$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l'=0}^{\infty} \frac{r_{min}^{l'}}{r_{max}^{l'+1}} \frac{4\pi}{2l+1} \sum_{m=-l'}^{l'} Y_{l'}^{m'}(\vartheta,\varphi) Y_{l'}^{m'*}(\vartheta',\varphi') \; ; \quad r_{max} = \max\{r, r'\} ; \; r_{min} = \min\{r, r'\}$$

1.

$$\rho(\vec{r}') = \rho(r') f(\vartheta',\varphi') = \rho(r') \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm} Y_l^m(\vartheta',\varphi')$$

$$\varphi(\vec{r}) = \iiint_{\mathbb{R}^3} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}' =$$

$$= \iiint_{\mathbb{R}^3} r'^2 \rho(r') \left(\sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm} Y_l^m(\vartheta',\varphi') \sum_{l'=0}^{\infty} \frac{r_{min}^{l'}}{r_{max}^{l'+1}} \frac{4\pi}{2l'+1} \sum_{m'=-l'}^{l'} Y_{l'}^{m'}(\vartheta,\varphi) Y_{l'}^{m'*}(\vartheta',\varphi') \sin\vartheta' d\varphi' d\vartheta' \right) dr' =$$

$$= \int_{\mathbb{R}^+} r'^2 \rho(r') \sum_{l=0}^{n} \sum_{m=-l}^{l} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} a_{lm} \frac{r_{min}^{l'}}{r_{max}^{l'+1}} \cdot \frac{4\pi}{2l'+1} Y_{l'}^{m'}(\vartheta,\varphi) \underbrace{\iint_{\mathbb{R}^2} Y_l^m(\vartheta',\varphi') Y_{l'}^{m'*}(\vartheta',\varphi') \sin\vartheta' d\varphi' d\vartheta'}_{\delta_{ll'}\delta_{mm'}} =$$

$$= \int_{\mathbb{R}^+} r'^2 \rho(r') \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm} \frac{r_{min}^{l}}{r_{max}^{l+1}} \cdot \frac{4\pi}{2l+1} Y_l^m(\vartheta,\varphi) =$$

$$= \sum_{l=0}^{n} \int_{\mathbb{R}^+} r'^2 \rho(r') \frac{r_{min}^{l}}{r_{max}^{l+1}} \cdot \frac{4\pi}{2l+1} \sum_{m=-l}^{l} a_{lm} Y_l^m(\vartheta,\varphi)$$

$$\rho(\vec{r}) = \rho_0 e^{-r} \cos^2\vartheta \, ;$$

2.

$$\iiint_{\mathbb{R}^3} \rho(\vec{r})\, d^3\vec{r} = \rho_0 \int_0^{2\pi} d\varphi \int_0^{\pi} \cos^2\vartheta \sin\vartheta \, d\vartheta \int_0^{+\infty} r^2 e^{-r} dr$$

$$\int_0^{+\infty} r^2 e^{-r} dr = -\int_0^{+\infty} r^2 d e^{-r} = \cancel{-e^{-r} r^2 \Big|_0^{+\infty}} + \int_0^{+\infty} e^{-r} \cdot 2r\, dr = -2 \int_0^{+\infty} r\, d e^{-r} =$$

$$= -2 \left\{ e^{-r} r \Big|_0^{+\infty} - \int_0^{+\infty} e^{-r} dr \right\} = 2 \int_0^{+\infty} e^{-r} dr = -2 e^{-r} \Big|_0^{+\infty} = (-2)(0-1) = 2$$

$$\int_0^{\pi} \cos^2\vartheta \sin\vartheta \, d\vartheta = -\int_0^{\pi} \cos^2\vartheta \, d\cos\vartheta = -\frac{1}{3}\cos^3\vartheta \Big|_0^{\pi} = -\frac{1}{3}(-1-1) = \frac{2}{3}$$

$$\Rightarrow \iiint_{\mathbb{R}^3} \rho(\vec{r})\, d^3\vec{r} = \rho_0 \cdot 2\pi \cdot \frac{2}{3} \cdot 2 = \boxed{\rho_0 \cdot \frac{8\pi}{3} = 1} \Rightarrow \rho_0 = \frac{3}{8\pi}$$

$$Y_0^0(\vartheta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

3.

$$Y_1^0(\vartheta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\vartheta$$

$$Y_1^{\pm1}(\vartheta,\varphi) = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\pm i\varphi}\sin\vartheta$$

$$Y_2^0(\vartheta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\vartheta - 1)$$

$$Y_2^{\pm1}(\vartheta,\varphi) = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{\pm i\varphi}\sin\vartheta\cos\vartheta$$

$$Y_2^{\pm2}(\vartheta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{\pm 2i\varphi}\sin^2\vartheta$$

$$\rho(\vec{r}) = \frac{3}{8\pi}e^{-r}\cos^2\vartheta = \rho(r)f(\vartheta,\varphi)$$

$$\Rightarrow \rho(r) = \frac{3}{8\pi}e^{-r}$$

$$f(\vartheta,\varphi) = \cos^2\vartheta = \sum_{l=0}^{n}\sum_{m=-l}^{l} a_{lm}Y_l^m(\vartheta,\varphi)$$

$$\cos^2\vartheta = a_{00}Y_0^0 + 0\cdot Y_1^{-1} + 0\cdot Y_1^0 + 0\cdot Y_1^{+1} + a_{20}Y_2^0 + 0\cdot Y_2^{-2} + 0\cdot Y_2^{-1} + 0\cdot Y_2^1 + 0\cdot Y_2^2$$

