Discrete Morse Theory for Computing Mayer-Vietoris Long Exact Sequences for Cellular Cosheaf Homology

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Abstract

In this paper, we discuss how discrete Morse theory can be used to simplify the computation of cosheaf homology groups and their intermediate maps in Mayer-Vietoris long exact sequences. We begin by introducing the notion of cosheaves and their associated chain complexes and homology groups. We then consider how we can restrict those objects to subcomplexes in order to obtain valid short exact sequences of the Meyer-Vietoris type. We then transfer discrete Morse theory to this context. This is where we derive a core result which ensures that the long exact sequences on cosheaf homology of the original chain complex are isomorphic to the long exact sequences on cosheaf homology of the corresponding smaller Morse chain complexes. Based on this, we justify how this can simplify the necessary computations. Finally, we discuss briefly generalisations beyond the standard Mayer-Vietoris sequence.

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1 Cosheaves, Chain Complexes and Homology

Throughout this paper, for a simplicial complex K, we will restrict our focus to cosheafs over K which take values in finite dimensional vector spaces over $\mathbb{R}$.

Definition 1.0.1 (adapted from [3]). *Let K be a finite simplicial complex, and let $(K, \geq)$ be the partially ordered set of simplices in K, ordered by the face relation. A* ***cosheaf*** *over K is a functor $\mathcal{C} : (K, \geq) \to \mathbf{vect}_{\mathbb{R}}$. That is, $\mathcal{C}$ assigns an real vector space $\mathcal{C}(\tau)$, the* ***costalk****, to each simplex τ of K, and a linear map $\mathcal{C}(\tau \geq \tau') : \mathcal{C}(\tau) \to \mathcal{C}(\tau')$, the* ***extension map****, to each face relation $\tau \geq \tau'$ in K. Furthermore,*

1. *$\mathcal{C}(\tau \geq \tau)$ is the identity on $\mathcal{C}(\tau)$*
2. *$\mathcal{C}(\tau' \geq \tau'') \circ \mathcal{C}(\tau \geq \tau') = \mathcal{C}(\tau \geq \tau'')$ holds for every triple of simplices $\tau \geq \tau' \geq \tau''$*

Introducing some further notation, for a given simplex τ and a set of indeces S, we shall denote by τ_S the simplex obtained from τ by removing the vertices at indeces S (if $S = \{i\}$ is a singleton, we sometimes write τ_{-i}). We shall further write $\tau' \lhd \tau$ if τ' is a face of τ with codimension one, and $[\tau' : \tau] \in \{0, +1, -1\}$ for the coefficient of τ' in the boundary of τ.

Definition 1.0.2. *Let τ, τ' be two simplices in K. The* ***scaled extension map*** *$\mathcal{C}_{\tau,\tau'} : \mathcal{C}(\tau) \to \mathcal{C}(\tau')$ between their costalks is defined as*

$$\mathcal{C}_{\tau,\tau'} = [\tau' : \tau] \cdot \mathcal{C}(\tau \geq \tau') = \begin{cases} +\mathcal{C}(\tau \geq \tau') & \text{if } \tau' = \tau_{-i} \text{ for even } i \\ -\mathcal{C}(\tau \geq \tau') & \text{if } \tau' = \tau_{-i} \text{ for odd } i \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.0.3. *We define the* ***k-chains of*** *K* ***with $\mathcal{C}$ coefficients*** *as the product*

$$\boldsymbol{C}_k(K, \mathcal{C}) = \prod_{\dim \tau = k} \mathcal{C}(\tau)$$

Definition 1.0.4. *We further define the* ***k-th boundary map of K with $\mathcal{C}$ coefficients*** *as the linear map*

$$\partial_k^{\mathcal{C}} : \boldsymbol{C}_k(K, \mathcal{C}) \to \boldsymbol{C}_{k-1}(K, \mathcal{C})$$

such that for each pair $\tau \geq \tau'$ with $\dim(\tau) = k$, $\dim(\tau') = k - 1$ the $\mathcal{C}(\tau) \to \mathcal{C}(\tau')$ block component of $\partial_k^{\mathcal{C}}$ is given by

$$\partial_k^{\mathcal{C}}|_{\tau,\tau'} = \mathcal{C}_{\tau,\tau'}$$

We proceed to showing that the boundary map induces a chain complex on $\{\boldsymbol{C}_k(K, \mathcal{C})\}_{k \geq 0}$.

Theorem 1.0.1. *$(\boldsymbol{C}_\bullet(K, \mathcal{C}), \partial_k^{\mathcal{C}})$ defines a chain complex.*

Proof. We need to show that $\partial_k^{\mathcal{C}} \circ \partial_{k+1}^{\mathcal{C}} = 0$ for all $k \geq 0$. We argue block-wise. To this end, choose any pair $\tau \geq \tau''$ with $\dim(\tau) = k + 1$, $\dim(\tau') = k - 1$, and consider the block $\mathcal{C}(\tau) \to \mathcal{C}(\tau'')$. We have

$$(\partial_k^{\mathcal{C}} \circ \partial_{k+1}^{\mathcal{C}})|_{\tau,\tau''} = \sum_{\tau'' \lhd \tau' \lhd \tau} [\tau' : \tau][\tau'' : \tau']\mathcal{C}(\tau' \geq \tau'') \circ \mathcal{C}(\tau \geq \tau')$$

$$
\begin{aligned}
&= \sum_{\tau'' \lhd \tau' \lhd \tau} [\tau' : \tau][\tau'' : \tau']\mathcal{C}(\tau \geq \tau'') && \text{by 1.0.1} \\
&= \big(\sum_{\tau'' \lhd \tau' \lhd \tau} [\tau' : \tau][\tau'' : \tau']\big)\mathcal{C}(\tau \geq \tau'')
\end{aligned}
$$

But τ'' is obtained from τ by removing two vertices from τ, say at indeces k and l. Then $\tau'' = \tau_{\{k,l\}}$ and $\tau' \in \{\tau_{\{k\}}, \tau_{\{l\}}\}$. Without loss of generality, assume $k > l$. Then $[\tau_{\{l\}} : \tau] = [\tau_{\{k,l\}} : \tau_{\{k\}}]$, but $[\tau_{\{k\}} : \tau] = -[\tau_{\{k,l\}} : \tau_{\{l\}}]$. It follows that

$$\sum_{\tau'' \lhd \tau' \lhd \tau} [\tau' : \tau][\tau'' : \tau'] = [\tau_{\{k,l\}} : \tau_{\{k\}}][\tau_{\{k\}} : \tau] + [\tau_{\{k,l\}} : \tau_{\{l\}}][\tau_{\{l\}} : \tau] = 0$$

We have shown that any block matrix is the zero map, and hence the full linear map $\partial_k^{\mathcal{C}} \circ \partial_{k+1}^{\mathcal{C}} : \mathcal{C}_{k+1}(K, \mathcal{C}) \to \mathcal{C}_{k-1}(K, \mathcal{C})$ is the zero map, as required. □

Definition 1.0.5. *For $k \geq 0$, we define the* ***k-th homology group of $C_\bullet(K, \mathcal{C})$*** *as the quotient vector space*

$$\boldsymbol{H}_k(K, \mathcal{C}) = \ker \partial_k^{\mathcal{C}} / \mathrm{img} \partial_{k+1}^{\mathcal{C}}$$

Before proceeding to construct Mayer-Vietoris type short exact sequences of cosheafs, let us illustrate a few simple examples of cosheaves, following [3].

Example 1.0.2 (Zero Cosheaf $\overline{0}_K$)**.** *The zero cosheaf of a simplicial complex K is obtained by assigning to each simplex the trivial (0-dimensional) vector space. The extension maps are then all trivial.*

Example 1.0.3 (Constant Cosheaf $\overline{\mathbb{R}}_K$)**.** *The constant cosheaf assigns to each simplex the one-dimensional costalk $\mathbb{R}$. The extension maps are then the identity $\mathbb{R} \to \mathbb{R}$.*

Example 1.0.4 (Skyscraper Cosheaf $\overline{Sk}_K$)**.** *The skyscraper cosheaf assigns the trivial (0-dimensional) costalk to all simplices except one, say τ, to which it assigns the one-dimensional costalk $\mathbb{R}$. The only non-zero extension map is the one associated to $\tau \geq \tau$, which is the identity $\mathbb{R} \to \mathbb{R}$.*

2 Meyer-Vietoris Type Short Exact Sequences for Cosheaves

Assume now that our simlicial complex K is the union of two sub-complexes, i.e. $K = L \cup M$. We denote their intersection as $I = L \cap M$. In this chapter, we construct a short exact sequence

$$0 \to \mathbf{C}_\bullet(I, \mathcal{C}|_I) \xrightarrow{p_\bullet} \mathbf{C}_\bullet(L, \mathcal{C}|_L) \oplus \mathbf{C}_\bullet(M, \mathcal{C}|_M) \xrightarrow{q_\bullet} \mathbf{C}_\bullet(K, \mathcal{C}) \to 0 \tag{2.1}$$

by defining chain maps $p_\bullet$ and $q_\bullet$. Let us first consider how to restrict a cosheaf to a subcomplex.

