

Mathematics Collection

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Abstract

Mathematics my boi

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Introduction

Chapter 1

Logic and Set Theory

1.1 Logic of Propositions

Definition 1.1.1: Binary Decision

A *binary* decision is a type of „question“ which can be in one of two states, true t or false f , hence the classification as binary (which comes from greek and means twofold).

1.2 Set Theory

Chapter 2

Algebra

2.1 Algebraic Structures

Motivation

Although the following sections may seem like a quite boring sea of compact, unreadable equations and proofs, it forms the essence of a modern understanding of most mathematics used today, since it formalizes the relations that sets form with operations acting upon them. This leads to structures like groups, rings or fields, which allow for the generalization of various viewpoints¹ into an abstract form.

Coming from High School, algebra may just sound like solving complicated equations, like $x^2 + 1 = 0$, which in fact forms the origin of the study.

2.1.1 Groups

Definition 2.1.1: Magmata

Let X be a set and $\circ: X^2 \rightarrow X$. We call (X, \circ) an algebraic structure, or magma, iff

$$\forall x_1, x_2 \in X: x_1 \circ x_2 \in X. \quad (2.1)$$

Algebraic structures are a very broad and basic structure, the fundamental idea of group theory and everything that is build upon it. A trivial simple example is $(\mathbb{N}, +)$, where $+$ denotes regular addition. Contrasting that, $(\mathbb{N}, -)$, with regular subtractions is not an algebraic structure, since $\exists x_1, x_2 \in \mathbb{N}: x_1 - x_2 \notin \mathbb{N}$, i.e. $1 - 2 \notin \mathbb{N}$. Hence given any set X paired with an operation $\circ: X^2 \rightarrow X$, one should prove that (X, \circ) forms an algebraic structure, in order to apply any general proofs or methods. If (X, \circ) is an algebraic structure, then we call X closed under \circ . There is no standardized notation for this circumstance, hence if this is the case, you might want to state closure under \circ at the beginning of your work. In order to prove that X is closed under \circ , one might negate [equation \(2.1\)](#), yielding

$$\exists x_1, x_2 \in X: x_1 \circ x_2 \notin X. \quad (2.2)$$

Hence, if proving (X, \circ) is an algebraic structure, we find a pair $x_1, x_2 \in X$, where $x_1 \circ x_2 \notin X$, i.e. $2 - 1$. Generally, when you need to prove such general properties, using a contradiction is the proper method, since finding a counterexample is most of the times a lot easier than proving said property for all elements of X .

Given an algebraic structure (X, \circ) , one might find triplets $(x_1, x_2, x_3) \in X^3$, where $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$.

Definition 2.1.2: Associativity and Semi-Groups

Let (X, \circ) be an algebraic structure. We call \circ associative, iff

$$\forall x_1, x_2, x_3 \in X: (x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3). \quad (2.3)$$

We then call (X, \circ) a semi- (or half-) group.

Semi-groups are the most basic structure, that are actually useful and have some² use in e.g. linear algebra³.

Theorem 2.1.1: Associativity of Set Operations

Let X be a set, then the following structures are semi-groups:

$$\begin{aligned} &(\mathcal{P}(X), \cap) \\ &(\mathcal{P}(X), \cup) \\ &(\mathcal{P}(X), \triangle) \end{aligned} \quad (2.4)$$

and $(\mathcal{P}(X), \setminus)$ is just algebraic.

¹although the most general viewpoint would be in fact Category Theory

²although very limited

³like $(\mathbb{R}^{m \times n}, \cdot)$ with the regular matrix multiplication

Proof for Theorem 2.1.1. We already proved in hat \cap , \cup and \triangle are associative, and \setminus is not. Hence we only have to prove that $\forall Y_1, Y_2 \in \mathcal{P}(X): Y_1 \circ Y_2 \in \mathcal{P}(X)$.

$$\begin{aligned}
Y_1 \cap Y_2 = Y & \quad (\forall y \in Y: y \in Y_1 \wedge y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2: y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y: y \in X \\
Y_1 \cup Y_2 = Y & \quad (\forall y \in Y: y \in Y_1 \vee y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2: y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y: y \in X \\
Y_1 \triangle Y_2 = Y & \quad (\forall y \in Y: (y \in Y_2 \setminus Y_1) \vee (y \in Y_1 \setminus Y_2)) \Rightarrow \forall y \in Y: y \in X \\
Y_1 \setminus Y_2 = Y & \quad \forall y \in Y: y \in Y_1 \Rightarrow \forall y \in Y: y \in X \\
Y_2 \setminus Y_1 = Y & \quad \forall y \in Y: y \in Y_2 \Rightarrow \forall y \in Y: y \in X
\end{aligned}$$

i.e. $(\mathcal{P}(X), \cap)$, $(\mathcal{P}(X), \cup)$, $(\mathcal{P}(X), \triangle)$ and $(\mathcal{P}(X), \setminus)$ are closed under their relative operation. q.e.d.

