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ON THE LOGIC OF THEORY CHANGE: PARTIAL MEET CONTRACTION AND REVISION FUNCTIONS

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Abstract. This paper extends earlier work by its authors on formal aspects of the processes of contracting a theory to eliminate a proposition and revising a theory to introduce a proposition. In the course of the earlier work, Gärdenfors developed general postulates of a more or less equational nature for such processes, whilst Alchourrón and Makinson studied the particular case of contraction functions that are maximal, in the sense of yielding a maximal subset of the theory (or alternatively, of one of its axiomatic bases), that fails to imply the proposition being eliminated.

In the present paper, the authors study a broader class, including contraction functions that may be less than maximal. Specifically, they investigate "partial meet contraction functions", which are defined to yield the intersection of some nonempty family of maximal subsets of the theory that fail to imply the proposition being eliminated. Basic properties of these functions are established: it is shown in particular that they satisfy the Gärdenfors postulates, and moreover that they are sufficiently general to provide a representation theorem for those postulates. Some special classes of partial meet contraction functions, notably those that are "relational" and "transitively relational", are studied in detail, and their connections with certain "supplementary postulates" of Gärdenfors investigated, with a further representation theorem established.

§1. Background. The simplest and best known form of theory change is *expansion*, where a new proposition (axiom), hopefully consistent with a given theory A, is set-theoretically added to A, and this expanded set is then closed under logical consequence. There are, however, other kinds of theory change, the logic of which is less well understood. One form is theory *contraction*, where a proposition x, which was earlier in a theory A, is rejected. When A is a code of norms, this process is known among legal theorists as the *derogation* of x from A. The central problem is to determine which propositions should be rejected along with x so that the contracted theory will be closed under logical consequence. Another kind of change is *revision*, where a proposition x, inconsistent with a given theory A, is added to A under the requirement that the revised theory be consistent and closed under logical consequence. In normative contexts this kind of change is also known as *amendment*.

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A basic formal problem for the processes of contraction and revision is to give a characterization of ideal forms of such change. In [3] and [4], Gärdenfors developed postulates of a more or less equational nature to capture the basic properties of these processes. It was also argued there that the process of revision can be reduced to that of contraction via the so-called Levi identity: if $A \dot{-} x$ denotes the contraction of A by x, then the revision of A by x, denoted $A \dot{+} x$, can be defined as $\mathrm{Cn}((A \dot{-} \neg x) \cup \{x\})$, where Cn is a given consequence operation.

In [2], Alchourrón and Makinson tried to give a more explicit construction of the contraction process, and hence also of the revision process via the Levi identity. Their basic idea was to choose $A \dot{-} x$ as a *maximal* subset of A that fails to imply x. Contraction functions defined in this way were called "choice contractions" in [2], but will here be more graphically referred to as "maxichoice contractions".

As was observed in [2], the maxichoice functions have, however, some rather disconcerting properties. In particular, maxichoice revision $\dot{+}$, defined from maxichoice contraction as above, has the property that for every theory A, whether complete or not, $A \dot{+} x$ will be complete whenever x is a proposition inconsistent with A. Underlying this is the fact, also noted in [2], that when A is a theory with $x \in A$, then for every proposition y, either $(x \vee y) \in A \dot{-} x$ or $(x \vee \neg y) \in A \dot{-} x$, where $\dot{-}$ is maxichoice contraction. The significance of these formal results is discussed briefly in [2], and in more detail in Gärdenfors [5] and Makinson [6].

The "inflation properties" that ensue from applying the maxichoice operations bring out the interest of looking at other formal operations that yield smaller sets as values. In this paper, we will start out from the assumption that there is a selection function γ that picks out a class of the "most important" maximal subsets of A that fail to imply x. The contraction $A \dot{-} x$ is then defined as the intersection of all the maximal subsets selected by γ. Functions defined in this way will be called *partial meet contraction* functions, and their corresponding revision functions will be called *partial meet revision* functions. It will be shown that they satisfy Gärdenfors' postulates, and indeed provide a representation theorem for those postulates. When constrained in suitable ways, by relations or, more restrictedly, by transitive relations, they also satisfy his "supplementary postulates", and provide another representation theorem for the entire collection of "basic" plus "supplementary" postulates.

Acquaintance with [6] will help the reader with overall perspective, but it is not necessary for technical details.

Some background terminology and notation: By a *consequence operation* we mean, as is customary, an operation Cn that takes sets of propositions to sets of propositions, such that three conditions are satisfied, for any sets X and Y of propositions: $X \subseteq \mathrm{Cn}(X)$, $\mathrm{Cn}(X) = \mathrm{Cn}(\mathrm{Cn}(X))$, and $\mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$ whenever $X \subseteq Y$. To simplify notation, we write $\mathrm{Cn}(x)$ for $\mathrm{Cn}(\{x\})$, where x is any individual proposition, and we also sometimes write $y \in \mathrm{Cn}(X)$ as $X \vdash y$. By a *theory*, we mean, as is customary, a set A of propositions that is closed under Cn; that is, such that $A = \mathrm{Cn}(A)$, or, equivalently, such that $A = \mathrm{Cn}(B)$ for some set B of propositions. As in [2], we assume that Cn includes classical tautological implication, is compact (that is, $y \in \mathrm{Cn}(X')$ for some finite subset X' of X whenever $y \in \mathrm{Cn}(X)$), and satisfies the rule of "introduction of disjunctions in the premises" (that is, y

$\in \mathrm{Cn}(X \cup \{x_1 \vee x_2\})$ whenever $y \in \mathrm{Cn}(X \cup \{x_1\})$ and $y \in \mathrm{Cn}(X \cup \{x_2\})$). We say that a set X of propositions is *consistent* (modulo Cn) iff for no proposition y do we have $y \,\&\, \neg y \in \mathrm{Cn}(X)$.

§2. Partial meet contraction. Let Cn be any consequence operation over a language, satisfying the conditions mentioned at the end of the preceding section, and let A be any set of propositions. As in [1] and [2], we define $A \perp x$ to be the set of all maximal subsets B of A such that $B \nvdash x$. The maxichoice contraction functions $\dot{-}$ studied in [2] put $A \dot{-} x$ to be an arbitrary element of $A \perp x$ whenever the latter is nonempty, and to be A itself in the limiting case that $A \perp x$ is empty. In the search for suitable functions with smaller values, it is tempting to try the operation $A \sim x$ defined as $\bigcap(A \perp x)$ when $A \perp x$ is nonempty, and as A itself in the limiting case that $A \perp x$ is empty. But as shown in Observation 2.1 of [2], this set is in general far too *small*. In particular, when A is a theory with $x \in A$, then $A \sim x = A \cap \mathrm{Cn}(\neg x)$. In other words, the only propositions left in $A \sim x$ when A is a theory containing x are those which are already consequences of $\neg x$ considered alone. And thus, as noted in Observation 2.2 of [2], if revision is introduced as usual via the Levi identity as $\mathrm{Cn}((A \sim \neg x) \cup \{x\})$, it reduces to $\mathrm{Cn}((A \cap \mathrm{Cn}(x)) \cup \{x\}) = \mathrm{Cn}(x)$, for any theory A and proposition x inconsistent with A. In other words, if we revise a theory A in *this* way to bring in a proposition x inconsistent with A, we get no more than the set of consequences of x considered alone—a set which is far too small in general to represent the result of an intuitive process of revision of A so as to bring in x.

Nevertheless, the operation of *meet contraction*, as we shall call $\sim$, is very useful as a point of reference. It serves as a natural *lower bound* on any reasonable contraction operation: any contraction operation $\dot{-}$ worthy of the name should surely have $A \sim x \subseteq A \dot{-} x$ for all A, x, and a function $\dot{-}$ satisfying this condition for a given A will be called *bounded over A*.

Following this lead, let A be any set of propositions and let γ be any function such that for every proposition x, $\gamma(A \perp x)$ is a nonempty subset of $A \perp x$, if the latter is nonempty, and $\gamma(A \perp x) = \{A\}$ in the limiting case that $A \perp x$ is empty. We call such a function a *selection function* for A. Then the operation $\dot{-}$ defined by putting $A \dot{-} x = \bigcap\gamma(A \perp x)$ for all x is called the *partial meet contraction over A determined by* γ. The intuitive idea is that the selection function γ picks out those elements in $A \perp x$ which are "most important" (for a discussion of this notion cf. Gärdenfors [5]) and then the contraction $A \dot{-} x$ contains the propositions which are common to the selected elements of $A \perp x$. *Partial meet revision* is defined via the Levi identity as $A \dot{+} x = \mathrm{Cn}((A \dot{-} \neg x) \cup \{x\})$. Note that the identity of $A \dot{-} x$ and $A \dot{+} x$ depends on the choice function γ, as well, of course, as on the underlying consequence operation Cn. Note also that the concept of partial meet contraction includes, as special cases, those of maxichoice contraction and (full) meet contraction. The former is partial meet contraction with $\gamma(A \perp x)$ a singleton; the latter is partial meet contraction with $\gamma(A \perp x)$ the entire set $A \perp x$. We use the same symbols $\dot{-}$ and $\dot{+}$ here as for the maxichoice operations in [2]; this should not cause any confusion.

Our first task is to show that all partial meet contraction and revision functions satisfy Gärdenfors' postulates for contraction and revision. We recall (cf. [2] and [6]) that these postulates may conveniently be formulated as follows:

($\dot{-}$1) $A \dot{-} x$ is a theory whenever A is a theory (closure).
($\dot{-}$2) $A \dot{-} x \subseteq A$ (inclusion).
($\dot{-}$3) If $x \notin \mathrm{Cn}(A)$, then $A \dot{-} x = A$ (vacuity).
($\dot{-}$4) If $x \notin \mathrm{Cn}(\varnothing)$, then $x \notin \mathrm{Cn}(A \dot{-} x)$ (success).
($\dot{-}$5) If $\mathrm{Cn}(x) = \mathrm{Cn}(y)$, then $A \dot{-} x = A \dot{-} y$ (preservation).
($\dot{-}$6) $A \subseteq \mathrm{Cn}((A \dot{-} x) \cup \{x\})$ whenever A is a theory (recovery).

The Gärdenfors postulates for revision may likewise be conveniently formulated as follows:

($\dot{+}$1) $A \dot{+} x$ is always a theory.
($\dot{+}$2) $x \in A + x$.
($\dot{+}$3) If $\neg x \notin \mathrm{Cn}(A)$, then $A \dot{+} x = \mathrm{Cn}(A \cup \{x\})$.
($\dot{+}$4) If $\neg x \notin \mathrm{Cn}(\varnothing)$, then $A \dot{+} x$ is consistent under Cn.
($\dot{+}$5) If $\mathrm{Cn}(x) = \mathrm{Cn}(y)$, then $A \dot{+} x = A \dot{+} y$.
($\dot{+}$6) $(A \dot{+} x) \cap A = A \dot{-} \neg x$, whenever A is a theory.

Our first lemma tells us that even the very weak operation of (full) meet contraction satisfies recovery.

LEMMA 2.1. *Let A be any theory. Then* $A \subseteq \mathrm{Cn}((A \sim x) \cup \{x\})$.

