LORDSINSTITUTEOFENGINEERINGANDTECHNOLOGY

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B.E-I SEMESTER, QUESTION BANK

MATHEMATICS-I 2023-24

(COMMON FOR ALL BRANCHES)

SAQ UNIT-I			
S.NO		**CO MAPPING**	**Bloom's Taxonomy Level**
1	Define limit of Sequence and give one example.	CO2	BTL1
2	Define Convergent, Divergent and Oscillatory Sequences with examples.	CO2	BTL1
3	Define Convergent Series, Divergent Series and Oscillatory Series .	CO2	BTL1
4	Test for Convergence of the Series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$	CO2	BTL4
5	Write the statement of P-Series and Give one Example.	CO2	BTL1
6	Test for Convergence of $\sum\frac{1}{2^n}$.	CO2	BTL4
7	Write the n^{th} term of the series $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+.......$	CO2	BTL2
8	Test the Convergence of $\sum\frac{2^n}{n^3}$	CO2	BTL3
9	Define D' Alembert Ratio Test.	CO2	BTL 1
10	Test the convergence of $\sum\frac{n}{3^n(n^2+2)}$.	CO2	BTL4

11	Test the convergence of $\sum(1+\frac{1}{\sqrt{n}})^{-n^{\frac{3}{2}}}$	CO2	BTL4
12	Test the Convergence of the following Series $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+$............	CO2	BTL4
13	Test the Convergence of Series $\sum\frac{cosn\pi}{(n^2+1)}$	CO2	BTL4
14	Test the convergence of the Series $\sum\frac{n!2^n}{n^n}$.	CO2	BTL4
15	Test for convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2^n+3}{3^n+1}\right)^{1/2}$	CO2	BTL4

LAQ UNIT - I			
1	Test for convergence of the series $\sum\sqrt{n^4+1}-n^2$.	CO2	BTL 4
2	Test for Convergence of the Series $\frac{1}{4.7.10}+\frac{4}{7.10.13}+\frac{9}{10.13.16}+$............	CO2	BTL 4
3	Test for convergence of the series $\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+--------$.	CO2	BTL5
4	Test for Convergence of the Series $\sum\frac{4.7..........(3n+1)}{1.2.3.........n}x^n$	CO2	BTL5
5	Find whether the following Series is Convergent or Divergent. $\sum_{n=1}^{\infty}\frac{\left[(n+1)x\right]^n}{n^{n+1}}$	CO2	BTL5
6	Test for Convergence of the Series $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n.3^n}$	CO2	BTL 4
7	Test for Convergence of the following Series $\frac{3.6.9......3n}{7.10.13.....(3n+4)}x^n$	CO2	BTL 4

8	Examine the absolute or conditional convergence of $3\left(\frac{1}{2}\right)-4\left(\frac{1}{2}\right)^2+5\left(\frac{1}{2}\right)^3-6\left(\frac{1}{2}\right)^4+---$.	CO2	BTL5
9	Test for absolute or conditional Convergence of the Series $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+-$, $0<x<1$.	CO2	BTL 4
10	Discuss the Convergence of the Series $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots\ldots\ldots\ldots$	CO2	BTL 5
11	Test for convergence of the series $\sum\frac{1.2.3\ldots\ldots\ldots n}{3.5.7\ldots\ldots(2n+1)}$	CO2	BTL4
12	Test for convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$	CO2	BTL4
13	Examine the following series for absolute or conditional convergence $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$	CO2	BTL4
14	Test for convergence of the series $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\ldots\ldots$	CO2	BTL4
15	Test for convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$	CO2	BTL4

unit 1

sequences & series.

1Q Define limit of a sequence and give one example

Ans Let $\{u_n\}$ be a sequence and $K \in R$, K is said to be limit of a sequence $\lim_{n \to \infty} u_n = k$.

$$Ex: \quad u_n = \frac{n^2+1}{2n^2+3}, \quad u_n = \frac{n+1}{2n+4}$$

2Q Define convergent, divergent and oscillatory sequence with example

Ans convergent sequence If $\lim_{n \to \infty} u_n = k$ then we say $\{u_n\}$ converges to k

$$\lim_{n \to \infty} \frac{1}{n^2} = 0 = \text{finite}$$

divergent sequence - A sequence is said to be divergent if its limit is infinite $\lim_{n \to \infty} u_n = \infty$, $\lim_{n \to \infty} n^2 = \infty$

oscillatory sequence - A sequence $\{u_n\}$ is said to be oscillatory if $\lim_{n \to \infty} u_n$ is not unique ex: $\lim_{n \to \infty} u_n = \left.\begin{matrix} +1 \\ -1 \end{matrix}\right\} \left\{\begin{matrix} \text{even} \\ \text{odd} \end{matrix}\right.$

3Q Define convergent series, divergent series, oscillatory series.

Ans Convergent series :- A series Σu_n is said to be convergent if

$\lim_{n\to\infty} u_n$ = finite, where u_n is sum of first 'n' terms of the series.

divergent series : A series Σu_n is said to be divergent if $\lim_{n\to\infty} S_n = +\infty$

Oscillatorty series.- A series Σu_n is said to be oscillatory if $\lim_{n\to\infty} S_n$ or $\lim_{n\to\infty} u_n$ is finite but not unique.

4Q Test for convergence of the series. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Ans Power of 'n' in Nr & Dr is different ∴ Apply comparision test.

step ① assume $V_n = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{q-p}}$

$\Rightarrow V_n = \frac{1}{n^{3/2}}$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0} \frac{\frac{\sqrt{n}}{n^2+1}}{1/n^{3/2}}$$

$$\lim_{\frac{1}{n}\to 0} \frac{\sqrt{n}}{n^2+1} \times \frac{n^{3/2}}{1}$$

$$= \lim_{\frac{1}{n}\to 0} \frac{\frac{\sqrt{n}}{\sqrt{n}} \times \frac{n\sqrt{n}}{n\sqrt{n}}}{\frac{n^2+1}{n^2}}$$

$$= \frac{1}{1+\frac{1}{n^2}} = \lim_{\frac{1}{n}\to 0} \frac{1}{1+0} = 1$$

$= 1 =$ finite.

p- test on $v_n = \frac{1}{n^{3/2}}$

$\frac{3}{2} > 1 \Rightarrow v_n$ is Convergent

by comparision Σu_n & Σv_n are Converge together or diverge together

$\therefore \Rightarrow u_n$ is also convergent.

$\boxed{5Q}$ write the statement of p-series and give example

$\boxed{Sol^n}$ Let $\Sigma u_n = \Sigma \frac{1}{n^p}$ be a sequence of +ve terms.

then $p>1 \Rightarrow u_n$ is convergent

$\Rightarrow p \doteq 1 \Rightarrow u_n$ is divergent

$$\Sigma \frac{1}{n^p} = 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots$$

$\boxed{6Q}$ Test for convergence of $\Sigma \frac{1}{2^n}$.

Sol^n $\Sigma \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

It is a geometric series.

$r = \frac{1}{2}$ = common ratio.

$r = \frac{1}{2} < 1 \Rightarrow u_n$ is convergent

$\boxed{7Q}$ write the n^{th} term of the series

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots$$

Sol^n n^{th} term $= \dfrac{x^n}{n(n+1)}$

$a+(n-1)d = n^{th}$ term.

8Q Test the convergence of $\Sigma \frac{2^n}{n^3}$.

Sol^n $u_n = \frac{2^n}{n^3}$, $(u_n)^{1/n} = \left(\frac{2^n}{n^3}\right)^{1/n}$

$$= \frac{2}{n^{3/n}}$$

$$\lim_{\frac{1}{n} \to 0} (u_n)^{1/n} = \lim_{\frac{1}{n} \to 0} \frac{2}{n^{3/n}} = \frac{2}{n^0} = 2$$

$2 > 1 \Rightarrow$ divergent

9Q Define D'Alembert Ratio Test

Sol^n If Σu_n is a series of +ve terms

then $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = k$,

$k > 1 \Rightarrow$ convergent

$k < 1 \Rightarrow$ divergent

$k = 1 \Rightarrow$ test fails.

$\boxed{10\,Q}$ Test the convergence of $\sum \frac{n}{3^n(n^2+2)}$

$\boxed{Sol^n}$

$$u_n = \frac{n}{3^n(n^2+2)}.$$

$$u_{n+1} = \frac{n+1}{3^{n+1}(n^2+1+2n+2)}.$$

$$= \frac{n+1}{3^n.3(n^2+2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n}{3^n(n^2+2)} \times \frac{3^n.3(n^2+2n+3)}{(n+1)}$$

$$\lim_{\frac{1}{n}\to 0} \frac{n/n \cdot 3\left(\frac{n^2+2n+3}{n^2}\right)}{\left(\frac{n^2+2}{n^2}\right)\left(\frac{n+1}{n}\right)}$$

$$= \frac{3\cdot(1+0+0)}{(1+0)(1+0)} = 3>1$$

$3>1 \Rightarrow$ convergent

by D Alemberts ratio test

11 Q Test the convergence of $\Sigma \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

Sol^n

$$u_n = \left[1+\frac{1}{\sqrt{n}}\right]^{-n\sqrt{n}}.$$

$$(u_n)^{1/n} = \left\{\left[1+\frac{1}{\sqrt{n}}\right]^{-n\sqrt{n}}\right\}^{1/n}.$$

$$(u_n)^{1/n} = \left[1+\frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}.$$

$$= \left\{\left[1+\frac{1}{\sqrt{n}}\right]^{+\sqrt{n}}\right\}^{-1} = \frac{1}{e} < 1$$

$\Rightarrow$ convergent

$$\because \left[1+\frac{1}{n}\right]^{n} = e$$

12 Q Test the convergence of the following series $\frac{3}{1} + \frac{4}{8} + \frac{5}{27} + \frac{6}{64} + \ldots$

Sol^n $\quad u_n = \dfrac{n+2}{n^3}, \quad V_n = \dfrac{1}{n^{3-1}}$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{V_n} = \lim_{\frac{1}{n}\to 0} \frac{n+2}{n^3} \times \frac{n^2}{1}$$

$$= \lim_{\frac{1}{n}\to 0} \frac{\frac{n+2}{n}}{\frac{n^3}{n^3}} \times \frac{n^2}{n^2}$$

$$= (1+0) = 1.$$

p- test on $V_n = \frac{1}{n^2} \Rightarrow$ Convergent

$\Rightarrow u_n$ is convergent.

$\boxed{13Q}$ Test the convergence of the series. $\sum \frac{\cos n\pi}{(n^2+1)}$.

$\boxed{Sol^n}$ $\frac{\cos n\pi}{n^2+1} = \frac{(-1)^n}{n^2+1}$

It is an alternating series
To test the alternating series we apply Leibnitz test.

$$\lim_{\frac{1}{n}\to 0} u_n = \frac{1}{n^2+1} = \lim_{\frac{1}{n}\to 0} \frac{1/n^2}{\frac{n^2+1}{n^2}} = 0$$

$u_n > u_{n+1}$

$\Rightarrow u_n - u_{n+1} \neq 0$

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \frac{\left[(n+1)^2+1\right] - n^2+1}{(n^2+1)\left[(n+1)^2+1\right]} \neq 0$$

$\Rightarrow u_n$ is convergent.

$\boxed{14Q}$ Test the Convergence of the series $\sum \frac{n!\, 2^n}{n^n}$.

$\boxed{Sol^n}$ $u_n = \frac{n!\, 2^n}{n^n}$, $u_{n+1} = \frac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}}$

$$= \frac{u_n}{u_{n+1}} = \frac{n!\, 2^n}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!\, 2^n \cdot 2}$$

$(n+1)! = (n+1)[n+1-1]!$

$\Rightarrow$

$$\frac{n!\, 2^n}{n^n} \times \frac{(n+1)^n (n+1)}{(n+1)\, n!\, 2^n \cdot 2}$$

$$= \left(\frac{n+1}{n}\right)^n = \lim_{\frac{1}{n}\to 0} \left[1+\frac{1}{n}\right]^n = e$$

$e > 1 \Rightarrow$ Convergent

$e = 2.7182$

15Q Test for convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$

Sol^n $u_n = \left\{ \frac{2^n(2^n+3)}{2^n} \times \left(\frac{3^n+1}{\cancel{3^n}}\right)^{-1} 3^n \right\}^{1/2}$

$$u_n = \left(\frac{1+\frac{3}{2^n}}{1+\frac{1}{3^n}}\right)^{1/2} \left(\frac{2^n}{3^n}\right)^{1/2}$$

$$v_n = \left(\frac{2}{3}\right)^{n/2}$$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{v_n} = \left(\frac{1+0}{1+0}\right)^{1/2} \times \left(\frac{2}{3}\right)^{n/2} \times \left(\frac{3}{2}\right)^{n/2}$$

$$= 1$$

$v_n = \left(\frac{2}{3}\right)^{n/2}$, $\quad r = \sqrt{\frac{2}{3}} < 1$

Geometric series convergent

LAQ

1Q Test for Convergence of the Series $\Sigma \sqrt{n^4+1} - n^2$

Ans $u_n = \dfrac{\sqrt{n^4+1} - n^2}{\sqrt{n^4+1} + n^2} \times \sqrt{n^4+1} + n^2$

$= \dfrac{\left(\sqrt{n^4+1}\right)^2 - (n^2)^2}{\sqrt{n^4+1} + n^2} = \dfrac{1}{\sqrt{n^4+1} + n^2}$

$V_n = \dfrac{1}{n^2}$

$\lim\limits_{\frac{1}{n} \to 0} \dfrac{u_n}{V_n} = \lim\limits_{\frac{1}{n} \to 0} \dfrac{1}{\sqrt{n^4+1} + n^2} \times \dfrac{n^2}{1}$

$\lim\limits_{\frac{1}{n} \to 0} \dfrac{n^2/n^2}{\sqrt{\dfrac{n^4+1}{n^4}} + \dfrac{n^2}{n^2}} = \dfrac{1}{\sqrt{1+0}+1} = \dfrac{1}{2}$

p-test on $V_n = \dfrac{1}{n^2}$

$P = 2 > 1 \Rightarrow$ convergent

$\Rightarrow$ by comparision. Test

u_n is convergent

$\boxed{2Q}$ Test the convergence of the series

$$\frac{1}{4\cdot 7\cdot 10}+\frac{4}{7\cdot 10\cdot 13}+\frac{9}{10\cdot 13\cdot 16}+\cdots$$

$4, 7, 10 \ldots$

$\underline{\underline{Sol^n}}$ $\quad u_n = \frac{1}{(3n+1)(3n+4)(3n+7)}$

$a+(n-1)d$

$4+(n-1)3$

$4+3n-3$

$3n+1$

power of "n" in Nr & Dr is different, we apply comparision test

$7, 10, 13, \ldots$

$7+(n-1)3$

$7+3n-3$

$\underline{3n+4}$

$$V_n = \frac{1}{n^{3-0}} = \frac{1}{n^3}.$$

$10, 13, 16, \ldots$

$10+(n-1)3$

$10+3n-3$

$\underline{3n+7}$

$$\lim_{\frac{1}{n}\to 0}\frac{u_n}{V_n} = \lim_{\frac{1}{n}\to 0}\frac{1}{(3n+1)(3n+4)(3n+7)}\times\frac{n^3}{1}$$

$$\lim_{\frac{1}{n}\to 0}\frac{n^3/n^3}{\left(\frac{3n+1}{n}\right)\left(\frac{3n+4}{n}\right)\left(\frac{3n+7}{n}\right)}$$

$$= \frac{1}{(3+0)(3+0)(3+0)} = \frac{1}{27}$$

p-test on $V_n = \frac{1}{n^3}$

$p = 3 > 1 \Rightarrow$ Convergent

$\Rightarrow u_n$ is convergent.

$\boxed{3Q}$ Test for convergence of the Series

$$\frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots$$

$\boxed{Sol^n}$

$$u_n = \frac{1.3.5 \cdots (2n-1)x^n}{2.4.6 \cdots 2n}$$

$$u_{n+1} = \frac{1.3.5 \cdots (2n-1)[2(n+1)-1]\,x^{n+1}}{2.4.6 \cdots 2n\,2[n+1]}$$

$$u_{n+1} = \frac{1.3.5 \cdots (2n-1)[2n+1]x^n.x}{2.4.6 \cdots 2n(2n+2)}$$

$$\frac{u_n}{u_{n+1}} = \frac{1.3.5 \cdots (2n-1)x^n}{2.4 \cdots 2n} \times \frac{2.4.6 \cdots 2n(2n+2)}{1.3.5 \cdots (2n-1)(2n+1)x^n.x}$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)\frac{1}{x},$$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{u_{n+1}} = \lim_{\frac{1}{n}\to 0} \frac{1}{x}\left[\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}}\right]$$

$$= \lim_{\frac{1}{n}\to 0} \frac{1}{x}\left[\cancel{\frac{2+\frac{2}{n}}{2+\frac{x}{n}}}\right]$$

$$\lim_{\frac{1}{n}\to 0} \Rightarrow \frac{1}{x}\left[\frac{2+\frac{2}{n}}{2+1/n}\right] = \frac{1}{x}\left[\frac{2+0}{2+0}\right.$$

$= \frac{1}{x} > 1 \Rightarrow$ convergent

$\frac{1}{x} < 1 \Rightarrow$ divergent

$\frac{1}{x} = 1 \Rightarrow$ test fails.

To solve further put $x = 1$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{\frac{1}{n} \to 0} \frac{1}{1}\left(\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}}\right) \Rightarrow \lim_{\frac{1}{n} \to 0} \left[\frac{2 + \frac{2}{n}}{2 + \frac{1}{n}}\right] = 1.$$

test fails.

Apply Rabee's test.

$$\lim_{\frac{1}{n} \to 0} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\lim_{\frac{1}{n} \to 0} n\left[\frac{2n+2}{2n+1} - 1\right]$$

$$\lim_{\frac{1}{n} \to 0} n\left[\frac{(2n+2) - (2n+1)}{2n+1}\right]$$

$$\lim_{\frac{1}{n} \to 0} n\left[\frac{1}{2n+1}\right] \Rightarrow \lim_{\frac{1}{n} \to 0} \frac{n}{n}\left[\frac{1}{\frac{2n+1}{n}}\right].$$

$$= \frac{1}{2} < 1$$

$\Rightarrow$ divergent

$\boxed{4\ Q}$ Test for convergence of the series. $\sum \frac{4 \cdot 7 \cdots (3n+1)}{1 \cdot 2 \cdots n} x^n$.

$\boxed{Sol^n}$ $$u_n = \frac{4 \cdot 7 \cdots (3n+1) x^n}{1 \cdot 2 \cdots n}$$

$$u_{n+1} = \frac{4 \cdot 7 \cdots (3n+1)[3(n+1)+1] x^{n+1}}{1 \cdot 2 \cdots n(n+1)}$$

$$u_{n+1} = \frac{4 \cdot 7 \cdots (3n+1)(3n+4) x^n \cdot x}{1 \cdot 2 \cdots n(n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4 \cdot 7 \cdots (3n+1) x^n}{1 \cdot 2 \cdots n} \times \frac{1 \cdot 2 \cdots n(n+1)}{4 \cdot 7 \cdots (3n+1)(3n+4) x^n \cdot x}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)}{(3n+4)x}$$

$$\lim_{\frac{1}{n} \to 0} \frac{1}{x}\left[\frac{\frac{n+1}{n}}{\frac{3n+4}{n}}\right] = \lim_{\frac{1}{n} \to 0} \frac{1}{x}\left[\frac{1+\frac{1}{n}}{3+\frac{4}{n}}\right]$$

$$= \frac{1}{3x} > 1 \Rightarrow \text{convergent}$$

$$= \frac{1}{3x} < 1 \Rightarrow \text{divergent}$$

$$= \frac{1}{3x} = 1 \Rightarrow \text{test fail}$$

To solve further

$$\frac{1}{3x} = 1 \implies 3x = 1$$

$$x = \frac{1}{3}$$

Put $x = \frac{1}{3}$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{\frac{1}{n} \to 0} \left(\frac{\frac{n+1}{n}}{\frac{3n+4}{n}} \right) \frac{1}{1/3} = 3 \lim_{\frac{1}{n} \to 0} \left[\frac{1 + \frac{1}{n}}{3 + \frac{4}{n}} \right]$$

$$= 3 \left[\frac{1+0}{3+0} \right] = 1. \text{ test fails.}$$

Apply Rabee's.

$$\lim_{\frac{1}{n} \to 0} n \left[\frac{3(n+1)}{3n+4} - 1 \right]$$

$$\lim_{\frac{1}{n} \to 0} n \left[\frac{3n+3-3n-4}{3n+4} \right]$$

$$\lim_{\frac{1}{n} \to 0} \frac{n}{n} \left[\frac{-1}{\frac{3n+4}{n}} \right]$$

$$\lim_{\frac{1}{n} \to 0} \left[\frac{-1}{3 + \frac{4}{n}} \right] = \frac{-1}{3} < 1$$

$\Rightarrow u_n$ is divergent

58) find whether the following series is convergent or divergent $\sum_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^{n+1}}$

Sol^n

$$u_n = \frac{[(n+1)x]^n}{n^n \cdot n^1}$$

$$(u_n)^{1/n} = \frac{\left(\{[n+1]x\}^n\right)^{1/n}}{n^{\frac{n+1}{n}}}$$

$$(u_n)^{1/n} = \frac{[n+1]x}{n^{1+\frac{1}{n}}} = \frac{[n+1]x}{n \cdot n^{1/n}}$$

$$\lim_{\frac{1}{n} \to 0} \left[\frac{n+1}{n}\right] x \cdot \left(\frac{1}{n}\right)^{1/n} \qquad \because \lim_{\frac{1}{n}\to 0} \left(\frac{1}{n}\right)^{1/n} = 1$$

$$\lim_{\frac{1}{n} \to 0} \left[1 + \frac{1}{n}\right] x \cdot 1$$

$= x > 1 \Rightarrow$ divergent

$x < 1 \Rightarrow$ convergent

To solve further put $x = 1$ in u_n

$$u_n = \frac{[(n+1)1]^n}{n^{n+1}}$$

Power of "n" in Nr & Dr is different we apply comparision test-

$$V_n = \frac{1}{\frac{n+1-n}{n}} = \frac{1}{n}.$$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{V_n} = \lim_{\frac{1}{n}\to 0} \left[\frac{(n+1)^n}{n^n \cdot n} \times n\right]$$

$$= \lim_{\frac{1}{n}\to 0} \left[\frac{n+1}{n}\right]^n$$

$$= \lim_{\frac{1}{n}\to 0} \left[1+\frac{1}{n}\right]^n = e = \text{finite}$$

p-test on $V_n = \frac{1}{n}$

$p=1 \Rightarrow V_n$ is convergent

$\Rightarrow$ by comparision test u_n is also convergent.

