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16

MODAL LOGIC IN MATHEMATICS

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1 INTRODUCTION

Formal modal logic is mostly mathematical in its methods, regardless of area of application. This Handbook presents a wide variety of mathematical techniques developed over decades of studying the intricate details of modal logic. Also included among relatively recent general purpose sources on the mathematics of modal logic are monographs [57, 75, 99, 114, 153] and a survey paper [115]. For that matter, the applications of mathematics in modal logic are overwhelming, while those in the dual category, the uses of modal logic in mathematics, are less numerous.

Mathematics normally finds a proper language and level of abstraction for the study of its objects. Propositional modal logic offers a new paradigm of applying logical methods: instead of using the traditional languages with quantification (first-order or higher-order) to describe a structure, look for an appropriate quantifier-free language with additional logic operators (modalities) that represent the phenomenon at hand. In a number of prominent cases, we end up with a logic-based language which is much richer than Boolean logic, and yet, unlike universal languages with quantification, does not fall under the scope of classical undecidability limitations. Modal logic often offers better decidability and complexity results than the rival first-order logic.

We adopt a strict approach as to what constitutes an application of modal logic in mathematics, i.e., we limit our attention to mathematical objects which existed independently of modal logic, rather than those developed for the needs of modal logic itself. This requirement is not by any means sufficient; after all, any class of binary relations in mathematics specifies some propositional modal logic which, however, does not automatically make these connections worthy of study. We consider only the cases in which a mathematical modality-like notion was developed and studied by mathematicians to the extent that the modal logical language and methods became pertinent. Neither is this requirement necessary; for example, elaborate algebraic models originally developed for the needs of logic (e.g., modal logic) are now deeply embedded into the corresponding field of mathematics and may well be regarded as a contribution of modal logic to mathematics. Fortunately, algebraic models for modal logic have been covered in Chapter 6 of this Handbook. Moreover, the present author has not been quite pedantic in carrying out even this imperfect approach; such important issues as topos models and the connection between modal logic and Grothendieck topology on categories were barely mentioned in this survey. Some of these topics were considered in Chapter 9 of this Handbook.

There are two major ideas that dominate the landscape of modal logic application in mathematics: Gödel's provability semantics and Tarski's topological semantics.

Gödel's use of modal logic to describe provability in the 1930s gave the first exact semantics of modality. This approach led to a comprehensive provability semantics for a broad class of modal logics, including the major ones: K, T, $\mathsf{K4}$, $\mathsf{S4}$, $\mathsf{S5}$, GL, Grz, and others. It also proved vital for such applications as the Brouwer-Heyting-Kolmogorov (intended) provability semantics for intuitionistic logic, for introducing justification into formal epistemology and tackling its logical omniscience problem, for introducing self-reference into combinatory logic and lambda-calculi, etc.

Another major use of modal logic in mathematics is the topological semantics suggested by Tarski and developed by Tarski and McKinsey in the 1940s. Here modal logic provides a natural high-level language for describing topology in a point-free manner. In addition to its natural mathematical appeal, this approach has evolved into an active research area with applications in dynamic systems, control systems, spatio-temporal reasoning, etc.

There has also been significant research activity in applying modal logic to set theory, which can be traced back to Solovay's work of the 1970s. We devote Section 7 to this issue.

The reader might perceive a certain bias towards provability logic in this survey. A possible explanation is that Gödel's provability semantics of modal logic is the oldest and arguably the most well-established tradition of applying modal logic to mathematics. It is perhaps more essential for proof theory and foundations than other applications of modal logic for the corresponding object areas of mathematics. This observation is not intended to discount other interpretations of modal logic considered here; we hope that this survey gives a fair assessment of their beauty and vast potential.

Among other recent surveys in this area, we recommend the article 'Provability logic' by Verbrugge in the Stanford Encyclopedia of Philosophy
`http://plato.stanford.edu/entries/logic-provability/`,
the handbook chapter 'Provability Logic' [25], and the forthcoming collection 'The Logic of Space' edited by Aiello, van Benthem, and Pratt-Hartmann.

2 SOME HISTORY

In his 1933 paper [109], Gödel chose the language of propositional modal logic to describe the basic logical laws of provability. According to his approach, $\Box F$ should be interpreted informally as

F is provable,

and the classical modal logic $\mathsf{S4}$ provides a system of plausible postulates for provability. Gödel's goal was to provide an exact interpretation of intuitionistic propositional logic within a classical modal logic of provability, hence giving classical meaning to the basic intuitionistic logical system.

This line of research attracted a great deal of attention in mathematics and eventually led to two distinct models of provability based on modal logics:

1. the Provability Logic GL, which was shown by Solovay to be the logic of Gödel's formal provability predicate;

2. Gödel's original logic $\mathsf{S4}$, which was shown by Artemov to be a forgetful projection of the Logic of Proofs LP.

These two models complement each other and cover a wide range of applications, from traditional proof theory to formal verification and epistemology.

The use of modal logic in topology was initially motivated by Kuratowski's axioms for topological spaces, which were recast in the manner of modal logic by Tarski in the late 1930s. Under this interpretation, the Boolean components were treated in the usual set theoretical way as subsets of a given topological set, whereas $\Box$ was interpreted as

the interior operator.

In their seminal paper of 1944 [187], McKinsey and Tarski proved that $\mathsf{S4}$ was the logic of any separable dense-in-itself metric space, in particular the real topological space $\mathbb{R}^n$, for each $n = 1, 2, 3, \ldots$. Among other known topological operators on sets,

the derived set operator,

regarded as the modality $\Diamond$, has been given a complete axiomatization in works by Esakia and Shehtman. The modal logic of topology developed into an area of research that included modal studies of other operators in topological spaces, modal logic of metric spaces, dynamic topological logic, spatio-temporal reasoning, etc., with applications outside the original mathematical core.

It was perhaps Solovay who initiated research in the application of modal logic to set theory in 1976 when he gave a complete axiomatization of such modalities as

true in all transitive models of ZF

and

true in all universes.

Hamkins and Löwe recently found a complete axiom system of the modality

true in all forcing extensions.

Studies of connections between infinitary modal logic and set theory initiated by Barwise and Moss in 1996 culminated in Baltag's Structural Theory of Sets STS, which considered

the canonical model of infinitary modal logic as the set theoretical universe.

3 TWO MODELS OF PROVABILITY

According to Brouwer, truth in intuitionistic mathematics means the existence of a proof. An axiom system for intuitionistic logic was suggested by Heyting in 1930; its full description may be found in the fundamental monographs [132, 149, 246]. By IPC, we infer Heyting's intuitionistic propositional calculus. In 1931–34, Heyting and Kolmogorov gave an informal description of the intended proof-based semantics for intuitionistic logic [130, 131, 132, 150], which is now referred to as the *Brouwer-Heyting-Kolmogorov* (*BHK*) *semantics*. According to the *BHK*-conditions, a formula is 'true' if it has a proof. Furthermore, a proof of a compound statement is connected to proofs of its parts in the following way:

- a proof of $A \wedge B$ consists of a proof of proposition A and a proof of proposition B,
- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,
- a proof of $A \rightarrow B$ is a construction transforming proofs of A into proofs of B,
- falsehood $\perp$ is a proposition which has no proof; $\neg A$ is shorthand for $A \rightarrow \perp$.

From a foundational point of view, it did not make much sense to understand the above 'proofs' as proofs in an intuitionistic system which those conditions were supposed to specify. So in 1933 ([109]), Gödel took the first step towards developing an exact semantics for intuitionism based on **classical provability**. Gödel considered the classical modal logic $\mathsf{S4}$ to be a calculus describing properties of provability in classical mathematics:

1. *Axioms and rules of classical propositional logic*,
2. $\Box(F \to G) \to (\Box F \to \Box G)$,
3. $\Box F \to F$,
4. $\Box F \to \Box\Box F$,
5. *Rule of necessitation*: $\dfrac{\vdash F}{\vdash \Box F}$.

Based on Brouwer's understanding of logical truth as provability, Gödel defined a translation $tr(F)$ of the propositional formula F in the intuitionistic language into the language of classical modal logic, i.e., $tr(F)$ was obtained by prefixing every subformula of F with the provability modality $\Box$. Informally speaking, when the usual procedure of determining classical truth of a formula is applied to $tr(F)$, it will test the provability (not the truth) of each of F's subformulas in agreement with Brouwer's ideas.

> Gödel's treatment of provability as modality in [109] has an interesting prehistory. In his letter to Gödel [263] of January 12, 1931, von Neumann actually used formal provability as a modal-like operator B and gave a shorter, modal-style derivation of the second Gödel's incompleteness theorem. Von Neumann freely used such modal logic features as the transitivity axiom $B(a) \to B(B(a))$, equivalent substitution, and the fact that the modality commutes with the conjunction '$\wedge$.' Even earlier, in 1928, Orlov published the paper [205] in Russian, in which he considered an informal modal-like operator of provability, introduced modal postulates (ii)–(v), and described the translation $tr(F)$ from propositional formulas to modal formulas. On the other hand, Orlov chose to base his modal system on a type of relevance logic; his system fell short of $\mathsf{S4}$.

From Gödel's results in [109], and the McKinsey-Tarski work on topological semantics for modal logic [188], it follows that the translation $tr(F)$ provides a proper embedding of the intuitionistic logic IPC into $\mathsf{S4}$, i.e., an embedding of IPC into classical logic extended by the provability operator.

THEOREM 1 (Gödel, McKinsey, Tarski). IPC *proves* $F \quad \Leftrightarrow \quad \mathsf{S4}$ *proves* $tr(F)$.

Still, Gödel's original goal of defining IPC in terms of classical provability was not reached, since the connection of $\mathsf{S4}$ to the usual mathematical notion of provability was not established. Moreover, Gödel noticed that the straightforward idea of interpreting modality $\Box F$ as *F is provable in a given formal system T* contradicted Gödel's second incompleteness theorem (cf. [62, 65, 90, 126, 240] for basic information concerning proof and provability predicates, as well as Gödel's incompleteness theorems).

Indeed, $\Box(\Box F \to F)$ can be derived in $\mathsf{S4}$ by the rule of necessitation from the axiom $\Box F \to F$. On the other hand, interpreting modality $\Box$ as the predicate $\mathsf{Provable}_T(\cdot)$ of formal provability in theory T and F as contradiction, i.e., $0 = 1$, converts this formula into a false statement that the consistency of T is internally provable in T:

$$\mathsf{Provable}_T\big(\lceil \mathit{Consis(T)} \rceil\big)\ .$$

To see this, it suffices to notice that the following formulas are provably equivalent in T:

$$\mathsf{Provable}_T(\lceil 0=1 \rceil) \to (0=1)\ , \quad \neg \mathsf{Provable}_T(\lceil 0=1 \rceil)\ , \quad \mathit{Consis(T)}\ .$$

Here $\lceil \varphi \rceil$ stands for the Gödel number of φ. Below we will omit Gödel number notation whenever it is safe, e.g., we will write $\mathsf{Provable}(\varphi)$ and $\mathsf{Proof}(n, \varphi)$ instead of $\mathsf{Provable}(\lceil \varphi \rceil)$ and $\mathsf{Proof}(n, \lceil \varphi \rceil)$.

The situation after Gödel's paper [109] can be described by the following figure where '$\hookrightarrow$' denotes a proper embedding:

$$\mathsf{IPC} \ \hookrightarrow\ \mathsf{S4} \ \hookrightarrow\ ? \ \hookrightarrow\ \mathit{CLASSICAL\ PROOFS}\ .$$

In a public lecture in Vienna in 1938 [110], Gödel suggested using the format of explicit proofs *t is a proof of F* for interpreting his provability calculus $\mathsf{S4}$, though he did not give a complete set of principles of the resulting logic of proofs. Unfortunately, Gödel's work [110] remained unpublished until 1995, when the Gödelian logic of proofs had already been axiomatized and supplied with completeness theorems connecting it to both $\mathsf{S4}$ and classical proofs.

