LORDS INSTITUTE OF ENGINEERING AND TECHNOLOGY

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B.E-II SEMESTER, QUESTION BANK, 2023

MATHEMATICS-II

(COMMON FOR ALL BRANCHES)

SAQ UNIT-I			
S.NO		**CO MAPPING**	**Bloom’s Taxonomy Level**
1	Find the rank of the matrix $A=\begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$	CO2	BTL1
2	Find the Eigen values of the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$	CO2	BTL2
3	Find the sum and product of the Eigen values of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	CO2	BTL2
4	Show that the vectors (1, 2, 3), (2, 3, 4), (0, 0, 1), (3, 4, 5) are Linearly Independent.	CO2	BTL4
5	Define rank of the matrix and give one example.	CO2	BTL2
6	Find the value of ‘k’ such that the rank of A = $\begin{bmatrix} 1 & -2 & 3 \\ 2 & k & 4 \\ -1 & 2 & 5 \end{bmatrix}$ is 2	CO2	BTL3

7	Find the rank of the matrix $$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$	CO2	BTL2
8	If A= $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, Find the Eigen values of A^3+7A^2+2A	CO2	BTL3
9	Convert the matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{bmatrix}$ in to echelon form.	CO2	BTL 3
10	Write the symmetric matrix for the Quadratic form, $$Q = x^2 + 2y^2 + 3z^2 - 2xy + 4yz + 6zx$$	CO2	BTL1
11	Examine linear independence of the given vectors (1,1,0,1); (1,1,1,1); (-1,1,1,1); (1,0,0,1)	CO2	BTL3
12	Discuss the nature of quadratic form $x^2 - y^2 + 4z^2 + 2yz + 6zx + 4xy$. Also find index and signature	CO2	BTL2
13	Write the matrix form and also the augmented matrix for the given system of equations $3x - y - z = 3, 2x - 8y + z = -5, x - 2y + 9z = 8$	CO2	BTL2
14	Define Eigen value and Eigen vector with example.	CO2	BTL1
15	Find the Eigen value corresponding to the Eigen vector $X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$	CO2	BTL2
SAQ **UNIT – II**			
1	Define Exact Differential Equation	CO-3	BTL1
2	Define the Integrating factor of non homogenous differential equation $\frac{dy}{dx} + p(x)y = Q(x)$	CO-3	BTL1
3	Write Riccati's equation and Clairaut's equation.	CO-3	BTL1

4	Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$	CO-3	BTL3
5	Define Orthogonal Trajectories of a given family of curve and write the procedure to find it in polar coordinates.	CO-3	BTL4
6	Solve $x\frac{dy}{dx} + y = \log x$	CO-3	BTL5
7	Find the general solution of $y = xp - p^3$ where $p = \frac{dy}{dx}$	CO-3	BTL2
8	Find the orthogonal trajectories of the family of curves $y = cx^2$ where c is a parameter.	CO-3	BTL3
9	Solve $xdy - ydx = (x^2 + y^2)dy$	CO-3	BTL 4
10	Find the solution of the differential equation $(y - x + 1)dy - (y + x + 2)dx = 0.$	CO-3	BTL3
11	Solve $y(2xy + e^x)dx = e^x dy$	CO-3	BTL4
12	Find the orthogonal trajectories of the family of curves $x^2 + 16y^2 = c$	CO-3	BTL3
13	Solve $\frac{dy}{dx} = e^x + y$	CO-3	BTL4
14	Find the orthogonal trajectories of the family of curves $r = c\theta^2$	CO-3	BTL3
15	Find an integrating factor of $(x^3 + y^3)dx - x^2ydy = 0$	CO-3	BTL2
SAQ **UNIT – III**			
1	Solve $y'' - y = 0$, when $y = 0$ and $y' = 2$ at $x = 0$	CO4	BTL3
2	Solve $(D^4 - 81)y = 0$.	CO4	BTL5
3	Solve $(D^4 + 8D^2 + 16)y = 0$.	CO4	BTL4
4	Find particular integral of $(D^2 - 4D + 4)y = e^{2x}$	CO4	BTL2
5	Find the solution of initial value problem $y'' + 4y' - 13y = 0$, $y(0) = y'(0) = 1.$	CO4	BTL4
6	Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$	CO4	BTL4

7	Find complimentary function of $(D^2-4D+3)y=\sin 3x\ \cos 2x$.	CO4	BTL2
8	Find the P.I of $(D^2+1)y=8e^{-x}$.	CO4	BTL2
9	Find the particular Integral of $(D^3-6D^2+11D-6)\ y=e^{-3x}$	CO4	BTL2
10	write in brief about the method of variation of parameters	CO4	BTL1
11	Define the terms i) Complementary function ii) Particular integral	CO4	BTL1
12	State Euler-Cauchy equation and brief method to solve it.	CO4	BTL1
13	Solve the D.E $D^2y=sin2x$	CO4	BTL4
14	Find the value of $\frac{1}{D+1}\ (x^2+1)$	CO4	BTL4
15	Find particular value of $\frac{1}{(D-2)(D-3)}e^{2x}$	CO4	BTL2
SAQ **UNIT-IV**			
1	Evaluate $\gamma\left(-\frac{5}{2}\right)$	CO5	BTL4
2	Define Beta function.	CO5	BTL1
3	State Gamma function	CO5	BTL1
4	State Rodrigue’s formula.	CO5	BTL1
5	State the relation between Beta and Gamma functions.	CO5	BTL1
6	Prove that $p_n(1)=1$	CO5	BTL4

7	Classify the singular points of the differential equation $x^2y'' - 5y' + 3x^2y = 0$	CO5	BTL2
8	Find the value of $\beta(\frac{9}{2},\frac{7}{2})$	CO5	BTL2
9	Prove that $\beta(m,n) = \beta(n,m)$	CO5	BTL4
10	Express f(x) $= 2x^3 - 6x^2 + 5x - 3$ in terms of Legendre polynomial $p_n(x)$.	CO5	BTL2
11	Prove that Γ(n+1)=n !	CO5	BTL4
12	Evaluate $\int_{-1}^{0}(1-x^2)^n\,dx$	CO5	BTL5
13	Express $3p_3(x) + 2p_2(x) + 4p_1(x) + 5p_0(x)$ as polynomial in x	CO5	BTL2
14	Determine the nature of point x=0 for equation $xy'' + ysinx = 0$	CO5	BTL2
15	Evaluate Γ(9/2)	CO5	BTL5
SAQ **UNIT-V**			
1	Find $L\{e^{-t}\sin 2t\}$	CO6	BTL2
2	State Convolution theorem of Laplace transforms.	CO6	BTL1
3	Find $L\{t\cos 2t\}$.	CO6	BTL2
4	Evaluate $L^{-1}\left\{\frac{2}{S^3}+\frac{1}{S^2}\right\}$.	CO6	BTL5
5	Find $L\{te^{-t}\}$	CO6	BTL1

6	Find inverse Laplace transform of $f(t)=t^2 \sinh t$	CO6	BTL2
7	Find $L\{t^3 e^{-4t}\}$	CO6	BTL2
8	Find $L^{-1}\left\{\dfrac{1}{(s+2)(s+3)}\right\}$	CO6	BTL2
9	Find the Laplace transform of $f(t)=\sin^2 t$.	CO6	BTL2
10	Define Unit Impulse function and write its Laplace transform.	**CO6**	BTL1
11	Evaluate $L[cos^2 t]$.	**CO6**	BTL5
12	Find $L^{-1}\left\{\dfrac{1}{(S+1)(S+2)}\right\}$	**CO6**	BTL2
13	Find $L\{t^3 e^{-4t}\}$	**CO6**	BTL3
14	Evaluate $L^{-1}[\frac{1}{s(s+1)}]$	**CO6**	BTL5
15	Find $L[t^3]$.	**CO6**	BTL2
LAQ **UNIT- I**			
1	Find the values of a, b such that the equation $2x + 3y + 5z = 9, 7x + 3y + 2z = 8,\ 2x + 3y + az = b$, has (i) No Solution (ii) Infinite Solutions (iii) Unique Solution.	CO2	BTL 4
2	Solve the system of equations x+3y+2z =0; 2x-y+3z = 0; 3x-5y+4z = 0; x+17y+4z = 0.	CO2	BTL 4
3	Verify Cayley-Hamilton Theorem and hence find the inverse of the matrix, where A= $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$	CO2	BTL5

4	Reduce the Quadratic form, $Q = 3x^2+3y^2+3z^2+2xy+2zx-2yz$ to Canonical form and find its nature, index and signature.	CO2	BTL5
5	Test for the consistency and solve, if consistent, the system of equations $x+y+z=3$, $3x-9y+2z=-4$, $5x-3y+4z=6$.	CO2	BTL5
6	Reduce the matrix $A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$ to Echelon form and hence find its rank. .	CO2	BTL 4
7	Verify Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and hence find A^5	CO2	BTL 4
8	Find the Eigen values and corresponding Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	CO2	BTL5
9	Verify Cayley-Hamilton for the matrix A= $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$	CO2	BTL 4
10	Find the rank of the matrix $A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$	CO2	BTL 5
11	Reduce the quadratic form to Canonical form $8{x_1}^2 + 7{x_2}^2 + 3{x_3}^2 + 12x_1x_2 + 4x_1x_3 - 8x_2x_3$	CO2	BTL 3
12	Show that sum of eigen values of a matrix is its trace and product of eigen values is its determinant.	CO2	BTL 4

13	Reduce the quadratic form $2x_1x_2 + 2x_1x_3 + 2x_2x_3$ to canonical form. Hence find Index, Signature and orthogonal transformation.	CO2	BTL 3
14	Find the Eigen values and corresponding Eigen vectors of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	CO2	BTL 3
15	Determine eigen values of i) A^2 ii)A^{-1} iii) $B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$	CO2	BTL 3
LAQ **UNIT - II**			
1	Solve $(\mathrm{x} - \mathrm{y}^2)\mathrm{dx} + 2\mathrm{xydy} = 0$	CO-3	BTL 4
2	Find the orthogonal trajectories of $\mathrm{r}^n \sin n\theta = \mathrm{c}$, where c is the parameter.	CO-3	BTL 6
3	Solve $\mathrm{x}\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \mathrm{x}^3\mathrm{y}^6$.	CO-3	BTL5
4	Solve the differential equation $\mathrm{ydx} - \mathrm{xdy} + \mathrm{e}^{\mathrm{x}}\mathrm{ydy} = 0$.	CO-3	BTL 4
5	Solve $y(2x^2y + e^x)dx = (e^x + y^3)dy$.	CO-3	BTL5
6	Solve the differential equation $y' + 4xy + xy^3 = 0$	CO-3	BTL 4
7	Solve $\frac{dy}{dx} + 2xy = 2x$.	CO-3	BTL5
8	Find the general solution of the Riccati's equation $y' = 3y^2 - (1 + 6\mathrm{x})\mathrm{y} + 3\mathrm{x}^2 + \mathrm{x} + 1$, if $y = x$ is a particular solution.	CO-3	BTL 4
9	Find the Orthogonal Trajectories of the family of cardioids $r=a(1-cos\theta)$	CO-3	BTL5
10	Find the general solution of the equation $\frac{\mathrm{dy}}{\mathrm{dx}} = 2\mathrm{x}y^2 + (1 - 4\mathrm{x})\mathrm{y} + 2\mathrm{x} - 1$, if y=1 is a particular solution.	CO-3	BTL 3
11	Solve $y(x + y)dx - \mathrm{x}^2\mathrm{dy} = 0$	CO-3	BTL 5

