LORDS INSTITUTE OF ENGINEERING AND TECHNOLOGY

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B.E-II SEMESTER, QUESTION BANK, 2023

MATHEMATICS-II

(COMMON FOR ALL BRANCHES)

SAQ UNIT-I			
S.NO		**CO MAPPING**	**Bloom's Taxonomy Level**
1	Find the rank of the matrix $A=\begin{bmatrix}-1 & 0 & 6\\ 3 & 6 & 1\\ -5 & 1 & 3\end{bmatrix}$	CO2	BTL1
2	Find the Eigen values of the matrix $A=\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$	CO2	BTL2
3	Find the sum and product of the Eigen values of the matrix $\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$	CO2	BTL2
4	Show that the vectors (1, 2, 3), (2, 3, 4), (0, 0, 1), (3, 4, 5) are Linearly Independent.	CO2	BTL4
5	Define rank of the matrix and give one example.	CO2	BTL2
6	Find the value of 'k' such that the rank of A = $\begin{bmatrix}1 & -2 & 3\\ 2 & k & 4\\ -1 & 2 & 5\end{bmatrix}$ is 2	CO2	BTL3

7	Find the rank of the matrix $$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$	CO2	BTL2
8	If A= $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, Find the Eigen values of A^3+7A^2+2A	CO2	BTL3
9	Convert the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{bmatrix}$ in to echelon form.	CO2	BTL 3
10	Write the symmetric matrix for the Quadratic form, $Q = x^2 + 2y^2 + 3z^2 - 2xy + 4yz + 6zx$	CO2	BTL1
11	Examine linear independence of the given vectors (1,1,0,1); (1,1,1,1); (-1,1,1,1); (1,0,0,1)	CO2	BTL3
12	Discuss the nature of quadratic form $x^2 - y^2 + 4z^2 + 2yz + 6zx + 4xy$. Also find index and signature	CO2	BTL2
13	Write the matrix form and also the augmented matrix for the given system of equations $3x - y - z = 3, 2x - 8y + z = -5, x - 2y + 9z = 8$	CO2	BTL2
14	Define Eigen value and Eigen vector with example.	CO2	BTL1
15	Find the Eigen value corresponding to the Eigen vector $X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$	CO2	BTL2
SAQ **UNIT – II**			
1	Define Exact Differential Equation	CO-3	BTL1
2	Define the Integrating factor of non homogenous differential equation $\frac{dy}{dx} + p(x)y = Q(x)$	CO-3	BTL1
3	Write Riccati’s equation and Clairaut’s equation.	CO-3	BTL1

4	Solve $(2x-y+1)dx+(2y-x-1)dy=0$	CO-3	BTL3
5	Define Orthogonal Trajectories of a given family of curve and write the procedure to find it in polar coordinates.	CO-3	BTL4
6	Solve $x\frac{dy}{dx}+y = \log x$	CO-3	BTL5
7	Find the general solution of $y=xp-p^3$ where $p=\frac{dy}{dx}$	CO-3	BTL2
8	Find the orthogonal trajectories of the family of curves $y=cx^2$ where c is a parameter.	CO-3	BTL3
9	Solve $xdy-ydx=(x^2+y^2)dy$	CO-3	BTL 4
10	Find the solution of the differential equation $(y-x+1)dy-(y+x+2)dx=0.$	CO-3	BTL3
11	Solve $y(2xy+e^x)dx=e^x dy$	CO-3	BTL4
12	Find the orthogonal trajectories of the family of curves $x^2+16y^2=c$	CO-3	BTL3
13	Solve $\frac{dy}{dx}=e^x+y$	CO-3	BTL4
14	Find the orthogonal trajectories of the family of curves $r=c\theta^2$	CO-3	BTL3
15	Find an integrating factor of $(x^3+y^3)dx-x^2ydy=0$	CO-3	BTL2
	SAQ **UNIT – III**		
1	Solve $y''-y=0,$ when $y=0$ and $y'=2$ at $x=0$	CO4	BTL3
2	Solve $(D^4-81)y=0$.	CO4	BTL5
3	Solve $(D^4+8D^2+16)y=0$.	CO4	BTL4
4	Find particular integral of $(D^2-4D+4)y=e^{2x}$	CO4	BTL2
5	Find the solution of initial value problem $y''+4y'-13y=0$, $y(0)=y'(0)=1.$	CO4	BTL4
6	Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0.$	CO4	BTL4

7	Find complimentary function of $(D^2 - 4D + 3)y = \sin 3x \ \cos 2x$.	CO4	BTL2
8	Find the P.I of $(D^2 + 1)y = 8e^{-x}$.	CO4	BTL2
9	Find the particular Integral of $(D^3 - 6D^2 + 11D - 6)\ y = e^{-3x}$	CO4	BTL2
10	write in brief about the method of variation of parameters	CO4	BTL1
11	Define the terms i) Complementary function ii) Particular integral	CO4	BTL1
12	State Euler-Cauchy equation and brief method to solve it.	CO4	BTL1
13	Solve the D.E $D^2y = sin2x$	CO4	BTL4
14	Find the value of $\frac{1}{D+1}\ (x^2 + 1)$	CO4	BTL4
15	Find particular value of $\frac{1}{(D-2)(D-3)}e^{2x}$	CO4	BTL2
SAQ UNIT-IV			
1	Evaluate $\gamma\left(-\frac{5}{2}\right)$	CO5	BTL4
2	Define Beta function.	CO5	BTL1
3	State Gamma function	CO5	BTL1
4	State Rodrigue's formula.	CO5	BTL1
5	State the relation between Beta and Gamma functions.	CO5	BTL1
6	Prove that $p_n(1) = 1$	CO5	BTL4

7	Classify the singular points of the differential equation $x^2y'' - 5y' + 3x^2y = 0$	CO5	BTL2
8	Find the value of $\beta(\frac{9}{2}, \frac{7}{2})$	CO5	BTL2
9	Prove that $\beta(m,n) = \beta(n,m)$	CO5	BTL4
10	Express f(x) $= 2x^3 - 6x^2 + 5x - 3$ in terms of Legendre polynomial $p_n(x)$.	CO5	BTL2
11	Prove that Γ(n+1)=n !	CO5	BTL4
12	Evaluate $\int_{-1}^{0}(1 - x^2)^n\, dx$	CO5	BTL5
13	Express $3p_3(x) + 2p_2(x) + 4p_1(x) + 5p_0(x)$ as polynomial in x	CO5	BTL2
14	Determine the nature of point x=0 for equation $xy'' + ysinx = 0$	CO5	BTL2
15	Evaluate Γ(9/2)	CO5	BTL5
SAQ **UNIT-V**			
1	Find $L\{e^{-t} \sin 2t\}$	CO6	BTL2
2	State Convolution theorem of Laplace transforms.	CO6	BTL1
3	Find $L\{t \cos 2t\}$.	CO6	BTL2
4	Evaluate $L^{-1}\left\{\frac{2}{S^3} + \frac{1}{S^2}\right\}$.	CO6	BTL5
5	Find $L\{te^{-t}\}$	CO6	BTL1

6	Find inverse Laplace transform of $f(t)=t^2\sinh t$	CO6	BTL2
7	Find $L\{t^3e^{-4t}\}$	CO6	BTL2
8	Find $L^{-1}\left\{\dfrac{1}{(s+2)(s+3)}\right\}$	CO6	BTL2
9	Find the Laplace transform of $f(t)=\sin^2 t$.	CO6	BTL2
10	Define Unit Impulse function and write its Laplace transform.	**CO6**	BTL1
11	Evaluate $L[cos^2t]$.	**CO6**	BTL5
12	Find $L^{-1}\left\{\dfrac{1}{(S+1)(S+2)}\right\}$	**CO6**	BTL2
13	Find $L\{t^3e^{-4t}\}$	**CO6**	BTL3
14	Evaluate $L^{-1}[\frac{1}{s(s+1)}]$	**CO6**	BTL5
15	Find $L[t^3]$.	**CO6**	BTL2
LAQ **UNIT- I**			
1	Find the values of a, b such that the equation $2x+3y+5z=9, 7x+3y+2z=8,\ 2x+3y+az=b$, has (i) No Solution (ii) Infinite Solutions (iii) Unique Solution.	CO2	BTL 4
2	Solve the system of equations x+3y+2z =0; 2x-y+3z = 0; 3x-5y+4z = 0; x+17y+4z = 0.	CO2	BTL 4
3	Verify Cayley-Hamilton Theorem and hence find the inverse of the matrix, where A= $\begin{bmatrix}2 & 3 & 4\\0 & 1 & 5\\0 & 0 & -1\end{bmatrix}$	CO2	BTL5

4	Reduce the Quadratic form, $Q = 3x^2+3y^2+3z^2+2xy+2zx-2yz$ to Canonical form and find its nature, index and signature.	CO2	BTL5
5	Test for the consistency and solve, if consistent, the system of equations $x+y+z=3,\ 3x-9y+2z=-4,\ 5x-3y+4z=6.$	CO2	BTL5
6	Reduce the matrix A= $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$ to Echelon form and hence find its rank. .	CO2	BTL 4
7	Verify Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and hence find A^5	CO2	BTL 4
8	Find the Eigen values and corresponding Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	CO2	BTL5
9	Verify Cayley-Hamilton for the matrix A= $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$	CO2	BTL 4
10	Find the rank of the matrix A= $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$	CO2	BTL 5
11	Reduce the quadratic form to Canonical form $8{x_1}^2 + 7{x_2}^2 + 3{x_3}^2 + 12x_1x_2 + 4x_1x_3 - 8x_2x_3$	CO2	BTL 3
12	Show that sum of eigen values of a matrix is its trace and product of eigen values is its determinant.	CO2	BTL 4

13	Reduce the quadratic form $2x_1x_2 + 2x_1x_3 + 2x_2x_3$ to canonical form. Hence find Index, Signature and orthogonal transformation.	CO2	BTL 3
14	Find the Eigen values and corresponding Eigen vectors of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	CO2	BTL 3
15	Determine eigen values of i) A^2 ii)A^{-1} iii) $B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$	CO2	BTL 3
LAQ **UNIT - II**			
1	Solve $(\mathrm{x} - \mathrm{y}^2)\mathrm{dx} + 2\mathrm{xydy} = 0$	CO-3	BTL 4
2	Find the orthogonal trajectories of $\mathrm{r}^{\mathrm{n}}\sin\mathrm{n}\theta = \mathrm{c}$, where c is the parameter.	CO-3	BTL 6
3	Solve $\mathrm{x}\,\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \mathrm{x}^3\mathrm{y}^6$.	CO-3	BTL5
4	Solve the differential equation $\mathrm{ydx} - \mathrm{xdy} + \mathrm{e}^{\mathrm{x}}\mathrm{ydy} = 0$.	CO-3	BTL 4
5	Solve $y(2x^2y + e^x)dx = (e^x + y^3)dy$.	CO-3	BTL5
6	Solve the differential equation $y' + 4xy + xy^3 = 0$	CO-3	BTL 4
7	Solve $\frac{dy}{dx} + 2xy = 2x$.	CO-3	BTL5
8	Find the general solution of the Riccati's equation $y' = 3y^2 - (1 + 6\mathrm{x})\mathrm{y} + 3\mathrm{x}^2 + \mathrm{x} + 1$, if $y = x$ is a particular solution.	CO-3	BTL 4
9	Find the Orthogonal Trajectories of the family of cardioids $r=a(1-cos\theta)$	CO-3	BTL5
10	Find the general solution of the equation $\frac{\mathrm{dy}}{\mathrm{dx}} = 2\mathrm{x}y^2 + (1 - 4\mathrm{x})\mathrm{y} + 2\mathrm{x} - 1$, if y=1 is a particular solution.	CO-3	BTL 3
11	Solve $y(x + y)dx - \mathrm{x}^2\mathrm{dy} = 0$	CO-3	BTL 5

12	Solve $\frac{dy}{dx} + xsin2y = x^3cos^2y$	CO-3	BTL 4
13	Find the Orthogonal Trajectories of the family of hypocycloids $x^{2/3} + y^{2/3} = a^{2/3}$	CO-3	BTL 3
14	Find the general and singular solution of the equation $y = xp + p^2$ where $p = \frac{dy}{dx}$	CO-3	BTL 3
15	Solve $ysin2xdx - (1 + y^2 + cos^2x)dy = 0$	CO-3	BTL 4
LAQ **UNIT - III**			
1	Find the general solution of $y'' + 3y' + 2y = 2e^x$	CO4	BTL3
2	Find the general solution of $(D^2 - 4D + 4)y = e^{2x}$	CO4	BTL 4
3	Solve $(D^2 + 9)y = sin3x.$	CO4	BTL4
4	Find the general solution of $(D^2 - 4)y = cos^2x$	CO4	BTL 4
5	Solve: $(D + 2)(D - 1)^2y = e^{-2x} + 2sinhx$	CO4	BTL4
6	Find the general solution of $(D^2 + 2D + 1)y = xcosx$	CO4	BTL4
7	Solve $(D^2 + 4)y = e^x + sin2x + cos2x$	CO4	BTL4
8	Solve $(D^3 - 1)y = e^x + sin3x + 2$	CO4	BTL4
9	Solve $y'' - 2y' + y = xe^x \sin x$	CO4	BTL4
10	Solve $(D^2 + 4)y = x^2 + 1 + \cos 2x$	CO4	BTL4
11	Find the general solution of $y'' + 4y' + 4y = 6e^{-2x}cos^2x$	CO4	BTL3
12	Apply method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \sec x$	CO4	BTL2
13	Find the particular integral of	CO4	BTL3

	$(D^2 - 6D + 9)y = x^2 + 2x$		
14	Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$	CO4	BTL4
15	Apply method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$	CO4	BTL3
LAQ **UNIT-IV**			
1	Evaluate $\frac{d}{dx}[\operatorname{erf}(\alpha x)]$	CO5	BTL5
2	Show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$.	CO5	BTL4
3	Prove that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx.$	CO5	BTL4
4	Define Gamma function and show that $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$	CO5	BTL4
5	Evaluate $\int_0^\infty \frac{x^{\frac{3}{2}}}{\sqrt{a^2+x^2}} dx$ using Beta and gamma function.	CO5	BTL5
6	Express $2x^3 + 3x^2 - x + 1$ in terms of Legendre's Polynomial.	CO5	BTL2
7	Prove that $erf(x) + erf_c(x) = 1$	CO5	BTL4
8	Find the power series solution of the differential equation $(1 - x^2)y'' - 2xy' + 2y = 0 \; about \; x = 0.$	CO5	BTL3
9	Express the following sum of the Legendre Polynomial in terms of x $8P_4(x) + 2P_2(x) + P_0(x).$	CO5	BTL2
10	Evaluate the improper Integral $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$	CO5	BTL5
11	Prove that $\beta(m,n)=2\int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta\; cos^{2n-1}\,\theta\, d\theta$	CO5	BTL4
12	Evaluate $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}}$	CO5	BTL5
13	Prove that $2^{2n-1}\Gamma(\mathrm{n})\Gamma\left(n + \frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}$	CO5	BTL4
14	Evaluate $\int_0^\infty 3^{-4x^2} dx$	CO5	BTL5

15	Evaluate $\int_0^1 x^m(1-x^2)^n dx$ in terms of beta function, where m,n are positive constants.	CO5	BTL4
	LAQ **UNIT-V**		
1	Find the Laplace transform of $f(t)=\frac{e^{-2t}\sin 3t}{t}$	CO6	BTL3
2	Using Laplace transform, solve the initial value problem $y''+y=e^t\sin(t)$, $y(0)=0=y'(0)$.	CO6	BTL 4
3	Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(S+1)(S+2)}\right\}$.	CO6	BTL3
4	Evaluate $L^{-1}\left\{\log\left(\frac{S-3}{S+3}\right)\right\}$	CO6	BTL 5
5	Find the Laplace transform of $f(t)=\frac{2\sin^2 t}{t}$.	CO6	BTL2
6	Find inverse Laplace transform of $\frac{S}{S^4+S^2+1}$.	CO6	BTL2
7	Show that $L\left\{e^{at}\right\}=\frac{1}{s-a}$	CO6	BTL4
8	Find the Laplace transform of the function $f(t)=\begin{cases}\sin t & if\ 0<t<\pi \\ 0 & if\ \pi<t<2\pi\end{cases}$ and the period of f(t) is 2π.	CO6	BTL3
9	Show that $L\{\sin at\}=\frac{a}{s^2+a^2}$	CO6	BTL4
10	Find the Laplace transform of $f(t)=e^{-t}(2\cos 5t-3\sin 5t)$.	CO6	BTL2
11	Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.	CO6	BTL5
12	Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t$, $y=\frac{dy}{dt}=0$, when t=0 using Laplace transform	CO6	BTL4
13	Using Convolution Theorem find $L^{-1}[\frac{1}{(s+1)(s+2)}]$	CO6	BTL3
14	Find$L^{-1}[\frac{1}{s(s^2+9)}]$	CO6	BTL2
15	Find $L^{-1}[\frac{1}{s(s+2)}]$	CO6	BTL2

M - I

Assignment - I

SAQ's :

1) Find the rank of the matrix

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

Sol: Given matrix is

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 3R_1$, $R_3 \rightarrow R_3 - 5R_1$

$$\sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 1 & -27 \end{bmatrix}$$

$R_3 \rightarrow 6R_3 - R_2$

$$\sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 0 & -181 \end{bmatrix}$$

$\therefore$ The no. of non-zero rows are 3.

Hence, Rank of the matrix is 3.

2) Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Sol: Given, $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

The charachteristic equation is

i.e., $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda)(2-\lambda) - (4)(3) = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0.$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0.$$

$$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 2) = 0.$$

$$\Rightarrow \lambda - 5 = 0 \;\&\; \lambda + 2 = 0$$

$$\Rightarrow \lambda = 5 \;\&\; \lambda = -2.$$

So, the eigen values are $-2, 5$.

3) Find the sum and product of the eigen values of the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

We know that

Sum of the Eigen Values of the matrix

$= \text{Trace}(A)$.

i.e., $\lambda_1 + \lambda_2 + \lambda_3 = \text{Trac}(A)$.

Now, $\text{Trac}(A) = 2+3+2 = 7$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 7$.

Similarly,

product of the Eigen Values of the matrix

$= \det(A)$.

$$\det(A) = 2(6-2) - 2(2-1) + 1(2-3)$$
$$= 2(4) - 2(1) + 1(-1)$$
$$= 8-2-1 = 8-3 = 5. \Rightarrow \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

4) Show that the vectors $(1,2,3), (2,3,4), (0,0,1), (3,4,5)$ are Linearly independent.

Sol:- $\left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right]$

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 4 & 0 \\ 3 & 4 & 1 & 5 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 3 & 4 & 1 & 5 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 3R_1 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & -2 & 1 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$a_1 + 2a_2 + 3a_4 = 0$$

$$-a_2 - 2a_4 = 0.$$

Let $\boxed{a_4 = K} \Rightarrow -a_2 - 2a_4 = 0 \Rightarrow -a_2 - 2K = 0$

$\Rightarrow -2K = a_2 \Rightarrow \boxed{a_2 = -2K}$

(3)

$$a_1 + 2a_2 + 3a_4 = 0$$

$$\Rightarrow a_1 + 2(-2K) + 3(K) = 0$$

$$\Rightarrow a_1 - 4K + 3K = 0$$

$$\Rightarrow a_1 - K = 0$$

$$\Rightarrow \boxed{a_1 = K}$$

$$\Rightarrow a_1 = K, \quad a_2 = -2K, \quad a_3 = 0, \quad a_4 = K.$$

$\Rightarrow [K, -2K, 0, K] \Rightarrow$ - Trivial.

$\Rightarrow$ Linearly Dependent.

for

$\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

(5) Define Rank of the matrix and give one example of matrix.

Sol:- Let A be an $m \times n$ matrix, and non zero then we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is zero and

(ii) there exists at least one r^{th} order minor of A which is not zero.