6Q Test for convergence of the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n3^n}$.

Sol^n

$$u_n = \frac{x^{n-1}}{n3^n}, \quad u_{n+1} = \frac{x^{n+1-1}}{(n+1)3^{n+1}}$$

$$u_{n+1} = \frac{x^n}{(n+1)3^n \cdot 3}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{3^n \cdot n} \times \frac{(n+1)\,3^n \cdot 3}{x^n}$$

$$= \frac{x^{n-1-n}\;3(n+1)}{n}$$

$$= \frac{3}{x}\left[\frac{n+1}{n}\right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{3}{x}\left[1+\frac{1}{n}\right] = \frac{3}{x}\left[1+0\right].$$

$= \frac{3}{x} > 1$ convergent

$= \frac{3}{x} < 1 \Rightarrow$ divergent

$\frac{3}{x} = 1 \Rightarrow$ test fails.

To solve further $\frac{3}{x} = 1$

$x = 3$ · put in u_n.

$$u_n = \frac{x^{n-1}}{3^n \cdot n} = \frac{3^{n-1}}{3^n \cdot n} = \frac{3^n \cdot 3^{-1}}{3^n \cdot n}$$

$= \frac{1}{3n}$, assume $V_n = \frac{1}{n}$

$$\lim_{\frac{1}{n}\to 0} \frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0} \frac{1}{3n} \times \frac{n}{1} = \frac{1}{3}.$$

p-test on V_n.

$V_n = \frac{1}{n}$, $p = 1 \Rightarrow$ divergent

$\therefore$ By comparision test

u_n is convergent

$\boxed{70}$ Test the convergence of the following series $\frac{3 \cdot 6 \cdot 9 \dots 3n\, x^n}{7 \cdot 10 \dots (3n+4)}$.

$\boxed{Sol^n}$

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n \cdot (3n+3)\, x^{n+1}}{7 \cdot 10 \dots (3n+4)[3(n+1)+4]}$$

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n(3n+3)\, x^n \cdot x}{7 \cdot 10 \dots (3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{3 \cdot 6 \cdot 9 \dots 3n\, x^n}{7 \cdot 10 \dots (3n+4)} \times \frac{7 \cdot 10 \dots (3n+4)(3n+7)}{3 \cdot 6 \cdot 10 \dots 3n(3n+3)\, x^n \cdot x}$$

$$= \frac{(3n+7)}{(3n+3)x} \Rightarrow \lim_{\frac{1}{n}\to 0} \frac{1}{x} \left[\frac{\frac{3n+7}{n}}{\frac{3n+3}{n}} \right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{x}\left[\frac{3+\frac{7}{n}}{3+\frac{3}{n}}\right] = \frac{1}{x}\left[\frac{3+0}{3+0}\right] = \frac{1}{x}$$

$\frac{1}{x} > 1 \Rightarrow$ convergent

$\frac{1}{x} < 1 \Rightarrow$ divergent

$\frac{1}{x} = 1 \Rightarrow$ test fails.

put $x = 1$ in $\frac{u_n}{u_{n+1}}$

$$= \lim_{\frac{1}{n}\to 0} \frac{1}{1}\left[\frac{3+\frac{7}{n}}{3+\frac{3}{n}}\right] = 1. \text{ test fails}$$

Apply Rabee's test

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{3n+7}{3n+3} - 1\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{(3n+7)-(3n+3)}{3n+3}\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{4}{3n+3}\right] \Rightarrow \lim_{\frac{1}{n}\to 0} \frac{n}{n}\left[\frac{4}{\frac{3n+3}{n}}\right]$$

$$\frac{4}{3+0} = \frac{4}{3} > 1 \Rightarrow \text{Convergent}$$

8Q Examine the absolute or conditional convergence of

$$3\tfrac{1}{2} - 4\left(\tfrac{1}{2}\right)^2 + 5\left(\tfrac{1}{2}\right)^3 - 6\left(\tfrac{1}{2}\right)^4 + \dots$$

Sol^n

$$u_n = \frac{n+2}{2^n}$$

$a+(n-1)d$

$3+(n-1)1$

It is an alternating series.

$$u_{n+1} = \frac{n+3}{2^{n+1}}$$

$$\lim_{\frac{1}{n}\to 0} (-1)^{n-1} \frac{(n+2)}{2^n} = 0$$

$$u_n > u_{n+1}$$

$$\Rightarrow u_n - u_{n+1} \neq 0$$

$$\frac{n+2}{2^n} - \frac{n+3}{2^n \cdot 2} \neq 0$$

$\Rightarrow$ Convergent.

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{n+2}{2^n} \times \frac{2^n \cdot 2}{(n+3)}$$

$$\lim_{\frac{1}{n}\to 0} 2\left[\frac{n+2}{n+3}\right] = \lim_{\frac{1}{n}\to 0} 2\left[\frac{\frac{n+2}{n}}{\frac{n+3}{n}}\right]$$

$$= \lim_{\frac{1}{n}\to 0} 2\left[\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right] = 2$$

$2 > 1 \Rightarrow$ convergent

u_n is convergent

$|u_n|$ is also convergent

absolute convergent.

9Q Test the convergence of the series $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \frac{x^4}{1+x^4} + \dots$

Sol^n It is an alternating series

$$u_n = \frac{(-1)^n x^n}{1+x^n}, \quad u_n = \frac{x^n}{1+x^n}$$

$$u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n(1+x^{n+1}) - (1+x^n)(x^{n+1})}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{\cancel{x^n} + x^{2n+n} - x^{n+1} - x^{2n+\cancel{n}}}{(1+x^n)(1+x^{n+1})} \neq 0.$$

$$\lim_{\frac{1}{n} \to 0} \frac{x^n}{1+x^n} = 0.$$

$\Rightarrow$ convergent

$\boxed{10g}$ Discuss the convergence of

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

$\boxed{Sol^n}$ $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}, \quad \begin{aligned} &a+(n-1)d \\ &0+(n-1)2 \\ &= 2n-2 \end{aligned}$$

$$u_{n+1} = \frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}.$$

$$= \frac{1}{x^2}\left[\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}\right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{x^2}\left[\frac{\frac{n+2}{n}\sqrt{\frac{n+1}{n}}}{\left(\frac{n+1}{n}\right)\sqrt{\frac{n}{n}}}\right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{x^2}\left[\frac{\left(1+\frac{2}{n}\right)\left(\sqrt{1+\frac{1}{n}}\right)}{\left(1+\frac{1}{n}\right)(1)}\right]$$

$$= \frac{1}{x^2}\frac{(1+0)\sqrt{1+0}}{(1+0)(1)} = \frac{1}{x^2}$$

$\frac{1}{x^2} > 1 \Rightarrow$ Convergent

$\frac{1}{x^2} < 1 \Rightarrow$ divergent

$\frac{1}{x^2} = 1 \Rightarrow$ test fails

$\frac{1}{x^2} = 1 \Rightarrow$ to solve further

put $x = 1$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{\frac{1}{n}\to 0}\frac{1}{1}\left[\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}\right]$$

$$= \frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}}{\left(1+\frac{1}{n}\right)\sqrt{\frac{n}{n}}} =$$

put $x = 1$ in u_n

$$u_n = \frac{1}{1^2} \frac{1}{(n+1)\sqrt{n}},$$

assume $V_n = \frac{1}{n\sqrt{n}}$.

$$\lim_{\frac{1}{n} \to 0} \frac{u_n}{V_n} = \lim_{\frac{1}{n} \to 0} \frac{1}{(n+1)\sqrt{n}} \times n\sqrt{n}$$

$$= \lim_{\frac{1}{n} \to 0} \left[\frac{\frac{n\sqrt{n}}{n\sqrt{n}}}{\left(\frac{n+1}{n}\right)\sqrt{\frac{n}{n}}} \right] = 1.$$

$$V_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

$\frac{3}{2} > 1 \Rightarrow$ convergent

$\Rightarrow$ by comparision u_n is also convergent

11Q Test for convergence of the series $\sum \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}$

Sol^n $u_{n+1} = \frac{1\cdot 2\cdot 3 \cdots n(n+1)}{3\cdot 5\cdot 7 \cdots (2n+1)[2(n+1)+1]}$

$$\frac{u_n}{u_{n+1}} = \frac{1\cdot 3 \cdots n}{3\cdot 5 \cdots (2n+1)} \times \frac{3\cdot 5 \cdots (2n+1)(2n+3)}{1\cdot 2 \cdots n(n+1)}$$

$$= \frac{2n+3}{n+1}$$

$$\lim_{\frac{1}{n}\to 0} \frac{\frac{2n+3}{n}}{\frac{n+1}{n}} = \frac{2+\frac{3}{n}}{1+\frac{1}{n}} = 2$$

$$\frac{u_n}{u_{n+1}} = 2 > 1 \Rightarrow \text{Convergent}$$

12Q Test the convergence of the series. $\sum_{n=1}^{\infty} (-1)^{n+1}\left[\sqrt{n+1} - \sqrt{n}\right]$

$$u_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}.$$

$$u_n = \frac{(\sqrt{n+1} - \sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}}$$

$$u_n = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$v_n = \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \times \frac{\sqrt{n}}{1}$$

$$\lim_{\frac{1}{n} \to 0} \left[\frac{\sqrt{n}/\sqrt{n}}{\sqrt{\frac{n+1}{n}} + \sqrt{\frac{n}{n}}} \right]$$

$$= \frac{1}{\sqrt{1+0} + \sqrt{1}} = \frac{1}{2}$$

$$v_n = \frac{1}{\sqrt{n}} \quad P = \frac{1}{2}$$

divergent

by C.T

$$|u_n| = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

is convergent

$$\lim_{\frac{1}{n} \to 0} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

$$u_n - u_{n+1} \neq 0$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{\sqrt{n+2} + \sqrt{n+1}} \neq 0$$

$$\frac{\sqrt{n+2} + \cancel{\sqrt{n+1}} - \cancel{\sqrt{n+1}} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+2} + \sqrt{n+1}} \neq 0$$

$\Rightarrow u_n$ is absolute convergent

$\boxed{13Q}$ Examine the following series for absolute or conditional convergence

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$$

$$u_n = \frac{n}{n^2+1}$$

$$\lim_{\frac{1}{n}\to 0} \frac{n/n^2}{\frac{n^2+1}{n^2}} = \frac{1/n}{1+\frac{1}{n^2}} = 0$$

$$u_n - u_{n+1} \neq 0$$

$$\frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} \neq 0$$

$$\frac{n(n^2+2n+2) - (n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)} \neq 0$$

u_n is convergent

$$|u_n| = \frac{n}{n^2+1}, \quad V_n = \frac{1}{n}$$

$$\frac{u_n}{\cancel{u_n}} = \frac{u_n}{V_n} = \frac{n}{n^2+1} \times \frac{n}{1}$$

$$\lim_{\frac{1}{n}\to 0} \frac{n^2}{n^2+1} = \lim_{\frac{1}{n}\to 0} \frac{n^2/n^2}{\frac{n^2+1}{n^2}}$$

$$= \frac{1}{1+\frac{1}{n^2}} = 1$$

$$V_n = \frac{1}{n}, \quad P = 1$$

$\Rightarrow V_n$ is divergent

$\Rightarrow U_n$ is divergent

$\Rightarrow U_n$ is conditional convergent

$\boxed{14\,Q}$ Test for convergence of the series.

$$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \cdots$$

$$U_n = \frac{x^n}{(2n-1)2n}$$

$$U_{n+1} = \frac{x^{n+1}}{[2(n+1)-1][2(n+1)]}$$

$$U_{n+1} = \frac{x^n \cdot x}{(2n+1)(2n+2)}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^n}{(2n-1)(2n)} \times \frac{(2n+1)(2n+2)}{x^n \cdot x}$$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{x}\left[\frac{(2n+1)(2n+2)}{(2n-1)(2n)}\right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{x}\left[\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(2-\frac{1}{n}\right)\left(\frac{2n}{n}\right)}\right] = \frac{1}{x}.$$

~~test~~ $\frac{1}{x} > 1 \Rightarrow$ convergent

$\frac{1}{x} < 1 \Rightarrow$ divergent

$\frac{1}{x} = 1 \Rightarrow$ test fails

put $\frac{1}{x} = 1 \Rightarrow x = 1.$

$$\lim_{\frac{1}{n}\to 0} \frac{1}{1}\left[\frac{2+\frac{1}{n}}{2-\frac{1}{n}} \times \frac{2+\frac{2}{n}}{\frac{2n}{n}}\right] = 1$$

Apply Rabee's.

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{(2n+1)(2n+2)}{(2n-1)(2n)} - 1\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{(2n+1)(2n+2) - (2n-1)(2n)}{(2n-1)(2n)}\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{\cancel{4n^2}+4n+2n+2-\cancel{4n^2}+2n}{4n^2-2n}\right]$$

$$\lim_{\frac{1}{n}\to 0} n\left[\frac{8n+2}{4n^2-2n}\right]$$

$$\lim_{\frac{1}{n}\to 0} \frac{n}{n}\left[\frac{\frac{8n+2}{n}}{\frac{4n^2-2n}{n^2}}\right]$$

$$= \frac{8}{4} = 2 > 1.$$

$\Rightarrow$ convergent

150 Test for Convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Sol^n

$$\lim_{\frac{1}{n}\to 0}\left[\frac{n}{3n+1}\right]^n$$

$$(u_n)^{1/n} = \left[\frac{n}{3n+1}\right]^{n/n}.$$

$$\lim_{\frac{1}{n}\to 0}\left[\frac{n/n}{\frac{3n+1}{n}}\right]$$

$$\lim_{\frac{1}{n}\to 0}\left[\frac{1}{3+\frac{1}{n}}\right]=\frac{1}{3}$$

$\frac{1}{3}<1 \Rightarrow$ convergent

unit II

SAQ

1Q Check whether Rolle's theorem can be applicable for $f(x) = |x|$ in $[-1, 1]$

Soln

$f(x) = |x|$

$f(x) = x \quad , \quad x \geq 0$

$\quad = -x \quad , \quad x < 0$

$f(x)$ exists $[-1, 1]$. $\lim_{x \to 0^-} (-x) = 0$ $\lim_{x \to 0^+} (x) = 0$

$\therefore f(x)$ is continuous in $[-1, 1]$

$$f'(x) = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x} = \underset{x \to 0^+}{Lt} \frac{x - 0}{x} = 1$$

$$f'(x) = \underset{x \to 0^-}{Lt} \frac{f(x) - f(0)}{x} = \underset{x \to 0^-}{Lt} \frac{-x - 0}{x} = -1$$

$\therefore f'(x)$ not unique

$f(x)$ is not derivable in $(-1, 1)$

Hence the theorem cannot be verified.

$\boxed{20}$ find a point at which the tangent to the curve $y = \ln x$ is parallel to the chord joining the point $(1, 0)$ and $(e, 1)$.

$\boxed{Sol^n}$ by Lagrange's mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$f(x) = \log x, \quad f'(x) = \frac{1}{x}, \quad f'(c) = \frac{1}{c}$$

$$a = 1, \quad b = e$$

$$\frac{1}{c} = \frac{\log(e) - \log(1)}{e - 1} =$$

$$\frac{1}{c} = \frac{1 - 0}{e - 1} \Rightarrow \frac{1}{c} = \frac{1}{e - 1}$$

$$c = e - 1, \quad 1 < (e - 1) < e$$

the point at which the tangent is parallel to the chord is $(e - 1)$

$\boxed{3Q}$ find the radius of curvature for the curve $y = x^2 - 6x + 10$ at $(3,1)$.

$\boxed{\text{Sol}^n}$ $y = x^2 - 6x + 10$

$$\frac{dy}{dx} = 2x - 6 = y'$$

$$= 2(3) - 6 = 0$$

$$y'' = \frac{d^2y}{dx^2} = 2$$

$$\text{radius} = \frac{[1 + (y')^2]^{3/2}}{y''}$$

$$= \frac{[1+0]^{3/2}}{2} = \frac{1}{2}$$

$\boxed{4Q}$ find the point on the curve $f(x) = x^2 - 2x$ in $[0,2]$ which the tangent is parallel to x-axis

$\underline{\text{Sol}^n}$ $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$

$f'(c) = 2c - 2$

$$2c - 2 = 0 \Rightarrow c = 1 \in (0, 2)$$

$\therefore$ The point on the curve at which the tangent is parallel to x-axis

$\boxed{5g}$ Obtain Taylor series expansion

$f(x) = \sin x$ at $x = \pi/4$

$\boxed{\text{Sol}^n}$ $f(x) = \sin x$, $f'(x) = \cos x$

$f(\frac{\pi}{4}) = \sin \pi/4$, $f'(\frac{\pi}{4}) = \cos \pi/4$

$= \frac{1}{2}$, $= 1/2$

$f''(x) = -\sin x$, $f''(\pi/4) = -\sin \pi/4$

$= -1/2$

Taylor series.

$$f(x) = f(\tfrac{\pi}{4}) + (x - \pi/4) f'(\pi/4) + \frac{(x - \pi/4)^2}{2!} f''(\pi/4) + \cdots$$

$$f(x) = \sin x = \frac{1}{\sqrt{2}} + (x - \pi/4)\frac{1}{\sqrt{2}} + \frac{(x - \pi/4)^2}{2!}(-1/\sqrt{2}) + \cdots$$

6 Q find "c" of Cauchy's mean value theorem $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ in $[a,b]$, where $0<a<b$

Soln

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$f'(x)=\frac{1}{2\sqrt{x}}$$

$$g'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1}$$

$$\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2}c^{-3/2}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$-\frac{c^{-1/2}}{c^{-3/2}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$-c^{-\frac{1}{2}+\frac{3}{2}} = -\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} \times \sqrt{ab}$$

$$c^{\frac{1}{2}} = \sqrt{ab}$$

$$\sqrt{c} = \sqrt{ab}$$

$$a < c < b$$

(7) verify Cauchy's mean value theorem for $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $(1, e)$

Sol^n

$f'(x) = \frac{1}{x}$, $g'(x) = \frac{-1}{x^2}$

$f'(c) = \frac{1}{c}$, $g'(c) = -\frac{1}{c^2}$

$f(x) = \log x$, $f(e) = \log e = 1$

$f(1) = \log 1 = 0$,

$g(x) = \frac{1}{x}$, $g(x) = \frac{1}{x}$

$g(1) = \frac{1}{1} = 1$, $g(e) = \frac{1}{e}$

$f(x)$ & $g(x)$ are continuous and derivable

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - 1}$$

$$-c = \frac{1}{\frac{1-e}{e}} = \frac{e}{-(1-e)}$$

$$c = \frac{e}{e-1} \in (1, e)$$

$\boxed{89}$ Calculate the radius of Curvature of $x^2+y^2=16$

Solⁿ

$$2x+2yy'=0$$

$$2(x+yy')=0$$

$$x=-yy'$$

$$y'=\frac{-y}{x}$$

$$y''=-\left[\frac{x(y')-y(1)}{x^2}\right]$$

$$y''=-\frac{\left[x\left(-\frac{y}{x}\right)-y\right]}{x^2}$$

$$y''=-\left[-\frac{2y}{x^2}\right]=\frac{2y}{x^2}$$

$$\therefore \rho=\frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

$$\rho=\frac{\left[1+(-y/x)^2\right]^{3/2}}{2y/x^2}$$

$$\rho = \frac{\left[(x^2+y^2)/x^2\right]^{3/2}}{\frac{2y}{x^2}}$$

$$\rho = \frac{(x^2+y^2)^{3/2}}{(x^2)^{3/2}} \times \frac{x^2}{2y}$$

$$\rho = \frac{(x^2+y^2)^{3/2}}{x^3} \cdot \left(\frac{x^2}{2y}\right)$$

$$\rho = \frac{(x^2+y^2)^{3/2}}{2xy}$$

9Q use Taylor series to express the polynomial $2x^3+7x^2+x-6$ in Powers of $(x-2)$.

Soln $f(x) = 2x^3+7x^2+x-6$

$f(2) = 2(2^3) + 7(2^2) + 2 - 6$

$= 16 + 28 + 2 - 6 = 40$

$f'(x) = 6x^2 + 14x + 1$

$f'(2) = 6(2^2) + 14(2) + 1$

$f'(2) = 24 + 28 + 1 = 53$

$f''(x) = 12x + 14$

$f''(2) = 12(2) + 14 = 24 + 14 = 38$

$f'''(x) = 12.$

Taylor series

$$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} + \cdots$$

$$= 40 + (x-2).(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-3)^3}{3!}(12) + \cdots$$

$\boxed{10}$ find the value of "c" by using generalized mean value theorem for $f(x) = e^{x}$, $g(x) = e^{-x}$ in $[3, 7]$.