12	Solve $\frac{dy}{dx} + xsin2y = x^3cos^2y$	CO-3	BTL 4
13	Find the Orthogonal Trajectories of the family of hypocycloids $x^{2/3} + y^{2/3} = a^{2/3}$	CO-3	BTL 3
14	Find the general and singular solution of the equation $\mathrm{y} = \mathrm{xp} + \mathrm{p}^2$ where $p = \frac{dy}{dx}$	CO-3	BTL 3
15	Solve $ysin2xdx - (1 + y^2 + cos^2x)dy = 0$	CO-3	BTL 4
LAQ **UNIT - III**			
1	Find the general solution of $y'' + 3y' + 2y = 2e^x$	CO4	BTL3
2	Find the general solution of $(D^2 - 4D + 4)y = e^{2x}$	CO4	BTL 4
3	Solve $(D^2 + 9)y = sin3x.$	CO4	BTL4
4	Find the general solution of $(D^2 - 4)y = cos^2x$	CO4	BTL 4
5	Solve: $(D + 2)(D - 1)^2y = e^{-2x} + 2sinhx$	CO4	BTL4
6	Find the general solution of $(D^2 + 2D + 1)y = xcosx$	CO4	BTL4
7	Solve $(D^2 + 4)y = e^x + sin2x + cos2x$	CO4	BTL4
8	Solve $(D^3 - 1)y = e^x + sin3x + 2$	CO4	BTL4
9	Solve $y'' - 2y' + y = xe^x \sin x$	CO4	BTL4
10	Solve $(D^2 + 4)y = x^2 + 1 + \cos 2x$	CO4	BTL4
11	Find the general solution of $y'' + 4y' + 4y = 6e^{-2x}cos^2x$	CO4	BTL3
12	Apply method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \sec x$	CO4	BTL2
13	Find the particular integral of	CO4	BTL3

	$(D^2-6D+9)y = x^2+2x$		
14	Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$	CO4	BTL4
15	Apply method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$	CO4	BTL3
	LAQ **UNIT-IV**		
1	Evaluate $\frac{d}{dx}[\text{erf}\,(\alpha x)]$	CO5	BTL5
2	Show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$.	CO5	BTL4
3	Prove that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$	CO5	BTL4
4	Define Gamma function and show that $$\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$	CO5	BTL4
5	Evaluate $\int_0^\infty \frac{x^{\frac{3}{2}}}{\sqrt{a^2+x^2}}dx$ using Beta and gamma function.	CO5	BTL5
6	Express $2x^3+3x^2-x+1$ in terms of Legendre's Polynomial.	CO5	BTL2
7	Prove that $$erf(x) + erf_c(x) = 1$$	CO5	BTL4
8	Find the power series solution of the differential equation $$(1-x^2)y'' - 2xy' + 2y = 0 \; about \; x = 0.$$	CO5	BTL3
9	Express the following sum of the Legendre Polynomial in terms of x $8P_4(x) + 2P_2(x) + P_0(x)$.	CO5	BTL2
10	Evaluate the improper Integral $\int_0^\infty \sqrt{x}\, e^{-x^2}dx$	CO5	BTL5
11	Prove that $\beta(m,n)$=2$\int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta\, cos^{2n-1}\,\theta\, d\theta$	CO5	BTL4
12	Evaluate $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}}$	CO5	BTL5
13	Prove that $$2^{2n-1}\Gamma(\mathrm{n})\Gamma\left(n+\frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}$$	CO5	BTL4
14	Evaluate $\int_0^\infty 3^{-4x^2}\,dx$	CO5	BTL5

15	Evaluate $\int_0^1 x^m(1-x^2)^n dx$ in terms of beta function, where m,n are positive constants.	CO5	BTL4
LAQ **UNIT-V**			
1	Find the Laplace transform of $f(t)=\frac{e^{-2t}\sin 3t}{t}$	CO6	BTL3
2	Using Laplace transform, solve the initial value problem $y'' + y = e^t \sin(t)$, $y(0)=0=y'(0)$.	CO6	BTL 4
3	Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(S+1)(S+2)}\right\}$.	CO6	BTL3
4	Evaluate $L^{-1}\left\{\log\left(\frac{S-3}{S+3}\right)\right\}$	CO6	BTL 5
5	Find the Laplace transform of $f(t)=\frac{2\sin^2 t}{t}$.	CO6	BTL2
6	Find inverse Laplace transform of $\frac{S}{S^4+S^2+1}$.	CO6	BTL2
7	Show that $L\left\{e^{at}\right\}=\frac{1}{s-a}$	CO6	BTL4
8	Find the Laplace transform of the function $f(t)=\begin{cases}\sin t & if\ 0<t<\pi \\ 0 & if\ \pi<t<2\pi\end{cases}$ and the period of f(t) is 2π.	CO6	BTL3
9	Show that $L\left\{\sin at\right\}=\frac{a}{s^2+a^2}$	CO6	BTL4
10	Find the Laplace transform of $f(t)=e^{-t}(2\cos 5t-3\sin 5t)$.	CO6	BTL2
11	Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.	CO6	BTL5
12	Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y = \sin t$, $y=\frac{dy}{dt}=0$, when t=0 using Laplace transform	CO6	BTL4
13	Using Convolution Theorem find $L^{-1}[\frac{1}{(s+1)(s+2)}]$	CO6	BTL3
14	Find$L^{-1}[\frac{1}{s(s^2+9)}]$	CO6	BTL2
15	Find $L^{-1}[\frac{1}{s(s+2)}]$	CO6	BTL2

M - I

Assignment - I

SAQ's :

1) Find the rank of the matrix

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

Sol: Given matrix is

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 1 & -27 \end{bmatrix}$$

$R_3 \to 6R_3 - R_2$

$$\sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 0 & -181 \end{bmatrix}$$

$\therefore$ The no. of non-zero rows are 3.

Hence, Rank of the matrix is 3.

2) Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Sol: Given, $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

The charachteristic equation is

i.e., $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda)(2-\lambda) - (4)(3) = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0.$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0.$$

$$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 2) = 0.$$

$$\Rightarrow \lambda - 5 = 0 \text{ \& } \lambda + 2 = 0$$

$$\Rightarrow \lambda = 5 \text{ \& } \lambda = -2.$$

So, the eigen values are $-2, 5$

3) Find the sum and product of the eigen values of the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

sol:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

We know that

Sum of the Eigen Values of the matrix $= \text{Trace}(A)$.

i.e., $\lambda_1 + \lambda_2 + \lambda_3 = \text{Trac}(A)$.

Now, $\text{Trac}(A) = 2 + 3 + 2 = 7$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 7$.

Similarly,

product of the Eigen Values of the matrix $= \det(A)$.

$$\det(A) = 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 2(4) - 2(1) + 1(-1)$$

$$= 8 - 2 - 1 = 8 - 3 = 5. \Rightarrow \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

4) Show that the vectors $(1,2,3), (2,3,4), (0,0,1), (3,4,5)$ are Linearly independent.

Sol:-

$$\left[\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix} \right]$$

$$a_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + a_2 \begin{bmatrix}2\\3\\4\end{bmatrix} + a_3 \begin{bmatrix}0\\0\\1\end{bmatrix} + a_4 \begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 4 & 0 \\ 3 & 4 & 1 & 5 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 3 & 4 & 1 & 5 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 3R_1 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & -2 & 1 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$a_1 + 2a_2 + 3a_4 = 0$$
$$-a_2 - 2a_4 = 0.$$

Let $\boxed{a_4 = K}$ $\Rightarrow -a_2 - 2a_4 = 0 \Rightarrow -a_2 - 2K = 0$

$\Rightarrow -2K = a_2 \Rightarrow \boxed{a_2 = -2K}$

(3)

$$a_1 + 2a_2 + 3a_4 = 0$$

$$\Rightarrow a_1 + 2(-2K) + 3(K) = 0$$

$$\Rightarrow a_1 - 4K + 3K = 0$$

$$\Rightarrow a_1 - K = 0$$

$$\Rightarrow \boxed{a_1 = K}$$

$$\Rightarrow a_1 = K, \; a_2 = -2K, \; a_3 = 0, \; a_4 = K.$$

$$\Rightarrow [K, -2K, 0, K] \Rightarrow$$ - Trivial.

$\Rightarrow$ Linearly Dependent.

for

$\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

⑤ Define Rank of the matrix and give one example of matrix.

Sol:- Let A be an $m \times n$ matrix, and non zero then we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is zero and

(ii) there exists at least one r^{th} order minor of A which is not zero.

Rank of A is denoted as $\rho(A)$.

eg:- Find the rank of the matrix. $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

Sol:- $|A| = -1(18-1) + 6(3+30)$

$= 181 \neq 0$

$\therefore \rho(A) = 3.$

⑥ Find the value of K if the rank $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & K & 4 \\ -1 & 2 & 5 \end{bmatrix}$ is 2.

sol: Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & K & 4 \\ -1 & 2 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & k & 4 \\ -1 & 2 & 5 \end{vmatrix} = 1(5k-8) - (-2)[10-(-4)] + 3[4-(-k)]$$

$$= 5k - 8 + 2(14) + 3(4+k)$$

$$= 5k - 8 + 28 + 12 + 3k$$

$$\begin{array}{r} 28 \\ 12 \\ \hline 32 \end{array}$$

$$\Rightarrow 8k + 32 = 0$$

$$8k = -32$$

$$k = -\frac{\cancel{32}\,4}{\cancel{8}\,1} \Rightarrow \boxed{k = -4}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix}$$

$$|A| = 1(48 - 40) - 2(36 - 28) + 3(30 - 28)$$

$$8 - 16 + 6 = 14 - 16 = -2 \neq 0$$

$$\boxed{\rho(A) = 3.}$$

Q.7 Find the rank of the matrix $\begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{vmatrix}$

Sol: Given: $A = \begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{vmatrix}$

$R_2 \rightarrow R_2 - \frac{1}{2} R_1$

$$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 5 & 2 & 4 & 3 \end{vmatrix}$$

$R_3 \rightarrow R_3 - \frac{5}{2} R_1$

$$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 0 & 9/2 & -1/2 & 1/2 \end{vmatrix}$$

$R_3 \rightarrow R_3 - \frac{9}{8} R_2$

$$= \begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -3/2 & 1/2 \\ 0 & 0 & 5/8 & 5/8 \end{vmatrix}$$

The row echelon form is obtained.

$\therefore$ The rank of matrix is 3.

8. If $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, Find the Eigen values of $A^3 + 7A^2 + 2A$

sol:- Characteristic eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(1-\lambda) - 0 = 0$$

$$-1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

eigen values of A are $1, -1$.

A^3 are $1, -1$

A^2 are $1, 1$.

eigen values of $(A^3 + 7A^2 + 2A)$ are

$$= (-1 + 7 - 2, \ 1 + 7 + 2)$$

$$= (4, 10)$$

∴ The eigen values are 4, 10 //

Q9. Convert the matrix $A = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{vmatrix}$ into echelon form.

Sol: Given: $A = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 5 \\ -1 & 2 & 3 \end{vmatrix}$.