Definition 2.0.1. *Let $L \subset K$ be a subcomplex, and let $\mathcal{C}$ be our familiar cosheaf on K. Then $\mathcal{C}|_L$ is given by*

$$\mathcal{C}|_L(\tau) = \mathcal{C}(\tau) \quad \forall\ \tau \in L \quad \textit{and} \quad \mathcal{C}|_L(\tau \geq \tau') = \mathcal{C}(\tau \geq \tau') \quad \forall\ \tau, \tau' \in L \textit{ with } \tau \geq \tau'$$

Note that L is a subcomplex, so if $\tau \geq \tau'$ in K and $\tau \in L$, then $\tau \geq \tau'$ in L. Conversely, since $L \subset K$, we know that if $\tau \geq \tau'$ in L then $\tau \geq \tau'$ in K. So the above restrictions are well-defined. Recall further the definitions of exact sequences of vector spaces.

Definition 2.0.2 (adapted from [3])**.** *A sequence of vector spaces and linear maps*

$$\cdots \xrightarrow{a_{k+2}} V_{k+1} \xrightarrow{a_{k+1}} V_k \xrightarrow{a_k} V_{k-1} \xrightarrow{a_{k-1}} \cdots$$

*is called **exact at k** if* $\ker a_k = \text{img } a_{k+1} \subset V_k$. *The sequence is **exact** if this holds for all* $k \geq 0$. *The sequence is called **short exact** if the* V_k *are trivial for all but three* k.

Definition 2.0.3 (adapted from [3])**.** *A **short exact sequence of chain complexes** consists of three chain complexes and two chain maps between them:*

$$(\boldsymbol{C_\bullet}, d_\bullet) \xrightarrow{\phi_\bullet} (\boldsymbol{C'_\bullet}, d'_\bullet) \xrightarrow{\psi_\bullet} (\boldsymbol{C''_\bullet}, d''_\bullet),$$

In addition, we require that for each $k \geq 0$, *the chain groups*

$$0 \longrightarrow \boldsymbol{C_k} \xrightarrow{\phi_k} \boldsymbol{C'_k} \xrightarrow{\psi_k} \boldsymbol{C''_k} \longrightarrow 0$$

form a short exact sequence of $\mathbb{F}$*-vector spaces.*

In our case, $\mathbb{F} = \mathbb{R}$, and the chain complexes are as in (2.1). We now define $p_k : \mathbf{C}_k(I, \mathcal{C}|_I) \to \mathbf{C}_k(L, \mathcal{C}|_L) \oplus \mathbf{C}_k(M, \mathcal{C}|_M)$ and $q_k : \mathbf{C}_k(L, \mathcal{C}|_L) \oplus \mathbf{C}_k(M, \mathcal{C}|_M) \to \mathbf{C}_k(K, \mathcal{C})$. Before doing so, we recall that for $p_\bullet$ and $q_\bullet$ to be valid chain maps, we require that, for all $k \geq 0$,

$$(\partial_k^{\mathcal{C}}|_L, \partial_k^{\mathcal{C}}|_M) \circ p_k = p_{k-1} \circ \partial_k^{\mathcal{C}}|_I \tag{2.2}$$

$$\partial_k^{\mathcal{C}} \circ q_k = q_{k-1} \circ (\partial_k^{\mathcal{C}}|_L, \partial_k^{\mathcal{C}}|_M). \tag{2.3}$$

where p_{-1} and q_{-1} should be viewed as identities on the trivial $\mathbb{R}$-vector space. Furthermore, we write $(\partial_k^{\mathcal{C}}|_L, \partial_k^{\mathcal{C}}|_M)$ for the boundary map $\partial_k^{\mathcal{C}}|_{L \oplus M}$, since it acts as the respective boundary map on the subspaces of the direct sum corresponding to L and M. We denote by γ_k, δ_k, ϵ_k and ζ_k the natural inclusions that furnish the below Vietoris-Meyer diamond:

$$\begin{tikzcd}
& \mathbf{C}_k(K, \mathcal{C}) & \\
\mathbf{C}_k(L, \mathcal{C}|_L) \arrow[ur, hook, "\gamma_k"] & & \mathbf{C}_k(M, \mathcal{C}|_M) \arrow[ul, hook', "\delta_k"'] \\
& \mathbf{C}_k(I, \mathcal{C}|_I) \arrow[ul, hook', "\epsilon_k"] \arrow[ur, hook, "\zeta_k"'] &
\end{tikzcd}$$

Using these, first specify the $\mathbb{R}$-linear map $q_\bullet$. Let v be an element of $\mathbf{C}_k(I, \mathcal{C}|_I)$. We define

$$p_k(v) = (\epsilon_k v, -\zeta_k v)$$

Similarly, for an element (v, u) in $\mathbf{C}_k(L, \mathcal{C}|_L) \oplus \mathbf{C}_k(M, \mathcal{C}|_M)$ we specify $q_\bullet$ as the map

$$q_k(v, u) = \gamma_k v + \delta_k u$$

Lemma 2.0.1. *Equation* (2.1) *defines a valid short exact sequence with the definitions of* $p_\bullet$ *and* $q_\bullet$ *given above.*

We first establish an intermediary result. In what follows, we shall denote by S_k the set of k-simplices in a simplicial complex S, so that $|S_k|$ denotes the number of k-simplices in S.

Lemma 2.0.2. *Let* $S \subset K$ *be a subcomplex of the simplicial complex* K. *Let* $\mathcal{C}$ *be a cosheaf on* K *an let* $C_k(S, \mathcal{C}|_S)$ *and* $C_k(K, \mathcal{C})$ *be the corresponding cosheaf chain complexes. Further, denote by* ι_k *the inclusion map* $\iota_k : C_k(S, \mathcal{C}|_S) \hookrightarrow C_k(K, \mathcal{C})$. *Then the inclusion operator commutes with the boundary operator, i.e.* $\iota^{k-1} \circ \partial_k^{\mathcal{C}}|_S = \partial_k^{\mathcal{C}} \circ \iota^k$.

Proof. As usual, consider the block component $\mathcal{C}(\tau) \to \mathcal{C}(\tau')$. There are two relevant cases.

Case 1: $\boldsymbol{\tau \in S_k, \tau' \in S_{k-1}}$. The equality is trivial, since the inclusion operates as the identity on both sides of the equality.

Case 2: $\boldsymbol{\tau \in S_k, \tau' \notin S_{k-1}}$. Note that S is a subcomplex. So it cannot be the case that $\tau' \triangleleft \tau$. Hence $[\tau' : \tau] = 0$, so that $\partial_k^{\mathcal{C}}|_{\tau,\tau'}$ acts as the the zero map on the RHS. On the LHS, since $\tau' \neq S_{k-1}$, the inclusion acts as the zero map. □

Proof. (*Lemma* 2.0.1) We first note that

$$0 \to \mathbf{C}_k(I, \mathcal{C}|_I) \xrightarrow{p_k} \mathbf{C}_k(L, \mathcal{C}|_L) \oplus \mathbf{C}_k(M, \mathcal{C}|_M) \xrightarrow{q_k} \mathbf{C}_k(K, \mathcal{C}) \to 0$$

is short exact for each $k \geq 0$. Indeed, we see that $\ker p_k = 0$ and img $p_k = \ker q_k$. It is also easy to see that q_k is surjective: Let v be an element in $\mathbf{C}_k(K, \mathcal{C})$. Denote by $|K_k|$, $|L_k|$ and $|M_k|$ the number of k-simpleces in K, L and M. Then $v = (v_1, \ldots, v_{|K_k|-|L_k|}, \ldots, v_{|M_k|}, \ldots, v_{|K_k|})$, where possibly $|K_k| - |L_k| = |M_k|$. But $\hat{v} = (v_1, \ldots, v_{|K_k|-|L_k|}, \ldots, v_{|M_k|}, 0, \ldots, 0)$ is in img γ_k and $\tilde{v} = (0, \ldots, 0, v_{|M_k|+1}, \ldots, v_{|K_k|})$ is in img δ_k. So $v \in$ img q_k as required. Note that here, without loss of generality, we assumed an implicit ordering of the factor spaces in $\mathbf{C}_k(K, \mathcal{C})$, so that the costalks from $K_k \setminus M_k$ appear first, then the costalks from $L_k \cap M_k$ and finally those from $K_k \setminus L_k$. Furthermore, when we expand an element in $v \in \mathbf{C}_k(K, \mathcal{C})$, each component v_i is itself a vector in one of the costalks making up the k-chain.