The provability semantics of $\mathsf{S4}$ was discussed in [62, 65, 70, 111, 158, 169, 176, 191, 197, 199, 200, 221, 222]. These works constitute a remarkable contribution to this area (cf. Section 4), however, they neither found the Gödelian logic of proofs nor provided $\mathsf{S4}$ with a provability interpretation capable of modeling the *BHK* semantics for intuitionistic logic. Comprehensive surveys of work on provability semantics for $\mathsf{S4}$ may be found in [16, 21, 25].

The Logic of Proofs LP was first reported in 1994 at a seminar in Amsterdam and at a conference in Münster. Complete proofs of the main theorems of the realizability of $\mathsf{S4}$ in LP, and about the completeness of LP with respect to the standard provability semantics were published in the technical report [14] in 1995. The foundational picture now is

$$\mathsf{IPC} \ \hookrightarrow\ \mathsf{S4} \ \hookrightarrow\ \mathsf{LP} \ \hookrightarrow\ \mathit{CLASSICAL\ PROOFS}\ .$$

The correspondence between intuitionistic and modal logics induced by Gödel's translation $tr(F)$ has been studied by Blok, Dummett, Esakia, Flagg, H. Friedman, Grzegorczyk, Kuznetsov, Lemmon, Maksimova, McKinsey, Muravitsky, Rybakov, Shavrukov, Tarski, and many others. A detailed survey of modal companions of intermediate (or superintuitionistic) logics is given in [74]; a brief one is in [75], Sections 9.6 and 9.8.

Gödel's 1933 paper [109] on a modal logic of provability left two natural open problems:

(A) Find a precise provability semantics for the modal logic $\mathsf{S4}$, which appeared to be 'a provability calculus without a provability semantics.'

(B) Find a modal logic of Gödel's predicate of formal provability $\mathsf{Provable}(x)$, which appeared to be 'a provability semantics without a calculus.'

The solution to problem (A) was obtained by Artemov through the Logic of Proofs LP (see above and Section 5).

Problem (B) was solved in 1976 by Solovay, who showed that the modal logic GL (a.k.a. G, L, K4.W, PRL) axiomatized all propositional properties of the provability predicate $\mathsf{Provable}(F)$ ([62, 65, 147, 241, 242]).

The provability logic GL is given by the following list of postulates:

1. *Axioms and rules of classical propositional logic*,
2. $\Box(F\rightarrow G)\rightarrow(\Box F\rightarrow\Box G)$,
3. $\Box(\Box F\rightarrow F)\rightarrow\Box F$,
4. $\Box F\rightarrow\Box\Box F$,
5. *Rule of necessitation*: $\dfrac{\vdash F}{\vdash \Box F}$.

Models (A) and (B) have quite different expressive capabilities. The logic GL formalizes Gödel's second incompleteness theorem $\neg\Box(\neg\Box\bot)$, Löb's theorem $\Box(\Box F\rightarrow F)\rightarrow\Box F$, and a number of other meaningful provability principles. However, proofs as objects are not present in this model. LP naturally contains typed λ-calculus, modal logic, and modal λ-calculus ([18, 19]). On the other hand, model (A) cannot express Gödel's incompleteness theorem.

Provability models (A) and (B) complement each other by addressing different areas of application. The provability logic GL finds applications in traditional proof theory (cf. Subsection 4.11). The Logic of Proofs LP targets areas of typed theories and programming languages, foundations of verification, formal epistemology, etc. (cf. Subsection 5.7)

4 PROVABILITY LOGIC

A significant step towards finding a modal logic of formal provability was made by Löb who formulated in [180], on the basis of previous work by Hilbert and Bernays from 1939 (see [133]), a number of natural modal-style properties of the formal provability predicate and observed that these properties were sufficient to prove Gödel's second incompleteness theorem. These properties, known as the *Hilbert-Bernays-Löb derivability conditions*, essentially coincide with postulates (2), (4), and (5) of the above formulation of GL, i.e., with the modal logic K4. Moreover, Löb found an important strengthening of the Gödel theorem. He established the validity of the following *Löb Rule* about formal provability:

$$\frac{\vdash \Box F\rightarrow F}{\vdash F}\ .$$

It was later noticed in (cf. [182]) that this rule can be formalized in arithmetic, which gave a valid law of formal provability known as *Löb's principle*:

$$\Box(\Box F\rightarrow F)\rightarrow\Box F\ .$$

This principle provided the last axiom of the provability logic GL, named after Gödel and Löb. Neither Gödel nor Löb formulated the logic explicitly, though they established the validity of the underlying arithmetical principles. Presumably, it was Smiley, whose work [238] on the foundations of ethics was the first to consider GL a modal logic.

Significant progress in the general understanding of the formalization of metamathematics, particularly in [90], inspired Kripke, Boolos, de Jongh, and others to look into the problem of modal axiomatization of the logic of provability. More specifically, the effort was concentrated on establishing GL's completeness with respect to the formal provability interpretation. Independently, a similar problem in an algebraic context was considered by Magari and his school in Italy (see [184]). A comprehensive account of these early developments in provability logic can be found in [66].

H. Friedman formulated the question of decidability of the letterless fragment of provability logic as his Problem 35 in [97]. This question, which happened to be much easier than the general case, was immediately answered by a number of people including Boolos [60], van Benthem, Bernardi, and Montagna. This result was apparently known to von Neumann as early as 1931 [263].

4.1 Solovay's completeness theorem

The modal logic of Gödel's predicate of formal provability $\mathsf{Provable}(x)$ was found in 1976 by Solovay.

Let $*$ be a mapping from the set of propositional letters to the set of arithmetical sentences. We call such a mapping an (arithmetical) *interpretation*. Given a standard provability predicate $\mathsf{Provable}(x)$ in Peano arithmetic PA, we can extend the interpretation $*$ to all modal formulas as follows:

- $\bot^* = \bot$; $\top^* = \top$;
- $*$ commutes with all Boolean connectives;
- $(\Box G)^* = \mathsf{Provable}(G^*)$.

The Hilbert-Bernays-Löb derivability conditions, together with the validity of Löb's principle, essentially mean that GL is sound with respect to the arithmetical interpretation.

PROPOSITION 2. *If* $\mathsf{GL} \vdash X$, *then for all interpretations* $*$, $\mathsf{PA} \vdash X^*$.

Solovay in [242] established that GL is also complete with respect to the arithmetical interpretation. Solovay also showed that the set of modal formulas expressing universally *true* principles of provability was axiomatized by a decidable extension of GL, which is usually denoted by S. The system S has the axioms

- all theorems of GL (a decidable set),
- $\Box X \rightarrow X$,

and *modus ponens* as the sole rule of inference.

THEOREM 3 (Solovay [242]).

(1) $\mathsf{GL} \vdash X$ *iff for all interpretations* $*$, $\mathsf{PA} \vdash X^*$;

(2) $\mathsf{S} \vdash X$ *iff for all interpretations* $*$, X^* *is true.*

For the proof of this theorem in [242], Solovay invented an elegant technique of embedding Kripke models into arithmetic. Variants and generalizations of this construction have been applied to obtain arithmetical completeness results for various logics with provability and interpretability semantics. An inspection of Solovay's construction shows that it works for all natural formal theories containing a rather weak *elementary arithmetic* EA (cf [25], section 3.1). Such robustness allows us to claim that GL is indeed a universal propositional logic of formal provability.

Whether or not Solovay's theorem can be extended to bounded arithmetic theories such as S^1_2 or S_2 remains an intriguing open question. Interesting partial results here were obtained by Berarducci and Verbrugge in [53].

Solovay's results and methods opened a new page in the development of provability logic. Several groups of researchers in the USA (Solovay, Boolos, Smoryński), the Netherlands (D. de Jongh, Visser), Italy (Magari, Montagna, Sambin, Valentini), and USSR (Artemov and his students), have started to work intensively in this area. An early textbook by Boolos [62], followed by Smoryński's [241], played an important educational role.

The following uniform version of Solovay's Theorem 3.1 was established independently by Artemov, Avron, Boolos, Montagna, and Visser [7, 8, 37, 63, 194, 253]:

there is an arithmetical interpretation $*$ *such that for each modal formula* X, $\mathsf{PA} \vdash X^*$ *iff* $\mathsf{GL} \vdash X$.

The main thrust of the research efforts in the wake of Solovay's Theorem was in the direction of generalizing Solovay's results to more expressive languages. Some of the problems that have received prominent attention are covered below.

4.2 Fixed point theorem

As an important early result on the application of modal logic to the study of the concept of provability in formal systems, a theorem stands out that was found independently by de Jongh and Sambin, who established that GL has the fixed point property (see [62, 65, 240, 241]). The de Jongh-Sambin fixed point theorem is a striking reproduction of Gödel's fixed point lemma in a propositional language free of coding, self-substitution functions, etc.

A modal formula $F(p)$ is said to be *modalized in* p if every occurrence of the sentence letter p in $F(p)$ is within the scope of $\Box$.

THEOREM 4 (de Jongh, Sambin). *For every modal formula* $F(p)$ *modalized in the sentence letter* p*, there is a modal formula* H *containing only sentence letters from* F*, not containing* p*, and such that* GL *proves*

$$H \leftrightarrow F(H) .$$

Moreover, any two solutions to this fixed-point equation with respect to F *are provably equivalent in* GL.

The uniqueness segment was also established by Bernardi in [54].

The proof actually provided an efficient algorithm that, given F, calculates its fixed point H. Here are some examples of F's and their fixed points H.

Modal formula $F(p)$	Its fixed point H
$\Box p$	$\top$
$\Box\neg p$	$\Box\bot$
$\neg\Box p$	$\neg\Box\bot$
$\neg\Box\neg p$	$\bot$
$q \wedge \Box p$	$q \wedge \Box q$

Perhaps the most famous fixed point of the above sort is given by the second Gödel incompleteness theorem. Indeed, consider $\neg\Box p$ as $F(p)$. By the above table, the corresponding fixed point H is $\neg\Box\bot$. Hence GL proves

(1) $\neg\Box\bot \rightarrow \neg\Box(\neg\Box\bot)$.

Since the arithmetical interpretation of $\neg\Box\bot$ for a given theory T is the consistency formula *Consis(T)*, this yields that (1) represents the formalized second Gödel incompleteness theorem:

if T is consistent, then T does not prove its consistency

and that this theorem is provable in T.

The fixed point theorem for GL allowed van Benthem [248] and then Visser [262] to interpret the modal μ-calculus in GL. Together with van Benthem's observation that GL is faithfully interpretable in μ-calculus [248], this relates two originally disjoint research areas.

4.3 First-order provability logics

The natural problem of axiomatizing first-order provability logic was first introduced by Boolos in [62, 64] as the major open question in this area. A straightforward conjecture that the first-order version of GL axiomatizes first-order provability logic was shown to be false by Montagna [196]. A final negative solution was given in papers by Artemov [9] and Vardanyan [252].

THEOREM 5 (Artemov, Vardanyan). *First-order provability logic is not recursively axiomatizable.*

In particular, Artemov showed that the set of the first-order modal formulas that are true under any arithmetical interpretation is not arithmetical. This proof used Tennenbaum's well-known theorem about the uniqueness of the recursive model of Peano arithmetic. Vardanyan showed that the set of first-order modal formulas that are provable in PA under any interpretation is Π^0_2-complete, thus not effectively axiomatizable. Independently but somewhat later, similar results were obtained by McGee in his Ph.D. thesis; they were never published.

Even more dramatically, [11] showed that first-order provability logics are sensitive to a particular formalization of the provability predicate and thus are not robustly defined.

The material on first-order provability logic is extensively covered in a textbook [65] and in a survey [147].

4.4 Intuitionistic provability logic

The question of generalizing Solovay's results from classical theories to intuitionistic ones, such as Heyting arithmetic HA, proved to be remarkably difficult. Visser in [253] found a number of nontrivial principles of the provability logic of HA. Similar observations were independently made by Gargov and Gavrilenko. In [255], a characterization and a decision algorithm for the letterless fragment of the provability logic of HA were obtained, thus solving an intuitionistic analog of Friedman's 35th problem.