Rank of A is denoted as $\rho(A)$.

eg:- Find the rank of the matrix. $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

Sol:- $|A| = -1(18-1) + 6(3+30)$

$= 181 \neq 0$

$\therefore \rho(A) = 3.$

⑥ Find the value of k if the rank $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & k & 4 \\ -1 & 2 & 5 \end{bmatrix}$ is 2.

sol: Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & k & 4 \\ -1 & 2 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & k & 4 \\ -1 & 2 & 5 \end{vmatrix} = 1(5k-8) - (-2)[10-(-4)] + 3[4-(-k)]$$

$$= 5k - 8 + 2(14) + 3(4+k)$$

$$= 5k - 8 + 28 + 12 + 3k$$

$$\begin{array}{r} 28 \\ 12 \\ \hline 32 \end{array}$$

$$\Rightarrow 8k + 32 = 0$$

$$8k = -32$$

$$k = -\frac{\cancel{32}^{4}}{\cancel{8}_{1}} \Rightarrow \boxed{k = -4}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix}$$

$$|A| = 1(48 - 40) - 2(36 - 28) + 3(30 - 28)$$

$$8 - 16 + 6 = 14 - 16 = -2 \neq 0$$

$$\boxed{\rho(A) = 3.}$$

Q.7 Find the rank of the matrix $\begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{vmatrix}$

Sol: Given: $A = \begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{vmatrix}$

$R_2 \to R_2 - \frac{1}{2} R_1$

$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 5 & 2 & 4 & 3 \end{vmatrix}$

$R_3 \to R_3 - \frac{5}{2} R_1$

$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 0 & 9/2 & -1/2 & 1/2 \end{vmatrix}$

$R_3 \to R_3 - \frac{9}{8} R_2$

$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 0 & 0 & 5/8 & 5/8 \end{vmatrix}$

The row echelon form is obtained.

$\therefore$ The rank of matrix is 3.

8. If $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, Find the Eigen values of $A^3 + 7A^2 + 2A$

sol:- Characteristic eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(1-\lambda) - 0 = 0$$

$$-1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

eigen values of A are $1, -1$.

A^3 are $1, -1$

A^2 are $1, 1$.

eigen values of $(A^3 + 7A^2 + 2A)$ are

$$= (-1 + 7 - 2, \quad 1 + 7 + 2)$$

$$= (4, 10)$$

$\therefore$ The eigen values are $4, 10$ //.

Q9. Convert the matrix $A = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{vmatrix}$ into echelon form.

Sol: Given: $A = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{vmatrix}$.

$R_1 \leftrightarrow R_2$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix}$$

$R_2 = R_2 - \frac{R_1}{2}$

$R_3 = R_3 + \frac{R_1}{2}$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{vmatrix}$$

$R_2 = \frac{R_2}{2}$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix}$$

$R_3 = R_3 - R_2$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

Q10 Examine linear independence of the given vectors
(1, 1, 0, 1); (1, 1, 1, 1); (-1, 1, 1, 1); (1, 0, 0, 1).

$$\left|\begin{array}{cccc} + & - & + & - \\ \hline 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right| = 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \left[1\left[(1-0)\right] - 1(1-0) + 1(0)\right] - \left[1(1-0) - 1(-\cancel{1}+\cancel{1}) + 1(0-1)\right]$$
$$- \left[1(0-0) - 1(0-1) + 1(0-1)\right]$$

$$= \cancel{1} - \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1}$$

$$= 0$$

$\therefore$ x_1, x_2, x_3, x_4 are linearly dependent

11. Examine linear independence of the given vectors $(1,1,0,1);\ (1,1,1,1);\ (-1,1,1,1);\ (1,0,0,1)$.

Sol:-

$$\begin{array}{cccc} + & - & + & - \end{array}$$
$$\left|\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right| = 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \Big[1\big[(1-0)\big] - 1(1-0) + 1(0)\Big] - \Big[1(1-0) - 1(\cancel{-1}+\cancel{1}) + 1(0-1)\Big]$$
$$- \Big[1(0-0) - 1(0-1) + 1(0-1)\Big]$$

$$= 1 - 1 - 1 + 1 - 1 + 1$$

$$= 0$$

$\therefore\ x_1, x_2, x_3, x_4$ are linearly dependent

12. Discuss the nature of quadratic form $x^2-y^2+4z^2+2yz+6zx+4xy$. also find index and signature

sol:- The given QF is

$$x^2-y^2+4z^2+2yz+6zx+4xy.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

characteristic eqn $|A-\lambda I|=0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)[-1-\lambda-4+\lambda] - 2[8-2\lambda-3] + 3[2+3+3\lambda]$$

$$= \lambda^2-3\lambda-5-\lambda^3+3\lambda^2+5\lambda-10+4\lambda+15+9\lambda=0$$

$$-\lambda^3+4\lambda^2+15\lambda=0$$

$$-\lambda(\lambda^2-4\lambda-15)=0$$

$$\lambda=0, \quad \lambda=\frac{4\pm\sqrt{16+60}}{2}$$

$$\lambda=0, \quad \lambda=\frac{4\pm\sqrt{76}}{2}$$

$\therefore$ The nature of Quadratic form is indefinite

index $= 1$. (no of +ve eigen values)

signature = no of +ve eigen values - no of -ve eigen value.

si. $= 1-1$

$= 0$

Q.13 Write the matrix form and also the Augmented matrix for the given system of eqns. $3x-y-z=3,\ 2x-8y+z=-5,$ $x-2y+9z=8.$ $\boxed{CO2 \mid BTL2}$

Sol: The given eqn's can be written in the matrix form as $AX=B$

Here $A=\begin{bmatrix} 3 & -1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$

Now, Augmented matrix

$$[A:B]=\left[\begin{array}{ccc:c} 3 & -1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

14. Define eigenvalues & eigen vectors with example.

Ans: Defn:- A non zero vector x is an eigen vector (or characteristic vector) of a square matrix A if there exists a scalar λ such that $Ax = \lambda x$. Then λ is an eigenvalue (or characteristic value) of A.

Eg:- consider $\cancel{A=} \begin{bmatrix} \text{[illegible]} & \text{[illegible]} \\ \text{[illegible]} & \text{[illegible]} \end{bmatrix}$ $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

char eqn $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0.$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$\lambda = 6, -1$ are eigen values.

then for $\lambda = -1$, eigen vectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ //

$\lambda = 6$ " " " $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ //

15Q. verify that $X = \begin{Bmatrix}1\\2\\1\end{Bmatrix}$ is one of the eigen vector of

$$\begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix}.$$

Sol :- $A = \begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix}$

Char eqn $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 1 & 3\\1 & 5-\lambda & 1\\3 & 1 & 1-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(5-\lambda)(1-\lambda)-1\} - 1[1-\lambda-3] + 3[1-15+3\lambda] = 0$$

$$\Rightarrow (1-\lambda)\{5-6\lambda+\lambda^2-1\} + \lambda + 2 - 42 + 9\lambda = 0$$

$$\Rightarrow (1-\lambda)\{\lambda^2-6\lambda+4\} + 10\lambda - 40 = 0.$$

$$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + 10\lambda - 40 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0.$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0.$$

$\lambda = 3 \Rightarrow 3^3 - 7(3)^2 + 36 = 0.$

$$\begin{array}{r|rrrr} \lambda = 3 & 1 & -7 & 0 & 36 \\ & 0 & 3 & -12 & -36 \\ \hline & 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\Rightarrow \lambda^2 - 4\lambda - 12 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-6) = 0. \Rightarrow \lambda = -2, 6.$$

$\therefore$ the eigen values are $\lambda = -2, 3, 6.$

find the values of a, b such that the equation $2x+3y+5z=9$, $7x+3y+2z=8$ $2x+3y+az=b$ has (i) No solution (ii) infinite solution (iii) unique solution

Sol The given systems of equation can be written as

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ b \end{bmatrix}$$

$$\therefore \frac{A}{B} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & 2 & 8 \\ 2 & 3 & a & b \end{bmatrix}$$

$R_2 \to 2R_2 - 7R_1$; $R_3 \to R_3 - R_1$

$$A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & a-5 & b-9 \end{bmatrix}$$

(i) Let $a=5$ and $b \neq 9$.

$$\Rightarrow A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & 0 & b-9 \end{bmatrix}$$

$\Rightarrow \rho(A/B) = 3$; $\rho(A) = 2$.

$\because \rho(A/B) \neq \rho(A)$

The system is inconsistent and it has no solution.

(ii) Let $a=5$ and $b=9$

$$A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow \rho(A/B) = 2;\ \rho(A) = 2.$

$\because \rho(A/B) = \rho(A) = 2 < n(3)$

$\Rightarrow$ The system is consistent and it has infinite number of solutions.

(iii) Let $a \neq 5$ and $b = 9$.

$\Rightarrow \rho(A/B) = 3;\ \rho(A) = 3.$

$\because \rho(A/B) = \rho(A) = r(3) = n(3)$

The system is consistent and it has an unique solution

2. Solve the system of equation $x+3y+2z=0$, $2x-y+3z=0$; $3x-5y+4z=0$; $x+17y+4z=0$.

Sol Given system of equation can be written in the form of

$$AX = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$; $R_3 \to R_3 - 2R_2$

i.e.

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ It is echelon form.

No of non-zero rows $= 2$.

$\Rightarrow \rho(A) = 2 = r$

$\Rightarrow n = 3$

$\because n > r$ i.e., we have to assign $n - r = 3 - 2 = 1$ arbitratory constant.

$\Rightarrow$ Non-trival solution.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 3y + 2z = 0 \quad \ldots (1)$.

$-7y - z = 0 \quad \ldots (2)$.

(2) $\Rightarrow -7y - z = 0$.

$-7y = z$.

$y = \frac{-z}{7}$.

Let $z = k \Rightarrow y = \frac{-k}{7}$.

(1) $\Rightarrow x + 3y + 2z = 0$.

$x + 3\left(\frac{-k}{7}\right) + 2k = 0$.

$x + \left(2 - \frac{3}{7}\right)k = 0$.

$x + \left(\frac{11}{7}\right)k = 0$.

$\therefore x = -\frac{11}{7}k$ //.

3. Verify cayley hamilton theorem and find the rank of the matrix where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

Sol> Characteristic matrix $= A - \lambda I$

$$\therefore A-\lambda I = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

characteristic equation $\Rightarrow |A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\,[0-(1-\lambda)\,(-1-\lambda)] - 0 + 0 = 0.$$

$$(2-\lambda)\,(-1(-1-\lambda+\lambda+\lambda^2))] = 0$$

$$(2-\lambda)\,(-\lambda^2+1) = 0.$$

$$-2\lambda^2+2+\lambda^3-\lambda$$

$$\therefore \lambda^3-2\lambda^2-\lambda+2 = 0.$$

By cayley hamilton theorem

$$A^3-2A^2-A+2 = 0.$$

$$A^2 = A\cdot A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 21 & 42 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

$LHS = A^3 - 2A^2 - A + 2I$

$$= \begin{bmatrix} 8 & 21 & 42 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-8-2+2 & 21-18-3-0 & 42-38-4+0 \\ 0-0-0+0 & 1-2-1+2 & 5-0-5+0 \\ 0-0-0+0 & 0-0-0+0 & -1-2+1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= RHS$

$\therefore LHS = RHS$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ It is echelon form.

$\rho(A) = 3$.

4. Reduce the quadratic form $Q = 3x^2 + 3y^2 + 3z^2 + 2xy + 2zx - 2yz$ to canonical form and find its nature, index and signature.

sol 2

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic matrix = $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristics equation $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)(3-\lambda)-1\right] + \left[3-\lambda+1\right] + 1\left[-1-3+\lambda\right] = 0$$

$$(3-\lambda)\left[9+\lambda^2-6\lambda-1\right] - 4 + \lambda - 4 + \lambda = 0$$

$$(3-\lambda)\left[\lambda^2-6\lambda+8\right] - 8 + 2\lambda = 0$$

$$3\lambda^2 - 18\lambda + 24 - \lambda^3 + 6\lambda^2 - 8\lambda - 8 + 2\lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0$$

$$\lambda^3 + 9\lambda^2 + 24\lambda - 16 = 0$$

$$
\begin{array}{c|cccc}
\lambda = 4 & 1 & -9 & 24 & -16 \\
 & 0 & 4 & -20 & 16 \\
\hline
\lambda = 1 & 1 & -5 & 4 & \boxed{0} \\
 & 0 & 1 & -4 & \\
\hline
\lambda = 4 & 1 & -4 & \boxed{0} & \\
 & 0 & 4 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}
$$

$\Rightarrow \lambda = 1, 4, 4.$

$\because$ All eigen values are positive

$\Rightarrow$ Nature of quadratic form is positive definite

consider $(A - \lambda I) X_i = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \cdots (1).$$

(1) eigen vector corresponding to $\lambda = 1$:

(1) $\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x + y + z = 0 \text{ — } (2)$

$x + 2y - z = 0 \text{ — } (3)$

$x - y + 2z = 0 \text{ — } (4)$

Solving (2) and (3).

$$\begin{array}{cccc} x & & y & 1 \\ 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{array}$$

$-1 \quad 2 \quad 1 \quad -1 \quad 2.$

$$\frac{x}{-1-2} = \frac{y}{1+2} = \frac{z}{4-1} = k$$

$$\frac{x}{-3} = \frac{y}{3} = \frac{z}{3} = k.$$

multiply by -3

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{-1} = k.$$

$\therefore x = k = y = -k;\ z = -k.$

$$\therefore X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ -k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(ii) eigen vector corresponding to $\lambda = 4$

$$(1) \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x+y+z=0 \quad —(5)$$
$$x-y-z=0 \quad —(6)$$
$$x-y-z=0 \quad —(7)$$

Solving 5 & 6.

$$\frac{x}{-1+1} = \frac{-y}{+1-1} = \frac{-z}{1-1} = k$$

$$\frac{x}{0} = \frac{-y}{0} = \frac{z}{0} = k$$

Let $z = k_1$ and $y = k_2$.

$(5) \Rightarrow \quad x-y-z=0$

$$x = y+z$$
$$x = k_1+k_2$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$B = [X_1 X_2 X_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$\therefore$ model matrix $(P) = [e_1 \; e_2 \; e_3]$

$$e_1 = \frac{x_1}{\|x_1\|} \; ; \; e_2 = \frac{x_2}{\|x_2\|} \; ; \; e_3 = \frac{x_3}{\|x_3\|}.$$

$$\|x_1\| = \sqrt{1^2+(-1)^2+(-1)^2} = \sqrt{3}$$

$$\|x_2\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}.$$

$$\|x_3\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}.$$

$$\therefore \; P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\therefore \; P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$P^{-1}AP = \text{Diagonal}(1,4,4) = D$.

$\therefore$ Canonical form $= Y^T D Y$.

$$Y^T = [y_1\ y_2\ y_3] \;;\quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow Y^T D Y = [y_1\ \ y_2\ \ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 4y_2^2 + 4y_3^2$$

$\therefore$ Index $s = 3$; Rank $= 3$.

$\therefore$ signature $= 2s - r = 2(3) - 3 = 3$.

5. Test for the consistency and solve if consistent system of equations

$$x + y + z = 3$$
$$3x - 9y + 2z = -4$$
$$5x - 3y + 4z = 6.$$

Sol: The given system of equation can be written as $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$$

$$A/B = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -9 & 2 & -4 \\ 5 & -3 & 4 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 5R_1$

$$A/B \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -12 & -1 & -13 \\ 0 & -8 & -1 & -9 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$A/B \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -12 & -1 & -13 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\rho(A/B) = 3$:

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow \rho(A) = 3$

$\therefore \rho(A) = \rho\left(\frac{A}{B}\right) = n(3) = n(3)$

$\Rightarrow$ The system is consistent and has an unique solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -13 \\ -1 \end{bmatrix}$$

$$\Rightarrow x + y + z = 3 \quad \text{—} \; (1)$$

$$-12y - z = -13 \quad \text{—} \; (2)$$

$$-z = -1 \Rightarrow z = 1$$

$(2) \Rightarrow -12y - 1 = -13 \Rightarrow -12y = -12.$

$$y = 1$$

$(1) \Rightarrow x + 1 + 1 = 3 \Rightarrow x = 1.$

$\therefore x = 1, y = 1, z = 1$ is the unique solution

6. Reduce the matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$ to echelon form and hence find its rank.

Sol: Given

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 9 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 + R_1$.

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & -7 & -13 & -5 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -13 & -5 \\ 0 & -7 & -14 & -3 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_4 \rightarrow 7R_4 + 6R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -13 & -5 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -130 \end{bmatrix}$$

$\therefore$ It is in echelon form.

$\rho(A) = 4.$

7 Verify cayley hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

sol: $\therefore$ characteristics matrix $= A - \lambda I$.

$\Rightarrow A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

characteristic equation $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0.$$

By cayley hamilton theorem

$A^2 - 7A + 6I = 0$

$$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}.$$

$LHS = A^2 - 7A + 6I$

$$= \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29-35+6 & 28-28+0 \\ 7-7+0 & 8-14+6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS.$$

Hence, cayley hamilton theorem is verified

8. find the eigen values and corresponding eigen vectors of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

sol characteristic of matrix $= A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$y = 0$

$z = 0$

Let $x = k$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{//}$$

9. Verify cayley hamilton theorem for the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

sol $\therefore$ characteristic matrix $= A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 2\lambda & 0 \\ 1 & 2 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristic equation $= |A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda) - 0\right) - 2(0-0) + 1(0-2+\lambda) = 0$$

$$(3-\lambda)\left[\lambda^2 - 5\lambda + 6\right] - 2 + \lambda = 0$$

$$3\lambda^2 - 15\lambda + 18 - \lambda^3 + 5\lambda^2 - 6\lambda - 2 + \lambda = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

characteristic equation $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 .$$

$$(1-\lambda)\left[-(1-\lambda)^2\right] = 0$$

$$(1-\lambda)\left[-(1^2+\lambda^2-2\lambda)\right] = 0$$

$$(1-\lambda)\left[-1-\lambda^2+2\lambda\right] = 0 .$$

$$-1-\lambda^2+2\lambda+\lambda+\lambda^3--2\lambda^2 = 0$$

$$\lambda^3-3\lambda^2+3\lambda-1 = 0 .$$

$\lambda=1$	1	-3	3	-1
	0	1	-2	1
$\lambda=1$	1	-2	1	$\boxed{0}$
	0	1	-1	
$\lambda=1$	1	-1	$\boxed{0}$	
	0	1		
	1	$\boxed{0}$		

The eigen values are 1, 1, 1

Consider $(A - \lambda I) X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

cayley hamilton theorem states that every square matrix satisfies its own characteristic equation

$$A^3 - 8A^2 + 20A - 16I = 0$$

$$LHS = A^3 - 8A^2 + 20A - 16I$$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix} - 8\begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix} + 20\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} - 16\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36 - 80 + 60 - 16 & 56 - 96 + 40 - 0 & 28 - 40 + 20 - 0 \\ 0 - 0 + 0 - 0 & 8 - 32 + 40 - 16 & 0 - 0 + 0 - 0 \\ 28 - 48 + 20 - 0 & 56 - 96 + 40 - 0 & 36 + 80 + 60 - 16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} RHS$$

$$LHS = RHS$$

Hence, cayley hamilton theorem is verified

10. find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Sol

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + R_1$; $R_3 \rightarrow R_3 - R_1$; $R_4 \rightarrow R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \longrightarrow 2R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is echelon form.

No of non zero rows $= 3$

$\therefore \rho(A) = 3$.

11) ~~Find~~ Reduce the quadratic form to canonical form $8x_1^2+7x_2^2+3x_3^2+12x_1x_2+4x_1x_3-8x_2x_3$

sol:

$$A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$$

char eqn is $|A-\lambda I|=0$

$$\begin{vmatrix}8-\lambda & -6 & 2\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix} \Rightarrow$$

$$=(8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+2]+0.5[24-(0.5)(7-\lambda)]=0$$

$$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+6(6\lambda-16)+0.5(0.5\lambda+20.5)=0$$

$$\cancel{-\lambda^3+18\lambda^2+44-25\lambda+66.75=0}$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda^2-18+45)=0$$

$$\lambda(\lambda^2-15\lambda-3\lambda+45]=0$$

$$\lambda[(\lambda-15)(\lambda-3)]=0$$

$$\therefore \lambda=0,3,15.$$

case(i):- $\lambda=0 \Rightarrow \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$8x_1-6x_2+2x_3=0$$

$$-6x_1+7x_2-4x_3=0$$

$$2x_1-4x_2+3x_3=0$$

$$\frac{x_1}{(7)(3)-(-4)(-4)}=\frac{-x_2}{-18+8}=\frac{x_3}{24-14}$$

$$\frac{x_1}{5}=\frac{\cancel{-}x_2}{\cancel{-}10}=\frac{x_3}{10} \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

where $\|x_1\| = \sqrt{1+4+4}$
$= \sqrt{9} = 3.$

case (ii) :- $\lambda = 3$,

$$A = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

$$\frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \frac{X_2}{\|X_2\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

$|||^{ly}$ $$\frac{X_3}{\|X_3\|} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \quad P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$PAP^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$\therefore$ The canonical form is $0y_1^2 + 3y_2^2 + 15y_3^2$.