$$= a_{00}\cdot\frac{1}{2}\sqrt{\frac{1}{\pi}} + a_{20}\cdot\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\vartheta - 1) =$$

$$= a_{00}\cdot\frac{1}{2}\sqrt{\frac{1}{\pi}} + a_{20}\cdot\frac{3}{4}\sqrt{\frac{5}{\pi}}\cos^2\vartheta - a_{20}\cdot\frac{1}{4}\sqrt{\frac{5}{\pi}}$$

$$\Rightarrow a_{20}\cdot\frac{3}{4}\sqrt{\frac{5}{\pi}} = 1 \quad \Rightarrow a_{20} = \frac{4}{3}\sqrt{\frac{\pi}{5}}$$

$$\Rightarrow a_{00}\cdot\frac{1}{2}\sqrt{\frac{1}{\pi}} - \frac{4}{3}\sqrt{\frac{\pi}{5}}\cdot\frac{1}{4}\sqrt{\frac{5}{\pi}} = 0,$$ т.е. $a_{00} = \frac{2}{3}\sqrt{\pi}$

$$a_{2,-2} = a_{2,-1} = a_{21} = a_{22} = 0$$

4.

$$\varphi(\vec{r}) = \sum_{l=0}^{n} \int_{\mathbb{R}^+} r'^2 \rho(r') \frac{r_{min}^{\,l}}{r_{max}^{\,l+1}} \cdot \frac{4\pi}{2l+1} \sum_{m=-l}^{l} a_{lm} Y_l^m(\vartheta, \varphi) \ , \quad \rho(r) = \frac{3}{8\pi} e^{-r}$$

$$a_{00} = \frac{2}{3}\sqrt{\pi} \ , \quad a_{20} = \frac{4}{3}\sqrt{\frac{\pi}{5}} \ , \quad a_{lm} = 0 \quad \forall l \neq 0,2 \ ; \ m \neq 0$$

$$\varphi(\vec{r}) = \int_{\mathbb{R}^+} r'^2 \frac{\cancel{3}}{8\cancel{\pi}} e^{-r'} \frac{1}{r_{max}} \cdot \frac{4\cancel{\pi}}{1} \cdot \frac{2}{\cancel{3}}\cancel{\sqrt{\pi}} \cdot \frac{1}{2}\cancel{\sqrt{\frac{1}{\pi}}}\, dr' +$$

$$+ \int_{\mathbb{R}^+} r'^2 \frac{\cancel{3}}{8\cancel{\pi}} e^{-r'} \frac{r_{min}^2}{r_{max}^3} \cdot \frac{4\cancel{\pi}}{5} \cdot \frac{\cancel{4}}{\cancel{3}}\cancel{\sqrt{\frac{\pi}{5}}} \cdot \frac{1}{\cancel{4}}\cancel{\sqrt{\frac{5}{\pi}}} (3\cos^2\vartheta - 1)\, dr' =$$

$$= \frac{4}{8} \int_{\mathbb{R}^+} r'^2 e^{-r'} \frac{1}{r_{max}}\, dr' + \frac{4}{5 \cdot 8} \int_{\mathbb{R}^+} r'^2 e^{-r'} \frac{r_{min}^2}{r_{max}^3} (3\cos^2\vartheta - 1)\, dr' =$$

$$\boxed{\varphi(\vec{r}) = \frac{1}{2} \int_{\mathbb{R}^+} r'^2 e^{-r'} \frac{1}{r_{max}}\, dr' + \frac{1}{10}(3\cos^2\vartheta - 1) \int_{\mathbb{R}^+} r'^2 e^{-r'} \frac{r_{min}^2}{r_{max}^3}\, dr'}$$

$$\int_1^{+\infty} \to \int_1^0 \qquad z = \frac{1}{z'} \Rightarrow dz = -\frac{1}{z'^2}dz' \Rightarrow dz' = -\frac{dz}{z^2}$$

5.

$$\left(z \vee z' \Leftrightarrow \frac{1}{z'} \vee \frac{1}{z}\right) \Rightarrow \max\{z, z'\} = \frac{1}{\min\{\frac{1}{z'}, \frac{1}{z}\}}$$

$$\min\{z, z'\} = \frac{1}{\max\{\frac{1}{z'}, \frac{1}{z}\}}$$

$$\int_{\mathbb{R}^+} z'^2 e^{-z'} \frac{1}{z_{max}} dz' = \left(\int_0^1 + \int_1^{+\infty}\right) z'^2 e^{-z'} \frac{1}{z_{max}} dz' =$$

$$= \int_0^1 z'^2 e^{-z'} \frac{1}{\max\{z, z'\}} dz' + \int_1^{+\infty} z'^2 e^{-z'} \frac{1}{\max\{z, z'\}} dz' =$$

$$= \int_0^1 z'^2 e^{-z'} \frac{1}{\max\{z, z'\}} dz' + \boxed{\int_0^1 \frac{1}{z^4} e^{-\frac{1}{z}} \min\{z, \frac{1}{z}\} dz}$$

$$\int_{\mathbb{R}^+} z'^2 e^{-z'} \frac{z_{min}^2}{z_{max}^3} dz' = \int_0^1 z' e^{-z'} \frac{z_{min}^2}{z_{max}^3} dz' + \boxed{\int_0^1 \frac{1}{z^4} e^{-\frac{1}{z}} \frac{\min\{z, \frac{1}{z}\}^3}{\max\{z, \frac{1}{z}\}^2} dz}$$