It remains to show that $q_\bullet$ and $p_\bullet$ are valid chain maps, i.e. that (2.2) and (2.3) are satisfied. We first show (2.2), using *Lemma* 2.0.2 throughout. We compute for some $v \in \mathbf{C}_k(I, \mathcal{C}|_I)$:

$$\begin{aligned}
(\partial_k^{\mathcal{C}}|_L, \partial_k^{\mathcal{C}}|_M) \circ p_k(v) &= (\partial_k^{\mathcal{C}}|_L(\epsilon_k v), \partial_k^{\mathcal{C}}|_M(\zeta_k v)) \\
&= (\epsilon_{k-1}\partial_k^{\mathcal{C}}|_I(v), \zeta_{k-1}\partial_k^{\mathcal{C}}|_I(v)) && \text{by } \textit{Lemma } 2.0.2 \\
&= p_{k-1} \circ \partial_k^{\mathcal{C}}|_I(v)
\end{aligned}$$

We now show (2.3). In this case, let $(v, u) \in \mathbf{C}_\bullet(L, \mathcal{C}|_L) \oplus \mathbf{C}_\bullet(M, \mathcal{C}|_M)$. We compute:

$$\begin{aligned}
\partial_k^{\mathcal{C}} \circ q_k(v, u) &= \partial_k^{\mathcal{C}}(\gamma_k v - \delta_k v) \\
&= \partial_k^{\mathcal{C}}(\gamma_k v) - \partial_k^{\mathcal{C}}(\delta_k u) && \text{by linearity} \\
&= \gamma_{k-1}\partial_k^{\mathcal{C}}(v) - \delta_{k-1}\partial_k^{\mathcal{C}}(u) && \text{by } \textit{Lemma } 2.0.2
\end{aligned}$$

$$= q_{k-1} \circ (\partial_k^{\mathcal{C}}|_L(v), \partial_k^{\mathcal{C}}|_M(u))$$

This completes the proof. □

3 Morse Chain Complexes

We now consider how the computation of homology groups $\boldsymbol{H}_\bullet(K, \mathcal{C})$, defined in 1.0.5, can be simplified by working with the homology groups of corresponding Morse chain complexes. We define partial matchings on a simplicial complex Σ and specify what it means for them to be acyclic and $\mathcal{C}$-compatible. First, we recall the notation $\sigma \lhd \tau$, which signifies that σ is a face of τ and $\dim \sigma = \dim \tau - 1$. Throughout this chapter 0 and 1 denote the (block-wise) zero and identity map respectively.

Definition 3.0.1 (adapted from [3])**.** *A* ***partial matching*** *on a simplicial complex K is a collection $\Sigma = \{(\sigma_\bullet \lhd \tau_\bullet)\}$ of simplex pairs in K such that if $\sigma \lhd \tau \in \Sigma$, then there is no other pair $\sigma' \lhd \tau' \in \Sigma$ s.t. $\sigma' \in \{\sigma, \tau\}$ or $\tau' \in \{\sigma, \tau\}$. Simplices in K are called* ***Σ-critical*** *if they do not appear in any pair in Σ.*

Definition 3.0.2 (adapted from [3])**.** *A* ***Σ-path*** *is a zigzag sequence of distinct simplices in K of the form*

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m),$$

where $(\sigma_i \lhd \tau_i) \in \Sigma \ \forall i \in \{0, \ldots, m\}$. The Σ-path is ***gradient*** *if either $m = 1$ or σ_1 is not a face of τ_m. The matching Σ is* ***acyclic*** *if all possible paths are gradient. We sometimes refer to σ_1 as σ_ρ and to τ_m as τ_ρ.*

Definition 3.0.3. *Given a cosheaf $\mathcal{C}$ defined on the simplicial complex K, an acyclic partial matching Σ on K is* ***$\mathcal{C}$-compatible*** *if the extension map $\mathcal{C}(\sigma \le \tau)$ is an isomorphism for every pair $(\sigma \lhd \tau)$ in Σ.*

Next, we hope to describe how working with Morse chain complexes, arising from a given $\mathcal{C}$-compatible acyclic partial matching, can drastically reduce the dimensionality of the homology computation. Before providing a description of Morse chain complexes in the cosheaf setting, we define $\mathcal{C}$-path weights.

Definition 3.0.4. *Let*

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m),$$

be a Σ-path. We define the ***$\mathcal{C}$-path weight*** *$w_{\mathcal{C}(\rho)}$ as the composite linear map*

$$(-1)^m \cdot \left[\mathcal{C}^{-1}_{\tau_m,\sigma_m} \circ \ldots \circ \mathcal{C}^{-1}_{\tau_2,\sigma_2} \circ \mathcal{C}_{\tau_1,\sigma_2} \circ \mathcal{C}^{-1}_{\tau_1,\sigma_1}\right]$$

The inverses in the above definition are well defined as Σ is assumed to be $\mathcal{C}$-compatible.

Definition 3.0.5. *Let $\mathcal{C}$ be a cosheaf over the simplicial complex K, with $\mathcal{C}$-compatible acyclic partial matching Σ. The **Morse complex of Σ with coefficients in $\mathcal{C}$** is a cochain complex*

$$(\boldsymbol{C}^{\Sigma}_{\bullet}(K,\mathcal{C}), \partial^{\mathcal{C},\Sigma}_{\bullet})$$

defined as follows. For each dimension $k \geq 0$,

1. *the vector space $\boldsymbol{C}^{\Sigma}_{k}(K,\mathcal{C})$ is given by the product of costalks $\prod_{\alpha\in\mathrm{crit}_k(\Sigma)} \mathcal{C}(\alpha)$ where $\mathrm{crit}_k(\Sigma) = \{\sigma \in K | \dim\sigma = k,\ \sigma \text{ is } \Sigma\text{-critical}\}$.*
2. *the linear map $\partial^{\mathcal{C},\Sigma}_{k} : \boldsymbol{C}^{\Sigma}_{k}(K,\mathcal{C}) \to \boldsymbol{C}^{\Sigma}_{k-1}(K,\mathcal{C})$ is represented by a block-matrix whose $\mathcal{C}(\tau) \to \mathcal{C}(\sigma)$ component is given by*

$$\partial^{\mathcal{C},\Sigma}_{k}|_{\tau,\sigma} = \mathcal{C}_{\tau,\sigma} + \sum_{\rho} \mathcal{C}_{\tau_\rho,\sigma} \circ w_{\mathcal{C}(\rho)} \circ \mathcal{C}_{\tau,\sigma_\rho}$$

where ρ ranges over all Σ-paths.