THEOREM 6 (Visser [255]). *The letterless fragment of the provability logic of* HA *is decidable.*

Some significant further results were obtained in [79, 135, 136, 137, 255, 258, 260, 261], but the general problem of axiomatizing the provability logic of HA remains a major open question.

4.5 Classification of provability logics

Solovay's theorems naturally led to the notion of *provability logic for a given theory T relative to a metatheory U*, which was suggested by Artemov in [7, 8] and Visser in [253]. This logic, denoted $\boldsymbol{PL}_T(U)$, is defined as the set of all propositional principles of provability in T that can be established by means of U. In particular, GL is the provability logic $\boldsymbol{PL}_T(U)$ with $U = T = \mathsf{PA}$, and Solovay's provability logic S from Theorem 3.2 corresponds to $T = \mathsf{PA}$ and U's being the set of all true sentences of arithmetic. The problem of describing all provability logics for a given theory T relative to a metatheory U, where T and U range over extensions of Peano arithmetic, has become known as the *classification problem for provability logics.* Each of these logics extends GL, hence can be represented in the form $\mathsf{GL}X$ which is GL with additional axioms X and modus ponens as the sole rule of inference. Within this notational convention, $\mathsf{S}{=}\mathsf{GL}\{\Box p \to p\}$. Consider sentences $F_n = \Box^{n+1}\bot \to \Box^n\bot$, for $n \in \omega$. In [8, 10, 254], the following three families of provability logics were found:

$$\mathsf{GL}_\alpha = \mathsf{GL}\{F_n \mid n \in \alpha\},\ \text{where } \alpha \subseteq \omega\ ;$$

$$\mathsf{GL}^-_\beta = \mathsf{GL}\{\bigvee_{n\notin\beta} \neg F_n\},\ \text{where } \beta \text{ is a confinite subset of } \omega\ ;$$

$$\mathsf{S}_\beta = \mathsf{S} \cap \mathsf{GL}^-_\beta,\ \text{where } \beta \text{ is a confinite subset of } \omega\ .$$

The families GL_α, GL^-_β and S_β are ordered by inclusion of their indices, and $\mathsf{GL}_\beta \subset \mathsf{S}_\beta \subset \mathsf{GL}^-_\beta$, for cofinite β.

In [10], the classification problem was reduced to finding all provability logics in the interval between GL_ω and S. In [143], Japaridze found a new provability logic Dzh in this interval,

$$\mathsf{Dzh} = \mathsf{GL}\{\neg\Box\bot, \Box(\Box p \vee \Box q) \to (\Box p \vee \Box q)\}\ .$$

He showed that Dzh is the provability logic of PA relative to $\mathsf{PA}+$ *formalized ω-consistency of* PA. This discovery produced one more provability logic series,

$$\mathsf{Dzh}_\beta = \mathsf{Dzh} \cap \mathsf{GL}^-_\beta,\ \text{where } \beta \text{ is a confinite subset of } \omega\ ,$$

with $\mathsf{GL}_\beta \subset \mathsf{Dzh}_\beta \subset \mathsf{S}_\beta \subset \mathsf{GL}^-_\beta$, for cofinite β.

The classification was completed by Beklemishev who showed in [42] that no more provability logics exist.

THEOREM 7 (Beklemishev [42]). *All provability logics occur in* GL_α, GL^-_β, S_β, *and* Dzh_β, *for* $\alpha, \beta \subseteq \omega$, *and* β *cofinite.*

The proof of Theorem 7 produced yet another provability interpretation of Dzh which was shown to be the provability logic of any Σ_1-sound but not sound theory relative to the set of all true sentences of arithmetic. For more details, see [25, 42, 50].

4.6 Provability logics with additional operators

Solovay's theorems have been generalized to various extensions of the propositional language by additional operators having arithmetical interpretations.

The most straightforward generalization is obtained by simultaneously considering several provability operators corresponding to different theories. Already in the simplest case of *bimodal provability logic*, the axiomatization of such logics turns out to be very difficult. The bimodal logics for many natural pairs of theories have been characterized in [43, 44, 73, 143, 241]. However, the general classification problem for bimodal provability logics for pairs of recursively enumerable extensions of PA remains a major open question.

Bimodal logic has been used to study relationships between provability and interesting related concepts such as the Mostowski operator, and Rosser, Feferman, and Parikh provabilities (see [179, 225, 226, 241, 256]). In a number of cases, Solovay-style arithmetical completeness theorems have been obtained. These results have their origin in an important paper by Guaspari and Solovay [123] (see also [241]). They consider an extension of the propositional modal language by a *witness comparison* operator allowing the formalization of Rosser-style arguments from his well-known proof of the incompleteness theorem [218]. Similar logics have since been used in [71, 72, 78], e.g., in the study of the speed-up of proofs.

4.7 Generalized provability predicates

A natural generalization of the provability predicate is given by the notion of n-*provability* which is, by definition, a provability predicate in the set of all true arithmetical Π_n-sentences. For $n = 0$, this concept coincides with the usual notion of provability. As was observed in [241], the logic of each individual n-provability predicate coincides with GL. A joint logic of n-provability predicates for $n = 0, 1, 2, \ldots$ contains the modalities [0], [1], [2], etc. The arithmetical interpretation of a formula in this language is defined as usual, except that now we require, for each $n \in \omega$, that $[n]$ be interpreted as n-provability.

The system GLP introduced by Japaridze [143, 144] is given by the following axioms and rules of inference.

1. *Axioms of* GL *for each operator* $[n]$,
2. $[m]\phi \rightarrow [n]\phi$, *for* $m \leq n$,
3. $\langle m\rangle\phi \rightarrow [n]\langle m\rangle\phi$, *for* $m < n$,
4. *Rule modus ponens*,
5. *Rule* $\phi \vdash [n]\phi$.

THEOREM 8 (Japaridze). GLP *is sound and complete with respect to the n-provability interpretation.*

Originally, Japaridze established in [143, 144] the completeness of GLP for an interpretation of modalities $[n]$ as the provability in arithmetic using not more than n nested applications of the ω-rule. Later, Ignatiev in [141] observed that Japaridze's theorem holds for the n-provability interpretation. Ignatiev also found normal forms for letterless formulas in GLP which play a significant role in Section 4.11 (where only the soundness of GLP is essential).

4.8 Interpretability and conservativity logics

Interpretability is one of the central concepts of mathematics and logic. A theory X is interpretable in Y iff the language of X can be translated into the language of Y in such a way that Y proves the translation of every theorem of X. For example, Peano arithmetic PA is interpretable in Zermelo-Fraenkel set theory ZF. The importance of this concept lies in its ability to compare theories of different mathematical character in different languages, e.g., set theory and arithmetic. The notion of interpretability was given a mathematical shape by Tarski in 1953 in [245]. There is not much known about interpretability in general. The modal logic approach provides insights into the structure of interpretability in special situations when X and Y are finite propositional-style extensions of a base theory containing a certain sufficient amount of arithmetic.

Visser, following Švejdar [243], introduced a binary modality $A \rhd B$ to stand for the arithmetization of the statement

$$\textit{the theory } T + A \textit{ interprets } T + B.$$

Interpretations here are understood in the standard sense of Tarski, and are limited to theories T containing a sufficient amount of arithmetic, and to propositional A's and B's. This new modality emulates provability $\Box F$ by $\neg F \rhd \bot$, and thus is more expressive than the ordinary $\Box$. The resulting *interpretability logic* substantially depends on the basis theory T.

The following logic IL is the collection of some basic interpretability principles valid in all reasonable theories: axioms and rules of GL plus

- $\Box(A \rightarrow B) \rightarrow A \rhd B$,
- $(A \rhd B \wedge B \rhd C) \rightarrow A \rhd C$,
- $(A \rhd C \wedge B \rhd C) \rightarrow (A \vee B) \rhd C$,
- $A \rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$,
- $\Diamond A \rhd A$.

(We assume here that the interpretability modality '$\rhd$' binds stronger than the Boolean connectives.)

For two important classes of theories T, the interpretability logic has been characterized axiomatically.

Let ILP be IL augmented by the principle

$$A \rhd B \;\rightarrow\; \Box(A \rhd B)\ .$$

THEOREM 9 (Visser [257]). *The interpretability logic of a finitely axiomatizable theory satisfying some natural conditions is* ILP.

In particular, the class of theories covered by this theorem includes the arithmetical theories $\mathsf{I}\Sigma_n$ for all $n = 1, 2, 3, \ldots$, the second-order arithmetic ACA_0, and the von Neumann-Gödel-Bernays theory GB of sets and classes.

Let ILM be IL augmented by Montagna's principle

$$A \rhd B \;\rightarrow\; (A \wedge \Box C) \rhd (B \wedge \Box C)\ .$$

The following theorem was established independently in [224] and [52].

THEOREM 10 (Shavrkurov, Berarducci). *The interpretability logic of essentially reflexive theories satisfying some natural conditions is* ILM.

In particular, this theorem states that ILM is the interpretability logic for Peano arithmetic PA and Zermelo-Fraenkel set theory ZF.

An axiomatization of the minimal interpretability logic, i.e., of the set of interpretability principles that hold over all reasonable arithmetical theories, is not known. Important progress in this area has been made by Goris and Joosten, who have found new universal interpretability principles (cf. [120, 148]). Yet more new interpretability principles have been found recently by Goris; they were discovered using the Kripke semantics and later shown sound for arithmetic.

The $\rhd$ modality has a related *conservativity* interpretation, which leads to the conservativity logics studied in [124, 125, 140]. Logics of *interpolability* and of *tolerance*, introduced by Ignatiev and Japaridze [80, 81, 142], have a related arithmetical interpretation, but a format which is different from that of interpretability logics; see [147] for an overview.

An excellent survey of interpretability logic is given in [259]; see also [147].

4.9 Magari algebras and propositional second-order provability logic

An algebraic approach to provability logic was initiated by Magari and his students [183, 184, 194, 195]. The *provability algebra* of a theory T, also called the *Magari algebra of* T, is defined as the set of T-sentences factorized modulo provable equivalence in T and equipped with the usual Boolean operations together with the provability operator mapping a sentence F to $\mathsf{Provable}_T(F)$.

Using the notion of provability algebra, one can impart a provability semantics to a representative subclass of propositional second-order modal formulas, i.e., modal formulas with quantifiers over arithmetical sentences. These are just first-order formulas over the provability algebra. For several years, the questions of decidability of the propositional second-order provability logic and of the first-order theory of the provability algebra of PA remained open (cf. [24]). Shavrukov in [227] provided a negative solution to both of these questions.

THEOREM 11 (Shavrukov [227]). *The first-order theory of the provability algebra of* PA *is mutually interpretable with the set of all true arithmetical formulas.*

This result was proved by one of the most ingenious extensions of Solovay's techniques.

4.10 'True and Provable' modality

A gap between the provability logic GL and S4 can be bridged to some extent by using the *strong provability* modality $\boxdot F$ which is interpreted as

$$(\boxdot F)^* = F^* \wedge \mathsf{Provable}(F^*) \ .$$

The reflexivity principle

$$\boxdot F \rightarrow F$$

is then vacuously provable, hence the strong provability modality is S4-compliant.

This approach has been explored in [61, 111, 170], where it was shown independently that the arithmetically complete modal logic of strong provability coincides with Grzegorczyk's logic Grz, which is the extension of S4 by the axiom

$$\Box(\Box(F \rightarrow \Box F) \rightarrow F) \rightarrow F \ .$$

The modality of strong provability has been further studied in [202, 203]; it played a significant role in introducing justification into formal epistemology (cf. [30, 31, 32]), as well as in the topological semantics for modal logic (cf. surveys [86, 100]).