$\boxed{\text{Sol}^n}$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

$$\frac{e^{c}}{-e^{-c}} = \frac{e^{7} - e^{3}}{e^{-7} - e^{-3}}.$$

$$-e^{c+c} = \frac{e^{7} - e^{3}}{\frac{1}{e^{7}} - \frac{1}{e^{3}}}$$

$$-e^{2c} = \frac{e^{7} - e^{3}}{\frac{e^{3} - e^{7}}{e^{7+3}}}$$

$$-e^{2c} = -\left[\frac{e^{7} - e^{3}}{e^{7} - e^{3}}\right] \times e^{10}$$

$$2c = 10$$

$$c = 5$$

11Q. State Cauchy's mean value theorem

Statement:- Let $f: [a,b] \to R$, $g: [a,b] \to R$

(i) $f(x)$ and $g(x)$ are continuous $[a,b]$

(ii) $f(x)$ and $g(x)$ are derivable (a,b)

then $\exists$ a point $c \in (a,b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

12Q. Write the expression for Taylor's series and Maclaurin series.

Statement $f: [a,b] \to R$ having partial derivatives.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Maclaurin series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

13Q using Rolle's mean value theorem of $g(x) = 8x^3 - 6x^2 - 2x + 1$, find the value of "c" defined over $(0,1)$

Sol^n $g'(c) = 24c^2 - 12c - 2$

$g'(c) = 0$

$24c^2 - 12c - 2 = 0$

$c = \frac{3 \pm \sqrt{21}}{12} = 0.6318 \in (0,1)$

$g(0) = g(1)$

$1 = 8 - 6 - 2 + 1 = 1$

$g(x)$ is continous and $g(x)$ is derivable.

14Q verify Lagrange's mean value theorem for $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$.

Sol^n $f(0) = (0-1)(0-2)(0-3) = -6$

$f(4) = (4-1)(4-2)(4-3) = 6$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(c) = 3c^2 - 12c + 11$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$3c^2 - 12c + 11 = \frac{6+6}{4-0} = \frac{12}{4} = 3$$

$$3c^2 - 12c + 11 = 3.$$

$$3c^2 - 12 + 8 = 0.$$

$$c = \frac{12 \pm \sqrt{144-96}}{6} = \frac{6 \pm 2\sqrt{3}}{3} \in (0,4)$$

150 find the coordinates of the centre of curvature at any point of the rectangular hyperbola $xy = c^2$

Sol^n $xy = c^2$, diff y w.r.to x

$xy' + y(1) = 0 \Rightarrow y' = -\frac{y}{x}$

$y'' = -\left[\frac{xy' - y(1)}{x^2}\right] = -\left[\frac{x(-\frac{y}{x}) - y}{x^2}\right]$

$y'' = -\frac{[-2y]}{x^2} = \frac{2y}{x^2}$

$X = x - \frac{y'[1+(y')^2]}{y''}$,

$X = x - \frac{-\frac{y}{x}\left[1 + \frac{y^2}{x^2}\right]}{\frac{2y}{x^2}}$

$X = x + \frac{\cancel{y}}{x}\left[\frac{x^2+y^2/\cancel{x^2}}{2\cancel{y}/\cancel{x^2}}\right]$

$X = x + \frac{(x^2+y^2)}{2x} \Rightarrow X = \frac{3x^2+y^2}{2x}$

$$Y = y + \frac{[1+(y')^2]}{y''}$$

$$Y = y + \frac{[1+(-y/x)^2]}{\frac{2y}{x^2}}$$

$$Y = y + \frac{[x^2+y^2]/x^2}{2y/x^2}$$

$$Y = \frac{3y^2+x^2}{2y}.$$

(X, Y) is coordinates of Centre of Curvature.

LAQ

1Q State and prove Rolle's Theorem.

Statement:- Let $f : [a,b] \to R$.

(i) $f(x)$ is continuous over $[a,b]$

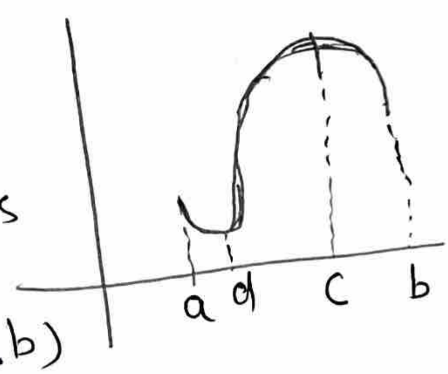
(ii) $f(x)$ is derivable over (a,b)

(iii) $f(a) = f(b)$ then $\exists$ a point $c \in (a,b) \ni f'(c) = 0$

Sol^n $f(x)$ is continuous over $[a,b]$

$\Rightarrow f(x)$ is bounded.

$\because$ Every continuous function is bounded.



It attains its bounds. (gLb, LUb)

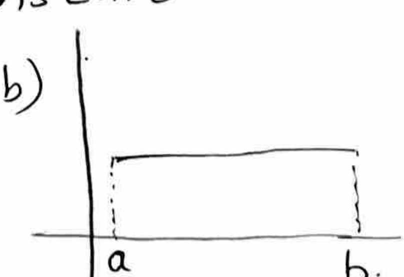
gLb = greatest Lower bound.

LUb = least upper bound.

Case (i) If $f(x)$ is constant

$\exists$ a point $c \in (a,b)$

$f'(c) = 0$



Case (ii) $f(x)$ is not constant

It has lower bound and upper bound.

Let $f(c)$ = maximum value $= M$.

$$f(c+h) \leq M \text{ for } h>0.$$

$$f(c+h) - f(c) \leq 0$$

$$M - M \leq 0.$$

If $h>0$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0$$

If $h<0$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0$$

$$f'(c) = 0$$

$\boxed{2Q}$ If $a<b$ prove that

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}.$$

hence deduce

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

$\boxed{\text{Sol}^n}$ by Lagrange's mean value theorem we have.

$$f'(c) = \frac{f(b) - f(a)}{b-a}. \quad \text{—— I}$$

Let $f(x) = \tan^{-1}x.$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(c) = \frac{1}{1+c^2}$$

Sub in —— I

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

given $\quad a < c < b.$

$$a^2 < c^2 < b^2$$

$$(1+a^2) < (1+c^2) < (1+b^2)$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}.$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}b - \tan^{-1}a > \frac{b-a}{1+a^2}$$

Let $a = \cancel{}1,\quad b = \frac{4}{3},\quad a = 1$

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} > \tan^{-1}\frac{4}{3} - \tan^{-1}1 > \frac{\frac{4}{3}-1}{1+1^2}$$

$$\frac{1}{3} \times \frac{9}{25} > \tan^{-1}\frac{4}{3} - \frac{\pi}{4} > \frac{1}{3(2)}$$

$$\frac{3}{25} + \frac{\pi}{4} > \tan^{-1}\frac{4}{3} > \frac{\pi}{4} + \frac{1}{6}$$

$\boxed{3Q}$ State and prove Lagrange's Mean value Theorem

$\boxed{\text{Sol}^n}$ statement: Let $f : [a,b] \longrightarrow R$

(i) $f(x)$ is continuous over $[a,b]$

(ii) $f(x)$ is derivable over (a,b)

$\exists$ a point $c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Proof Assume $g(x) = f(x) + Ax$

To check $g(x)$ satisfies all conditions of Rolle's Theorem.

(i) $g(x) = f(x) + Ax$

is a polynomial, every polynomial is continuous.

$\therefore$ $g(x)$ is continuous over $[a,b]$.

(ii) $g'(x) = f'(x) + A(1)$

$g'(x)$ exist

$\therefore$ $g(x)$ is derivable over (a,b).

(iii) $g(a) = g(b)$

$f(a) + Aa = f(b) + Ab$.

$$f(b) - f(a) = Aa - Ab.$$

$$f(b) - f(a) = -A(b-a)$$

$$\frac{f(b)-f(a)}{b-a} = -A$$

$g(x)$ satisfied all conditions of Rolle's theorem

$\therefore \exists$ a point $c \in (a,b) \ni g'(c) = 0$

$$g'(c) = f'(c) + A = 0$$
$$f'(c) = -A$$
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$\boxed{40}$ verify Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$

$\boxed{\text{Sol}^n}$ $f(x)$ is continuous. $[0, \pi]$

e^x is continuous.

$\sin x$ is continuous.

$f'(x) = e^x \sin x + e^x \cos x$

$f'(x)$ exist

$\therefore f'(x)$ is derivable. $(0, \pi)$

$f(0) = f(\pi)$

$e^0 \sin 0 = e^\pi \sin \pi \quad \because \sin \pi = 0$

$0 = 0$

$f(x)$ satisfied all conditions of Rolle's theorem $\exists$ a point

$c \in (a, b) \ni f'(c) = 0$

$e^c \sin c + e^c \cos c = 0$

$e^c [\sin c + \cos c] = 0$

$\cos c = -\sin c$

$\left| \frac{\sin c}{\cos c} \right| = \left| -\tan c \right| = 1$

$= \tan \frac{\pi}{4} = 1.$

$a < c < b \Rightarrow 0 < \frac{\pi}{4} < \pi$

5Q) If $f(x) = \log x$ $g(x) = x^2$ in $[a,b]$ with $b > a > 1$ using Cauchy's theorem prove that $\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$

Solⁿ) by Cauchy's we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

$$f'(x) = \frac{1}{x}, \quad g'(x) = 2x$$

$$f'(c) = \frac{1}{c}, \quad g'(c) = 2c$$

$$\frac{\frac{1}{c}}{2c} = \frac{\log b - \log a}{b^2 - a^2}$$

$$\frac{1}{2c^2} = \frac{\log b - \log a}{(b+a)(b-a)}$$

$$\frac{a+b}{2c^2} = \frac{\log b - \log a}{b-a}$$

$\boxed{6Q}$ State and prove Cauchy's mean value theorem.

Proof assume $h(x) = f(x) + A g(x)$, $[a,b]$

check whether $h(x)$ satisfy all conditions of Rolle's theorem.

$h(x)$ is continous over $[a,b]$

$\because$ $h(x)$ is sum of two continuous functions.

$\Rightarrow$ $h'(x) = f'(x) + A g'(x)$

$\Rightarrow$ $h(x)$ is derivable.

$$h(a) = h(b)$$

$$f(a) + A g(a) = f(b) + A g(b)$$

$$-A g(b) + A g(a) = f(b) - f(a)$$

$$-A[g(b) - g(a)] = f(b) - f(a)$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = -A$$

$h(x)$ satisfies all conditions of

Rolle's theorem

$\therefore \exists$ a point $c \in (a,b) \ni$

$h'(c) = 0$

$$h'(c) = f'(c) + A\,g'(c) = 0$$

$$f'(c) = -A\,g'(c)$$

$$\frac{f'(c)}{g'(c)} = -A$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Statement: of Cauchy's.

Let $f: [a,b] \to R$, $g: [a,b] \to R$.

(i) $f(x)$ and $g(x)$ are continuous over $[a,b]$

(ii) $f(x)$ and $g(x)$ are derivable over (a,b)

then $\exists$ a point $c \in (a,b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$\boxed{70}$ verify Rolle's Theorem for $f(x)$

$f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$.

$\boxed{\text{Sol}^n}$ (i) $\sin x$ is continuous function $[0, \pi]$

e^x is continuous function $[0, \pi]$

(ii) $\sin x$ and e^x are differentiable $(0, \pi)$

$$f(0) = \frac{\sin 0}{e^0} = 0, \quad f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$$f'(x) = \frac{e^x \cos x - \sin x \, e^x}{(e^x)^2}$$

$$= \frac{(\cos x - \sin x)\cancel{e^x}}{(e^x) \cdot \cancel{e^x}} = \frac{\cos x - \sin x}{e^x}$$

$$f'(c) = 0 \Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

$$\cos c = \sin c \Rightarrow \frac{\sin c}{\cos c} = 1$$

$$\tan c = 1 = \tan \pi/4$$

$$0 < \frac{\pi}{4} < \pi$$

$\boxed{89}$ S.T $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$

and hence deduce that

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{2} - \frac{x^3}{48} \cdots\cdots$$

$\boxed{\text{Sol}^n}$

$$\log(1+e^x) = f(x)$$

$$\log(1+e^0) = f(0) =$$

$$\log(1+1) = \log 2$$

$$f'(x) = \frac{e^x}{1+e^x}$$

$$f'(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \qquad d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$f''(0) = \frac{(1+e^0)e^0 - e^0 \cdot e^0}{(1+e^0)^2} = \frac{2-1}{2^2} = \frac{1}{4}.$$

$$f''(x) = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x)e^x \cdot e^x}{(1+e^x)^4}$$

$$f'''(0) = \frac{(1+e^0)^2 e^0 - 2(1+e^0)e^0 \cdot e^0}{(1+e^0)^4}$$

$$= \frac{(2)^2 - 2(2)}{(1+1)^4} = 0$$

Mac laurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log(1+e^x) = \log 2 + x \frac{1}{2} + \frac{x^2}{2}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \cdots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$$

$\boxed{9Q}$ find the Eqn of circle of Curvature at the point $(a/4, a/4)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

<u>sol^n</u> given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0$$

$$\cancel{\tfrac{1}{2}}\left[\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}y'\right]=0$$

$$\frac{1}{\sqrt{x}}=-\frac{1}{\sqrt{y}}y'$$

$$y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\sqrt{\frac{a/4}{a/4}}=-1$$

$d\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}$, $\quad y''=-\left[\dfrac{\sqrt{x}\,\frac{1}{2\sqrt{y}}y'-\frac{1}{2\sqrt{x}}\sqrt{y}}{(\sqrt{x})^2}\right]$

$$y''=-\frac{\left[\frac{1}{2}\sqrt{\frac{a/4}{a/4}}(-1)-\frac{1}{2}\sqrt{\frac{a/4}{a/4}}\right]}{a/4}$$

$$=-\left(-\frac{1}{2}-\frac{1}{2}\right)\times\frac{4}{a}$$

$$y''=\frac{4}{a}$$

Centre of Curvature (X, Y)

$$X = x - \frac{y'[1+(y')^2]}{y''}$$

$$X = \frac{a}{4} - \frac{(-1)[1+(-1)]^2}{4/a}$$

$$X = \frac{a}{4} + \frac{a}{4}(2] = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$Y = y + \frac{[1+(y')^2]}{y''}$$

$$Y = \frac{a}{4} + \frac{[1+(-1)^2]}{4/a}$$

$$Y = \frac{a}{4} + \frac{a}{4}(2) = \frac{3a}{4}$$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} = \text{radius.}$$

$$= \frac{[1+(-1)^2]^{3/2}}{4/a} = \frac{a}{\cancel{4}_{2}} 2\sqrt{2} = \frac{a}{\sqrt{2}}$$

Eqn of circle of curvature

$$(x-X)^2+(y-Y)^2=\rho^2$$

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2$$

10Q Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3+y^3=3axy$.

Soln given $x^3+y^3=3axy$

$$3x^2+3y^2y'=3a[xy'+y(1)]$$

$$\cancel{3}[x^2+y^2y']=\cancel{3}a[xy'+y]$$

$$y^2y'-axy'=ay-x^2$$

$$y'=\frac{ay-x^2}{y^2-ax}$$

$$y'=-\left(\frac{x^2-ay}{y^2-ax}\right)$$

$$y' = + \frac{\left[\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)\right]}{a\left(\frac{3a}{2}\right) - \left(\frac{9a^2}{4}\right)}$$

$$y' = - \frac{\left[\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)\right]}{\left[\frac{9a^4}{4} - a\left(\frac{3a}{2}\right)\right]} = -1.$$

$$y'' = - \left[\frac{(y^2 - ax)(2x - ay') - (x^2 - ay)(2yy' - a)}{(y^2 - ax)^2}\right]$$

$$y'' = - \frac{\left[\cancel{\left(\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)\right)}\left(2\left(\frac{3a}{2}\right) - a(-1)\right) - \cancel{\left(\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)\right)}\left(2\left(\frac{3a}{2}\right)(-1) - a\right)\right]}{\left[\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)\right]^{\cancel{2}}}$$

$$y'' = - \frac{[4a + 4a]}{\frac{9a^2}{4} - \frac{3a^2}{2}} = \frac{8a}{\frac{9a^2 - 6a^2}{4}}$$

$$= \frac{32a}{3a^2} = \frac{32}{3a}$$

radius $\rho = \dfrac{[1+(y')^2]^{3/2}}{y''}$

$= \dfrac{[1+(-1)^2]^{3/2}}{\frac{32}{3a}}$

$= \dfrac{3a(\cancel{2}\sqrt{2})}{\cancel{32}} = \dfrac{3a(\sqrt{2})}{16}$

$\dfrac{3a\cancel{\sqrt{2}}}{8\sqrt{2}\cancel{\sqrt{2}}} = \dfrac{3a}{8\sqrt{2}}$

$\boxed{11Q}$ S.T the radius of Curvature of the cycloid $x = a(t+\sin t)$
$y = a(1-\cos t)$ is $4a \cdot \cos(t/2)$

Sol^n $x = a(t+\sin t)$, $y = a(1-\cos t)$

$\dfrac{dx}{dt} = a(1+\cos t)$ $\dfrac{dy}{dt} = a\sin t$

$y' = \dfrac{dy/dt}{dx/dt} = \dfrac{dy}{dx} = \dfrac{a\sin t}{a(1+\cos t)}$

$$y' \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/_2 \cos t/_2}{2\cos^2 t/_2} = \tan t/_2$$

$$y'' = \sec^2 \tfrac{t}{2} \left(\tfrac{1}{2}\right) \frac{dt}{dx}$$

$$= \sec^2 \tfrac{t}{2} \left(\tfrac{1}{2}\right) \cdot \frac{1}{a(1+\cos t)}$$

$$= \frac{1}{2a} \sec^2 t/_2 \times \frac{1}{2\cos^2(t/_2)}$$

$$\frac{1}{2\cdot 2a} \sec^4 t/_2 = \frac{1}{4a} \sec^4 t/_2 .$$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} = \frac{[1+\tan^2 t/_2]^{3/2}}{\frac{1}{4a} \sec^4 t/_2}$$

$$4a \cdot \frac{[\sec^2 t/_2]^{3/2}}{\sec^4 t/_2} = 4a \cdot \frac{\sec^3 t/_2}{\sec^4 t/_2}$$

$$4a \cdot \frac{1}{\sec t/_2} = \underline{4a(\cos t/_2)}$$

$\boxed{12\ Q}$ find the radius of curvature of $x = at^2,\ y = 2at$

$\boxed{\text{Sol}^n}$ $\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$

$$y' = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y'' = \frac{-1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2} \times \frac{1}{2at}$$

$$y'' = \frac{-1}{2at^3}$$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''}$$

$$\rho = \left| \frac{[1+\frac{1}{t^2}]^{3/2}}{-\frac{1}{2at^3}} \right|$$

$$\rho = \left| [(t^2+1)/t^2]^{3/2} \times 2at^3 \right|$$

$$\rho = \underline{(t^2+1)^{3/2} \cdot 2a}$$

13) Q) S.T the radius of curvature for the rectangular parabola $xy=c^2$ is $\frac{(x^2+y^2)^{3/2}}{2c^2}$

Sol^n

$$xy = c^2 \text{ (given)}$$

$$xy' + y(1) = 0$$

$$y' = -\frac{y}{x}$$

$$y'' = -\left[\frac{xy' - y(1)}{x^2}\right]$$

$$y'' = -\left[\frac{x\left(-\frac{y}{x}\right) - y}{x^2}\right]$$

$$y'' = \frac{2y}{x^2}$$

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

$$= \frac{\left[1+\left(-\frac{y}{x}\right)^2\right]^{3/2}}{2y/x^2}$$

$$\rho = \frac{[x^2+y^2]^{3/2}}{(x^2)^{3/2}} \times \frac{x^2}{2y}$$

$$= \frac{[x^2+y^2]^{3/2}}{x^3} \times \frac{x^2}{2y}$$

$$= \frac{[x^2+y^2]^{3/2}}{2xy} = \frac{[x^2+y^2]^{3/2}}{2c^2}$$

$\boxed{14Q}$ find the radius of Curvature of $r = a(1-\cos\theta)$.