12(Q) S.T. sum of eigen values of a matrix is its trace and product of eigen values is its determinant

sol:- (i) $tr(A) = \sum_{i=1}^{\infty} \lambda_i$

(ii) $|A| = \prod_{i=1}^{\infty} \lambda_i$

Proof:- Char eqn $|A - \lambda I| = 0$.

$(-\lambda)^m + \alpha_{m-1}(-\lambda)^{m-1} + \cdots + \alpha_1(-\lambda) + \alpha_0 = 0 \rightarrow$ polynomial form.

solve for $\alpha_0, \lambda = 0$

$$\alpha_0 = |A - (0)I| = |A|$$

solve for α_{m-1}.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{21}-\lambda & \cdots & a_{2m} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn}-\lambda \end{vmatrix}.$$

Determinant produces all products of m terms of A such that exactly 1 element from each row and each column

Only 1 way to achieve λ^{m-1}! The product of the diagonal elements

$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{mn} - \lambda$

$\alpha_{m-1} = a_{11} + a_{22} + \cdots + a_{mn} = tr(A)$

$\lambda_1, \lambda_2 \cdots \lambda_m$ eigen values are the roots of Characteristic eqn.

$(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_m - \lambda) = 0$

$$\alpha_0 = \prod_{i=1}^{m} \lambda_i = |A|$$

$$\alpha_{m-1} = \sum_{i=1}^{m} \lambda_i = tr(A)$$

13
(5) Reduce the quadratic form to canonical form
$2xy + 2yz + 2zx$ or $2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

sol:- Given $2xy + 2yz + 2zx$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$\lambda = 2, -1, -1 \Rightarrow$ Eigen values.

case(1):- $\lambda = 2$.

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + z = 0 \quad -(1)$$
$$x - 2y + z = 0 \quad -(2)$$
$$x + y - 2z = 0 \quad -(3)$$

$$\frac{x}{1+2} = \frac{-y}{-2-1} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{+y}{+3} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k \text{ (say)}. \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case (ii) :- $\lambda = -1$

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$

Let $x = 1,\ y = 0$ $\qquad$ $n - r$

$z = -1$ $\qquad$ $3 - 1 = 2$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

case (iii) :- $\lambda = -1$.

$\therefore$ ~~X_3 is orthogonal to X_1~~

Let $x = 1,\ y = 1 \Rightarrow x + y + z = 0$

$z = -2$.

$$\therefore X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$\therefore X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0.$$

$\|X_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ $\quad$, $\|X_3\| = \sqrt{1^2 + (-2)^2 + 1^2}$

$\|X_2\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$ $\qquad\qquad = \sqrt{6}$

$$P = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right]$$

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Now $PAP^T =$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now canonical form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$

$$\underline{\underline{2x^2 - y^2 - z^2 = 0}}$$

Index = no of +ve eigen values = 1

Signature = (no of +ve eigen values) - (no of -ve eigen)

$= 1 - (2) = -1$

Nature is indefinite. //

148) Find the eigen value & eigen vector of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

char eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \quad -①$$

$$= (1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right] - 1\left[(1-\lambda)-1\right] + 1\left[1-(1-\lambda)\right]$$

$$= (1-\lambda)\left[\cancel{1}-\lambda-\lambda+\lambda^2-\cancel{1}\right] - \left[\cancel{1}-\lambda-\cancel{1}\right] + 1\left[\cancel{1}-\cancel{1}-\lambda\right]$$

$$= (1-\lambda)(\lambda^2-2\lambda) + \lambda + \lambda$$

$$\Rightarrow \lambda^2 - 2\cancel{\lambda} - \lambda^3 + 2\lambda^2 + 2\cancel{\lambda} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda - 3) = 0$$

$$-\lambda^2 = 0, \quad \lambda - 3 = 0$$

$$\lambda = 0 \quad \lambda = 3$$

$\therefore \lambda = 0, 0, 3$. are eigen values.

case(i) :- Put $\lambda = 0$ in eqn ①

consider $(A - \lambda I) X_i = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x+y+z=0 \quad - (2)$$
$$x+y+z=0 \quad - (3)$$
$$x+y+z=0 \quad - (4)$$

$$\begin{array}{cccc} & x & y & z \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x}{1-1} = \frac{y}{1-1} = \frac{z}{1-1} = k.$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $x+y+z=0$

let $z=k_1$, $y=k_2$

$$x+k_1+k_2=0$$

$$x=-k_1-k_2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1-k_2 \\ k_1 \\ k_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) :- Put $\lambda=3$ in (1).

$(A-\lambda I)x_i = 0.$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 1 & & 1 & & -2 & & 1 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x}{1+2} = \frac{y}{1+2} = \frac{z}{4-1} = k \text{ (say)}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

multiply by 3 $\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(15) Determine eigen values of (i) A^2 (ii) A^{-1} (iii) $B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$

Sol: Given $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$

Char eqn $A - \lambda I = \begin{pmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{pmatrix}$

$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$
$\Rightarrow 16 - 10\lambda + \lambda^2 + 8 = 0$
$\Rightarrow \lambda^2 - 10\lambda + 24 = 0.$
$\Rightarrow \lambda^2 - 6\lambda - 4\lambda + 24 = 0$
$\Rightarrow \lambda(\lambda-6) - 4(\lambda-6) = 0$
$(\lambda-4)(\lambda-6) = 0$
$\Rightarrow \lambda = 4, 6$ are the eigen values of 'A'

(i) eigen values of A^2 are $4^2, 6^2 \Rightarrow 16, 36$
(ii) eigen values of A^{-1} are $4^{-1}, 6^{-1} \Rightarrow 1/4, 1/6.$
(iii) eigen values of $B = 2A^2 - \frac{1}{2}A + 3I$ are.

$\Rightarrow 2(16) - \frac{1}{2}(4) + 3(1)$
$= 32 - 2 + 3 = 33.$

and $2(36) - \frac{1}{2}(6) + 3(1)$
$= 72 - 3 + 3 = 72.$

$\therefore$ eigen values of B are 33, 72.

Unit II

SAQ.

1Q. Define Exact differential Equation

Ans An Equation of the form.

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

if it satisfy the condition.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Then general solution

$$\int M\,dx + \int_{ffx} N\,dy = c$$

2Q. Define Integrating factor of non-homogenous differential Eqn.

$$\frac{dy}{dx} + Py = Q(x)$$

Ans Integrating factor $= e^{\int P dx}$

general soln $y(I\cdot F) = \int Q(I\cdot F)\,dx + c$

3Q write Riccati & clairuret's Eqn.

Ans An Eqn of the form.

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

where $y = v(x)$ is the Particular soln.

· Clairaut's Eq^n.

$$y = xy' + f(y')$$

$\boxed{4Q}$ Solve $(2x-y+1)dx + (2y-x-1)dy = 0$

Sol^n $M = 2x-y+1$, $N = 2y-x-1$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1$$

This is an exact.

$$\int (2x-y+1)dx + \int (1+2y)dy = c$$

$\longrightarrow \int(2x-y+1)dx + \int(2y-1)dy =$

$$\cancel{2}\frac{x^2}{\cancel{2}} - xy + x + \cancel{2}\frac{y^2}{\cancel{2}} - y = C$$

$\boxed{5Q.}$ Define orthogonal Trajectories of a given family of curve and write the procedure to find it in polar form

$\boxed{Ans}$ A curve which cuts the family of curves orthogonally (at right angles) then that curve is called orthogonal Trajectory of the family of curves.

Procedure of polar form of (O.T)

1) differentiate $^{=}r^{2}$ w.r. to θ $\left(\frac{dr}{d\theta}\right)$

2) eliminate the constant

3) replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

6Q. Solve $x\frac{dy}{dx}+y=\log x$

Ans $\div x$

$$\frac{dy}{dx}+\frac{y}{x}=\frac{\log x}{x}$$

this is linear diff Eqn.

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x} = x$$

$$g.s = y(x) = \int\left(\frac{\log x}{\cancel{x}}\cdot\cancel{x}\right)dx + c$$

$$= y(x) = x\log x - x + c$$

7Q find the general soln of $y = xp - p^3$

Soln diff $^{=}y^{2}$ w.r. to x

$$\frac{dy}{dx} = x\frac{dp}{dx} + p(1) - 3p^2\frac{dp}{dx}$$

$$p = \frac{dp}{dx}(x-3p^2)+p$$

$$p-p = \frac{dp}{dx}(x-3p^2)$$

$$0 = \frac{dp}{dx}(x-3p^2)$$

$$x = 3p^2$$

$$y = xp - p^3$$

$$y = p(3p^2) - p^3$$

$$\left.\begin{aligned} y &= 2p^3 \\ x &= 3p^2 \end{aligned}\right\} \text{gen sol}^n$$

$\boxed{8\,Q}$ find the orthogonal Trajectory of the family of Curves $y = cx^2$ where "c" is a Parameter

$\boxed{Sol^n}$

$$y = cx^2 \quad \text{—①}$$

$$\frac{dy}{dx} = c(2x)$$

$$C = \frac{dy}{dx}\left(\frac{1}{2x}\right)$$

sub in — ①

$$y = \frac{x^2}{2x}\frac{dy}{dx}$$

$$y = \frac{x}{2}\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = -\frac{x}{2}\frac{dx}{dy}$$

$$\int y\,dy = -\frac{1}{2}\int x\,dx + k$$

$$\frac{y^2}{\cancel{2}} = -\frac{1}{\cancel{2}}\frac{x^2}{2} + k$$

$\boxed{9Q}$ solve $x\,dy - y\,dx = (x^2+y^2)dy$

$\boxed{sol^n}$ by Inspection Method

$$\frac{x\,dy - y\,dx}{x^2+y^2} = dy$$

$$\int d\tan^{-1}\left(\frac{y}{x}\right) = \int dy + k$$

$$\tan^{-1}(y/x) = y + k$$

$\boxed{10Q}$ find the sol^n of diff Eq^n.

$$(y-x+1)dy-(y+x+2)dx=0$$

$\boxed{Sol^n}$ $M=-y-x-2,\ N=y-x+1$

$$\frac{\partial M}{\partial y}=-1,\quad \frac{\partial N}{\partial x}=-1$$

is an exact.

$$\int(-y-x-2)dx+\int y\,dy=c$$

$$-yx-\frac{x^2}{2}-2x+\frac{y^2}{2}=c$$

$\boxed{11Q}$ solve $y(2xy+e^x)dx=e^x dy$

$\boxed{Sol^n}$ $[2xy^2+e^x y]\,dx=e^x dy$

$$e^x y\,dx-e^x dy=-2xy^2dx$$

$$\div\ y^2$$

$$\frac{e^x y\,dx-e^x dy}{y^2}=-2x\,dx$$

$$\int d\left(\frac{e^x}{y}\right)=-2\int x\,dx$$

$$e^x/y=-2\frac{x^2}{2}+k$$

12Q find the O.T of $x^2+16y^2=c$

Ans diff "y^2" w.r. to "x^2"

$$2x+16(2)y\frac{dy}{dx}=0$$

$$x+16y\frac{dy}{dx}=0$$

$$x=-16y\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x=+16y\frac{dx}{dy}.$$

$$\int\frac{dx}{x}=\frac{1}{16}\int\frac{dy}{y}$$

$$\log x=\frac{1}{16}\log y+\log c$$

13Q. solve $\frac{dy}{dx}=e^x+y$

Soln $\frac{dy}{dx}-y=e^x$

$P=-1,\quad Q=e^x$

$$e^{-\int 1 dx} = e^{-x}$$

$$g.s = y(I.F) = \int Q(I.F)dx + c$$

$$y\,e^{-x} = \int e^{x} e^{-x} dx + c$$

$$y\,e^{-x} = \int 1\,dx + c$$

$$y\,e^{-x} = x + c$$

$\boxed{149}$ find the orthogonal Trajectory of family of curves $r = c\theta^2$ — ①

$\boxed{Sol^n}$ $\frac{dr}{d\theta} = c(2\theta)$

$$c = \frac{dr}{d\theta} \times \frac{1}{2\theta}$$

Sub in ①

$$r = \frac{1}{2\theta} \frac{dr}{d\theta} (\theta^2)$$

$$r = \frac{1}{2} \frac{dr}{d\theta} (\theta)$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{\cancel{d\theta}}{\cancel{\theta}} = \frac{-\cancel{r}^2}{\cancel{2}} \frac{\cancel{d\theta}}{\cancel{dr}}$$

$$r = \frac{1}{2}\theta\left(-r^2 \frac{d\theta}{dr}\right)$$

$$1 = \frac{1}{2}(-\theta)\frac{d\theta}{dr}\cdot r$$

$$\int \frac{dr}{r} = -\frac{1}{2}\int \theta \, d\theta + k$$

$$\log r = -\frac{1}{2}\frac{\theta^2}{2} + k$$

$\boxed{15Q}$. find Integrating factor of

$$(x^3+y^3)\,dx - x^2y\,dy = 0$$

$\boxed{Sol^n}$ this is the form of homogenous diff Eqn.

$$\text{Integrating factor } \frac{1}{Mx+Ny}$$

$$= \frac{1}{(x^3+y^3)x+(-x^2y)y} = \frac{1}{x^4+y^3x-x^2y^2}$$

Unit II

LAQ

1.Q Solve $(x-y^2)dx+2xy\,dy=0$

Sol^n $M\,dx+N\,dy=0$

$M=x-y^2, \quad N=2xy$

$\frac{\partial M}{\partial y}=-2y, \quad \frac{\partial N}{\partial x}=2y$

This is non-exact diff Eqn.

$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-2y-2y=-4y$

$\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-4y}{2xy}=-\frac{2}{x}=\text{I.F}$

$\boxed{\text{I.F}=1/x^2}=-2\log_e x \Leftarrow e^{-\int 2/x\,dx}$

Multiply the given Eqn by I.F

$\left(\frac{x-y^2}{x^2}\right)dx+\frac{2xy}{x^2}dy=0$

$\left(\frac{1}{x}-\frac{y^2}{x^2}\right)dx+\frac{2}{x}y\,dy=0$

$\frac{\partial M}{\partial y}=-\frac{2y}{x^2}, \quad \frac{\partial N}{\partial x}=-\frac{2y}{x^2}$

Now exact

$$\int\left(\frac{1}{x}-\frac{y^2}{x^2}\right)dx+\int\frac{2y}{x}dy=k$$

$$\log x+\frac{y^2}{x}+0=k$$

$\boxed{2Q}$ find orthogonal Trajectory of $r^n \sin n\theta = c$ where "c" is a parameter -

$\boxed{Sol^n}$ $\log(r^n \sin n\theta)=\log c$

$$\log r^n+\log\sin n\theta=\log c$$

$$n\log r+\log\sin n\theta=\log c$$

diff "r" w.r.to θ

$$\frac{n}{r}\frac{dr}{d\theta}+\frac{n\cos n\theta}{\sin n\theta}=0$$

$$\frac{dr}{d\theta}\left(\frac{1}{r}\right)=-\cot n\theta$$

Replace $\frac{dr}{d\theta}=-r^2\frac{d\theta}{dr}$

$$\frac{-\cancel{r^2}\,d\theta}{\cancel{(r)}\,dr}=-\cot n\theta$$

$$r\frac{d\theta}{dr} = \cot n\theta$$

$$\int \frac{dr}{r} = \int \tan n\theta \, d\theta + \log k$$

$$\log r = \log\left|\frac{\sec n\theta}{n}\right| + \log k$$

$\boxed{3Q}$ Solve $x\frac{dy}{dx} + y = x^3y^6$

$\boxed{\text{Sol}^n}$ $\div x$

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^6 \quad \text{—} (1)$$

This is Bernoulli's.

$\div y^6$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \quad \text{—} (2)$$

~~This~~ Let $y^{-5} = v$

$$-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}$$

Sub in — (2)

$$-\frac{1}{5}\frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\frac{dv}{dx} - \frac{5}{x}v = -x^2(5)$$

This is linear form

$P = \frac{5}{x}$, $e^{\int \frac{5}{x} dx} = e^{-5\log x} = x^{-5}$

$V(I.F) = \int Q \cdot (I.F) dx + c$

$V(x^{-5}) = \int 5x^2 x^{-5} dx + c$

$y^{-5} x^{-5} = 5 \frac{x^{-3+1}}{-3+1} + c$

$\boxed{4Q}$ solve

$y(2x^2y + e^x)dx = (e^x + y^3)dy$

$\boxed{Sol^n}$ $2x^2y^2dx + e^x y dx - e^x dy = y^3 dy$

$\div y^2$

$2x^2 dx + \left(\frac{e^x y dx - e^x dy}{y^2} \right) = y dy$

$2\int x^2 dx + \int d\left(\frac{e^x}{y}\right) = \int y dy + k$

$2\frac{x^3}{3} + \frac{e^x}{y} = \frac{y^2}{2} + k.$

$\boxed{5Q}$ Solve the diff Eqn

$y' + 4xy + xy^3 = 0$

$y' + 4xy = -xy^3$

This is the form of Bernoulli's

$\boxed{\div y^3}$

$y^{-3}\dfrac{dy}{dx} + 4xy^{-2} = -x$

put $V = y^{-2}$

$\dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx}$

$-\dfrac{1}{2}\dfrac{dv}{dx} = y^{-3}\dfrac{dy}{dx}$

$-\dfrac{1}{2}\dfrac{dv}{dx} + 4xV = -x$

$\times$ by (2)

$\dfrac{dv}{dx} - 8xV = 2x$

linear form

$P = -8x, \quad Q = 2x$

$e^{-\int 8x\,dx} = e^{-\frac{8x^2}{2}} = e^{-4x^2}$

$V e^{-4x^2} = 2\int . x . e^{-4x^2} dx + k$

$y^{-2} e^{-4x^2} = 2\int x . e^{t} \dfrac{dt}{-8x} + k$

$-4x^2 = t$

$-8x\,dx = dt$

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$$y^{-2} e^{-4x^2} = -\frac{1}{4}[e^t] + k$$

$$y^{-2} e^{-4x^2} = -\frac{1}{4} e^{-4x^2} + k$$

$\boxed{7Q}$ $\frac{dy}{dx} + 2xy = 2x$

$\boxed{Sol^n}$ This is the form of linear

$$P = 2x, \quad Q = 2x$$

$$I.F = e^{\int P dx} = e^{2\int x dx} = e^{2x^2/2} = e^{x^2}$$

$$y e^{x^2} = 2\int . x e^{x^2} dx + c$$

$$x^2 = t$$

$$2x dx = dt$$

$$y e^t = 2\int x . e^t \frac{dt}{2x} + C$$

$$y e^t = \int e^t dt + c$$

$$y e^{x^2} = e^{x^2} + c$$

$\boxed{8Q}$ Find the general sol^n of the Riccati diff Eq^n.

$$y' = 3y^2 - (1+6x)y + 3x^2 + x + 1$$

Sol^n $\quad y = x + \frac{1}{z}$ (assume).

$$y' = 1 - \frac{1}{z^2}\frac{dz}{dx}$$

Sub in the given Eq^n Eq^n.

$$1 - \frac{1}{z^2}\frac{dz}{dx} = 3\left(x+\frac{1}{z}\right)^2 - (1+6x)\left(x+\frac{1}{z}\right) + 3x^2 + x + 1$$

$$1 - \frac{1}{z^2}\frac{dz}{dx} = 3x^2 + \frac{3}{z^2} + \frac{6x}{z} - x - \frac{6x}{z} - \frac{1}{z} - 6x^2 + 3x^2 + x + 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{3}{z^2} - \frac{1}{z}$$

$$-\frac{1}{\cancel{z^2}}\frac{dz}{dx} = \frac{3-z}{\cancel{z^2}}$$

$$-\frac{dz}{dx} = 3 - z$$

$$\frac{dz}{dx} = -3 + z$$

$$\frac{dz}{dx} - z = -3$$

$$P = -1, \quad Q = -3$$

$$e^{-\int 1\,dx} = e^{-x}$$

$$z\, e^{-x} = -3\int e^{-x}\,dx + c$$

$$z\, e^{-x} = -3\frac{e^{-x}}{-1} + c$$

$$z e^{-x} = 3 e^{-x} + c \Rightarrow z = 3 + c e^{x}$$

$$y = x + \frac{1}{z}$$

$$y = x + \frac{1}{3 + c e^{x}}$$

$\boxed{9\,Q}$ find the orthogonal Trajectory of cardiods $r = a(1-\cos\theta)$ — (1)

Sol^n $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a(\sin\theta)$$

$$a = \frac{dr}{d\theta}\left(\frac{1}{\sin\theta}\right)$$

sub in $\rightarrow$ (1)

$$r = \frac{1}{\sin\theta}\frac{dr}{d\theta}(1-\cos\theta)$$

replace $\frac{dr}{d\theta}$ (by) $\frac{-r^2 d\theta}{dr}$

$$r = -r^2\frac{d\theta}{dr}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\frac{dr}{r} = -\left(\frac{1-\cos\theta}{\sin\theta}\right)d\theta$$

$$\frac{dr}{r} = -\left[\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right] d\theta$$

$$\frac{dr}{r} = \left(\frac{-1}{\sin\theta} + \cot\theta\right) d\theta$$

$$\int \frac{dr}{r} = \int (-\operatorname{cosec}\theta + \cot\theta)\, d\theta$$

$$\int \frac{dr}{r} = -\int \operatorname{cosec}\theta\, d\theta + \int \cot\theta\, d\theta + \log k$$

$$\log r = -\log|\operatorname{cosec}\theta - \cot\theta| + \log|\sin\theta| + \log k$$

$\boxed{10\,Q}$ find the general sol^n of the Equation. $\frac{dy}{dx} = 2xy^2 + (1-4x)y + 2x - 1$ If $y = 1$ is a particular sol^n

$\boxed{Sol^n}$ Let $y = 1 + \frac{1}{z}$ be the sol^n.

