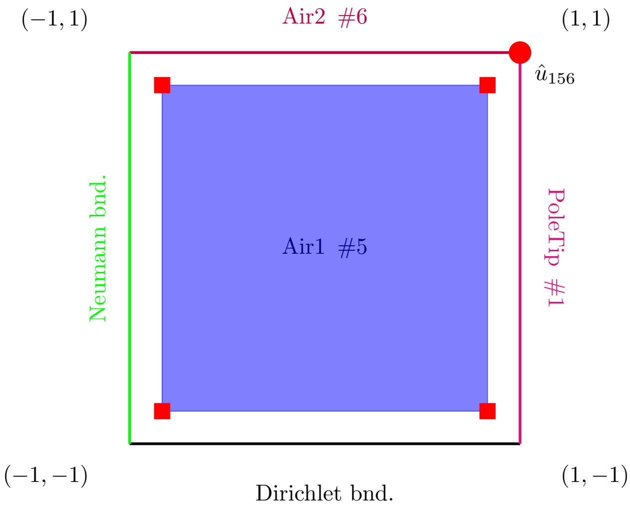
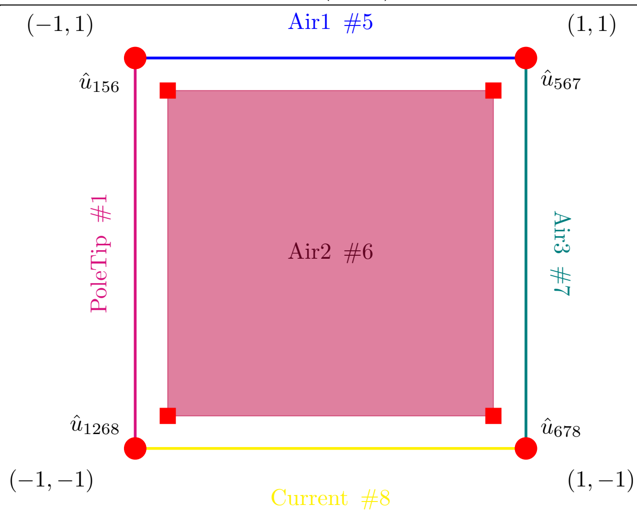
Ansatz function for domain solution: $v(x, y) = (x + 1)(1 - x)(1 - y)$ Interface functions:

- $f_{41}(x) = \frac{1}{2}(x + 1)(1 - x)(1 + x)$ and $f_{14}(x) = -\frac{1}{2}(x - 1)(1 - x)(1 + x)$
- $f_{43}(y) = \frac{1}{2}(x + 1)(1 - y)$ and $f_{34}(y) = -\frac{1}{2}(1 - x)(x - 1)$
- $f_{134}(x, y) = (1 + x)(1 + y)$ and

Better Quadrupole

Pole Tip (# 1)	Iron Yoke(# 2)
	
$B_{i,2}(x)$ with $\Xi = [-1,1]$ $B_{i,1}(y)$ with $\Xi = [-1,1]$ $v(x,y) = (x+1)(1-x)(y+1)(1-y)$	$B_{i,2}(x)$ with $\Xi = [-1,1]$ $B_{i,1}(y)$ with $\Xi = \left[-1,-\frac{1}{3},\frac{1}{3},1\right]$ $v(x,y) = (x+1)(1-x)(y+1)(1-y)$
$f_{12}(x) = (x+1)(1-x) \cdot -\frac{1}{2}(y-1)\, u_{12}^{NN}(x)$ $f_{15}(x) = (x+1)(1-x) \cdot \frac{1}{2}(y+1)\, u_{15}^{NN}(x)$ $f_{16}(x) = (y+1)(1-y) \quad \cdot \quad \frac{1}{2}(x+1)\, u_{16}^{NN}(y)$	$f_{21}(x) = (x+1)(1-x) \cdot \frac{1}{2}(y+1)\, u_{12}^{NN}(x)$ $f_{23}(x) = (x+1)(1-x) \cdot -\frac{1}{2}(y-1)\, u_{23}^{NN}(x)$ $f_{28}(x) = (y+1)(1-y) \cdot \left(\frac{1}{2}(x+1)\, u_{28}^{NN}(y) + \ldots\right)$
$f_{156}(x,y) = \hat{u}_{156}\left((x+1)(y+1)\right)^2$ $f_{1268}(x,y) = \hat{u}_{1268}\left((x+1)(1-y)\right)^2$	$f_{1268}(x,y) = \hat{u}_{1268}\left((x+1)(y+1)\right)^2$ $f_{238}(x,y) = \hat{u}_{238}\left((x+1)(1-y)\right)^2$