PROOF. In the limiting case that $x \notin A$ we have $A \sim x = A$ and we are done. Suppose $x \in A$. Then, by Observation 2.1 of [2], we have $A \sim x = A \cap \mathrm{Cn}(\neg x)$ so it will suffice to show $A \subseteq \mathrm{Cn}((A \cap \mathrm{Cn}(\neg x)) \cup \{x\})$. Let $a \in A$. Then since A is a theory, $\neg x \vee a \in A$. Also $\neg x \vee a \in \mathrm{Cn}(\neg x)$, so $\neg x \vee a \in A \cap \mathrm{Cn}(\neg x)$, so since Cn includes tautological implication, $a \in \mathrm{Cn}((A \cap \mathrm{Cn}(\neg x)) \cup \{x\})$. □

COROLLARY 2.2. *Let $\dot{-}$ be any function on pairs A, x. Let A be any theory. If $\dot{-}$ is bounded over A, then $\dot{-}$ satisfies recovery over A.*

OBSERVATION 2.3. *Every partial meet contraction function $\dot{-}$ satisfies the Gärdenfors postulates for contraction, and its associated partial meet revision function satisfies the Gärdenfors postulates for revision.*

PROOF. It is easy to show (cf. [3] and [4]) that the postulates for revision can all be derived from those for contraction via the Levi identity. So we need only verify the postulates for contraction. Closure holds, because when A is a theory, so too is each $B \in A \perp x$, and the intersection of theories is a theory; inclusion is immediate; vacuity holds because when $x \notin \mathrm{Cn}(A)$ then $A \perp x = \{A\}$ so $\gamma(A \perp x) = \{A\}$; success holds because when $x \notin \mathrm{Cn}(\varnothing)$ then by compactness, as noted in Observation 2.2 of [1], $A \perp x$ is nonempty and so $A \dot{-} x = \bigcap \gamma(A \perp x) \nvdash x$; and preservation holds because the choice function is defined on families $A \perp x$ rather than simply on pairs A, x, so that when $\mathrm{Cn}(x) = \mathrm{Cn}(y)$ we have $A \perp x = A \perp y$, so that $\gamma(A \perp x) = \gamma(A \perp y)$. Finally, partial meet contraction is clearly bounded over any set A, and so by Corollary 2.2 satisfies recovery. □

In fact, we can also prove a converse to Observation 2.3, and show that for theories, the Gärdenfors postulates for contraction *fully characterize* the class of partial meet contraction functions. To do this we first establish a useful general lemma related to 7.2 of [2].

LEMMA 2.4. *Let A be a theory and x a proposition. If $B \in A \perp x$, then $B \in A \perp y$ for all $y \in A$ such that $B \nvdash y$.*

PROOF. Suppose $B \in A \perp x$ and $B \nvdash y$, $y \in A$. To show that $B \in A \perp y$ it will suffice to show that whenever $B \subset B' \subseteq A$, then $B' \vdash y$. Let $B \subset B' \subseteq A$. Since $B \in A \perp$

x we have $B' \vdash x$. But also, since $B \in A \perp x$, $A \perp x$ is nonempty, so $A \sim x = \bigcap(A \perp x) \subseteq B$; so, using Lemma 2.1, $A \subseteq \mathrm{Cn}(B \cup \{x\}) \subseteq \mathrm{Cn}(B' \cup \{x\}) = \mathrm{Cn}(B')$, so since $y \in A$ we have $B' \vdash y$. □

OBSERVATION 2.5. *Let $\dot{-}$ be a function defined for sets A of propositions and propositions x. For every theory A, $\dot{-}$ is a partial meet contraction operation over A iff $\dot{-}$ satisfies the Gärdenfors postulates ($\dot{-}$1)–($\dot{-}$6) for contraction over A.*

PROOF. We have left to right by Observation 2.3. For the converse, suppose that $\dot{-}$ satisfies the Gärdenfors postulates over A. To show that $\dot{-}$ is a partial meet contraction operation, it will suffice to find a function such that:

(i) $\gamma(A \perp x) = \{A\}$ in the limiting case that $A \perp x$ is empty,

(ii) $\gamma(A \perp x)$ is a nonempty subset of $A \perp x$ when $A \perp x$ is nonempty, and

(iii) $A \dot{-} x = \bigcap \gamma(A \perp x)$.

Put $\gamma(A \perp x)$ to be $\{A\}$ when $A \perp x$ is empty, and to be $\{B \in A \perp x : A \dot{-} x \subseteq B\}$ otherwise. Then (i) holds immediately. When $A \perp x$ is nonempty, then $x \notin \mathrm{Cn}(\varnothing)$ so by the postulate of success $A \dot{-} x \nvdash x$, so, using compactness, $\gamma(A \perp x)$ is nonempty, and clearly $\gamma(A \perp x) \subseteq A \perp x$, so (ii) also holds. For (iii) we have the inclusion $A \dot{-} x \subseteq \bigcap \gamma(A \perp x)$ immediately from the definition of γ. So it remains only to show that $\bigcap \gamma(A \perp x) \subseteq A \dot{-} x$.

In the case that $x \notin A$ we have by the postulate of vacuity that $A \dot{-} x = A$, so the desired conclusion holds trivially. Suppose then that $x \in A$, and suppose $a \notin A \dot{-} x$; we want to show that $a \notin \bigcap \gamma(A \perp x)$. In the case $a \notin A$, this holds trivially, so we suppose that $a \in A$. We need to find a $B \in A \perp x$ with $A \dot{-} x \subseteq B$ and $a \notin B$. Since $\dot{-}$ satisfies the postulate of recovery, and $a \in A$, we have $(A \dot{-} x) \cup \{x\} \vdash a$. But, by hypothesis, $a \notin A \dot{-} x = \mathrm{Cn}(A \dot{-} x)$ by the postulate of closure, so since Cn includes tautological implication and satisfies disjunction of premises, $(A \dot{-} x) \cup \{\neg x\} \nvdash a$, so $A \dot{-} x \nvdash x \vee a$. Hence by compactness there is a $B \in A \perp (x \vee a)$ with $A \dot{-} x \subseteq B$. Since $B \in A \perp (x \vee a)$ we have $B \nvdash x \vee a$, so $a \notin B$. And also since $B \nvdash x \vee a$ we have $B \nvdash x$, so, by Lemma 2.4, and the hypothesis that $x \in A$, we have $B \in A \perp x$, and the proof is complete. □

A corollary of Observation 2.5 is that whenever $\dot{-}$ satisfies the Gärdenfors postulates for contraction over a theory A, then it is bounded over A. However, this consequence can also be obtained, under slightly weaker conditions, by a more direct argument. We first note the following partial converse of Lemma 2.1.

LEMMA 2.6. *Let A be any theory. Then for every set B and every $x \in A$, if $A \subseteq \mathrm{Cn}(B \cup \{x\})$, then $A \sim x \subseteq \mathrm{Cn}(B)$.*

PROOF. Suppose $x \in A$, $A \subseteq \mathrm{Cn}(B \cup \{x\})$, and $a \in A \sim x$; we want to show that $a \in \mathrm{Cn}(B)$. Since A is a theory and $x \in A$ we have $A \sim x = \mathrm{Cn}(\neg x) \cap A$ by Observation 2.1 of [2]; so $\neg x \vdash a$, so $B \cup \{\neg x\} \vdash a$. But also since $a \in A \sim x \subseteq A \subseteq \mathrm{Cn}(B \cup \{x\})$ we have $B \cup \{x\} \vdash a$, so by disjunction of premises and the fact that Cn includes tautological implication, we have $a \in \mathrm{Cn}(B)$. □

OBSERVATION 2.7. *Let $\dot{-}$ be any function on pairs A, x. Let A be a theory. If $\dot{-}$ satisfies closure, vacuity and recovery over A, then $\dot{-}$ is bounded over A.*

PROOF. Suppose $\dot{-}$ satisfies closure, vacuity and recovery over A. Let x be any proposition; we need to show $A \sim x \subseteq A \dot{-} x$. In the case $x \notin A$ we have trivially $A \sim x = A \dot{-} x$ by vacuity. In the case $x \in A$ we have $A \subseteq \mathrm{Cn}((A \dot{-} x) \cup \{x\})$ by recovery, so $A \sim x \subseteq \mathrm{Cn}(A \dot{-} x) = A \dot{-} x$ by Lemma 2.6 and closure. □

§3. Supplementary postulates for contraction and revision. Gärdenfors [5] has suggested that revision should also satisfy two further "supplementary postulates", namely:

($\dotplus$7) $A \dotplus (x \& y) \subseteq \mathrm{Cn}((A \dotplus x) \cup \{y\})$ for any theory A,

and its conditional converse:

($\dotplus$8) $\mathrm{Cn}((A \dotplus x) \cup \{y\}) \subseteq A \dotplus (x \& y)$ for any theory A, provided that $\neg y \notin A \dotplus x$.

Given the presence of the postulates ($\dot{-}$1)–($\dot{-}$6) and ($\dotplus$1)–($\dotplus$6), these two supplementary postulates for $\dotplus$ can be shown to be equivalent to various conditions on $\dot{-}$. Some such conditions are given in [5]; these can however be simplified, and one particularly simple pair, equivalent respectively to ($\dotplus$7) and ($\dotplus$8), are:

($\dot{-}$7) $(A \dot{-} x) \cap (A \dot{-} y) \subseteq A \dot{-} (x \& y)$ for any theory A.

($\dot{-}$8) $A \dot{-} (x \& y) \subseteq A \dot{-} x$ whenever $x \notin A \dot{-} (x \& y)$, for any theory A.

OBSERVATION 3.1. *Let $\dot{-}$ be any partial meet contraction operation over a theory A. Then it satisfies ($\dot{-}$7) iff it satisfies ($\dotplus$7).*

PROOF. We recall that $\dotplus$ is defined by the Levi identity $A \dotplus x = \mathrm{Cn}((A \dot{-} \neg x) \cup \{x\})$. Let A be any theory and suppose that ($\dot{-}$7) holds for all x and y. We want to show that ($\dotplus$7) holds for all x and y. Let

$$w \in A \dotplus (x \& y) = \mathrm{Cn}((A \dot{-} \neg(x \& y)) \cup \{x \& y\}).$$

We need to show that

$$\begin{aligned} w \in \mathrm{Cn}((A \dotplus x) \cup \{y\}) &= \mathrm{Cn}(\mathrm{Cn}((A \dot{-} \neg x) \cup \{x\}) \cup \{y\}) \\ &= \mathrm{Cn}((A \dot{-} \neg x) \cup \{x \& y\}) \end{aligned}$$

by general properties of consequence operations. Noting that

$$\mathrm{Cn}(\neg x) = \mathrm{Cn}(\neg(x \& y) \& (\neg x \vee y)),$$

it will suffice by condition ($\dot{-}$7) to show that

$$w \in \mathrm{Cn}((A \dot{-} \neg(x \& y)) \cup \{x \& y\}) \quad \text{and} \quad w \in \mathrm{Cn}((A \dot{-} (\neg x \vee y)) \cup \{x \& y\}).$$

But the former is given by hypothesis, so we need only verify the latter. Now by the former, we have $w \in \mathrm{Cn}(A \cup \{x \& y\})$, so it will suffice to show that

$$A \cup \{x \& y\} \subseteq \mathrm{Cn}((A \dot{-} (\neg x \vee y)) \cup \{x \& y\}).$$

But clearly $x \& y \in \mathrm{RHS}$, and moreover since $x \& y \vdash y \vdash \neg x \vee y$ we have by recovery that $A \subseteq \mathrm{RHS}$, and we are done.