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

Sub y, y' in the given Eq^n

$$-\frac{1}{z^2}\frac{dz}{dx} = 2x\left(1 + \frac{1}{z}\right)^2 + (1-4x)\left(1 + \frac{1}{z}\right) + 2x - 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \cancel{2x} + \frac{2x}{z^2} + \cancel{\frac{4x}{z}} + 1 - \cancel{\frac{4x}{z}} - 4x + \frac{1}{z} + \cancel{2x} - 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{2x}{z^2} + \frac{1}{z}$$

$$-\cancel{\frac{1}{z^2}}\frac{dz}{dx} = \frac{2x+z}{\cancel{z^2}}$$

$$-\frac{dz}{dx} = 2x + z$$

$$-\frac{dz}{dx} - z = 2x$$

$$\frac{dz}{dx} + z = -2x$$

$$P = 1, \quad Q = -2x$$

$$e^{\int 1 dx} = e^x$$

$$z e^x = -2\int x e^x dx + c$$

$$z e^x = -2\left[e^x(x-1)\right] + c$$

$$z = -2(x-1) + c e^{-x}$$

$y = 1 + \frac{1}{z}$ be the sol^n

$$y = 1 + \frac{1}{-2(x-1) + c e^{-x}}$$

$\boxed{11Q}$ Solve $y(x+y)\,dx - x^2 dy = 0$

Sol^n $(xy + y^2)\,dx - x^2 dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = xy + y^2, \quad N = -x^2$$

$$\frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = -2x$$

Non - exact

$$\frac{1}{Mx + Ny} = \frac{1}{(xy+y^2)x + (-x^2 y)}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \cancel{x + 2y + 2x}$$

$$\frac{1}{Mx + Ny} = \frac{1}{xy^2} = I \cdot F$$

xly the Eqn by. I.F

$$\left(\frac{xy + y^2}{xy^2}\right)dx - \frac{x^2}{xy^2}\,dy = 0$$

$$\int\left(\frac{1}{y} + \frac{1}{x}\right)dx - \int\frac{x}{y^2}\,dy = C$$

$$\frac{x}{y} + \log x + 0 = C$$

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$\boxed{12Q}$

Solve. $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ — (1)

$\boxed{Sol^n}$ $\div \cos^2 y$

$$\sec^2 y \frac{dy}{dx} + x\left[\frac{2\sin y\cos y}{\cos^2 y}\right] = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3 \quad \text{— (2)}$$

put $v = \tan y$

$$\frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$$

Sub in — (2)

$$\frac{dv}{dx} + 2xv = x^3$$

$$P = 2x, \quad Q = x^3$$

$$e^{2\int x\,dx} = e^{2x^2/2} = e^{x^2}$$

$$v e^{x^2} = \int x \cdot x^2 e^{x^2} dx + c$$

$$x^2 = t, \quad 2x\,dx = dt$$

$$\tan y\, e^t = \int x \cdot t e^t \frac{dt}{2x} + k$$

$$\tan y\, e^t = \int t e^t dt + k$$

$$\tan y\,(e^t) = e^t(t-1) + k$$

$$\tan y\; e^{x^2} = e^{x^2}(x^2-1) + k$$

13Q find the O.T of family of Curves.

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Sol^n diff $\bar{y}$ w.r.to x

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Replace y' by $\left(-\frac{1}{y'}\right)$

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$$

$$x^{-1/3} = y^{-1/3}\frac{dx}{dy}$$

$$\int y^{1/3}\,dy = \int dx\,(x^{1/3}) + k$$

$$\frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + k$$

(14) Q.

Sol^n $y = xp + p^2$ $\quad p = y'$

$$\frac{dy}{dx} = x\frac{dp}{dx} + 1 \cdot p + 2p\frac{dp}{dx}$$

$$p = \frac{dp}{dx}(x+2p) + p$$

$$p - p = \frac{dp}{dx}(x+2p)$$

$$0 = \frac{dp}{dx}(x+2p)$$

$$x + 2p = 0$$

$$x = -2p$$

$$y = xp + p^2$$

$$y = -2pp + p^2$$

$$\left.\begin{aligned} y &= -p^2 \\ x &= -2p \end{aligned}\right\} \text{general sol}^n.$$

$$x^2 = 4p^2$$

$$y = -\frac{x^2}{4} \text{ singular sol}^n.$$

15Q Solve

$y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$

Sol^n $M = y \sin 2x$

$M = 2y \sin x \cos x$

$\frac{\partial M}{\partial y} = 2 \sin x \cos x$

$N = -1 - y^2 \cancel{+} \cos^2 x$

$\frac{\partial N}{\partial x} = -2 \cos x (-\sin x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

$\int y \cdot (\sin 2x)\, dx + \int -y^2 dy = c$

$y \left[-\frac{\cos 2x}{2} \right] - \frac{y^3}{3} = c$

LAQ 4

4Q Solve $y dx - x dy + e^x y^2 d\cancel{x} = 0$

Sol^n $\frac{y dx - x dy}{y^2} + e^x dx = 0$

$\int d\left(\frac{x}{y}\right) + \int e^x dx = c$

$x/y + e^x = c$

1). Solve $y'' - y = 0$, when $y = 0$ & $y' = 2$ at $x = 0$.

Sol: The operator form is

$$(D^2 - 1)y = 0$$

The A.E is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$.

$$\therefore y(x) = y_c = c_1 e^x + c_2 e^{-x}. \quad \text{---} (*)$$

when $y = 0$ at $x = 0$

$$\Rightarrow y(x) = c_1 e^x + c_2 e^{-x}$$

$$\Rightarrow y(0) = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 = 0$$

$$\Rightarrow c_1 + c_2 = 0 \quad \text{---} (1)$$

Now. Diff $(*)$ w.r. to x.

$$y'(x) = c_1 e^x - c_2 e^{-x}.$$

when $y'(0) = 2$.

$$\Rightarrow y'(0) = c_1 e^0 - c_2 e^{-0} = 2$$

$$\Rightarrow c_1 - c_2 = 2 \quad \text{---} (2)$$

Solving Eq (1) & (2), we get

$$c_1 = 1 \quad \& \quad c_2 = -1.$$

$$\therefore y(x) = e^x - e^{-x}.$$

2) Solve $(D^4 - 81)y = 0$

Sol:- The Auxilary Equation is

$$f(m) = m^4 - 81 = 0.$$

$$(m^2)^2 - (9)^2 = 0.$$

$$\Rightarrow (m^2-9)(m^2+9) = 0$$

$$\Rightarrow m^2 - 9 = 0 \text{ \& } m^2+9 = 0.$$

$$\Rightarrow m^2 = 9 \text{ \& } m^2 = -9.$$

$$\Rightarrow m = 3, -3 \text{ \& } m = \pm 3i. \Rightarrow m = 3i, -3i.$$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{3ix} + c_4 e^{-3ix}. \text{ or}$$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-3x} + e^{0x}[c_1 \cos 3x + c_2 \sin 3x]$$

3) Solve $(D^4 + 8D^2 + 16)y = 0.$

Sol:- The A.E is

$$f(m) = m^4 + 8m^2 + 16 = 0$$

$$\Rightarrow (m^2+4)(m^2+4) = 0$$

$$\Rightarrow (m^2+4)^2 = 0$$

$$\Rightarrow m^2+4 = 0 \text{ \& } m^2+4 = 0.$$

$$m^2 = -4 \text{ \& } m^2 = -4$$

$$m = \pm 2i \text{ \& } m = \pm 2i$$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cdot \cos 2x + c_4 \sin 2x.$$

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4) Find the particular integral of

$$(D^2 - 4D + 4)y = e^{2x}.$$

Sol:- The Auxilary Equation is

$$f(m) = m^2 - 4m + 4 = 0.$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2.$$

$\therefore$ The roots are real & Equal.

$$y_c = (c_1 + c_2 x)e^{2x}.$$

$$P.I = y_p = \frac{1}{D^2-4D+4} \cdot e^{2x} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2}{2!} e^{2x}.$$

$$\Rightarrow y_p = \frac{x^2}{2} \cdot e^{2x}.$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2 x)e^{2x} + \frac{x^2}{2} \cdot e^{2x}.$$

---

5) Find the solution of initial Value problem $y'' + 4y' - 13 \cdot y = 0$, $y(0) = y'(0) = 1$.

Sol:- The given D.E is

$$(D^2 + 4D - 13)y = 0.$$

The A.E is $m^2 + 4m - 13 = 0$

$$m = -2 \pm \sqrt{17}.$$

6) Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Sol:- The operator form is

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

& The Auxilary Equation is

$$f(m) = m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m+1)(m^2 + 5m + 6) = 0.$$

$m=-1$	1	6	11	6
	0	-1	-5	-6
$m=-2$	1	5	6	$\boxed{0}$
	0	-2	-6	
$m=-3$	1	3	$\boxed{0}$	
	0	-3		
	1	0.		

$$\Rightarrow (m+1)\left[m(m+2) + 3(m+2)\right] = 0$$

$$\Rightarrow (m+1)\left[(m+2)(m+3)\right] = 0$$

$$\Rightarrow (m+1)(m+2)(m+3) = 0$$

$$\Rightarrow m+1 = 0,\ m+2 = 0,\ m+3 = 0$$

$$\Rightarrow m = -1,\ m = -2,\ m = -3.$$

$\therefore$ The roots are real & Distinct.

$$\therefore\ y_c = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}.$$

7) find complementary function of

$$(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$$

Sol: The Auxiliary Equation is

$f(m) = 0$

$\Rightarrow (m^2 - 4m + 3) = 0$.

$\Rightarrow m^2 - 4m + 3 = 0$

$\Rightarrow m^2 - 3m - m + 3 = 0$

$\Rightarrow m(m-3) - 1(m-3) = 0$

$\Rightarrow (m-3)(m-1) = 0$

$\Rightarrow m - 3 = 0$ & $m - 1 = 0$

$\Rightarrow m = 3$ and $m = 1$

$\Rightarrow m = 1, 3$.

$\therefore$ The roots are real & distinct.

$\therefore y_c = c_1 e^{x} + c_2 e^{3x}$.

8) find the P.I. of $(D^2+1)y = 8e^{-x}$. (4)

Sol:- The A.E is

$f(m) = 0 \Rightarrow m^2+1=0 \Rightarrow m^2=-1.$

$\Rightarrow m = \pm i$

$\therefore$ The roots are complex congugate numbers.

$\therefore y_c = e^{0x}\left[c_1 \cos x + c_2 \sin x\right]$

$y_p = \frac{1}{D^2+1}(8e^{-x}) = 8.\frac{1}{D^2+1}e^{-x}.$

$\Rightarrow y_p = 8.\frac{1}{(-1)^2+1}e^{x} = 8.\frac{1}{1+1}e^{x}$ $\boxed{D=a=-1}$

$\Rightarrow y_p = 8.\frac{1}{2}.e^{x}.$

$\Rightarrow y_p = 4e^{x}.$

$\therefore$ The particular integral is $4e^{x}$.

$\therefore$ The $y = y_c + y_p = c_1 \cos x + c_2 \sin x + 4e^{x}$

9) Find the particular Integral of

$$(D^3 - 6D^2 + 11 \cdot D - 6) \cdot y = e^{-3x}.$$

Sol:- The A.E is

$$f(m) = 0.$$

$$\Rightarrow m^3 - 6m^2 + 11m - 6 = 0.$$

$$\Rightarrow (m-1)(m^2 - 5m + 6) = 0$$

$$\Rightarrow (m-1)(m-2)(m-3) = 0.$$

$$\Rightarrow m-1 = 0, \; m-2 = 0, \; m-3 = 0$$

$$\Rightarrow m = 1, 2, 3.$$

$\therefore$ The roots are real & distinct.

$$\therefore \; y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

$$y_p = P.I = \frac{e^{-3x}}{D^3 - 6D^2 + 11D - 6} = \frac{e^{-3x}}{(-3)^3 - 6(-3)^2 + 11(-3) - 6}$$

$$= \frac{e^{-3x}}{-120}$$

$$\Rightarrow y_p = \frac{e^{-3x}}{-120}$$

$$\Rightarrow y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{e^{-3x}}{120}$$

10) Write in brief about the method of Variation of Parameters.

Ans: These are the steps briefly about the method of Variation of Parameters.

1). Reduce the given equation to the standard form, if necessary.

2). Find the solution of

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Q \cdot y = 0.$$

and let the solution be

$$C.F = c_1 \cdot u(x) + c_2 \cdot v(x).$$

3). Take $P.I = y_p = A \cdot u + B \cdot v$,
where A and B are functions of x.

4). Find $W(u,v) = u \cdot \frac{dv}{dx} - v \cdot \frac{du}{dx}$

5). Find A and B, using

$$A = -\int \frac{v \cdot R\, dx}{W(u,v)} = -\int \frac{v \cdot R \cdot dx}{u\frac{dv}{dx} - v \cdot \frac{du}{dx}}$$

$$B = \int \frac{u \cdot R\, dx}{W(u,v)} = \int \frac{u \cdot R \cdot dx}{u\frac{dv}{dx} - v \cdot \frac{du}{dx}}.$$

6). Write the general solution of the given equation as
$y = y_c + y_p$.

i.e., $y = c_1 \cdot u(x) + c_2 \cdot v(x) + A(x) \cdot u(x) + B(x) \cdot v(x)$
where c_1 and c_2 are constants.

11) Define the terms

i) Complementary Function

ii) Particular Integral

Sol: (i) If $y = y_c$ is the general solution of $f(D)y = 0$, then we know that y_c is the complementary function (C.F). of $f(D)y = Q(x)$.

(ii). If the equation is $f(D)y = Q(x)$.

Then the particular integral,

$$y_p = P.I = \frac{1}{f(D)} \cdot Q(x).$$

---

12) State Euler - Cauchy equation and brief method to solve it.

Sol:- The second order Euler - Cauchy equation is of the form

$$ax^2 \cdot y'' + bxy' + cy = 0 \quad \text{(or)}$$

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \cdot \frac{dy}{dx} + a_0 \cdot y = g(x)$$

when $g(x) = 0$, then the above equation is called the homogeneous Euler - Cauchy equation.

Eg: $x^2 y'' - 9xy' + 25 \cdot y = 0$.

Brief method :

1). Let us assume that $y = y(x) = x^{r}$ be the solution of a given differentiation equation. where r is a constant to be determined.

2). Fill the above formula for y in the differential equation and simplify.

3). Solve the obtained polynomial equation for r.

---

13) Solve the D.E. $D^2 y = \sin 2x$.

Sol:- The A.E is $m^2 = 0$

$$\Rightarrow m = 0, 0.$$

$$\therefore y_c = c_1 e^{0x} + c_2 e^{0x}$$

$$\Rightarrow y_c = c_1 + c_2.$$

$$P.I = y_p = \frac{1}{f(D)} \cdot \sin 2x$$

$$= \frac{1}{D^2} \cdot \sin 2x = \frac{1}{D} \cdot \frac{1}{D} (\sin 2x)$$

$$= \frac{1}{D} \int \sin 2x \, dx = \frac{1}{D} \left[- \frac{\cos 2x}{2} \right]$$

$$= \int \left(- \frac{\cos 2x}{2} \right) dx = -\frac{1}{2} \int \cos 2x \, dx$$

$$= -\frac{1}{2} \left[\frac{\sin 2x}{2} \right] + c = -\frac{1}{4} \cdot \sin 2x + c.$$

14) Find the Value of $\frac{1}{D+1}[x^2+1]$

Sol:- $\frac{1}{D+1}(x^2+1)$

$\Rightarrow \frac{1}{1+D}(x^2+1)$

$\Rightarrow (1+D)^{-1}[x^2+1]$

$\Rightarrow [1-D+D^2+$ Higher order terms$][x^2+1]$

$\Rightarrow (1-D+D^2)(x^2+1)$

$= (1-D+D^2)(x^2) + (1-D+D^2)(1)$

$= x^2 - D(x^2) + D^2(x^2) + 1 - D(1) + D^2(1)$

$= x^2 - 2x + 2 + 1 + 0 + 0$

$= x^2 - 2x + 2.$

---

15) Find the particular Value of

$\frac{1}{(D-2)(D-3)} \cdot e^{2x}.$

Sol: $\frac{1}{(D-2)(D-3)} \cdot e^{2x} = \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right]$

Now, $\frac{1}{D-3}e^{2x} = e^{3x} \cdot \int e^{2x} \cdot e^{-3x} dx = e^{3x} \int e^{-x} dx$

$\frac{1}{D-3}e^{2x} = e^{3x} \cdot [-e^{-x}] = -e^{3x-x} = -e^{2x}.$

$\left[\because \frac{1}{(D-\beta)(D-\alpha)} \cdot Q = \frac{1}{(D-\beta)}\left[e^{\alpha x}\int Q e^{-\alpha x} dx\right]\right]$

$$\therefore \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right] = \frac{1}{D-2}\left[-e^{2x}\right]$$

$$= -e^{2x}\int e^{2x}\cdot e^{-2x}\,dx$$

$$= -e^{2x}\int e^{2x-2x}\,dx = -e^{2x}\int e^{0}\,dx$$

$$= -e^{2x}\cdot\int dx = -e^{2x}\cdot x = -x\cdot e^{2x}.$$

M - II - UNIT - 3 - LAQ's.

1) Find the general solution of.

$$y'' + 3y' + 2y = 2e^{x}.$$

Sol: The Auxilary Equation is.

$$f(m) = 0 \Rightarrow m^2 + 3m + 2 = 0.$$

$$\Rightarrow m^2 + m + 2m + 2 = 0$$

$$\Rightarrow m(m+1) + 2(m+1) = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m+1 = 0 \text{ \& } m+2 = 0$$

$$\Rightarrow m = -1 \text{ \& } m = -2$$

$$\Rightarrow m = -1, -2.$$

$$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

$$P.I = y_p = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{(D^2+3D+2)} \cdot 2e^{x}$$

$$= 2 \cdot \frac{1}{(D^2+3D+2)} e^{x}$$

$$\boxed{D = a = 1}$$

$$= 2 \cdot \frac{1}{(1)^2 + 3(1) + 2} e^{x} = 2 \cdot \frac{1}{1+3+2} e^{x}$$

$$= 2 \cdot \frac{1}{6} \cdot e^{x} = \frac{1}{3} e^{x}$$

$$\therefore y_p = \frac{1}{3} e^{x}.$$

$$G.S = y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{3} e^{x}.$$

2) Find the general solution of

$$(D^2-4D+4)y = e^{2x}.$$

Sol: The Auxilary Equation is

$$f(m)=0 \Rightarrow m^2-4m+4=0.$$

$$\Rightarrow (m-2)^2=0. \Rightarrow m=2,2.$$

$\therefore$ The roots are real & equal.

$$y_c = (c_1+c_2x)e^{2x}.$$

$$P.I = y_p = \frac{e^{2x}}{D^2-4D+4} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2}{2!}e^{2x}.$$

$$\therefore y_p = \frac{x^2}{2}\cdot e^{2x}.$$

$$\therefore y = y_c+y_p = (c_1+c_2x)e^{2x}+\frac{x^2}{2}\cdot e^{2x}.$$

---

3) Solve $(D^2+9)y = \sin 3x$.

Sol: The given equation is

$$(D^2+9)y = \sin 3x.$$

Let $f(D) = D^2+9.$

A.E is $f(m)=0 \Rightarrow m^2+9=0$

$$\Rightarrow m^2=-9 \Rightarrow m=\pm 3i.$$

$$\Rightarrow m = 3i, -3i.$$

$\therefore$ The roots are complex conjugate numbers.