Iron Yoke Right Middle (# 3)	Iron Yoke Right Lower (# 4)
	
$B_{i,1}(x)$ with $\Xi = [-1,1]$ $B_{i,2}(y)$ with $\Xi = [-1,1]$ $v(x,y) = (x+1)(1-x)(y+1)(1-y)$	$B_{i,1}(x)$ with $\Xi = [-1,1]$ $B_{i,2}(y)$ with $\Xi = [-1,1]$ $v(x,y) = (x+1)(y+1)(1-y)$
$f_{34}(x,y) = (y+1)(1-y) \cdot \frac{1}{2}(x+1)\, u_{34}^{NN}(y)$ $f_{38}(x,y) = (x+1)(1-x) \cdot \frac{1}{2}(y+1)\, u_{38}^{NN}(x)$ $f_{32}(x,y) = (y+1)(1-y) \cdot -\frac{1}{2}(x-1)\, u_{23}^{NN}(y)$	– $f_{47}(x,y) = (x+1) \cdot \frac{1}{2}(y+1)\, u_{47}^{NN}(x)$ $f_{43}(x,y) = (y+1)(1-y) \cdot -\frac{1}{2}(x-1)\, u_{34}^{NN}(y)$
$f_{238}(x,y) = \hat{u}_{238}\,((1-x)(y+1))^2$ $f_{3478}(x,y) = \hat{u}_{3478}\,((x+1)(y+1))^2$	– $f_{3478}(x,y) = \hat{u}_{3478}\,((1-x)(y+1))^2$