For the converse, suppose that ($\dotplus$7) holds for all x, y. Let $a \in (A \dot{-} x) \cap (A \dot{-} y)$; we need to show that $a \in A \dot{-} (x \& y)$. Noting that

$$\mathrm{Cn}(x) = \mathrm{Cn}(\neg((\neg x \vee \neg y) \& \neg x)),$$

we have

$$\begin{aligned} a \in A \dot{-} \neg((\neg x \vee \neg y) \& \neg x) &\subseteq A \dotplus ((\neg x \vee \neg y) \& \neg x) \\ &\subseteq \mathrm{Cn}((A \dotplus (\neg x \vee \neg y)) \cup \{\neg x\}). \end{aligned}$$

A similar reasoning gives us also $a \in \mathrm{Cn}((A \dotplus (\neg x \vee \neg y)) \cup \{\neg y\})$. So applying disjunction of premises and the fact that Cn includes tautological implication, we have

$$a \in \mathrm{Cn}(A \dotplus (\neg x \vee \neg y)) = A \dotplus (\neg x \vee \neg y) = \mathrm{Cn}((A \dot{-} (x \,\&\, y)) \cup \{\neg(x \,\&\, y)\}).$$

But by recovery we also have $a \in \mathrm{Cn}((A \dot{-} (x \,\&\, y)) \cup \{x \,\&\, y\})$, so, again using disjunction of premises,

$$a \in \mathrm{Cn}(A \dot{-} (x \,\&\, y)) = A \dot{-} (x \,\&\, y)$$

by closure, and we are done. □

OBSERVATION 3.2. *Let* $\dot{-}$ *be any partial meet contraction function over a theory* A. *Then it satisfies* $(\dot{-}8)$ *iff it satisfies* $(\dotplus 8)$.

PROOF. Let A be a theory and suppose that $(\dot{-}8)$ holds for all x and y. We want to show that $(\dotplus 8)$ holds for all x and y. Noting that $\mathrm{Cn}(\neg x) = \mathrm{Cn}((\neg x \vee \neg y) \,\&\, \neg x)$ we have $A \dot{-} \neg x = A \dot{-} ((\neg x \vee \neg y) \,\&\, \neg x)$. But also, supposing for $(\dotplus 8)$ that $\neg y \notin A \dotplus x = \mathrm{Cn}((A \dot{-} \neg x) \cup \{x\})$, we have $\neg x \vee \neg y \notin A \dot{-} \neg x$. We may thus apply $(\dot{-}8)$ to get

$$A \dot{-} \neg x = A \dot{-} ((\neg x \vee \neg y) \,\&\, \neg x) \subseteq A \dot{-} (\neg x \vee \neg y) = A \dot{-} \neg(x \,\&\, y).$$

This inclusion justifies the inclusion in the following chain, whose other steps are trivial:

$$\begin{aligned} \mathrm{Cn}((A \dotplus x) \cup \{y\}) &= \mathrm{Cn}(\mathrm{Cn}((A \dot{-} \neg x) \cup \{x\}) \cup \{y\}) \\ &= \mathrm{Cn}((A \dot{-} \neg x) \cup \{x \,\&\, y\}) \subseteq \mathrm{Cn}((A \dot{-} \neg(x \,\&\, y)) \cup \{x \,\&\, y\}) \\ &= A \dotplus (x \,\&\, y). \end{aligned}$$

For the converse, suppose $(\dotplus 8)$ holds for all x and y, and suppose $x \notin A \dot{-} (x \,\&\, y)$. Then clearly

$$x \notin \mathrm{Cn}(A \dot{-} (x \,\&\, y) \cup \{\neg x \vee \neg y\}) = A \dotplus \neg(x \,\&\, y),$$

so we may apply $(\dotplus 8)$ to get

$$\begin{aligned} \mathrm{Cn}((A \dotplus \neg(x \,\&\, y)) \cup \{\neg x\}) &\subseteq A \dotplus (\neg(x \,\&\, y) \,\&\, \neg x) = A \dotplus \neg x \\ &= \mathrm{Cn}((A \dot{-} x) \cup \{\neg x\}). \end{aligned}$$

Thus, since $A \dot{-} (x \,\&\, y)$ is included in the leftmost term of this series, we have

$$A \dot{-} (x \,\&\, y) \subseteq \mathrm{Cn}((A \dot{-} x) \cup \{\neg x\}).$$

But using recovery we also have $A \dot{-} (x \,\&\, y) \subseteq A \subseteq \mathrm{Cn}((A \dot{-} x) \cup \{x\})$, so by disjunction of premises and the fact that Cn includes tautological implication, we have $A \dot{-} (x \,\&\, y) \subseteq \mathrm{Cn}(A \dot{-} x) = A \dot{-} x$ by closure, as desired. □

We end this section with some further observations on the powers of $(\dot{-}7)$ and $(\dot{-}8)$. Now postulate $(\dot{-}7)$ does not tell us that $A \dot{-} x$ and $A \dot{-} y$, considered separately, are included in $A \dot{-} (x \,\&\, y)$. But it goes close to it, for it does yield the following "partial antitony" property.

OBSERVATION 3.3. *Let* $\dot{-}$ *be any partial meet contraction function over a theory* A. *Then* $\dot{-}$ *satisfies* $(\dot{-}7)$ *iff it satisfies the condition*

$(\dot{-}\mathrm{P})$ $(A \dot{-} x) \cap \mathrm{Cn}(x) \subseteq A \dot{-} (x \,\&\, y)$ *for all* x *and* y.

PROOF. Suppose ($\dot{-}$7) is satisfied. Suppose $w \in A \dot{-} x$ and $x \vdash w$; we want to show that $w \in A \dot{-} (x \,\&\, y)$. If $x \notin A$ or $y \notin A$, then trivially $A \dot{-} (x \,\&\, y) = A$, so $w \in A \dot{-} (x \,\&\, y)$. So suppose that $x \in A$ and $y \in A$. Now

$$A \dot{-} (x \,\&\, y) = A \dot{-} ((\neg x \vee y) \,\&\, x),$$

so by ($\dot{-}$7) it will suffice to show that $w \in A \dot{-} (\neg x \vee y)$ and $w \in A \dot{-} x$. We have the latter by supposition. As for the former, recovery gives us $A \dot{-} (\neg x \vee y) \cup \{\neg x \vee y\} \vdash x$, so $A \dot{-} (\neg x \vee y) \cup \{\neg x\} \vdash x$, so $A \dot{-} (\neg x \vee y) \vdash x \vdash w$, so $w \in A \dot{-} (\neg x \vee y)$.

For the converse, suppose ($\dot{-}$P) is satisfied, and suppose $w \in (A \dot{-} x) \cap (A \dot{-} y)$; we want to show that $w \in A \dot{-} (x \,\&\, y)$. Since $w \in A \dot{-} x$, we have $x \vee w \in A \dot{-} x$, and so since $x \vdash x \vee w$, ($\dot{-}$P) gives us $x \vee w \in A \dot{-} (x \,\&\, y)$. Similarly, $y \vee w \in A \dot{-} (x \,\&\, y)$. Hence $w \vee (x \,\&\, y) = (x \vee w) \,\&\, (y \vee w) \in A \dot{-} (x \,\&\, y)$. But by recovery, $A \dot{-} (x \,\&\, y) \cup \{x \,\&\, y\} \vdash w$, so $w \vee \neg(x \,\&\, y) \in A \dot{-} (x \,\&\, y)$. Putting these together gives us $w \in A \dot{-} (x \,\&\, y)$ as desired. □

Condition ($\dot{-}$8) is related to another condition, which we shall call the *covering condition*:

($\dot{-}$C) For any propositions x, y, $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$ or $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} y$.

OBSERVATION 3.4. *Let $\dot{-}$ be any partial meet contraction function over a theory A. If $\dot{-}$ satisfies ($\dot{-}$8) over A, then it satisfies the covering condition ($\dot{-}$C) over A.*

PROOF. Let x and y be propositions. In the case $x \,\&\, y \in \mathrm{Cn}(\varnothing)$ we have, say, $x \in \mathrm{Cn}(\varnothing)$; so $A \dot{-} (x \,\&\, y) = A = A \dot{-} x$ and we are done. In the case $x \,\&\, y \notin \mathrm{Cn}(\varnothing)$, then by success we have $x \,\&\, y \notin A \dot{-} (x \,\&\, y)$, so either $x \notin A \dot{-} (x \,\&\, y)$ or $y \notin A \dot{-} (x \,\&\, y)$, so by ($\dot{-}$8) either $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$ or $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} y$. □

However, the converse of Observation 3.4 fails. For as we shall show at the end of the next section, there is a theory, finite modulo Cn, with a partial meet contraction over A that satisfies the covering condition (and indeed also supplementary postulate ($\dot{-}$7)), but that does not satisfy ($\dot{-}$8). Using Observation 3.4, it is easy to show that when A is a theory and $\dot{-}$ satisfies postulates ($\dot{-}$1)–($\dot{-}$6), then ($\dot{-}$8) can equivalently be formulated as $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$ whenever $x \notin A \dot{-} y$.

In [2], it was shown that whilst the *maxichoice* operations do not in general satisfy ($\dot{+}$7) and ($\dot{+}$8), they do so when constrained by a relational condition of "orderliness". Indeed, it was shown that for the maxichoice operations, the conditions ($\dot{+}$7), ($\dot{+}$8), and orderliness are mutually equivalent, and also equivalent to various other conditions. Now as we have just remarked, in the general context of partial meet contraction, ($\dot{-}$7) does not imply ($\dot{-}$8), and it can also be shown by an example (briefly described at the end of next section) that the converse implication likewise fails. The question nevertheless remains whether there are relational constraints on the partial meet operations that correspond, perfectly or in part, to the supplementary postulates ($\dot{-}$7) and ($\dot{-}$8). That is the principal theme of the next section.

§4. Partial meet contraction with relational constraints. Let A be a set of propositions and γ a selection function for A. We say that γ is *relational* over A iff there is a relation $\leq$ over 2^A such that for all $x \notin \mathrm{Cn}(\varnothing)$, $\leq$ *marks off* $\gamma(A \perp x)$ in the sense that the following identity, which we call the *marking off* identity, holds:

$$\gamma(A \perp x) = \{B \in A \perp x: B' \leq B \text{ for all } B' \in A \perp x\}.$$

Roughly speaking, γ is relational over A iff there is some relation that marks off the elements of $\gamma(A \perp x)$ as the *best* elements of $A \perp x$, whenever the latter is nonempty. Note that in this definition, $\leq$ is required to be fixed for all choices of x; otherwise all partial meet contraction functions would be trivially relational. Note also that the definition does not require any special properties of $\leq$ apart from being a relation; if there is a transitive relation $\leq$ such that for all $x \notin \mathrm{Cn}(\varnothing)$ the marking off identity holds, then γ is said to be *transitively relational* over A. Finally, we say that a partial meet contraction function $\dot{-}$ is relational (transitively relational) over A iff it is determined by some selection function that is so. "Some", because a single partial meet contraction function may, in the infinite case, be determined by two distinct selection functions. In the finite case, however, this cannot happen, as we shall show in Observation 4.6.