Sol^n $r' = a(0-(-\sin\theta)) = a\sin\theta$

$r'' = a\cos\theta$

$$\rho = \frac{[r^2+(r')^2]^{3/2}}{r^2+2r_1^2-rr_2}$$

$$\left[r^2+(r')^2\right]^{3/2} = \left[a^2(1-\cos\theta)^2+a^2\sin^2\theta\right]^{3/2}$$

$$= (a^2)^{3/2}\left[1+\cos^2\theta-2\cos\theta+\sin^2\theta\right]^{3/2}$$

$$= a^3\left[2-2\cos\theta\right]^{3/2}$$

$$= a^3(2)^{3/2}\left[1-\cos\theta\right]^{3/2}$$

$$a^3\, 2\sqrt{2}\left[2\sin^2\theta/2\right]^{3/2}$$

$$= 2\sqrt{2}\, a^3\, 2\sqrt{2}\sin^3\theta/2$$

$$= 8a^3\sin^3\theta/2$$

$$r^2+2r_1^2-r r_2$$

$$= a^2(1-\cos\theta)^2+2(a\sin\theta)^2-a(1-\cos\theta)(a\cos\theta)$$

$$= a^2\left[1+\cos^2\theta-2\cos\theta+2\sin^2\theta-\cos\theta+\cos^2\theta\right]$$

$$= a^2\left[3-3\cos\theta\right] = a^2(3)\left[1-\cos\theta\right]$$

$$= a^2(3)\left(2\sin^2\theta/2\right)$$

$$= 6a^2\sin^2\theta/2$$

$$\rho = \frac{8a^3\sin^3\theta/2}{6a^2\sin^2\theta/2} = \frac{4}{3}a\sin\theta/2$$

$\boxed{15Q}$ find the radius of Curvature for $y^2 = \frac{a^2(a-x)}{x}$ at $(a,0)$

$$y^2 = \frac{a^2(a-x)}{x}$$

$$xy^2 = a^2(a-x)$$

$$xy^2 = a^3 - a^2x$$

$$x\,2yy' + y^2 \cdot 1 = -a^2$$

$$y' = \frac{-a^2 - y^2}{2xy}$$

$$\text{at } (a,0) = y' = \infty$$

find $\frac{dx}{dy} = \frac{2xy}{-a^2-y^2} = 0$

$$x'' = -2\left[\frac{(a^2+y^2)[x \cdot 1 + y(x')] - xy(2y)}{(a^2+y^2)^2}\right]$$

$$x'' = -2\,\frac{[(a^2+0)[a+0] - 0]}{(a^2+0)^2}$$

$$x'' = -2[a^3/a^4] = -\frac{2}{a}$$

$$\rho = \frac{[1+(x')^2]^{3/2}}{x''} = \left|\frac{(1+0)^{3/2}}{-2/a}\right| = \frac{a}{2}$$

UNIT - 3

1. Evaluate the $\lim_{(x,y)\to(1,1)} \frac{x(y-1)}{y(x-1)}$, if it exists?

sol: $\lim_{x\to1}\left[\lim_{y\to1} \frac{x(y-1)}{y(x-1)}\right]$

$\lim_{x\to1}\left[\frac{x(1-1)}{1(x-1)}\right] = 0$ — (1)

$\lim_{y\to1}\left[\lim_{x\to1} \frac{x(y-1)}{y(x-1)}\right] = \lim_{y\to1}\left[\frac{1(y-1)}{y(1-1)}\right]$

$= \lim_{y\to1} \frac{y-1}{0}$

$= \infty$ — (2)

$\therefore$ (1) $\neq$ (2)

limit does not exist.

2. S.T $\lim_{(x,y)\to(0,0)}\left[\frac{x^3y}{x^6+y^2}\right]$ does not exist?

sol: $\lim_{x\to0}\left[\lim_{y\to0} \frac{x^3y}{x^6+y^2}\right]$

$\lim_{x\to0}\left[\frac{x^3(0)}{x^6+0}\right] = 0$ — (1)

$\lim_{y\to0}\left[\lim_{x\to0} \frac{x^3y}{x^6+y^2}\right] = 0$ — (2).

Let $y = mx \Rightarrow \lim_{x\to0}\left[\lim_{y\to mx} \frac{x^3y}{x^6y^2}\right]$

$$= \lim_{x\to 0}\left[\frac{x^3 . m(x)}{x^6+(mx)^2}\right] = 0 \quad — (3)$$

Let $y = mx^2$

$$= \lim_{x\to 0}\left[\frac{x^3 m x^2}{x^6+(mx^2)^2}\right] = 0 \quad — (4)$$

Let $y = mx^3$

$$= \lim_{x\to 0}\left[\frac{x^3 . m x^3}{x^6+(mx^3)^2}\right] = \lim_{x\to 0}\frac{\cancel{x^6} . m}{\cancel{x^6} + m^2 \cancel{x^6}}$$

$$= \frac{m}{1+m^2} \quad — (5)$$

$\therefore$ (1) = (2) = (3) = (4) $\neq$ (5).

$\therefore$ limit does not exist.

3. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ then P.T $xU_x + yU_y = 1$

Sol:

$$u = \log_e\frac{(x^2+y^2)}{x+y}.$$

$$\left[\begin{array}{l} U_x = \frac{\partial u}{\partial x} \\ U_y = \frac{\partial u}{\partial y} \end{array}\right.$$

$$e^u = \frac{x^2+y^2}{x+y}$$

which is homogeneous fn of degree "1" in xy.

By Euler we have.

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 1 . e^u.$$

$$xe^u . \frac{\partial u}{\partial x} + ye^u . \frac{\partial u}{\partial y} = e^u.$$

$$\cancel{e^u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \cancel{e^u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

$$xU_x + yU_y = 1 \ //.$$

4. Discuss the continuity of the fn $f(x,y)=\frac{(x-y)^2}{x^2+y^2}$, $(x,y)\neq(0,0)$; 0, $(x,y)=(0,0)$. ?

sol:- Given
$$f(x,y)=\begin{cases}\frac{(x-y)^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{x\to0}\left[\lim_{y\to0}\frac{(x-y)^2}{x^2+y^2}\right]=\lim_{x\to0}\left[\frac{(x-0)^2}{x^2+0}\right]=\lim_{x\to0}\left[\frac{\cancel{x^2}}{\cancel{x^2}}\right]=\underline{1}$$

$$\lim_{y\to0}\left[\lim_{x\to0}\frac{(x-y)^2}{x^2+y^2}\right]=\lim_{y\to0}\left[\frac{(0-y)^2}{0^2+y^2}\right]=\lim_{y\to0}\frac{\cancel{y^2}}{\cancel{y^2}}=1.$$

$\therefore$ limit of the fn exist & value is unique $=1$

$\Rightarrow$ It is continuous

5. Define homogeneous fn with example?

sol:- If "u" is a H.F of degree 'n' in x & y then
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

eg:- If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ P.T. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$

$$\tan u=\frac{x^3+y^3}{x-y}.$$
$$=\frac{x^{\cancel{3}\,2}\left[1+\frac{y^3}{x^3}\right]}{\cancel{x}\left[1-\frac{y}{x}\right]}=x^2 f\left(\frac{y}{x}\right).$$

is a H.F of degree '2' in x & y.

$\therefore$ By Euler $x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u.$

$$x\sec^2u\frac{\partial u}{\partial x}+y\sec^2u\frac{\partial u}{\partial y}=2\tan u.$$

7* If $x+y+z=u$, $y+z=uv$, $z=uvw$ then evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol: $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x+y+z=u$, $y+z=uv$, $z=uvw$

$x+uv=u$ $\qquad$ $y+uvw=uv$

$x=u-uv$ $\qquad$ $y=uv-uvw$

$\dfrac{\partial x}{\partial u} = 1-v$, $\dfrac{\partial x}{\partial v} = -u$, $\dfrac{\partial x}{\partial w} = 0$

$\dfrac{\partial y}{\partial u} = v-vw$, $\dfrac{\partial y}{\partial v} = u-uw$, $\dfrac{\partial y}{\partial w} = -uv$

$\dfrac{\partial z}{\partial u} = vw$, $\dfrac{\partial z}{\partial u} = uw$, $\dfrac{\partial z}{\partial w} = uv$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[u^2v - \cancel{u^2vw} + \cancel{u^2vw}] + u[uv^2 - uv^2w + uv^2w] + 0$$

$$= (1-v)(u^2v) + u(uv^2)$$

$$= uv^2 - \cancel{uv^2} + \cancel{u^2v^2}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = uv^2$$

*8 If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol:-
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = x^2 - 2y$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2$, $\frac{\partial u}{\partial z} = 0$

$v = x + y + z$

$\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 1$, $\frac{\partial v}{\partial z} = 1$

$w = x - 2y + 3z$

$\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial y} = -2$, $\frac{\partial w}{\partial z} = 3$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$= 2x(3+2) + 2(3-1) + 0$

$= 2x(5) + 2(2)$

$= 10x + 4$.

9 Obtain Taylor's series for $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ at the point $(-1,1)$.

Sol

$$f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$$

$$x_0 = -1, \quad y_0 = 1$$

$$x = x_0 + h \qquad y = y_0 + k$$

$$x = -1 + h \qquad y = 1 + k$$

$$h = x + 1 \qquad k = y - 1$$

Taylor's Series:-

$$f(x,y) = f(x_0,y_0) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \right) + \cdots$$

$$f(x_0 y_0) = 2(-1)^2 - (-1)(1) + (1)^2 + 3(-1) - 4(1) + 1$$

$$= 2 + 1 + 1 - 3 - 4 + 1$$

$$= -2.$$

$$\frac{\partial f}{\partial x} = 4x - y + 3. \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(-1,1)} = 4(-1) - (1) + 3 = -4 - 1 + 3 = -2$$

$$\frac{\partial f}{\partial y} = -x + 2y - 4 \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(-1,1)} = -(-1) + 2(1) - 4 = 1 + 2 - 4 = -1$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \; ; \quad \frac{\partial^2 f}{\partial y^2} = 2 \; , \quad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$f(x,y) = -2 + \left[(x+1)(-2) + (y-1)(-1)\right] + \frac{1}{2!}\left[(x+1)^2(4) + 2(x+1)(y-1)(-1) + (y-1)^2(2)\right] + \cdots$$

$$\Rightarrow f(x,y) = -2 + [-2x-2-y+1] + \frac{1}{2!}\left[4(x+1)^2 - 2(x+1)(y-1) + 2(y-1)^2\right] + \cdots$$

$$= -2 + [-2x-y-1] + \frac{1}{2!}\{4x^2+8x+4-2xy+2x-2y+2$$

$$+2y^2-4y+2] + \cdots$$

$$= -2 - [2x+y+1] + \frac{1}{2!}[4x^2+2y^2+10x-6y-2xy+8] + \cdots$$

10.) Write a working rule to find the maximum and minimum values of $f(x,y)$.

Sol ① Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate it to zero to get two equations

$\frac{\partial f}{\partial x} = 0$ — ① $\quad \frac{\partial f}{\partial y} = 0$ — ② solve equation ① + ② and obtain the values of x and y as $(a_1,b_1)(a_2,b_2)(a_3,b_3)\cdots$ called as stationary points

② Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$, find r, s, t at the stationary points obtained in the step-1

③ i) If $rt - s^2 > 0$, $r > 0$ at (a_1,b_1) then $f(x,y)$ is said to have minimum value at (a_1,b_1) and minimum value is $f(a_1,b_1)$

ii) If $rt-s^2>0$, $r<0$ at (a_1,b_1) then $f(x,y)$ is said to have maximum value at (a_1,b_1) and maximum value is $f(a_1,b_1)$

iii) If $rt-s^2<0$, no maxima and minima exist and the point is called saddle point

iv) If $rt-s^2=0$, no conclusion can be drawn further investigation is ~~not~~ required.

14) Write about Lagrange's method of multipliers.

→ Working Rule:-

To find the extreme values for the function $f(x,y,z)$ where x,y,z are not all independent and are connected by a relation $\phi(x,y,z)=0$

① ~~Form~~ the Langranges function $F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$ — ①.

where ~~d~~ λ is called Langranges multiplier.

② Differentiating eq ① partially w.r.t x,y,z and equating it to zero we have

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \quad \text{—— ②}$$

$$\frac{\partial F}{\partial y}=\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \quad \text{—— ③}$$

$$\frac{\partial F}{\partial z}=\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \quad \text{—— ④}$$

③ Solving equation ②, ③, ④ and the given relation we get the values of x,y,z. Hence the stationary points

(12) Find the maximum and minimum of the function $f(x) = x^5 - 3x^4 +$

Sol

$f(x) = x^5 - 3x^4 + 5$

diff w.r.t x

$f'(x) = 5x^4 - 12x^3$

$f'(x) = 0$

$5x^4 - 12x^3 = 0$

$x^3(5x - 12) = 0$

$x^3 = 0, \quad 5x - 12 = 0$

$x = 0, \quad x = 12/5$

$f''(x) = 20x^3 - 36x^2$

for $x = 0$, $f''(x) = 0$

$\therefore x = 0$ no maxima & minimam exist

for $x = 12/5$

$f''(x) = 20\left(\frac{12}{5}\right)^3 - 36\left(\frac{12}{5}\right)^2$

$f''(x) = 4 \times \frac{12^3}{25} - 36 \times \frac{12 \times 12}{25}$

$f''(x) = 276.48 - 207.36 \Rightarrow 69.12 > 0$

$\therefore f''(x)$ is minimum at $x = 12/5$, & minimum value is

$f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^5 - 3\left(\frac{12}{5}\right)^4 + 5$

$= -14.9$//.

(4)

(13) Discuss the maximum and minimum of $x^2+y^2+6x+12$

Sol: Given $f = x^2+y^2+6x+12$

diff w.r.t 'x, y'

$$\frac{\partial f}{\partial x} = 2x+6 \quad \text{——— (1)}$$

$$\frac{\partial f}{\partial y} = 2y \quad \text{——— (2)}$$

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0$$

$$2x+6=0 \qquad 2y=0$$

$$x = \frac{-6}{2} \qquad 2y=0$$

$$x=-3 \qquad y=0$$

$(-3,0)$ are the stationary points

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$rt-s^2 = (2)(2)-(0)^2 = 4, \quad r=2>0.$

$rt-s^2>0$, $r>0$ the function has minimum value at $(-3,0)$

$$f(-3,0) = (-3)^2+(0)^2+6(-3)+12$$

$$= 9-18+12$$

$$= 3.$$

$\therefore$ minimum value is 3

(14) Examine for max & min value of the fn

$f(x,y) = x^2 - 3xy + y^2 + 2x$.

Sol: $f(x,y) = x^2 - 3xy + y^2 + 2x$

Diff w.r.t x, y.

$\frac{\partial f}{\partial x} = 2x - 3y + 2 = 0$; $\frac{\partial f}{\partial y} = -3x + 2y = 0$

$\Rightarrow 2x - 3y = -2$ — (1) $\Rightarrow -3x + 2y = 0$ — (2)

solving (1) ×3 & (2) ×2.

$$6\cancel{x} - 9y = -6$$
$$-\cancel{6x} + 4y = 0$$
$$-5y = -6 \Rightarrow y = \frac{6}{5}$$

from (1) $\Rightarrow 2x - 3\left(\frac{6}{5}\right) = -2$

$$2x = -2 + \frac{18}{5}$$
$$\cancel{2}x = \frac{\cancel{8}}{5}$$
$$x = 4/5$$

∴ The stationary pt is $\left(\frac{4}{5}, \frac{6}{5}\right)$

$r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = -3$, $t = \frac{\partial^2 f}{\partial y^2} = 2$.

$rt - s^2 = (2)(2) - (-3)^2$

$= 4 - 9$

$= -5 < 0$

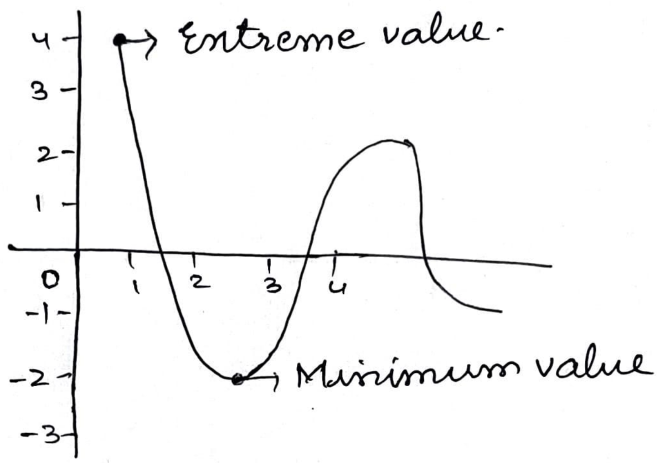
No max & min exist & the point is called saddle pt

15 Define the terms

(i) Extremum (ii) Saddle point

Ans (i) Extremum :-

An Extremum (or extreme value) of a fn is a point at which a maximum or minimum value of the fn is obtained at some internal.



(ii) Saddle point :- A point where the derivative of the fn is zero but the derivatives does not change sign is known as saddle point.

* Obtain the total derivative of $z = \tan^{-1}\left(\frac{x}{y}\right)$; $(x,y) \neq (0,0)$ where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

Sol:- $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$z = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y} = \frac{\frac{1}{\cancel{y}}}{\frac{y^2+x^2}{y^{\cancel{2}}}} = \frac{y}{y^2+x^2}.$$

$$\cancel{\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) = \frac{-\frac{x}{y^2}}{\frac{y^2+x^2}{y^2}}} = \frac{-x}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}} \cdot x\left(-\frac{1}{y^2}\right)$$

$$x = e^t - e^{-t}$$

$$\frac{dx}{dt} = e^t + e^{-t} = y.$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{x}{\cancel{y^2}}}{\frac{y^2+x^2}{\cancel{y^2}}} = \frac{-x}{x^2+y^2}$$

$$y = e^t + e^{-t} \Rightarrow \frac{dy}{dt} = e^t - e^{-t} = x.$$

$$\frac{dz}{dt} = \frac{y}{x^2+y^2} = \frac{y}{x^2+y^2}(y) + \frac{-x}{x^2+y^2}(x)$$

$$\frac{dz}{dt} = \frac{y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

Properties of Jacobians :-

1) If $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

2) If $u = u(x,y)$, $v = v(x,y)$ & $x = f(r,t)$, $y = g(r,t)$ then

$$\frac{\partial(u,v)}{\partial(r,t)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,t)}.$$

2** If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, S.T. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Sol:-
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = \frac{yz}{x}$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$, $\frac{\partial u}{\partial y} = \frac{z}{x}$, $\frac{\partial u}{\partial z} = \frac{y}{x}$

$v = \frac{zx}{y}$

$\frac{\partial v}{\partial x} = \frac{z}{y}$, $\frac{\partial v}{\partial y} = \frac{-zx}{y^2}$, $\frac{\partial v}{\partial z} = \frac{x}{y}$.

$w = \frac{xy}{z}$

$\frac{\partial w}{\partial x} = \frac{y}{z}$, $\frac{\partial w}{\partial y} = \frac{x}{z}$, $\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 0 - \frac{\cancel{z}}{\cancel{x}}\left(-\frac{2\cancel{x}}{\cancel{z}}\right) + \frac{\cancel{y}}{\cancel{x}}\left(\frac{2\cancel{x}}{\cancel{y}}\right)$$

$$= 2+2 = \underline{\underline{4}}$$

3Q) If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = ut^3$.

ST $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Sol: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$u = \sin^{-1}(x-y)$

$\frac{\partial}{\partial x}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{du}{dx} = \frac{1}{\sqrt{1-(x-y)^2}}(+1)$

$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1)$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1)$$

$x = 3t, \quad y = 4t^3$

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 12t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3) - \frac{1}{\sqrt{1-(x-y)^2}}12t^2$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}} = \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2(1-t^2)}}$$

$$= \frac{3\cancel{(1-4t^2)}}{\cancel{(1-4t^2)}\sqrt{1-t^2}} = \frac{3}{\sqrt{1-t^2}}$$

$-16t^6+24t^4-9t^2+1$

$-16t^6$

$9t^2-16t^6+24t^4+1$

$-16t^6$

$(1-4t^2)^2(1-t^2)$

$(1+16t^4-8t^2)$

Qu. If $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$, Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol:- If $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$, $J' = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ $= JJ' = 1$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} ; \quad u = \frac{2yz}{x}$$

$$\frac{\partial u}{\partial x} = \frac{-2yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{2z}{x}, \quad \frac{\partial u}{\partial z} = \frac{2y}{x}$$

$v = 3zx/y$

$$\frac{\partial v}{\partial x} = \frac{3z}{y}, \quad \frac{\partial v}{\partial y} = \frac{-3zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{3x}{y}$$

$$w = \frac{4xy}{z}$$

$$\frac{\partial w}{\partial x} = \frac{4y}{z}, \quad \frac{\partial w}{\partial y} = \frac{4x}{z}, \quad \frac{\partial w}{\partial z} = -\frac{4xy}{z^2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{z} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix}$$

$$= -\frac{2yz}{x^2}\left[\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right] - \frac{2z}{x}\left[-\frac{12xyz}{yz^2} - \frac{12xy}{yz}\right]$$

$$+ \frac{2y}{z}\left[\frac{12xz}{yz} + \frac{12xyz}{y^2z}\right]$$

$$= -\frac{2yz}{x^2}\left(\frac{12x^2}{yz} - \frac{12x^2}{yz}\right) - \frac{2z}{x}\left(-\frac{12x}{z} - \frac{12x}{z}\right) + \frac{2y}{x}\left(\frac{12x}{y} + \frac{12x}{y}\right)$$

$$= -\frac{2yz}{x^2}(0) - \frac{2\cancel{z}}{\cancel{x}}\left(-\frac{24\cancel{x}}{\cancel{z}}\right) + \frac{2\cancel{y}}{\cancel{x}}\left(\frac{24\cancel{x}}{\cancel{y}}\right)$$

$$J' = 2(24) + 2(24) = 96$$

$$JJ' = 1 \Rightarrow J = \frac{1}{J'} = \frac{1}{96}$$

$$\underline{\underline{J = \frac{1}{96}}}$$

5 ** Examine whether the fn are functionally dependent $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ ~~$x+y$~~. If so find the relation between them.

$\frac{\partial y}{\partial x}$

Sol:- $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$\because$ If $J\left(\frac{u,v}{x,y}\right) = 0$, then u & v is F.D.

If $J\left(\frac{u,v}{x,y}\right) \neq 0$, then u & v is F.I.

$$u = \frac{x+y}{1-xy}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} = \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2} = \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$v = \tan^{-1}x + \tan^{-1}y.$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= \underline{\underline{0}}.$$

$\therefore$ The given fn's are functionally dependent

wkt $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$v = \tan^{-1}u$ or $\underline{\underline{\tan v = u}}$.

6 **

If $u = x + 3y^2 + z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$

Evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$.