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$$y_c = e^{0x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

$$P.I = y_p = \frac{1}{D^2+9} \cdot \sin 3x.$$

$$\left[D^2 = -a^2 = -3^2 = -9\right]$$

$$= \frac{1}{-9+9} \cdot \sin 3x = \frac{1}{0} \cdot \sin 3x.$$

$$= \frac{-x}{2\times 3} \cos 3x = -\frac{x}{6} \cos 3x.$$

$$\left[\because \frac{1}{D^2+a^2} \sin ax = -\frac{x \cos ax}{2a}\right]$$

$$y_p = -\frac{x}{6} \cos 3x.$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{6} \cdot \cos 3x.$$

4) Find the general solution of

$$(D^2-4)y = \cos^2 x.$$

Sol:- The given equation is

$$(D^2-4)y = \cos^2 x.$$

Let $f(D) = D^2-4$

The A.E. is $m^2-4=0.$

$$\Rightarrow m^2 = 4 \Rightarrow m = 2, -2$$

$\therefore$ The roots are real and distinct.

$$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}.$$

$$P.I = y_p = \frac{1}{D^2-4}\cdot(\cos^2 x)$$

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$= \frac{1}{D^2-4}\left[\frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} + \frac{1}{D^2-4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{e^{0x}}{D^2-4} + \frac{\cos 2x}{D^2-4}\right]$$

$$\boxed{\left[\because D = a = 0 \quad \& \quad D^2 = -a^2 = -2^2\right]}$$

$$= \frac{1}{2}\left[\frac{e^{0x}}{(0)^2-4} + \frac{\cos 2x}{-2^2-4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} + \frac{\cos 2x}{-4-4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} + \frac{\cos 2x}{-8}\right]$$

$$y_p = -\frac{1}{8} - \frac{\cos 2x}{16}.$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{\cos 2x}{16}.$$

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5) Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

Sol:- The A.E is

$$(m+2)(m-1)^2 = 0.$$

$$\Rightarrow m+2=0 \ \& \ (m-1)^2 = 0$$

$$\Rightarrow m=-2 \ \& \ m=1,1.$$

$$\Rightarrow y_c = c_1 e^{-2x} + (c_2 + c_3 x)e^x.$$

$$P.I = y_p = \frac{1}{(D+2)(D-1)^2}\cdot\left[e^{-2x} + 2\sinh x\right]$$

$$= \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + 2\cdot\frac{e^x - e^{-x}}{2}\right]$$

$\boxed{\because \sinh x = \frac{e^x - e^{-x}}{2}}$

$$= \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + e^x - e^{-x}\right]$$

$$P.I_1 = \frac{1}{(D+2)(D-1)^2}\left(e^{-2x}\right) = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right]$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-3)^2}e^{-2x}\right]$$

$\boxed{\because D=a=-2}$

$$= \frac{1}{D+2}\left[\frac{1}{9}\cdot e^{-2x}\right] = \frac{1}{9}\left[\frac{1}{D+2}e^{-2x}\right] = \frac{1}{9}\left[\frac{1}{-2+2}e^{-2x}\right]$$

$$= \frac{1}{9}\left[\frac{1}{0}\cdot e^{-2x}\right] \text{ (Case failure)}$$

$\boxed{\because D=a=-2}$

$$\Rightarrow \frac{1}{9}\left[x\cdot e^{-2x}\right] = P.I_1$$

$$PI_2 = \frac{1}{(D+2)(D-1)^2} \cdot e^x$$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{D+2}(e^x)\right]$$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^x\right] = \frac{1}{(D-1)^2}\left[\frac{1}{3}e^x\right] \qquad \boxed{\because D=a=1}$$

$$= \frac{1}{3} \cdot \frac{1}{(D-1)^2} \cdot e^x = \frac{1}{3} \cdot \frac{1}{(1-1)^2} e^x = \frac{1}{3} \cdot \frac{1}{0} \cdot e^x$$

(case-failure)

$$\Rightarrow \frac{1}{3} \cdot \frac{x}{2(D-1)} \cdot e^x = \frac{1}{3} \cdot \frac{x}{2(1-1)} e^x = \frac{1}{3} \cdot \frac{x}{0} \cdot e^x$$

(case-failure)

$$\Rightarrow \frac{1}{3} \cdot \frac{x^2}{2} \cdot e^x = \frac{1}{6} x^2 \cdot e^x = PI_2.$$

$$PI_3 = \frac{1}{(D+2)(D-1)^2} \cdot e^{-x}.$$

$$= \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{(1)(-2)^2} e^{-x} = \frac{1}{+4} e^{-x}$$

$$[\because D = a = -1]$$

$$PI_3 = -\frac{e^x}{2}.$$

$$\therefore \; y_p = PI_1 + PI_2 + PI_3 = \frac{1}{9}\left[x e^{-2x}\right] + \frac{1}{6}x^2 e^x - \frac{e^x}{4}.$$

$$\therefore \; y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{1}{9} \cdot x e^{-2x} + \frac{1}{6} x^2 e^x - \frac{e^x}{4}$$

6) Find the general solution of

$$(D^2+2D+1)\cdot y = x\cdot \cos x.$$

Sol:- The given equation is

$$(D^2+2D+1)\, y = x\cdot\cos x.$$

$\Rightarrow$ Let $f(D) = D^2+2D+1$

The A.E is $f(m)=0$.

$\Rightarrow m^2+2m+1=0. \Rightarrow (m+1)^2=0.$

The roots are real & equal $\Rightarrow m=-1,-1.$

$\Rightarrow y_c = (c_1+c_2 x)\, e^{-x}.$

$$y_p = \frac{x\cdot\cos x}{D^2+2D+1} = \left[x - \frac{1}{f(D)}\cdot f'(D)\right]\cdot\frac{1}{f(D)}\cos x.$$

$$= \left[x - \frac{1}{D^2+2D+1}\cdot 2(D+1)\right]\cdot\frac{1}{D^2+2D+1}\cdot\cos x$$

$$= \left[x - \frac{1}{(D+1)^2}\cdot 2(D+1)\right]\cdot\frac{1}{D^2+2D+1}\cdot\cos x$$

$$= \left[x - \frac{1}{(D+1)^2}\cdot 2(D+1)\right]\frac{1}{-1+2D+1}\cos x$$

$\left[\because D^2 = -a^2 = -1^2 = -1\right]$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{2D}\cos x = \left[x - \frac{2}{D+1}\right]\cdot\frac{\sin x}{2}$$

$\left[\because \frac{1}{D}(\cos x) = \int \cos x = \sin x\right]$

$$= \frac{x}{2}\sin x - \frac{\sin x}{D+1}$$

$$= \frac{x}{2}\cdot\sin x - \frac{D-1}{D^2-1}\cdot\sin x.$$

$$= \frac{x\sin x}{2} - \frac{D-1}{-1-1}\cdot\sin x.$$

$$= \frac{x\cdot\sin x}{2} - \frac{(D-1)\cdot\sin x}{-2}$$

$$= \frac{x\sin x}{2} - \frac{D(\sin x) - \sin x}{-2}$$

$$= \frac{x\sin x}{2} - \frac{(\cos x - \sin x)}{-2}$$

$$= \frac{x\cdot\sin x}{2} + \frac{\cos x + \sin x}{2}$$

$$\therefore\ y = y_c + y_p$$

$$\Rightarrow y = (C_1 + C_2 x)e^{-x} + \frac{x}{2}\cdot\sin x + \frac{1}{2}[\cos x + \sin x]$$

7) Solve $(D^2+4)y = e^x + \sin 2x + \cos 2x$.

Sol: The given D.E is

$$(D^2+4)y = e^x + \sin 2x + \cos 2x.$$

Let $f(D) = D^2+4$

The A.E is $m^2+4=0$

$$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i.$$

∴ The roots are complex conjugate numbers.

$$y_c = c_1 \cdot \cos 2x + c_2 \sin 2x.$$

$$P.I = y_p = \frac{1}{D^2+4}\left[e^x + \sin 2x + \cos 2x\right]$$

$$\Rightarrow y_p = \frac{1}{D^2+4} \cdot e^x + \frac{1}{D^2+4} \sin 2x + \frac{1}{D^2+4} \cos 2x.$$

$$\Rightarrow y_p = PI_1 + PI_2 + PI_3.$$

$$\Rightarrow PI_1 = \frac{1}{D^2+4} \cdot e^x = \frac{1}{1^2+4} e^x = \frac{1}{1+4} e^x = \frac{1}{5} e^x$$ $[\because D = a = 1]$

$$PI_2 = \frac{1}{D^2+4} \cdot \sin 2x = \frac{1}{D^2+2^2} \sin 2x = \frac{1}{-2^2+2^2} \sin 2x$$

(case failure)

$$PI_2 = -\frac{x}{2 \cdot 2} \cos 2x$$ $\left[\because \frac{\sin ax}{D^2+a^2} = -\frac{x}{2a} \cos ax\right]$

$$PI_2 = -\frac{x}{4} \cdot \cos 2x.$$

$$PI_3 = \frac{\cos 2x}{D^2+4} = \frac{\cos 2x}{D^2+2^2} = \frac{\cos 2x}{-2^2+2^2}$$

(case-failure).

$$PI_3 = \frac{x}{2\cdot 2}\cdot \sin 2x \qquad \left[\because \frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax\right]$$

$$PI_3 = \frac{x}{4}\cdot \sin 2x.$$

$$\therefore \; y = y_c + y_p$$

$$\Rightarrow y = y_c + PI_1 + PI_2 + PI_3.$$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x + \frac{x}{4}\sin 2x.$$

---

8) Solve $(D^3-1)y = e^x + \sin 3x + 2.$

Sol:- The given D.E is

$$(D^3-1)y = e^x + \sin 3x + 2$$

The A.E is $m^3 - 1 = 0$

$$\Rightarrow (m-1)(m^2+m+1) = 0$$

$$\Rightarrow m-1 = 0 \;\; \& \;\; m^2+m+1 = 0$$

$$\Rightarrow m = 1 \;\; \&$$

$$m^2+m+1 = 0$$

$$\Rightarrow a = 1,\; b = 1,\; c = 1$$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4\times1\times1}}{2\times1} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

$$\Rightarrow y_c = c_1 e^{x} + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I = y_p = \frac{1}{D^3-1}\left[e^{x} + \sin 3x + 2\right]$$

$$= \frac{1}{D^3-1}\cdot e^{x} + \frac{1}{D^3-1}\sin 3x + \frac{1}{D^3-1}(2)\cdot e^{0x}$$

$$PI = PI_1 + PI_2 + PI_3.$$

Now, $PI_1 = \frac{1}{D^3-1}e^{x} = \frac{1}{1^3-1} = \frac{1}{1-1} = \frac{1}{0}$.

case failure. $\boxed{D=a=1}$

$$\therefore \quad PI_1 = \frac{x}{1!}e^{x} = \underline{\underline{xe^{x}}}$$

$$PI_2 = \frac{1}{D^3-1}\sin 3x.$$

$$= \frac{1}{D^2\cdot D-1}\sin 3x.$$

$$\boxed{D^2 = -a^2 = -3^2 \\ = -9}$$

$$= \frac{1}{-9D-1}\sin 3x$$

$$= \frac{1}{-(1+9D)}\sin 3x$$

$$= -\frac{(1-9D)}{(1-9D)(1+9D)}\sin 3x$$

$$= -\frac{(1-9D)\sin 3x}{1-81D^2} = -\frac{(1-9D)\sin 3x}{1-81(-3^2)}$$

$$= -\frac{[\sin 3x - 9\cdot D(\sin 3x)]}{1-81(-9)}$$

$$= \frac{-(\sin 3x - 9\cdot D(\sin 3x))}{1+729}$$

$$= \frac{-[\sin 3x - 9\cdot[3\cos 3x]]}{730}.$$

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$$= -\frac{(\sin 3x - 27\cos 3x)}{730}.$$

$$= -\frac{1}{730}\left[\sin 3x - 27\cos 3x\right]$$

$$PI_3 = \frac{1}{D^3-1}\cdot 2\cdot e^{0x} = 2\cdot\frac{1}{D^3-1}e^{0x}.$$

$$= 2\cdot\frac{1}{0-1} = \frac{2}{-1} = -2.$$

$$y_p = PI_1 + PI_2 + PI_3$$

$$\Rightarrow y_p = xe^x - \frac{1}{730}\left[\sin 3x - 27\cos 3x\right] - 2.$$

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^x + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$+ xe^x - \frac{1}{730}\left[\sin 3x - 27\cos 3x\right] - 2.$$

9) Solve $y'' - 2y' + y = xe^x \sin x$.

Sol:- The A.E is $f(m) = 0$

$$\Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)(m-1) = 0$$

$$\Rightarrow (m-1) = 0 \;\&\; (m-1) = 0$$

$$\Rightarrow m = 1, 1$$

$\therefore$ The roots are real & Equal.

$$y_c = [c_1 + c_2 x] e^x.$$

$$y_p = P.I = \frac{1}{D^2 - 2D + 1}\left[x e^x . \sin x\right]$$

$$= e^x . \frac{x \sin x}{[(D+1)-1]^2} = e^x . \frac{x \sin x}{[(D+1)-1]^2}$$

$$= e^x . \frac{x \sin x}{D^2} = e^x . \frac{1}{D^2}[x \sin x]$$

$$= e^x . \frac{1}{D}\left[\int x \sin x\right] = e^x . \frac{1}{D}\left[x \int \sin x \, dx - \int \left[\frac{d}{dx}(x) . \int \sin x \, dx\right] dx\right]$$

$$= e^x . \frac{1}{D}\left[x . (-\cos x) - \int 1 . (-\cos x) . 1 . dx\right]$$

$$= e^x . \frac{1}{D}\left[-x . \cos x + \sin x\right]$$

$$= e^x \left[-\int x \cos x \, dx + \int \sin x \, dx \right]$$

$$= e^x \left[-\left\{ x \sin x - \int 1 \cdot \sin x \right\} - \cos x \right]$$

$$= e^x \cdot \left[-\left\{ x \cdot \sin x + \cos x \right\} - \cos x \right]$$

$$= e^x \cdot \left[-x \sin x - \cos x - \cos x \right]$$

$$= e^x \left[-x \sin x - 2 \cos x \right]$$

$$= -x e^x \cdot \sin x - 2 \cdot e^x \cdot \cos x$$

$$y_p = -x e^x \sin x - 2 e^x \cos x.$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = [c_1 + c_2 x] e^x + [x e^x \sin x + 2 e^x \cos x]$$

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10) Solve $(D^2+4)y = x^2+1+\cos 2x$.

Sol:- The given D.E is

$$(D^2+4)y = x^2+1+\cos 2x$$

The A.E is $f(m)=0$

$$\Rightarrow m^2+4=0$$

$$\Rightarrow m^2=-4 \Rightarrow m=\pm 2i.$$

$$y_c = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$y_p = \frac{1}{D^2+4}\left[x^2+1+\cos 2x\right]$$

$$= \frac{1}{D^2+4}\cdot x^2 + \frac{1}{D^2+4}\cdot 1\cdot e^{0x} + \frac{1}{D^2+4}\cos 2x.$$

$$y_p = PI_1 + PI_2 + PI_3$$

$$\Rightarrow PI_1 = \frac{1}{D^2+4}(x^2) = \frac{1}{4\left[1+\frac{D^2}{4}\right]}(x^2)$$

$$= \frac{1}{4}\cdot\frac{1}{\left(1+\frac{D^2}{4}\right)}(x^2) = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}\cdot(x^2)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}\right](x^2) \qquad \left[\because (1+D)^{-1} = 1-D+D^2+\cdots\right]$$

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$$= \frac{1}{4}\left[x^2 - \frac{D^2}{4}(x^2)\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{2}{4}\right] = \frac{1}{4}\left[x^2 - \frac{1}{2}\right]$$

$$PI_2 = \frac{1}{D^2+4} e^{0x} = \frac{1}{0^2+4} = \frac{1}{4}.$$

$$PI_3 = \frac{1}{D^2+4}(\cos 2x)$$

$$\boxed{D^2 = -2^2 = -4}$$

$$= \frac{1}{-2^2+2^2}[\cos 2x] = \frac{1}{0}\cos 2x$$

(Case failure).

$$= \frac{x}{2D}[\cos 2x]$$

$$= \frac{x}{2D}(\cos 2x)$$

$$= \frac{x}{2} \cdot \frac{1}{D}(\cos 2x)$$

$$= \frac{x}{2}\int \cos 2x.$$

$$= \frac{x}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{x}{4}\sin 2x$$

$$PI_3 = \frac{x}{4}(\sin 2x).$$

$$y_p = P.I = PI_1 + PI_2 + PI_3.$$

$$\Rightarrow y_p = \frac{1}{4}\left[x^2 - \frac{1}{2}\right] + \frac{1}{4} + \frac{x}{4}(\sin 2x).$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{1}{4} + \frac{x}{4}(\sin 2x).$$

11) Find the general solution of

$$y'' + 4y' + 4y = 6 \cdot e^{-2x} \cdot \cos^2 x.$$

Sol: The given D.E is

$$y'' + 4y' + 4y = 6 \cdot e^{-2x} \cdot \cos^2 x.$$

$\cos 2x = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x = \cos 2x + 1$

$\Rightarrow \cos^2 x = \frac{1}{2}[1 + \cos 2x]$

$f(m) = 0$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow m = -2, -2$ .. The roots are real & Equal.

$\therefore y_c = [c_1 + c_2 x] e^{-2x}$.

$$\therefore y_p = \frac{1}{D^2 + 4D + 4}\left[6 e^{-2x} \cdot \cos^2 x\right]$$

$$= \frac{1}{D^2 + 4D + 4}\left[6 \cdot e^{-2x} \cdot \frac{1}{2}(1 + \cos 2x)\right]$$

$$= \frac{1}{D^2 + 4D + 4}\left[3 \cdot e^{-2x} \cdot (1 + \cos 2x)\right]$$

$\boxed{\text{Put } D = D - 2}$

$$= 3 \cdot e^{-2x} \frac{1}{(D-2)^2 + 4(D-2) + 4} \cdot [1 + \cos 2x]$$

$$= 3e^{-2x} \cdot \frac{1}{D^2}[1 + \cos 2x]$$

$$\Rightarrow y_p = 3e^{-2x} \cdot \frac{1}{D} \cdot \frac{1}{D}(1+\cos 2x)$$

$$= 3e^{-2x} \cdot \frac{1}{D} \int [1+\cos 2x]\, dx$$

$$= 3e^{-2x} \cdot \frac{1}{D}\left[x + \frac{\cos 2x}{2}\right]$$

$$= 3e^{-2x} \cdot \int \left[x + \frac{\cos 2x}{2}\right] dx$$

$$= 3e^{-2x} \cdot \left[\frac{x^2}{2} + \frac{\sin 2x}{2 \cdot 2}\right]$$

$$= 3e^{-2x}\left[\frac{x^2}{2} + \frac{\sin 2x}{4}\right]$$

$$\Rightarrow y_p = \frac{3}{2} e^{-2x}\left[x^2 + \frac{\sin 2x}{2}\right]$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2 x)e^{-2x} + \frac{3}{2}e^{-2x}\left[x^2 + \frac{\sin 2x}{2}\right]$$

12) Apply method of Variation of Parameters to solve $\frac{d^2y}{dx^2}+y=\sec x$.