Strong provability plays a certain foundational role: it provides an exact provability-based model for intuitionistic logic IPC. Indeed, by Grzegorczyk's result from [122], Gödel's translation *tr* specifies an exact embedding of IPC into Grz (cf. Theorem 1):

$$\mathsf{IPC} \textit{ proves } F \quad \Leftrightarrow \quad \mathsf{Grz} \textit{ proves } tr(F) \ .$$

However, the foundational significance of this reduction for intuitionistic logic is somewhat limited by a nonconstructive meaning of strong provability as 'classically true and formally provable,' which seems incompatible with the intended intuitionistic semantics. The aforementioned embedding does not bring us closer to the *BHK* semantics for IPC either. For more discussion on these matters, see [12, 16, 171].

4.11 Applications

The methods of modal provability logic are applicable to the study of fragments of Peano arithmetic.

Using provability logic methods, Beklemishev in [45] answered a well-known question: what kind of computable functions could be proved to be total in the fragment of PA where induction is restricted to Π_2-formulas without parameters? He showed that these functions coincide with those that are primitive recursive. In general, provability logic analysis substantially clarified the behavior of parameter-free induction schemata.

Later results [46, 48] revealed a deeper connection between provability logic and traditional proof-theoretic questions, such as consistency proofs, ordinal analysis, and independent combinatorial principles. In [48], Beklemishev gave an alternative proof of

Gentzen's famous theorem on the proof of the consistency of PA by transfinite induction up to the ordinal ϵ_0.

In [47] (cf. also surveys [25, 49]), Beklemishev suggested a simple PA-independent combinatorial principle called *the Worm Principle*, which is derived from Japaridze's polymodal extension GLP of provability logic (cf. Section 4.7). Finite words in the alphabet of natural numbers will be called *worms.* The Worm Principle asserts the termination of any sequence $w_0, w_1, w_2, \ldots$ of worms inductively constructed according to the following two rules. Suppose $w_m = x_0 \ldots x_n$, then

1. if $x_n = 0$, then $w_{m+1} := x_0 \ldots x_{n-1}$ (the head of the worm is cut away);
2. if $x_n > 0$, set $k := \max\{i < n : x_i < x_n\}$ and let $w_{m+1} = x_0 \ldots x_k(x_{k+1} \ldots x_{n-1}(x_n - 1))^{m+1}$ (the head of the worm decreases by one, and the part after position k is appended to the worm m times).

Clearly, the emerging sequence of worms is fully determined by the initial worm w_0. For example, consider a worm $w_0 = 2031$. Then the sequence looks as follows:

$w_0 = 2031$
$w_1 = 203030$
$w_2 = 20303$
$w_3 = 20302222$
$w_4 = 2030222122212221222212221$
$w_5 = 2030(22212221222122212220)^6$
$\ldots$

THEOREM 12 (Beklemishev [47]).

(1) *For any initial worm w_0, there is an m such that w_m is empty.*

(2) *The previous statement is unprovable in Peano arithmetic* PA. *In fact, Statement 1 is equivalent to the 1-consistency of* PA.

For other PA-independent principles, cf. [244].

Japaridze used a technique from the area of Provability Logic to investigate fundamental connections between provability, computability, and truth in his work on Computability Logic [145, 146].

The Logic of Proofs (Section 5) with its applications also emerged from studies in Provability Logic.

5 LOGIC OF PROOFS

The source of difficulties in provability interpretation of modality lies in the implicit nature of the existential quantifier $\exists$. Consider, for instance, the reflection principle in PA, i.e., all formulas of type $\mathsf{Provable}(F) \to F$. By Gödel's second incompleteness theorem, this principle is not provable in PA, since the consistency formula $\mathsf{Con}(\mathsf{PA})$ coincides with a special case of the reflection principle, namely $\mathsf{Provable}(\perp) \to \perp$. The formula $\mathsf{Provable}(F)$ is $\exists x \mathsf{Proof}(x, F)$ where $\mathsf{Proof}(x, y)$ is Gödel's *proof predicate*

$$x \textit{ is (a code of) a proof of a formula (having code) } y \,.$$

Assuming $\mathsf{Provable}(F)$ does not yield pointing to any specific proof of F, since this x may be a nonstandard natural number which is not a code of any actual derivation in PA.

For proofs represented by explicit terms, the picture is very different. The principle of *explicit reflection* $\mathsf{Proof}(p, F) \rightarrow F$ is provable in PA for each specific derivation p. Indeed, if $\mathsf{Proof}(p, F)$ holds, then F is evidently provable in PA, and so is the formula $\mathsf{Proof}(p, F) \rightarrow F$. Otherwise, if $\mathsf{Proof}(p, F)$ does not hold, then $\neg\mathsf{Proof}(p, F)$ is true and provable, therefore $\mathsf{Proof}(p, F) \rightarrow F$ is also provable.

This observation suggests a remedy: representing proofs by terms t in the proof formula $\mathsf{Proof}(t, F)$ instead of implicit representation of proofs by existential quantifiers in the provability formula $\exists x \mathsf{Proof}(x, F)$. As we have already mentioned, Gödel suggested using the format of explicit proof terms for the interpretation of $\mathsf{S4}$ as early as 1938, but that paper remained unpublished until 1995 ([110]). Independently, the study of explicit modal logics was initiated in [14, 33, 34, 35, 247]. The Logic of Proofs may be regarded an instance of Gabbay's Labelled Deductive Systems (cf. [98]).

Proof polynomials are terms built from *proof variables* $x, y, z, \ldots$ and *proof constants* $a, b, c, \ldots$ by means of three operations: *application* '$\cdot$' (binary), *union* '$+$' (binary), and *proof checker* '!' (unary). The language of *Logic of Proofs* LP is the language of classical propositional logic supplemented by a new rule for building formulas, namely for each proof polynomial p and formula F, there is a new formula $p{:}F$ denoting 'p is a proof of F.' It is also possible to read this language type-theoretically: formulas become types, and $p{:}F$ denotes 'term p has type F.' We assume also that '$t{:}$' and '$\neg$' bind stronger than '$\wedge, \vee$' which, in turn, bind stronger than '$\rightarrow$.'

Axioms and inference rules of LP:

1. *Axioms of classical propositional logic*
2. $t{:}(F \rightarrow G) \rightarrow (s{:}F \rightarrow (t \cdot s){:}G)$ (*application*)
3. $t{:}F \rightarrow F$ (*reflection*)
4. $t{:}F \rightarrow !t{:}(t{:}F)$ (*proof checker*)
5. $s{:}F \rightarrow (s+t){:}F, \quad t{:}F \rightarrow (s+t){:}F$ (*sum*)
6. *Rule modus ponens*
7. $\vdash c{:}A$*, where* A *is from* 1-5*, and* c *is a proof constant* (*constant specification*)

As one can see from the principles of LP, constants denote proofs of axioms. The application operation corresponds to the internalized *modus ponens* rule: for each s and t, a proof $s \cdot t$ is a proof of all formulas G such that s is a proof of $F \rightarrow G$ and t is a proof of F for some F. The sum '$s + t$' of proofs s and t is a proof which proves everything that either s or t does. Finally, '!' is interpreted as a universal program for checking the correctness of proofs, which given a proof t, produces a proof that t proves F ([14, 16]). In [17], it was noted that proof polynomials represent the whole set of possible operations on proofs for a propositional language. It was shown that any operation on proofs which is invariant with respect to a choice of a normal proof system and which can be specified in a propositional language can be realized by a proof polynomial.

In what follows, '$\vdash$' denotes derivability in LP unless stated otherwise. By a *constant specification* $\mathcal{CS}$, we mean a set of formulas $\{c_1{:}A_1, c_2{:}A_2, \ldots\}$ where each A_i is an axiom from 1–5 of LP, and each c_i is a proof constant. By default, with each derivation in LP, we associate a constant specification $\mathcal{CS}$ introduced in this derivation by the use of the rule of constant specification.

One of the basic properties of LP is its capability of internalizing its own derivations. The weak form of this property yields the following admissible rule for LP ([14, 16]):

$$\textit{if} \vdash F, \textit{ then} \vdash p{:}F \textit{ for some proof polynomial } p\ .$$

This rule is a translation of the well-known necessitation rule of modal logic

$$\frac{\vdash F}{\vdash \Box F}$$

into the language of explicit proofs. The following more general *internalization rule* holds for LP: *if*

$$A_1, \ldots, A_n \vdash B\ ,$$

then it is possible to construct a proof polynomial $t(x_1, \ldots, x_n)$ *such that*

$$x_1{:}A_1, \ldots, x_n{:}A_n \vdash t(x_1, \ldots, x_n){:}B$$

One might notice that the Curry-Howard isomorphism covers only a simple instance of the proof internalization property where all of $A_1, \ldots, A_n, B$ are purely propositional formulas containing no proof terms. For the Curry-Howard isomorphism basics, see, e.g., [108].

The decidability of LP was established by Mkrtychev in [193]. Kuznets in [168] obtained an upper bound Σ_2^p on the satisfiability problem for LP-formulas in Mkrtychev models (cf. Section 5.3). This bound was lower than the known upper bound *PSPACE* on the satisfiability problem in $\mathsf{S4}$ (under the assumption that $\Sigma_2^p \neq \mathit{PSPACE}$). A possible explanation of why LP wins in complexity over $\mathsf{S4}$ is that the satisfiability test for LP is somewhat similar to type checking, i.e., checking the correctness of assigning types (formulas) to terms (proofs), which is known to be relatively easy in classical cases.

Milnikel in [190] established Π_2^p-completeness of LP for some natural classes of constant specifications, including so-called injective ones, when each constant denotes a proof of not more than one axiom. Π_2^p-hardness for the whole LP remains on open problem.

N. Krupski in [159] established the disjunctive property for LP:

$$\textit{if } \mathsf{LP} \vdash s{:}F \vee t{:}G, \textit{ then } \mathsf{LP} \vdash s{:}F \textit{ or } \mathsf{LP} \vdash t{:}G.$$

5.1 Arithmetical completeness

The Logic of Proofs LP is sound and complete with respect to the natural provability semantics. By a *proof system* we mean a provably in PA decidable predicate $\mathsf{Proof}(x, y)$ that enumerates all theorems of PA, i.e.,

$$\mathsf{PA} \vdash \varphi \quad \text{iff} \quad \mathsf{Proof}(n, \varphi) \ \text{ holds for some } n\ ,$$

together with computable functions $\mathbf{m}(x, y)$, $\mathbf{a}(x, y)$ and $\mathbf{c}(x)$ which satisfy identities for '$\cdot$,' '+,' and '!' respectively, i.e., for all arithmetical formulas φ, ψ and all natural numbers k, n the following holds:

$$\mathsf{Proof}(k, \varphi \to \psi) \wedge \mathsf{Proof}(n, \varphi) \to \mathsf{Prf}(\mathbf{m}(k, n), \psi)$$

$$\mathsf{Proof}(k,\varphi)\rightarrow\mathsf{Proof}(\mathbf{a}(k,n),\varphi),\qquad \mathsf{Proof}(n,\varphi)\rightarrow\mathsf{Proof}(\mathbf{a}(k,n),\varphi)$$
$$\mathsf{Proof}(k,\varphi)\rightarrow\mathsf{Proof}(\mathbf{c}(k),\mathsf{Proof}(k,\varphi)).$$

The class of proof systems includes the Gödelian proof predicate in PA

x is a Gödel number of a derivation in PA *containing a formula with a Gödel number y*

with obvious choice of operations $\mathbf{m}(x,y)$, $\mathbf{a}(x,y)$ and $\mathbf{c}(x)$. In particular, $\mathbf{a}(n,m)$ is the concatenation of proofs n and m, and $\mathbf{c}$ is a computable function that given a Gödel number of a proof n, returns the Gödel number $\mathbf{c}(n)$ of a proof, containing formulas $\mathsf{Proof}(n,\varphi)$ for all φ's such that $\mathsf{Proof}(n,\varphi)$ holds.