It is useful to verify that indeed this gives a valid chain complex. Recall that we know from Theorem 1.0.1 that for the case $\Sigma = \emptyset$, $(\boldsymbol{C}^{\Sigma}_{\bullet}(K,\mathcal{C}), \partial^{\mathcal{C},\Sigma}_{\bullet})$ indeed is a chain complex. This is our base case. We may now proceed by induction, which requires us to show that $(\boldsymbol{C}^{\Sigma}_{\bullet}(K,\mathcal{C}), \partial^{\mathcal{C},\Sigma}_{\bullet})$ is a valid chain complex for $\Sigma = \{(\sigma \triangleleft \tau)\}$ for a single pair $(\sigma \triangleleft \tau)$. The only Σ-path is then $\rho = (\sigma \triangleleft \tau)$. We need to show that:

$$(\partial^{\mathcal{C},\Sigma}_{k-1} \circ \partial^{\mathcal{C},\Sigma}_{k})|_{\alpha,\beta} = 0 \quad \text{for all } \alpha,\ \beta \text{ with } \dim\alpha = k \text{ and } \dim\beta = k-2.$$

In the expansions of this and the following proofs, we sum over all ω regardless of dimensionality. Yet, only those ω with $\beta \leq \omega \leq \alpha$ can yield non-zero contributions to the sum by the construction of the scaled extension map. We compute as follows:

$$\begin{aligned}
(\partial^{\mathcal{C},\Sigma}_{k-1} \circ \partial^{\mathcal{C},\Sigma}_{k})|_{\alpha,\beta} =& \sum_{\omega\in K\setminus\{\sigma,\tau\}} (\mathcal{C}_{\omega,\beta} - \mathcal{C}_{\tau,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\omega,\sigma}) \circ (\mathcal{C}_{\alpha,\omega} - \mathcal{C}_{\tau,\omega} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\sigma}) \\
=& \sum_{\omega\in K\setminus\{\sigma,\tau\}} (\mathcal{C}_{\omega,\beta} \circ \mathcal{C}_{\alpha,\omega} \\
& - \mathcal{C}_{\omega,\beta} \circ \mathcal{C}_{\tau,\omega} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\sigma} - \mathcal{C}_{\tau,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\omega,\sigma} \circ \mathcal{C}_{\alpha,\omega}) \\
& + \mathcal{C}_{\tau,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\omega,\sigma} \circ \mathcal{C}_{\tau,\omega} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\sigma} \\
=& - (\mathcal{C}_{\sigma,\beta} \circ \mathcal{C}_{\alpha,\sigma} + \mathcal{C}_{\tau,\beta} \circ \mathcal{C}_{\alpha,\tau}) \\
& - [\sum_{\omega\in K\setminus\{\sigma,\tau\}} \mathcal{C}_{\omega,\beta} \circ \mathcal{C}_{\tau,\omega}] \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\sigma} - \mathcal{C}_{\tau,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ [\sum_{\omega\in K\setminus\{\sigma,\tau\}} \mathcal{C}_{\omega,\sigma} \circ \mathcal{C}_{\alpha,\omega})] \\
=& - (\mathcal{C}_{\sigma,\beta} \circ \mathcal{C}_{\alpha,\sigma} + \mathcal{C}_{\tau,\beta} \circ \mathcal{C}_{\alpha,\tau}) \\
& + [\mathcal{C}_{\sigma,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma}] \circ \mathcal{C}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\sigma} + \mathcal{C}_{\tau,\beta} \circ \mathcal{C}^{-1}_{\tau,\sigma} \circ [\mathcal{C}_{\tau,\sigma} \circ \mathcal{C}_{\alpha,\tau}] \\
=& 0
\end{aligned}$$

So indeed 3.0.5 defines a valid chain complex. Here we used the fact that $\mathcal{C}_{\sigma,\tau}$ is the boundary operator from Theorem 1.0.1, so that $\sum_{\omega\in K}(\mathcal{C}_{\omega,\beta} \circ \mathcal{C}_{\alpha,\omega}) = 0$ and $\mathcal{C}_{\sigma,\sigma} = \mathcal{C}_{\tau,\tau} = 0$. In the third equality, we used the fact that $\mathcal{C}_{\omega,\sigma} \circ \mathcal{C}_{\tau,\omega} = 0$ since $\sigma \triangleleft \tau$ and so $[\omega : \tau] = 0$ or $[\sigma : \omega] = 0$. To see now how the Morse complex simplifies the computation of the homology groups $\boldsymbol{H}_\bullet(K, \mathcal{C})$, we state and then prove the following theorem:

Theorem 3.0.1. *Let $\mathcal{C}$ be a cosheaf on a simplicial complex K and let Σ be a $\mathcal{C}$-compatible acyclic partial matching on K. Then the Morse chain complex $(\boldsymbol{C}_\bullet^\Sigma(K,\mathcal{C}), \partial_\bullet^{\mathcal{C},\Sigma})$ is chain homotopy equivalent to $(\boldsymbol{C}_\bullet(K,\mathcal{C}), \partial_\bullet^{\mathcal{C}})$, according to the below definitions.*

Definition 3.0.6 (adapted from [3])**.** *A **chain homotopy** $\eta_\bullet$ between chain maps $\psi_\bullet, \psi_\bullet$: $(\boldsymbol{C}_\bullet, d_\bullet) \to (\boldsymbol{C}'_\bullet, d'_\bullet)$ is a collection of $\mathbb{R}$-linear maps $\eta_k : \boldsymbol{C}_k \to \boldsymbol{C}'_{k+1}$ which satisfy*

$$\phi_k - \psi_k = \eta_{k-1} \circ d_k + d'_{k+1} \circ \eta_k$$

for each $k \geq 0$.

Definition 3.0.7 (adapted from [3])**.** *A pair of chain complexes is said to be **chain homotopy equivalent** if there are two chain maps*

$$\Psi_\bullet : (\boldsymbol{C}_\bullet, d_\bullet) \to (\boldsymbol{C}'_\bullet, d'_\bullet) \text{ and } \Phi_\bullet : (\boldsymbol{C}'_\bullet, d'_\bullet) \to (\boldsymbol{C}_\bullet, d_\bullet)$$

along with chain homotopies

$$\eta_\bullet : 1_{(\boldsymbol{C}_\bullet, d_\bullet)} \implies \Phi_\bullet \circ \Psi_\bullet \text{ and } \eta'_\bullet : \Psi_\bullet \circ \Phi_\bullet \implies 1_{(\boldsymbol{C}'_\bullet, d'_\bullet)}.$$

Here, $1_{(\boldsymbol{C}_\bullet, d_\bullet)}$ is the identity map on $(\boldsymbol{C}_\bullet, d_\bullet)$ and $1_{(\boldsymbol{C}'_\bullet, d'_\bullet)}$ is the identity map on $(\boldsymbol{C}'_\bullet, d'_\bullet)$.

Before proving Theorem 3.0.1, we first prove an intermediate lemma.

Lemma 3.0.2. *Let Σ be an acyclic partial matching on K containing only one pair $(\sigma \triangleleft \tau)$. Then the simplicial chain complex $(\boldsymbol{C}_\bullet(K,\mathcal{C}), \partial_\bullet^{\mathcal{C}})$ is chain homotopy equivalent to the Morse complex $(\boldsymbol{C}_\bullet^\Sigma(K,\mathcal{C}), \partial_\bullet^{\mathcal{C},\Sigma})$*

Proof. To this end, we first need to define suitable chain maps between $(\boldsymbol{C}_\bullet(K,\mathcal{C}), \partial_\bullet^{\mathcal{C}})$ and $(\boldsymbol{C}_\bullet^\Sigma(K,\mathcal{C}), \partial_\bullet^{\mathcal{C},\Sigma})$. Define $\Psi_k : \boldsymbol{C}_k(K,\mathcal{C}) \to \boldsymbol{C}_k^\Sigma(K,\mathcal{C})$ to have components $\mathcal{C}(\alpha) \to \mathcal{C}(\beta)$, where $\alpha \in K$ and $\beta \in K \setminus \{\sigma, \tau\}$, given by

$$\Psi_k|_{\alpha,\beta} = \begin{cases} -\mathcal{C}_{\tau,\beta} \circ \mathcal{C}_{\tau,\sigma}^{-1} & \text{if } \alpha = \sigma \\ 1 & \text{if } \alpha = \beta \neq \sigma \\ 0 & \text{otherwise} \end{cases}$$

Similarly, define $\Phi_k : \boldsymbol{C}_k^\Sigma(K,\mathcal{C}) \to C_k(K,\mathcal{C})$ to have components $\mathcal{C}(\beta) \to \mathcal{C}(\alpha)$, given by

$$\Phi_k|_{\beta,\alpha} = \begin{cases} \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\beta,\sigma} & \text{if } \alpha = \tau \\ 1 & \text{if } \alpha = \beta \neq \tau \\ 0 & \text{otherwise} \end{cases}$$

We will verify only that $\Psi_\bullet$ is a valid chain map. The argument for $\Phi_\bullet$ is analogous.