$R_1 \leftrightarrow R_2$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix}$$

$R_2 = R_2 - \frac{R_1}{2}$

$R_3 = R_3 + \frac{R_1}{2}$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{vmatrix}$$

$R_2 = \frac{R_2}{2}$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix}$$

$R_3 = R_3 - R_2$

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

Q10 Examine linear independence of the given vectors

$(1,1,0,1)$; $(1,1,1,1)$; $(-1,1,1,1)$; $(1,0,0,1)$.

$$\left|\begin{pmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{0} & \overset{-}{1} \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right| = 1\begin{vmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{1} \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \left[1[(1-0)] - 1(1-0) + 1(0)\right] - \left[1(1-0) - 1(-\cancel{1}+\cancel{1}) + 1(0-1)\right]$$
$$- \left[1(0-0) - 1(0-1) + 1(0-1)\right]$$

$$= \cancel{1} - \cancel{1} - \cancel{1} + 1 - \cancel{1} + 1$$

$$= 0$$

$\therefore$ x_1, x_2, x_3, x_4 are linearly dependent

11. Examine linear independence of the given vectors

(1, 1, 0, 1); (1, 1, 1, 1); (-1, 1, 1, 1); (1, 0, 0, 1).

sol:-

$$\begin{array}{cccc} + & - & + & - \end{array}$$

$$\left|\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right| = 1\begin{vmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{1} \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \left[1\left[(1-0)\right] - 1(1-0) + 1(0)\right] - \left[1(1-0) - 1(-\cancel{1}+\cancel{1}) + 1(0-1)\right]$$
$$- \left[1(0-0) - 1(0-1) + 1(0-1)\right]$$

$$= \cancel{1} - \cancel{1} - 1 + 1 - 1 + 1$$

$$= 0$$

$\therefore$ x_1, x_2, x_3, x_4 are linearly dependent

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12. Discuss the nature of quadratic form $x^2-y^2+4z^2+2yz+6zx+4xy$. also find index and signature

sol:- The given QF is

$$x^2-y^2+4z^2+2yz+6zx+4xy.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

characteristic eqn $|A-\lambda I|=0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)[-1-\lambda-4+\lambda] - 2[8-2\lambda-3] + 3[2+3+3\lambda]$$

$$= \lambda^2-3\lambda-5-\lambda^3+3\lambda^2+5\lambda-10+4\lambda+15+9\lambda=0$$

$$-\lambda^3+4\lambda^2+15\lambda=0$$

$$-\lambda(\lambda^2-4\lambda-15)=0$$

$$\lambda=0, \quad \lambda=\frac{4\pm\sqrt{16+60}}{2}$$

$$\lambda=0, \quad \lambda=\frac{4\pm\sqrt{76}}{2}$$

$\therefore$ The nature of Quadratic form is indefinite

index $=1$ (no of +ve eigen values)

signature = no of +ve eigen values - no of -ve eigen value.

$$= 1-1$$

$$= 0$$

Q.13 Write the matrix form and also the Augmented matrix for the given system of eqns. $3x - y - z = 3$, $2x - 8y + z = -5$, $x - 2y + 9z = 8$. $\boxed{CO2 \mid BTL2}$

Sol: The given eqn's can be written in the matrix form as $AX = B$

Here $A = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$

Now, Augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 3 & -1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

14. Define eigenvalues & eigen vectors with example.

Ans:- Defn:- A non zero vector x is an eigen vector (or characteristic vector) of a square matrix A if there exists a scalar λ such that $Ax = \lambda x$. Then λ is an eigenvalue (or characteristic value) of A.

Eg:- consider $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

char eqn $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0.$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$\lambda = 6, -1$ are eigen values.

then for $\lambda = -1$, eigen vectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ //

$\lambda = 6$ " " " $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ //

15Q. verify that $X = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$ is one of the eigen vector of

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Sol :- $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Char eqn $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(5-\lambda)(1-\lambda)-1\} - 1[1-\lambda-3] + 3[1-15+3\lambda] = 0$$

$$\Rightarrow (1-\lambda)\{5-6\lambda+\lambda^2-1\} + \lambda + 2 - 42 + 9\lambda = 0$$

$$\Rightarrow (1-\lambda)\{\lambda^2-6\lambda+4\} + 10\lambda - 40 = 0.$$

$$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + 10\lambda - 40 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0.$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0.$$

$$\lambda = 3 \Rightarrow 3^3 - 7(3)^2 + 36 = 0.$$

$$\lambda = 3 \begin{array}{|cccc} 1 & -7 & 0 & 36 \\ 0 & 3 & -12 & -36 \\ \hline 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\Rightarrow \lambda^2 - 4\lambda - 12 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-6) = 0. \Rightarrow \lambda = -2, 6.$$

$\therefore$ the eigen values are $\lambda = -2, 3, 6.$

find the values of a, b such that the equation $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+az=b$ has (i) No solution (ii) infinite solution (iii) unique solution

Sol The given systems of equation can be written as

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ b \end{bmatrix}$$

$$\therefore \frac{A}{B} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & 2 & 8 \\ 2 & 3 & a & b \end{bmatrix}$$

$$R_2 \rightarrow 2R_2 - 7R_1; \; R_3 \rightarrow R_3 - R_1$$

$$A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & a-5 & b-9 \end{bmatrix}$$

(i) Let $a = 5$ and $b \neq 9$. $\Rightarrow A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & 0 & b-9 \end{bmatrix}$

$\Rightarrow P(A/B) = 3; \; P(A) = 2.$

$\because P(A/B) \neq P(A)$

The system is inconsistent and it has no solution.

(ii) Let $a = 5$ and $b = 9$.

$$A/B \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow \rho(A/B) = 2;\ \rho(A) = 2.$

$\because \rho(A/B) = \rho(A) = 2 < n(3)$

$\Rightarrow$ The system is ~~in~~consistent and it has infinite number of solutions.

(iii) Let $a \neq 5$ and $b = 9$.

$\Rightarrow \rho(A/B) = 3;\ \rho(A) = 3.$

$\because \rho(A/B) = \rho(A) = r(3) = n(3)$

The system is consistent and it has an unique solution

2. Solve the system of equation $x+3y+2z=0$, $2x-y+3z=0$; $3x-5y+4z=0$; $x+17y+4z=0$.

Sol: Given system of equation can be written in the form of

$AX = B$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$; $R_3 \to R_3 - 2R_2$

i.e. $$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ It is echelon form.

No of non-zero rows $= 2$.

$\Rightarrow \rho(A) = 2 = r$

$\Rightarrow n = 3$

$\because n > r$ i.e., we have to assign $n - r = 3 - 2 = 1$ arbitratory constant.

$\Rightarrow$ Non-trival solution.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 3y + 2z = 0 \quad \text{---- (1)}.$

$-7y - z = 0 \quad \text{---- (2)}.$

(2) $\Rightarrow -7y - z = 0.$

$-7y = z.$

$y = \frac{-z}{7}.$

Let $z = k \Rightarrow y = \frac{-k}{7}.$

(1) $\Rightarrow x + 3y + 2z = 0.$

$x + 3\left(\frac{-k}{7}\right) + 2k = 0.$

$x + \left(2 - \frac{3}{7}\right)k = 0.$

$x + \left(\frac{11}{7}\right)k = 0.$

$\therefore x = -\frac{11}{7}k$ //.

3. Verify cayley hamilton theorem and find the rank of the matrix where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

Sol: characteristic matrix $= A - \lambda I$

$$\therefore A-\lambda I = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

characteristic equation $\Rightarrow |A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\left[0-(1-\lambda)(-1-\lambda)\right]-0+0=0.$$

$$(2-\lambda)\left(-1(-1-\lambda+\lambda+\lambda^2)\right]=0$$

$$(2-\lambda)(-\lambda^2+1)=0.$$

$$-2\lambda^2+2+\lambda^3-\lambda$$

$$\therefore \lambda^3-2\lambda^2-\lambda+2=0.$$

By cayley hamilton theorem

$$A^3-2A^2-A+2=0.$$

$$A^2=A\cdot A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3=A^2\cdot A = \begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 21 & 42 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$LHS = A^3 - 2A^2 - A + 2I$$

$$= \begin{bmatrix} 8 & 21 & 42 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-8-2+2 & 21-18-3-0 & 42-38-4+0 \\ 0-0-0+0 & 1-2-1+2 & 5-0-5+0 \\ 0-0-0+0 & 0-0-0+0 & -1-2+1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= RHS$

$\therefore LHS = RHS$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ It is echelon form.

$\rho(A) = 3$.

4. Reduce the quadratic form $Q = 3x^2 + 3y^2 + 3z^2 + 2xy + 2zx - 2yz$ to canonical form and find its nature, index and signature.

Sol
2

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic matrix $= A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristics equation $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)(3-\lambda) - 1\right] + \left[3-\lambda+1\right] + 1\left[-1-3+\lambda\right] = 0.$$

$$(3-\lambda)\left[9+\lambda^2-6\lambda-1\right] - 4 + \lambda - 4 + \lambda = 0$$

$$(3-\lambda)\left[\lambda^2 - 6\lambda + 8\right] - 8 + 2\lambda = 0$$

$$3\lambda^2 - 18\lambda + 24 - \lambda^3 + 6\lambda^2 - 8\lambda - 8 + 2\lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0$$

$$\lambda^3 + 9\lambda^2 + 24\lambda - 16 = 0$$

$$
\begin{array}{c|cccc}
\lambda = 4 & 1 & -9 & 24 & -16 \\
 & 0 & 4 & -20 & 16 \\
\hline
\lambda = 1 & 1 & -5 & 4 & \boxed{0} \\
 & 0 & 1 & -4 & \\
\hline
\lambda = 4 & 1 & -4 & \boxed{0} & \\
 & 0 & 4 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}
$$

$\Rightarrow \lambda = 1, 4, 4$.

$\because$ All eigen values are positive

$\Rightarrow$ Nature of quadratic form is positive definite

Consider $(A - \lambda I) X_i = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

(i) eigen vector corresponding to $\lambda = 1$:

(1) $\Rightarrow$ $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + y + z = 0 \quad \text{--- (2)}$$
$$x + 2y - z = 0 \quad \text{--- (3)}$$
$$x - y + 2z = 0 \quad \text{--- (4)}$$

Solving (2) and ③.

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ 2 & 1 & 1 & 2 & 1 & 1 & \\ 1 & 2 & -1 & 1 & 2 & -1 & \\ & -1 & 2 & 1 & -1 & 2 & \end{array}$$

$$\frac{x}{-1-2} = \frac{y}{1+2} = \frac{z}{4-1} = k$$

$$\frac{x}{-3} = \frac{y}{3} = \frac{z}{3} = k.$$

multiply by -3

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{-1} = k.$$

$\therefore x = k = y = -k ; z = -k.$

$$\therefore X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ -k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(ii) eigen vector corresponding to $\lambda = 4$

$$(1) \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x+y+z=0 \quad \text{—} (5)$$

$$x-y-z=0 \quad \text{—} (6)$$

$$x-y-z=0 \quad \text{—} (7)$$

Solving 5 & 6.

$$\frac{x}{-1+1} = \frac{-y}{+1-1} = \frac{-z}{1-1} = k.$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0} = k.$$

Let $z = k_1$ and $y = k_2$.

(5) $\Rightarrow$ $x - y - z = 0$

$x = y + z$

$x = k_1 + k_2$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$B = [X_1 X_2 X_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$\therefore$ model matrix $(P) = [e_1 \; e_2 \; e_3]$

$$e_1 = \frac{x_1}{\|x_1\|} \; ; \; e_2 = \frac{x_2}{\|x_2\|} \; ; \; e_3 = \frac{x_3}{\|x_3\|}.$$

$$\|x_1\| = \sqrt{1^2+(-1)^2+(-1)^2} = \sqrt{3}.$$

$$\|x_2\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}.$$

$$\|x_3\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}.$$

$$\therefore \; P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\therefore \; P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$P^{-1}AP = \text{Diagonal }(1, 4, 4) = D$.

$\therefore$ Canonical form $= Y^T D Y$.

$$Y^T = [y_1 \; y_2 \; y_3] \; ; \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow Y^T D Y = [y_1 \; y_2 \; y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 4y_2^2 + 4y_3^2$$

$\therefore$ Index $s = 3$; Rank $= 3$.

$\therefore$ signature $= 2s - r = 2(3) - 3 = 3$.

5. Test for the consistency and solve if consistent system of equations

$$x + y + z = 3$$
$$3x - 9y + 2z = -4$$
$$5x - 3y + 4z = 6.$$

Sol The given system of equation can be written as $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$$

$$A/B = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -9 & 2 & -4 \\ 5 & -3 & 4 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 5R_1$

$$A/B \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -12 & -1 & -13 \\ 0 & -8 & -1 & -9 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$A/B \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -12 & -1 & -13 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\rho(A/B) = 3$:

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow \rho(A) = 3$.