An arithmetical interpretation $*$ is determined by a choice of proof system as well as an interpretation of proof variables and constants by numerals (denoting proofs), and propositional variables by arithmetical sentences. Boolean connectives are understood in the same way in LP and PA, and a formula $p{:}F$ is interpreted as an arithmetical formula $\mathsf{Proof}(p^*,F^*)$.

> This kind of provability semantics is referred to as *call-by-value semantics*; it was introduced in [15] and used in [16, 18, 29, 119, 270]. A more sophisticated *call-by-name semantics* of the language of LP was introduced in [14] and used in [160, 161, 235, 269]. Under the call-by-name semantics, proof polynomials are interpreted as Gödel numbers of definable provably recursive arithmetical terms. Call-by-value interpretations may be regarded as a special case of call-by-name interpretations since numerals are definable provably recursive arithmetical terms.

For a given constant specification $\mathcal{CS}$, an interpretation $*$ is called a *$\mathcal{CS}$-interpretation* if all formulas from $\mathcal{CS}$ are true under a given $*$. The following arithmetical completeness theorem has been established in [14] for the call-by-name semantics and in [15] for the call-by-value semantics (see also articles [16, 18]):

THEOREM 13 (Artemov [14, 15]). *A formula F is derivable in* LP *with a given constant specification $\mathcal{CS}$ iff* $\mathsf{PA}\vdash F^*$*, for any $\mathcal{CS}$-interpretation $*$.*

This theorem stands if one replaces '$\mathsf{PA}\vdash F^*$' by 'F^* holds in the standard model of arithmetic.'

In his recent paper [119], Goris showed that LP is sound and complete with respect to the call-by-value semantics of proofs in Buss's weak arithmetic S^1_2, thus showing that proof polynomials can be realized by *PTIME*-computable operations on proofs. Note that the corresponding question for the Provability Logic GL remains a major open problem.

The logic of single-conclusion proofs was described by V. Krupski in [160, 161]. This system does not correspond to any normal modal logic.

5.2 Realization Theorem

Another major feature of the Logic of Proofs is its ability to realize all S4-derivable formulas by restoring corresponding proof polynomials inside all occurrences of modality. This fact may be expressed by the following realization theorem ([14, 16]). By a *forgetful projection* of an LP-formula F, we understand a modal formula obtained by replacing all assertions $t{:}(\cdot)$ in F by $\Box(\cdot)$.

THEOREM 14 (Artemov [14]). *$\mathsf{S4}$ is the forgetful projection of LP.*

That the forgetful projection of LP is $\mathsf{S4}$-compliant is a straightforward observation. The converse has been established in [14, 16] by presenting an algorithm which substitutes proof polynomials for all occurrences of modalities in a given cut-free Gentzen-style $\mathsf{S4}$-derivation of a formula F, thereby producing a formula F^r derivable in LP. The original realization algorithms from [14, 16] were exponential. Brezhnev and Kuznets in [69] offered a realization algorithm of $\mathsf{S4}$ into LP which is polynomial in the size of a cut-free derivation in $\mathsf{S4}$. The lengths of realizing proof polynomials can be kept quadratic in the length of the original cut-free $\mathsf{S4}$-derivation.

Here is an example of an $\mathsf{S4}$-derivation realized as an LP-derivation in the style of Theorem 14. There are two columns in the table below. The first is a Hilbert-style $\mathsf{S4}$-derivation of a modal formula $\Box A\vee\Box B\rightarrow\Box(\Box A\vee B)$. The second column displays corresponding steps of an LP-derivation resulted in an LP-proof of a formula

$$x{:}A\vee y{:}B\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$$

with constant specification

$$\{\ a{:}(x{:}A\rightarrow x{:}A\vee B),\quad b{:}(B\rightarrow x{:}A\vee B)\ \}\ .$$

	Derivation in $\mathsf{S4}$	Derivation in LP
1.	$\Box A\rightarrow\Box A\vee B$	$x{:}A\rightarrow x{:}A\vee B$
2.	$\Box(\Box A\rightarrow\Box A\vee B)$	$a{:}(x{:}A\rightarrow x{:}A\vee B)$
3.	$\Box\Box A\rightarrow\Box(\Box A\vee B)$	$!x{:}x{:}A\rightarrow(a{\cdot}!x){:}(x{:}A\vee B)$
4.	$\Box A\rightarrow\Box\Box A$	$x{:}A\rightarrow !x{:}x{:}A$
5.	$\Box A\rightarrow\Box(\Box A\vee B)$	$x{:}A\rightarrow(a{\cdot}!x){:}(x{:}A\vee B)$
5′.		$(a{\cdot}!x){:}(x{:}A\vee B)\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$
5″.		$x{:}A\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$
6.	$B\rightarrow\Box A\vee B$	$B\rightarrow x{:}A\vee B$
7.	$\Box(B\rightarrow\Box A\vee B)$	$b{:}(B\rightarrow x{:}A\vee B)$
8.	$\Box B\rightarrow\Box(\Box A\vee B)$	$y{:}B\rightarrow(b{\cdot}y){:}(x{:}A\vee B)$
8′.		$(b{\cdot}y){:}(x{:}A\vee B)\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$
8″.		$y{:}B\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$
9.	$\Box A\vee\Box B\rightarrow\Box(\Box A\vee B)$	$x{:}A\vee y{:}B\rightarrow(a{\cdot}!x{+}b{\cdot}y){:}(x{:}A\vee B)$

Extra steps 5′, 5″, 8′, and 8″ are needed in the LP case to reconcile different internalized proofs of the same formula: $(a{\cdot}!x){:}(x{:}A\vee B)$ and $(b{\cdot}y){:}(x{:}A\vee B)$. The resulting realization respects Skolem' idea that negative occurrences of existential quantifiers (here over proofs hidden in the modality of provability) are realized by free variables whereas positive occurrences are realized by functions of those variables.

Switching from the provability format to the language of specific witnesses reveals hidden self-referentiality of modal logic, i.e., the necessity of using proof assertions of the form $t{:}F(t)$, where t occurs in the very formula $F(t)$ of which it is a proof. A recent result by Kuznets in [69] shows that self-referentiality is an intrinsic feature of the modal logic approach to provability in general.

THEOREM 15 (Kuznets [69]). *Self-referential constant specifications of the sort $c{:}A(c)$ are necessary for realization of the modal logic $\mathsf{S4}$ in the Logic of Proofs LP.*

In particular, the S4-theorem

$$\neg\Box\neg(S\to\Box S)$$

cannot be realized in LP without self-referential constant specifications of the sort $c{:}A(c)$.

Systems of proof polynomials for other classical modal logics K, K4, D, D4, T were described in [67, 68]. The case of S5 = S4 + $(\neg\Box F\to\Box\neg\Box F)$ was special because of the presence of negative information about proofs and its connections to formal epistemology. The paper by Artemov, Kazakov, and Shapiro [29] introduced a system of proof terms for S5, and established realizability of the logic S5 by these terms, decidability, and completeness of the resulting logic of proofs.

5.3 Fitting Models

The main idea of epistemic semantics for LP can be traced back to Mkrtychev and Fitting. It consists of augmenting Boolean or Kripke models with an *evidence function*, which assigns 'admissible evidence' terms to a statement before deciding its truth value.

Fitting models are defined as follows. A *frame* is a structure (W,R), where W is a non-empty set of *possible worlds* and R is a binary reflexive and transitive *evidence accessibility* relation on W. Given a frame (W,R), a *possible evidence function* $\mathcal{E}$ is a mapping from worlds and proof polynomials to sets of formulas. We can read $F\in\mathcal{E}(u,t)$ as

'F is one of the formulas for which t serves as possible evidence in world u.'

An *evidence function* is a possible evidence function which respects the intended meanings of the operations on proof polynomials, i.e., for all proof polynomials s and t, for all formulas F and G, and for all $u,v\in W$, each of the following hold:

1. *Monotonicity*: uRv implies $\mathcal{E}(u,t)\subseteq\mathcal{E}(v,t)$;
2. *Closure*:
   - *Application*: $F\to G\in\mathcal{E}(u,s)$ and $F\in\mathcal{E}(u,t)$ implies $G\in\mathcal{E}(u,s\cdot t)$;
   - *Inspection*: $F\in\mathcal{E}(u,t)$ implies $t{:}F\in\mathcal{E}(u,!t)$;
   - *Sum*: $\mathcal{E}(u,s)\cup\mathcal{E}(u,t)\subseteq\mathcal{E}(u,s+t)$.

A model is a structure $\mathcal{M}=(W,R,\mathcal{E},\Vdash)$ where (W,R) is a frame, $\mathcal{E}$ is an evidence function on (W,R), and $\Vdash$ is an arbitrary mapping from sentence variables to subsets of W. Given a model $\mathcal{M}=(W,R,\mathcal{E},\Vdash)$, the forcing relation $\Vdash$ is extended from sentence variables to all formulas by the following rules. For each $u\in W$:

1. $\Vdash$ respects connectives ($u\Vdash F\wedge G$ iff $u\Vdash F$ and $u\Vdash G$, $u\Vdash\neg F$ iff $u\nVdash F$, etc.);
2. $u\Vdash t{:}F$ iff $F\in\mathcal{E}(u,t)$ and $v\Vdash F$ for every $v\in W$ with uRv.

We consider the modality $\Box$, associated with the evidence accessibility relation R. In this terms, the last item of the above definition can be recast as

$2'$. $u\Vdash t{:}F$ iff $u\Vdash\Box F$ and t is an admissible evidence for F at u.

Mkrtychev models are Fitting models with singleton W's. LP was shown to be sound and complete with respect to both Mkrtychev models ([193]) and Fitting models ([91, 93]). Fitting models were adapted for multi-agent epistemic setting in [20, 30, 32, 92] and became the standard semantics for justification logics.

5.4 Joint logics of proofs and provability

The problem of finding a joint logic of proofs and provability has been a natural next step, since there are principles that can only be formulated in a mixed language of formal provability and explicit proofs. For example, the modal principle of negative introspection $\neg\Box F \rightarrow \Box\neg\Box F$ is not valid in the provability semantics; neither is a purely explicit version of negative introspection $\neg(x : F) \rightarrow t(x) : \neg(x : F)$. However, a mixed explicit-implicit principle $\neg(t{:}F) \rightarrow \Box\neg(t{:}F)$ is valid in the standard provability semantics.

The complete joint system of provability and explicit proofs without operations on proof terms, system B, was found in [13]. This system describes those principles that have a pure logical character and do not depend on any specific operations of proofs.

The postulates of B consist of those of GL together with the following new principles:

- A1. $t{:}F \rightarrow F$,
- A2. $t{:}F \rightarrow \Box t{:}F$,
- A3. $\neg t{:}F \rightarrow \Box\neg t{:}F$,
- RR. *Rule of reflection*: $\dfrac{\vdash \Box F}{\vdash F}$.

THEOREM 16 (Artemov [13]). B *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

The problem of joining two models of provability, GL and LP, into one model can be specified as that of finding an arithmetically complete logic containing postulates of both GL and LP and closed under internalization.

The first solution to this problem was offered by Yavorskaya (Sidon) who found an arithmetically complete system of provability and explicit proofs, LPP, containing both GL and LP (cf. [235, 269]). Along with natural extensions of principles and operations from GL and LP, LPP contains additional operations '$\Uparrow$' and '$\Downarrow$' which were used to secure the internalization property of LPP. The operation '$\Uparrow$' given a proof t of F, returns a proof $\Uparrow t$ of $\mathsf{Provable}(F)$. The operation '$\Downarrow$' takes a proof t of $\mathsf{Provable}(F)$ and returns a proof $\Downarrow t$ of F. The set of postulates of LPP consists of those of GL and LP together with A2, A3, and RR from B, plus two new principles:

- A4. $t{:}F \rightarrow (\Uparrow t){:}\Box F$,
- A5. $t{:}\Box F \rightarrow (\Downarrow t){:}F$.