Case 1: $\boldsymbol{\alpha = \sigma}$. We compute

$$(\partial_k^{\mathcal{C},\Sigma} \circ \Psi_k)|_{\alpha,\beta} = \sum_{\omega \in K \setminus \{\sigma,\tau\}} (\mathcal{C}_{\omega,\beta} + \mathcal{C}_{\tau,\beta} \circ \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\omega,\sigma}) \circ (-\mathcal{C}_{\tau,\omega} \circ \mathcal{C}_{\tau,\alpha}^{-1})$$

$$
\begin{aligned}
=&\Big[\sum_{\omega\in K\setminus\{\sigma,\tau\}} \mathcal{C}_{\omega,\beta}\circ(-\mathcal{C}_{\tau,\omega})\Big]\circ\mathcal{C}_{\tau,\sigma}^{-1}\\
&\underbrace{\quad+\sum_{\omega\in K\setminus\{\sigma,\tau\}}(\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\omega,\sigma})\circ(-\mathcal{C}_{\tau,\omega}\circ\mathcal{C}_{\tau,\alpha}^{-1})}_{=0\text{ as }\sigma\lhd\tau}\\
=&\mathcal{C}_{\sigma,\beta}
\end{aligned}
$$

and

$$
(\Psi_{k-1}\circ\partial_k^{\mathcal{C}})|_{\alpha,\beta}=\sum_{\omega\in K}\Psi_{k-1}|_{\omega,\beta}\circ(\mathcal{C}_{\sigma,\omega}+\mathcal{C}_{\tau,\omega}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\sigma,\sigma})=\mathcal{C}_{\sigma,\beta}
$$

Case 2: $\boldsymbol{\alpha\neq\sigma}$. We compute

$$
\begin{aligned}
(\partial^{\mathcal{C},\Sigma}\circ\Psi_k)|_{\alpha,\beta}=&\sum_{\omega\in K\setminus\{\sigma,\tau\}}\big(\mathcal{C}_{\omega,\beta}+\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\omega,\sigma}\big)\circ\Psi|_{\alpha,\omega}\\
=&\big(\mathcal{C}_{\alpha,\beta}+\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma}\big)\circ\Psi|_{\alpha,\alpha}\\
=&\mathcal{C}_{\alpha,\beta}+\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
(\Psi_{k-1}\circ\partial_k^{\mathcal{C}})|_{\alpha,\beta}=&\sum_{\omega\in K}\Psi_{k-1}|_{\omega,\beta}\circ(\mathcal{C}_{\alpha,\omega}+\mathcal{C}_{\tau,\omega}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma})\\
=&-\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma}-\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\tau,\sigma}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma}\\
&+\sum_{\omega\in K\setminus\{\sigma\}}\big(\Psi_{k-1}|_{\omega,\beta}\big)\circ(\mathcal{C}_{\alpha,\omega}+\mathcal{C}_{\tau,\omega}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma})\\
=&\mathcal{C}_{\alpha,\beta}+\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma}
\end{aligned}
$$

Now note that $\psi_\bullet\circ\phi_\bullet$ is the identity map on $\boldsymbol{C}_\bullet^\Sigma(K,\mathcal{C})$. Indeed, we have

$$
\begin{aligned}
\big(\Psi_k\circ\Phi_k\big)|_{\alpha,\alpha'}=&\sum_{\beta\in K}\Psi_k|_{\beta,\alpha'}\circ\Phi_k|_{\alpha,\beta}\\
=&\sum_{\beta\in K}\begin{cases}-\mathcal{C}_{\tau,\alpha'}\circ\mathcal{C}_{\tau,\sigma}^{-1} & (\beta=\sigma)\\ 1 & (\beta=\alpha'\neq\tau)\\ 0 & \text{(otherwise)}\end{cases}\circ\begin{cases}\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma} & (\beta=\tau)\\ 1 & (\beta=\alpha\neq\sigma)\\ 0 & \text{(otherwise)}\end{cases}\\
=&\sum_{\beta\in K}\begin{cases}0\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\alpha,\sigma} & (\beta=\tau)\\ -\mathcal{C}_{\tau,\alpha'}\circ\mathcal{C}_{\tau,\sigma}^{-1}\circ 0 & (\beta=\sigma)\\ 1\circ 1 & (\beta=\alpha=\alpha')\\ 0\circ 0 & \text{(otherwise)}\end{cases}\\
=&1
\end{aligned}
$$

The reversed composition is non-trivial, given by:

$$
\begin{aligned}
\big(\Phi_k\circ\Psi_k\big)|_{\alpha,\alpha'}=&\sum_{\beta\in K\setminus\{\sigma,\tau\}}\Phi_k|_{\beta,\alpha'}\circ\Psi_k|_{\alpha,\beta}\\
=&\sum_{\beta\in K\setminus\{\sigma,\tau\}}\begin{cases}\mathcal{C}_{\tau,\sigma}^{-1}\circ\mathcal{C}_{\beta,\sigma} & (\alpha'=\tau)\\ 1 & (\alpha'=\beta)\\ 0 & \text{(otherwise)}\end{cases}\circ\begin{cases}-\mathcal{C}_{\tau,\beta}\circ\mathcal{C}_{\tau,\sigma}^{-1} & (\alpha=\sigma)\\ 1 & (\alpha=\beta)\\ 0 & \text{(otherwise)}\end{cases}
\end{aligned}
$$

$$= \sum_{\beta \in K \setminus \{\sigma,\tau\}} \begin{cases} -\mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\beta,\sigma} \circ \mathcal{C}_{\tau,\beta} \circ \mathcal{C}_{\tau,\sigma}^{-1} & (\alpha = \sigma, \alpha' = \tau) \\ 1 \circ (-\mathcal{C}_{\tau,\alpha'}) \circ \mathcal{C}_{\tau,\sigma}^{-1} & (\alpha = \sigma, \beta = \alpha') \\ \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\alpha,\sigma} \circ 1 & (\alpha' = \tau, \beta = \alpha) \\ 1 & (\alpha' = \alpha = \beta) \\ 0 & (\text{otherwise}) \end{cases}$$

$$= \begin{cases} -\mathcal{C}_{\tau,\alpha'} \circ \mathcal{C}_{\tau,\sigma}^{-1} & (\alpha = \sigma, \alpha' \notin \{\sigma,\tau\}) \\ \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\alpha,\sigma} & (\alpha \notin \{\sigma,\tau\}, \alpha' = \tau) \\ 1 & (\alpha' = \alpha \notin \{\sigma,\tau\}) \\ 0 & (\text{otherwise}) \end{cases}$$

In removing the first case over the last equality, we used the fact that $\mathcal{C}_{\beta,\sigma} \circ \mathcal{C}_{\tau,\beta} = 0$, since we cannot have $\sigma \lhd \beta \lhd \tau$. While the first chain homotopy, $\eta_\bullet : 1_{(\boldsymbol{C}^\Sigma_\bullet(K,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma})} \implies \Psi_\bullet \circ \Phi_\bullet$ is trivially the zero map, the second homotopy $\eta'_\bullet : 1_{(\boldsymbol{C}^\bullet(K,\mathcal{C}),\partial_\bullet^{\mathcal{C}})} \implies \Phi_\bullet \circ \Psi_\bullet$ is non-trivial. We require

$$\Phi_k \circ \Psi_k - 1 = \partial_{k+1}^{\mathcal{C}} \circ \eta'_k - \eta'_{k-1} \circ \partial_k^{\mathcal{C}}.$$

To satisfy this, we can simply choose

$$\eta'_k|_{\alpha,\alpha'} = \begin{cases} -\mathcal{C}_{\tau,\sigma}^{-1} & \text{if } \alpha = \sigma \text{ and } \alpha' = \tau \\ 0 & \text{otherwise} \end{cases}$$

which completes the proof. □

Having proved *Lemma* 3.0.2, it is now relatively straight forward to deduce *Theorem* 3.0.1. Using the fact that Σ is acyclic, we can apply *Lemma* 3.0.2 iteratively, removing all pairs $(\sigma \lhd \tau) \in \Sigma$ in *any* order. To see this, consider for another pair $(\sigma' \lhd \tau')$ how the corresponding block entry has changed between $\partial_k^{\mathcal{C}}$ and $\partial_k^{\mathcal{C},\Sigma}$. That is, we compare $\partial_k^{\mathcal{C}}|_{\tau',\sigma'}$ and $(\partial^{\mathcal{C},\Sigma} \circ \Psi_k)|_{\tau',\sigma'}$. The former is just $\mathcal{C}_{\tau',\sigma'}$ and the latter is $\mathcal{C}_{\tau',\sigma'} - \mathcal{C}_{\tau,\sigma'} \circ \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\tau',\sigma}$, and their difference is simply:

$$\mathcal{C}_{\tau,\sigma'} \circ \mathcal{C}_{\tau,\sigma}^{-1} \circ \mathcal{C}_{\tau',\sigma} = \frac{[\sigma : \tau'] \cdot [\sigma' : \tau]}{[\sigma : \tau]} \mathcal{C}(\tau \geq \sigma') \circ \mathcal{C}(\tau \geq \sigma)^{-1} \circ \mathcal{C}(\tau' \geq \sigma)$$

But the coefficient in the above difference is only non-zero if the path $\sigma \lhd \tau \rhd \sigma' \lhd \tau'$ in Σ is not gradient, contradicting the acyclicity of Σ. This means that our augmentation of the boundary matrix, for any pair $(\sigma \lhd \tau) \in \Sigma$, does not augment the block entry for another pair $(\sigma' \lhd \tau') \in \Sigma$. This enables us to apply *Lemma* 3.0.2 iteratively, and we may conclude Theorem 3.0.1.