$\therefore \rho(A) = \rho\left(\frac{A}{B}\right) = n(3) = n(3)$

$\Rightarrow$ The system is consistent and has an unique solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -13 \\ -1 \end{bmatrix}$$

$\Rightarrow x + y + z = 3$ —— (1)

$- 12y - z = -13$ —— (2)

$-z = -1 \Rightarrow z = 1$

(2) $\Rightarrow -12y - 1 = -13 \Rightarrow -12y = -12$

$y = 1$

(1) $\Rightarrow x + 1 + 1 = 3 \Rightarrow x = 1$

$\therefore x = 1, y = 1, z = 1$ is the unique solution

6. Reduce the matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$ to echelon form and hence find its rank.

Sol: given

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 9 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - 3R_1$; $R_4 \rightarrow R_4 + R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & -7 & -13 & -5 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -13 & -5 \\ 0 & -7 & -14 & -3 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_4 \to 7R_4 + 6R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -13 & -5 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -130 \end{bmatrix}$$

$\therefore$ It is in echelon form

$\rho(A) = 4$.

7 Verify cayley hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

sol $\therefore$ characteristics matrix $= A - \lambda I$.

$\Rightarrow A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

characteristic equation $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

By cayley hamilton theorem

$$A^2 - 7A + 6I = 0$$

$$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \cancel{-7} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}.$$

$$LHS = A^2 - 7A + 6I$$

$$= \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29-35+6 & 28-28+0 \\ 7-7+0 & 8-14+6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS.$$

Hence, cayley hamilton theorem is verified

8. find the eigen values and corresponding eigen vectors of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

sol: characteristic of matrix $= A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$y = 0$

$z = 0$

Let $x = k$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ //

9. Verify cayley hamilton theorem for the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

sol: $\therefore$ characteristic matrix $= A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 2 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristic equation $= |A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\,[(2-\lambda)(3-\lambda) - 0) - 2(0-0) + 1(0-2+\lambda) = 0$$

$$(3-\lambda)\,[\lambda^2 - 5\lambda + 6] - 2 + \lambda = 0$$

$$3\lambda^2 - 15\lambda + 18 - \lambda^3 + 5\lambda^2 - 6\lambda - 2 + \lambda = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

characteristic equation $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[-(1-\lambda)^2\right] = 0$$

$$(1-\lambda)\left[-(1^2+\lambda^2-2\lambda)\right] = 0$$

$$(1-\lambda)\left[-1-\lambda^2+2\lambda\right] = 0.$$

$$-1-\lambda^2+2\lambda+\lambda+\lambda^3 - -2\lambda^2 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$$

$\lambda=1$	1	-3	3	-1
	0	1	-2	1
$\lambda=1$	1	-2	$+1$	$\boxed{0}$
	0	1	-1	
$\lambda=1$	1	-1	$\boxed{0}$	
	0	1		
	1	$\boxed{0}$		

The eigen values are 1, 1, 1

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

cayley hamilton theorem states that every square matrix satisfies its own characteristic equation

$$A^3 - 8A^2 + 20A - 16I = 0$$

$$LHS = A^3 - 8A^2 + 20A - 16I$$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix} - 8 \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 16 & 12 & 10 \end{bmatrix} + 20 \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} - 16 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36-80+60-16 & 56-96+40-0 & 28-40+20-0 \\ 0-0+0-0 & 8-32+40-16 & 0-0+0-0 \\ 28-48+20-0 & 56-96+40-0 & 36+80+60-16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} RHS$$

$$LHS = RHS$$

Hence, cayley hamilton theorem is verified

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10. find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

sol

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + R_1$; $R_3 \longrightarrow R_3 - R_1$; $R_4 \longrightarrow R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \longrightarrow 2R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is echelon form.

No of non zero rows $= 3$

$\therefore \rho(A) = 3$.

11) ~~Find~~ Reduce the quadratic form to canonical form $8x_1^2+7x_2^2+3x_3^2+12x_1x_2+4x_1x_3-8x_2x_3$

sol:

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

char eqn is $|A-\lambda I|=0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} \Rightarrow$$

$$= (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+2]+0.5[24-(0.5)(7-\lambda)]=0$$

$$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+6(6\lambda-16)+0.5(0.5\lambda+20.5)=0$$

$$\cancel{-\lambda^3+18\lambda^2+44-25\lambda+66.75=0}$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda^2-18+45)=0$$

$$\lambda(\lambda^2-15\lambda-3\lambda+45]=0$$

$$\lambda[(\lambda-15)(\lambda-3)]=0$$

$$\therefore \lambda=0,3,15.$$

case(i) :- $\lambda=0 \Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$8x_1-6x_2+2x_3=0$$

$$-6x_1+7x_2-4x_3=0$$

$$2x_1-4x_2+3x_3=0$$

$$\frac{x_1}{(7)(3)-(-4)(-4)} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14}$$

$$\frac{x_1}{5} = \frac{\cancel{-}x_2}{\cancel{-}10} = \frac{x_3}{10} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

where $\|x_1\| = \sqrt{1+4+4}$
$= \sqrt{9} = 3.$

case (ii) :- $\lambda = 3$,

$$A = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

$$\frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \frac{X_2}{\|X_2\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

III^{ly} $$\frac{X_3}{\|X_3\|} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \quad P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$PAP^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$\therefore$ The canonical form is $0y_1^2 + 3y_2^2 + 15y_3^2$.

12Q S.T. Sum of eigen values of a matrix is its trace ~~an~~ and product of eigen values is its determinant

Sol:- (i) $tr(A) = \sum_{i=1}^{\infty} \lambda i$

(ii) $|A| = \prod_{i=1}^{\infty} \lambda i$

Proof:- Char eqn $|A - \lambda I| = 0$.

$(-\lambda)^m + \alpha_{m-1}(-\lambda)^{m-1} + \dots + \alpha_1(-\lambda) + \alpha_0 = 0$ → polynomial form.

solve for α_0, $\lambda = 0$

$$\alpha_0 = |A - (0)I| = |A|$$

solve for α_{m-1}.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{21}-\lambda & \cdots & a_{2m} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn}-\lambda \end{vmatrix}.$$

Determinant produces all products of m terms of A such that exactly 1 element from each row and each column

Only 1 way to achieve λ^{m-1}! The product of the diagonal elements

$(a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{mn}-\lambda$

$\alpha_{m-1} = a_{11} + a_{22} + \cdots + a_{mn} = tr(A)$

$\lambda_1, \lambda_2 \dots \lambda_m$ eigen values are the roots of characteristic eqn.

$(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_m - \lambda) = 0$

$$\alpha_0 = \prod_{i=1}^{m} \lambda_i = |A|$$

$$\alpha_{m-1} = \sum_{i=1}^{m} \lambda_i = tr(A)$$

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5) Reduce the quadratic form to canonical form $2xy + 2yz + 2zx$ or $2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

sol:- Given $2xy + 2yz + 2zx$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$\lambda = 2, -1, -1 \Rightarrow$ Eigen values.

case(1):- $\lambda = 2$.

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + z = 0 \quad -(1)$$

$$x - 2y + z = 0 \quad -(2)$$

$$x + y - 2z = 0 \quad -(3)$$

$$\frac{x}{1+2} = \frac{-y}{-2-1} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{+y}{+3} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k \text{ (say)} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case (ii) :- $\lambda = -1$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

Let $x = 1,\ y = 0$ $\qquad n - r$

$z = -1$ $\qquad 3 - 1 = 2$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

case (iii) :- $\lambda = -1$.

$\therefore$ ~~X_3 is orthogonal to X_1~~

Let $x = 1,\ y = 1 \Rightarrow x + y + z = 0$

$z = -2$.

$$\therefore X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$\therefore X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0.$$

$\|X_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, $\quad \|X_3\| = \sqrt{1^2 + (-2)^2 + 1^2}$

$\|X_2\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$ $\qquad = \sqrt{6}$

$$P = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right]$$

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Now $PAP^T =$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now canonical form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$

$$\underline{2x^2 - y^2 - z^2 = 0}$$

Index = no of +ve eigen values = 1

Signature = (no of +ve eigen values) - (no of -ve eigen)

$= 1 - (2) = -1$

Nature is indefinite. //

14) Find the eigen value & eigen vector of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

char eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \quad -①$$

$$= (1-\lambda)[(1-\lambda)(1-\lambda)-1] - 1[(1-\lambda)-1] + 1[1-(1-\lambda)]$$

$$= (1-\lambda)[\cancel{1}-\lambda-\lambda+\lambda^2-\cancel{1}] - [\cancel{1}-\lambda-\cancel{1}] + 1\{\cancel{1}-\cancel{1}-\lambda\}$$

$$= (1-\lambda)(\lambda^2-2\lambda) + \lambda + \lambda$$

$$\Rightarrow \lambda^2 - 2\cancel{\lambda} - \lambda^3 + 2\lambda^2 + 2\cancel{\lambda} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda - 3) = 0$$

$$-\lambda^2 = 0, \quad \lambda - 3 = 0$$

$$\lambda = 0 \quad \lambda = 3$$

$\therefore \lambda = 0, 0, 3$. are eigen values.

case(i):- Put $\lambda = 0$ in eqn ①

consider $(A - \lambda I) X_i = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y + z = 0 \quad - (2)$$

$$x + y + z = 0 \quad - (3)$$

$$x + y + z = 0 \quad - (4)$$

$$\begin{array}{cccc} & x & y & z \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x}{1-1} = \frac{y}{1-1} = \frac{z}{1-1} = k.$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $x + y + z = 0$

let $z = k_1$, $y = k_2$

$$x + k_1 + k_2 = 0$$

$$x = -k_1 - k_2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) :- Put $\lambda = 3$ in (1).

$(A - \lambda I)x_i = 0.$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\begin{array}{cccccc} & x & & y & z & \\ 1 & & 1 & -2 & & 1 \\ -2 & & 1 & 1 & & -2 \end{array}$$

$$\frac{x}{1+2} = \frac{y}{1+2} = \frac{z}{4-1} = k \text{ (say)}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

multiply by 3 $\Rightarrow$ $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(15) Determine eigen values of (i) A^2 (ii) A^{-1} (iii) $B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$

<u>Sol</u>:- Given $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$

Char eqn $A - \lambda I = \begin{pmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{pmatrix}$

$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$

$\Rightarrow 16 - 10\lambda + \lambda^2 + 8 = 0$

$\Rightarrow \lambda^2 - 10\lambda + 24 = 0.$

$\Rightarrow \lambda^2 - 6\lambda - 4\lambda + 24 = 0$

$\Rightarrow \lambda(\lambda - 6) - 4(\lambda - 6) = 0$

$(\lambda - 4)(\lambda - 6) = 0$

$\Rightarrow \lambda = 4, 6$ are the eigen values of 'A'

(i) eigen values of A^2 are $4^2, 6^2 \Rightarrow 16, 36$

(ii) eigen values of A^{-1} are $4^{-1}, 6^{-1} \Rightarrow 1/4, 1/6.$

(iii) eigen values of $B = 2A^2 - \frac{1}{2}A + 3I$ are.

~~Bex~~ $\Rightarrow 2(16) - \frac{1}{\cancel{2}}(\cancel{4}^{2}) + 3(1)$

$= 32 - 2 + 3 = 33.$

and $2(36) - \frac{1}{\cancel{2}}(\cancel{6}^{3}) + 3(1)$

$= 72 - 3 + 3 = 72.$

~~a eigenvalu~~

$\therefore$ eigen values of B are 33, 72.

unit II

SA Q.

1Q. Define Exact differential Equation

Ans An Equation of the form.

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

if it satisfy the condition.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Then general solution

$$\int M\,dx + \int_{ffx} N\,dy = c$$

2Q. Define Integrating factor of non-homogenous differential Eqn.

$$\frac{dy}{dx} + Py = Q(x)$$

Ans Integrating factor $= e^{\int P dx}$

general sol^n $y(I \cdot F) = \int Q(I \cdot F)\,dx + c$

3Q write Riccati & clairet's Eq^n.

Ans An Eq^n of the form.

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x).$$

where $y = v(x)$ is the Particular sol^n.

