Mathematics Collection

Moritz Mossböck
Peter Waldert

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Abstract

Mathematics my boi

Contents

sec:intro

Introduction

Chapter 1

Logic and Set Theory

chp:logic_set

sec:logic

1.1 Logic of Propositions

Definition 1.1.1: Binary Decision

def:binary_decision

A *binary* decision is a type of „question“ which can be in one of two states, true t or false f, hence the classification as binary (which comes from greek and means twofold).

1.2 Set Theory

sec:set_theory

Chapter 2

Linear Algebra

chp:linear_algebra

sec:alg_structures

2.1 Algebraic Structures

ssec:alg_motivation

Motivation

Although the following sections may seem like a quite boring sea of compact, unreadable equations and proofs, it forms the essence of a modern understanding of most mathematics used today, since it formalizes the relations that sets form with operations acting upon them. This leads to structures like groups, rings or fields, which allow for the generalization of various viewpoints[1] into an abstract form.

Coming from High School, algebra may just sound like solving complicated equations, like $x^2 + 1 = 0$, which in fact forms the origin of the study.

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Improvement 1:
add more and better motivation

ssec:groups

2.1.1 Groups

Definition 2.1.1: Magmata

def:alg_struct

Let X be a set and $\circ\colon X^2 \to X$. We call $(X, \circ)$ an algebraic structure, or magma, iff

$$\forall x_1, x_2 \in X : x_1 \circ x_2 \in X. \tag{2.1}$$

{eq:def_alg_struct}

Algebraic structures are a very broad and basic structure, the fundamental idea of group theory and everything that is build upon it. A trivial simple example is $(\mathbb{N}, +)$, where $+$ denotes regular addition. Contrasting that, $(\mathbb{N}, -)$, with regular subtractions is not an algebraic structure, since $\exists x_1, x_2 \in \mathbb{N}\colon x_1 - x_2 \notin \mathbb{N}$, i.e. $1 - 2 \notin \mathbb{N}$. Hence given any set X paired with an operation $\circ\colon X^2 \to X$, one should prove that $(X, \circ)$ forms an algebraic structure, in order to apply any general proofs or methods. If $(X, \circ)$ is an algebraic structure, then we call X closed under $\circ$. There is no standardized notation for this circumstance, hence if this is the case, you might want to state closure under $\circ$ at the beginning of your work. In order to prove that X is closed under $\circ$, one might negate equation (2.1), yielding

$$\exists x_1, x_2 \in X : x_1 \circ x_2 \notin X. \tag{2.2}$$

{eq:neg_def_alg_struct}

Hence, if proving $(X, \circ)$ is an algebraic structure, we find a pair $x_1, x_2 \in X$, where $x_1 \circ x_2 \notin X$, i.e. $2 - 1$. Generally, when you need to prove such general properties, using a contradiction is the proper method, since finding a counterexample is most of the times a lot easier than proving said property for all elements of X.

Given an algebraic structure $(X, \circ)$, one might find triplets $(x_1, x_2, x_2) \in X^3$, where $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$.

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Improvement 2:
write more intro about associativity

Definition 2.1.2: Associativity and Semi-Groups

def:associativity

Let $(X, \circ)$ be an algebraic structure. We call $\circ$ associative, iff

$$\forall x_1, x_2, x_3 \in X : (x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3). \tag{2.3}$$

{eq:def_associativity}

We then call $(X, \circ)$ a semi- (or half-) group.

Semi-groups are the most basic structure, that are actually useful and have some[2] use in e.g. linear algebra[3].

Theorem 2.1.1: Associativity of Set Operations

thm:set_op_assoc

Let X be a set, then the following structures are semi-groups:

$$\begin{gathered}(\mathcal{P}(X), \cap)\\ (\mathcal{P}(X), \cup)\\ (\mathcal{P}(X), \triangle)\end{gathered} \tag{2.4}$$

{eq:assoc_set_ops}

and $(\mathcal{P}(X), \setminus)$ is just algebraic.

[1] although the most general viewpoint would be in fact Category Theory
[2] although very limited
[3] like $(\mathbb{F}^{m\times n}, \cdot)$ with the regular matrix multiplication

proof:assoc_set_ops

Proof for thm:set_op_assoc *Theorem 2.1.1*. We already proved in that $\cap$, $\cup$ and $\triangle$ are associative, and $\setminus$ is not. Hence we only have to prove that $\forall Y_1, Y_2 \in \mathcal{P}(X) \colon Y_1 \circ Y_2 \in \mathcal{P}(X)$.

