Divide And Conquer

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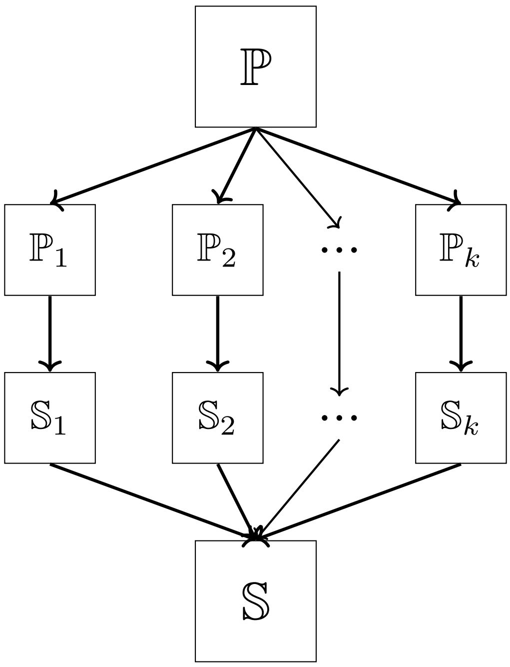
Abstract

Suppose we got a bigger problem, call it P of size n. Sometimes bigger problem are really daunting and we end up doing nothing. Instead, why don't we split the problem into multiple pieces. and solve them and combine the result. Its like we split the work into several pieces and assign to workers independently. So, each worker doesn't feel so hard about solving the problem. So we employ *DivideAndConquer* strategy.

1 Introduction



As discussed earlier, we are dividing a task into several pieces and solve them individually and independently.

1.1 Conditions for Divide and Conquer

1. All the sub-problems should be the same task as the main problem.
2. There should be an known method to combine the solutions at least

Since all the sub-problems are the same task as the parent problem, This is a recursive algorithm.

1.2 Applications of Divide and Conquer

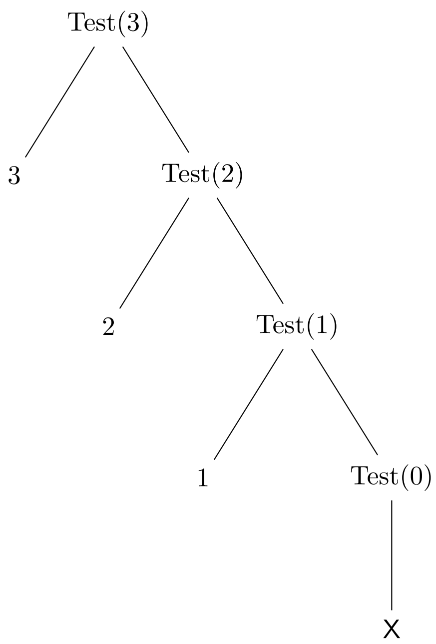
1. Binary search
2. Finding Minimum and Maximum
3. Merge Sort
4. Quick Sort
5. Stressen's Matrix Multiplication

2 Recurrence Relation

To analyze the time complexity of divide and conquer algorithms, we often use recurrence relations. For example:

```
void Test(int n){
    if (n > 0){
        printf("%d", n);
        Test(n-1);
    }
}

Test(3)
```



Since the only statement in the function is printf, which take $O(1)$ time, at each recursive step $O(1)$ is done. If $Test(n)$ is executed, the number of recursive calls are $n+1$, the number of printf calls are n. So the time complexity of this algorithm is $On(n)$

This won't be possible in every algorithm to open the function and count them and make assumption about n cases. Mathematically, That is not sufficient. Thus, Recurrence Relation Come into play.

Lets take the same code we used before:

Algorithm 1 Simple Printing with recursion

procedure TEST(int n) ▷ Assume it takes $T(n)$
    if $n > 0$ **then**
        $print(n)$ ▷ $O(1)$
        $Test(n-1)$ ▷ $T(n-1)$
    end if
end procedure

So,

$$T(n) = \begin{cases} T(n-1)+1 & n>0, \\ 1 & n=0 \end{cases}$$

$$\begin{aligned}
\because \quad T(n) &= T(n-1)+1, \\
T(n-1) &= T(n-2)+1, \\
\text{Substituting(3) in (1)},& \\
T(n) &= [T(n-2)+1]+1 = T(n-2)+2, \\
T(n) &= [T(n-3)+1]+2 = T(n-3)+3, \\
T(n) &= \quad \vdots \quad + \quad \vdots, \\
T(n) &= T(n-k)+k, \\
\because \quad T(0) &= 1, \\
\text{if (k = n)},& \\
T(n) &= T(0)+n, \\
T(n) &= 1+n.
\end{aligned}$$