Clairaut's Eqn.

$$y' = xy' + f(y')$$

$\boxed{4Q}$ Solve $(2x-y+1)dx + (2y-x-1)dy = 0$

Soln $M = 2x-y+1$, $N = 2y-x-1$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1$$

This is an exact.

$$\int (2x-y+1)dx + \int 1+2y\,dy = c$$

$\longrightarrow \int(2x-y+1)dx + \int(2y-1)dy =$

$$\cancel{2}\frac{x^2}{\cancel{2}} - xy + x + \cancel{2}\frac{y^2}{\cancel{2}} - y = C$$

$\boxed{5Q.}$ Define orthogonal Trajectories of a given family of curve and write the procedure to find it in polar form

$\boxed{Ans}$ A curve which cuts the family of curves orthogonally (at right angles) then that curve is called orthogonal Trajectory of the family of curves.

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Procedure of polar form of (O.T)

1) differentiate $\bar{\bar{r}}^2$ w.r.to θ $\left(\frac{dr}{d\theta}\right)$

2) eliminate the constant

3) replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$\boxed{6Q}$. Solve $x\frac{dy}{dx}+y=\log x$

$\boxed{Ans}$ $\div x$

$$\frac{dy}{dx}+\frac{y}{x}=\frac{\log x}{x}$$

this is linear diff Eqn.

$$I.F = e^{\int p\,dx} = e^{\int \frac{1}{x}dx}$$

$$= e^{\log x} = x$$

$$g.s = y(x) = \int\left(\frac{\log x}{\cancel{x}}\cdot\cancel{x}\right)dx + c$$

$$= y(x) = x\log x - x + c$$

$\boxed{7Q}$ find the general sol^n of $y = xp - p^3$

sol^n diff $\bar{\bar{y}}^2$ w.r.to x

$$\frac{dy}{dx} = x\frac{dp}{dx} + p(1) - 3p^2\frac{dp}{dx}$$

$$p = \frac{dp}{dx}(x - 3p^2) + p$$

$$p - p = \frac{dp}{dx}(x - 3p^2)$$

$$0 = \frac{dp}{dx}(x - 3p^2)$$

$$x = 3p^2$$

$$y = xp - p^3$$

$$y = \cancel{p}(3p^2) - p^3$$

$$\left.\begin{aligned} y &= 2p^3 \\ x &= 3p^2 \end{aligned}\right\} \text{gen sol}^n$$

$\boxed{8\ Q}$ find the orthogonal Trajectory of the family of Curves $y = cx^2$ where "c" is a parameter

$\boxed{Sol^n}$ $y = cx^2$ —— (1)

$$\frac{dy}{dx} = c(2x)$$

$$C = \frac{dy}{dx}\left(\frac{1}{2x}\right)$$

sub in — ①

$$y = \frac{x^2}{2x}\frac{dy}{dx}$$

$$y = \frac{x}{2}\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = -\frac{x}{2}\frac{dx}{dy}$$

$$\int y\,dy = -\frac{1}{2}\int x\,dx + k$$

$$\frac{y^2}{\cancel{2}} = -\frac{1}{\cancel{2}}\frac{x^2}{2} + k$$

$\boxed{9Q}$ solve $x\,dy - y\,dx = (x^2+y^2)dy$

$\boxed{sol^n}$ by Inspection Method

$$\frac{x\,dy - y\,dx}{x^2+y^2} = dy$$

$$\int d\tan^{-1}\left(\frac{y}{x}\right) = \int dy + k$$

$$\tan^{-1}(y/x) = y + k$$

10Q find the sol^n of diff Eq^n.

$$(y-x+1)\,dy-(y+x+2)\,dx=0$$

Sol^n $M=-y-x-2$, $N=y-x+1$

$$\frac{\partial M}{\partial y}=-1, \quad \frac{\partial N}{\partial x}=-1$$

is an exact.

$$\int(-y-x-2)\,dx+\int y\,dy=c$$

$$-yx-\frac{x^2}{2}-2x+\frac{y^2}{2}=c$$

11Q solve $y(2xy+e^x)\,dx=e^x\,dy$

Sol^n $[2xy^2+e^x y]\,dx=e^x\,dy$

$$e^x y\,dx-e^x\,dy=-2xy^2\,dx$$

$\div\ y^2$

$$\frac{e^x y\,dx-e^x\,dy}{y^2}=-2x\,dx$$

$$\int d\left(\frac{e^x}{y}\right)=-2\int x\,dx$$

$$e^x/y=-2\frac{x^2}{2}+k$$

$\boxed{12Q}$ find the O.T of $x^2+16y^2=c$

$\boxed{\text{Ans}}$ diff "y" w.r.to "x"

$$2x+16(2)y\frac{dy}{dx}=0$$

$$x+16y\frac{dy}{dx}=0$$

$$x=-16y\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x=+16y\frac{dx}{dy}.$$

$$\int\frac{dx}{x}=\frac{1}{16}\int\frac{dy}{y}$$

$$\log x=\frac{1}{16}\log y+\log c$$

$\boxed{13Q}$. solve $\frac{dy}{dx}=e^x+y$

$\boxed{\text{Sol}^n}$ $\frac{dy}{dx}-y=e^x$

$$P=-1,\quad Q=e^x$$

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$$e^{-\int 1 dx} = e^{-x}$$

$$g.s = y(I.F) = \int Q(I.F)\,dx + c$$

$$y\,e^{-x} = \int e^{x} e^{-x}\,dx + c$$

$$y\,e^{-x} = \int 1\,dx + c$$

$$\underline{y\,e^{-x} = x + c}$$

$\boxed{149}$ find the orthogonal Trajectory of family of curves $r = c\theta^2$ — ①

$\boxed{Sol^n}$

$$\frac{dr}{d\theta} = c(2\theta)$$

$$c = \frac{dr}{d\theta} \times \frac{1}{2\theta}$$

Sub in ①

$$r = \frac{1}{2\theta}\frac{dr}{d\theta}(\theta^2)$$

$$r = \frac{1}{2}\frac{dr}{d\theta}(\theta)$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$r = \frac{1}{2}\theta\left(-r^2\frac{d\theta}{dr}\right)$$

$$\frac{\cancel{d\theta}}{\cancel{\theta}} = \frac{-\cancel{r}^2}{\cancel{2}}\frac{\cancel{d\theta}}{\cancel{dr}}$$

$$1 = \frac{1}{2}(-\theta)\frac{d\theta}{dr}\cdot r$$

$$\int \frac{dr}{r} = -\frac{1}{2}\int \theta\, d\theta + k$$

$$\log r = -\frac{1}{2}\frac{\theta^2}{2} + k$$

$\boxed{15Q}$. find Integrating factor of

$$(x^3+y^3)\,dx - x^2y\,dy = 0$$

$\boxed{Sol^n}$ This is the form of homogenous diff Eq^n.

Integrating factor $\dfrac{1}{Mx+Ny}$

$$= \frac{1}{(x^3+y^3)x + (-x^2y)y} = \frac{1}{x^4+y^3x - x^2y^2}$$

Unit II

LAQ

$\boxed{1.Q}$ solve $(x-y^2)\,dx + 2xy\,dy = 0$

$\boxed{\text{Sol}^n}$ $M\,dx + N\,dy = 0$

$M = x - y^2$, $N = 2xy$

$\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 2y$

This is non-exact diff Eqn.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y - 2y = -4y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y}{2xy} = -\frac{2}{x} = I.F$$

$$\boxed{I.F = 1/x^2} = -2\log_e x \Leftarrow e^{-\int 2/x\,dx}$$

Multiply the given Eqn by I.F

$$\left(\frac{x-y^2}{x^2}\right)dx + \frac{2xy}{x^2}\,dy = 0$$

$$\left(\frac{1}{x} - \frac{y^2}{x^2}\right)dx + \frac{2}{x}y\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-2y}{x^2}, \quad \frac{\partial N}{\partial x} = \frac{-2y}{x^2}$$

Now exact

$$\int\left(\frac{1}{x}-\frac{y^2}{x^2}\right)dx+\int\frac{2y}{x}dy=k$$

$$\log x+\frac{y^2}{x}+0=k$$

$\boxed{2Q}$ find orthogonal Trajectory of $r^n \sin n\theta = c$ where "c" is a parameter -

$\boxed{Sol^n}$ $\log(r^n \sin n\theta) = \log c$

$$\log r^n + \log \sin n\theta = \log c$$

$$n\log r + \log \sin n\theta = \log c$$

diff "r" w.r.to θ

$$\frac{n}{r}\frac{dr}{d\theta}+\frac{n\cos n\theta}{\sin n\theta}=0$$

$$\frac{dr}{d\theta}\left(\frac{1}{r}\right)=-\cot n\theta$$

Replace $\frac{dr}{d\theta}=-r^2\frac{d\theta}{dr}$

$$\frac{-\cancel{r^2}\,d\theta}{\cancel{(r)}\,dr}=-\cot n\theta$$

$$r \frac{d\theta}{dr} = \cot n\theta$$

$$\int \frac{dr}{r} = \int \tan n\theta \, d\theta + \log k$$

$$\log r = \log \left|\frac{\sec n\theta}{n}\right| + \log k$$

$\boxed{3Q}$ Solve $x \frac{dy}{dx} + y = x^3 y^6$

$\boxed{\text{sol}^n}$ $\div\ x$

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \quad \text{—(1)}$$

This is Bernoulli's.

$\div\ y^6$

$$y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \quad \text{—(2)}$$

~~This~~ Let $y^{-5} = v$

$$-5 y^{-6} \frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-6} \frac{dy}{dx} = -\frac{1}{5} \frac{dv}{dx}$$

Sub in —(2)

$$-\frac{1}{5} \frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\frac{dv}{dx} - \frac{5}{x} v = -x^2 (5)$$

This is linear form

$$P = \frac{5}{x}, \quad e^{\int \frac{5}{x} dx} = e^{-5\log x} = x^{-5}$$

$$V(I.F) = \int Q \cdot (I.F)\, dx + c$$

$$V(x^{-5}) = \int 5x^2 x^{-5}\, dx + c$$

$$y^{-5} x^{-5} = 5 \frac{x^{-3+1}}{-3+1} + c$$

$\boxed{4Q}$ solve

$$y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$$

$\boxed{Sol^n}$

$$2x^2y^2\,dx + e^x y\,dx - e^x\,dy = y^3\,dy$$

$$\div\ y^2$$

$$2x^2\,dx + \left(\frac{e^x y\,dx - e^x\,dy}{y^2}\right) = y\,dy$$

$$2\int x^2\,dx + \int d\left(\frac{e^x}{y}\right) = \int y\,dy + k$$

$$2\frac{x^3}{3} + \frac{e^x}{y} = \frac{y^2}{2} + k.$$

$\boxed{5Q}$ Solve the diff Eqn

$$y' + 4xy + xy^3 = 0$$

$$y' + 4xy = -xy^3$$

this is the form of Bernoulli's

$\boxed{\div y^3}$

$$y^{-3}\frac{dy}{dx} + 4xy^{-2} = -x$$

put $V = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx} = y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx} + 4xV = -x$$

$\times$ by (2)

$$\frac{dv}{dx} - 8xV = 2x$$

linear form

$$p = -8x, \quad Q = 2x$$

$$e^{-\int 8x\,dx} = e^{-\frac{8x^2}{2}} = e^{-4x^2}$$

$$V e^{-4x^2} = 2\int . x . e^{-4x^2} dx + k$$

$$y^{-2}e^{-4x^2} = 2\int x . e^{t} \frac{dt}{-8x} + k$$

$$-4x^2 = t$$

$$-8x\,dx = dt$$

$$y^{-2} e^{-4x^2} = -\frac{1}{4}[e^t] + k$$

$$y^{-2} e^{-4x^2} = -\frac{1}{4} e^{-4x^2} + k$$

7Q $\frac{dy}{dx} + 2xy = 2x$

Sol^n This is the form of linear

$$P = 2x, \quad Q = 2x$$

$$I.F = e^{\int p\,dx} = e^{2\int x\,dx} = e^{2x^2/2} = e^{x^2}$$

$$y e^{x^2} = 2\int x e^{x^2} dx + c$$

$$x^2 = t$$

$$2x\,dx = dt$$

$$y e^t = 2\int x \cdot e^t \frac{dt}{2x} + C$$

$$y e^t = \int e^t dt + c$$

$$y e^{x^2} = e^{x^2} + c$$

8Q find the general sol^n of the Riccati diff Eq^n.