Relationality is linked with supplementary postulate $(\dot{-}7)$, and transitive relationality even more closely linked with the conjunction of $(\dot{-}7)$ and $(\dot{-}8)$. Indeed, we shall show, in the first group of results of this section, that a partial meet contraction function $\dot{-}$ is transitively relational iff $(\dot{-}7)$ and $(\dot{-}8)$ are both satisfied. In the later part of this section we shall describe the rather more complex relationship between relationality and $(\dot{-}7)$ considered alone. It will be useful to consider various further conditions, and two that are of immediate assistance are:

(γ7) $\gamma(A \perp x \,\&\, y) \subseteq \gamma(A \perp x) \cup \gamma(A \perp y)$, for all x and y.

(γ8) $\gamma(A \perp x) \subseteq \gamma(A \perp x \,\&\, y)$ whenever $A \perp x \cap \gamma(A \perp x \,\&\, y) \neq \varnothing$.

As with $(\dot{-}8)$, it is easy to show that when A is a theory and γ is a selection function over A, then (γ8) can equivalently be formulated as

$$\gamma(A \perp x) \subseteq \gamma(A \perp x \,\&\, y) \quad \text{whenever } A \perp x \cap \gamma(A \perp y) \neq \varnothing.$$

The following lemma will also be needed throughout the section.

LEMMA 4.1. *Let A be any theory and $x, y \in A$. Then $A \perp (x \,\&\, y) = A \perp x \cup A \perp y$.*

PROOF. We apply Lemma 2.4. When $B \in A \perp (x \,\&\, y)$, then $B \nvdash x \,\&\, y$ so $B \nvdash x$ or $B \nvdash y$, so by 2.4 either $B \in A \perp x$ or $B \in A \perp y$. Conversely, if $B \in A \perp x$ or $B \in A \perp y$, then $B \nvdash x \,\&\, y$ so, by 2.4 again, $B \in A \perp (x \,\&\, y)$. □

OBSERVATION 4.2. *Let A be a theory and $\dot{-}$ a partial meet contraction function over A determined by a selection function γ. If γ satisfies the condition* (γ7), *then $\dot{-}$ satisfies* $(\dot{-}7)$, *and if it satisfies* (γ8), *then $\dot{-}$ satisfies* $(\dot{-}8)$.

PROOF. Suppose (γ7) holds. Then we have:

$$\begin{aligned}(A \dot{-} x) \cap (A \dot{-} y) &= \textstyle\bigcap \gamma(A \perp x) \cap \bigcap \gamma(A \perp y) && \text{since } \gamma \text{ determines } \dot{-}\\ &= \textstyle\bigcap (\gamma(A \perp x) \cup \gamma(A \perp y)) && \text{by general set theory}\\ &\subseteq \textstyle\bigcap \gamma(A \perp (x \,\&\, y)) && \text{using condition } (\gamma 7)\\ &= A \dot{-} (x \,\&\, y).\end{aligned}$$

Suppose now that (γ8) holds, and suppose $x \notin A \dot{-} (x \,\&\, y)$; that is, $x \notin \bigcap \gamma(A \perp x \,\&\, y)$. We need to show that $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$. In the case $x \notin A$ we have $A \dot{-} (x \,\&\, y) = A = A \dot{-} x$. So suppose $x \in A$. Since $x \notin \bigcap \gamma(A \perp x \,\&\, y)$ there is a $B \in \gamma(A \perp x \,\&\, y)$ with $B \nvdash x$, so, by Observation 2.4, $B \in A \perp x$ and thus $B \in A \perp x \cap \gamma(A \perp x \,\&\, y)$. Applying ($\gamma$8) we have $\gamma(A \perp x) \subseteq \gamma(A \perp x \,\&\, y)$, so $A \dot{-} (x \,\&\, y) = \bigcap \gamma(A \perp x \,\&\, y) \subseteq \bigcap \gamma(A \perp x) = A \dot{-} x$ as desired. □

OBSERVATION 4.3. *Let A be any theory and γ a selection function for A. If γ is relational over A then γ satisfies the condition* (γ7), *and if γ is transitively relational over A, then γ satisfies the condition* (γ8).

PROOF. In the cases that $x \in \mathrm{Cn}(\varnothing)$, $y \in \mathrm{Cn}(\varnothing)$, $x \notin A$ and $y \notin A$, both (γ7) and (γ8) hold trivially, so we may suppose that $x \notin \mathrm{Cn}(\varnothing)$, $y \notin \mathrm{Cn}(\varnothing)$, $x \in A$ and $y \in A$.

Suppose γ is relational over A, and suppose $B \in \gamma(A \perp x \& y)$. Now $\gamma(A \perp x \& y) \subseteq A \perp x \& y = A \perp x \cup A \perp y$, so $B \in A \perp x$ or $B \in A \perp y$; consider the former case, as the latter is similar. Let $B' \in A \perp x$. Then $B' \in A \perp x \cup A \perp y = A \perp x \& y$, and so $B' \leq B$ since $B \in \gamma(A \perp x \& y)$ and γ is relational over A; and thus, by relationality again, $B \in \gamma(A \perp x) \subseteq \gamma(A \perp x) \cup \gamma(A \perp y)$, as desired.

Suppose now that γ is transitively relational over A, and suppose $A \perp x \cap \gamma(A \perp x \& y) \neq \varnothing$. Suppose for reductio ad absurdum that there is a $B \in \gamma(A \perp x)$ with $B \notin \gamma(A \perp x \& y)$. Since $B \in \gamma(A \perp x) \subseteq A \perp x \subseteq A \perp x \& y$ by Lemma 4.1, whilst $B \notin \gamma(A \perp x \& y)$, we have by relationality that there is a $B' \in A \perp x \& y$ with $B' \not\leq B$. Now by the hypothesis $A \perp x \cap \gamma(A \perp x \& y) \neq \varnothing$, there is a $B'' \in A \perp x$ with $B'' \in \gamma(A \perp x \& y)$. Hence by relationality $B' \leq B''$ and also $B'' \leq B$. Transitivity gives us $B' \leq B$ and thus a contradiction. □

When A is a theory and γ is a selection function for A, we define γ^*, the *completion* of γ, by putting $\gamma^*(A \perp x) = \{B \in A \perp x : \bigcap \gamma(A \perp x) \subseteq B\}$ for all $x \notin \mathrm{Cn}(\varnothing)$, and $\gamma^*(A \perp x) = \gamma(A \perp x) = \{A\}$ in the limiting case that $x \in \mathrm{Cn}(\varnothing)$. It is easily verified that γ^* is also a selection function for A, and determines the same partial meet contraction function as γ does. Moreover, we clearly have $\gamma(A \perp x) \subseteq \gamma^*(A \perp x) = \gamma^{**}(A \perp x)$ for all x. This notion is useful in the formulation of the following statement:

OBSERVATION 4.4. *Let A be any theory, and $\dot{-}$ a partial meet contraction function over A, determined by a selection function γ. If $\dot{-}$ satisfies the conditions* ($\dot{-}$7) *and* ($\dot{-}$8) *then γ^* is transitively relational over A.*

PROOF. Define the relation $\leq$ over 2^A as follows: for all $B, B' \in 2^A$, $B' \leq B$ iff either $B' = B = A$, or the following three all hold:

(i) $B' \in A \perp x$ for some $x \in A$.

(ii) $B \in A \perp x$ and $A \dot{-} x \subseteq B$ for some $x \in A$.

(iii) For all x, if $B', B \in A \perp x$ and $A \dot{-} x \subseteq B'$, then $A \dot{-} x \subseteq B$.

We need to show that the relation is transitive, and that it satisfies the marking off identity $\gamma^*(A \perp x) = \{B \in A \perp x : B' \leq B \text{ for all } B' \in A \perp x\}$ for all $x \notin \mathrm{Cn}(\varnothing)$.

For the identity, suppose first that $B \in \gamma^*(A \perp x) \subseteq A \perp x$ since $x \notin \mathrm{Cn}(\varnothing)$. Let $B' \in A \perp x$; we need to show that $B' \leq B$. If $x \notin A$ then $B' = B = A$ so $B' \leq B$. Suppose that $x \in A$. Then clearly conditions (i) and (ii) are satisfied. Let y be any proposition, and suppose $B', B \in A \perp y$ and $A \dot{-} y \subseteq B'$; we need to show that $A \dot{-} y \subseteq B$. Now by covering, which we have seen to follow from ($\dot{-}$8), either $A \dot{-} (x \& y) \subseteq A \dot{-} x$ or $A \dot{-} (x \& y) \subseteq A \dot{-} y$. And in the latter case $A \dot{-} (x \& y) \subseteq A \dot{-} y \subseteq B' \in A \perp x$ so $x \notin A \dot{-} (x \& y)$; so by ($\dot{-}$8) again $A \dot{-} (x \& y) \subseteq A \dot{-} x$. Thus in either case $A \dot{-} (x \& y) \subseteq A \dot{-} x$. Now suppose for reductio ad absurdum that there is a $w \in A \dot{-} y$ with $w \notin B$. Then $y \vee w \in A \dot{-} y$ and so since $y \vdash y \vee w$ we have by ($\dot{-}$7) using Observation 3.3 that $y \vee w \in A \dot{-} (x \& y) \subseteq A \dot{-} x = \bigcap \gamma^*(A \perp x) \subseteq B$; so $y \vee w \in B$. But also since $B \in A \perp y$ and $w \notin B$ and $w \in A$, we have $B \cup \{w\} \vdash y$, so $\neg w \vee y \in B$. Putting these together gives us $(y \vee w) \& (y \vee \neg w) \in B$, so $y \in B$, contradicting $B \in A \perp y$.

For the converse, suppose $B \notin \gamma^*(A \perp x)$ and $B \in A \perp x$; we need to find a $B' \in A \perp x$ with $B' \not\leq B$. Clearly the supposition implies that $x \in A$, so $B \neq A$. Since $B \in A \perp x$, the latter is nonempty, so $\gamma^*(A \perp x)$ is nonempty; let B' be one of its elements. Noting that $B', B \in A \perp x$, $B' \in \gamma^*(A \perp x)$, but $B \notin \gamma^*(A \perp x)$, we see that condition (iii) fails, so that $B' \not\leq B$, as desired.

Finally, we check out transitivity. Suppose $B'' \leq B'$ and $B' \leq B$; we want to show that $B'' \leq B$. In the case that $B = A$ then clearly since $B' \leq B$ we have $B' = B = A$, and thus since $B'' \leq B'$ we have $B'' = B' = A$, so $B'' = B = A$ and $B'' \leq B$. Suppose for the principal case that $B \neq A$. Then since $B' \leq B$, clearly $B' \neq A$. Since $B' \leq B$ we have $B \in A \perp w$ and $A \dot{-} w \subseteq B$ for some $w \in A$, so (ii) is satisfied. Since $B'' \leq B'$ we have $B'' \in A \perp w$ for some $w \in A$, so (i) is satisfied. It remains to verify (iii). Suppose $B'', B \in A \perp y$ and $A \dot{-} y \subseteq B''$; we need to show that $A \dot{-} y \subseteq B$. First, note that since $B \neq A$ by the condition of the case, we have $y \in A$. Also, since $B'' \leq B'$ and $B' \neq A$, there is an $x \in A$ with $B' \in A \perp x$ and $A \dot{-} x \subseteq B'$. Since $x, y \in A$ we have by Lemma 4.1 that $A \perp x \,\&\, y = A \perp x \cup A \perp y$, so $B'', B', B \in A \perp x \,\&\, y$. Now by covering, either $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} y$ or $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$. The former case gives us $A \dot{-} (x \,\&\, y) \subseteq B''$, so since $B'' \leq B'$ and $B' \neq A$ we have $A \dot{-} (x \,\&\, y) \subseteq B'$, so again since $B' \leq B$ and $B \neq A$ we have $A \dot{-} (x \,\&\, y) \subseteq B$. Likewise, the latter case gives us $A \dot{-} (x \,\&\, y) \subseteq B'$, so since $B' \leq B$ and $B \neq A$ we have $A \dot{-} (x \,\&\, y) \subseteq B$. Thus in either case, $A \dot{-} (x \,\&\, y) \subseteq B$. Now let $w \in A \dot{-} y$; we need to show that $w \in B$. Since $w \in A \dot{-} y$ we have $y \vee w \in A \dot{-} y$; so by $(\dot{-}7)$ and Observation 3.3, since $y \vee w \in \mathrm{Cn}(y)$, we have $y \vee w \in A \dot{-} (x \,\&\, y) \subseteq B$. Hence $B \cup \{\neg y\} \vdash w$. But since $B \in A \perp y$ and $w \in A$, we also have $B \cup \{y\} \vdash w$, so $B \vdash w$ and thus $w \in B$ as desired. □

COROLLARY 4.5. *Let A be any theory, and $\dot{-}$ a partial meet contraction function over A determined by a selection function γ. Then $\dot{-}$ is transitively relational over A iff $\dot{-}$ satisfies both* $(\dot{-}7)$ *and* $(\dot{-}8)$.