Given (\mathbb{N}, \cdot) , where \cdot is the common multiplication, we observe that there exists pairs $(a, b) \in \mathbb{N}^2$, where $a \cdot b = b$, which is the case for $a = 1$. We call 1 the left-neutral element of \cdot . In contrast, 1 is right-neutral if $b \cdot 1 = b$.

Definition 2.1.3: Neutral Element and Monoids

Let (X, \circ) be a semi-group. We call $e \in X$ neutral, if it is left- and right neutral, i.e. $\forall x \in X: e \circ x = x = x \circ e$. If X has a neutral element e , we call (X, \circ) a monoid.

Theorem 2.1.2: Uniqueness of Neutral Elements

Let (X, \circ) be a monoid with neutral element e , then e is unique.

Proof for Theorem 2.1.2. Given a monoid (X, \circ) with neutral elements $e, e' \in X$ it follows:

$$\begin{aligned}
e' &= e' \circ e = e \\
e &= e \circ e' = e'
\end{aligned}$$

q.e.d.

Chapter 3

Real Analysis

3.1 Numbers and Sequences

Although we already introduced the concept of fields in the number system we use today, i.e. the real numbers \mathbb{R} are typically introduced in lectures about real analysis. Hence we follow this approach, as the concept of the reals encapsulates a good part of what we are trying to establish here.

3.1.1 Establishing a Number System

A number system is a set of numbers, typically infinite, which contains objects called numbers. The origin of numbers lies in the counting of objects. Based on our observations on fields, we want to apply the properties of a field to our number system.

Definition 3.1.1: Axioms of PEANO

Let $s: \mathbb{N} \rightarrow \mathbb{N}$ be the successor function:

- $1 \in \mathbb{N}$
- $\forall n \in \mathbb{N}: \exists n' \in \mathbb{N}: s(n) = s'$
- $\nexists n \in \mathbb{N}: s(n) = 1$
- $\forall m, n \in \mathbb{N}: s(m) = s(n) \Rightarrow m = n$
- $1 \in X \wedge \forall x \in X: s(x) \in X \Rightarrow \mathbb{N} \subseteq X$

The successor function s is injective and produces the natural numbers by mapping $n \in \mathbb{N}$ to $n + 1$, since the successor of n is $n + 1$. These axioms lead directly to the method of Proof by Induction.

Definition 3.1.2: Induction

Let $\mathcal{P}(n)$ be a predicate where n is element of some (ordered) index-family I , where $\forall k \in I: \mathcal{P}(k)$. If we can prove that $\forall n \in \mathbb{N}: \mathcal{P}(n) \Rightarrow \mathcal{P}(s(n))$, then $\exists \sigma: J \rightarrow \mathbb{N}$.

Proofs by induction are very powerful, since we can simply check whether $\mathcal{P}(n) \Rightarrow \mathcal{P}(n + 1)$ for $n \in J$. A very common example is the gaussian sum-formula:

Theorem 3.1.1: Gaussian Sum Formula

Let $n \in \mathbb{N}$, then

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad (3.1)$$

Proof. We first find $n_0 \in \mathbb{N}: \mathcal{P}(n_0)$. Hence we test $n_0 = 1$:

$$\begin{aligned} \sum_{k=1}^1 k &= 1 \\ \frac{2}{2} &= 1 \end{aligned}$$

This checks out, so we may proof $\mathcal{P}(n) \Rightarrow \mathcal{P}(n + 1)$, i.e.

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \Rightarrow \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

We use the fact, that we can split up sums into their summands:

$$\sum_{k=1}^{n+1} k = n + 1 + \sum_{k=1}^n k = n + 1 + \frac{n(n+1)}{2} = \frac{2n + 2 + n^2 + n}{2} = \frac{(n+1)(n+2)}{2}$$

Where the last equality follows by $(n+1)(n+2) = n^2 + 3n + 2$.

q.e.d.