Finally, Nogina in [30, 201] noticed that operations '$\Uparrow$' and '$\Downarrow$' along with A4 and A5 are in certain sense redundant and offered a simpler system, GLA, which is an arithmetically complete logic in a joint language of GL and LP, containing postulates of both GL and LP, and closed under internalization. The system GLA is presented in [30, 201] by the set of postulates of GL and LP augmented by the principles:

- $t{:}F \rightarrow \Box F$,
- $\neg t{:}F \rightarrow \Box\neg t{:}F$,
- $t{:}\Box F \rightarrow F$.

and *Rule of reflection* RR.

THEOREM 17.

(1) (Yavorskaya (Sidon) [235, 269]). LPP *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

(2) (Nogina [30, 201]). GLA *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

It was the system GLA, which served in [30, 32] as a prototype of justification logics (cf. Subsection 5.7).

5.5 Quantified logics of proofs

The arithmetical provability semantics for the logic of proofs may be naturally generalized to first-order language and to the language of LP with quantifiers over proofs. Both possibilities of enhancing the expressive power of LP were investigated and in both cases, axiomatizability questions have been answered negatively.

THEOREM 18.

(1) (Artemov, Yavorskaya (Sidon) [36]). *The first-order logic of proofs is not recursively enumerable.*

(2) (Yavorsky [271]). *The logic of proofs with quantifiers over proofs is not recursively enumerable.*

An interesting decidable fragment of the first-order logic of the standard proof predicate was found in [270].

5.6 Intuitionistic logic of proofs

The problem of building the intuitionistic logic of proofs has two distinct parts. Firstly, one has to answer the question about the propositional logical principles that axiomatize HA-tautologies in the propositional language enriched by atoms *u is a proof of F* without operations on proof terms, i.e. when u is a variable. The resulting basic logic of proofs reflects purely logical principles of the chosen format. Secondly, one has to pick systems of operations on proofs and study the corresponding intuitionistic logics of proofs. The first of the above problems was solved by Artemov and Iemhoff in [27] where the *Basic Intuitionistic Logic of Proofs*, iBLP, was introduced and found to be arithmetically complete with respect to the semantics of proofs in HA. The paper essentially uses technique and results by de Jongh [77], Smoryński [239], de Jongh and Visser's work on a basis for admissible rules in IPC (circa 1991, cf. [137]), Artemov & Strassen [33] and Artemov [13], Ghilardi [106, 107], Iemhoff [136, 138, 139].

The completeness proof presented in [27] is also interesting because it is the first result in this area for constructive theories; the corresponding problem for the provability logic of Heyting arithmetic HA is still open (Section 4.4).

5.7 Applications

Here we will list some conceptual applications of the Logic of Proofs.

1. *Existential semantics for modal logic.* Proof polynomials and LP represent an exact *existential semantics* for mainstream modal logic. Initially, Gödel regarded the modality $\Box F$ as the provability assertion, i.e.,

there exists a proof for F.

Thus, according to Gödel, modality is an informal Σ_1-sentence, i.e., the one which consists of an existential quantifier (here over proofs) followed by a decidable condition. Such an understanding of modality is typical of 'naive' semantics for a wide range of epistemic and provability logics. Nonetheless, before LP was discovered, major modal logics lacked a mathematical semantics of an existential character. The exception to the rule is the arithmetical provability interpretation for the Provability Logic GL, which still cannot be extended to the major modal logics $\mathsf{S4}$ and $\mathsf{S5}$.

Almost 30 years after the first work by Gödel on the subject, a semantics of a *universal* character was discovered for modal logic, namely Kripke semantics. Modality in that semantics is read informally as the sentence:

in each possible situation, F holds.

Such a reading of modality naturally appears in dynamic and temporal logics aimed at describing computational processes, states of which usually form a (possibly branching) Kripke structure. Universal semantics has been playing a prominent role in modal logic. However, it is not the only possible semantical tool in the study and application of modality. The existential semantics of realizability by proof polynomials can also be useful for foundations and application of modal logic. For more discussion on the existential semantics for modal logic, see [22].

2. *Justification Logic.* A major area of application of the Logic of Proofs is epistemology. The books [89, 189] serve as an excellent introduction to the mathematical logic of knowledge.

Plato's celebrated tripartite definition of knowledge as *justified true belief* is generally regarded in mainstream epistemology as a set of necessary conditions for the possession of knowledge. Due to Hintikka, the 'true belief' components have been fairly formalized by means of modal logic and its possible worlds semantics. The remaining 'justification' condition has received much attention in epistemology (cf., for example, [59, 105, 116, 129, 174, 177, 178, 204]), but lacked formal representation. The issue of finding a formal epistemic logic with justification has also been discussed in [247]. Such a logic contains assertions of the form $\Box F$ (*F is known*), along with those of the form $t\,{:}\,F$ (*t is a justification for F*). Justification was introduced into formal epistemology in [20, 30, 31, 32] by combining Hintikka-style epistemic modal logic with justification calculi arising from the Logic of Proofs LP. The generic name for this kind of systems is *Justification Logic*.

3. *Logical omniscience problem.* The traditional Hintikka-style modal logic approach to knowledge has the well-known defect of *logical omniscience*, which is an unrealistic feature that an agent knows all logical consequences of his/her assumptions ([87, 88, 134, 198, 206, 207]). Justification Logic addresses the issue of logical omniscience in a natural way. The paper [28] suggests looking at the logical omniscience as a complexity issue

and offers the following Logical Omniscience Test (LOT): an epistemic system E is not logically omniscient if for any valid in E knowledge assertion $\mathcal{A}$ of type '*F is known*' there is a proof of F in E, the complexity of which is bounded by some polynomial in the length of $\mathcal{A}$. The usual epistemic modal logics are logically omniscient (modulo some common complexity assumptions). On the other hand, Justification Logic is logically omniscient w.r.t. the usual (implicit) knowledge and are not logically omniscient w.r.t. the evidence-based knowledge.

4. *Justified Knowledge.* Justification Logic was used in [20, 23] to offer a new approach to *common knowledge.* A modal operator $J\varphi$ for *justified knowledge* introduced in [20, 23] is defined as a forgetful projection of justification assertions $t{:}\varphi$. Hence the intended meaning of $J\varphi$ is

there is an access to an explicit evidence for φ .

In particular, justified knowledge J was shown in [5, 20, 23] to provide a lighter, constructive version of common knowledge and can be used as such in solving specific problems.

6 MODAL LOGIC OF SPACE

The application of modal logic to topology has a rather long history. The idea of a simple 'algebraic calculus' suitable for proving some topological theorems dates back to Kuratowski [163]. A somewhat similar idea was proposed earlier by Riesz in [216]. A. Robinson in [217] put the problem of developing a topological model theory in the same manner as the classical first-order model theory. Classical first-order logic is insufficient for topology because here one usually deals both with points and sets, hence some fragments of second-order logic should be involved. Topological model theory in this style was developed in [95, 96].

The modal logic approach to topology lies within the same mathematical tradition. Modal calculi can also be interpreted in certain weak fragments of second-order logic. However, modal logics of interest are usually decidable and have a good mathematical structure with respect to both model theory and proof theory. All these features bring into topology some specific logical tools and results.

The use of modal logic in topology was initially motivated by Kuratowski's axioms. Let $\mathcal{T} = \langle \boldsymbol{X}, \mathbb{I}\rangle$ be a topological space, where $\boldsymbol{X}$ is a set of points and $\mathbb{I}$ the interior operation. In terms of the interior ($\mathbb{I}$) and Boolean operations, modal topological principles look as follows:

A1. $\mathbb{I}(Y \cap Z) = \mathbb{I}Y \cap \mathbb{I}Z$;
A2. $\mathbb{I}Y = \mathbb{I}\mathbb{I}Y$;
A3. $\mathbb{I}Y \subseteq Y$;
A4. $\mathbb{I}\boldsymbol{X} = \boldsymbol{X}$

Here Y, Z are subsets of $\boldsymbol{X}$. These axioms can be viewed as identities in the language of Boolean algebras with an extra functional symbol $\mathbb{I}$. They define the variety of so-called *topo-Boolean algebras* (a.k.a. *interior algebras* or *closure algebras*, the latter used by McKinsey and Tarski), and in an obvious way every topological space $\mathcal{T}$ corresponds to a topo-Boolean algebra, which is the powerset of $\mathcal{T}$ with the interior (or closure) operation acting on the subsets of $\mathcal{T}$. The above axioms can also be written as propositional modal formulas: Boolean operations should be replaced by the corresponding propositional connectives, and $\mathbb{I}$ with the modal connective $\Box$. Thus we obtain the following modal axiom schemes:

A1. $\Box(A \wedge B) = \Box A \wedge \Box B$,
A2. $\Box A \rightarrow \Box\Box A$,
A3. $\Box A \rightarrow A$,
A4. $\Box\top$,

which are the well-known postulates of $\mathsf{S4}$. This property, noticed in the late 1930s by Tarski, and independently by Stone and Tang, is rather surprising because Lewis' original motivation of $\mathsf{S4}$ was purely logical, and Gödel's provability interpretation of $\mathsf{S4}$ was also of a logical character.

The topological interpretation can be modified to fit other modal logics, namely, one can consider *neighborhood frames*. By definition, such a frame $\mathcal{F} = (\boldsymbol{X}, \mathbb{I})$ is a set $\boldsymbol{X}$ together with an operation $\mathbb{I}$ on its subsets. Then U is called a neighborhood of x if $x \in \mathbb{I}U$. Given a valuation φ which sends proposition letters to subsets of X, we can extend it to all modal formulas as follows:

$\varphi(\neg A) = \boldsymbol{X} \setminus \varphi(A)$,
$\varphi(A \wedge B) = \varphi(A) \cap \varphi(B)$,
$\varphi(A \vee B) = \varphi(A) \cup \varphi(B)$,
$\varphi(\Box A) = \mathbb{I}\varphi(A)$.

The same definition can be given in terms of a forcing relation (or the truth at a point): a formula A is called true at w under interpretation φ if $w \in \varphi(A)$; this is also denoted by $w \Vdash A$. Now the above conditions for extending φ can be written as follows:

$w \Vdash \neg A$ iff $w \nVdash A$,
$w \Vdash A \wedge B$ iff ($w \Vdash A$ and $w \Vdash B$),
$w \Vdash A \vee B$ iff ($w \Vdash A$ or $w \Vdash B$),
$w \Vdash \Box A$ iff $\{y \mid y \Vdash A\}$ is a neighborhood of w.

So $\Box A$ can be read as A *is locally true* [220]. A formula A is called valid in $\mathcal{F}$ (notation: $\mathcal{F} \Vdash A$) if $\varphi(A) = \boldsymbol{X}$ for any valuation φ.

The set $\mathbf{L}(\mathcal{F}) := \{A \mid \mathcal{F} \Vdash A\}$ is called the modal logic of $\mathcal{F}$. Logics of this form are called neighborhood (N-) complete. All well-known modal logics are N-complete. Moreover, they can be presented as logics of familiar topological spaces.

The following theorem is a classical result in this area:

THEOREM 19 (McKinsey, Tarski [187]). *Let $\mathcal{M}$ be a separable dense-in-itself metric space. Then $\mathbf{L}(\mathcal{M}) = \mathsf{S4}$.*

In particular, $\mathbf{L}(\mathbb{R}^n) = \mathsf{S4}$, for each $n = 1, 2, 3, \ldots$.

A simplified proof of topological completeness of $\mathsf{S4}$ with respect to the Cantor space was obtained in [192]. Simplified proofs of completeness of $\mathsf{S4}$ with respect to the real line $\mathbb{R}$ were given in [56, 237].

However, there exist N-incomplete modal logics, even among the extensions of $\mathsf{S4}$. Such examples can be found in [103, 104, 228, 233]. This fact is perhaps counter to the naive intuition: it turns out that there exist systems of topo-Boolean identities that do not correspond to any particular topo-Boolean algebra of a topological space — every such algebra satisfies some other identities that are non-derivable from the original system. This is indeed an incompleteness phenomenon at the level of propositional modal logic akin to those in arithmetical theories.