PROOF. If $\dot{-}$ satisfies $(\dot{-}7)$ and $(\dot{-}8)$ then, by 4.4, γ^* is transitively relational, so since γ^* determines $\dot{-}$, the latter is transitively relational. Conversely, if $\dot{-}$ is transitively relational, then γ' is transitively relational for some γ' that determines $\dot{-}$; so, by 4.3, γ' satisfies $(\gamma7)$ and $(\gamma8)$; so, by 4.2, $\dot{-}$ satisfies $(\dot{-}7)$ and $(\dot{-}8)$. □

This result is the promised representation theorem for the collection of "basic" plus "supplementary" postulates. Since this collection of postulates can be independently motivated (cf. Gärdenfors [3]), there is strong reason to focus on transitively relational partial meet contraction functions as an ideal representation of the intuitive process of contraction.

Note that Observation 4.4 and its corollary give us a sufficient condition for the transitive relationality of γ^*, and thus of $\dot{-}$, rather than of γ itself. The question thus arises: when can we get the latter? We shall show that in the finite case the passage from γ to $\dot{-}$ is injective, so that $\gamma = \gamma^*$, where γ is any selection function that determines $\dot{-}$. By the *finite case*, we mean the case where A is finite modulo Cn; that is, where the equivalence relation defined by $\mathrm{Cn}(x) = \mathrm{Cn}(y)$ partitions A into finitely many cells.

OBSERVATION 4.6. *Let A be any theory finite modulo* Cn, *and let γ and γ' be selection functions for A. For every proposition x, if $\gamma(A \perp x) \neq \gamma'(A \perp x)$, then $\bigcap\gamma(A \perp x) \neq \bigcap\gamma'(A \perp x)$.*

SKETCH OF PROOF. Suppose $B \in \gamma(A \perp x)$, but $B \notin \gamma'(A \perp x)$. Then clearly $x \in A$ and $x \notin \mathrm{Cn}(\varnothing)$. Since A is finite (we identify A with its quotient structure), so is B; put b to be the conjunction of its elements. Then it is easy to check that $b \in B$ but $b \notin B'$ for all $B' \in \gamma'(A \perp x)$. Put $c = \neg b \vee x$: then it is easy to check that $c \notin B \supseteq \bigcap \gamma(A \perp x)$, but $c \in B'$ for all $B' \in \gamma'(A \perp x)$; that is, $c \in \bigcap \gamma'(A \perp x)$. □

COROLLARY 4.7. *Let A be any theory finite modulo* Cn, *and $\dot{-}$ a partial meet contraction function over A determined by a selection function γ. If $\dot{-}$ satisfies conditions* ($\dot{-}$7) *and* ($\dot{-}$8), *then γ is transitively relational over A.*

PROOF. Immediate from 4.4 and 4.6. □

We turn now to the question of the relation of condition ($\dot{-}$7), considered alone, to relationality; and here the situation is rather more complex and less satisfying. Now we have from Observation 4.2 that when $\dot{-}$ is determined by γ, then if γ satisfies (γ7), then $\dot{-}$ satisfies ($\dot{-}$7), and it is not difficult to show, by an argument similar to that of 4.6, that:

OBSERVATION 4.8. *If A is a theory finite modulo* Cn, *and $\dot{-}$ a partial meet contraction function over A determined by a selection function γ, then $\dot{-}$ satisfies* ($\dot{-}$7) *iff γ satisfies* (γ7). *Also, $\dot{-}$ satisfies* ($\dot{-}$8) *iff γ satisfies* (γ8).

But on the other hand, even in the finite case, (γ7) does *not* imply the relationality of γ or of $\dot{-}$:

OBSERVATION 4.9. *There is a theory A, finite modulo* Cn, *with a partial meet contraction function $\dot{-}$ over A, determined by a selection function γ, such that $\dot{-}$ satisfies* (γ7), *but $\dot{-}$ is not relational over A.*

SKETCH OF PROOF. Take the sixteen-element Boolean algebra, take an atom a_0 of this algebra, and put A to be the principal filter determined by a_0. This will be an eight-element structure, lattice-isomorphic to the Boolean algebra of eight elements. We take Cn in the natural way, putting $\mathrm{Cn}(X) = \{x : \bigwedge X \leq x\}$. We label the eight elements of A as $a_0, \dots, a_7$, where a_0 is already defined, a_1, a_2, a_3 are the three atoms of A (not of the entire Boolean algebra), a_4, a_5, a_6 are the three dual atoms of A, and a_7 is the greatest element of A (i.e. the unit of the Boolean algebra). For each $i \leq 7$, we write $!a_i$ for $\{a_j \in A : a_i \leq a_j\}$. We define γ by putting $\gamma(A \perp a_7) = \gamma(A \perp \mathrm{Cn}(\varnothing)) = \{A\} = !a_0$ as required in this limiting case, $\gamma(A \perp a_j) = A \perp a_j$ for all j with $1 \leq j < 7$, and $\gamma(A \perp a_0) = \{!a_1\}$. Then it is easy to verify that for all $a_i \notin \mathrm{Cn}(\varnothing)$, $\gamma(A \perp a_i)$ is a nonempty subset of $A \perp a_i$, so γ is a selection function for A. By considering cases we easily verify (γ7) (and thus also by 4.2 ($\dot{-}$7)); and by considering the role of $!a_2$ it is easy to verify that γ (and hence by 4.6, $\dot{-}$ itself) is not relational over A. □

The question thus arises whether there is a condition on $\dot{-}$ or on γ that is equivalent to the relationality of $\dot{-}$ or of γ respectively. We do not know of any such condition for $\dot{-}$, but there is one for γ, of an infinitistic nature. It is convenient, in this connection, to consider a descending series of conditions, as follows:

(γ7:∞) $A \perp x \cap \bigcap_{i \in I}\{\gamma(A \perp y_i)\} \subseteq \gamma(A \perp x)$, whenever $A \perp x \subseteq \bigcup_{i \in I}\{A \perp y_i\}$.

(γ7:N) $A \perp x \cap \gamma(A \perp y_1) \cap \cdots \cap \gamma(A \perp y_n) \subseteq \gamma(A \perp x)$, whenever $A \perp x \subseteq A \perp y_1 \cup \cdots \cup A \perp y_n$, for all $n \geq 1$.

(γ7:2) $A \perp x \cap \gamma(A \perp y_1) \cap \gamma(A \perp y_2) \subseteq \gamma(A \perp x)$, whenever $A \perp x \subseteq A \perp y_1 \cup A \perp y_2$.

(γ7:1) $A \perp x \cap \gamma(A \perp y) \subseteq \gamma(A \perp x)$, whenever $A \perp x \subseteq A \perp y$.

OBSERVATION 4.10. *Let A be any theory and γ a selection function over A. Then γ*

is relational over A iff $(\gamma7{:}\infty)$ *is satisfied. Moreover, we have* $(\gamma7{:}\infty) \rightarrow (\gamma7{:}N) \leftrightarrow (\gamma7{:}2) \rightarrow (\gamma7{:}1) \leftrightarrow (\gamma7)$. *On the other hand,* $(\gamma7{:}1)$ *does not imply* $(\gamma7{:}2)$, *even in the finite case; although in the finite case,* $(\gamma7{:}N)$ *is equivalent to* $(\gamma7{:}\infty)$.

SKETCH OF PROOF. Writing $(\gamma\mathbf{R})$ for "γ is relational over A", we show first that $(\gamma\mathrm{R}) \rightarrow (\gamma7{:}\infty)$. Suppose $(\gamma\mathrm{R})$, and suppose $A \perp x \subseteq \bigcup_{i\in I}\{A \perp y_i\}$. Suppose $B \in A \perp x \cap \bigcap_{i\in I}\{\gamma(A \perp y_i)\}$. We need to show that $B \in \gamma(A \perp x)$. Since $B \in \gamma(A \perp y_i)$ for all $i \in I$, we have by relationality that $B' \leq B$ for all $B' \in A \perp y_i$, for all $i \in I$; so, by the supposition, $B' \leq B$ for all $B' \in A \perp x$. Hence, since $B \in A \perp x$, so that also $x \notin \mathrm{Cn}(\varnothing)$, we have by relationality that $B \in \gamma(A \perp x)$. To show the converse $(\gamma7{:}\infty) \rightarrow (\gamma\mathrm{R})$, suppose $(\gamma7{:}\infty)$ holds, and define $\leq$ over 2^A by putting $B' \leq B$ iff there is an x with $B \in \gamma(A \perp x)$ and $B' \in A \perp x$; we need to verify the marking off identity. The left to right inclusion of the marking off identity is immediate. For the right to left, suppose $B \in A \perp x$ and for all $B' \in A \perp x$, $B' \leq B$. Then by the definition of $\leq$, for all $B_i \in \{B_i\}_{i\in I} = A \perp x$ there is a y_i with $B \in \gamma(A \perp y_i)$ and $B_i \in A \perp y_i$. Since $B_i \in A \perp y_i$ for all $B_i \in A \perp x$, we have $A \perp x \subseteq \bigcup_{i\in I}\{A \perp y_i\}$, so we may apply $(\gamma7{:}\infty)$. But clearly $B \in A \perp x \cap \bigcap_{i\in I}\{\gamma(A \perp y_i)\}$. Hence by $(\gamma7{:}\infty)$ we have $B \in \gamma(A \perp x)$, as desired.

The implications $(\gamma7{:}\infty) \rightarrow (\gamma7{:}N) \rightarrow (\gamma7{:}2) \rightarrow (\gamma7{:}1)$ are trivial, as is the equivalence of $(\gamma7{:}\infty)$ to $(\gamma7{:}N)$ in the finite case. To show that $(\gamma7{:}2)$ implies the more general $(\gamma7{:}N)$, it suffices to show that for all $n \geq 2$, $(\gamma7{:}n) \rightarrow (\gamma7{:}n+1)$: this can be done using the fact that when $y_n, y_{n+1} \in A$, $A \perp y_n \cup A \perp y_{n+1} = A \perp (y_n \,\&\, y_{n+1})$ by Lemma 4.1.