$$y' = 3y^2 - (1+6x)y + 3x^2 + x + 1$$

Soln $y = x + \frac{1}{z}$ (assume).

$$y' = 1 - \frac{1}{z^2}\frac{dz}{dx}$$

Sub in the given Eqn Eqn.

$$1 - \frac{1}{z^2}\frac{dz}{dx} = 3\left(x+\frac{1}{z}\right)^2 - (1+6x)\left(x+\frac{1}{z}\right) + 3x^2+x+1$$

$$1 - \frac{1}{z^2}\frac{dz}{dx} = 3x^2 + \frac{3}{z^2} + \frac{6x}{z} - x - \frac{6x}{z} - \frac{1}{z} - 6x^2 + 3x^2 + x + 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{3}{z^2} - \frac{1}{z}$$

$$-\frac{1}{\cancel{z^2}}\frac{dz}{dx} = \frac{3-z}{\cancel{z^2}}$$

$$-\frac{dz}{dx} = 3 - z$$

$$\frac{dz}{dx} = -3 + z$$

$$\frac{dz}{dx} - z = -3$$

$$P = -1, \quad Q = -3$$

$$e^{-\int 1\,dx} = e^{-x}$$

$$z\,e^{-x} = -3\int e^{-x}\,dx + c$$

$$z\,e^{-x} = -3\frac{e^{-x}}{-1} + c$$

$$z e^{-x} = 3e^{-x} + c \Rightarrow z = 3 + ce^{x}$$

$$y = x + \frac{1}{z}$$

$$y = x + \frac{1}{3+ce^{x}}$$

$\boxed{9Q}$ find the orthogonal Trajectory of cardiods $r = a(1-\cos\theta)$ — ①

Sol^n

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a(\sin\theta)$$

$$a = \frac{dr}{d\theta}\left(\frac{1}{\sin\theta}\right)$$

sub in → ①

$$r = \frac{1}{\sin\theta}\frac{dr}{d\theta}(1-\cos\theta)$$

replace $\frac{dr}{d\theta}$ (by) $\frac{-r^2 d\theta}{dr}$

$$r = -r^2\frac{d\theta}{dr}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\frac{dr}{r} = -\left(\frac{1-\cos\theta}{\sin\theta}\right)d\theta$$

$$\frac{dr}{r} = -\left[\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right] d\theta$$

$$\frac{dr}{r} = \left(\frac{-1}{\sin\theta} + \cot\theta\right) d\theta$$

$$\int \frac{dr}{r} = \int \left(-\operatorname{cosec}\theta + \cot\theta\right) d\theta$$

$$\int \frac{dr}{r} = -\int \operatorname{cosec}\theta \, d\theta + \int \cot\theta \, d\theta + \log k$$

$$\log r = -\log|\operatorname{cosec}\theta - \cot\theta| + \log|\sin\theta| + \log k$$

$\boxed{10\ Q}$ find the general sol^n of the Equation. $\frac{dy}{dx} = 2xy^2 + (1-4x)y + 2x - 1$

If $y = 1$ is a particular sol^n

$\boxed{\text{Sol}^n}$ Let $y = 1 + \frac{1}{z}$ be the sol^n.

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

Sub y, y' in the given Eq^n

$$-\frac{1}{z^2}\frac{dz}{dx} = 2x\left(1+\frac{1}{z}\right)^2 + (1-4x)\left(1+\frac{1}{z}\right) + 2x - 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \cancel{2x} + \frac{2x}{z^2} + \cancel{\frac{4x}{z}} + 1 - \frac{4x}{z} - 4x + \frac{1}{z} + \cancel{2x} - 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{2x}{z^2} + \frac{1}{z}$$

$$-\cancel{\frac{1}{z^2}}\frac{dz}{dx} = \frac{2x+z}{\cancel{z^2}}$$

$$-\frac{dz}{dx} = 2x + z$$

$$-\frac{dz}{dx} - z = 2x$$

$$\frac{dz}{dx} + z = -2x$$

$$P = 1, \quad Q = -2x$$

$$e^{\int 1\,dx} = e^{x}$$

$$z e^{x} = -2\int x e^{x}\,dx + c$$

$$z e^{x} = -2\left[e^{x}(x-1)\right] + c$$

$$z = -2(x-1) + c e^{-x}$$

$y = 1 + \frac{1}{z}$ be the solⁿ

$$y = 1 + \frac{1}{-2(x-1) + c e^{-x}}$$

$\boxed{119}$ Solve $y(x+y)\,dx - x^2\,dy = 0$

Solⁿ $(xy + y^2)\,dx - x^2\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = xy + y^2, \quad N = -x^2$$

$$\frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = -2x$$

Non - exact

$$\frac{1}{Mx + Ny} = \frac{1}{(xy+y^2)x + (-x^2 y)}$$

$$\cancel{\frac{\partial M}{\partial y}} - \cancel{\frac{\partial N}{\partial x}} = \cancel{x + 2y + 2x}.$$

$$\frac{1}{Mx+Ny} = \frac{1}{xy^2} = I.F$$

xly the Eqn by. I.F

$$\left(\frac{xy + y^2}{xy^2}\right)dx - \frac{x^2}{xy^2}\,dy = 0$$

$$\int\left(\frac{1}{y} + \frac{1}{x}\right)dx - \int \frac{x}{y^2}\,dy = C$$

$$\frac{x}{y} + \log x + 0 = C$$

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$\boxed{12Q}$

solve. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ — ①

$\boxed{Sol^n}$ $\div \cos^2 y$

$$\sec^2 y\frac{dy}{dx} + x\left[\frac{2\sin y\cos y}{\cos^2 y}\right] = x^3$$

$$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3 \quad \text{— ②}$$

put $v = \tan y$

$$\frac{dv}{dx} = \sec^2 y\,\frac{dy}{dx}$$

Sub in — ②

$$\frac{dv}{dx} + 2xv = x^3$$

$$P = 2x,\quad Q = x^3$$

$$e^{2\int x\,dx} = e^{2x^2/2} = e^{x^2}$$

$$v\,e^{x^2} = \int x\cdot x^2\,e^{x^2}\,dx + c$$

$$x^2 = t,\quad 2x\,dx = dt$$

$$\tan y\,e^t = \int x\cdot t\,e^t\,\frac{dt}{2x} + k$$

$$\tan y\,e^t = \int t e^t\,dt + k$$

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$$\tan y\,(e^t) = e^t(t-1) + k$$

$$\tan y\; e^{x^2} = e^{x^2}(x^2-1) + k$$

13 Q) find the O.T of family of Curves.

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Soln diff $\bar{y}$ w.r.to x

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Replace y' by $\left(-\frac{1}{y'}\right)$

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$$

$$x^{-1/3} = y^{-1/3}\frac{dx}{dy}$$

$$\int y^{1/3}\,dy = \int dx\,(x^{1/3}) + k$$

$$\frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + k$$

14)

Sol^n $y = xp + p^2$ $p = y'$

$$\frac{dy}{dx} = x\frac{dp}{dx} + 1\cdot p + 2p\frac{dp}{dx}$$

$$p = \frac{dp}{dx}(x+2p) + p$$

$$p - p = \frac{dp}{dx}(x+2p)$$

$$0 = \frac{dp}{dx}(x+2p)$$

$$x + 2p = 0$$

$$x = -2p$$

$$y = xp + p^2$$

$$y = -2pp + p^2$$

$$\left.\begin{aligned} y &= -p^2 \\ x &= -2p \end{aligned}\right\}$$ general sol^n.

$$x^2 = 4p^2$$

$$y = -\frac{x^2}{4}$$ singular sol^n.

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15Q Solve

$$y\sin 2x\,dx - (1+y^2+\cos^2 x)\,dy = 0$$

Sol^n $M = y\sin 2x$

$M = 2y\sin x\cos x$

$$\frac{\partial M}{\partial y} = 2\sin x\cos x$$

$N = -1 - y^2 \cancel{+} \cos^2 x$

$$\frac{\partial N}{\partial x} = -2\cos x(-\sin x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$\int y\cdot(\sin 2x)\,dx + \int -y^2 dy = c$$

$$y\left[-\frac{\cos 2x}{2}\right] - \frac{y^3}{3} = c$$

LAQ 4

4Q Solve $y\,dx - x\,dy + e^x y^2 d\cancel{x} = 0$

Sol^n

$$\frac{y\,dx - x\,dy}{y^2} + e^x dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int e^x dx = c$$

$$x/y + e^x = c$$

1). Solve $y'' - y = 0$, when $y = 0$ & $y' = 2$ at $x = 0$.

Sol: The operator form is

$$(D^2 - 1)y = 0$$

The A.E is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$.

$$\therefore \; y(x) = y_c = c_1 e^x + c_2 e^{-x} \quad \text{---} \; (*)$$

when $y = 0$ at $x = 0$

$$\Rightarrow y(x) = c_1 e^x + c_2 e^{-x}$$

$$\Rightarrow y(0) = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 = 0$$

$$\Rightarrow \quad c_1 + c_2 = 0 \quad \text{---} \; (1)$$

Now. Diff (*) w.r. to x.

$$y'(x) = c_1 e^x - c_2 e^{-x}.$$

when $y'(0) = 2$.

$$\Rightarrow y'(0) = c_1 e^0 - c_2 e^{-0} = 2$$

$$\Rightarrow \quad c_1 - c_2 = 2 \quad \text{---} \; (2)$$

Solving Eq (1) & (2), we get

$$c_1 = 1 \quad \& \quad c_2 = -1.$$

$$\therefore \; y(x) = e^x - e^{-x}.$$

2). Solve $(D^4 - 81)y = 0$

Sol:- The Auxilary Equation is

$$f(m) = m^4 - 81 = 0.$$

$$(m^2)^2 - (9)^2 = 0.$$

$$\Rightarrow (m^2-9)(m^2+9) = 0$$

$$\Rightarrow m^2 - 9 = 0 \quad \& \quad m^2 + 9 = 0.$$

$$\Rightarrow m^2 = 9 \quad \& \quad m^2 = -9.$$

$$\Rightarrow m = 3, -3 \quad \& \quad m = \pm 3i. \Rightarrow m = 3i, -3i.$$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{3ix} + c_4 e^{-3ix} \cdot \text{ or}$$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-3x} + e^{0x}[c_1 \cos 3x + c_2 \sin 3x] \text{ //}$$

3) Solve $(D^4 + 8D^2 + 16)y = 0.$

Sol:- The A.E is

$$f(m) = m^4 + 8m^2 + 16 = 0$$

$$\Rightarrow (m^2+4)(m^2+4) = 0$$

$$\Rightarrow (m^2+4)^2 = 0$$

$$\Rightarrow m^2 + 4 = 0 \quad \& \quad m^2 + 4 = 0.$$

$$m^2 = -4 \quad \& \quad m^2 = -4$$

$$m = \pm 2i \quad \& \quad m = \pm 2i$$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cdot \cos 2x + c_4 \sin 2x.$$

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4) Find the particular integral of

$$(D^2-4D+4)y = e^{2x}.$$

Sol:- The Auxilary Equation is

$$f(m) = m^2-4m+4 = 0.$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2.$$

$\therefore$ The roots are real & Equal.

$$y_c = (c_1 + c_2 x)e^{2x}.$$

$$P.I = y_p = \frac{1}{D^2-4D+4} \cdot e^{2x} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2}{2!} e^{2x}.$$

$$\Rightarrow y_p = \frac{x^2}{2} \cdot e^{2x}.$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2 x)e^{2x} + \frac{x^2}{2} \cdot e^{2x}.$$

5) Find the solution of initial value problem $y'' + 4y' - 13 \cdot y = 0$, $y(0) = y'(0) = 1$.