Sol: Given that the D.E is

$$(D^2+1)y=\sec x \quad , \; \& \; R=\sec x.$$

The A.E is $m^2+1=0$.

$$\Rightarrow m^2=-1 \Rightarrow m=\pm i.$$

$$\Rightarrow y_c=e^{0x}\left[c_1\cos x+c_2\sin x\right]$$

$$\Rightarrow y_c=c_1\cdot\cos x+c_2\cdot\sin x.$$

$$\Rightarrow y_c=c_1\cdot u+c_2\cdot v.$$

where $u=\cos x$ & $v=\sin x$.

$$u'=\frac{du}{dx}=-\sin x \quad \& \quad \frac{dv}{dx}=\cos x=v'$$

$$\therefore A=-\int\frac{v\cdot R}{uv'-u'v}\,dx=-\int\frac{\sin x\cdot\sec x}{\cos x\cdot\cos x-(-\sin x)(\sin x)}\,dx$$

$$A=-\int\frac{\sin x\cdot\frac{1}{\cos x}}{\cos^2x+\sin^2x}\,dx=-\int\frac{\frac{\sin x}{\cos x}}{1}\,dx$$

$$\Rightarrow A=-\int\frac{\sin x}{\cos x}\,dx=\log|\cos x|.$$

$$B = \int \frac{u \cdot R}{uv' - u'v}\, dx.$$

$$\Rightarrow B = \int \frac{\cos x \cdot \sec x}{1}\, dx = \int \cos x \cdot \sec x \, dx.$$

$$B = \int \cos x \cdot \frac{1}{\cos x}\, dx = \int 1 \, dx = x + c$$

$$P.I = y_p = A \cdot u + B \cdot V$$

$$\Rightarrow y_p = \log|\cos x| \cos x + x \cdot \sin x.$$

$$\Rightarrow y_p = (\cos x)(\log|\cos x|) + x \cdot \sin x.$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x + (\cos x)(\log|\cos x|) + x \cdot \sin x$$

13) Find the particular integral of

$$(D^2-6D+9)y = x^2+2x.$$

Sol: The given D.E is

$$(D^2-6D+9)y = x^2+2x.$$

The A.E is

$$m^2-6m+9=0$$

$$\Rightarrow (m-3)^2=0 \Rightarrow m=3,3.$$

$$\therefore y_c = (c_1+c_2x)e^{3x}.$$

$$y_p = P.I = \frac{1}{[D^2-6D+9]}[x^2+2x]$$

$$= \frac{1}{9\left[1+\frac{D^2-6D}{9}\right]}(x^2+2x)$$

$$= \frac{1}{9}\left[1+\frac{D^2-6D}{9}\right]^{-1}[x^2+2x]$$

$$= \frac{1}{9}\left[1-\left(\frac{6D-D^2}{9}\right)\right]^{-1}(x^2+2x)$$

$$= \frac{1}{9}\left[1+\left[\frac{6D-D^2}{9}\right]+\left[\frac{6D-D^2}{9}\right]^2\right](x^2+2x)$$

$$= \frac{1}{9}\left[x^2+2x+\frac{6}{9}D(x^2+2x)-\frac{1}{9}D^2(x^2+2x)+\frac{36}{81}D^2(x^2+2x)\right]$$

$$= \frac{1}{9}\left[x^2+2x+\frac{6}{9}(2x+2)-\frac{1}{9}(2)+\frac{36}{81}(2)\right]$$

$$= \frac{1}{9}\left[x^2+2x+\frac{4x}{3}+\frac{4}{3}-\frac{2}{9}+\frac{8}{9}\right]$$

$$= \frac{1}{9}\left[x^2+\frac{10}{3}x+2\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[x^2+\frac{10}{3}x+2\right]$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = [c_1+c_2x]e^{3x}+\frac{1}{9}\left[x^2+\frac{10}{3}x+2\right].$$

14) Solve $x^2 \cdot \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx} - 3y = x^2 \cdot \log x$.

Sol: The given D.E is

$$\left[x^2 D^2 - x D - 3\right] y = x^2 \cdot \log x.$$

Now, it is in the form of Cauchy's homogeneous Linear D.E.

Put $x = e^z \Rightarrow \log x = z$.

$x D = D$, $x^2 D^2 = D(D-1)$

$$\Rightarrow \left[D(D-1) - D - 3\right] y = e^{2z} \cdot z$$

$$\Rightarrow \left[D^2 - D - D - 3\right] y = e^{2z} \cdot z$$

$$\Rightarrow \left[D^2 - 2D - 3\right] y = e^{2z} \cdot z.$$

The A.E is $m^2 - 2m - 3 = 0$

$$\Rightarrow (m+1)(m-3) = 0$$

$$\Rightarrow m = -1, 3.$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}.$$

$$P.I = y_p = \frac{1}{D^2 - 2D - 3}\left(z \cdot e^{2z}\right)$$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} \cdot z$$ $\boxed{\text{Put } D = D+2}$

$$= e^{2z} \frac{1}{D^2 + 2D - 3} \cdot z$$

$$= e^{2z} \cdot \frac{1}{-3\left[1 - \frac{D^2+2D}{3}\right]} \cdot z.$$

$$= -\frac{e^{2z}}{3} \cdot \frac{1}{\left[1 - \frac{2D+D^2}{3}\right]} \cdot z.$$

$$= -\frac{e^{2z}}{3} \cdot \left[1 - \frac{2D+D^2}{3}\right]^{-1} \cdot (z)$$

$$= -\frac{e^{2z}}{3}\left[z + \frac{2}{3}\right]$$

$$y_p = -\frac{x^2}{3}\left[\log x + \frac{2}{3}\right]$$

$\boxed{z = \log x}$ $\boxed{x = e^z}$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{3}\left[\log x + \frac{2}{3}\right]$$

---

15) Apply method of Variation of Parameters to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

Sol: Given D.E is $(D^2+1)y = \operatorname{cosec} x$.

& $R = \operatorname{cosec} x$

The A.E is $m^2+1 = 0$.

$$\Rightarrow m^2 = -1 \Rightarrow m = \pm i.$$

$$y_c = c_1 \cdot \cos x + c_2 \sin x.$$

By the method of Variation of parameters

$$y_c = c_1 \cdot u + c_2 \cdot v$$

where $u = \cos x$ & $v = \sin x$.

$u' = -\sin x$ & $v' = \cos x$.

$$A = -\int \frac{v \cdot R}{uv' - v \cdot u'} \cdot dx.$$

$$= -\int \frac{\sin x \cdot \csc x}{\cos^2 x + \sin^2 x}\, dx \quad = -\int 1\, dx = -x$$

$$A = -x.$$

$$\text{\&} \quad B = \int \frac{u \cdot R}{uv' - vu'} \cdot dx$$

$$= \int \frac{\cos x \cdot \csc x}{\cos^2 x + \sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin x}\, dx = \log|\sin x|$$

$$\therefore \quad B = \log|\sin x|.$$

$$y_p = P.I = A \cdot u + B \cdot V$$

$$= (-x) \cdot \cos x + \log|\sin x| \cdot (\sin x)$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \log|\sin x|$$

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UNIT-IV SAQ Question Bank

Q1 Evaluate $\gamma\left(-\frac{5}{2}\right)$

CO	BTL
CO1	3

Sol: We know that $\gamma(n) = \frac{\gamma(n+1)}{n}$

$$\therefore \gamma\left(-\tfrac{5}{2}\right) = \frac{\gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\tfrac{2}{5}\,\gamma\left(\tfrac{-3}{2}\right)$$

$$= -\frac{2}{5}\,\frac{\gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \left(-\tfrac{2}{5}\right)\left(-\tfrac{2}{3}\right)\gamma\left(-\tfrac{1}{2}\right)$$

$$= \tfrac{4}{15}\,(-2)\,\frac{\gamma\left(-\frac{1}{2}+1\right)}{\cancel{-\frac{1}{2}}} = \frac{-8}{15}\,\gamma\left(\tfrac{1}{2}\right)$$

$$= -\frac{8}{15}\sqrt{\pi} \qquad \left\{\because \gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right\}$$

Q.2 State Beta function

CO1	BTL5

Sol. The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the "beta function" & is denoted by $B(m,n)$. $m>0, n>0$

i.e $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

Q.3 State Gamma function

Sol. The definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is called Gamma function & is denoted as $\gamma(n)$

$$\gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx.$$

Q.4: State Rodrigues formula. $\boxed{CO5 | BTL1}$

Sol. The relation $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

is known as "Rodrigues Formula".

Q.5. State the relation between Beta & Gamma function. $\boxed{CO5 | BTL1}$

Sol: $B(m,n) = \frac{\gamma_m \gamma_n}{\gamma_{m+n}}$ or $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m>0, n>0.$

Q.6. Prove that $P_n(1) = 1$ $\boxed{CO4 | BTL5}$

Sol. we know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

If $x=1$

$$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$((1-t)^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$(1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\Rightarrow 1+t+t^2+\dots+t^n = \sum_{n=0}^{\infty} t^n P_n(1)$$

Comparing coefficients of t^n on b/s,

$$\Rightarrow P_n(1) = 1$$

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7Q. prove that $erf(x) + erf_c(x) = 1$.

Soln By the defn we have.

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du + \frac{2}{\pi} \int_x^{\infty} e^{-u^2} du.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du.$$

$$= erf(\infty) = 1.$$

$$\therefore erf(x) + erf_c(x) = 1.$$

Q.8 Find the value of $B(9/2, 7/2)$

Sol → we know $B(m,n) = \frac{\gamma_m \gamma_n}{\gamma_{m+n}}$

$$B(9/2, 7/2) = \frac{\gamma(9/2)\gamma(7/2)}{\gamma(9/2 + 7/2)} = \frac{\frac{1}{2}\gamma(7/2)\, 5/2\, \gamma(5/2)}{\gamma(16/2)}$$

$$= \frac{\frac{5}{2}\, 7/2 \left[\frac{5}{2}\gamma(3/2)\, \frac{3}{2}\gamma(3/2)\right]}{7!}$$

$$= \frac{3/2 (5/2)^2 (7/2) \left[3/2\, \gamma(3/2) \cdot \frac{1}{2}\gamma(1/2)\right]}{7!}$$

$$= \frac{\left(\frac{1}{2}\right)(3/2)^2 (5/2)^2 (7/2) \left[\frac{1}{2}\gamma(1/2) \cdot \sqrt{\pi}\right]}{7!}$$

$$= \frac{\left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right)^2 \left(\frac{5}{2}\right)^2 \left(\frac{7}{2}\right) \pi}{7!}$$

Q.9. Prove That $B(m,n) = B(n,m)$ $\boxed{CO4 | BTL4}$

Sol: we know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

let $(1-x) = y$ $\quad | x \to 0, y \to 1$

$\Rightarrow dx = -dy \quad x \to 1, y \to 0$

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$\boxed{B(m,n) = B(n,m)}$$

Q.10 Express $f(x) = 2x^3 - 6x^2 + 5x - 3$ in terms of Legendre Polynomial $P_n(x)$ CO4 | BTL

Sol. By Rodrigues Formula $P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}}\cancel{2}x$$

$$P_1(x) = x \quad \text{or} \quad \boxed{x = P_1(x)}$$

$$P_2(x) = \frac{1}{2!2^2}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8_2}\frac{d}{dx}\cancel{2}(x^2-1)\cdot\cancel{2}x$$

$$P_2(x) = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1)$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow 3x^2 - 1 = 2P_2(x)$$

$$3x^2 = 2P_2(x) + 1$$

$$\boxed{x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)}$$

$$P_3(x) = \frac{1}{3!\,2^3}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{6\times 8}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5 - 12x^3 + 6x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4 - 36x^2 + 6)$$

$$= \frac{1}{48}(120x^3 - 72x)$$

$$= \frac{\cancel{24}\,1}{\cancel{48}\,2}(5x^3 - 3x)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow 5x^3 - 3x = 2P_3(x)$$

$$5x^3 = 2P_3(x) + 3x$$

$$\boxed{x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)}$$

Substituting x, x^2, x^3 in the given Polynomial, we get

$$f(x) = 2\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] - \cancel{3}6\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + 5P_1(x) - 3P_0(x)$$

$$\Rightarrow f(x) = \frac{4}{5}P_3(x) + \frac{6}{5}P_1(x) - 4P_2(x) - 2P_0(x) + 5P_1(x) - 3P_0(x)$$

$$\therefore f(x) = \frac{4}{5}P_3(x) + \frac{31}{5}P_1(x) - 4P_2(x) - 5P_0(x)$$

Q.11 Prove that $\Gamma(n+1) = n!$ [CO4 | BTL2]

Sol: we know $\Gamma(n+1) = n(n-1)\Gamma(n-1)$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$\vdots$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\cdots 1$$

$$\boxed{\Gamma(n+1) = n!}, \ (n = 0,1,2,\cdots)$$

Q.12 Evaluate $\int_{-1}^{0} (1-x^2)^n dx$.

Sol: Let $x^2 = t$ | $x \to -1, t \to 1$

$2x\,dx = dt$ | $x \to 0, t \to 0$

$$\int_1^0 (1-t)^n \frac{dt}{2x} = \int_1^0 (1-t)^n \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_1^0 (1-t)^n t^{-1/2} dt$$

$$= \frac{-1}{2}\int_0^1 t^{1/2-1}(1-t)^{n-1} dt$$

$$= -\frac{1}{2}B\left(\frac{1}{2}, n\right) = -\frac{1}{2}\frac{\Gamma(1/2)\Gamma(n)}{\Gamma(n+1/2)}$$

Q.13 Express as Polynomial in x,
$3P_3(x)+2P_2(x)+4P_1(x)+5P_0(x)$ $\boxed{CO4|BTL3}$

Sol. We know $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

$P_3(x)=\frac{1}{2}(5x^3-3x)$

substituting in given eqn.

$\frac{3}{2}(5x^3-3x)+(3x^2-1)+4x+5$

$\Rightarrow \frac{15}{2}x^3-\frac{9}{2}x+3x^2-1+4x+5$

$\Rightarrow \frac{15}{2}x^3+3x^2+\frac{1}{2}x+4$

Q.14. Determine the nature of Point $x=0$
for eqn. $xy''+y\sin x=0$ $\boxed{CO4|BTL2}$

Sol:- Given $xy''+y\sin x=0$

$\Rightarrow y''+\frac{\sin x}{x}y=0$

$y''+P(x)y'+Q(x)y=0$

$P(x)=0$, $Q(x)=\frac{\sin x}{x}$

for $x=0$, $Q(x)=1$

$\therefore x=0$ is a regular singular Point.

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Q.15. Evaluate $\Gamma(9/2)$

Sol $\Gamma(9/2) = \frac{7}{2}\Gamma(7/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma(5/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma(3/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

Q1 Evaluate $\frac{d}{dx}(erf(ax))$ $\boxed{CO4 | BTL4}$

Sol. By definition, $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$
$= \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$

We know $e^{-u^2} = 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots$

$\left\{ \because e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right\}$.

$\therefore erf(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right) du$

$\Rightarrow erf(ax) = \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{10} - \frac{u^7}{42} + \cdots\right]_0^{ax}$

$\Rightarrow erf(ax) = \frac{2}{\sqrt{\pi}} \left[ax - \frac{a^3x^3}{3} + \frac{a^5x^5}{10} - \frac{a^7x^7}{42} + \cdots\right]$

Now $\frac{d}{dx}[erf(ax)] = \frac{2}{\sqrt{\pi}} \cdot \frac{d}{dx}\left[ax - \frac{a^3x^3}{3} + \frac{a^5x^5}{10} - \frac{a^7x^7}{42} + \cdots\right]$

$= \frac{2}{\sqrt{\pi}} \left[a - \frac{\cancel{3}a^3x^2}{\cancel{3}} + \frac{\cancel{5}a^5x^4}{\cancel{10}_2} - \frac{\cancel{7}a^7x^6}{\cancel{42}_6} + \cdots\right]$

$= \frac{2a}{\sqrt{\pi}} \left[1 - a^2x^2 + \frac{a^4x^4}{2} - \frac{a^6x^6}{3!} + \cdots\right]$

$= \frac{2a}{\sqrt{\pi}} \left[1 - \frac{(a^2x^2)}{1!} + \frac{(a^2x^2)^2}{2!} - \frac{(a^2x^2)^3}{3!} + \cdots\right]$

$\therefore \frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$

Q.2 Show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$. $\boxed{CO4 | BTLS}$

Sol: By def^n. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $x = \frac{1}{1+y}$ | $x \to 0, y \to \infty$; $x \to 1, y \to 0$

$dx = -\frac{1}{(1+y)^2} dy$

$\therefore B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \left(-\frac{1}{(1+y)^2}\right) dy$

$= \int_0^\infty \left(\frac{1}{1+y}\right)^{m+1} \left(\frac{\cancel{1}+y-\cancel{1}}{1+y}\right)^{n-1} dy$

$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

{$\because$ ~~Bd~~ is symmetric}

$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Q.3. Prove that $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ $\boxed{CO4 | BTLS}$

Sol:- we know that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \to (1)$

now, $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$, let $x = 1/y$ | $x \to 1, y \to 1$

$dx = -\frac{1}{y^2} dy$ | $x \to \infty\ y \to 0$

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$$= \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{1}{y^{m-1}\left(\frac{y+1}{y}\right)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^n \cdot \cancel{y^m}}{\cancel{y^m} \cdot y^{-1} (1+y)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\therefore B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\boxed{B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.}$$

Q.4 Define Gamma function & show that $\Gamma(1/2) = \sqrt{\pi}$. | CO4 | BTL4 |

Sol- The definite Integral $\int_0^\infty e^{-x} x^{n-1} dx$ is defined as gamma function & is denoted as

$$\gamma_n \text{ or } \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

To show that $\Gamma(1/2) = \sqrt{\pi}$.

we know that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Let $m = n = 1/2$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1/2+1/2)} = \left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 \rightarrow (A)$$

we know that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2}$$

$$= \cancel{2}\left[\pi/\cancel{2} - 0\right]$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi \quad \{\text{using } (A)\}$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Q.5. Evaluate $\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}dx$ using Beta & gamma function.

CO5	BTL5

Sol: $\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}dx.$

$$= \int_{x=0}^{\infty} \frac{x^{3/2}}{a\sqrt{1+x^2/a^2}}dx \rightarrow (1)$$

Put $\frac{x^2}{a^2} = y \Rightarrow x^2 = a^2 y$
$x = a\sqrt{y}$

Diff. b/s w.r.t x

$2x\,dx = a^2 dy$ If $x=0, y=0$
If $x=\infty, y=\infty$

$x\,dx = \frac{a^2}{2}dy$

$dx = \frac{a^2}{2}\frac{dy}{x}$

$dx = \frac{a^2}{2}\frac{dy}{a\sqrt{y}} = \frac{a}{2}y^{-1/2}$

substitute in (1)

$$\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}dx = \int_{y=0}^{\infty} \frac{(a\sqrt{y})^{3/2}}{a\sqrt{1+y}} a/2\, y^{-1/2} dy$$

$$= \int_0^\infty \frac{a^{3/2}}{2} \frac{y^{3/4} \cdot y^{-1/2}}{(1+y)^{1/2}} dy$$

$$= \frac{a^{3/2}}{2} \int_0^\infty \frac{y^{1/4}}{(1+y)^{1/2}} dy$$

$$= \frac{a^{3/2}}{2} \int_0^{\infty} \frac{y^{5/4-1}}{(1+y)^{5/4+(-3/4)}} dy$$

$$= \frac{a^{3/2}}{2} B\left(5/4, -3/4\right) \quad \left\{\because B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx\right.$$

$$= \frac{a^{3/2}}{2} \frac{\Gamma(5/4)\,\Gamma(-3/4)}{\Gamma(5/4 - 3/4)}$$

$$= \frac{a^{3/2}}{2} \frac{\frac{1}{4}\Gamma(1/4)\,\Gamma(-3/4)}{\Gamma(1/2)} \quad \left\{\because \Gamma(n) = \frac{\Gamma(n+1)}{n}\right\}$$

$$= \frac{a^{3/2}}{8} \frac{\Gamma(1/4)\,\Gamma(-3/4)}{\sqrt{\pi}} \quad \because \Gamma_{1/2} = \sqrt{\pi}.$$

$$= \frac{a^{3/2}}{8} \frac{\Gamma(1/4)\,\Gamma(-3/4)}{\sqrt{\pi}}.$$

Q.6 Express $2x^3+3x^2-x+1$ in terms of Legendre Polynomial. |CO4|BTL5|

Sol: we know that By Rodrigues Formula

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\boxed{P_0(x) = 1}$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2} \times 2x$$

$$\boxed{P_1(x) = x}$$

$$P_2(x) = \frac{1}{2!\,2^2} \frac{d^2}{dx^2}(x^2-1)^2$$

$$P_2(x) = \frac{1}{8_2} \frac{d}{dx} 2(x^2-1) \cdot 2x$$

$$P_2(x) = \frac{1}{2} \frac{d}{dx}(x^3 - x)$$

$$P_2(x) = \frac{1}{2}(3x^2-1)^2$$

$$2P_2(x) = 3x^2-1$$

$$\Rightarrow 3x^2 = 2P_1(x) \cancel{+} 1 = 2P_2(x) + P_0(x)$$

$$\Rightarrow x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$P_3(x) = \frac{1}{3!2^3}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5 - 12x^3 + 6x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4 - 36x^2 + 6)$$

$$= \frac{1}{48}(120x^3 - 72x)$$

$$= \frac{\cancel{24}}{\cancel{48}_2}(5x^3 - 3x)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow 2P_3(x) = 5x^3 - 3x$$

$$\Rightarrow 5x^3 = 2P_3(x) + 3x$$

$$\Rightarrow \cancel{5}x^3 = \frac{\cancel{2}}{5}(2P_3(x) + 3x)$$

$$\Rightarrow x^3 = \frac{1}{5}\left[2P_3(x) + 3x\right]$$

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Substituting x, x^2, x^3, we get

$$2x^3+3x^2-x+1 = 2\left[\frac{1}{5}\left(2P_3(x)+3P_1(x)\right)\right] + 3\left[\frac{1}{3}\left(2P_2(x)+P_0(x)\right)\right]$$

$$- P_1(x) + P_0(x)$$

$$\boxed{2x^3+3x^2-x+1 = \frac{4}{5}P_3(x) + 2P_2(x) + \frac{1}{5}P_1(x) + 2P_0(x)}$$

Q.7 First Recurrence Relation for $P_n(x)$.
Prove that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$\boxed{CO5 \mid BTL3}$

Sol:- We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Differentiating Partially w.r.t 't' on b/s

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(1-2xt+t^2)^{-1}(x-t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} x t^n P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - \sum_{n=0}^{\infty} 2xn t^n P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x)$$

Equating coeff of t^n on b/s.