Sol:-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = x + 3y^2 \cancel{+} z^3$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 6y, \quad \frac{\partial u}{\partial z} = -3z^2.$$

$v = 4x^2yz$.

$$\frac{\partial v}{\partial x} = 8xyz, \quad \frac{\partial v}{\partial y} = 4x^2z, \quad \frac{\partial v}{\partial z} = 4x^2y.$$

$w = 2z^2 - xy$.

$$\frac{\partial w}{\partial x} = -y, \quad \frac{\partial w}{\partial y} = -x, \quad \frac{\partial w}{\partial z} = 4z.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$\left.\frac{\partial(u,v,w)}{\partial(x,y,z)}\right\}_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(0-4) + 6(0+4) + 0$$

$$= -4 + 24$$

$$= \underline{\underline{20}}$$

80) Find the first 3 terms of the taylor's series of $f(x,y) = e^x \cos y$ around $(0,0)$.

sol:- $f(x,y) = e^x \cos y$, $f(0,0) = e^0 \cos 0 = 1$,

$f_x = e^x \cos y = e^0 \cos 0 = 1$

$f_{xx} = e^x \cos y = e^0 \cos 0 = 0$

$f_{xxx} = e^x \cos y = e^0 \cos 0 = 1$

$f_y = e^x[-\sin y] = e^0(-\sin 0) = 0$

$f_{yy} = -e^x(\cos y) = -1$

$f_{yyy} = -e^x[-\sin y] = e^0(\sin 0) = 1$

$f_{xy} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}(-e^x \sin y)$

$= -e^0 \sin y = 0$

$f_{xxy} = \frac{\partial}{\partial x}(f_{xy}) = -\frac{\partial}{\partial x}(e^x \sin y)$

$= -(e^x \sin y) = 0$

$f_{xyy} = \frac{\partial}{\partial x}(f_{yy}) = \frac{\partial}{\partial x}(-e^x \cos y) = -1.$

$$e^x \cos y = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

$$= 1 + [x(1) + y(0)] + \frac{1}{2!}[x^2(1) + y^2(-1) + 2xy(0)] + \frac{1}{3!}[x^3(1) + y^3(1) + 3x^2y(0) + 3xy^2(-1)] +$$

9

** Evaluate the first three terms of the taylor's series of the fn $f(x,y) = e^x \sin y$ around $(0,0)$.

Sol:- $f(x,y) = e^x \sin y$ at $(0,0)$

$x_0 = 0$, $y_0 = 0$

$h = x$, $k = y$.

We have Taylor's series.

$$f(x,y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + \right.$$

terms
n of

$$+ y^2\frac{\partial^2 f}{\partial y^2}\Big) + \frac{1}{3!}\left(x^3\frac{\partial^3 f}{\partial x^3} + 3x^2y\frac{\partial^3 f}{\partial x^2 y} + 3xy^2\frac{\partial^3 f}{\partial x\partial y^2} + y^3\frac{\partial^3 f}{\partial y^3}\right) + \cdots$$

$$f(x,y) = e^x\sin y \Rightarrow f(0,0) = 0$$

$$\frac{\partial f}{\partial x} = e^x\sin y \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = e^x\cos y \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^x\sin y \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = e^x\cos y \Rightarrow \left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x\sin y \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = e^x\sin y \Rightarrow \left(\frac{\partial^3 f}{\partial x^3}\right)_{(0,0)} = 0$$

$$\left(\frac{\partial^3 f}{\partial x^2\partial y}\right) = e^x\cos y \Rightarrow \left(\frac{\partial^3 f}{\partial x^2\partial y}\right)_{(0,0)} = 1$$

$$\left(\frac{\partial^3 f}{\partial x\partial y^2}\right) = -e^x\sin y \Rightarrow \left(\frac{\partial^3 f}{\partial x\partial y^2}\right)_{(0,0)} = 0.$$

$$\frac{\partial^2 f}{\partial y^3} = -e^x\cos y \Rightarrow \left(\frac{\partial^3 f}{\partial y^3}\right)_{(0,0)} = -1$$

$$\therefore f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)]$$
$$+ \frac{1}{3!}[x^2(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)] + \cdots$$

$$f(x,y) = y + xy + \frac{1}{6}(3x^2y - y^3) + \cdots$$

10

** Find the stationary point of $u(x,y) = \sin x \cdot \sin y \cdot \sin(x+y)$ where $0 < x < \pi$, $0 < y < \pi$ & Find the max u.

Sol:- $u(x,y) = \sin x \sin y \sin(x+y)$

Diff w.r.t 'x' & 'y'.

$$\frac{\partial u}{\partial x} = \sin y \,[\sin x \sin(x+y)]$$

$$= \sin y \,[\sin x \cdot \cos(x+y) + \cos x \sin(x+y)]$$

$$= \sin y \,[\sin(2x+y)]$$

$$\frac{\partial u}{\partial y} = \sin x \,[\sin y \cdot \sin(x+y)]$$

$$= \sin x \,[\sin y \cos(x+y) + \cos y \sin(x+y)]$$

$$= \sin x \sin(x+2y)]$$

$$\frac{\partial u}{\partial x} = \sin y(\sin(2x+y)] = 0 ; \quad \frac{\partial u}{\partial y} = 0$$

$\sin y\,[\sin(2x+y)] = 0$, $\quad \sin x . \sin(x+2y) = 0$

$y = 0 ; \pm\pi$, $\quad x = 0, \pm\pi$

$2x+y = 0$ $\quad x+2y = 0$

$2x+y = \pm\pi$ — ① $\quad x+2y = \pm\pi$ — ②

1

solving ① $\times 2$ & ②.

$$4x + 2y = 2\pi$$
$$x + 2y = \pi$$
$$\overline{3x = \pi}$$
$$x = \pi/3$$

from ① $\Rightarrow 2\left(\frac{\pi}{3}\right) + y = \pi$

$$y = \pi - \frac{2\pi}{3}$$
$$y = \pi/3$$

$(\pi/3, \pi/3)$ $(-\pi/3, -\pi/3)$ are the stationary points

$r = \frac{\partial^2 f}{\partial x^2} = \sin y \cos(2x+y)(2) = \sin(\pi/3)\left[\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)(2)\right.$

$= \frac{\sqrt{3}}{\cancel{2}}(-1)(\cancel{2}) = -\sqrt{3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = \sin y \cos(2x+y) + \cos y \sin(2x+y)$

$= \sin(2x+2y) = \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$t = \frac{\partial^2 f}{\partial y^2} = \sin x \cos(x+2y)(2)$

$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)(2)$

$= \frac{\sqrt{3}}{\cancel{2}}(-1)(\cancel{2}) = -\sqrt{3}.$

$\therefore rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

$r = -\sqrt{3} < 0$

$u(x,y)$ is maximum & the max value is

$u\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{3} \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$

11Q. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Sol: Let (x, y, z) be a point on the surface $xyz^2 = 2$ — (1)

distance from origin $d = \sqrt{x^2 + y^2 + z^2}$

$f = d^2 = x^2 + y^2 + z^2$; $\phi = xyz^2 - 2$

lagranges fn:

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$.

$F = x^2 + y^2 + z^2 + \lambda(xyz^2 - 2)$ — (2)

Diff w.r.t. x, y, z.

$\frac{\partial F}{\partial x} = 2x + \lambda yz^2$, $\frac{\partial F}{\partial y} = 2y + \lambda xz^2$, $\frac{\partial F}{\partial z} = 2z + 2\lambda xyz$

$2x + \lambda yz^2 = 0$ — (3) , $2y + \lambda xz^2 = 0$ — (4) $2z + 2\lambda xyz = 0$ — (5)

from (3) $\lambda = \frac{-2x}{yz^2}$ — (i)

from (4) $\lambda = -\frac{2y}{xz^2}$ — (ii)

from (5) $\lambda = \frac{-2\cancel{z}}{2xy\cancel{z}} \Rightarrow \lambda = \frac{-1}{xy}$

from (i), (ii) $\Rightarrow \cancel{-}\frac{\cancel{2}x}{y\cancel{z^2}} = \cancel{-}\frac{\cancel{2}y}{x\cancel{z^2}} \Rightarrow \frac{x}{y} = \frac{y}{x} \Rightarrow x^2 = y^2$

$x = y$

from (ii) & (iii) $\Rightarrow -\frac{2y}{\cancel{x}z^2} = \frac{-1}{\cancel{x}y} \Rightarrow 2y^2 = z^2 \Rightarrow y^2 = \frac{z^2}{2}$

$\Rightarrow z^2 = 2y^2$

$\Rightarrow z = 2x^2$

from (1) $\Rightarrow xyz^2 = 2$

$(x)(x)(2x^2) = 2$

$\cancel{2}x^4 = \cancel{2} \Rightarrow x^4 = 1$

$\therefore x = 1, y = 1, z = \sqrt{2}$

$\therefore$ The pt on the surface is $(1, 1, \sqrt{2})$ & the shortest distance from origin is $d = \sqrt{1^2 + 1^2 + 2}$

$= \sqrt{4} = 2$

12

** Discuss the max & min $f(x,y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$?

sol:- $f(x,y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$.

Diff w.r.t 'x' & 'y'.

$\frac{\partial f}{\partial x} = 8x + 4y - 10 = 0$, $\frac{\partial f}{\partial y} = 4y + 4x - 2 = 0$

$8x + 4y = 10$, $4x + 4y = 2$.

solving ① & ②.

$$8x + 4y = 10$$
$$4x + 4y = 2$$
$$\underline{- \quad - \quad\quad -}$$
$$4x = 8$$
$$x = 2 \quad \text{sub in ①.}$$

$$8(2) + 4(y) = 10$$
$$4y = 10 - 16$$
$$\cancel{4}^{2} y = -\cancel{6}^{3}$$
$$y = -3/2$$

The stationary pt is $(2, -3/2)$

$$\overset{l}{r} = \frac{\partial^2 f}{\partial x^2} = 8, \quad \overset{m}{s} = \frac{\partial^2 f}{\partial x \partial y} = 4, \quad \overset{n}{t} = \frac{\partial^2 f}{\partial y^2} = 4$$

$ln - m^2$

$$rt - s^2 = (8)(4) - 4^2 = 36 - 16 = 16 > 0$$
$$\overset{l}{r} = 8 > 0$$

$\therefore f(x,y)$ has minimum value at $(2, -3/2)$

min value is $f(2, -3/2) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$

$$= 4(2)^2 + 2\left(\frac{-3}{2}\right)^2 + 4(2)\left(\frac{-3}{2}\right) - 10(2) - \cancel{2}\left(\frac{-3}{\cancel{2}}\right) - 3$$

$$= 4(4) + 2\left(\frac{9}{4}\right) + 4\left(-\frac{6}{2}\right) - 20 + 3 - 3$$

$$= 16 + \frac{9}{2} - 12 - 20$$

$$= 16 + \frac{9}{2} - 32$$

$$= \frac{32 + 9 - 64}{2} =$$

$$f(2, -3/2) = -\frac{23}{2} //$$

where λ and μ are the lagrange multipliers and proceed as above.

13 ** Find the maximum and minimum values of the $x+y+z$ subject to condition

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

sol:- $f(x,y,z)$, $\phi = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$

Lagrange's function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= x+y+z+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$ — ①

D.w.r.t 'x' & 'y' & 'z':

$\frac{\partial F}{\partial x} = 1-\frac{\lambda}{x^2}=0$ — ②

$\frac{\partial F}{\partial y} = 1-\frac{\lambda}{y^2}=0$ — ③

$\frac{\partial F}{\partial z} = 1-\frac{\lambda}{z^2}=0$ — ④

From ② $\Rightarrow \frac{\lambda}{x^2}=1 \Rightarrow x^2=\lambda$

$x=\pm\sqrt{\lambda}$

③ $\Rightarrow \frac{\lambda}{y^2}=1 \Rightarrow y^2=\lambda \Rightarrow y=\pm\sqrt{\lambda}$

④ $\Rightarrow \frac{\lambda}{z^2}=1 \Rightarrow z^2=\lambda \Rightarrow z=\pm\sqrt{\lambda}$.

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \Rightarrow \frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}=1$

$\frac{3}{\sqrt{\lambda}}=1 \Rightarrow \lambda=9$.

$x = \pm\sqrt{9}$, $y = \pm\sqrt{9}$, $z = \pm\sqrt{9}$

$x = \pm 3$ $y = \pm 3$, $z = \pm 3$.

$(3,3,3)$ $(-3,-3,-3)$ are the stationary pts

at $(3,3,3)$, $f(x,y,z) = 3+3+3 = 9$

at $(3,-3,-3)$, $f(x,y,z) = -3-3-3 = 9$

max value at $(3,3,3) = 9$

min value at $(-3,-3,-3) = -9$.

14 ** Find the min value at $x^2+y^2+z^2$ with the constraint $x+y+z = 3a$.

Sol: $f = x^2+y^2+z^2$, $\phi = x+y+z-3a$

Lagrange's Function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= x^2+y^2+z^2 + \lambda(x+y+z-3a)$ — (1)

(1) wrt $x, y,$ & z..

$\frac{\partial F}{\partial x} = 2x+\lambda = 0 \Rightarrow 2x = -\lambda \Rightarrow x = -\lambda/2$

$\frac{\partial F}{\partial y} = 2y+\lambda = 0 \Rightarrow 2y = -\lambda \Rightarrow y = -\lambda/2$

$\frac{\partial F}{\partial z} = 2z+\lambda = 0 \Rightarrow 2z = -\lambda \Rightarrow z = -\lambda/2$

$x+y+z = 3a$.

$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a$.

$-\frac{\cancel{3}\lambda}{2} = \cancel{3}a$

$\lambda = -2a$.

$$x = \frac{2a}{2}, \quad y = \frac{2a}{2}, \quad z = \frac{2a}{2}$$

$$x = a, \quad y = a, \quad z = a$$

(a,a,a) are the stationary points

at $(a,a,a) = a^2 + a^2 + a^2 = 3a^2$

$\therefore$ Hence the min value is $3a^2$

15 ** Find the volume of the greatest rectangular parallelopipe that can be inscribed in the ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

sol: Let $2x, 2y, 2z$ be the length, breadth & height of the rectangular parallelopipe that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — (1)

then the centeroid of the parallelopipe coincide with the centre of the ellipsoid $(0,0,0)$ & the corners of the parallelopipe lie on the surface of the ellipsoid

$$\text{volume} = l \times b \times h$$

$$= 8xyz$$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Lagranges Function

$$F(x,y,z) = f(x,y,z) + \lambda\{\phi(x,y,z)\}$$

$$F = 8xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right] \quad \text{—— (2)}$$

Dw.r.t x, y, z.

$$\frac{\partial F}{\partial x} = 8yz + \frac{2x\lambda}{a^2} = 0 \Rightarrow \cancel{8}^{4} yz = -\frac{\cancel{2}^{1} x\lambda}{a^2} \Rightarrow \lambda = -\frac{4a^2yz}{x}$$

$$\frac{\partial F}{\partial y} = 8xz + \frac{2y\lambda}{b^2} = 0 \Rightarrow \lambda = -\frac{4b^2xz}{y}$$

$$\frac{\partial F}{\partial z} = 8xy + \frac{2z\lambda}{c^2} = 0 \Rightarrow \lambda = -\frac{4c^2xy}{z}$$

Equating

$$-\frac{\cancel{4}a^2y\cancel{z}}{x} = -\frac{\cancel{4}b^2x\cancel{z}}{y}, \qquad -\frac{\cancel{4}b^2\cancel{x}z}{y} = -\frac{\cancel{4}c^2\cancel{x}y}{z}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \qquad\qquad \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

sub in (1)

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

III^{ly} $y = \frac{b}{\sqrt{3}}, \; z = \frac{c}{\sqrt{3}}$

$\therefore \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is the stationary point

$\therefore$ The max volume $V = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$

$$= \frac{8abc}{3\sqrt{3}}.$$

SAQ UNIT-IV

1. Evaluate $\int_0^1 \int_0^3 (x^2+y^2)\,dy\,dx$

Sol: $\int_0^1 \left(\int_0^3 (x^2+y^2)\,dy \right) dx$

$= \int_0^1 \left(\int_0^3 x^2\,dy + \int_0^3 y^2\,dy \right) dx$

$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_0^3 dx = \int_0^1 \left(3x^2 + \frac{27}{3} - 0 \right) dx = \int_0^1 (3x^2+9)\,dx$

$= \left(3\frac{x^3}{3} + 9x \right)_0^1 = 1+9 = 10$

2. Evaluate $\int_0^{\pi} \int_0^1 x\cos(xy)\,dy\,dx$

Sol: $= \int_0^{\pi} \int_0^1 x\cos(xy)\,dy\,dx$

$= \int_0^{\pi} \left(\int_0^1 x\cos(xy)\,dy \right) dx$ $\quad \boxed{\therefore \int \cos ax\,dx = \frac{\sin ax}{a}}$

$= \int_0^{\pi} \left(\left(\cancel{x} \frac{\sin xy}{\cancel{x}} \right)_0^1 \right) dx$

$= \int_0^{\pi} (\sin x - 0)\,dx = \int_0^{\pi} \sin x\,dx = (-\cos x)_0^{\pi}$

$= -\cos\pi + \cos 0$

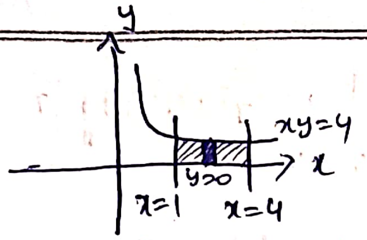
$= -(-1) + 1$

$= 1+1$

$= 2$

3. Evaluate $\iint x^2\,dx\,dy$ over the region bounded by hyperbola $xy=4, y=0,$ $x=1, x=4$

$\iint x^2 \, dx\, dy$



$x \to 1-4$

$y \to 0 - 4/x$

$= \int_1^4 \int_0^{4/x} x^2 \, dy\, dx = \int_1^4 x^2 (y)_0^{4/x} dx = \int_1^4 x^2 \left(\frac{4}{x}\right) dx = 4\int_1^4 x\, dx$

$= 4\left(\frac{x^2}{2}\right)_1^4 = 4\left(\frac{16}{2} - \frac{1}{2}\right) = \cancel{4}^{2}\left(\frac{15}{\cancel{2}}\right) = 30$

4. Evaluate $\int_0^2 \int_0^x e^{(x+y)} \, dx\, dy$

Sol: $\int_0^2 \int_0^x e^{(x+y)} \, dx\, dy = \int_0^2 \int_0^x e^{x+y} \, dy\, dx$

$= \int_0^2 e^x (e^y)_0^x \, dx = \int_0^2 e^x (e^x - e^0) dx = \int_0^2 e^x (e^x - 1)\, dx$

$= \int_0^2 e^{2x} - e^x \, dx = \left(\frac{e^{2x}}{2} - e^x\right)_0^2 = \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$

$= \frac{e^4}{2} - e^2 + \frac{1}{2}$

$= \frac{e^4 - 2e^2 + 1}{2}$

5. Evaluate $\int_0^2 \int_0^x y\,dy\,dx$

$$\int_0^2 \int_0^x y\,dy\,dx = \int_0^2 \left(\frac{y^2}{2}\right)_0^x dx = \int_0^2 \left(\frac{x^2}{2}\right) = \left(\frac{x^3}{2\times 3}\right)_0^2 = \frac{2^3}{6} = \frac{8}{6}$$

$$= \frac{4}{3}$$

6. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Sol: $= \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 x^2\sqrt{x} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\,dx$$

$$= \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\,dx$$

$$= \left(\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{(5/2)3} - \frac{4x^4}{\frac{12}{3}}\right)_0^1 = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{2}{7} + \frac{2-5}{15}$$

$$= \frac{2}{7} - \frac{1}{5} = \frac{10-7}{35} = \frac{3}{35}$$

7. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

Sol: $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

$$= \int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2}dx\right]dy.$$

we know $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, using gamma function.

$$\Rightarrow \int_0^\infty e^{-y^2}\frac{\sqrt{\pi}}{2}\,dy$$

$$= \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2}dy$$

$$= \frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2} \qquad \left(\text{Illy } \int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}\right)$$

$$= \frac{\pi}{4}.$$

8. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} 6\,dz\,dy\,dx$

Soln:

$$6\int_0^1\int_0^{1-x}\int_0^{1-x-y} 6\,dz\,dy\,dx$$

$$= 6\int_0^1\int_0^{1-x}(z)_0^{1-x-y}\,dy\,dx$$

$$= 6\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$= 6\int_0^1\left(y-xy-\frac{y^2}{2}\right)_0^{(1-x)}dx = 6\int_0^1 1-x-x(1-x)-\frac{(1-x)^2}{2}\,dx$$

$$= 6\int_0^1 1-x-x+x^2-\frac{1-2x+x^2}{2}\,dx = 6\int_0^1 1-2x+x^2-\frac{1-2x+x^2}{2}\,dx$$

$$= 6\int_0^1 \frac{2-4x+2x^2-1+2x-x^2}{2}\,dx = \frac{\overset{3}{\cancel{6}}}{\cancel{2}}\int_0^1 x^2-2x+1\,dx = 3\left(\frac{x^3}{3}-\frac{\cancel{2}x^2}{\cancel{2}}+x\right)_0^1$$

$$= \cancel{3}\left(\frac{1}{\cancel{3}}-\cancel{1}+\cancel{1}\right) = 1$$ ans

9. Evaluate $\int_0^{\frac{\pi}{6}}\int_0^{1}\int_{-2}^{3} y\sin z\, dx\, dy\, dz$

Sol:- $\int_0^{\frac{\pi}{6}}\int_0^{1}\int_{-2}^{3} y\sin z\, dx\, dy\, dz$

$\int_0^{\frac{\pi}{6}}\int_0^{1} y\sin z\,(x)_{-2}^{3}\, dy\, dz$

$\int_0^{\frac{\pi}{6}}\int_0^{1} y\sin z\,(3+2)\, dy\, dz = 5\int_0^{\frac{\pi}{6}} \sin z\left(\frac{y^2}{2}\right)_0^1 dz$

$= 5\int_0^{\frac{\pi}{6}} \sin z\left(\frac{1}{2}\right) dz = \frac{5}{2}\int_0^{\frac{\pi}{6}} \sin z\, dz = \frac{5}{2}(-\cos z)_0^{\frac{\pi}{6}}$