Kripke semantics can be regarded as a particular case of neighborhood semantics. In fact, given a Kripke frame (W, R), one can build the neighborhood frame $\mathbf{N}(W, R) = (W, \mathbb{I})$ where $\mathbb{I}U := \{x \mid R(x) \subseteq U\}$, so that validities in these two frames are the same. Hence, Kripke-completeness implies N-completeness. The converse is not true; there exist topological spaces with Kripke-incomplete modal logics [103, 104, 228, 233].

For topological semantics of first-order modal logic, see Chapter 9 this Handbook. Modal logics of product topologies were studied in [181, 219, 251]. Gabelaia's master's thesis [100] is a very informative source on modal logic and topology.

6.1 Other operators in topological spaces

The modality $\Box$ can be interpreted in topological spaces not only as the interior, but in some other natural ways. There are other known topological operators on sets that are not expressible in terms of Boolean operation and interior, e.g., taking the derived set $d(X)$ which is the set of all limit points of X [164]. It turns out that the corresponding 'derivational modal logics' of natural classes of topological spaces are not among the most popular modal logics, with one noticeable exception: the derivational modal logic of Cantor's scattered topological spaces turned out to be the Provability Logic GL.

DEFINITION 20. Let C be a class of topological spaces. We understand by $\mathbf{Ld}(C)$ the *derivational modal logic of* C, i.e., the set of propositional formulas with the modality $\Diamond$ interpreted as the derived set operator d that hold in all $\mathcal{T}$'s from C. By $\mathsf{wK4}$ ('weak $\mathsf{K4}$') we understand the modal logic $\mathsf{K} + (p \wedge \Box p) \rightarrow \Box\Box p$, and by $\mathsf{D4}$ the logic $\mathsf{K4} + \neg\Box\bot$.

By the modality $\Box^+ F$, we mean $F \wedge \Box F$.

THEOREM 21.

(1) (Esakia [83, 85, 86]). $\mathsf{wK4}$ *is the derivational logic of the class of all topological spaces.*

(2) (Shehtman [230]). *For* $n > 1$, $\mathbf{Ld}(\mathbb{R}^n) = \mathsf{D4} + \Box[(p\wedge\Box p)\vee(\neg p\wedge\Box\neg p)] \rightarrow \Box p\vee\Box\neg p$.

(3) (Shehtman [232]). $\mathbf{Ld}(\mathbb{R}) = \mathsf{D4} + \Box(\Box^+ F_1 \vee \Box^+ F_2 \vee \Box^+ F_3) \rightarrow (\Box\neg F_1 \vee \Box\neg F_2 \vee \Box\neg F_3)$, *where*

$$F_i = p_i \wedge \bigwedge_{j\neq i} \neg p_j \ .$$

Note that the derivational modal logics of $\mathbb{R}$ and $\mathbb{R}^n$ for $n > 1$ are different. Shehtman in [230] also found that derivational modal logics of $\mathbb{Q}$, Cantor's discontinuum $\mathcal{C}$, as well as any 0-dimensional separable dense-in-itself metric space are all equal to $\mathsf{D4}$. Further results on axiomatization and definability of derivational logics can be found in [55].

A topological space is called *scattered* if it has no dense-in-itself non-empty subsets. Let α be an ordinal. We view α as a topological space with its interval topology. Then it is known that every ordinal α is a scattered space ([186]).

THEOREM 22.

(1) (Esakia [84, 86]). GL *is the derivational logic of the class of all scattered spaces.*

(2) (Abashidze [1], Blass [58]). GL *is the derivational logic of* α, *for any specific ordinal* $\alpha \geq \omega^\omega$.

These theorems demonstrate that Gödel's consistency operator *Con(F)*, stating that

F is consistent with Peano arithmetic , and Cantor's topological derived set operator d on scattered spaces have the same set of propositional identities.

6.2 Adding the universal modality

Topological spaces may be considered Boolean algebras with several extra operations, and this leads to different polymodal logics. The basic modal language can be expanded by other modal connectives. For example, one can add the universal modality $[\forall]$, with the following interpretation:

$$w \Vdash [\forall]A \quad \text{iff} \quad x \Vdash A \quad \text{holds for any } x \in \boldsymbol{X}\,.$$

The new language is more expressive: in fact, the formula

$$(\mathrm{AC}) \;:=\; [\forall](\Box p \vee \Box\neg p) \rightarrow [\forall]p \vee [\forall]\neg p$$

is valid exactly in connected spaces, but connectedness cannot be expressed in the basic language. Moreover, the following analogs of the classical McKinsey–Tarski Theorem 19 hold. Let

$\mathsf{S4U} = \mathsf{S4}(\text{for } \Box) + \mathsf{S5}(\text{for } [\forall]) + [\forall]p \rightarrow \Box p$,

$\mathsf{S4UC} = \mathsf{S4U} + (\mathrm{AC})$.

Let also $\mathbf{LU}(C)$ denote the logic of a class C in the expanded language with $\Box$ and $[\forall]$.

THEOREM 23.

(1) (Goranko, Passy [117]). $\mathsf{S4U} = \mathbf{LU}$*(all topological spaces).*

(2) (Shehtman [231]). *If $\mathcal{X}$ is a separable dense-in-itself metric space, then*

$$\mathbf{LU}(\mathcal{X}) = \mathsf{S4UC}\,.$$

Some refinement of Shehtman's result (2) can be found in [250] and Chapter 9 of this Handbook. It was shown in [117, 231] that S4U and S4UC have the finite model property, and so they are decidable. As for complexity, S4U is known to be *PSPACE*-complete [6].

An interesting feature is that many *mereotopological relations* between spatial regions (such as 'X is disconnected from Y' or 'X is a (non)tangential proper part of Y') arising in geographical information systems and qualitative spatial representation and reasoning can be expressed within S4U. For example, spatial regions of the *region connection calculus* RCC-8 [51, 82, 211, 212] are interpreted as regular closed subsets of a topological space, and hence can be represented by S4U-formulas of the form $\Diamond\Box X$. The binary relations of RCC-8 can be captured using the universal modality, for instance, $[\forall](\Diamond\Box X \rightarrow \Diamond\Box Y)$ says that region X is a part of region Y. RCC-8 is *NP*-complete whereas the satisfiability problem for BRCC-8 (RCC-8 with Boolean operations on regions) in the Euclidean spaces is *PSPACE*-complete, that is, of the same complexity as S4U itself ([101, 266]).

6.3 Modal logic of metric spaces

The first paper on modal logic for metric spaces was, perhaps, the McKinsey and Tarski paper [187], though there were no special modalities for distances there. First-order modal logics for metric spaces were considered in [121]. Modal logics containing specific metric modalities

- $\exists^{<a}$ (or $\exists^{\leq a}$) for 'somewhere in the sphere of radius a excluding (or including) the boundary,' where a is a positive rational number;
- $\exists^{<a}_{>b}$ for 'somewhere at distance d with $b < d < a$,' where $b < a$ are positive rational numbers,

were introduced and studied in [166, 267, 268]. In particular, Wolter and Zakharyaschev in [268] introduced the *modal logic of metric and topology*, MT, in the language containing $\Box$ and $[\forall]$, along with the metric modalities $\exists^{<a}$ and $\exists^{\leq a}$.

THEOREM 24 (Wolter, Zakharyaschev [268]).
(1) MT *is decidable and EXPTIME-complete over arbitrary metric spaces.*
(2) MT *is decidable over the one-dimensional Euclidean space* $\mathbb{R}$.
(3) MT *over* $\mathbb{R}^2$ *with the Euclidean metric is undecidable.*

For a survey of other results and further research directions cf. [165].

6.4 Modal logic of dynamic topology

One more class of natural mathematical objects, the *topological dynamic systems*, became a subject of modal logic studies. Two independently working groups can be credited for its origin in 1997: one at Stanford (Kremer, Mints, and Rybakov), and one at Cornell (Artemov, Davoren, and Nerode). We will start by observing the results of the latter, since their approach was more general.

The basic model under consideration is a topological dynamic system $\langle \mathcal{T}, f\rangle$ consisting of a topological space $\mathcal{T} = \langle \boldsymbol{X}, \mathbb{I}\rangle$ and a total function f mapping $\boldsymbol{X}$ to $\boldsymbol{X}$. The corresponding bimodal logic consists of good old $\mathsf{S4}$ with its standard topological interpretation in $\langle \boldsymbol{X}, \mathbb{I}\rangle$, together with a unary modality $\bigcirc$ similar to the one called *the next* or *tomorrow* in temporal logic. A temporal logic was first introduced in [208, 209, 210, 264, 265].

The interpretation of the Boolean connectives is set theoretical in $\boldsymbol{X}$, $\Box$ is interpreted as the interior operation $\mathbb{I}$ on $\mathcal{T}$, and $\bigcirc Y$ is interpreted as $f^{-1}Y$, i.e., the inverse image of Y with respect to f. Hence, the interpretation of $\bigcirc$ reflects the idea of the 'next' temporal operator: $\bigcirc Y$ is the set of points of $\boldsymbol{X}$ which will land in Y 'tomorrow,' after f acts on them once.

DEFINITION 25. The basic system $\mathsf{S4F}$ of the dynamic topological logic is $\mathsf{S4}$ together with two temporal principles:

$\bigcirc(A \to B) \to (\bigcirc A \to \bigcirc B)$,
$\bigcirc(\neg A) \leftrightarrow \neg\bigcirc A$,

and the Rule of necessitation for $\bigcirc$: $\dfrac{\vdash A}{\vdash \bigcirc A}$.

The expressive power of $\mathsf{S4F}$ suffices to capture the Hoare implication $A \to \bigcirc B$, stating that with a precondition A after action f, the condition B will hold. One of the main motivations for the authors of [26] to introduce and study dynamic topological logic was to devise a logic tool for analysis of classical and hybrid control systems, where the control function is not necessarily continuous. This line of work has been pursued by Davoren in her dissertation [76], and in subsequent works.

Dynamic systems with continuous function f have been given special treatment. The bimodal language of dynamic topological logic naturally expresses continuity via the principle

Cont: $\bigcirc\Box A \to \Box\bigcirc A$,

reflecting the definition of a continuous mapping as one where an inverse image of an open set is open. Consider the logic

$$\mathsf{S4C} = \mathsf{S4F} + \textbf{Cont}\ .$$

THEOREM 26 (Artemov, Davoren, Nerode [26]).
(1) S4F *is sound and complete with respect to the class of all dynamic systems* $\langle \mathcal{T}, f\rangle$,
(2) S4C *is sound and complete with respect to the class of all dynamic systems* $\langle \mathcal{T}, f\rangle$ *where* f *is continuous on* $\mathcal{T}$.

In addition, S4F and S4C enjoy cut-elimination and the finite model property w.r.t. the corresponding class of Kripke models.

It follows from the proof that S4C is also sound and complete w.r.t. continuous dynamic systems with Alexandrov spaces (the topological equivalents of Kripke frames). Slavnov in [236] and independently Kremer and van Benthem (cf. [156, 237]) showed that the analog of the McKinsey-Tarski completeness theorem does not hold here: S4C is not complete with respect to the real topology over $\mathbb{R}$. In [237], the following weaker form of the McKinsey-Tarski theorem for S4C was established by Slavnov:

if F *is not provable in* S4C, *then* F *has a countermodel in* $\mathbb{R}^n$ *for an appropriate* n .

There is no complete axiomatization known for continuous dynamic systems over $\mathbb{R}^n$ for any specific n.