Sol:- The given D.E is

$$(D^2+4D-13)y = 0.$$

The A.E is $m^2+4m-13 = 0$

$$m = -2 \pm \sqrt{17}.$$

6) Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Sol:- The operator form is

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

& The Auxilary Equation is

$$f(m) = m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m+1)(m^2 + 5m + 6) = 0.$$

$$\begin{array}{c|cccc} m=-1 & 1 & 6 & 11 & 6 \\ & 0 & -1 & -5 & -6 \\ \hline m=-2 & 1 & 5 & 6 & \boxed{0} \\ & 0 & -2 & -6 & \\ \hline m=-3 & 1 & 3 & \boxed{0} & \\ & 0 & -3 & & \\ \hline & 1 & 0. & & \end{array}$$

$$\Rightarrow (m+1)\left[m(m+2) + 3(m+2)\right] = 0$$

$$\Rightarrow (m+1)\left[(m+2)(m+3)\right] = 0$$

$$\Rightarrow (m+1)(m+2)(m+3) = 0$$

$$\Rightarrow m+1 = 0, \quad m+2 = 0, \quad m+3 = 0$$

$$\Rightarrow m = -1, \quad m = -2, \quad m = -3.$$

$\therefore$ The roots are real & Distinct.

$$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}.$$

7) Find complementary Function of

$(D^2-4D+3)y = \sin 3x \cdot \cos 2x$.

Sol: The Auxiliary Equation is

$f(m)=0$

$\Rightarrow (m^2-4m+3)=0.$

$\Rightarrow m^2-4m+3=0$

$\Rightarrow m^2-3m-m+3=0$

$\Rightarrow m(m-3)-1(m-3)=0$

$\Rightarrow (m-3)(m-1)=0$

$\Rightarrow m-3=0$ & $m-1=0$

$\Rightarrow m=3$ and $m=1$

$\Rightarrow m=1,3.$

$\therefore$ The roots are real & distinct.

$\therefore y_c = c_1e^x + c_2e^{3x}.$

8) find the P.I. of $(D^2+1)y = 8e^{-x}$. (4)

Sol:- The A.E is

$f(m) = 0 \Rightarrow m^2+1=0 \Rightarrow m^2 = -1.$

$\Rightarrow m = \pm i$

$\therefore$ The roots are complex congugate numbers.

$\therefore y_c = e^{0x}\left[c_1 \cos x + c_2 \sin x\right]$

$y_p = \frac{1}{D^2+1}(8e^{-x}) = 8 \cdot \frac{1}{D^2+1} e^{-x}.$

$\Rightarrow y_p = 8 \cdot \frac{1}{(-1)^2+1} e^{x} = 8 \cdot \frac{1}{1+1} e^{x}$ $\boxed{D=a=-1}$

$\Rightarrow y_p = 8 \cdot \frac{1}{2} \cdot e^{x}.$

$\Rightarrow y_p = 4e^{x}.$

$\therefore$ The particular integral is $4e^{x}$.

& The $y = y_c + y_p = c_1 \cos x + c_2 \sin x + 4e^{x}$

9) Find the particular Integral of
$(D^3-6D^2+11\cdot D-6)\cdot y = e^{-3x}$.

Sol:- The A.E is

$f(m) = 0$.

$\Rightarrow m^3-6m^2+11m-6=0$.

$\Rightarrow (m-1)(m^2-5m+6)=0$

$\Rightarrow (m-1)(m-2)(m-3)=0$.

$\Rightarrow m-1=0, m-2=0, m-3=0$

$\Rightarrow m=1, 2, 3$.

$\therefore$ The roots are real & distinct.

$\therefore y_c = c_1e^{x}+c_2e^{2x}+c_3e^{3x}$.

$y_p = P.I = \dfrac{e^{-3x}}{D^3-6D^2+11D-6} = \dfrac{e^{-3x}}{(-3)^3-6(-3)^2+11(-3)-6}$.

$= \dfrac{e^{-3x}}{-120}$

$\Rightarrow y_p = \dfrac{e^{-3x}}{-120}$

$\Rightarrow y = y_c+y_p = c_1e^{x}+c_2e^{2x}+c_3e^{3x}-\dfrac{e^{-3x}}{120}$

10) Write in brief about the method of Variation of Parameters.

Ans: These are the steps briefly about the method of Variation of Parameters.

1). Reduce the given equation to the standard form, if necessary.

2). Find the solution of

$$\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Q\cdot y = 0.$$

and let the solution be

$$C.F = c_1 \cdot u(x) + c_2 \cdot v(x).$$

3). Take $P.I = y_p = A\cdot u + B\cdot v$,

where A and B are functions of x.

4). Find $W(u,v) = u\cdot\frac{dv}{dx} - v\cdot\frac{du}{dx}$.

5). Find A and B, using

$$A = -\int \frac{v.R\,dx}{W(u,v)} = -\int \frac{v.R.dx}{u\frac{dv}{dx} - v.\frac{du}{dx}}$$

$$B = \int \frac{u.R\,dx}{W(u,v)} = \int \frac{u.R.dx}{u\frac{dv}{dx} - v.\frac{du}{dx}}.$$

6). Write the general solution of the given equation as $y = y_c + y_p$.

i.e., $y = c_1\cdot u(x) + c_2\cdot v(x) + A(x).u(x) + B(x).v(x)$

where c_1 and c_2 are constants.

11) Define the terms

i) Complementary function

ii) Particular integral

Sol: (i) If $y = y_c$ is the general solution of $f(D)y = 0$, then we know that y_c is the complementary function (C.F). of $f(D)y = Q(x)$.

(ii). If the equation is $f(D)y = Q(x)$.

Then the particular integral,

$$y_p = P.I = \frac{1}{f(D)} \cdot Q(x).$$

---

12) State Euler - Cauchy equation and brief method to solve it.

Sol:- The second order Euler - Cauchy equation is of the form

$$ax^2 \cdot y'' + bxy' + cy = 0 \quad \text{(or)}$$

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \cdot \frac{dy}{dx} + a_0 \cdot y = g(x)$$

when $g(x) = 0$, then the above equation is called the homogeneous Euler - Cauchy equation.

Eg: $x^2 y'' - 9xy' + 25 \cdot y = 0$.

Brief method :

1). Let us assume that $y = y(x) = x^{r}$ be the solution of a given differentiation equation. where r is a constant to be determined.

2). Fill the above formula for y in the differential equation and simplify.

3). Solve the obtained polynomial equation for r.

---

13) Solve the D.E. $D^2 y = \sin 2x$.

Sol:- The A.E is $m^2 = 0$

$$\Rightarrow m = 0, 0.$$

$$\therefore y_c = c_1 e^{0x} + c_2 e^{0x}$$

$$\Rightarrow y_c = c_1 + c_2.$$

$$P.I = y_p = \frac{1}{f(D)} \cdot \sin 2x$$

$$= \frac{1}{D^2} \cdot \sin 2x = \frac{1}{D} \cdot \frac{1}{D} (\sin 2x)$$

$$= \frac{1}{D} \int \sin 2x \, dx = \frac{1}{D} \left[- \frac{\cos 2x}{2} \right]$$

$$= \int \left(- \frac{\cos 2x}{2} \right) dx = -\frac{1}{2} \int \cos 2x \, dx$$

$$= -\frac{1}{2} \left[\frac{\sin 2x}{2} \right] + c = -\frac{1}{4} \cdot \sin 2x + c.$$

14) Find the Value of $\frac{1}{D+1}[x^2+1]$

Sol:- $\frac{1}{D+1}(x^2+1)$

$\Rightarrow \frac{1}{1+D}(x^2+1)$

$\Rightarrow (1+D)^{-1}[x^2+1]$

$\Rightarrow [1-D+D^2+\text{Higher order terms}][x^2+1]$

$\Rightarrow (1-D+D^2)(x^2+1)$

$= (1-D+D^2)(x^2) + (1-D+D^2)(1)$

$= x^2 - D(x^2) + D^2(x^2) + 1 - D(1) + D^2(1)$

$= x^2 - 2x + 2 + 1 + 0 + 0$

$= x^2 - 2x + 2.$

---

15) Find the particular Value of

$\frac{1}{(D-2)(D-3)} \cdot e^{2x}.$

Sol: $\frac{1}{(D-2)(D-3)} \cdot e^{2x} = \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right]$

Now, $\frac{1}{D-3}e^{2x} = e^{3x} \cdot \int e^{2x} \cdot e^{-3x}\,dx = e^{3x}\int e^{-x}\,dx$

$\frac{1}{D-3}e^{2x} = e^{3x} \cdot [-e^{-x}] = -e^{3x-x} = -e^{2x}.$

$\left[\because \frac{1}{(D-\beta)(D-\alpha)} \cdot Q = \frac{1}{(D-\beta)}\left[e^{\alpha x}\int Q e^{-\alpha x}dx\right]\right]$

$$\therefore \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right] = \frac{1}{D-2}\left[-e^{2x}\right]$$

$$= -e^{2x}\int e^{2x}\cdot e^{-2x}\,dx$$

$$= -e^{2x}\int e^{2x-2x}\,dx = -e^{2x}\int e^{0}\,dx$$

$$= -e^{2x}\cdot\int dx = -e^{2x}\cdot x = -x\cdot e^{2x}.$$

M - II - UNIT - 3 - LAQ's.

1) Find the general solution of.

$$y'' + 3y' + 2y = 2e^{x}.$$

Sol: The Auxilary Equation is.

$f(m) = 0 \Rightarrow m^2 + 3m + 2 = 0.$

$\Rightarrow m^2 + m + 2m + 2 = 0$

$\Rightarrow m(m+1) + 2(m+1) = 0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m+1 = 0$ & $m+2 = 0$

$\Rightarrow m = -1$ & $m = -2$

$\Rightarrow m = -1, -2.$

$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x}.$

$P.I = y_p = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{(D^2+3D+2)} \cdot 2e^{x}$

$= 2 \cdot \frac{1}{(D^2+3D+2)} e^{x}$ $\boxed{D = a = 1}$

$= 2 \cdot \frac{1}{(1)^2 + 3(1) + 2} e^{x} = 2 \cdot \frac{1}{1+3+2} e^{x}$

$= 2 \cdot \frac{1}{6} \cdot e^{x} = \frac{1}{3} e^{x}$

$\therefore y_p = \frac{1}{3} e^{x}.$

$G.S = y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{3} e^{x}.$

2) Find the general solution of

$$(D^2-4D+4)y = e^{2x}.$$

Sol: The Auxilary Equation is

$f(m)=0 \Rightarrow m^2-4m+4=0.$

$\Rightarrow (m-2)^2=0. \Rightarrow m=2,2.$

$\therefore$ The roots are real & equal.

$$y_c = (c_1+c_2 x)e^{2x}.$$

$$P.I = y_p = \frac{e^{2x}}{D^2-4D+4} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2}{2!}e^{2x}.$$

$$\therefore y_p = \frac{x^2}{2}.e^{2x}.$$

$$\therefore y = y_c + y_p = (c_1+c_2x)e^{2x} + \frac{x^2}{2}.e^{2x}.$$

---

3) Solve $(D^2+9)y = \sin 3x.$

Sol: The given equation is

$(D^2+9)y = \sin 3x.$

Let $f(D) = D^2+9.$

A.E is $f(m)=0 \Rightarrow m^2+9=0$

$\Rightarrow m^2=-9 \Rightarrow m=\pm 3i.$

$\Rightarrow m = 3i, -3i.$

$\therefore$ The roots are complex conjugate numbers.