$= \frac{5}{2}\left(-\cos\frac{\pi}{6} + \cos 0\right) = \frac{5}{2}\left(-\frac{\sqrt{3}}{2}+1\right) = \frac{5}{2}\left(\frac{-\sqrt{3}+2}{2}\right)$

$= \frac{5}{2}\left(\frac{2-\sqrt{3}}{2}\right) = \frac{5}{4}(2-\sqrt{3})$ //

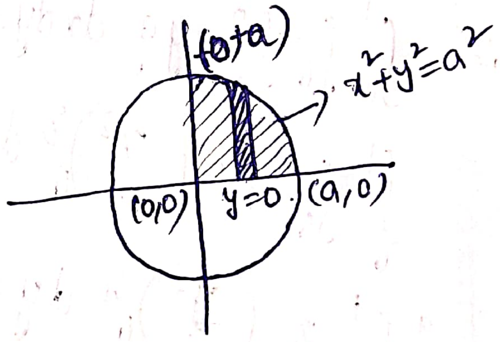
10. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$

Sol $x^2+y^2=a^2$

$C(0,0)\ r=a$

$x \rightarrow 0 - a$

$y \rightarrow 0 - \sqrt{a^2-x^2}$



area $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dx\, dy$

$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dx\, dy = \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$

$= \int_0^a \left(\frac{x\,(\cancel{\sqrt{a^2-x^2}})^{\cancel{2}}}{2}\right) dx = \int_0^a x\left(\frac{a^2-x^2}{2}\right) dx$

$$= \int_0^a \frac{xa^2 - x^3}{2}\, dx$$

$$= \frac{1}{2}\int_0^a xa^2 - x^3 dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}a^2 - \frac{x^4}{4}\right)_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right).$$

$$= \frac{1}{2}\left(\frac{a^4}{4}\right)$$

$$= \frac{a^4}{8}.$$

11. Evaluate $I = \int_0^1 \int_0^2 \int_0^3 xyz \, dx dy dz.$

Sol: $\int_0^1 \int_0^2 \int_0^3 xyz \, dx dy dz$

$\int_0^1 \int_0^2 yz \left(\frac{x^2}{2}\right)_0^3 dy dz = \int_0^1 \int_0^2 yz\left(\frac{9}{2}\right) dy\, dz$

$\frac{9}{2}\int_0^1 z\left(\frac{y^2}{2}\right)_0^2 dy\, dz$

$= \frac{9}{2}\int_0^1 z\left(\frac{4}{2}\right) dz$

$= \frac{9}{\cancel{2}} \times \cancel{2} \int_0^1 z\, dz = 9\left(\frac{z^2}{2}\right)_0^1 = 9\left(\frac{1}{2}\right) = \frac{9}{2}$

12. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dy\, dx$

Sol: $= \int_0^5 \left(\int_0^{x^2} x(x^2+y^2)\, dy \right) dx$

$= \int_0^5 \left(\int_0^{x^2} x^3 + xy^2 \, dy \right) dx$

$= \int_0^5 \left(x^3 y + \frac{xy^3}{3} \right)_0^{x^2} dx$

$= \int_0^5 \left(x^5 + \frac{x^7}{3} - 0 \right) dx$

$= \left(\frac{x^6}{6} + \frac{x^8}{24} \right)_0^5 = \frac{5^6}{6} + \frac{5^8}{24} = \frac{5^6 \times 4 + 5^8}{24} = \frac{5^6(4+25)}{24}$

$= 5^6 \left(\frac{29}{24} \right)$

13. Evaluate $\int_0^4 \int_{y^2/4}^{y} \frac{y}{(x^2+y^2)}\, dx\, dy$

Sol: $= \int_0^4 \left(\int_{y^2/4}^{y} \left(\tan^{-1}\frac{x}{y} \right)_{y^2/4}^{y} \right) dy$

$= \int_0^4 \left(\tan^{-1}(1) - \tan^{-1}\left(\frac{y^2/4}{y} \right) \right) dy$

$= \int_0^4 \left(\frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4} \right) \right) dy \longrightarrow (1)$

Consider $\int \tan^{-1}\left(\frac{y}{4} \right) \cdot 1\, dy$

$\tan^{-1}\left(\frac{y}{4}\right)\int 1\,dy - \int \left(\frac{d}{dy}\tan^{-1}\left(\frac{y}{4}\right)\int 1\,dy\right) dy$

$= \tan^{-1}\left(\frac{y}{4}\right) y - \int \frac{1}{1+(y/4)^2}\,\frac{y}{4}\,dy$

$= y\tan^{-1}\left(\frac{y}{4}\right) - \int \frac{4y}{4^2+y^2}\,dy$

$= y\tan^{-1}\left(\frac{y}{4}\right) - 2\int \frac{2y}{4^2+y^2}\,dy$

$= y\tan^{-1}\left(\frac{y}{4}\right) - 2\int \frac{2y}{16+y^2}\,dy$

$= y\tan^{-1}\left(\frac{y}{4}\right) - 2\log(16+y^2)$

Eq ① becomes

$= \int_0^4 \frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right) dy$

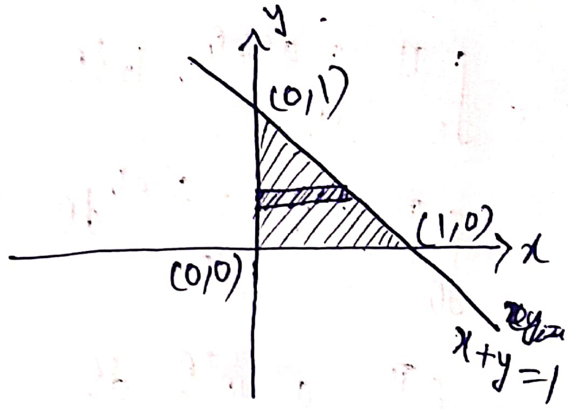
$= \left[\frac{\pi}{4}y - y\tan^{-1}\left(\frac{y}{4}\right) + 2\log(16+y^2)\right]_0^4$

$= \frac{\pi}{\cancel{4}}(\cancel{4}) - 4\tan^{-1}(1) + 2\log 32 - 2\log 16$

$= \cancel{\pi} - \cancel{\pi} + 2\log 32 - 2\log 16$

$= 2\left(\log\frac{32}{16}\right) = 2\log 2 = \log 4$ //

14. Evaluate $\iint (x^2+y^2)\, dx\, dy$ in the positive quadrant for which $x+y \le 1$



Sol

$x \to 0-(1-y)$

$y \to 0-1$

$$= \int_0^1 \int_0^{(1-y)} x^2+y^2 \, dx\, dy$$

$$= \int_0^1 \left(\int_0^{(1-y)} (x^2+y^2\, dx) \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + y^2 x \right)_0^{(1-y)} dy$$

$$= \int_0^1 \frac{(1-y)^3}{3} + y^2(1-y)\, dy$$

$$= \int_0^1 \frac{1-3(1)^2(y)+3(1)(y^2)-y^3}{3} + y^2 - y^3 \, dy$$

$$= \int_0^1 \frac{1-3y+3y^2-y^3}{3} + y^2 - y^3 \, dy$$

$$= \int_0^1 \frac{(1-3y+3y^2-y^3)}{3} + y^2 - y^3 \, dy$$

$$= \int_0^1 \frac{(1-3y+3y^2-y^3+3y^2-3y^3)}{3} \, dy$$

$$= \frac{1}{3} \int_0^1 (1-3y+6y^2-4y^3)\, dy$$

$$= \frac{1}{3}\left(y - \frac{3y^2}{2} + \frac{\cancel{6}^{2} y^3}{\cancel{3}} - \frac{\cancel{4}y^4}{\cancel{4}} \right)_0^1$$

$$= \frac{1}{3}\left(\cancel{1} - \frac{3}{2} + 2 - \cancel{1}\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$$

15. Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta$

$= \int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta$

$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$

$= \int_0^{\pi} \frac{a^2\sin^2\theta}{2}\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi} \sin^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi} \sin^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta$

$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$

$= \frac{a^2}{4}(\pi - 0)$

$= \frac{\pi a^2}{4}$ //

LAQ's

1. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta$

Sol: $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, d\theta$

$\int_0^{\pi}\left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)}\cos\theta\, d\theta$

$= \int_0^{\pi}\frac{a^3(1+\cos\theta)^3}{3}\cos\theta\, d\theta$

$= \frac{a^3}{3}\int_0^{\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\cos\theta\, d\theta$

$= \frac{a^3}{3}\int_0^{\pi}\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta\, d\theta$

$= \frac{a^3}{3}\int_0^{\pi}\cos\theta+3\left(\frac{1+\cos 2\theta}{2}\right)+\frac{3}{4}\left(\cos 3\theta+3\cos\theta\right)+\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$= \frac{a^3}{3}\int_0^{\pi}\cos\theta+\frac{3}{2}+\frac{3}{2}\cos 2\theta+\frac{3}{4}\cos 3\theta+\frac{9}{4}\cos\theta+$
$\frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right) d\theta$

$= \frac{a^3}{3}\int_0^{\pi}\frac{13}{4}\cos\theta+\frac{3}{2}+\frac{3}{2}\cos 2\theta+\frac{3}{4}\cos 3\theta+\frac{1}{4}+$
$\frac{1}{2}\cos 2\theta+\frac{1+\cos 4\theta}{8}\, d\theta$

$= \frac{a^3}{3}\int_0^{\pi}\left(\frac{13}{4}\cos\theta+\frac{3}{2}+\frac{3}{2}\cos 2\theta+\frac{3}{4}\cos 3\theta+\frac{1}{4}+\frac{1}{2}\cos 2\theta\right.$
$\left.+\frac{1}{8}+\frac{1}{8}\cos 4\theta\right) d\theta$

$$= \frac{a^3}{3} \int_0^{\pi} \left(\frac{13}{4} \cos\theta + \frac{15}{8} + 2\cos 2\theta + \frac{3}{4} \cos 3\theta + \frac{1}{8} \cos 4\theta \right) d\theta$$

$$= \frac{a^3}{3} \left[\frac{13}{4} \sin\theta + \frac{15}{8} \theta + 2 \frac{\sin 2\theta}{2} + \frac{3}{4} \frac{\sin 3\theta}{3} + \frac{1}{8} \frac{\sin 4\theta}{4} \right]_0^{\pi}$$

$$= \frac{a^3}{\cancel{3}} \left[\frac{\overset{5}{\cancel{15}}}{8} \pi \right]$$

$$= \frac{5\pi a^3}{8}$$

2. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$

Sol: $\int_0^4 \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx \qquad \left[\because \int e^{ax}dx = \frac{e^{ax}}{a}\right]$

$\int_0^4 xe^x - x\,dx$

$= \left[\left(xe^x - \int 1\cdot e^x dx\right) - \frac{x^2}{2}\right]_0^4$

$= \left[xe^x - e^x - \frac{x^2}{2}\right]_0^4$

$= 4e^4 - e^4 - \cancel{\frac{16}{2}}^{\,8} + 1$

$= 4e^4 - e^4 - 8 + 1$

$= 3e^4 - 7$

3. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates

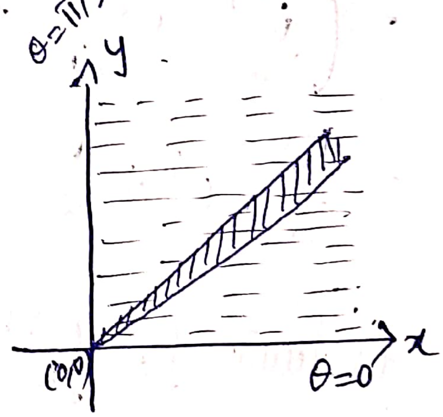
$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$x = 0$, $y = 0$

$x = \infty$ $y = \infty$

$r \to 0 - \infty$

$\theta \to 0 - \frac{\pi}{2}$



$\int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta$

multiply & divide by -2

$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}(-2r\,dr\,d\theta) = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(e^{-r^2}\right)_0^\infty d\theta$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (e^{-\infty} - e^0)\,d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (d\theta) = \frac{1}{2}(\theta)_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Evaluate $\int_0^2 \int_0^x e^{(x+y)}\,dy\,dx$.

$$\int_0^2 \int_0^x e^{(x+y)}\,dy\,dx$$

$$\int_0^2 \int_0^x e^x \cdot e^y\,dy\,dx$$

$$\int_0^2 e^x (e^y)_0^x\,dx$$

$$= \int_0^2 e^x(e^x - e^0)\,dx = \int_0^2 e^x(e^x - 1)\,dx = \int_0^2 e^{2x} - e^x\,dx$$

$$= \left(\frac{e^{2x}}{2} - e^x\right)_0^2 = \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$$

$$= \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{e^4 - 2e^2 + 1}{2}$$

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2+y^2)}\,dy\,dx$

$$\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2}\,dy\,dx$$

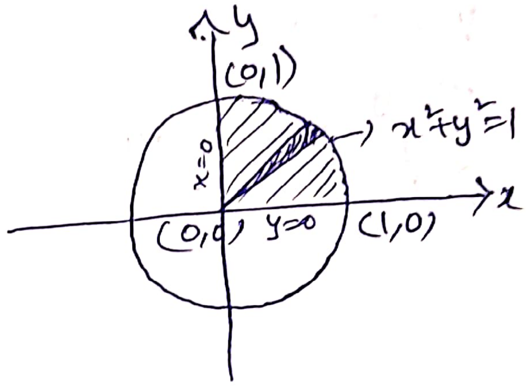
$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)_0^{\sqrt{1+x^2}}\right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}}\right) - 0 \, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} 1 \, dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} \, dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx = \frac{\pi}{4} \left(\sinh^{-1} x\right)_0^1 = \frac{\pi}{4} \left(\sinh^{-1} 1\right)$$

6Q. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$ by changing to polar coordinates.



Sol:- $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy.$

$x = r\cos\theta \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta.$

$x = 0 \quad x = \sqrt{1-y^2}$

$y = 0 \quad y = 1$

$x^2+y^2 = 1$

$r^2 = 1$

$r = 1$

$r \to 0 - 1$

$\theta \to 0 - \pi/2.$

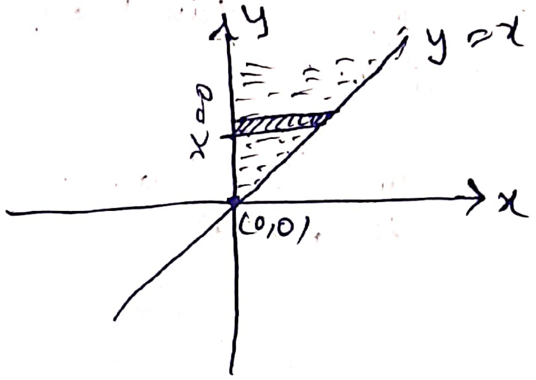
$$= \int_0^{\pi/2} \int_0^1 r^2 \, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 r^3 \, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^1 d\theta \quad \int_0^{\pi/2} \frac{1}{4} \, d\theta = \frac{1}{4} \left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4} \left[\frac{\pi}{2}\right] = \frac{\pi}{8}.$$

79) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Sol: $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$.



$x=0 \quad x=\infty$

$y=x \quad y=\infty$

$x \to 0 - y$ (By considering horizontal strip)

$y \to 0 - \infty$.

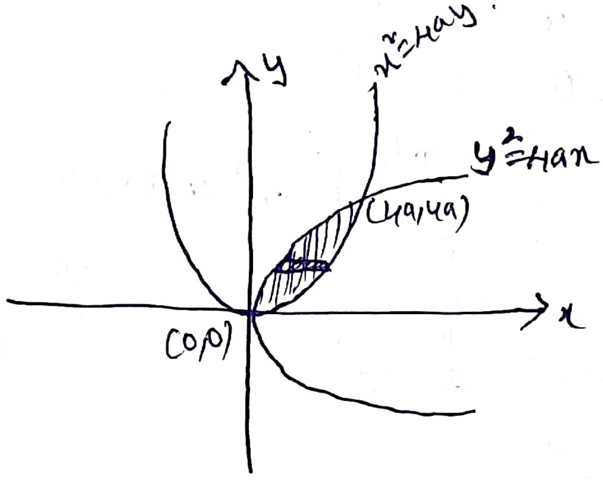
$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y}[x]_0^y\,dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}}(\cancel{y})\,dy$$

$$= \int_0^\infty e^{-y}\,dy = (-e^{-y})_0^\infty$$

$$= 0+1 = 1$$

80) Evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

Sol: $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$.



$x=0 \quad x=4a$

$y=\frac{x^2}{4a} \quad y=2\sqrt{ax}$.

$\Rightarrow x^2=4ay \quad y^2=4ax$.

$x \to y^2/4a - 2\sqrt{ay}$ (By considering horizontal strip)

$y \to 0 - 4a$

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy \cdot = \int_0^{4a} (x)_{y^2/4a}^{2\sqrt{ay}}\, dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

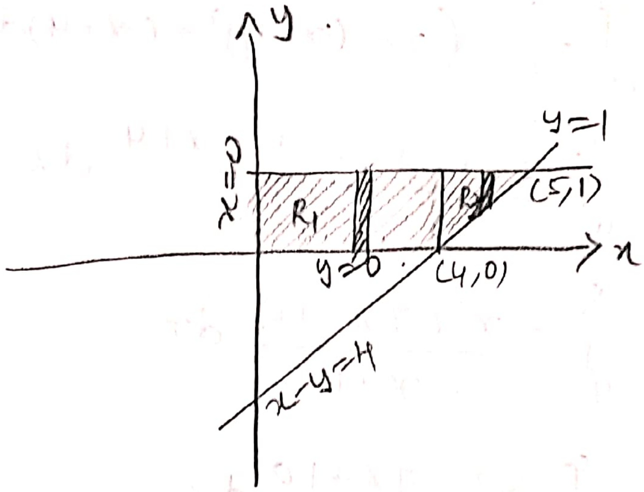
$$= \int_0^{4a} 2\sqrt{a}\sqrt{y} - \frac{1}{4a}y^2\,dy = \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right)_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\cdot(4a)^{3/2} - \frac{1}{\cancel{4a}}\frac{(4a)^{\cancel{3}\,2}}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{a^2}{3}(32-16) = \frac{16a^2}{3}.$$

9. Evaluate $\int_0^1\int_0^{y+4} \frac{2y+1}{x+1}\, dx\, dy$ by changing the order of integration.

Sol: $\int_0^1\int_0^{y+4} \frac{2y+1}{x+1}\, dx\, dy$



$x=0 \quad x=y+4$

$y=0 \quad x-y=4$

$y=1 \quad y=1$

$\iint_R = \iint_{R_1} + \iint_{R_2}$

Region: 1

$x \to 0-4$

$y \to 0-1$

$\int_0^4\int_0^1 \frac{2y+1}{x+1}\, dy\, dx$

$\int_0^4 \frac{1}{x+1}\left(\frac{2y^2}{2}+y\right)_0^1 dx$

$\int_0^4 \frac{1}{x+1}(2)\, dx$

$2\left(\log(x+1)\right)_0^4$

$2\log 5 - 0 = 2\log 5$

Region 2:

$x \to 4-5$

$y \to x-4-1$

$\int_4^5\int_{x-4}^1 \frac{2y+1}{x+1}\, dy\, dx.$

$$\int_4^5 \frac{1}{x+1}\left[\frac{xy^2}{2}+y\right]_{x-4}^{1} dx$$

$$\int_4^5 \frac{1}{x+1}\left(2-(x-4)^2-(x-4)\right)dx$$

$$\int_4^5 \frac{2-x^2+8x-16-x+4}{x+1}\,dx$$

$$\int_4^5 \frac{-x^2+7x-10}{x+1}\,dx$$

$$-\int_4^5 \frac{x^2-7x+10}{x+1}\,dx$$

By division method

$$x+1 \overline{)\, x^2-7x+10 \,(} \, x-8$$
$$\underline{x^2+x}$$
$$-8x+10$$
$$-8x-8$$
$$+ \quad +$$
$$\underline{18}$$

$$= -\int_4^5 x-8+\frac{18}{x+1}\,dx$$

$$= -\left[\frac{x^2}{2}-8x+18\log(x+1)\right]_4^5$$

$$= -\left[\frac{25}{2}-40+18\log 6-\cancel{\frac{16}{2}}^{8}+32-18\log 5\right]$$

$$= -\left[\frac{25}{2}-16+18\log 6-18\log 5\right]$$

$$\iint_R = \iint_R + \iint_{R_2} = 2\log 5-\frac{25}{2}+16-18\log 6+18\log 5$$

$$= 20\log 5+\frac{7}{2}-18\log 6$$

10Q. Evaluate $\iiint (xy+yz+zx)\,dx\,dy\,dz$ where v is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$.

Sol:- $\iiint (xy+yz+zx)\,dx\,dy\,dz$.

$x \to 0-1$
$y \to 0-2$
$z \to 0-3$.

$$\int_0^3\int_0^2\int_0^1 (xy+yz+zx)\,dx\,dy\,dz = \int_0^3\int_0^2 \left(\frac{x^2}{2}y + xyz + z\frac{x^2}{2}\right)_0^1 dy\,dz$$

$$= \int_0^3\int_0^2 \left(\frac{1}{2}y + yz + \frac{1}{2}z\right) dy\,dz = \int_0^3 \left(\frac{1}{2}\frac{y^2}{2} + z\frac{y^2}{2} + \frac{1}{2}zy\right)_0^2 dz$$

$$= \int_0^3 \left(\frac{1}{\cancel{2}}\times\cancel{2} + z\times 2 + \frac{1}{\cancel{2}}\cancel{2}z\right) dz = \int_0^3 1 + 2z + z\,dz$$

$$= \int_0^3 1+3z\,dz = \left(z + 3\frac{z^2}{2}\right)_0^3 = 3 + 3\left(\frac{9}{2}\right)$$

$$= 3 + \frac{27}{2} = \frac{6+27}{2}$$

$$= \frac{33}{2}$$

11Q. Evaluate $\int_0^{\ln c}\int_0^{\ln b}\int_0^{\ln a} e^{x+y+z}\,dx\,dy\,dz$.