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Change 1:
add reference for proof

$$
\begin{aligned}
&Y_1 \cap Y_2 = Y && (\forall y \in Y \colon y \in Y_1 \wedge y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2 \colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y \colon y \in X \\
&Y_1 \cup Y_2 = Y && (\forall y \in Y \colon y \in Y_1 \vee y \in Y_2) \wedge (\forall y_1 \in Y_1, y_2 \in Y_2 \colon y_1 \in X \wedge y_2 \in X) \Rightarrow \forall y \in Y \colon y \in X \\
&Y_1 \triangle Y_2 = Y && (\forall y \in Y \colon (y \in Y_2 \setminus Y_1) \vee (y \in Y_1 \setminus Y_2)) \Rightarrow \forall y \in Y \colon y \in X \\
&Y_1 \setminus Y_2 = Y && \forall y \in Y \colon y \in Y_1 \Rightarrow \forall y \in Y \colon y \in X \\
&Y_2 \setminus Y_1 = Y && \forall y \in Y \colon y \in Y_2 \Rightarrow \forall y \in Y \colon y \in X
\end{aligned}
$$

i.e. $(\mathcal{P}(X), \cap)$, $(\mathcal{P}(X), \cup)$, $(\mathcal{P}(X), \triangle)$ and $(\mathcal{P}(X), \setminus)$ are closed under their relative operation. q.e.d.

Given $(\mathbb{N}, \cdot)$, where $\cdot$ is the common multiplication, we observe that there exists pairs $(a, b) \in \mathbb{N}^2$, where $a \cdot b = b$, which is the case for $a = 1$. We call 1 the left-neutral element of $\cdot$. In contrast, 1 is right-neutral if $b \cdot 1 = b$.

Definition 2.1.3: Neutral Element and Monoids

def:monoids

Let $(X, \circ)$ be a semi-group. We call $e \in X$ neutral, if it is left- and right neutral, i.e. $\forall x \in X \colon e \circ x = x = x \circ e$. If X has a neutral element e, we call $(X, \circ)$ a monoid.

Theorem 2.1.2: Uniqueness of Neutral Elements

thm:neutral_unique

Let $(X, \circ)$ be a monoid with neutral element e, then e is unique.

Proof for thm:neutral_unique *Theorem 2.1.2*. Given a monoid $(X, \circ)$ with neutral elements $e, e' \in X$ it follows:

$$
\begin{aligned}
e' &= e' \circ e = e \\
e &= e \circ e' = e'
\end{aligned}
$$

q.e.d.

Chapter 3

Real Analysis

chp:real_analysis

sec:numb_sequ

3.1 Numbers and Sequences

Although we already introduced the concept of fields in , the number system we use today, i.e. the real numbers $\mathbb{R}$ are typically introduced in lectures about real analysis. Hence we follow this approach, as the concept of the reals encapsulates a good part of what we are trying to establish here.

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Change 2:
add reference to chapter 2

ssec:est_numb_sys

3.1.1 Establishing a Number System

A number system is a set of numbers, typically infinite, which contains objects called numbers. The origin of numbers lies in the counting of objects. Based on our observations on fields, we want to apply the properties of a field to our number system.

Definition 3.1.1: Axioms of PEANO

def:peano

Let $s\colon \mathbb{N} \rightarrowtail \mathbb{N}$ be the successor function:

- $1 \in \mathbb{N}$
- $\forall n \in \mathbb{N}\colon \exists n' \in \mathbb{N}\colon s(n) = s'$
- $\nexists n \in \mathbb{N}\colon s(n) = 1$
- $\forall m, n \in \mathbb{N}\colon s(m) = s(n) \Rightarrow m = n$
- $1 \in X \wedge \forall x \in X : s(x) \in X \Rightarrow \mathbb{N} \subseteq X$

The successor function s is injective and produces the natural numbers by mapping $n \in \mathbb{N}$ to $n + 1$, since the successor of n is $n + 1$. These axioms lead directly to the method of Proof by Induction.

Definition 3.1.2: Induction

def:induction

Let $\mathcal{P}(n)$ be a predicate where n is element of some (ordered) index-family I, where $\forall k \in I : \mathcal{P}(k)$. If we can prove that $\forall n \in \mathbb{N}\colon \mathcal{P}(n) \Rightarrow \mathcal{P}(s(n))$, then $\exists \sigma\colon J \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N}$.

Proofs by induction are very powerful, since we can simply check whether $\mathcal{P}(n) \Rightarrow \mathcal{P}(n + 1)$ for $n \in J$. A very common example is the gaussian sum-formula:

Theorem 3.1.1: Gaussian Sum Formula

thm:gauss_sum

Let $n \in N$, then

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{3.1}$$

{eq:gauss_sum}

Proof. We first find $n_0 \in \mathbb{N}\colon \mathcal{P}(n_0)$. Hence we test $n_0 = 1$:

$$\sum_{k=1}^{1} k = 1$$
$$\frac{2}{2} = 1$$

This checks out, so we may proof $\mathcal{P}(n) \Rightarrow \mathcal{P}(n + 1)$, i.e.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \Rightarrow \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

We use the fact, that we can split up sums into their summands:

$$\sum_{k=1}^{n+1} k = n + 1 + \sum_{k=1}^{n} k = n + 1 + \frac{n(n+1)}{2} = \frac{2n + 2 + n^2 + n}{2} = \frac{(n+1)(n+2)}{2}$$

Where the last equality follows by $(n + 1)(n + 2) = n^2 + 3n + 2$. q.e.d.