This algorithm is O(n)

Lets look Another example

Algorithm 2 Simple Printing with recursion

```
procedure TEST(int n)                        ▷ Assume it takes T(n)
    if n > 0 then
        for i ← 0 to n − 1 do                ▷ n − 1
            print(n)                         ▷ Everything inside for will be T(n) = n
        end for
        Test(n − 1)                          ▷ T(n − 1)
    end if
end procedure
```

So,

$$T(n) = T(n-1)+2n+2$$

We need to take Asymptotic Notation

$$T(n) \simeq T(n-1)+n$$

$$T(n) = \begin{cases} T(n-1)+n & n>0, \\ 1 & n=0 \end{cases}$$

$$\begin{aligned}
\because \quad T(n) &= T(n-1)+n, \\
T(n-1) &= T(n-2)+n-1, \\
\text{Substituting(3) in (1)},& \\
T(n) &= [T(n-2)+n-1]+n = T(n-2)+(n-1)+n, \\
T(n) &= [T(n-3)+n-2]+(n-1)+n = T(n-3)+(n-2)+(n-1)+n, \\
T(n) &= \quad \vdots \quad + \quad \vdots, \\
T(n) &= T(n-k)+(n-(k-1))+\cdots+(n-1)+n, \\
\because \quad T(0) &= 1, \\
\text{if (k = n)},& \\
T(n) &= T(0)+\frac{n(n+1)}{2} = 1+\frac{n(n+1)}{2}, \\
T(n) &\approx \frac{n(n+1)}{2} \approx \frac{n^2+n}{2}
\end{aligned}$$

This algorithm is $O(n^2)$

Lets look Another example

Algorithm 3 Simple Printing with recursion

```
procedure TEST(int n)                                        ▷ Assume it takes T(n)
    if n > 0 then
        for i ← 0 to n − 1 step 2 · i do                    ▷ This is
log(n) + 1, Wecanalsocalculatewithouttakingthisintoconsideration
            print(n)                                         ▷ This takeslog n
        end for
        Test(n − 1)                                          ▷ T(n − 1)
    end if
end procedure
```

Although I know we excluded many of the time complexties simply because we wont using it in asymtotic equation

After taking Asymptotic Notation

$$T(n) = T(n-1) + \log n \tag{1}$$

$$T(n) = \begin{cases} T(n-1) + \log n & n > 0, \\ 1 & n = 0 \end{cases} \tag{2}$$

$$\because \quad T(n) = T(n-1) + \log n$$

$$T(n-1) = T(n-2) + \log(n-1) \tag{3}$$

Substitutingeqn (3) in (1),

$$\begin{aligned} T(n) &= [T(n-2) + \log(n-1)] + \log(n) \\ &= T(n-2) + \log(n-1) + \log n \\ &= T(n-2) + \log(n^2 - n), \\ T(n) &= [T(n-3) + n - 2] + (n-1) + n \\ &= T(n-3) + \log(n-2) + \log(n-1) + \log n \\ &= T(n-3) + \log((n-2) \cdot (n-1) \cdot (n)) \end{aligned}$$

$$T(n) = T(n-k) + \log\left(\prod_{i=0}^{k-1}(n-i)\right) \tag{4}$$

$$T(n) = T(n-k) + \log\left(\frac{n!}{n-(k-2)!}\right)$$

$$\because \quad T(0) = 1,$$

if (k = n),

$$T(n) = T(0) + \log\left(\frac{n!}{2}\right) = 1 + \cdot \left(\frac{n!}{2}\right)$$

$$T(n) \approx \log n! \approx \log n!$$

Upper bound of $\log n!$ is $O(n \cdot \log n)$

This algorithm is $O(n \cdot \log n)$

OR

From eqn (4) :

$$T(n) = T(n-k) + \log\left(\prod_{i=0}^{k-1}(n-i)\right)$$

By Binomial Theorem,

$$T(n) = T(n-k) + \log(n^k + \cdots)$$

$$T(n) \approx T(n-k) + k \cdot \log n$$

$$\because \quad T(0) = 1,$$

if (k = n),

$$T(n) = T(0) + n \cdot logn$$

This algorithm is $O(n \cdot \log n)$