Sol:- $\int_0^{\ln c}\int_0^{\ln b}\int_0^{\ln a} e^{x+y+z}\,dx\,dy\,dz$.

$$= \int_0^{\ln c}\int_0^{\ln b} e^{y+z}\left(e^x\right)_0^{\ln a} dy\,dz.$$

$$= \int_0^{\ln c}\int_0^{\ln b} e^{y+z}\left(e^{\ln a} - e^0\right) dy\,dz.$$

$$= \int_0^{\ln c}\int_0^{\ln b} e^{y+z}(a-1)\,dy\,dz = (a-1)\int_0^{\ln c} e^{z}\left(e^{y}\right)_0^{\ln b} dz.$$

$$= (a-1)\int_0^{\ln c} e^{z}\left(e^{\ln b} - e^{0}\right) dz = (a-1)\int_0^{\ln c} e^{z}(b-1)\,dz$$

$$= (a-1)(b-1)\int_0^{\ln c} e^{z}\,dz = (a-1)(b-1)\left(e^{z}\right)_0^{\ln c}$$

$$= (a-1)(b-1)\left(e^{\ln c} - e^{0}\right)$$

$$= (a-1)(b-1)(c-1)$$

12Q) Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming to spherical coordinates.

Sol:- $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$.

Spherical coordinates.

$x = \rho\sin\theta\cos\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\theta$

$dx\,dy\,dz = \rho^2\sin\theta\, d\rho\, d\theta\, d\phi$

$\rho \to 0-1$

$\theta \to 0-\pi$

$\phi \to 0-2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\theta\, d\rho\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\right)_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\sin\theta\, d\theta\, d\phi = \frac{1}{3}\int_0^{2\pi}\left(-\cos\theta\right)_0^{\pi} d\phi$$

$$= \frac{1}{3}\int_0^{2\pi} (-\cos\pi + \cos 0)\, d\phi$$

$$= \frac{1}{3}\int_0^{2\pi} (1+1)\, d\phi = \frac{2}{3}(\phi)_0^{2\pi} = \frac{2}{3}(2\pi) = \frac{4\pi}{3}$$

13Q. Find the volume of the unit sphere $x^2+y^2+z^2=1$

Sol:- Volume $= \iiint dx\,dy\,dz$.

In spherical coordinates.

$dx\,dy\,dz = \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$

$\rho \to 0-1$

$\theta \to 0-\pi$

$\phi \to 0-2\pi$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\right)_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\sin\theta\, d\theta\, d\phi = \frac{1}{3}\int_0^{2\pi} (-\cos\theta)_0^{\pi}\, d\phi$$

$$= \frac{1}{3}\int_0^{2\pi} (-\cos\pi + \cos 0)\, d\phi = \frac{1}{3}\int_0^{2\pi} (+1+1)\, d\phi$$

$$= \frac{2}{3}(\phi)_0^{2\pi} = \frac{2}{3}(2\pi) = \frac{4\pi}{3}.$$

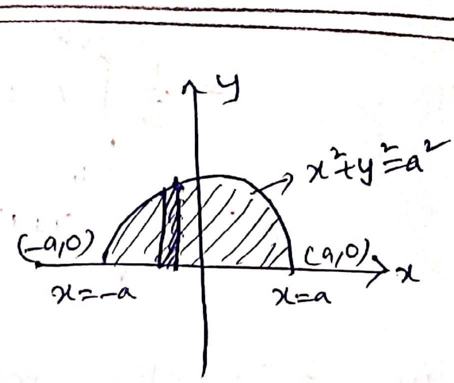
$\therefore$ Volume of unit sphere $= \frac{4\pi}{3}$

14Q. Transform the following to Cartesian form and hence evaluate. $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$.

Sol:- $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$.

$x = r\cos\theta \quad y = r\sin\theta$.

$dx\,dy = r\,dr\,d\theta$.



$r \to 0 - a$

$\theta \to 0 - \pi$.

By considering the vertical strip.

$x \to -a$ to a.

$y \to 0$ to $\sqrt{a^2-x^2}$

$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\,d\theta = \int_0^{\pi}\int_0^{a} (r\cos\theta)(r\sin\theta)(r\,dr\,d\theta)$$

$$= \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy \, dy\,dx.$$

$$= \int_{-a}^{a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = \int_{-a}^{a} x\left(\frac{a^2-x^2}{2}\right) dx.$$

$$= \frac{1}{2}\int_{-a}^{a} a^2x - x^3 \, dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{-a}^{a}$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} - \frac{a^4}{2} + \frac{a^4}{4}\right] = 0.$$

15Q. Evaluate $\displaystyle\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$.

Sol:- $$\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^3}{3}\right)_{a(1-\cos\theta)}^{a} d\theta.$$

$$= \int_0^{\pi/2}\left(\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3}\right) d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} 1 - (1-\cos\theta)^3 \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} 1 - (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta) \, d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \cancel{1} - \cancel{1} + 3\cos\theta - 3\cos^2\theta + \cos^3\theta \, d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - 3\cos^2\theta + \cos^3\theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - 3\left(\frac{1 \pm \cos 2\theta}{2}\right) + \frac{1}{4}(\cos 3\theta + 3\cos\theta) \, d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta \cdot d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \frac{15}{4}\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta \, d\theta$$

$$= \frac{a^3}{3} \left[\frac{15}{4}\sin\theta - \frac{3}{2}\theta - \frac{3}{2}\frac{\sin 2\theta}{2} + \frac{1}{4}\frac{\sin 3\theta}{3} \right]_0^{\pi/2}.$$

$$= \frac{a^3}{3} \left\{ \frac{15}{4}(1) - \frac{3}{2}\left(\frac{\pi}{2}\right) - 0 + \frac{1}{12}(-1) \right\}$$

$$= \frac{a^3}{3} \left\{ \frac{15}{4} - \frac{1}{12} - \frac{3\pi}{4} \right] = \frac{a^3}{3} \left[\frac{45 - 1 - 9\pi}{12} \right]$$

$$= \frac{a^3}{3} \left\{ \frac{44 - 9\pi}{12} \right] = \frac{a^3}{36}(44 - 9\pi)$$

UNIT-05

SAQ :-

1. If $\bar{a}$ is a constant vector and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then evaluate $\operatorname{div}(\vec{a} \times \vec{r})$

Sol:- $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{a} \times \bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\bar{a} \times \bar{r} = \bar{i}(a_2 z - a_3 y) - \bar{j}(a_1 z - a_3 x) + \bar{k}(a_1 y - a_2 x)$$

$$\operatorname{div}(\bar{a} \times \bar{r}) = \nabla \cdot (\bar{a} \times \bar{r})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left((a_2 z - a_3 y)\bar{i} - (a_1 z - a_3 x)\bar{j} + (a_1 y - a_2 x)\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) - \frac{\partial}{\partial y}(a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0.$$

---

2. Prove that $\operatorname{curl}(\operatorname{grad} \bar{f}) = 0$ where f is a differentiable scalar field.

Sol:- $\operatorname{grad} f = \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$$\operatorname{curl}(\operatorname{grad} f) = \nabla \times \operatorname{grad} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) - \bar{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)$$

$$= 0$$

3. Find ∇f if $f = \log_e(x^2+y^2+z^2)$.

Sol:

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}\frac{2x}{x^2+y^2+z^2} + \bar{j}\frac{2y}{x^2+y^2+z^2} + \bar{k}\frac{2z}{x^2+y^2+z^2}$$

$$= \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{x^2+y^2+z^2}$$

4. Evaluate $\int \bar{v}\cdot dr$ when $\vec{v} = x\bar{i} + y\bar{j} + z\bar{k}$ and c is the line segment from $A(1,2,2)$ to $B(3,6,6)$.

Sol:

$$\int \bar{v}\cdot d\bar{r} = \int_A^B (x\bar{i}+y\bar{j}+z\bar{k})\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$$

$$= \int_{(1,2,2)}^{(3,6,6)} x\,dx + y\,dy + z\,dz$$

Let $x = t \quad y = 2t \quad z = 2t$

$dx = dt \quad dy = 2dt \quad dz = 2dt.$

$t \to 1-3$

$$= \int_1^3 t\,dt + 4t\,dt + 4t\,dt = \int_1^3 9t\,dt = 9\left(\frac{t^2}{2}\right)_1^3$$

$$= 9\left(\frac{9}{2} - \frac{1}{2}\right)$$

$$= 9\left(\frac{\cancel{8}^{4}}{\cancel{2}}\right) = \underline{\underline{36}}.$$

5. Find the unit normal vector to the surface $f(x,y,z) = x^2y - y^2z - xyz$ at $P(1,-1,0)$.

Sol:- $\hat{n} = \dfrac{\nabla f}{|\nabla f|}$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i(2xy - yz) + j(x^2 - 2yz - xz) + k(-y^2 - xy)$$

$$(\nabla f)_{(1,-1,0)} = -2i + j$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{-2i+j}{\sqrt{4+1}} = \frac{-2i+j}{\sqrt{5}}$$

6. Find $\operatorname{div}\bar{f}$ where $\bar{f} = \operatorname{grad}(x^3+y^3+z^3+3xyz)$.

Sol:- $\operatorname{grad}(x^3+y^3+z^3+3xyz)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3+y^3+z^3+3xyz)$$

$$= \bar{i}(3x^2+3yz)+\bar{j}(3y^2+3xz)+\bar{k}(3z^2+3xy)$$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2+3yz)+\frac{\partial}{\partial y}(3y^2+3xz)+\frac{\partial}{\partial z}(3z^2+3xy)$$

$$= 6x+6y+6z$$

$$\operatorname{div}\bar{F} = 6(x+y+z).$$

7. State Green's theorem on a plane.

Ans. Green's Theorem :-

If s is a closed region in xy-plane bounded by a simple closed curve c and if M and N are continious functions of x and y having continious derivative in the region. R then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where

'c' is transversed in positive direction (anticlock wise).

8. Show that $V = 3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k$ is solenoidal.

Sol:- $\text{div}\,\bar{V} = \nabla \cdot \bar{V}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(3y^4z^2\bar{i} + 4x^3z^2\bar{j} + 3x^2y^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial x}(-3x^2y^2)$$

$$= 0$$

$\text{div}\,\bar{V} = 0$

$\therefore \bar{V}$ is solenoidal.

---

9. Find the directional derivative of the function $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of $2i - j - k$.

Sol:- Directional Derivative $= \nabla\phi \cdot \hat{e}$

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y + 8xz)$$

$$(\nabla\phi)_{(1,-2,-1)} = \bar{i}(+4+4) + \bar{j}(-1) + \bar{k}(-2-8)$$

$$= 8\bar{i} - \bar{j} - 10\bar{k}$$

$$\hat{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i} - \bar{j} - \bar{k}}{\sqrt{4+1+1}} = \frac{2\bar{i} - \bar{j} - \bar{k}}{\sqrt{6}}$$

Directional derivative $= \nabla\phi \cdot \hat{e}$

$$= (8\bar{i} - \bar{j} - 10\bar{k}) \cdot \frac{(2\bar{i} - \bar{j} - \bar{k})}{\sqrt{6}} = \frac{16 + 1 + 10}{\sqrt{6}}$$

$$= \frac{27}{\sqrt{6}}.$$

10. State Gauss divergence theorem.

Gauss divergence theorem:-

Let s be a closed surface enclosing a volume 'V' if $\bar{F}$ is continuously differentiable vector point function then

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds$$

11. If $\bar{a}$ is a constant vector and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then evaluate $\operatorname{curl}(\vec{a} \times \vec{r})$.

Sol:-

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\bar{a} \times \bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \bar{i}(a_2 z - a_3 y) - \bar{j}(a_1 z - a_3 x) + \bar{k}(a_1 y - a_2 x)$$

$$\operatorname{curl}(\bar{a}\times\bar{r}) = \nabla\times(\bar{a}\times\bar{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2z-a_3y & -a_1z+a_3x & a_1y-a_2x \end{vmatrix}$$

$$= i(a_1+a_1) - j(a_2-a_2) + k(a_3+a_3)$$

$$= 2a_1\bar{i} + 2a_2\bar{j} + 2a_3\bar{k}$$

$$= 2(a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$

$$= 2\bar{a}$$

12. Prove that curl(grad f) = 0, where f is a differentiable Scalar field.

Sol:-

$$\operatorname{grad} f = \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\operatorname{curl}(\operatorname{grad} f) = \nabla\times\operatorname{grad} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right) - j\left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right) + k\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right)$$

$$= 0$$

13. State Green's theorem.

Ans. **Green's theorem:-**

If S is a closed region in xy-plane bounded by a simple closed curve c and if M and N are continious functions of x and y having continious derivative in the region R then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

where

'c' is transversed in positive direction (anticlockwise).

14. Stake stoke's theorem.

Stoke's theorem:-

Let s be a open surface bounded by a closed non intersecting curve C, if $\bar{F}$ is any differentiable vector point function then

$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds.$$

15. Show that $\overline{V} = 12x\hat{i} - 15y\hat{j} + k$ is irrotational.

Sol:- Give $\overline{V} = 12x\bar{i} - 15y\bar{j} + \bar{k}$

$$\nabla \times \overline{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 12x & -15y & 1 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(0-0)$$

$$= 0$$

$\therefore$ $\overline{V}$ is irrotational.

UNIT-05 (LAQ)

1. Evaluate $\oint x\,dy - y\,dx$ where c is the triangle with vertices $(0,0)$ $(2,0)$ and $(0,1)$ using greens theorem.

Sol:- By Greens theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

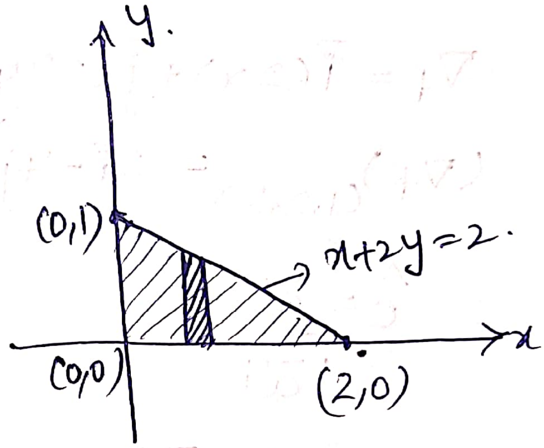
$$M = -y \qquad N = x$$

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint (1+1)\,dx\,dy = \iint 2\,dx\,dy$$

$x \to 0-2$

$y \to 0 - \frac{2-x}{2}$



$$= \int_0^2 \int_0^{2-x/2} 2\,dy\,dx$$

$$= 2\int_0^2 (y)_0^{\frac{2-x}{2}}\,dx$$

$$= 2\int_0^2 \frac{2-x}{2}\,dx = \left(2x - \frac{x^2}{2}\right)_0^2 = 4-2 = 2$$

$$\therefore \oint_C x\,dy - y\,dx = 2$$

2. Find the Directional Derivative of $f(x,y,z) = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line $\overline{PQ}$ where Q is the point $(5,0,4)$.

Sol:- $P(1,2,3)$ $Q(5,0,4)$

$\overline{PQ} = \overline{OQ} - \overline{OP}$

$\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$

$\overline{OQ} = 5\bar{i} + 4\bar{k}$

$\bar{a} = \overline{PQ} = (5\bar{i} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k})$

$\bar{a} = 4\bar{i} - 2\bar{j} + \bar{k}$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$\nabla f = \bar{i}(2x) + \bar{j}(-2y) + \bar{k}(4z)$

$(\nabla f)_{(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k}$

$$\hat{e} = \frac{\bar{a}}{|\bar{a}|}$$

$$= \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16+4+1}}$$

$$= \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$$

Direction Derivative $= \nabla f \cdot \hat{e}$

$$= (2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot \frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} \Rightarrow \frac{28}{\sqrt{21}}$$

3. Find the angle between the surfaces $x^2+y^2+z^2=9$, $z+3=x^2+y^2$ at $(-2,1,2)$.

Sol:-

$\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = x^2+y^2-z-3$

$\bar{n_1} = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$

$\bar{n_1} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$

$(\bar{n_1})_{(-2,1,2)} = -4\bar{i}+2\bar{j}+4\bar{k}$

$\bar{n_2} = \nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$

$= \bar{i}(2x)+\bar{j}(2y)+\bar{k}(-1)$

$(\bar{n_2})_{(-2,1,2)} = -4\bar{i}+2\bar{j}-\bar{k}$

$\cos\theta = \frac{\bar{n_1}\cdot\bar{n_2}}{|\bar{n_1}||\bar{n_2}|} = \frac{(-4i+2j+4k)\cdot(-4i+2j-k)}{\sqrt{16+4+16}\sqrt{16+4+1}}$

$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$

$= \frac{8}{3\sqrt{21}}$

$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

4. Find the constants a, b, c such that $\bar{F}=(2x+3y+az)\bar{i}+(bx+2y+3z)\bar{j}+(2x+cy+3z)\bar{k}$ is irrotational, find the scalar function f such that $\bar{F}=\nabla f$.

Sol:- Given $\text{curl}\,\bar{F}=0$

$$\nabla\times\bar{F}=\begin{vmatrix} \bar{i}\;\;\bar{j}\;\;\bar{k} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}=0$$

$\Rightarrow i(c-3)-j(2-a)+k(b-3)=0$

$c-3=0,\qquad -2+a=a,\qquad b-3=0$

$c=3,\qquad a=2,\qquad b=3$

Since $\bar{F}$ is irrotational $\exists$ a scalar potential f such that $\bar{F}=\nabla f$

$(2x+3y+2z)\bar{i}+(3x+2y+3z)\bar{j}+(2x+3y+3z)\bar{k}$

$=i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x}=2x+3y+2z \longrightarrow (1)$

$\frac{\partial f}{\partial y}=3x+2y+3z \longrightarrow (2)$

$\frac{\partial f}{\partial z}=2x+3y+3z \longrightarrow (3)$

on Integrating (1), (2) and (3)

$$f = \frac{2x^2}{2} + 3xy + 2xz + c_1(y,z)$$

$$f = 3xy + \frac{2y^2}{2} + 3yz + c_2(x,z)$$

$$f = 2xz + 3yz + \frac{3z^2}{2} + c_3(x,y)$$

$$\therefore f = x^2 + y^2 + \frac{3z^2}{2} + 3xy + 2xz + 3yz + c.$$

5. Find constant a and b so that the surface $ax^2 - 2bxyz = (a+4)x$ will be orthogonal to the surface $4x^2y + z^2 = 4$ at $(1,-1,2)$.

Sol:-

$$\phi_1 = ax^2 - 2byz - (a+4)x$$

$$\phi_2 = 4x^2y + z^2 - 4$$

$$\frac{\overline{n_1} \cdot \overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|} = 0 \qquad \begin{bmatrix} \theta = 90^\circ \\ \cos 90 = 0 \end{bmatrix}$$

$$\overline{n_1} \cdot \overline{n_2} = 0 \longrightarrow (1)$$

$$\overline{n_1} = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$\overline{n_1} = i(2ax - a - 4) + j(-2bz) + k(-2by)$$

$$(\overline{n_1})_{(1,-1,2)} = (a-4)\bar{i} - 4b\bar{j} + 2b\bar{k}$$

$$\bar{n_2} = \nabla \phi_2 = i\frac{\partial \phi_2}{\partial x} + j\frac{\phi_2}{\partial y} + k\frac{\partial \phi_2}{\partial z}$$

$$\bar{n_2} = i(8xy) + j(4x^2) + k(2z)$$

$$(\bar{n_2})_{(1,-1,2)} = -8\bar{i} + 4\bar{j} + 4\bar{k}$$

from equation (1)

$$((a-4)\bar{i} - 4b\bar{j} + 2b\bar{k}) \cdot (-8\bar{i} + 4\bar{j} + 4\bar{k})$$

$$-8(a-4) - 16b + 8b = 0$$

$$-a + 4 - 2b + b = 0$$

$$\Rightarrow a + b = 4 \longrightarrow (2)$$

sub the point $(1,-1,2)$ in $\phi_1 = ax^2 - 2byz - (a+4)x$

$$\Rightarrow a(1)^2 - 2b(-1)(2) = (a+4)(1)$$

$$\Rightarrow a + 4b = a + 4$$

$$\Rightarrow 4b = \cancel{a} + 4 - \cancel{a} \Rightarrow 4b = 4$$

$$\boxed{b = 1}$$

sub $b=1$ in equation (2)

$$\Rightarrow a + 1 = 4$$

$$\Rightarrow \boxed{a = 3}$$

$$\therefore a = 3, b = 1$$

6. Find angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$.

Sol:- $\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = x^2+y^2-z-3$

$$\overline{n_1} = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$\overline{n_1} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$$(\overline{n_1})_{(2,-1,2)} = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$\overline{n_2} = \nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y) + k(-1)$$

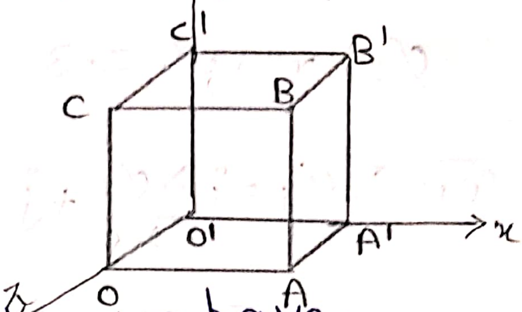
$$(\overline{n_2})_{(2,-1,2)} = 4\bar{i} - 2\bar{j} - k$$

$$\cos\theta = \frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\overline{n_2}|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

7. Verify Gauss divergence theorem for $F=(x^3-yz)i-2x^2yj+zk$ taken over the surface of the cube bounded by the planes $x=y=z=a$ and coordinate planes.