To show that $(\gamma7{:}1) \rightarrow (\gamma7)$, recall from 4.1 that when $x, y \in A$, then $A \perp x \,\&\, y = A \perp x \cup A \perp y$; so $A \perp x \subseteq A \perp x \,\&\, y$, and so, by $(\gamma7{:}1)$, $(A \perp x) \cap \gamma(A \perp x \,\&\, y) \subseteq \gamma(A \perp x)$. Similarly $(A \perp y) \cap \gamma(A \perp x \,\&\, y) \subseteq \gamma(A \perp y)$. Forming unions on left and right, distributing on the left, and applying 4.1 gives us $\gamma(A \perp x \,\&\, y) \subseteq \gamma(A \perp x) \cup \gamma(A \perp y)$ as desired.

To show conversely that $(\gamma7) \rightarrow (\gamma7{:}1)$, suppose $(\gamma7)$ is satisfied, suppose $A \perp x \subseteq A \perp y$ and consider the principal case that $x, y \in A$. Then using compactness we have $y \vdash x$, so $\mathrm{Cn}(y) = \mathrm{Cn}(x \,\&\, (\neg x \vee y))$, so by $(\gamma7)$

$$\gamma(A \perp y) \subseteq \gamma(A \perp x) \cup \gamma(A \perp \neg x \vee y),$$

so $A \perp x \cap \gamma(A \perp y) \subseteq \gamma(A \perp x) \cup \gamma(A \perp \neg x \vee y)$. The verification is then completed by showing that $A \perp x$ is disjoint from $\gamma(A \perp \neg x \vee y)$.

Finally, to show that $(\gamma7{:}1)$ does not imply $(\gamma7{:}2)$, even in the finite case, consider the same example as in the proof of Observation 4.9. We know from that proof that this example satisfies $(\gamma7)$ and thus also $(\gamma7{:}1)$, but that γ is not relational over A, so by earlier parts of this proof, $(\gamma7{:}\infty)$ fails, so by finiteness $(\gamma7{:}N)$ fails, so $(\gamma7{:}2)$ fails. Alternatively, a direct counterinstance to $(\gamma7{:}2)$ in this example can be obtained by putting $x = a_0$, $y_1 = a_1$, and $y_2 = a_2$. □

§5. Remarks on connectivity. It is natural to ask what the consequences are of imposing connectivity as well as transitivity on the relation that determines a selection function. Perhaps surprisingly, it turns out that in the infinite case it adds very little, and in the finite case nothing at all. This is the subject of the present section.

Let A be a set of propositions and γ a selection function for A. We say that γ is *connectively relational* over A iff there is a relation that is connected over 2^A such that for all $x \notin \mathrm{Cn}(\varnothing)$, the marking off identity of §4 holds. And a partial meet contraction function is called connectively relational iff it is determined by some selection function that is so.

We note as a preliminary that it suffices to require connectivity over the much smaller set $U_A = \bigcup_{x \in A}\{A \perp x\}$. For suppose that $\leq$ is connected over U_A. Put $\leq_0$ to be the restriction of $\leq$ to U_A; then $\leq_0$ will still be connected over U_A. Then put $\leq_1$ to be $\leq_0 \cup ((2^A - U_A) \times 2^A)$. Clearly $\leq_1$ will be connected over 2^A. Moreover, if $\leq$ satisfies the marking off identity, so does $\leq_1$.

Indeed, when $\leq$ is transitive, it suffices to require connectivity on the even smaller set $U_\gamma = \bigcup\{\gamma(A \perp x): x \in A, x \notin \mathrm{Cn}(\varnothing)\}$. For here likewise we can define $\leq_0$ as the restriction of $\leq$ to U_γ, and then define $\leq_1$ to be $\leq_0 \cup ((2^A - U_\gamma) \times 2^A)$. Then clearly $\leq_1$ is connected over 2^A, and is transitive if $\leq$ is transitive; and we can easily check, using the transitivity of $\leq$, that if $\leq$ satisfies the marking off identity for γ, so does $\leq_1$.

OBSERVATION 5.1. *Let A be any theory and $\dot{-}$ a partial meet contraction function over A. Then $\dot{-}$ is transitively relational iff it is transitively and connectively relational.*

PROOF. Suppose that $\dot{-}$ is determined by the transitively relational selection function γ. Then by 4.2 and 4.3, $\dot{-}$ satisfies the conditions ($\dot{-}$7) and ($\dot{-}$8), so the conditions of Observation 4.4 hold and the relation $\leq$ defined in its proof is transitive and satisfies the marking off identity for γ^*. By the above remarks, to show that $\dot{-}$ is transitively and connectively relational it suffices to show that $\leq$ is connected over the set U_{γ^*}.

Let $B', B \in U_{\gamma^*}$ and suppose $B' \not\leq B$. Since $B', B \in U_{\gamma^*}$, conditions (i) and (ii) of the definition of $\leq$ in the proof of 4.4 are satisfied for both B' and B. Hence since $B' \not\leq B$ we have by (iii) that there is an x with $B', B \in A \perp x$, $A \dot{-} x \subseteq B'$ and $A \dot{-} x \not\subseteq B$. But since $A \dot{-} x \subseteq B' \in A \perp x$ we have by the definition of γ^* that $B' \in \gamma^*(A \perp x)$, so by the marking off identity for γ^*, $\leq$ as verified in the proof of 4.4, since $B \in A \perp x$ we have $B \leq B'$ as desired. □

In the case that A is finite modulo Cn, this result can be both broadened and sharpened: Broadened to apply to relationality in general rather than only to transitive relationality, and sharpened to guarantee connectivity over U_A of any given relation under which the selection function γ is relational, rather than merely connectivity, as above, of a specially constructed relation under which the closure γ^* of γ is relational.

OBSERVATION 5.2. *Let A be a theory finite modulo* Cn, *and let $\dot{-}$ be a partial meet contraction function over A, determined by a selection function γ. Suppose that γ is relational, with the relation $\leq$ satisfying the marking off identity. Then $\leq$ is connected over U_A.*

PROOF. Let $B', B \in U_A = \bigcup_{x \in A}\{A \perp x\}$. Since A is finite modulo Cn, there are b', $b \in A$ with $A \perp b' = \{B'\}$ and $A \perp b = \{B\}$—for example, put b to be the disjunction of all (up to equivalence modulo Cn) the elements $a \in A$ such that $B \nvdash a$. Now $A \perp b' \& b = A \perp b' \cup A \perp b = \{B', B\}$ by Observation 4.1, and so since γ is a selection function, $\gamma(A \perp b' \& b)$ is a nonempty subset of $\{B', B\}$, which implies that either B' or B is in $\gamma(A \perp b' \& b)$. In the former case we have $B \leq B'$, and in the latter case we have the converse. □

COROLLARY 5.3. *Let A be a theory finite modulo* Cn, *and let $\dot{-}$ be a partial meet contraction function over A. Then $\dot{-}$ is relational iff it is connectively relational.*

PROOF. Immediate from 5.2. □

§6. Maxichoice contraction functions and factoring conditions on $A \dot{-} (x \& y)$. The first topic of this section will be a brief investigation of the consequences of the following rather strong *fullness* condition:

($\dot{-}$F) If $y \in A$ and $y \notin A \dot{-} x$, then $\neg y \vee x \in A \dot{-} x$, for any theory A.

From the results in Gärdenfors [4], it follows that if $\dot{-}$ is a partial meet contraction function, then this condition (there called (-6)) is equivalent with the following condition (called (21) in Gärdenfors [4]) on partial meet revision functions:

($\dot{+}$F) If $y \in A$ and $y \notin A \dot{+} x$, then $\neg y \in A \dot{+} x$, for any theory A.

The strength of the condition ($\dot{-}$F) is shown by the following simple representation theorem:

OBSERVATION 6.1. *Let $\dot{-}$ be any partial meet contraction function over a theory A. Then $\dot{-}$ satisfies* ($\dot{-}$F) *iff $\dot{-}$ is a maxichoice contraction function.*

PROOF. Suppose $\dot{-}$ satisfies ($\dot{-}$F). Suppose B, $B' \in \gamma(A \perp x)$ and assume for contradiction that $B \neq B'$. There is then some $y \in B'$ such that $y \notin B$. Hence $y \notin A \dot{-} x$ and since $y \in A$ it follows from ($\dot{-}$F) that $\neg y \vee x \in A \dot{-} x$. Hence $\neg y \vee x \in B'$, but since $y \in B'$ it follows that $x \in B'$, which contradicts the assumption that $B' \in A \perp x$. We conclude that $B = B'$ and hence that $\dot{-}$ is a maxichoice contraction function.

For the converse, suppose that $\dot{-}$ is a maxichoice contraction function and suppose that $y \in A$ and $y \notin A \dot{-} x$. Since $A \dot{-} x = B$ for some $B \in A \perp x$, it follows that $y \notin B$. So by the definition of $A \perp x$, $x \in \text{Cn}(B \cup \{y\})$. By the properties of the consequence operation we conclude that $\neg y \vee x \in B = A \dot{-} x$, and thus ($\dot{-}$F) is satisfied.

In addition to this representation theorem for maxichoice contraction functions, we can also prove another one based on the following *primeness* condition.

($\dot{-}$Q) For all $y, z \in A$ and for all x, if $y \vee z \in A \dot{-} x$, then either $y \in A \dot{-} x$ or $z \in A \dot{-} x$.

OBSERVATION 6.2. *Let $\dot{-}$ be any partial meet contraction function over a theory A. Then $\dot{-}$ satisfies* ($\dot{-}$Q) *iff $\dot{-}$ is a maxichoice contraction function.*

PROOF. Suppose first that $\dot{-}$ is a maxichoice function and suppose $y, z \in A$, $y \notin A \dot{-} x$ and $z \notin A \dot{-} x$. Then by maximality, $A \dot{-} x \cup \{y\} \vdash x$ and $A \dot{-} x \cup \{z\} \vdash x$, so $A \dot{-} x \cup \{y \vee z\} \vdash x$. But since say $y \in A$ and $y \notin A \dot{-} x$, we have $x \notin \text{Cn}(\varnothing)$, so $x \notin A \dot{-} x$. Thus $y \vee z \notin A \dot{-} x$, which shows that ($\dot{-}$Q) is satisfied.

For the converse, suppose that ($\dot{-}$Q) is satisfied. By Observation 6.1, it suffices to derive ($\dot{-}$F). Suppose $y \in A$ and $y \notin A \dot{-} x$. We need to show that $\neg y \vee x \in A \dot{-} x$. Now $(y \vee \neg y) \vee x \in \text{Cn}(\varnothing)$, and so $(y \vee \neg y) \vee x = y \vee (\neg y \vee x) \in A \dot{-} x$. Also by hypothesis $y \in A$, and since $y \notin A \dot{-} x$ we have $x \in A$, so $\neg y \vee x \in A$. We can now apply the primeness condition ($\dot{-}$Q) and get either $y \in A \dot{-} x$ or $\neg y \vee x \in A \dot{-} x$. By hypothesis, the former fails, so the latter holds and ($\dot{-}$F) is verified.

With the aid of these results we shall now look at three "factoring" conditions on the contraction of a conjunction from a theory A. They are

Decomposition ($\dot{-}$D). For all x and y, $A \dot{-} (x \& y) = A \dot{-} x$ or $A \dot{-} (x \& y) = A \dot{-} y$.