It remains to deduce that the homology groups $\boldsymbol{H}_k^\Sigma(K,\mathcal{C})$, corresponding to our Morse chain complex $(\boldsymbol{C}_\bullet^\Sigma(K,\mathcal{C}), \partial_\bullet^{\mathcal{C},\Sigma})$ are isomorphic to the homology groups $\boldsymbol{H}_k(K,\mathcal{C})$ of the original complex. To this end, we appeal to the functoriality of homology. In particular, we have

$$\boldsymbol{H}_k(\Psi \circ \Phi) = \boldsymbol{H}_k\Psi \circ \boldsymbol{H}_k\Phi \text{ and } \boldsymbol{H}_k(\Phi \circ \Psi) = \boldsymbol{H}_k\Phi \circ \boldsymbol{H}_k\Psi \quad \forall k \geq 0.$$

But we have just shown that $\Psi_\bullet \circ \Phi_\bullet$ is chain homotopic to $1_{(\boldsymbol{C}^\Sigma_\bullet(K,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma})}$ and $\Phi_\bullet \circ \Psi_\bullet$ is chain homotopic to $1_{(\boldsymbol{C}^\bullet(K,\mathcal{C}),\partial_\bullet^{\mathcal{C}})}$. We can now invoke the following result:

Lemma 3.0.3 (adapted from [3])**.** *If general chain maps* $\xi_\bullet, \nu_\bullet : (\boldsymbol{C}_\bullet, d_\bullet) \to (\boldsymbol{C}'_\bullet, d'_\bullet)$ *are chain homotopic, then their induced maps on homology coincide, i.e.,*

$$\boldsymbol{H}_k\xi = \boldsymbol{H}_k\nu$$

for each dimension $k \geq 0$.

It follows that $(\boldsymbol{H}_k\Psi)^{-1} = \boldsymbol{H}_k\Phi$ and $(\boldsymbol{H}_k\Phi)^{-1} = \boldsymbol{H}_k\Psi$. In other words, there exists an isomorphism between $\boldsymbol{H}_k^{\Sigma}(K,\mathcal{C})$ and $\boldsymbol{H}_k(K,\mathcal{C})$ for each k. Note further that this makes our chain maps $\Psi_\bullet$ and $\Phi_\bullet$ quasi-isomorphisms between $(\boldsymbol{C}_\bullet^{\Sigma}(K,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma})$ and $(\boldsymbol{C}_\bullet(K,\mathcal{C}),\partial_\bullet^{\mathcal{C}})$.

Computational Considerations In practice, to obtain a basis for the homology groups, we would like to obtain a basis for img $\partial_k^{\mathcal{C}}$ and ker $\partial_{k-1}^{\mathcal{C}}$ for each $k \geq 0$. This involves, for any chosen basis of $\boldsymbol{C}_k(K,\mathcal{C})$ and $\boldsymbol{C}_{k-1}(K,\mathcal{C})$, computing the Smith normal form of $\partial_k^{\mathcal{C}}$ (cf.[3]) from which we can read out the basis for ker $\partial_k^{\mathcal{C}}$ and img $\partial_k^{\mathcal{C}}$. For a map $A : \mathbb{R}^n \to \mathbb{R}^m$, obtaining the Smith normal form has complexity $\mathcal{O}(\min(n,m)mn)$. For $A = \partial_k^{\mathcal{C}}$ on the original cosheaf chain complex, we have $n = \dim \boldsymbol{C}_k = \sum_{\dim\tau=k}\dim\mathcal{C}(\tau)$ and $m = \dim \boldsymbol{C}_{k-1} = \sum_{\dim\tau=k-1}\dim\mathcal{C}(\tau)$. This reduces to $n = \dim \boldsymbol{C}_k^{\Sigma} = \sum_{\tau\in\mathrm{crit}_k(\Sigma)}\dim\mathcal{C}(\tau)$ and $m = \dim \boldsymbol{C}_{k-1}^{\Sigma} = \sum_{\tau\in\mathrm{crit}_{k-1}(\Sigma)}\dim\mathcal{C}(\tau)$, when we work with the isomorphic homology groups of the Morse chain complex $\boldsymbol{C}_\bullet^{\Sigma}(K,\mathcal{C})$. Hence, by working with the Morse chain complex, we may achieve a reduction in complexity which is approximately cubic in the number of Σ-critical k-simplices at each dimension k.

4 Meyer-Vietoris Long Exact Sequences of Morse Chain Complexes

Having proved that the homology groups of the Morse complex are isomorphic to the homology groups of the original cosheaf chain complex, we now aim to simplify the computation of the long exact sequences corresponding to (2.1) by resorting to an isomorphic Morse representation. To ensure that the inclusions in the Mayer-Vietoris diamond are retained when passing to the Morse complexes, we need to impose further restrictions on the acyclic partial matching Σ. To this end, we will establish the following theorem.

Theorem 4.0.1. *Let the short exact sequence* (2.1) *with $q_\bullet$ and $p_\bullet$ be given. Let Σ be a $\mathcal{C}$-compatible acyclic partial matching on our simplicial complex $K = M \cup L$. Define the Morse chain complexes $\mathcal{M}_\bullet^K = (\boldsymbol{C}_\bullet^{\Sigma}(K,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma})$, $\mathcal{M}_\bullet^L = (\boldsymbol{C}_\bullet^{\Sigma}(L,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma}|_L)$, $\mathcal{M}_\bullet^M = (\boldsymbol{C}_\bullet^{\Sigma}(M,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma}|_M)$, and $\mathcal{M}_\bullet^I = (\boldsymbol{C}_\bullet^{\Sigma}(I,\mathcal{C}),\partial_\bullet^{\mathcal{C},\Sigma}|_I)$. If additionally we require, for all pairs $(\sigma \lhd \tau) \in \Sigma$, that $\sigma \in M \implies \tau \in M$ and $\sigma \in L \implies \tau \in L$, then there exist chain maps $p'_\bullet$ and $q'_\bullet$ such that*

$$0 \to \mathcal{M}_\bullet^I \xrightarrow{p'_\bullet} \mathcal{M}_\bullet^L \oplus \mathcal{M}_\bullet^M \xrightarrow{q'_\bullet} \mathcal{M}_\bullet^K \to 0 \tag{4.1}$$

is a valid short exact sequence.

First note that the conditions of the theorem also imply, for each pair $\sigma \lhd \tau$ in Σ, that if $\sigma \in I$, then $\tau \in I$. Throughout the proof, for a subcomplex $S \subset K$ we denote by S_Σ the set of Σ^S-critical indeces, where $\Sigma^S \subset \Sigma$ excludes any pairs $(\sigma \lhd \tau) \in \Sigma$ where one of σ, τ is not in S. The Morse complexes of $M_\bullet^L$, $M_\bullet^K$ and $M_\bullet^I$ (in Theorem 4.0.1) should also be understood to have been constructed w.r.t. Σ^L, Σ^M and Σ^I.

Proof. We first show that under the assumptions made, $L_\Sigma \cup M_\Sigma = K_\Sigma$ and $L_\Sigma \cap M_\Sigma = I_\Sigma$.