Sol:- $F=(x^3-yz)\bar{i}-2x^2y\bar{j}+z\bar{k}$

$x=y=z=a$

By Gauss Divergence theorem then we have

$$\int_S \bar{F}\cdot\bar{n}\,ds=\int_V \nabla\cdot\bar{F}\,dv$$

$$\int_S \bar{F}\cdot\bar{n}\,ds=\int_{OABC}+\int_{OAA'O'}+\int_{OO'C'C}+\int_{C'B'BC}+\int_{O'A'B'C'}+\int_{A'ABB'}$$

S_1: O'A'B'C' : Its in xy plane

$$\iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$\bar{n}=-\bar{k}$

$\bar{F}\cdot\bar{n}=-z$ $\qquad \int_{S_1} z\,dx\,dy=0$ $\quad$ ($\because z=0$ in xy plane)

S_2: OAA'O' :- Its in xz plane.

$$\iint_R \bar{F}\bar{n}\cdot\frac{dx\,dz}{|n\cdot j|}$$ $\qquad \bar{n}=-\bar{j}$

$\bar{F}\cdot\bar{n}=2x^2y$

$\int_{S_2} 2x^2y\,dx\,dz=0$ $\qquad$ ($\because y=0$ in xz plane)

$S_3 = OO'C'C$ Its in yz plane.

$$\iint_{R3} F\cdot\bar{n}\,\frac{dydz}{|n\cdot i|} \qquad \bar{n} = -i$$

$$\bar{F}\cdot\bar{n} = -(x^3 - yz)$$

$$-\int_{S3} x^3 - yz\,dydz = \int_{S3} yz\,dydz = \iint_R yz\,dydz.$$

$$= \int_0^a\int_0^a yz\,dydz$$

$$= \int_0^a y\left(\frac{z^2}{2}\right)_0^a dy \Rightarrow \frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^a = \frac{a^4}{4}$$

$S_4 : C'B'BC$; Let the surface be projected in xz plane.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dxdz}{|n\cdot j|}.$$

$$\bar{n} = \bar{j}$$

$$\bar{F}\cdot\bar{n} = -2x^2y$$

$$= -2\iint x^2y\,dxdz$$

$$= -2a\int_0^a\int_0^a x^2\,dxdz$$

$$= -2a\int_0^a x^2 (z)_0^a\,dx$$

$$= -2a^2\int_0^a x^2\,dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a$$

$$= -2a^2\left(\frac{a^3}{3}\right) \Rightarrow -\frac{2a^5}{3}$$

S_5: OABC :- Let the surface be projected in xy plane.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$\bar{n} = \bar{k}$$

$$\bar{F}\cdot\bar{n} = z = a$$

$$\int_0^a\int_0^a a\,dx\,dy = a\int_0^a (y)_0^a\,dx$$

$$= a^2(x)_0^a \Rightarrow a^3$$

S_6 : AA'BB' :- Let the surface be projected in yz plane.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{n} = \bar{i}$$

$$\bar{F}\cdot\bar{n} = x^3 - yz = a^3 - yz$$

$$\int_0^a\int_0^a (a^3 - yz)\,dy\,dz$$

$$= \int_0^a \left(a^3z - y\frac{z^2}{2}\right)_0^a dy \Rightarrow \int_0^a a^4 - \frac{a^2 y}{2}\,dy = \left(a^4 y - \frac{a^2}{2}\frac{y^2}{2}\right)_0^a$$

$$= a^5 - \frac{a^4}{4}$$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = 0 + 0 + \cancel{\frac{a^4}{4}} - \frac{2a^5}{3} + a^3 + a^5 - \cancel{\frac{a^4}{4}}$$

$$= \frac{a^5}{3} + a^3$$

<u>R.H.S</u> $= \int_V \nabla\cdot\bar{F}\,dv$

$$\nabla\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)$$

$$= 3x^2 - 2x^2 + 1 \Rightarrow x^2 + 1$$

$$\int_V \nabla\cdot F\,dv = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz$$

$$= \int_0^a\int_0^a \left(\frac{a^3}{3} + a\right) dy\,dz = \left(\frac{a^3}{3} + a\right)\int_0^a (y)_0^a\,dz$$

$$= \left(\frac{a^3}{3} + a\right) a \int_0^a dz$$

$$= a\left(\frac{a^3}{3} + a\right)(z)_0^a \Rightarrow a^2\left(\frac{a^3}{3} + a\right)$$

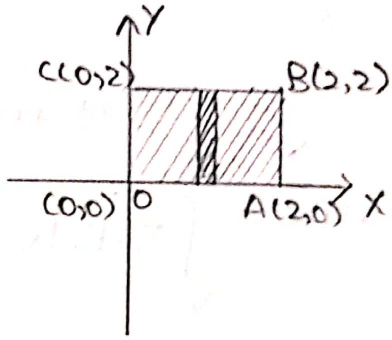
$$\Rightarrow \frac{a^5}{3} + a^3$$

$\therefore$ LHS = RHS ; Hence the theorem is verified

8. Verify Green's theorem in the plane for

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$ where C is a square with vertices $(0,0), (2,0); (2,2)$ and $(0,2)$.

Sol:- By Green's theorem



$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

L.H.S

$$\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $(0,0)(2,0)$

$x \to 0-2 \quad y = 0 \Rightarrow dy = 0$

$$\int_{OA} (x^2 - 0)\,dx = \int_0^2 x^2\,dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along AB $(2,0)(2,2)$

$x = 2 \Rightarrow dx = 0, \quad y \to 0-2$

$$\int_{AB} 0 + (y^2 - 4y)\,dy = \int_0^2 y^2 - 4y\,dy$$

$$= \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2 = \frac{8}{3} - 8 \Rightarrow \frac{-16}{3}$$

Along BC (2,2) (0,2)

$x \to 2-0$, $y=2 \Rightarrow dy=0$

$$\int_{BC} x^2-8x\,dx = \int_2^0 x^2-8x\,dx = \left(\frac{x^3}{3}-\frac{8x^2}{2}\right)_0^2$$

$$= 0-\frac{8}{3}+16 \Rightarrow \frac{40}{3}$$

Along CO (0,2) (0,0)

$x=0 \Rightarrow dx=0 \quad y \to 2-0$

$$\int_{CO} y^2 dy = \int_2^0 y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = 0-\frac{8}{3} = \frac{-8}{3}$$

L.H.S $= \oint_C M\,dx + N\,dy = \frac{8}{3}-\frac{16}{3}+\frac{40}{3}-\frac{8}{3} = \frac{24}{3} = 8$

R.H.S $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$

$M = x^2-xy^3 \qquad N = y^2-2xy$

$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$

$$\iint_R (-2y+3xy^2)\,dx\,dy$$

$x \to 0-2 \quad y \to 0-2$

$$\int_0^2\int_0^2 -2y+3xy^2\,dx\,dy$$

$$= \int_0^2 \left(-2yx + \frac{3x^2}{2}y^2\right)_0^2 dy$$

$$= \int -4y + 6y^2 dy \Rightarrow \left(-4\frac{y^2}{2} + 6\frac{y^3}{3}\right)_0^2$$

$$\Rightarrow -8 + 16 = 8$$

$\therefore$ L.H.S = R.H.S

Hence the theorem is verified.

9. prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2 + y^2 + z^2$

diff p.w.r.t x, y and z

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 r^n = \nabla(\nabla r^n)$$

$$= \nabla\left(i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}\right)$$

$$= \nabla \sum i\frac{\partial}{\partial x} r^n = \nabla \sum i n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \nabla \sum i n r^{n-1}\frac{x}{r}$$

$$= \nabla \sum i n r^{n-2} x$$

$$= \nabla \sum n r^{n-2} x i = \nabla \left(n r^{n-2} (xi + yj + zk) \right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \left(n r^{n-2} (xi + yj + zk) \right)$$

$$= n \sum \frac{\partial}{\partial x} r^{n-2} . x$$

$$= n \sum (n-2) r^{n-3} \frac{\partial r}{\partial x} + r^{n-2}$$

$$= n \sum (n-2) r^{n-3} \frac{x}{r} . x + r^{n-2}$$

$$= n \sum (n-2) r^{n-4} x^2 + r^{n-2}$$

$$= n \left[(n-2) r^{n-4} (x^2 + y^2 + z^2) + 3 r^{n-2} \right]$$

$$= n \left[(n-2) r^{n-4} r^2 + 3 r^{n-2} \right]$$

$$= n r^{n-2} [n - 2 + 3]$$

$$= n r^{n-2} (n+1)$$

$$= n(n+1) r^{n-2}$$

10. (a) prove that $\nabla(A \cdot B) = (B \cdot \nabla) + (A \cdot \nabla)B + B \times (\nabla \cdot A) + A \times (\nabla \times B)$.

Sol: ~~div~~

$$\bar{a} \times \text{curl}(\bar{b}) = \bar{a} \times (\nabla \times \bar{b}) = \bar{a} \times \sum \bar{i} \times \frac{\partial \bar{b}}{\partial x} = \sum \bar{a} \times \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$= \sum \left\{ \left(a \cdot \frac{\partial \bar{b}}{\partial x} \right) \bar{i} - (\bar{a} \cdot \bar{i}) \frac{\partial \bar{b}}{\partial x} \right\} = \sum \bar{i} \left\{ \bar{a} \frac{\partial \bar{b}}{\partial x} \right\} - \left\{ \bar{a} \cdot \sum i \frac{\partial}{\partial x} \right\} \bar{b}$$

$\therefore \bar{a} \times \text{curl}\,\bar{b} = \sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} \longrightarrow (1)$

Similarly, $\bar{b} \times \text{curl}\,\bar{a} = \sum \bar{i}\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a} \longrightarrow (2)$

(1) + (2) gives

$$\bar{a} \times \text{curl}\,\bar{b} + \bar{b} \times \text{curl}\,\bar{a} = \sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} + \sum \bar{i}\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a}$$

$$\Rightarrow \bar{a} + \text{curl}\,\bar{b} + \bar{b} \times \text{curl}\,\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a}$$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x} + \bar{b}\frac{\partial \bar{a}}{\partial x}\right) \Rightarrow \sum \bar{i}\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$$

$$= \nabla(\bar{a}\cdot\bar{b}) = \text{grad}\,(\bar{a}\cdot\bar{b}).$$

---

b. Prove that $\text{div}(a \times b) = b\cdot\text{curl}\,a - a\,\text{curl}\,b$.

Sol:- $\text{div}(\bar{a}\times\bar{b}) = \sum \bar{i}\frac{\partial}{\partial x}(\bar{a}\times\bar{b}) = \sum \bar{i}\left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$

$$= \sum \bar{i}\left(\frac{\partial \bar{a}}{\partial x} \times \bar{b}\right) + \sum \bar{i}\left(\bar{a} \times \frac{\partial \bar{b}}{\partial x}\right) = \sum\left(\bar{i} \times \frac{\partial \bar{a}}{\partial x}\right)\cdot\bar{b} - \sum\left(\bar{i} \times \frac{\partial \bar{b}}{\partial x}\right)\bar{a}$$

$$= (\nabla \times \bar{a})\cdot\bar{b} - (\nabla \times \bar{b})\bar{a} = \bar{b}\,\text{curl}\,\bar{a} - \bar{a}\,\text{curl}\,\bar{b}$$

Hence the theorem.

11. Evaluate $\int F \cdot dx$ where $F = x^2 i + y^2 j$ and curve $y = x^2$ in the xy-plane from $(0,0)$ to $(1,1)$.

Sol:-

$$\int \bar{F} \cdot d\bar{r} = \int (x^2 i + y^2 j) \cdot (dx\,i + dy\,j)$$

$$= \int x^2 dx + y^2 dy$$

given $y = x^2$

$dy = 2x\,dx \quad x \to 0-1$

$$= \int_0^1 x^2 dx + x^4 (2x\,dx)$$

$$= \int_0^1 (x^2 + 2x^5)\,dx \Rightarrow \left(\frac{x^3}{3} + \cancel{2}\frac{x^6}{\cancel{6}_3}\right)_0^1$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

12. Find the work done by the force $F = (3x^2-6xy)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$ in moving article from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve C; $x=t,\ y=t^2,\ z=t^3$.

Sol:- work done $= \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C \left((3x^2-6xy)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k} \right) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k}).$$

$$= \int_C (3x^2-6xy)\,dx + (2y+3xz)\,dy + (1-4xyz^2)\,dz$$

$x=t$ $\quad$ $y=t^2$ $\quad$ $z=t^3$

$dx=dt$ $\quad$ $dy=2t\,dt$ $\quad$ $dz=3t^2\,dt$

$t \to 0-1$

$$= \int_0^1 (3t^2-6t^3)\,dt + (2t^2+3t^4)2t\,dt + (1-4t^9)3t^2\,dt$$

$$= \int_0^1 3t^2-6t^3+4t^3+6t^5+3t^2-12t^{11}\,dt$$

$$= \int_0^1 6t^2-2t^3+6t^5-12t^{11}\,dt$$

$$= \left(\cancel{6}^{2}\frac{t^3}{\cancel{3}} - \cancel{2}\frac{t^4}{\cancel{4}_2} + \cancel{6}\frac{t^6}{\cancel{6}} - \cancel{12}\frac{t^{12}}{\cancel{12}} \right)_0^1 = 2 - \frac{1}{2} + \cancel{1} - \cancel{1}$$

$$= \frac{3}{2}.$$

13. Use Gauss divergence theorem to evaluate $\iint (yz^2 i + zx^2 j + 2z^2 k) ds$, where S is the closed surface bounded by the xy- plane and the upper half of the sphere $x^2+y^2+z^2=z^2$ above this plane.

Sol:- Given $\iint (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}) \cdot \bar{n}\, ds$

$$x^2+y^2+z^2=1$$

By Gauss divergence theorem

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_V \nabla\cdot\bar{F}\, dv$$

$$\bar{F} = yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}$$

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k})$$

$$= \frac{\partial}{\partial x} yz^2 + \frac{\partial}{\partial y} zx^2 + \frac{\partial}{\partial z} 2z^2$$

$$= 4z$$

$$\iint (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k})\cdot\bar{n}\, ds = \iiint 4z\, dx\, dy\, dz$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$$

$r \to 0-1$

$\theta \to 0-\pi$ $\qquad \phi \to 0-2\pi$

$$= 4\int_0^1\int_0^\pi\int_0^{2\pi} r\cos r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$= 4\int_0^1\int_0^\pi\int_0^{2\pi} r^3\sin\theta\cos\theta \, dr\, d\theta\, d\phi$$

$$= 4\int_0^1\int_0^\pi \frac{r^4}{4}\sin\theta\cos\theta \, d\theta\, d\phi$$

$$= \frac{\cancel{2}}{\cancel{4}_2}\int_0^1\int_0^\pi (r^4)_0^1 \sin 2\theta \, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^1\int_0^\pi \sin 2\theta \, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^1 \left(\frac{-\cos 2\theta}{2}\right)_0^\pi d\phi$$

$$= \frac{1}{2}\int_0^1 \left(\frac{-\cos 2\pi}{2} + \frac{\cos 2(0)}{2}\right) d\phi$$

$$= \frac{1}{2}\int_0^1 \left(\frac{-1}{2} + \frac{1}{2}\right) d\phi$$

$$= 0$$

14. Verify Greens theorem in plane for $\oint (3x^2-8y^2)dx + (4y-6xy)dy$ where c is the region bounded by $y=\sqrt{x}$ and $y=x^2$.

Sol:- By Green's theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$$

Given

$$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)dy$$

$M = 3x^2-8y^2 \qquad N = 4y-6xy$

L.H.S $\Rightarrow \oint_C M dx + N dy = \oint_C (3x^2-8y^2)dx + (4y-6xy)dy$

$$\oint_C = \int_{OA} + \int_{AO}$$

<u>Along OA</u>

$(0,0)\ (1,1)$

$$\int_{OA} (3x^2-8y^2)dx + (4y-6xy)dy$$

$x^2 = y$

$2x\,dx = dy$

$$\int_0^1 (3x^2-8x^4)dx + (4x^2-6x^3)(2x\,dx).$$

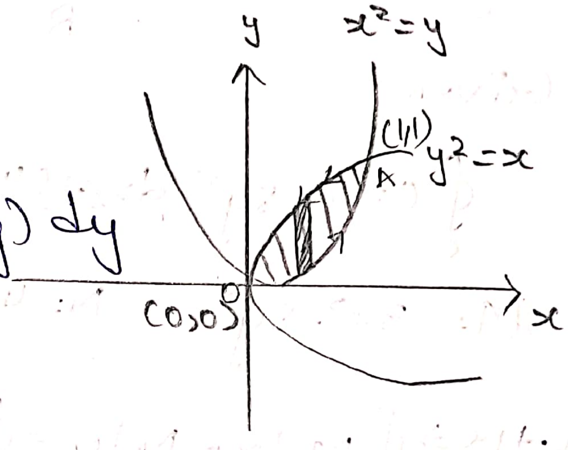
$$\int_0^1 (3x^2-8x^4+8x^3-12x^4)\,dx$$

$$\int_0^1 3x^2+8x^4-20x^4\,dx = \left(\frac{3x^3}{3}+\frac{8x^4}{4}-\frac{20x^5}{5}\right)_0^1$$

$$= 1+2-4 = -1$$

Along AO

(1,1) (0,0)



$$\int_{AO} (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$x=y^2$

$dx = 2y\,dy$

$$\int_1^0 (3y^4-8y^2)(2y\,dy)+(4y-6y^3)\,dy$$

$$\int_1^0 (6y^5-16y^3+4y-6y^3)\,dy$$

$$\int_1^0 4y-22y^3+6y^5\,dy$$

$$\left[\frac{4y^2}{2}-\frac{22y^4}{4}+\frac{6y^6}{6}\right]_1^0$$

$$= 0-\left(2-\frac{11}{2}+1\right) = -\left(3-\frac{11}{2}\right) = \frac{5}{2}$$

$\therefore$ L.H.S $\oint_C M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

R.H.S

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3x^2 - 8y^2$ $\qquad$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\qquad$ $\frac{\partial N}{\partial x} = -6y$

$$\iint_R -6y + 16y = \iint_R 10y\,dx\,dy$$

$$= 10\iint y\,dx\,dy$$

$x \to 0\text{-}1 \quad y \to x^2 - \sqrt{x}$

$$= 10\int_0^1\int_{x^2}^{\sqrt{x}} y\,dy\,dx$$

$$= 10\int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx \Rightarrow 10\int_0^1 \left(\frac{x}{2} - \frac{x^4}{2}\right) dx$$

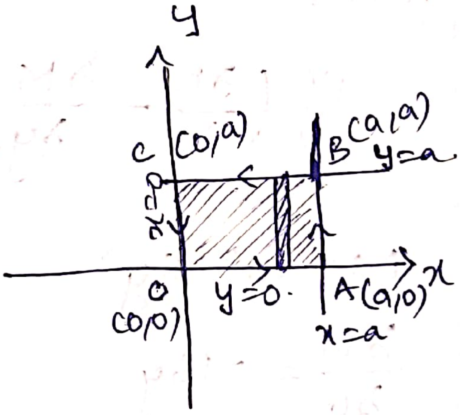
$$= 10\left(\frac{x^2}{4} - \frac{x^5}{10}\right) = 10\left(\frac{1}{4} - \frac{1}{10}\right)$$

$$= 10\left(\frac{5-2}{20}\right) = \frac{3}{2}$$

$\therefore$ L.H.S = R.H.S.

Hence theorem is verified.

15. Verify stokes theorem for the function $F = x^2 i + xyj$ integrated round the square in the plane $z=0$ whose sides are along the line $x=0$, $y=0$, $x=a$, $y=a$.



Sol:- $\bar{F} = x^2 i + xyj$

By strokes theorem, we have

$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \text{curl} \bar{F} \cdot \bar{n}\, ds$$

L.H.S :- $\oint_C \bar{F} \cdot d\bar{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

$$= \int (x^2 i + xyj) \cdot (dx i + dy j)$$

$$= \int_C x^2 dx + xy\, dy$$

Along OA :- (0,0) (a,0)

$x \to 0-a$ $y=0$

$dy = 0$

$$\int_{OA} x^2 dx = \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB :- (a,0) (a,a)

$x=a$ $y \to a-0$

$dx = 0$

$$\int_{AB} ay\,dy = a\int_0^a y\,dy = a\left(\frac{y^2}{2}\right)_0^a = \frac{a^3}{2}$$

Along BC:- $(a,a)(0,a)$

$x \to a-0,\ y=a$

$dy=0$

$$\int_{BC} x^2\,dx = \int_a^0 x^2\,dx = \left(\frac{x^3}{3}\right)_a^0 = 0 - \frac{a^3}{3} = \frac{-a^3}{3}$$

Along CO:- $(0,a)(0,0)$

$x=0 \quad y \to a-0$

$dx=0$

$$\int_{CO} 0 = 0$$

$$\oint_C \bar{F}\cdot\bar{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 \Rightarrow \frac{a^3}{2}$$

R.H.S = $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$

$\Rightarrow \text{curl}\,\bar{F} = \nabla \times \bar{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(y)$$

$\text{curl}\,\bar{F} = y\bar{k} \cdot \bar{n} = \bar{k}$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint \text{curl}\,\bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_0^a\int_0^a y\,dx\,dy$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^a dx$$

$$= \int_0^a \frac{a^2}{2}\,dx$$

$$= \frac{a^2}{2}(x)_0^a = \frac{a^3}{2}$$

$\therefore$ L.H.S = R.H.S

Hence theorem is Verified.