Intersection ($\dot{-}$I). For all x and y in A, $A \dot{-} (x \& y) = A \dot{-} x \cap A \dot{-} y$.

Ventilation ($\dot{-}$V). For all x and y, $A \dot{-} (x \& y) = A \dot{-} x$ or $A \dot{-} (x \& y) = A \dot{-} y$ or $A \dot{-} (x \& y) = A \dot{-} x \cap A \dot{-} y$.

These bear some analogy to the very processes of maxichoice, full meet, and partial meet contraction respectively, and the analogy is even more apparent if we express the factoring conditions in their equivalent n-ary forms:

$$A \dot{-} (x_1 \& \cdots \& x_n) = A \dot{-} x_i \quad \text{for some } i \leq n;$$

$$A \dot{-} (x_1 \& \cdots \& x_n) = \bigcap_{i \leq n} \{A \dot{-} x_i\} \quad \text{whenever } x_1, \ldots, x_n \in A;$$

$$A \dot{-} (x_1 \& \cdots \& x_n) = \bigcap_{i \in I} \{A \dot{-} x_i\} \quad \text{for some } I, \text{ where } \varnothing \neq I \subseteq \{1, \ldots, n\}.$$

This analogy of formulation corresponds indeed to quite close relationships between the three kinds of contraction process, on the one hand, and the three kinds of factorization on the other. We shall state the essential relationships first, to give a clear overall picture, and group the proofs together afterwards.

First, the relationship between maxichoice contraction and decomposition. In [2] it was shown that if A is a theory and $\dot{-}$ is a maxichoice contraction function over A, then decomposition is equivalent to each of ($\dot{-}$7) and ($\dot{-}$8). In the more general context of partial meet contraction functions these equivalences between the conditions break down, and it is decomposition ($\dot{-}$D) that emerges as the strongest among them:

OBSERVATION 6.3. *Let A be a theory and $\dot{-}$ a partial meet contraction function over A. Then the following conditions are equivalent*:

(a) $\dot{-}$ *satisfies* ($\dot{-}$D).

(b) $\dot{-}$ *is a maxichoice contraction function and satisfies at least one of* ($\dot{-}$7) *and* ($\dot{-}$8).

(c) $\dot{-}$ *is a maxichoice contraction function and satisfies both of* ($\dot{-}$7) *and* ($\dot{-}$8).

(d) $\dot{-}$ *satisfies* ($\dot{-}$WD).

(e) $\dot{-}$ *is a maxichoice contraction function and satisfies* ($\dot{-}$C).

Here ($\dot{-}$WD) is the *weak decomposition* condition: for all x and y, $A \dot{-} x \subseteq A \dot{-} x \& y$ or $A \dot{-} y \subseteq A \dot{-} x \& y$.

The relationship of full meet contraction to the intersection condition ($\dot{-}$I) is even more direct. This is essentially because a full meet contraction function, as defined at the beginning of §2, is always transitively relational, and so always satisfies ($\dot{-}$7) and ($\dot{-}$8). For since $\gamma(A \perp x) = A \perp x$ for all $x \notin \mathrm{Cn}(\varnothing)$, γ is determined via the marking off identity by the total relation over 2^A or over $\bigcup_x \{A \perp x\}$.

OBSERVATION 6.4. *Let A be a theory and $\dot{-}$ a partial meet contraction function over A. Then the following conditions are equivalent*:

(a) $\dot{-}$ *satisfies* ($\dot{-}$I).
(b) $\dot{-}$ *satisfies* ($\dot{-}$M).
(c) $\dot{-}$ *is a full meet contraction function.*

Here ($\dot{-}$M) is the *monotony condition*: for all $x \in A$, if $x \vdash y$ then $A \dot{-} x \subseteq A \dot{-} y$. This result gives us a representation theorem for full meet contraction. Note, as a point of detail, that whereas decomposition and ventilation are formulated for arbitrary propositions x and y, the intersection and monotony conditions are formulated under the restriction that x and y (respectively, x) are in A. For if $x \notin A$, then $x \& y \notin A$, so $A \dot{-} (x \& y) = A$ whilst $A \dot{-} x \cap A \dot{-} y = A \cap A \dot{-} y = A \dot{-} y \neq A$ if $y \in A$ and $y \notin \mathrm{Cn}(\varnothing)$.

Of the three factoring conditions, ventilation ($\dot{-}$V) is clearly the most "general" and the weakest. But it is still strong enough to imply the "supplementary postulates" ($\dot{-}$7) and ($\dot{-}$8):

OBSERVATION 6.5. *Let A be a theory and $\dot{-}$ a partial meet contraction function over A. Then $\dot{-}$ satisfies ($\dot{-}$V) iff $\dot{-}$ satisfies both ($\dot{-}$7) and ($\dot{-}$8).*

PROOF OF OBSERVATION 6.3. We know by the chain of equivalences in §8 of [2] that if $\dot{-}$ is a maxichoice contraction function then the conditions ($\dot{-}$7), ($\dot{-}$8) and ($\dot{-}$D) are mutually equivalent. This already shows the equivalence of (b) and (c), and also shows that they imply (a). (d) is a trivial consequence of (a). To prove the equivalence of (a)–(d) it remains to show that (d) implies (b).

Suppose that $\dot{-}$ satisfies ($\dot{-}$WD). Clearly it then satisfies ($\dot{-}$7), so we need only verify that $\dot{-}$ is a maxichoice function, for which it suffices by Observation 6.1 to verify ($\dot{-}$F); that is, that whenever $y \in A$ and $y \notin A \dot{-} x$ then $\neg y \vee x \in A \dot{-} x$. Suppose for reductio ad absurdum that $y \in A, y \notin A \dot{-} x$ and $\neg y \vee x \notin A \dot{-} x$. Note that this implies that $x \in A$. Now $\mathrm{Cn}(x) = \mathrm{Cn}((x \vee y) \& (x \vee \neg y))$, so by ($\dot{-}$WD) we have $A \dot{-} (x \vee y) \subseteq A \dot{-} x$ or $A \dot{-} (x \vee \neg y) \subseteq A \dot{-} x$. In the former case, $\neg y \vee x \notin A \dot{-} (x \vee y)$. But by recovery $A \dot{-} (x \vee y) \cup \{x \vee y\} \vdash x$, so $A \dot{-} (x \vee y) \cup \{y\} \vdash x$, so $\neg y \vee x \in A \dot{-} (x \vee y)$, giving a contradiction. And in the latter case, $y \notin A \dot{-} (x \vee \neg y)$, whereas by recovery $A \dot{-} (x \vee \neg y) \cup \{x \vee \neg y\} \vdash y$, so $A \dot{-} (x \vee \neg y) \cup \{\neg y\} \vdash y$, so $y \in A \dot{-} (x \vee \neg y)$, again giving a contradiction.

Finally, it must be shown that (e) is equivalent with (a)–(d). First note that it follows immediately from Observation 3.4 that (c) entails (e). To complete the proof we show that (e) entails (b). In the light of Observation 6.1 it suffices to show that ($\dot{-}$F) and ($\dot{-}$C) together entail ($\dot{-}$8). To do this assume that $x \notin A \dot{-} x \& y$. We want to show that $A \dot{-} x \& y \subseteq A \dot{-} x$. In the case when $x \notin A$, this holds trivially; so suppose that $x \in A$. It then follows from ($\dot{-}$F) that $\neg x \vee (x \& y) \in A \dot{-} x \& y$, so $\neg x \vee y \in A \dot{-} (x \& y) = A \dot{-} (\neg x \vee y) \& x$. By ($\dot{-}$C), $A \dot{-} x \& y = A \dot{-} (\neg x \vee y) \& x \subseteq A \dot{-} \neg x \vee y$ or $A \dot{-} x \& y = A \dot{-} (\neg x \vee y) \& x \subseteq A \dot{-} x$. Since the second case is the desired inclusion, it will suffice to show that the first case implies the second. Suppose $A \dot{-} x \& y \subseteq A \dot{-} \neg x \vee y$. Then, since $\neg x \vee y \in A \dot{-} x \& y$, we have $\neg x \vee y \in A \dot{-} \neg x \vee y$, so by ($\dot{-}$4) $\neg x \vee y \in \mathrm{Cn}(\varnothing)$. But this means that $\mathrm{Cn}(x \& y) = \mathrm{Cn}(x)$, so $A \dot{-} x \& y = A \dot{-} x$ by ($\dot{-}$5), and we are done. □

The last part of this proof shows that for maxichoice contraction functions the converse of Observation 3.4 also holds.

PROOF OF OBSERVATION 6.4. Suppose first that $\dot{-}$ is a full meet contraction function. We show that ($\dot{-}$I) is satisfied. If $x \in \mathrm{Cn}(\varnothing)$ or $y \in \mathrm{Cn}(\varnothing)$ then the desired equation holds trivially. Suppose that $x, y \notin \mathrm{Cn}(\varnothing)$, and suppose that $x, y \in A$. Then we may apply Observation 4.1 to get

$$A \dot{-} (x \,\&\, y) = \bigcap \{A \perp x \,\&\, y\} = \bigcap \{A \perp x \cup A \perp y\} = \bigcap \{A \perp x\} \cap \bigcap \{A \perp y\}$$
$$= A \dot{-} x \cap A \dot{-} y,$$

so that $\dot{-}$ satisfies the intersection condition.

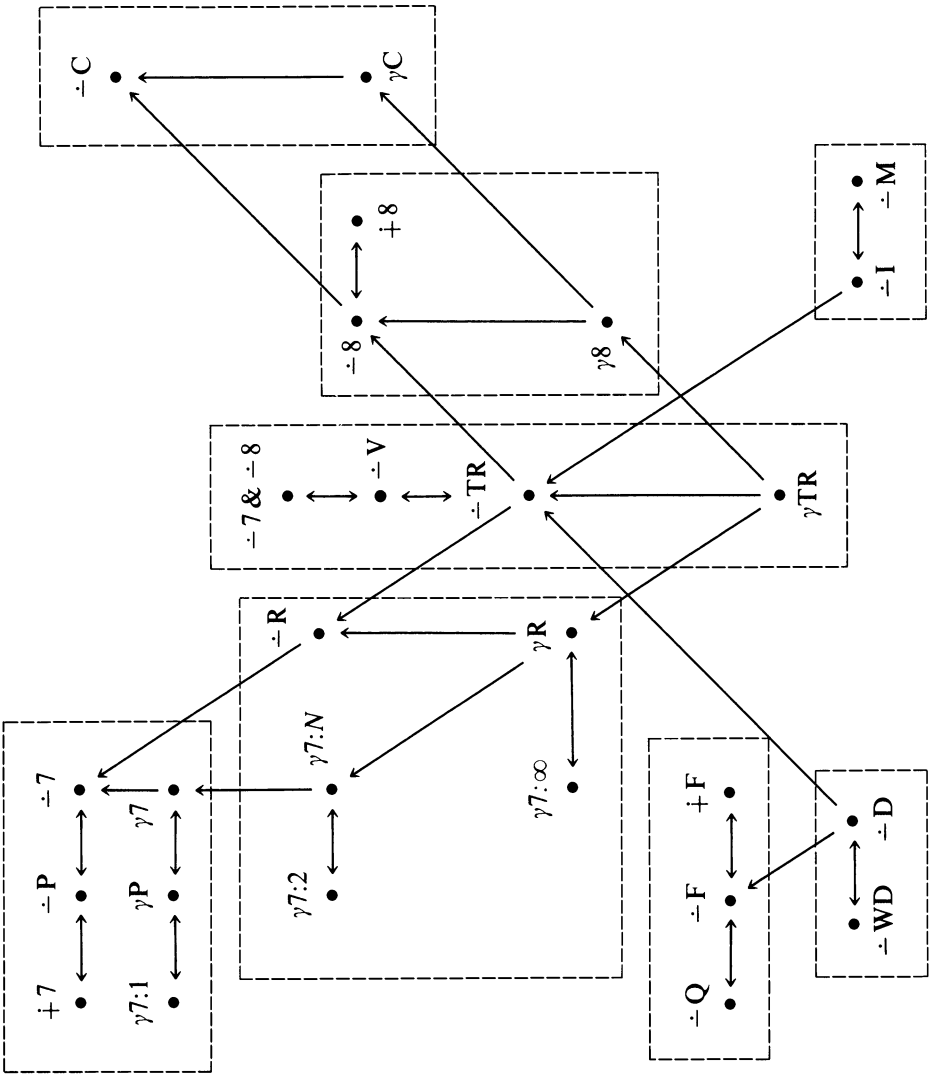
Trivially, intersection implies monotony. Suppose that $\dot{-}$ satisfies monotony; to prove (c) we need to show that $A \dot{-} x = A \sim x$, for which it clearly suffices to show that $A \dot{-} x \subseteq A \sim x$ in the light of Observation 2.7. In the case $x \notin A$ this holds trivially. In the case $x \in A$ we have by Observation 2.1 of [2] that $A \sim x = A \cap \mathrm{Cn}(\neg x)$, so we need only show $A \dot{-} x \subseteq \mathrm{Cn}(\neg x)$. Suppose $y \in A \dot{-} x$. Then by ($\dot{-}$M), since $x \in A$ and $x \vdash x \vee y$, we have $y \in A \dot{-} (x \vee y)$, so $x \vee y \in A \dot{-} (x \vee y)$; so, by the postulate ($\dot{-}$4), $x \vee y \in \mathrm{Cn}(\varnothing)$, so that $y \in \mathrm{Cn}(\neg x)$ as desired. □