$L_{\Sigma} \cup M_{\Sigma} = K_{\Sigma}$ Suppose $\sigma \in L_\Sigma \cup M_\Sigma$. Then σ is Σ^M-critical in M or Σ^L critical in L. Without loss of generality, suppose σ is Σ^M-critical in M. Suppose for a contradiction that σ is not Σ-critical in K. Then either $(\tau \lhd \sigma) \in \Sigma$ or $(\sigma \lhd \tau) \in \Sigma$ for some τ. Suppose first that $(\tau \lhd \sigma) \in \Sigma$. Then since M is a subcomplex, $\tau \in M$, so $(\tau \lhd \sigma) \in \Sigma^M$, a contradiction. Suppose now that $(\sigma \lhd \tau) \in \Sigma$ for some τ. Then by our additional assumption, $\tau \in M$, so $(\sigma \lhd \tau) \in \Sigma^M$, again a contradiction. So σ is Σ-critical in K. Clearly, $K_\Sigma \subset (L_\Sigma \cup M_\Sigma)$, since $M \cup L = K$, $\Sigma^L \subset \Sigma$ and $\Sigma^M \subset \Sigma$.

$L_{\Sigma} \cap M_{\Sigma} = I_{\Sigma}$ Suppose $\sigma \in I_\Sigma$, i.e. σ is Σ^I-critical in I. Suppose for a contradiction, and without loss of generality, that σ is not Σ^M-critical in M. Then either $(\tau \lhd \sigma) \in \Sigma^M$ or $(\sigma \lhd \tau) \in \Sigma^M$ for some τ. But $\sigma \in M \cap L$, so by our additional assumption and by the fact that M and L are subcomplexes, in both cases $\tau \in M \cap L$, and so in both cases the pair is in Σ^I, a contradiction. Now suppose $\sigma \in L_\Sigma \cap M_\Sigma$. Clearly, $\sigma \in I_\Sigma$, since otherwise σ is in some pair in $\Sigma^I = \Sigma^M \cap \Sigma^L$ which is a contradiction.

It immediately follows that there are inclusions $\gamma'_k, \delta'_k, \epsilon'_k, \zeta'_k$ that furnish the following Meyer-Vietoris diamond.

$$\begin{tikzcd}
 & \mathcal{M}_k^K & \\
\mathcal{M}_k^L \arrow[ur, hook, "\gamma'_k"] & & \mathcal{M}_k^M \arrow[ul, hook', "\delta'_k"'] \\
 & \mathcal{M}_k^I \arrow[ul, hook, "\epsilon'_k"] \arrow[ur, hook', "\zeta'_k"'] &
\end{tikzcd}$$

We may now specify our chain maps $p'_\bullet : \mathcal{M}_\bullet^I \to \mathcal{M}_\bullet^M \oplus \mathcal{M}_\bullet^L$ and $q'_\bullet : \mathcal{M}_\bullet^M \oplus \mathcal{M}_\bullet^L \to \mathcal{M}_\bullet^K$ analogously to those used in *Lemma* 2.0.1:

$$p'_k(v) = (\epsilon'_k v, -\zeta'_k v)$$
$$q'_k(v, u) = \gamma'_k v + \delta'_k u$$

These are valid chain maps, making (4.1) a valid short exact sequence, mutatis mutandis in *Lemma* 2.0.1 and the proof given there. □

We now want to appeal to a category theoretical result that establishes the **naturality** of the induced homomorphism in the following sense, following [4]. We first note that there is a category $\mathcal{S}$ whose objects are short exact sequences of chain complexes (viewed in an abelian category $\mathcal{C}$) and whose morphisms are commutative diagrams of the following form:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "p_\bullet"] \arrow[d, "r_\bullet"] & B \arrow[r, "q_\bullet"] \arrow[d, "l_\bullet"] & C \arrow[r] \arrow[d, "m_\bullet"] & 0 \\
0 \arrow[r] & A' \arrow[r, "p'_\bullet"] & B' \arrow[r, "q'_\bullet"] & C' \arrow[r] & 0
\end{tikzcd} \tag{*}$$

There is an analogous category $\mathcal{L}$ of long exact sequences in $\mathcal{C}$. In this setting, we can now apply the following theorem, which is basically an extension of the snake lemma in [3].

Theorem 4.0.2 (adapted from [4])**.** *For every short exact sequence in $\mathcal{S}$ there is a long exact sequence in $\mathcal{L}$, and for every map $(*)$ of short exact sequences there is a commutative diagram*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \boldsymbol{H}_k(A) & \xrightarrow{\boldsymbol{H}_k p} & \boldsymbol{H}_k(B) & \xrightarrow{\boldsymbol{H}_k q} & \boldsymbol{H}_k(C) & \xrightarrow{D_k} & \boldsymbol{H}_{k-1}(A) & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle \boldsymbol{H}_k r} & & \big\downarrow{\scriptstyle \boldsymbol{H}_k l} & & \big\downarrow{\scriptstyle \boldsymbol{H}_k m} & & \big\downarrow{\scriptstyle \boldsymbol{H}_{k-1} l} & & \\ \cdots & \longrightarrow & \boldsymbol{H}_k(A') & \xrightarrow{\boldsymbol{H}_k p'} & \boldsymbol{H}_k(B') & \xrightarrow{\boldsymbol{H}_k q'} & \boldsymbol{H}_k(C') & \xrightarrow{D'_k} & \boldsymbol{H}_{k-1}(A') & \longrightarrow & \cdots \end{array} \tag{**}$$

We now invoke Theorem 3.0.1 and *Lemma* 3.0.3 to justify the existence of quasi-isomorphisms

$$\begin{aligned} &r_\bullet : (\boldsymbol{C}_\bullet(K, \mathcal{C}), \partial^{\mathcal{C}}_\bullet) \to \mathcal{M}^K_\bullet, \\ &l_\bullet : (\boldsymbol{C}_\bullet(M, \mathcal{C}), \partial^{\mathcal{C}}_\bullet|_M) \oplus (\boldsymbol{C}_\bullet(L, \mathcal{C}), \partial^{\mathcal{C}}_\bullet|_L) \to \mathcal{M}^M_\bullet \oplus \mathcal{M}^L_\bullet, \\ &m_\bullet : (\boldsymbol{C}_\bullet(I, \mathcal{C}), \partial^{\mathcal{C}}_\bullet|_I) \to \mathcal{M}^I_\bullet. \end{aligned}$$

We can use these to furnish a commutative diagram of type (*) between (2.1) and (4.1), and by the naturality of the induced isomorphism (Theorem 4.0.2) a corresponding commutative diagram of type (**), shown below. Since $r_\bullet$, $l_\bullet$ and $m_\bullet$ are quasi-isomorphisms, all of $\boldsymbol{H}_k r$, $\boldsymbol{H}_k l$ and $\boldsymbol{H}_k m$ are isomorphisms. This establishes the main result of this paper, namely that the Mayer-Vietoris long exact sequences of homology groups derived from the original cosheaf chain complex and those derived from its Morse representation are isomorphic.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \boldsymbol{H}^\Sigma_{k+1}(K) & \xrightarrow{\boldsymbol{H}_k p} & \boldsymbol{H}_k(L) \oplus \boldsymbol{H}_k(M) & \xrightarrow{\boldsymbol{H}_k q} & \boldsymbol{H}_{k-1}(K) & \xrightarrow{D_k} & \boldsymbol{H}_{k-1}(I) & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle \boldsymbol{H}_k r} & & \big\downarrow{\scriptstyle \boldsymbol{H}_k l} & & \big\downarrow{\scriptstyle \boldsymbol{H}_k m} & & \big\downarrow{\scriptstyle \boldsymbol{H}_{k-1} l} & & \\ \cdots & \longrightarrow & \boldsymbol{H}^\Sigma_{k+1}(K) & \xrightarrow{\boldsymbol{H}_k p'} & \boldsymbol{H}^\Sigma_k(L) \oplus \boldsymbol{H}^\Sigma_k(M) & \xrightarrow{\boldsymbol{H}_k q'} & \boldsymbol{H}^\Sigma_k(K) & \xrightarrow{D'} & \boldsymbol{H}^\Sigma_{k-1}(I) & \longrightarrow & \cdots \end{array}$$

5 Computational Considerations

By the presented theorems, any computation for the full long exact sequence of cosheaf homology groups can simply be conducted on the cosheaf homology groups of the Morse chain complex, with the same results. But how does using the Morse version of the long exact sequence simplify the computation of the connecting homomorphism? To see this, we outline the steps by which $D'_k([z]) \in H^\Sigma_{k-1}(I)$ is obtained from $[z] \in H^\Sigma_k(I)$, where $[z]$ denotes the equivalence class of z in H^Σ_k as a quotient vector space. Note first that $[z] \in H^\Sigma_k(I)$ means $z \in \ker \partial^{\mathcal{C},\Sigma}_k \subset \mathcal{M}^K_k$. Let such a z be given. We proceed as follows (cf. [3]):

1. We *find* $(u, v) \in \mathcal{M}^M_k \oplus \mathcal{M}^L_k$ s.t. $q'(u, v) = z$, which exists by surjectivity of q'.
2. Since q' is a chain map and $z \in \ker \partial^{\mathcal{C},\Sigma}_k$, we have

$$q'_{k-1} \circ \partial^{\mathcal{C},\Sigma}_k|_{M \oplus L}(u, v) = \partial^{\mathcal{C},\Sigma}_k \circ q'_k(u, v) = 0$$

   i.e. $\partial^{\mathcal{C},\Sigma}_k|_{M \oplus L}(u, v) \in \ker q'_{k-1}$.