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = 2xnP_n(x) + xP_n(x) - (n-1)P_{n-1}(x) - P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Q.8 Find the Power Series solution of the differential eqn $(1-x)^2 y'' - 2xy' + 2y = 0$ about $x=0$ $\boxed{CO4 | BTL5}$

Sol:- $(1-x^2)y'' - 2xy' + 2y = 0 \rightarrow ①$

$P_0(x) = 1-x^2$, At $x=0 \Rightarrow 1-x^2 \neq 0$

$\therefore x=0$ is ordinary Point of ①

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$ be a solution of ① $\rightarrow ②$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$

Substitute in ①, we get

$(1-x^2)(2a_2 + 6a_3 x + 12a_4 x^2 \ldots) - 2x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) + 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0$

comparing the coefficients

x^0 (constants) $\Rightarrow 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

$x^1 \Rightarrow 6a_3 - 2a_1 + 2a_1 = 0 \Rightarrow 6a_3 = 0 \Rightarrow a_3 = 0$

$x^2 \Rightarrow -\cancel{2a_2} + 12a_4 - 4a_2 + \cancel{2a_2} = 0$

$\Rightarrow {}_3\cancel{12}a_4 = \cancel{4}a_2$

$a_4 = \frac{1}{3} a_2 = \frac{1}{3}(-a_0)$

Substitute in (3), we get

$$y = a_0 + a_1x + (-a_0)x^2 + 0x^3 + \left(-\frac{a_0}{3}\right)x^4 + \cdots$$

$$y = a_0\left(1 - x^2 - \frac{x^4}{3} + \cdots\right) + a_1x$$

where a_0 & a_1 are arbitary Constant.

Q.9 Express the following sum of the legendre Polynomial in terms of x, $8P_4(x) + 2P_2(x) + P_0(x)$. CO4 | BTL4

Sol: we know By Rodrigues formula,

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\rightarrow P_0(x) = \frac{1}{2^0 \cdot 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1 \Rightarrow \underline{P_0(x) = 1}$$

$$\rightarrow P_1(x) = \frac{1}{2^1 \cdot 1!}\frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}}\cancel{2}x = x \Rightarrow \underline{P_1(x) = x.}$$

$$\rightarrow P_2(x) = \frac{1}{2^2 \cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{\cancel{8}_2}\frac{d}{dx}\cancel{2}(x^2-1)\cdot\cancel{2}x$$

$$P_2(x) = \frac{1}{2}\frac{d}{dx}(x^3 - x) = \underline{\frac{1}{2}(3x^2-1)}$$

$$\rightarrow P_3(x) = \frac{1}{2^3 \cdot 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{8\times6}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5 - 12x^3 + 9x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4 - 36x^2 + 9)$$

$$= \frac{1}{48}(120x^3 - 72x) = \frac{\cancel{24}}{\cancel{48}_2}(5x^3 - 3x)$$

$$\boxed{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

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$$\rightarrow P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{\cancel{16}\times 24}\frac{d^3}{dx^3}\cancel{4}(x^2-1)^3 \times 2x$$

$$= \frac{1}{48}\frac{d^3}{dx^3}x(x^6-3x^4+3x^2-1)$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^7-3x^5+3x^3-x)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(7x^6-15x^4+9x^2-1)$$

$$= \frac{1}{48}\frac{d}{dx}(42x^5-60x^3+18x)$$

$$= \frac{6}{48}\frac{d}{dx}(7x^5-10x^3+3x)$$

$$= \frac{6}{\cancel{48}\,8}(35x^4-30x^2+3)$$

$$\boxed{P_4(x) = \frac{1}{8}(35x^4-30x^2+3)}$$

$$\therefore 8P_4(x)+2P_2(x)+P_0(x) = \frac{\cancel{8}}{\cancel{8}}(35x^4-30x^2+3) + \frac{\cancel{2}}{\cancel{2}}(3x^2-1) + 1$$

$$= 35x^4-30x^2+3+3x^2\cancel{-1}\cancel{+1}$$

$$= 35x^4-27x^2+3 \text{ // Ans.}$$

Q.10 Evaluate the Improper Integral,

$$\int_0^\infty \sqrt{x}\cdot e^{-x^2}\,dx. \qquad \boxed{CO4/BTL5}$$

Sol: Given $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx$

Let $x^2 = t \Rightarrow x = \sqrt{t}$ | $x \to 0,\ t \to 0$; $x \to \infty,\ t \to \infty$

$2x\,dx = dt$

$dx = \dfrac{dt}{2\sqrt{t}}$

$$\therefore \int_0^\infty e^{-t}\cdot t^{1/4}\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/4}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{3/4-1}\,dt$$

$$= \frac{1}{2}\,\Gamma(3/4)$$

Q.11 Prove that $B(m,n) = 2\int_0 \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

Sol. By Beta function def.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$dx = \sin 2\theta\,d\theta$

If $x = 0 \Rightarrow \sin^2\theta = 0 \Rightarrow \theta = 0$

If $x = 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \pi/2$.

$$\therefore B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \sin 2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \, (\cos^2\theta)^{n-1} \cdot 2\sin\theta \cdot \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \, \cos^{2n-2+1} d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1} d\theta$$

Q.12 Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$. $\boxed{CO4|BTL5}$

Sol: $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 x^2 (1-x^5)^{-1/2} dx$

$$= \int_0^1 \frac{x^2}{x^4} (1-x^5)^{-1/2} x^4 dx$$

$$= \int_0^1 x^{-2} (1-x^5)^{-1/2} x^4 dx$$

Put $x^5 = y \Rightarrow 5x^4 dx = dy$

$x = y^{1/5}$ or $x^4 dx = \frac{dy}{5}$

when $x=1$, $y=1$

when $x=0$, $y=0$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 y^{-2/5} (1-y)^{-1/2} \frac{dy}{5}$$

$$= \frac{1}{5} \int_0^1 y^{3/5 - 1} (1-y)^{\frac{1}{2}-1} dy$$

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{1-x^5}} = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)}$$

"OR"

Let $x^5 = y \Rightarrow x = y^{1/5}$

Diff. b/s ~~w.r.t.~~

~~$5x^4 dx = dy$~~ $dx = \frac{1}{5} y^{-4/5} dy$

~~or~~ when $x=0,\ y=0$

when $x=1,\ y=1$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx = \int_0^1 \frac{y^{2/5}}{(1-y)^{1/2}} \cdot \frac{1}{5} y^{-4/5} dy$$

$$= \frac{1}{5}\int_0^1 \frac{y^{2/5-4/5}}{(1-y)^{1/2}} dy$$

$$= \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2} dy$$

$$= \frac{1}{5}\int_0^1 y^{3/5-1}(1-y)^{1/2-1} dy$$

comparing with $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right).}$$

Q.13. Prove that $2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}) = \Gamma(2n)\sqrt{\pi}$

CO4 | BTL4

Sol. By def, we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

or

$$\int_0^1 x^{n-1}(1-x)^{m-1}dx = B(m,n) = \frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m)} \rightarrow (1)$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

when $x\to 0,\ \theta\to 0$

$x\to 1,\ \theta\to \pi/2$

From eqn (1)

$$\int_0^{\pi/2} \sin^{2n-2}\theta \cos^{2m-2}\theta \,(2\sin\theta\cos\theta)\, d\theta = \frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m)}$$

$$\text{or } \int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma(n)\cdot\Gamma(m)}{2\Gamma(n+m)} \rightarrow (2)$$

Put $m = \frac{1}{2}$ in eqn (2), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(n)\cdot\Gamma(\frac{1}{2})}{2\Gamma(n+\frac{1}{2})} = \frac{\sqrt{\pi}\cdot\Gamma(n)}{2\;\Gamma(n+\frac{1}{2})} \rightarrow (3)$$

now put $m = n$ in eqn (2), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{[\Gamma(n)]^2}{2\Gamma(2n)}$$

$$\text{or } \frac{[\Gamma(n)]^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1} \, d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} 2\theta \, d\theta$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{1}{2} \int_0^{\pi/2} \sin^{2n-1}\phi \, d\phi \quad (\text{Put } 2\theta = \phi)$$

$$= \frac{1}{2^{2n}} \cdot 2 \int_0^{\pi/2} \sin^{2n-1}\phi \, d\phi$$

$$\Rightarrow \frac{[\Gamma(n)]^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}} \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \; \{\text{using } (3)\}$$

$$\Rightarrow \boxed{2^{2n-1}\Gamma(n)\cdot\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\cdot\Gamma(2n)}$$

//

Q.14 Evaluate $\int_0^{\infty} 3^{-4x^2} dx$.

Let $I = \int_0^{\infty} 3^{-4x^2} dx$

Put $3^{-4x^2} = e^{-t}$

Taking log on b/s, we get

$\log 3^{-4x^2} = \log e^{-t}$

$\cancel{-}4x^2 \log 3 = \cancel{-}t \Rightarrow 4x^2 \log 3 = t$

$\Rightarrow x^2 = \frac{t}{4\log 3} \Rightarrow x = \frac{\sqrt{t}}{2\sqrt{\log 3}} = \frac{(t)^{1/2}}{2(\log 3)^{1/2}}$

Diff. b/s w.r.t x,

$dx = \frac{t^{-1/2}}{4\sqrt{\log 3}} dt$

$\therefore I = \int_0^{\infty} \frac{e^{-t} \cdot t^{-1/2}}{4\sqrt{\log 3}} dt$

$\Rightarrow I = \frac{1}{4\sqrt{\log 3}} \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt$

$\Rightarrow \boxed{I = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}}$ $\left\{ \because \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt = \sqrt{\pi} \right\}$

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15Q. Evaluate $\int_0^1 x^m (1-x^2)^n$

Soln: Let $x^2 = t \Rightarrow x = t^{1/2}$

$2x\,dx = dt$

$dx = \frac{dt}{2x} = \frac{dt}{2t^{1/2}}$

U.L $= 0 \Rightarrow t = 0$

L.L $= 1 \Rightarrow t = 1$

Subst in Above eqn.

$$\int_0^1 x^m (1-x^2)^n = \int_0^1 \left(t^{1/2}\right)^m (1-t)^n \frac{dt}{2t^{1/2}}$$

$$= \frac{1}{2}\int_0^1 t^{m/2}\, t^{-1/2} (1-t)^n\, dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{m-1}{2}} (1-t)^n\, dt$$

$$\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$m = 2m - 1$

Hence

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1} (1-x)^{n+1-1}\, dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, n+1\right)$$

M-II – UNIT – 5 – Q.B.

SAQ & LAQ's

(1) Find $L\{e^{-t} \cdot \sin 2t\}$.

Sol:- $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

Here $a = 2$.

$L\{\sin 2t\} = \dfrac{2}{s^2+2^2} = \dfrac{2}{s^2+4}$

Now, $L\{e^{-t} \cdot \sin 2t\} = \left[\dfrac{2}{s^2+4}\right]_{s \to s+1}$

$= \dfrac{2}{(s+1)^2+4} = \dfrac{2}{s^2+2s+1+4} = \dfrac{2}{s^2+2s+5}$.

$\therefore L\{e^{-t} \cdot \sin 2t\} = \dfrac{2}{s^2+2s+5}$.

(2) State convolution Theorem of Laplace Transforms.

Sol:- If $f(s)$ & $g(s)$ are Laplace transform of $F(t)$ & $G(t)$ respectively,

i.e., if $L^{-1}[f(s)] = F(t)$ & $L^{-1}[g(s)] = G(t)$,

then $L^{-1}\{f(s) * g(s)\} = \int_0^t F(u) \cdot G(t-u)\, du$.

3). Find $L\{t \cdot \cos 2t\}$

Sol:- $L\{t^n \cdot f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n}[f(t)]$.

$\Rightarrow L\{t^1 \cdot \cos 2t\} = (-1)^1 \cdot \frac{d^1}{ds^1}[\cos 2t]$

$L[\cos 2t] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4} = \frac{s}{s^2+a^2}$.

$\Rightarrow L[t \cdot \cos 2t] = (-1) \cdot \frac{d}{ds} \cdot \left[\frac{s}{s^2+4}\right]$

$= (-1)\frac{-s^2+4}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$.

$\boxed{\frac{u'v-uv'}{v^2}}$

4) Evaluate

$L^{-1}\left[\frac{2}{s^3} + \frac{1}{s^2}\right]$

Sol:- $2 \cdot L^{-1}\left[\frac{1}{s^3}\right] + L^{-1} \cdot \left[\frac{1}{s^2}\right]$

$= 2 \cdot \frac{t^2}{2!} + t \cdot = t^2 + t$.

$\left[\because L^{-1}\left[\frac{1}{s^3}\right] = \right.$

$\bar{f}(s) = \frac{1}{s^{n+1}} = \frac{1}{s^{2+1}} = \bar{f}(s)$

$L^{-1}[\bar{f}(s)] = f(t)$

$= \frac{t^n}{n!}$

(5) find $L\{t \cdot e^{-t}\}$

Sol:- $L\{e^{-t}\} = \frac{1}{s+1} = \bar{f}(s)$

$$L\{t^1 \cdot e^{-t}\} = (-1)^n \cdot \frac{d^n}{ds^n}[\bar{f}(s)]$$

$$= (-1)^1 \frac{d^1}{ds^1}\left[\frac{1}{s+1}\right]$$

$$= (-1) \cdot \frac{d}{ds}(s+1)^{-1}$$

$$= (-1) \cdot (-1) \cdot (s+1)^{-1-1}$$

$$= (-1) \cdot (-1) \cdot (s+1)^{-2}(1) = (s+1)^{-2} = \frac{1}{(s+1)^2}$$

(6) find ~~inverse~~ Laplace transform of

$f(t) =$ ~~[illegible]~~ $t \cdot e^{3t} \cdot \sin 2t$

Sol:- we know that

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$\therefore$ By first shifting Th.

$$L\{e^{3t} \cdot \sin 2t\} = \frac{2}{(s-3)^2+4} = \frac{2}{s^2-6s+13}$$

$$\Rightarrow L\{t \cdot e^{3t} \cdot \sin 2t\} = (-1) \cdot \frac{d}{ds}\left[\frac{2}{s^2-6s+13}\right]$$

$$= (-2) \cdot \frac{(-1)}{(s^2-6s+13)^2} (2s-6)$$

$$= \frac{4(s-3)}{(s^2-6s+13)^2}.$$

(7) Find $L\{t^3 \cdot e^{-4t}\}$

Sol:— $L\{e^{-4t}\} = \frac{1}{s+4}$

$$L\{t^3 \cdot e^{-4t}\} = (-1)^3 \cdot \frac{d^3}{ds^3}\left[\frac{1}{s+4}\right]$$

$$= (-1) \cdot \frac{d^3}{ds^3}[s+4]^{-1} = (-1) \cdot \frac{d^2}{ds^2} \cdot \frac{d}{ds}(s+4)^{-1}$$

$$= (-1) \cdot \frac{d^2}{ds^2} (-1) \cdot [s+4]^{-1-1} (1)$$

$$= \frac{d^2}{ds^2}[s+4]^{-2} = \frac{d}{ds} \cdot \frac{d}{ds}[s+4]^{-2}$$

$$= \frac{d}{ds} \cdot (-2)(s+4)^{-2-1} = \frac{d}{ds}\left[(-2)(s-4)^{-3}\right]$$

$$= (-2)(-3)(s-4)^{-3-1}$$

$$= 6 \cdot (s-4)^{-4} = \frac{6}{(s-4)^4}$$

3

(8) Find $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right]$

Sol:- $\frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$. —(1)

$1 = A(s+3) + B(s+2) = As + 3A + Bs + 2B$

$\Rightarrow$ ~~$1 = A(s+3) + B(s+2)$~~

$\Rightarrow 1 = (A+B)s + 3A + 2B$

$A + B = 0$

$3A + 2B = 1$

$\Rightarrow$ $\left.\begin{array}{l} 3A + 3B = 0 \\ 3A + 2B = 1 \end{array}\right\}$

solving the above two eq's, we get

$B = -1$ & $A = 1$

$= \frac{1}{s+2} - \frac{1}{s+3}$.

$\Rightarrow L^{-1}\left[\frac{1}{(s+2)(s+3)}\right] = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right]$

$\Rightarrow e^{-2t} - e^{-3t}$.

(9) Find the Laplace transform of

$f(t) = \sin^2 t$.

Sol:- $L[\sin^2 t]$

$\Rightarrow L\left[\frac{1-\cos 2t}{2}\right]$

$= \frac{1}{2} \cdot L[1 - \cos 2t]$

$= \frac{1}{2} \cdot \left[L[1] - L[\cos 2t]\right]$

$= \frac{1}{2} \cdot \left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{1}{2} \cdot \frac{4}{s(s^2+4)} = \frac{2}{s(s^2+4)}$

$\cos 2A = 1 - 2\sin^2 A$.

$\cos 2A - 1 = -2\sin^2 A$.

$\Rightarrow 2\sin^2 A = 1 - \cos 2A$

$\Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$

$\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$

(10) Explain Unit impulse function. (Dirac Delta function).

Sol:- The unit impulse function is considered as the limiting form of the function.

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a. \\ 0 & \text{for } t \neq a. \end{cases}$$

Such that $\int_0^\infty \delta(t-a)\,dt = 1$. $(a \geq 0)$

(11) Evaluate $L\{\cos^2 t\}$.