<table>
<tr><th>Air1 (# 5)</th><th>Air2 (# 6)</th></tr>
<tr><td>
</td><td>
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$B_{i,2}(y)$ with $\Xi = [-1,1]$
$v(x,y) = (1-x)(y+1)(1-y)$</td><td>$B_{i,1}(x)$ with $\Xi = [-1,1]$
$B_{i,1}(y)$ with $\Xi = [-1,1]$
$v(x,y) = (x+1)(1-x)(y+1)(1-y)$</td></tr>
<tr><td>$f_{56}(x,y) = (1-x)\cdot\frac{1}{2}(y+1)\,u_{56}^{NN}(x)$
$f_{51}(x,y) = (y+1)(1-y)\cdot\frac{1}{2}(x+1)\,u_{15}^{NN}(y)$
–
–</td><td>$f_{65}(x,y) = (x+1)(1-x)\cdot\frac{1}{2}(y+1)\,u_{56}^{NN}(x)$
$f_{67}(x,y) = (y+1)(1-y)\cdot\frac{1}{2}(x+1)\,u_{67}^{NN}(y)$
$f_{68}(x,y) = (x+1)(1-x)\cdot-\frac{1}{2}(y-1)\,u_{68}^{NN}(x)$
$f_{61}(x,y) = (y+1)(1-y)\cdot-\frac{1}{2}(x-1)\,u_{16}^{NN}(y)$</td></tr>
<tr><td>$f_{156}(x,y) = \hat{u}_{156}\,((x+1)(y+1))^2$
$f_{567}(x,y) = \hat{u}_{567}\,((1-x)(y+1))^2$
–
–</td><td>$f_{567}(x,y) = \hat{u}_{567}\,((x+1)(y+1))^2$
$f_{678} = \hat{u}_{678}\,((x+1)(1-y))^2$
$f_{1268} = \hat{u}_{1268}\,((1-x)(1-y))^2$
$f_{156} = \hat{u}_{156}\,((1-x)(y+1))^2$</td></tr>
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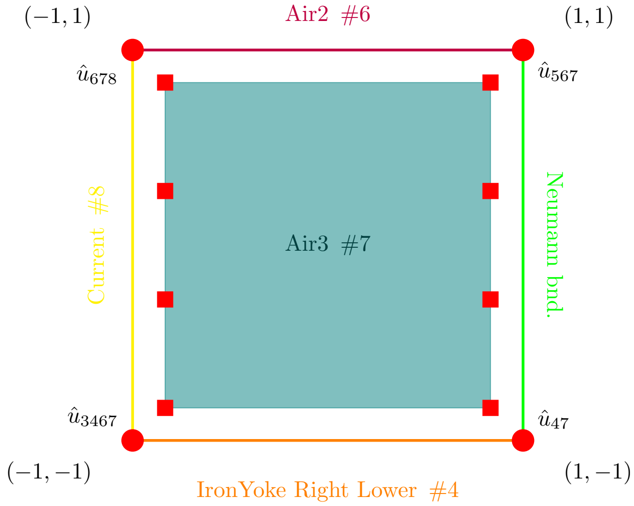
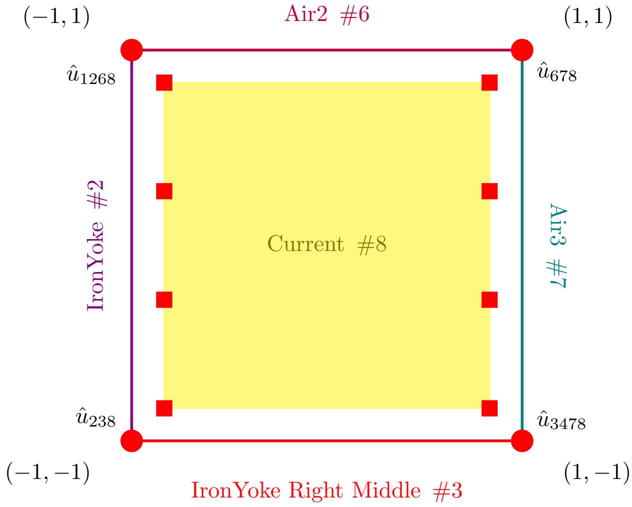
<table>
<tr><th>Air3 (# 7)</th><th>Current (# 8)</th></tr>
<tr><td>
</td><td>
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<tr><td>$B_{i,1}(x)$ with $\Xi = [-1, 1]$
$B_{i,1}(y)$ with $\Xi = \left[-1, -\frac{1}{3}, \frac{1}{3}, 1\right]$
$v(x,y) = (x+1)(y+1)(1-y)$</td><td>$B_{i,1}(x)$ with $\Xi = [-1, 1]$
$B_{i,1}(y)$ with $\Xi = \left[-1, -\frac{1}{3}, \frac{1}{3}, 1\right]$
$v(x,y) = (x+1)(1-x)(y+1)(1-y)$</td></tr>
<tr><td>$f_{76}(x,y) = (x+1) \cdot \frac{1}{2}(y+1)\, u_{67}^{NN}(x)$
–
$f_{74}(x,y) = (x+1) \cdot -\frac{1}{2}(y-1)\, u_{47}^{NN}(x)$
$f_{78} = (y+1)(1-y) \cdot \left(-\frac{1}{2}(x-1)\, u_{78}^{NN}(y) + \ldots\right)$</td><td>$f_{86}(x,y) = (x+1)(1-x) \cdot \frac{1}{2}(y+1)\, u_{68}^{NN}(x)$
$f_{87}(x,y) = (y+1)(1-y) \cdot \left(\frac{1}{2}(x+1)\, u_{78}^{NN}(y) + \ldots\right)$
$f_{83}(x,y) = (x+1)(1-x) \cdot -\frac{1}{2}(y-1)\, u_{38}^{NN}(x)$
$f_{82} = (y+1)(1-y) \cdot \left(-\frac{1}{2}(x-1)\, u_{28}^{NN}(y) + \ldots\right)$</td></tr>
<tr><td>$f_{567}(x,y) = \hat{u}_{567}\left((x+1)(y+1)\right)^2$
–
$f_{3467} = \hat{u}_{3467}\left((1-x)(1-y)\right)^2$
$f_{678} = \hat{u}_{678}\left((1-x)(y+1)\right)^2$</td><td>$f_{678}(x,y) = \hat{u}_{678}\left((x+1)(y+1)\right)^2$
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