3. By the exactness of 4.1, this means that in fact $p'_{k-1}(x) = \partial_k^{\mathcal{C},\Sigma}|_{M\oplus L}(u, v)$ for some $x \in \ker \partial_{k-1}^{\mathcal{C},\Sigma}|_I \subset \mathcal{M}_{k-1}^I$. This gives $D'_k([z]) = [x]$.

In **step 1**, we may obtain some suitable u and v by choosing all dimensions corresponding to costalks of simplices in $M_k \cap L_k$ to be zero in u. Then u and v are uniquely determined. Similarly, in **step 3**, since $q'_{k-1}(x) = (\iota'_3 x, \iota'_4 x)$, x can be obtained by reading out the relevant dimensions. Both computations are loosely bounded by $\mathcal{O}(\dim \mathcal{M}_k^K)$.

Step 2 requires us to compute $(\partial_k^{\mathcal{C},\Sigma}|_M, \partial_k^{\mathcal{C},\Sigma}|_L)(u, v)$. Now this computation is a linear map between $\mathcal{M}_k^M \oplus \mathcal{M}_k^L$ and $\mathcal{M}_{k-1}^M \oplus \mathcal{M}_{k-1}^L$. The product of an $m \times n$ matrix and an n-dimensional vector has complexity $\mathcal{O}(mn)$, at worst. In our case, the complexity can be estimated as $\mathcal{O}((\dim \mathcal{M}_{k-1}^M \cdot \dim \mathcal{M}_k^M) + (\dim \mathcal{M}_{k-1}^L \cdot \dim \mathcal{M}_k^L))$. But $\dim \mathcal{M}_k = \sum_{\tau \in \mathrm{crit}_k(\Sigma)} \dim \mathcal{C}(\tau)$, so by expanding the acyclic partial matching Σ, the computation of the boundary map simplifies.

Throughout, we assumed that Σ is given. For the context of cellular sheaf cohomology, the `scythe` algorithm in [2] provides a way of obtaining an acyclic partial matching Σ along with the corresponding Morse cochain complex for a given original sheaf cochain complex. We hypothesise that a similar algorithm could facilitate the computation of Σ for the case of cosheaf chain complexes. Note that such an algorithm of course incurs a computational overhead (the complexity of which can be found in [2]), but it automatically produces representations of the boundary maps $\partial_k^{\mathcal{C},\Sigma}$ which are used in computing the Morse homology groups (cf. chapter 3) and in step 2 above. Otherwise, the proof of Theorem 3.0.1 is constructive for obtaining these boundary maps from the original boundary map. We may take $\mathcal{O}(|\Sigma| pd)$ as a rough complexity estimate for obtaining the new (global) boundary matrix $\partial^{\mathcal{C},\Sigma}$, where we follow [2] in defining p as the maximum number of codimension one faces of any vertex in K and $d < \infty$ as the maximum dimensionality of any cosheaf assigned to a simplex in K. The estimate results from at most pd eliminations necessary for each application of *Lemma* 3.0.2 per pair in Σ, ignoring constant factors.

6 Generalisations

We now briefly consider how our framework can be extended to the case where $K = L_1 \cup L_2 \cup \cdots \cup L_n$, where each L_i is a subcomplex. We will present a model within which the short exact sequence in (2.1) is a special case when $n = 2$ by adapting the presentation given in [1] to our context. First, we introduce the following direct sums for $m \geq 1$

$$\boldsymbol{C}_{m,k} = \bigoplus_{i_j < i_{j+1}} \boldsymbol{C}_k(L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}, \mathcal{C})$$

Denote by $\iota_{k,i_1,\ldots,\hat{i}_j,\ldots,i_m}$ the inclusion

$$\iota_{k,i_1,\ldots,\hat{i}_j,\ldots,i_m} : \boldsymbol{C}_k(L_{i_1} \cap \cdots \cap L_{i_j} \cap \cdots \cap L_{i_m}, \mathcal{C}) \hookrightarrow \boldsymbol{C}_k(L_{i_1} \cap \cdots \cap L_{i_{j-1}} \cap L_{i_{j+1}} \cdots \cap L_{i_m}, \mathcal{C})$$

We then define a generalized q-map $q_{m,k} : \boldsymbol{C}_{m,k} \to \boldsymbol{C}_{m-1,k}$ componentwise:

$$q_{m,k}|_{(i_1,\ldots,i_m)\to(j_1,\ldots,j_{m-1})} = \begin{cases} (-1)^r \iota_{k,i_1,\ldots,\hat{i}_r,\ldots,i_m} & \text{if } (i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_m) = (j_1,\ldots,j_{m-1}) \\ 0 & \text{otherwise} \end{cases}$$

The generalization of the p-map $p_k : \boldsymbol{C}_{1,k} \to \boldsymbol{C}_k(K, \mathcal{C})$ is straightforward:

$$p(v_1, \ldots, v_n) = \sum_i^n \iota_{k,i} v_i$$

where here the inclusion $\iota_{k,i}$ means $\boldsymbol{C}_k(L_i, \mathcal{C}) \hookrightarrow \boldsymbol{C}_k(K, \mathcal{C})$. If we restrict our discussion to the case with all triple intersections empty, the relevant $\boldsymbol{C}_{m,k}$ are:

$$\boldsymbol{C}_{1,k} = \bigoplus_i \boldsymbol{C}_k(L_i, \mathcal{C}) \text{ and } \boldsymbol{C}_{2,k} = \bigoplus_{i<j} \boldsymbol{C}_k(L_i \cap L_j, \mathcal{C}).$$

For $n = 2$, we obtain the familiar case in (2.1). For $n = 3$, we obtain $q_{2,k}$ and p_k as:

$$q_{2,k}(v_{12}, v_{13}, v_{23}) = (\iota_{k,1,\hat{2}} v_{12} - \iota_{k,1,\hat{3}} v_{13}, -\iota_{k,\hat{1},2} v_{12} - \iota_{k,2,\hat{3}} v_{23}, -\iota_{k,\hat{1},3} v_{13} + \iota_{k,\hat{2},3} v_{23})$$

$$p_k(v_1, v_2, v_3) = \iota_{k,1} v_1 + \iota_{k,2} v_2 + \iota_{k,3} v_3$$

When triple intersections are empty, it is easy to check the usual conditions which make $0 \to \boldsymbol{C}_{2,\bullet} \to \boldsymbol{C}_{1,\bullet} \to \boldsymbol{C}_\bullet(K, \mathcal{C}) \to 0$ a short exact sequence. The case with non-empty triple-intersections yields a sequence $0 \to \boldsymbol{C}_{m,\bullet} \to \boldsymbol{C}_{m-1,\bullet} \to \cdots \to \boldsymbol{C}_\bullet(K, \mathcal{C}) \to 0$ with at most n non-trivial chain groups at each dimension k and which is exact as long as the number of intersections are finite, which we assumed. For the case with empty triple intersections, the Morse representation and most results should follow suit, where again we must avoid pairs in the acyclic partial matching that split across two subcomplexes, as discussed in Theorem 4.0.1. The case with non-empty triple intersection does not yield an obvious short exact sequence to which we could apply the snake lemma, making it an interesting direction for further research.

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