Sol! →

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$L[\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right] = \frac{1}{2} \cdot L[1+\cos 2t]$$

$$= \frac{1}{2} \cdot L[1] + L[\cos 2t]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right] = \frac{1}{2}\left[\frac{2s^2+4}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{2(s^2+2)}{s(s^2+4)}\right]$$

$$= \frac{s^2+2}{s(s^2+4)}.$$

12. Find $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$

Sol:- By partial fraction

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$\frac{1}{\cancel{(s+1)(s+2)}} = \frac{A(s+2)+B(s+1)}{\cancel{(s+1)(s+2)}}$$

$$1 = A(s+2)+B(s+1)$$

put $s=-1$

$$1 = A(-1+2)+B(-1+1)$$

$$1 = A(1)+0$$

$$\boxed{A=1}$$

put $s=-2$

$$1 = A(-2+2)+B(-2+1)$$

$$1 = 0+B(-1)$$

$$\boxed{B=-1}$$

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

13. Find $L\{t^3 e^{-4t}\}$

Sol: $L\{e^{-4t}\} = \frac{1}{s+4}$

By multiplication property

$$L\{t^3 e^{-4t}\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+4}\right) \quad \left[\because L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[f(s)]\right]$$

$$= -1 \frac{d^2}{ds^2}\left(\frac{d}{ds}(s+4)^{-1}\right)$$

$$= -1\cdot(-1)\frac{d^2}{ds^2}(s+4)^{-2}$$

$$= 1\frac{d}{ds}\left(\frac{d}{ds}(s+4)^{-2}\right) = 1\cdot(-2)\frac{d}{ds}(s+4)^{-3}$$

$$= -2(-3)(s+4)^{-4}$$

$$= \frac{6}{(s+4)^4}$$

14. Find $L^{-1}\left\{\frac{1}{s(s+1)}\right\}$

sol:- $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

By division Rule

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^t = \left[\frac{e^{-t} - e^{0}}{-1}\right]$$

$$= \frac{e^{-t} - 1}{-1} = 1 - e^{-t}$$

15. Find $L\{t^3\}$

sol:- $L\{t^3\} = \frac{3!}{s^4}$

$$= \frac{3 \times 2 \times 1}{s^4}$$

$$= \frac{6}{s^4}$$

$s^2+6s+34$

1. Find the Laplace transform of

$$f(t) = \frac{e^{-2t}\sin 3t}{t}$$

We know that

$$L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\sin 3t}{t}\right\} = 3\int_s^\infty \frac{1}{s^2+9}\,ds$$ (By division property)

$$= 3\int_s^\infty \frac{1}{s^2+3^2}\,ds$$

$$= \left[3\cdot\frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) \qquad \left[\tan^{-1}(x) + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$= \cot^{-1}\left(\frac{s}{3}\right)$$

$$L\left\{\frac{\sin 3t}{t}\right\} = \cot^{-1}\left(\frac{s}{3}\right)$$

$$L\left\{e^{-2t}\,\frac{\sin 3t}{t}\right\} = \cot^{-1}\left(\frac{s+2}{3}\right)$$

2 Using Laplace Transform solve the initial value problem

$y'' + y = e^t \sin t$, $y(0) = 0 = y'(0)$

Sol: Given

$y'' + y = e^t \sin t$

Taking L.T on both sides

$L\{y''\} + L\{y\} = L\{e^t \sin t\}$

$\left[\because L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0) \right]$

$s^2 L\{y\} - s y(0) - y'(0) + L\{y\} = \dfrac{1}{(s-1)^2 + 1^2}$

using initial condition

$y(0) = 0$, $y'(0) = 0$

$s^2 L\{y\} - 0 - 0 + L\{y\} = \dfrac{1}{(s-1)^2 + 1^2}$

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$$L\{y\}\left[s^2+1\right] = \frac{1}{s^2+1-2s+1}$$

$$L\{y\} = \frac{1}{s^2-2s+2} \cdot \frac{1}{s^2+1}$$

$$\frac{1}{(s^2-2s+2)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

$$\frac{1}{\cancel{(s^2-2s+2)(s^2+1)}} = \frac{(As+B)(s^2-2s+2)+(Cs+D)(s^2+1)}{\cancel{(s^2-2s+2)(s^2+1)}}$$

$$1 = (As+B)(s^2-2s+2)+(Cs+D)(s^2+1)$$

$$1 = As^3-2As^2+2As+Bs^2-2Bs+2B$$
$$+Cs^3+Cs+Ds^2+D$$

equating the co-efficient

s^3 co-efficient $A+C=0$

$A=-C \rightarrow (1)$

s^2 " $-2A+B+D=0 \rightarrow (2)$

$2A-2B+C=0 \rightarrow (3)$

s " $2B+D=1 \rightarrow (4)$

constants

$2A-2B+C=0$

$$2\cancel{C}+B+D=0$$
$$-2\cancel{C}-2B+C=0$$
$$\overline{-B+C+D=0}$$

$-4B+B+D=0$

$-3B+D=0$

$2B+D=1$

$\overline{-5B=-1}$

$B=1/5$

$A=2/5$

$C=-2/5$

$2(-C)-2B+C=0$

$-2C-2B+C=0$

$-2B-C=0$

$-2B=C \quad B=-\frac{C}{2}$

$-2(C)+\cancel{2B}\,C+C=0$

$\cancel{2B}+ \quad B+\cancel{D}=-2C$

$2B+\cancel{D}=1$

$\overline{-B=-2C-1}$

$-2C-2B+C=0$

$-C-2B=0$

$-2B=C$

$-2B=-A$

$A=2B$

$A = 2/5,\ B = 1/5\ \ C = -2/5\ \ D = 3/5$

sub in eqn (1)

$$L\{y\} = \frac{\frac{2}{5}s + \frac{1}{5}}{s^2+1} + \frac{-\frac{2}{5}s + \frac{3}{5}}{s^2-2s+2}$$

$$y = L^{-1}\left(\frac{\frac{2}{5}s + \frac{1}{5}}{s^2+1}\right) + L^{-1}\left(\frac{-\frac{2}{5}s + \frac{3}{5}}{s^2-2s+2}\right)$$

$$= \frac{2}{5}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right) +$$

$$\left(-\frac{1}{5}L^{-1}\left(\frac{2(s-1)-1}{s}\right)\right)$$

$$-\frac{1}{5}\left(L^{-1}\left[\frac{2s-3}{(s-1)^2+1}\right]\right)$$

$$-\frac{1}{5}\left(L^{-1}\left(\frac{2(s-1)-1}{(s-1)^2+1}\right)\right)$$

$$= \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{1}{5}e^t(\cos t - \sin t)$$

5) Evaluate $L^{-1}\left\{\log\left(\frac{s-3}{s+3}\right)\right\}$

Let $\bar{f}(s) = \log\left(\frac{s-3}{s+3}\right)$

$= \log(s-3) - \log(s+3)$

diff w.r.to "s" on both side

$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\log(s-3) - \frac{d}{ds}\log(s+3)$

$= \frac{1}{s-3} - \frac{1}{s+3}$

Taking I. L. Transform on both sides

$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}$

$= -t.f(t) = e^{3t} - e^{-3t}$

$f(t) = \frac{e^{-3t} - e^{3t}}{t}$

convolution theorem,

s^4

5. Find the Laplace transform of $f(t) = \frac{2\sin^2 t}{t}$

Sol: Given

$$f(t) = \frac{2\sin^2 t}{t}$$

$$\left[\begin{array}{l} \cos 2\theta = 1 - 2\sin^2\theta \\ 2\sin^2\theta = 1 - \cos 2\theta \end{array}\right.$$

W.K.T

$$L\{1-\cos 2t\}$$
$$= L\{1\} - L\{\cos 2t\}$$
$$= \frac{1}{s} - \frac{s}{s^2+2^2}$$

By division property

$$L\{f(t)\} = L\left\{\frac{2\sin^2 t}{t}\right\} = L\left\{\frac{1-\cos 2t}{t}\right\}$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{2s}{2(s^2+2^2)}\right) ds$$

$$= \left| \log s - \log(s^2+4) \right|_s^\infty$$

$$= \left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$$

$$= \left[\log \frac{s}{(s^2+4)^{1/2}} \right]_s^\infty$$

$$= \lim_{s \to \infty} \log \frac{\cancel{s}}{\cancel{s}\left(1+\frac{4}{s^2}\right)^{1/2}} - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= \log 1 - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= 0 - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= \log \frac{(s^2+4)^{1/2}}{s}$$

$$= \log \left(\frac{s^2+4}{s^2} \right)^{1/2}$$

$$= \frac{1}{2} \log \left(\frac{s^2}{s^2} + \frac{4}{s^2} \right)$$

$$= \frac{1}{2} \log \left(1 + \frac{4}{s^2} \right)$$

$$= \frac{1}{2} \log \left(1 + 4s^{-2} \right)$$

(6) Find the inverse L.T of $\frac{s}{s^4+s^2+1}$

Sol:- Let $\bar{f}(s) = \frac{s}{s^4+s^2+1} =$

$$= \frac{s}{(s^4+2s^2+1)-s^2} = \frac{s}{(s^2+1)^2-s^2}$$

$\left[a^2-b^2 = (a+b)(a-b) \right]$

$$= \frac{s}{(s^2+s+1)(s^2+1-s)}$$

$$= \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1} \longrightarrow (A)$$

Using partial fraction

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

$$s = As^3 - As^2 + As + Bs^2 - Bs + B + Cs^3 + Cs^2 + Cs + Ds^2 + Ds + D$$

on equating the co-efficient of

i) $s^3 \Rightarrow 0 = A + C \rightarrow (1)$

ii) $s^2 \Rightarrow 0 = -A + B + C + D \rightarrow (2)$

iii) $s \Rightarrow 1 = A - B + C + D \rightarrow (3)$

$1 = -B + D \quad (A + C = 0)$ (from (1))

iv) Constants $0 = B + D \Rightarrow (4)$

from (2)

$-A + C = 0 \rightarrow (5)$

from (1), (5)

$$\begin{array}{r} \cancel{A} + C = 0 \\ -\cancel{A} + C = 0 \\ \hline 2C = 0 \\ C = 0 \end{array}$$

from (1) $A + 0 = 0$

Also,

$$\begin{array}{r} \cancel{B} + D = 0 \\ -\cancel{B} + D = 1 \\ \hline 2D = 1 \\ D = 1/2 \end{array}$$

$B + D = 0$

$B = -D$

$B = -1/2$

from (A)

$$\frac{s}{s^4 + s^2 + 1} = -\frac{1}{2} \cdot \frac{1}{s^2 + s + 1} + \frac{1}{2} \cdot \frac{1}{s^2 - s + 1}$$

Taking I.L.T on both sides

$$L^{-1}\{\bar{f}(s)\} = -\frac{1}{2} L^{-1}\left\{\frac{1}{s^2 + s + 1}\right\} + \frac{1}{2}\left\{\frac{1}{s^2 - s + 1}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s^2 + s + \frac{1}{4}) + (1 - 1/4)}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s^2 - s + 1/4) + (1 - 1/4)}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\}$$

By shifting property

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = -\frac{1}{2} e^{-t/2} L^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\} + \frac{1}{2} e^{t/2} L^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\}$$

$$= -\frac{1}{2} \cdot e^{-t/2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2}t + \frac{1}{2} e^{t/2} \frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2}t$$

$$= -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} e^{-t/2} \cdot \sin \frac{\sqrt{3}}{2} t + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} e^{t/2} \sin \frac{\sqrt{3}}{2} t$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(e^{t/2} - e^{-t/2} \right) \times \frac{2}{2}$$

$$\left[\frac{e^{t} - e^{-t}}{2} = \sinh t \right]$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(\frac{e^{t/2} - e^{-t/2}}{2} \right)$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \cdot \sinh \frac{t}{2}$$

(7) To prove

$L\{e^{at}\} = \frac{1}{s-a}$

We know that

$L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot f(t) dt$

Here $f(t) = e^{at}$

$\therefore L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at} dt$

$= \int_0^{\infty} e^{-(s-a)t} dt \quad \left[\begin{matrix} -st+at \\ -(s-a)t \end{matrix}\right]$

$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$

$= \frac{1}{-(s-a)} (e^{-\infty} - e^{0})$

$= \frac{1}{-(s-a)} (0-1) = \frac{1}{s-a}$

8) Find the L.T of the function

$$f(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases}$$ and

the period of $f(t)$ is 2π

$$L\{f(t)\} = \frac{\int_0^{t} e^{-st} f(t)\,dt}{1-e^{-st}}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} (0)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt \right]$$

$$\left[\because \int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt) \right]$$

$a = -s,\ b = 1$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{(-s)^2+1^2} (-s \sin t - 1 \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{e^{-s\pi}}{s^2+1} (s \cdot \sin \pi + \cos \pi) - (0 + \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{e^{-s\pi}}{s^2+1} (0+1) + \frac{1}{s^2+1} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1^2} \right)$$

$$= \frac{1}{1^2-(e^{-\pi s})^2} \left(\frac{e^{-\pi s}+1}{s^2+1} \right)$$

$$= \frac{1}{\cancel{(1+e^{-\pi s})}(1-e^{-\pi s})} \cdot \left(\frac{\cancel{e^{-\pi s}+1}}{s^2+1}\right)$$

$$= \frac{1}{(1-e^{-\pi s})(s^2+1)}$$

Q:- 9) show that $L\{\sin at\} = \frac{a}{s^2+a^2}$

Sol:- we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

here $f(t) = \sin at$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$$

$$\left[\because \int_0^\infty e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]\right]$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s \sin at - a\cos at]\right]_0^\infty$$

$$= \frac{e^{-\infty}}{s^2+a^2}(-s\sin\infty - a\cos\infty) - \frac{e^{-0}}{s^2+a^2}[-s\sin 0 - a\cos 0]$$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

$$= \frac{a}{s^2+a^2}$$

* ——— * ———

LAQ:-

1.9) Find the Laplace transform of

$f(t) = \bar{e}^{t}(2\cos 5t - 3\sin 5t)$

$\therefore L\{2\cos 5t - 3\sin 5t\}$

$= 2L\{\cos 5t\} - 3L\{\sin 5t\}$ [By using Laplace Transform]

$= 2 \cdot \frac{s}{s^2+5^2} - 3\frac{5}{s^2+5^2}$

$= \frac{2s}{s^2+5^2} - \frac{15}{s^2+5^2}$

$= \frac{2s-15}{s^2+5^2}$

By shifting property

$L\{\bar{e}^{3t}(2\cos 5t - 3\sin 5t)\}$

$= \frac{2(s+3)-15}{(s+3)^2+5^2}$

$= \frac{2s+6-15}{s^2+6s+9+25} = \frac{2s-9}{s^2+6s+34}$

$= \frac{2s-9}{s^2+6s+34}$

convolution theorem :-

statement :- If $L^{-1}\{\bar{f}(s)\} = f(t)$, $L^{-1}\{\bar{g}(s)\} = g(t)$ then

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\,du$$

(1) Applying convolution theorem, solve $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

sol: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+a^2}$

By I.L.T on both sides

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$$

Also, $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\cdot\sin at = g(t)$

Using convolution theorem,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\,du$$

$$\left[\because\ 2\cos A\sin B = \sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2a}\int_0^t 2\cos au\cdot\sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \sin a(\cancel{u}+t-\cancel{u}) - \sin a(u-t+u)\}\,du$$

$$= \frac{1}{2a}\int_0^t\{\sin at - \sin a(2u-t)\}\,du$$

$$= \frac{1}{2a}\left[\sin at\int_0^t 1\,du - \int_0^t \sin a(2u-t)\,du\right]$$

$$= \frac{1}{2a}\left[\sin at\,|u|_0^t - \left(-\frac{\cos a(2u-t)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] \qquad \left[\begin{array}{l}\because \cos a(2t-t) \\ \cos at \\ \cos a(0-t) \\ \cos a(t) \\ = \cos at\end{array}\right]$$

$$= \frac{t\sin at}{2a}$$

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12). Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t$ with $y=\frac{dy}{dt}=0$ when $t=0$ using Laplace transform.

Sol:- Given equation can be written as

$y''+2y'-3y=\sin t$ with $y(0)=0$ & $y'(0)=0$

Taking Laplace Transform on both sides

$L\{y''\}+2L\{y'\}-3L\{y\}=L\{\sin t\}$

$s^2L\{y\}-sy(0)-y'(0)+2[sL\{y\}-y(0)]-3L\{y\}=\frac{1}{s^2+1}$

using initial conditions, we get $y(0)=0$, $y'(0)=0$

$\Rightarrow s^2L\{y\}-0-0+2[sL\{y\}-0]-3L\{y\}=\frac{1}{s^2+1}$

$L\{y\}[s^2+2s-3]=\frac{1}{s^2+1}$

$\Rightarrow L\{y\}[(s+3)(s-1)]=\frac{1}{s^2+1}$

$$L\{y\}=\frac{1}{(s^2+1)(s+3)(s-1)} \rightarrow (1)$$

$$\frac{1}{(s^2+1)(s+3)(s-1)}=\frac{As+B}{s^2+1}+\frac{C}{s+3}+\frac{D}{s-1} \rightarrow (2)$$

$\Rightarrow 1=(As+B)(s+3)(s-1)+C(s-1)(s^2+1)+D(s+3)(s^2+1) \rightarrow (3)$

$\Rightarrow 1=As(s^2+2s-3)+B(s^2+2s-3)+C(s^3+s-s^2-1)+D(s^3+s+3s^2+3)$

$\Rightarrow 1=A(s^3+2s^2-3s)+B(s^2+2s-3)+C(s^3-s^2+s-1)+D(s^3+3s^2+s+3) \rightarrow (4)$

for $s=-3$ in eqn (3)

$1=C(-3-1)(9+1)$

$\Rightarrow 1=-40C$

$\boxed{C=-1/40}$

for $s=1$, from (3) $\Rightarrow 1=D(1+3)(1+1)$

$1=8D \Rightarrow \boxed{D=1/8}$

Equating the co-efficients of s^3 & s^2, we get

$A+C+D=0$ & $2A+B-C+3D=0$

$\Rightarrow A-\frac{1}{40}+\frac{1}{8}=0$

$\Rightarrow A=\frac{1}{40}-\frac{1}{8}=\frac{1-5}{40}=\frac{-1}{40}$

$\boxed{A=-\frac{1}{10}}$

$$\Rightarrow 2\left(-\frac{1}{10}\right)+B+\frac{1}{40}+\frac{3}{8}=0$$

$$\Rightarrow B=\frac{1}{5}-\frac{1}{40}-\frac{3}{8}$$

$$=\frac{8-1-15}{40}=-\frac{1}{5}$$

$$\boxed{B=-\frac{1}{5}}$$

Sub the values of A, B, C & D in eqn (2) & using in eqn (1) we get

$$L\{y\}=\frac{-\frac{1}{10}s-\frac{1}{5}}{s^2+1}+\frac{-\frac{1}{40}}{s+3}+\frac{\frac{1}{8}}{s-1}$$

$$L\{y\}=-\frac{1}{10}\left(\frac{s}{s^2+1}\right)-\frac{1}{5}\left(\frac{1}{s^2+1}\right)-\frac{1}{40}\left(\frac{1}{s+3}\right)+\frac{1}{8}\left(\frac{1}{s-1}\right)$$

$$y=L^{-1}\left[-\frac{1}{10}\left(\frac{s}{s^2+1}\right)-\frac{1}{5}\left(\frac{1}{s^2+1}\right)-\frac{1}{40}\left(\frac{1}{s+3}\right)+\frac{1}{8}\left(\frac{1}{s-1}\right)\right]$$

$$=-\frac{1}{10}L^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{1}{40}L^{-1}\left\{\frac{1}{s+3}\right\}+\frac{1}{8}L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\Rightarrow y=-\frac{1}{10}\cos t-\frac{1}{5}\sin t-\frac{1}{40}e^{-3t}+\frac{1}{8}e^{t}$$

$$f(t) = \frac{e^{-3t} - e^{3t}}{t}$$

13 Q) Using convolation theorem,

find $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$

Sol: Given

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\}$$

Let $\bar{f}(s) = \frac{1}{s+1}$ and $\bar{g}(s) = \frac{1}{s+2}$

Taking I.L.T on both sides

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} = g(t)$$

using convolution theorem

$$L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} = \int_0^t f(u).g(t-u)\,du$$

$$= \int_0^t e^{-u} e^{-2(t-u)} \, du$$

$$= \int_0^t e^{-u} \cdot e^{-2t} \cdot e^{2u} \, du$$

$$= e^{-2t} \int_0^t e^{u} \, du \qquad = e]$$

$$= e^{-2t} \left(e^{u}\right)_0^t$$

$$= e^{-2t} \left(e^{t} - e^{0}\right)$$

$$= e^{-2t} \left(e^{t} - 1\right)$$

$$= e^{-2t} \cdot e^{t} - e^{-2t}$$

$$= e^{-t} - e^{-2t}$$

2 Using Laplace transform, solve

$y'' ... = e^{t} \sin(t)$, y

14) Find $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

Sol: Given

$$L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} \quad \left[\because L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t L^{-1}\{\bar{f}(s)\}\,dt\right]$$

$$= \int_0^t L^{-1}\left\{\frac{1}{s^2+3^2}\right\} dt$$

$$= \frac{1}{3}\int_0^t \sin 3t \, dt$$

$$= \frac{1}{3}\left(-\frac{\cos 3t}{3}\right)_0^t$$

$$= -\frac{1}{9}\left[\cos 3t - \cos(0)\right]$$

$$= -\frac{1}{9}\left[\cos 3t - 1\right]$$

$$= -\frac{\cos 3t}{9} + \frac{1}{9}$$

$$= \frac{1}{9} - \frac{\cos 3t}{9} = \frac{1-\cos 3t}{9}$$

15. Find $L^{-1}\left\{\frac{1}{s(s+2)}\right\}$

Sol: Given

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \int_0^t e^{-2t}\,dt$$

$$= \left[\frac{e^{-2t}}{-2}\right]_0^t = \left[\frac{e^{-2t} - e^{0}}{-2}\right]$$

$$= \frac{e^{-2t} - 1}{-2}$$

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\} = \frac{1 - e^{-2t}}{2}$$

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6. Find the L.T of $f(t) = t^2 \sinh t$

Sol:- $L\{t^2 \sinh t\} = L\{t^2 (\frac{e^t - e^{-t}}{2})\}$

$= \frac{1}{2}\{L(e^t t^2) - L(e^{-t} t^2)\}$

$L\{t^2\} = \frac{n!}{s^{n+1}} = \frac{2}{s^3}$

By shifting property

$L\{e^t \cdot t^2\} = \frac{2}{(s-1)^3}$

$L\{e^{-t} \cdot t^2\} = \frac{2}{(s+1)^3}$

$\therefore L\{t^2 \sinh t\} = \frac{1}{2}\{\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\}$

$= \frac{1}{2}\{\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\}$ //