Proof of Observation 6.5. For the left to right implication, suppose $\dot{-}$ satisfies ($\dot{-}$V). Then ($\dot{-}$7) holds immediately. For ($\dot{-}$8), let x and y be propositions and suppose $x \notin A \dot{-} (x \,\&\, y)$; we need to show that $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$. In the case that $x \notin A$ this holds trivially, so we suppose $x \in A$. Now $\mathrm{Cn}(x \,\&\, y) = \mathrm{Cn}(x \,\&\, (\neg x \vee y))$, so by ($\dot{-}$V) $A \dot{-} (x \,\&\, y)$ is identical with one of $A \dot{-} x$, $A \dot{-} (\neg x \vee y)$ or $(A \dot{-} x) \cap (A \dot{-} (\neg x \vee y))$. In the first and last cases we have the desired inclusion, so we need only show that the middle case is impossible. Now by recovery, $A \dot{-} (\neg x \vee y) \cup \{\neg x \vee y\} \vdash x$, so $A \dot{-} (\neg x \vee y) \cup \{\neg x\} \vdash x$, so $x \in A \dot{-} (\neg x \vee y)$. But by hypothesis, $x \notin A \dot{-} (x \,\&\, y)$, so $A \dot{-} (x \,\&\, y) \neq A \dot{-} (\neg x \vee y)$, as desired.

The converse can be proven via the representation theorem (Observation 4.4), but it can also be given a direct verification as follows. Suppose that $\dot{-}$ satisfies ($\dot{-}$7) and ($\dot{-}$8), and suppose that $A \dot{-} (x \,\&\, y) \neq A \dot{-} x$ and $A \dot{-} (x \,\&\, y) \neq A \dot{-} y$; we want to show that $A \dot{-} (x \,\&\, y) = A \dot{-} x \cap A \dot{-} y$. By ($\dot{-}$7) it suffices to show that $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x \cap A \dot{-} y$, so it suffices to show that $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$ and $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} y$. By ($\dot{-}$C), which we know by 3.4 to be an immediate consequence of ($\dot{-}$8), we have at least one of these inclusions. So it remains to show that under our hypotheses either inclusion implies the other. We prove one; the other is similar.

Suppose for reductio ad absurdum that $A \dot{-} (x \,\&\, y) \subseteq A \dot{-} x$ but $A \dot{-} (x \,\&\, y) \nsubseteq A \dot{-} y$. Since by hypothesis $A \dot{-} (x \,\&\, y) \neq A \dot{-} x$, we have $A \dot{-} x \nsubseteq A \dot{-} (x \,\&\, y)$, so there is an $a \in A \dot{-} x$ with $a \notin A \dot{-} (x \,\&\, y)$. Since $A \dot{-} (x \,\&\, y) \nsubseteq A \dot{-} y$, we have by ($\dot{-}$8) that $y \in A \dot{-} (x \,\&\, y)$. Hence since $a \notin A \dot{-} (x \,\&\, y)$ we have $\neg y \vee a \notin A \dot{-} (x \,\&\, y)$. Hence by ($\dot{-}$7), $\neg y \vee a \notin A \dot{-} x$ or $\neg y \vee a \notin A \dot{-} y$. But since $a \in A \dot{-} x$ the former alternative is impossible. And the second alternative is also impossible, since by recovery $A \dot{-} y \cup \{y\} \vdash a$, so that $\neg y \vee a \in A \dot{-} y$. □

§7. A diagram for the implications. To end the paper, we summarize the "implication results" of §§4 and 6 in a diagram. The conditions are as named in previous pages with in addition ($\dot{-}$R) and ($\dot{-}$TR), meaning that $\dot{-}$ is relational, respectively transitively relational, over A, and (γR) and (γTR), meaning that γ is. ($\dot{-}$C) is the covering condition of Observation 3.4; (γC) is its analogue $\gamma(A \perp x) \subseteq \gamma(A \perp x \,\&\, y)$ or $\gamma(A \perp y) \subseteq \gamma(A \perp x \,\&\, y)$. ($\dot{-}$P) is the partial antitony condition of


∸C
γC
∸8
∔8
γ8
∸I
∸M
∸7 & ∸8
∸V
∸TR
γTR
∸R
γR
γ7:N
γ7:∞
γ7:2
∸7
∸P
∔7
γ7
γP
γ7:1
∔F
∸F
∸Q
∸D
∸WD

3.3; and (γP) is its obvious analogue $\gamma(A \perp x \& y) \cap A \perp x \subseteq \gamma(A \perp x)$. Conditions are understood to be formulated for an arbitrary theory A, selection function γ for A, and partial meet contraction function $\dot{-}$ over A determined by γ. Arrows are of course for implications, and conditions grouped into the same box are mutually equivalent in the finite case. Conversely, conditions in separate boxes are known to be nonequivalent, even for the finite case. The diagram should be read as a map of an ordering, but *not* as a lattice: a "$\bigvee$" alignment does not necessarily mean that the bottom condition is equivalent to the conjunction of the other two. In some cases, it is—for example ($\dot{-}$TR) = ($\dot{-}$7) & ($\dot{-}$8) = ($\dot{-}$V), as proven in Observations 4.5 and 6.5; and again ($\dot{-}$D) = ($\dot{-}$F) & ($\dot{-}$8), as shown in Observation 6.3. But ($\dot{-}$7) & ($\dot{-}$C) is known *not* to be equivalent to ($\dot{-}$TR), and (γR) & ($\dot{-}$TR) may perhaps not be equivalent to (γTR). Finally, implications and nonimplications that follow from others by transitivity have not been written into the diagram, but are left as understood. Implications concerning connectivity from §5 have been omitted from the diagram, to avoid overcluttering.

All the general implications (arrows) have been proven in the text, or are immediate. The finite case equivalences issue from the injection result of Observation 4.6, and several were noted in Observation 4.10. Of the finite case nonequivalences, a first example serving to separate (γ7) from ($\dot{-}$R) was given in Observation 4.9, from which it follows immediately that (γ7) does not in the finite case imply ($\dot{-}$TR). The other nonequivalences need other examples, which we briefly sketch.

For the second example, take A to be the eight-element theory of Observation 4.9, but define γ as follows: In the limiting case of a_7, we put $\gamma(A \perp a_7) = \{!a_0\}$ as required by the fact that $a_7 \in \mathrm{Cn}(\varnothing)$; put $\gamma(A \perp a_j) = A \perp a_j$ for all j with $2 \leq j < 7$; put $\gamma(A \perp a_1) = \{!a_3\}$; and put $\gamma(A \perp a_0) = \{!a_1, !a_3\}$. Then it can be verified that the partial meet contraction function $\dot{-}$ determined by γ satisfies ($\dot{-}$C), and so by finiteness also (γC), but not ($\dot{-}$8) and so a fortiori not ($\dot{-}$TR).

For the third example, take A as before, and put $\gamma(A \perp a_7) = \{!a_0\}$ as always; put $\gamma(A \perp a_1) = \{!a_2\}$; and put $\gamma(A \perp a_i) = A \perp a_i$ for all other a_i. It is then easy to check that this example satisfies ($\dot{-}$8) but not ($\dot{-}$7), and so a fortiori not ($\dot{-}$R) and not ($\dot{-}$TR).

For the fourth and last example, take A as before, and put $\leq$ to be the least reflexive relation over 2^A such that $!a_1 \leq !a_2$, $!a_2 \leq !a_3$, $!a_3 \leq !a_2$ and $!a_3 \leq !a_1$. Define γ from $\leq$ via the marking off identity, and put $A \dot{-} x = \bigcap \gamma(A \perp x)$. Then it is easy to check that γ is a selection function for A, so that ($\dot{-}$R) holds. But ($\dot{-}$C) fails; in particular when $x = a_1$ and $y = a_2$ we can easily verify that $A \dot{-} (x \& y) \nsubseteq A \dot{-} x$ and $A \dot{-} (x \& y) \nsubseteq A \dot{-} y$. Hence, a fortiori, ($\dot{-}$8) and ($\dot{-}$TR) also fail.

Added in proof. The authors have obtained two refinements: the arrow ($\dot{-}$D) $\to$ ($\dot{-}$TR) of the diagram on page 528 can be strengthened to ($\dot{-}$D) $\to$ (γTR); the implication ($\gamma 7:\infty$) $\to$ ($\gamma 7:N$) of Observation 4.10 can be strengthened to an equivalence. The former refinement is easily verified using the fact that any *maxichoice* contraction function over a theory is determined by a *unique* selection function over that theory. The latter refinement can be established by persistent use of the compactness of Cn.

Observation 4.10 so refined implies that for a theory A and selection function γ over A, γ is relational over A iff ($\gamma 7{:}2$) holds. This raises an interesting *open question*, a positive answer to which would give a representation theorem for relational partial meet contraction, complementing Corollary 4.5: Can condition ($\gamma 7{:}2$) be expressed as a condition on the contraction operation $\dot{-}$ determined by γ?

We note that a rather different approach to contraction has been developed by Alchourrón and Makinson in *On the logic of theory change: safe contraction*, to appear in ***Studia Logica***, vol. 44 (1985), the issue dedicated to Alfred Tarski; the relationship between the two approaches is studied by the same authors in *Maps between some different kinds of contraction function: the finite case*, also to appear in ***Studia Logica***, vol. 44 (1985).

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