Mathematics Collection

Moritz Mossböck
Peter Waldert

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Abstract

Mathematics my boi

Contents

Introduction

sec:intro

Chapter 1

Logic and Set Theory

chp:logic_set

1.1 Logic of Propositions

sec:logic

def:binary_decision

Definition 1.1.1: Binary Decision

A *binary* decision is a type of „question“ which can be in one of two states, true t or false f, hence the classification as binary (which comes from greek and means twofold).

1.2 Set Theory

sec:set_theory

Chapter 2

Linear Algebra

chp:linear_algebra

2.1 Algebraic Structures

sec:alg_structures

Motivation

ssec:alg_motivation

Although the following sections may seem like a quite boring sea of compact, unreadable equations and proofs, it forms the essence of a modern understanding of most mathematics used today, since it formalizes the relations that sets form with operations acting upon them. This leads to structures like groups, rings or fields, which allow for the generalization of various viewpoints[1] into an abstract form.

Coming from High School, algebra may just sound like solving complicated equations, like $x^2 + 1 = 0$, which in fact forms the origin of the study.

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Improvement 1:
add more and better motivation

2.1.1 Groups

ssec:groups

def:alg_struct

Definition 2.1.1: Algebraical Structure

Let X be a set and $\circ\colon X^2 \to X$. We call $(X, \circ)$ an algebraic structure, iff

{eq:def_alg_struct}eq:def_alg_struct

$$\forall x_1, x_2 \in X\colon x_1 \circ x_2 \in X. \tag{2.1}$$

Algebraic structures are a very broad and basic structure, the fundamental idea of group theory and everything that is build upon it. A trivial simple example is $(\mathbb{N}, +)$, where $+$ denotes regular addition. Contrasting that, $(\mathbb{N}, -)$, with regular subtractions is not an algebraic structure, since $\exists x_1, x_2 \in \mathbb{N}\colon x_1 - x_2 \notin \mathbb{N}$, i.e. $1 - 2 \notin \mathbb{N}$. Hence given any set X paired with an operation $\circ\colon X^2 \to X$, one should prove that $(X, \circ)$ forms an algebraic structure, in order to apply any general proofs or methods. If $(X, \circ)$ is an algebraic structure, then we call X closed under $\circ$. There is no standardized notation for this circumstance, hence if this is the case, you might want to state closure under $\circ$ at the beginning of your work. In order to prove that X is closed under $\circ$, one might negate equation (2.1), yielding

{eq:neg_def_alg_struct}eq:neg_def_alg_struct

$$\exists x_1, x_2 \in X\colon x_1 \circ x_2 \notin X. \tag{2.2}$$

Hence, if proving $(X, \circ)$ is an algebraic structure, we find a pair $x_1, x_2 \in X$, where $x_1 \circ x_2 \notin X$, i.e. $2 - 1$. Generally, when you need to prove such general properties, using a contradiction is the proper method, since finding a counterexample is most of the times a lot easier than proving said property for all elements of X.

Given an algebraic structure $(X, \circ)$, one might find triplets $(x_1, x_2, x_2) \in X^3$, where $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$.

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Improvement 2:
write more intro about associativity

def:associativity

Definition 2.1.2: Associativity and Half-Groups

Let $(X, \circ)$ be an algebraic structure. We call $\circ$ associative, iff

{eq:def_associativity}eq:def_associativity

$$\forall x_1, x_2, x_3 \in X\colon (x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3). \tag{2.3}$$

We then call $(X, \circ)$ a half-group.

Half groups are the most basic structure, that are actually useful and have some[2] use in e.g. linear algebra[3].

thm:set_op_assoc

Theorem 2.1.1: Associativity of Set Operations

Let X be a set, then the following structures are half-groups:

{eq:assoc_set_ops}eq:assoc_set_ops

$$\begin{gathered}(\mathcal{P}(X), \cap)\\ (\mathcal{P}(X), \cup)\\ (\mathcal{P}(X), \triangle)\end{gathered} \tag{2.4}$$

and $(\mathcal{P}(X), \setminus)$ is just algebraic.

[1] although the most general viewpoint would be in fact Category Theory
[2] although very limited
[3] like $(\mathbb{F}^{m \times n}, \cdot)$ with the regular matrix multiplication

`proof:assoc_set_ops`

Proof for Theorem 2.1.1. We already proved in that $\cap$, $\cup$ and $\triangle$ are associative, and $\setminus$ is not. Hence we only have to prove that $\forall Y_1, Y_2 \in \mathcal{P}(X)\colon Y_1 \circ Y_2 \in \mathcal{P}(X)$.

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Change 1:
add reference for proof

$$
\begin{aligned}
Y_1 \cap Y_2 = Y \qquad & (\forall y \in Y : y \in Y_1 \wedge y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2\colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y : y \in X \\
Y_1 \cup Y_2 = Y \qquad & (\forall y \in Y : y \in Y_1 \vee y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2\colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y : y \in X \\
Y_1 \triangle Y_2 = Y \qquad & (\forall y \in Y : (y \in Y_2 \setminus Y_1) \vee (y \in Y_1 \setminus Y_2)) \Rightarrow \forall y \in Y : y \in X \\
Y_1 \setminus Y_2 = Y \qquad & \forall y \in Y : y \in Y_1 \Rightarrow \forall y \in Y : y \in X \\
Y_2 \setminus Y_1 = Y \qquad & \forall y \in Y : y \in Y_2 \Rightarrow \forall y \in Y : y \in X
\end{aligned}
$$

□