Mathematics Collection

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Since July 2022



Mathematics my boi

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ec:intro

Introduction

Chapter 1

Logic and Set Theory

logic_set

sec:logic

1.1 Logic of Propositions

Definition 1.1.1: Binary Decision

def:binary_decision

A binary decision is a type of "question" which can be in one of two states, true t or false f, hence the classification as binary (which comes from greek and means twofold).

1.2 Set Theory

t_theory

Chapter 2

Linear Algebra

r_algebra

tructures

otivation

2.1 Algebraic Structures

Although the following sections may seem like a quite boring sea of compact, unreadable equations and proofs, it forms the essence of a modern understanding of most mathematics used today, since it formalizes the relations that sets form with operations acting upon them. This leads to structures like groups, rings or fields, which

Coming from High School, algebra may just sound like solving complicated equations, like $x^2 + 1 = 0$, which in fact forms the origin of the study.

allow for the generalization of various viewpoints¹ into an abstract form.

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Improvement 1:
add more and better motivation

ec:groups

2.1.1 Groups

Motivation

Definition 2.1.1: Algebraical Structure

Let
$$X$$
 be a set and $\circ: X^2 \to X$. We call (X, \circ) an algebraic structure, iff

$$\forall x_1, x_2 \in X \colon x_1 \circ x_2 \in X. \tag{2.1}$$
{eq:def_alg_struct}

Algebraic structures are a very broad and basic structure, the fundamental idea of group theory and everything that is build upon it. A trivial simple example is $(\mathbb{N},+)$, where + denotes regular addition. Contrasting that, $(\mathbb{N},-)$, with regular subtractions is not an algebraic structure, since $\exists x_1, x_2 \in \mathbb{N} \colon x_1 - x_2 \notin \mathbb{N}$, i.e. $1-2 \notin \mathbb{N}$. Hence given any set X paired with an operation $\circ \colon X^2 \to X$, one should prove that (X,\circ) forms an algebraic structure, in order to apply any general proofs or methods. If (X,\circ) is an algebraic structure, then we call X closed under \circ . There is no standardized notation for this circumstance, hence if this is the case, you might want to state closure under \circ at the beginning of your work. In order to prove that X is closed under \circ , one might negate equation (2.1), yielding

$$\exists x_1, x_2 \in X \colon x_1 \circ x_2 \notin X. \tag{2.2}$$

Hence, if proving (X, \circ) is an algebraic structure, we find a pair $x_1, x_2 \in X$, where $x_1 \circ x_2 \notin X$, i.e. 2-1. Generally, when you need to prove such general properties, using a contradiction is the proper method, since finding a counterexample is most of the times a lot easier than proving said property for all elements of X.

Given an algebraic structure (X, \circ) , one might find triplets $(x_1, x_2, x_2) \in X^3$, where $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$.

Definition 2.1.2: Associativity and Half-Groups



{eq:def_associativity}

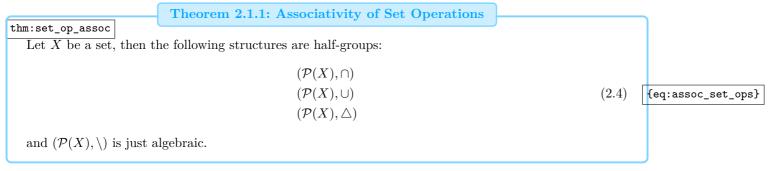
def:associativity

Let (X, \circ) be an algebraic structure. We call \circ associative, iff

$$\forall x_1, x_2, x_3 \in X : (x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3). \tag{2.3}$$

We then call (X, \circ) a half-group.

Half groups are the most basic structure, that are actually useful and have some² use in e.g. linear algebra³.



¹although the most general viewpoint would be in fact Category Theory

²although very limited

 $^{^3 \}text{like } (\mathbb{F}^{m \times n}, \cdot)$ with the regular matrix multiplication

c_set_ops

Proof for thm: set_op_assoc Theorem 2.1.1. We already proved in that \cap , \cup and \triangle are associative, and \setminus is not. Hence we only have to prove that $\forall Y_1, Y_2 \in \mathcal{P}(X) \colon Y_1 \circ Y_2 \in \mathcal{P}(X)$.

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Change 1:
add reference for proof

 $\begin{array}{ll} Y_1 \cap Y_2 = Y & (\forall y \in Y \colon y \in Y_1 \wedge y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2 \colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y \colon y \in X \\ Y_1 \cup Y_2 = Y & (\forall y \in Y \colon y \in Y_1 \vee y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2 \colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y \colon y \in X \\ Y_1 \triangle Y_2 = Y & (\forall y \in Y \colon (y \in Y_2 \setminus Y_1) \vee (y \in Y_1 \setminus Y_2)) \Rightarrow \forall y \in Y \colon y \in X \\ Y_1 \setminus Y_2 = Y & \forall y \in Y \colon y \in Y_1 \Rightarrow \forall y \in Y \colon y \in X \\ Y_2 \setminus Y_1 = Y & \forall y \in Y \colon y \in Y_2 \Rightarrow \forall y \in Y \colon y \in X \end{array}$