Dynamic systems $\langle \mathcal{T}, f\rangle$ with continuous f became a starting point for [154, 155, 157]. Consider the logic

$$\mathsf{S4}\bigcirc = \mathsf{S4C} + \Box\bigcirc A \to \bigcirc\Box A\ .$$

THEOREM 27 (Kremer, Mints, Rybakov [156, 157]). S4$\bigcirc$ *is sound and complete w.r.t. the following classes of dynamic systems* $\langle \mathcal{T}, f\rangle$*:*
(a) f *is a homeomorphism;*
(b) $\mathcal{T}$ *is an Alexandrov space,* f *is a homeomorphism;*
(c) $\mathcal{T}$ *is a real topology* $\mathbb{R}^n$*,* f *is a homeomorphism;*
(d) $\mathcal{T}$ *is a unit ball* $\mathcal{B}^n$*,* f *is a measure preserving homeomorphism.*

As was shown in [156], S4$\bigcirc$ has the finite model property, hence it is decidable.

The systems S4F, S4C, and S4$\bigcirc$ (along with so-called *temporal-over-topological fragment* of the dynamic topological logic from [156]), basically exhaust the list of known axiomatizability results in dynamic topology. The papers [76, 155, 157] in addition to $\Box$ and $\bigcirc$, consider the S4-type modality *henceforth*, $*$, to be borrowed from temporal logic, with an apparent goal of capturing some asymptotic behavior of the function f in a dynamic system $\langle \mathcal{T}, f\rangle$. The formal topological interpretation φ of $*B$ is

$$\varphi(*B) = \bigcap_{n\geq 0} f^{-n}\varphi(B)\ ,$$

which specifies the set of points $X \subseteq \varphi(B)$ that never leave $\varphi(B)$ under f, f^2, f^3, etc. The dual of $*$ is the modality $\sharp$, such that $\sharp B$ is interpreted as

$$\varphi(\sharp B) = \bigcup_{n \geq 0} f^{-n}\varphi(B) \ ,$$

which gives the set of points X that are either in $\varphi(B)$, or reach $\varphi(B)$ under at least one of the iterations f, f^2, f^3, etc. This third modality $*$ considerably extends the expressive power of the dynamic topological logic, making it closer to applications in dynamic systems and control theory. However, this expressive power seems to ruin good algorithmic behavior of dynamic topological logic, as is shown in the following theorem.

THEOREM 28 (Konev, Kontchakov, Wolter, Zakharyaschev [151]). *Let $\mathcal{M}$ be one of the following classes of dynamic systems $\langle \mathcal{T}, f \rangle$:*

(a) *f is a homeomorphism;*
(b) *$\mathcal{T}$ is the class of all Alexandrov spaces, f is a homeomorphism;*
(c) *$\mathcal{T}$ is a real topology $\mathbb{R}^n$, f is a homeomorphism;*
(d) *$\mathcal{T}$ is a unit ball $\mathcal{B}^n$, f is a measure preserving homeomorphism.*

Then the set of valid formulas in the language with $\{\Box, \bigcirc, *\}$ that are valid in $\mathcal{M}$ is not recursively enumerable. All these logics are different.

The proof is by reduction of the Post correspondence problem.

In addition, [151] considers logics for dynamic systems $\langle \mathcal{T}, f \rangle$, where $\mathcal{T}$ is a metric space and f an isometric function. The modal operator for topological interior $\Box$ is replaced by distance operators of the form $\exists^{\leq a}$ 'somewhere in the ball of radius a,' for a positive rational a. In contrast to the topological case, the resulting logic turns out to be decidable, but not bounded in time by any elementary function.

A follow-up paper [102] showed (using more general results on products of modal logics with expanding domains) that the dynamic topological logic interpreted in topological spaces with *continuous* functions was decidable if the number of function iterations was assumed to be *finite*, however, not in primitive recursive time. The decidability proof was based on Kruskal's tree theorem, and the proof of non-primitive recursiveness was established by reduction of the reachability problem for lossy channel systems. Note that the dynamic topological logics interpreted in topological spaces with finite iterations of homeomorphisms are not recursively enumerable.

Quite recently, by encoding the ω-reachability problem for lossy channel systems it was shown in [152] that the dynamic topological logic over some natural spaces with continuous functions is undecidable.

THEOREM 29 (Konev, Kontchakov, Wolter, Zakharyaschev [152]). *The set of formulas in the language with $\{\Box, \bigcirc, *\}$ that are valid in any of the following classes:*

(a) *all continuous dynamic systems $\langle \mathcal{T}, f \rangle$,*
(b) *continuous dynamic systems $\langle \mathcal{T}, f \rangle$ where $\mathcal{T}$ is the class of all Alexandrov spaces,*
(c) *continuous dynamic systems $\langle \mathcal{T}, f \rangle$ where $\mathcal{T}$ is a real topology $\mathbb{R}^n$, is undecidable. All these logics are different.*

This gives a solution to one of the major open problems in the area.

The remaining challenging open questions here are:

1. the decidability and axiomatizability of the dynamic topological logic in the language with $\{\Box, \bigcirc\}$ for the class of continuous dynamic systems over real topological spaces $\mathbb{R}_n$ for fixed $n = 1, 2, 3, \ldots$;

2. the axiomatizability of the dynamic topological logic in the language with $\{\Box, \bigcirc, *\}$ for the class of all continuous dynamic systems.

6.5 Other geometric notions

A number of other fundamental geometrical notions have been connected to corresponding extensions of modal logic in [2, 3, 4]. The paper [3] considered different topological and geometric structures such as connectedness, affine structure, convexity, etc., and proposed a number of languages extending the usual modal language in order to describe these structures. Some authors studied modal logics of such geometric notions as incidence, parallelism, orthogonality, and such structures as projective and affine planes. Precise references and details can be found, e.g., in [38].

The *logic of comparative similarity*, CSL, with the sole metric operator $\leftleftarrows$ for '*closer*' was introduced and investigated in [234]: $X \leftleftarrows Y$ is the set of all points of a given metric space that are closer to set X than to set Y. Despite its apparent simplicity, this langauge is quite impressive. In particular, the topological interior and closure operators as well as the universal modality can be expressed in terms of $\leftleftarrows$.

In all, the above papers contributed to making spatial and spatio-temporal reasoning a lively and actively developing area. Once again, we will refer the reader to the forthcoming collection 'The Logic of Space,' edited by Aiello, van Benthem, and Pratt-Hartmann.

6.6 Modal logic of spacetime

The Minkowski spacetime, together with the causal ($\prec$) and chronological ($\preceq$) accessibility relations, constitute Kripke-style frames which naturally have corresponding modal logics. Knowing such modal logics provide additional understanding of Minkowski's spacetime that forms the basis of Einstein's special theory of relativity. The mathematical problem of finding modal logics for chronological future modality was solved by Goldblatt [112] and Shehtman [229]; the modal logic of the chronological relation $\preceq$ turned out to be $\mathsf{S4.2}{=}\mathsf{S4}{+}\Diamond\Box F \rightarrow \Box\Diamond F$. A similar problem for causal future modality was solved by Shapirovsky and Shehtman in [223].

6.7 Topoi

Yet another incarnation of the topological semantics is given by interpreting intuitionistic modality in Grothendieck topology on a category and sheaf theory. Such an interpretation was suggested by Lawvere [173]; a relevant axiomatic system was suggested by Goldblatt in [113]. See the survey [115] and Chapter 9 of this Handbook for exact formulations and discussion. For a different connection between modalities and topos theory relying on geometric morphisms, also see Chapter 9 of this Handbook. An interesting topos-theoretic approach to modality can be found in the works of Reyes and his collaborators [172, 185, 213, 214, 215].

6.8 Universal algebra

A new research thrust in which using modal logic on classical mathematical structures makes a good sense was suggested by Goranko and Vakarelov in [118]. They have developed a uniform approach to axiomatizing various classes of traditional algebraic structures in modal logic, using the fact that difference modality is naturally definable there.

7 MODALITIES IN SET THEORY

We start with two theorems by Solovay, both published in [65], Chapter 13. These theorems gave a modal characterization of the notions of truth in all transitive models of ZF and truth in all models V_κ, where κ is inaccessible.

Let φ be a function that assigns to each propositional letter a sentence of the language of set theory. For each modal formula A, we define its interpretation, $\varphi(A)$ as follows: φ commutes with Boolean connectives, and $\varphi(\Box A)$ is the sentence of ZF that translates '$\varphi(A)$ holds in all transitive models of ZF.'

Let I be the system of modal logic that results when the principle

$$\Box(\Box A \to \Box B) \vee \Box(\Box B \to (A \wedge \Box A))$$

is added to GL as a new axiom schema.

A *universe* is a set V_κ, where κ is inaccessible. All such V_κ's are models of ZF (cf. [162]). Let ψ be defined as φ before, except that we now define $\psi(\Box A)$ as the sentence of ZF that translates '$\psi(A)$ holds in all universes.' Let J be GL plus the principle

$$\Box(\Box A \to B) \vee \Box((B \wedge \Box B) \to A)\ .$$

THEOREM 30 (Solovay, cf. [65]).

(1) $\mathsf{I} \vdash A$ *iff* $\mathsf{ZF}\vdash \varphi(A)$, *for all* φ *that translate* $\Box A$ *as 'A holds in all transitive models of* ZF*.'*

(2) $\mathsf{J} \vdash A$ *iff* $\mathsf{ZF}\vdash \psi(A)$, *for all* ψ *that translate* $\Box A$ *as 'A holds in all universes.'*

A strong connection between modal logic and non-well-founded sets has been provided by Barwise and Moss in [41] and Baltag in [39, 40]. Suppose one takes ordinary modal logic over some fixed set of atomic sentences and then considers the full infinitary propositional language generated by this. The resulting language has conjunctions and disjunctions of all sets of sentences, and this itself is a proper class of sentences. In addition to this, one can also consider the language with Boolean combinations of at most κ sentences, where κ is a cardinal number. A *pointed model* is a Kripke model with a distinguished point. Bisimulations between pointed models are ordinary bisimulations which relate the distinguished points. Barwise and Moss proved that for every pointed graph (X, x), there is a single sentence $\phi_{X,x}$ which characterizes (X, x) in the sense that for all (Y, y), $(Y, y) \models \phi_{X,x}$ iff (Y, y) is bisimilar to (X, x). The countable case of the Barwise-Moss result had been proved earlier in [249]. It also has roots in infinitary model theory: the Scott sentences there are essentially the same as the characterizing sentences for modal logic. The reason why these results are of interest in non-well-founded set theory is that one way to think about non-well-founded sets is as equivalence classes of pointed models, where the equivalence relation is just the maximum bisimulation. Incidentally, the presence of atomic sentences in the various modal logics then corresponds to the presence of *urelements* in the various set theories.

Viewing the canonical model as a structure for set theory will not give anything like a model of standard ZF because it would have a universal set. However, one can use the model to obtain a new set theory. This is what Baltag did in [39, 40]. His system STS (Structural Theory of Sets) contains a strengthening of Aczel's AFA axiom, expressed in terms of modal descriptions. Baltag's axiom SAFA, Super-Antifoundation, implies that every maximally consistent class in the infinitary modal logic characterizes some set. STS also has applications to paradoxes and to the 'large/small' distinction in set theory.

Fitting and Smullyan's book [94] is a development of forcing used in independence results, presented in the language of modal logic. The authors use modal terms to explicate many of the combinatorial issues in forcing. Forcing is not usually presented in this way, although it seems quite natural to do so, and they explore a number of affinities between modal logic and forcing.

The paper [58] by Blass presents a set theoretical interpretation of possibility and necessity, based on infinite combinatorics. This is set theoretically meaty, and the focus is on consistency results for infinite combinatorics.

Hamkins' paper [127] introduces the forcing interpretation of modal logic. The focus of the paper, however, is on the Maximality Principle, and it does not use much modal logic beyond observing that the Maximality Principle is equivalent to S5 under the forcing interpretation. The Ph.D. dissertation of Hamkins' student Leibman [175] explores the forcing interpretation of modal logic a bit further. In a recent paper [128] by Hamkins and Löwe, it was proved that the ZFC-provable modal validities for this interpretation are exactly S4.2. There are a large number of open questions in this area.

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