$$y_c = e^{0x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

$$P.I = y_p = \frac{1}{D^2+9} \cdot \sin 3x.$$

$$\boxed{D^2 = -a^2 = -3^2 = -9}$$

$$= \frac{1}{-9+9} \cdot \sin 3x = \frac{1}{0} \cdot \sin 3x.$$

$$= \frac{-x}{2\times 3} \cos 3x = -\frac{x}{6} \cos 3x.$$

$$\left[\because \frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}\right]$$

$$y_p = -\frac{x}{6} \cos 3x.$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{6} \cdot \cos 3x.$$

4) Find the general solution of

$$(D^2-4)y = \cos^2 x.$$

Sol:- The given equation is

$$(D^2-4)y = \cos^2 x.$$

Let $f(D) = D^2-4$

The A.E. is $m^2-4=0$.

$$\Rightarrow m^2 = 4 \Rightarrow m = 2, -2$$

$\therefore$ The roots are real and distinct.

$$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}.$$

$$P.I = y_p = \frac{1}{D^2-4}\cdot(\cos^2 x)$$

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$= \frac{1}{D^2-4}\left[\frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} + \frac{1}{D^2-4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{e^{0x}}{D^2-4} + \frac{\cos 2x}{D^2-4}\right]$$

$$\boxed{\left[\because D = a = 0 \ \& \ D^2 = -a^2 = -2^2\right]}$$

$$= \frac{1}{2}\left[\frac{e^{0x}}{(0)^2-4} + \frac{\cos 2x}{-2^2-4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} + \frac{\cos 2x}{-4-4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} + \frac{\cos 2x}{-8}\right]$$

$$y_p = -\frac{1}{8} - \frac{\cos 2x}{16}.$$

$$\therefore\ y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{\cos 2x}{16}.$$

5) Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

Sol:- The A.E is

$$(m+2)(m-1)^2 = 0.$$

$$\Rightarrow m+2=0 \ \& \ (m-1)^2=0$$

$$\Rightarrow m=-2 \ \& \ m=1,1.$$

$$\Rightarrow y_c = c_1 e^{-2x} + (c_2 + c_3 x)e^x.$$

$$P.I = y_p = \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + 2\sinh x\right]$$

$$= \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + 2\cdot\frac{e^x - e^{-x}}{2}\right]$$

$$\boxed{\because \sinh x = \frac{e^x - e^{-x}}{2}}$$

$$= \frac{1}{(D+2)(D-1)^2}\left[e^{-2x} + e^x - e^{-x}\right]$$

$$P.I_1 = \frac{1}{(D+2)(D-1)^2}(e^{-2x}) = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right]$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-3)^2}e^{-2x}\right]$$

$$\boxed{\because D=a=-2}$$

$$= \frac{1}{D+2}\left[\frac{1}{9}\cdot e^{-2x}\right] = \frac{1}{9}\left[\frac{1}{D+2}e^{-2x}\right] = \frac{1}{9}\left[\frac{1}{-2+2}e^{-2x}\right]$$

$$= \frac{1}{9}\left[\frac{1}{0}\cdot e^{-2x}\right] \text{ (case failure)}$$

$$\boxed{\because D=a=-2}$$

$$\Rightarrow \frac{1}{9}\left[x\cdot e^{-2x}\right] = P.I_1$$

$$PI_2 = \frac{1}{(D+2)(D-1)^2} \cdot e^x$$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{D+2}(e^x)\right]$$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^x\right] = \frac{1}{(D-1)^2}\left[\frac{1}{3}e^x\right] \qquad \boxed{\because D=a=1}$$

$$= \frac{1}{3} \cdot \frac{1}{(D-1)^2} \cdot e^x = \frac{1}{3} \cdot \frac{1}{(1-1)^2} e^x = \frac{1}{3} \cdot \frac{1}{0} \cdot e^x$$

(case-failure)

$$\Rightarrow \frac{1}{3} \cdot \frac{x}{2(D-1)} \cdot e^x = \frac{1}{3} \cdot \frac{x}{2(1-1)} e^x = \frac{1}{3} \cdot \frac{x}{0} \cdot e^x$$

(case-failure)

$$\Rightarrow \frac{1}{3} \cdot \frac{x^2}{2} \cdot e^x = \frac{1}{6} x^2 \cdot e^x = \underline{PI_2}.$$

$$PI_3 = \frac{1}{(D+2)(D-1)^2} \cdot e^{-x}.$$

$$= \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{(1)(-2)^2} e^{-x} = \frac{1}{+4} e^{-x}$$

$$[\because D=a=-1]$$

$$PI_3 = -\frac{e^x}{2}.$$

$$\therefore y_p = PI_1 + PI_2 + PI_3 = \frac{1}{9}\left[x e^{-2x}\right] + \frac{1}{6} x^2 e^x - \frac{e^x}{4}.$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{1}{9} \cdot x e^{-2x} + \frac{1}{6} x^2 e^x - \frac{e^x}{4}$$

6) Find the general solution of (4)

$$(D^2+2D+1)\cdot y = x\cdot\cos x.$$

Sol:- The given equation is

$$(D^2+2D+1)\,y = x\cdot\cos x.$$

$\Rightarrow$ Let $f(D) = D^2+2D+1$

The A.E is $f(m)=0.$

$\Rightarrow m^2+2m+1=0. \Rightarrow (m+1)^2=0.$

The roots are real & equal $\Rightarrow m=-1,-1.$

$$\Rightarrow y_c = (c_1+c_2x)\,e^{-x}.$$

$$y_p = \frac{x\cdot\cos x}{D^2+2D+1} = \left[x - \frac{1}{f(D)}\cdot f'(D)\right]\cdot\frac{1}{f(D)}\cos x.$$

$$= \left[x - \frac{1}{D^2+2D+1}\cdot 2(D+1)\right]\cdot\frac{1}{D^2+2D+1}\cdot\cos x$$

$$= \left[x - \frac{1}{(D+1)^2}\cdot 2(D+1)\right]\cdot\frac{1}{D^2+2D+1}\cdot\cos x$$

$$= \left[x - \frac{1}{(D+1)^2}\cdot 2(D+1)\right]\frac{1}{-1+2D+1}\cos x \qquad \left[\because D^2=-a^2 = -1^2=-1\right]$$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{2D}\cos x = \left[x-\frac{2}{D+1}\right]\cdot\frac{\sin x}{2} \qquad \left[\because \frac{1}{D}(\cos x) = \int\cos x = \sin x\right]$$

$$= \frac{x}{2}\sin x - \frac{\sin x}{D+1}$$

$$= \frac{x}{2}.\sin x - \frac{D-1}{D^2-1}.\sin x.$$

$$= \frac{x\sin x}{2} - \frac{D-1}{-1-1}.\sin x.$$

$$= \frac{x.\sin x}{2} - \frac{(D-1).\sin x}{-2}$$

$$= \frac{x\sin x}{2} - \frac{D(\sin x) - \sin x}{-2}$$

$$= \frac{x\sin x}{2} - \frac{(\cos x - \sin x)}{-2}$$

$$= \frac{x.\sin x}{2} + \frac{\cos x + \sin x}{2}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2 x)e^{-x} + \frac{x}{2}.\sin x + \frac{1}{2}[\cos x + \sin x]$$

7) Solve $(D^2+4)y = e^x + \sin 2x + \cos 2x$.

Sol: The given D.E is

$$(D^2+4)y = e^x + \sin 2x + \cos 2x.$$

Let $f(D) = D^2+4$

The A.E is $m^2+4=0$

$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$.

$\therefore$ The roots are complex conjugate numbers.

$$y_c = c_1 \cdot \cos 2x + c_2 \sin 2x.$$

$$P.I = y_p = \frac{1}{D^2+4}\left[e^x + \sin 2x + \cos 2x\right]$$

$$\Rightarrow y_p = \frac{1}{D^2+4} \cdot e^x + \frac{1}{D^2+4} \sin 2x + \frac{1}{D^2+4} \cos 2x.$$

$$\Rightarrow y_p = PI_1 + PI_2 + PI_3.$$

$$\Rightarrow PI_1 = \frac{1}{D^2+4} \cdot e^x = \frac{1}{1^2+4} e^x = \frac{1}{1+4} e^x = \frac{1}{5} e^x \quad \left[\because D = a = 1\right]$$

$$PI_2 = \frac{1}{D^2+4} \cdot \sin 2x = \frac{1}{D^2+2^2} \sin 2x = \frac{1}{-2^2+2^2} \sin 2x$$

(case failure)

$$PI_2 = -\frac{x}{2 \cdot 2} \cos 2x \qquad \left[\because \frac{\sin ax}{D^2+a^2} = -\frac{x}{2a} \cos ax\right]$$

$$PI_2 = -\frac{x}{4} \cdot \cos 2x.$$

$$PI_3 = \frac{\cos 2x}{D^2+4} = \frac{\cos 2x}{D^2+2^2} = \frac{\cos 2x}{-2^2+2^2}$$

(case failure).

$$PI_3 = \frac{x}{2\cdot 2} \cdot \sin 2x \qquad \left[\because \frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax\right]$$

$$PI_3 = \frac{x}{4} \cdot \sin 2x.$$

$\therefore\ y = y_c + y_p$

$\Rightarrow y = y_c + PI_1 + PI_2 + PI_3.$

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x + \frac{x}{4} \sin 2x.$

8) Solve $(D^3-1)y = e^x + \sin 3x + 2.$

Sol:- The given D.E is

$(D^3-1)y = e^x + \sin 3x + 2$

The A.E is $m^3 - 1 = 0$

$\Rightarrow (m-1)(m^2+m+1) = 0$

$\Rightarrow m-1=0$ & $m^2+m+1=0$

$\Rightarrow m=1$ &

$m^2+m+1=0$

$\Rightarrow a=1,\ b=1,\ c=1$

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$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4\times1\times1}}{2\times1} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

$$\Rightarrow y_c = c_1 e^x + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I = y_p = \frac{1}{D^3-1}\left[e^x + \sin 3x + 2\right]$$

$$= \frac{1}{D^3-1} \cdot e^x + \frac{1}{D^3-1}\sin 3x + \frac{1}{D^3-1}(2) \cdot e^{0x}.$$

$$PI = PI_1 + PI_2 + PI_3.$$

Now, $PI_1 = \frac{1}{D^3-1}e^x = \frac{1}{1^3-1} = \frac{1}{1-1} = \frac{1}{0}.$

case failure. $\boxed{D = a = 1}$

$$\therefore \; PI_1 = \frac{x}{1!}e^x = \underline{\underline{xe^x}}$$

$$PI_2 = \frac{1}{D^3-1}\sin 3x.$$

$$= \frac{1}{D^2 \cdot D-1}\sin 3x.$$

$$\boxed{\begin{aligned} D^2 &= -a^2 = -3^2 \\ &= -9 \end{aligned}}$$

$$= \frac{1}{-9D-1}\sin 3x$$

$$= \frac{1}{-(1+9D)}\sin 3x$$

$$= -\frac{(1-9D)}{(1-9D)(1+9D)}\sin 3x$$

$$= -\frac{(1-9D)\sin 3x}{1-81D^2} = -\frac{(1-9D)\sin 3x}{1-81(-3^2)}$$

$$= -\frac{[\sin 3x - 9 \cdot D(\sin 3x)]}{1-81(-9)}$$

$$= \frac{-(\sin 3x - 9 \cdot D(\sin 3x))}{1+729}$$

$$= \frac{-[\sin 3x - 9 \cdot [3\cos 3x]]}{730}.$$

$$= \frac{-(\sin 3x - 27 \cos 3x)}{730}.$$

$$= -\frac{1}{730}\left[\sin 3x - 27 \cos 3x\right]$$

$$PI_3 = \frac{1}{D^3-1} \cdot 2 \cdot e^{0x} = 2 \cdot \frac{1}{D^3-1} e^{0x}.$$

$$= 2 \cdot \frac{1}{0-1} = \frac{2}{-1} = -2.$$

$$y_p = PI_1 + PI_2 + PI_3$$

$$\Rightarrow y_p = xe^x - \frac{1}{730}\left[\sin 3x - 27 \cos 3x\right] - 2.$$

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^x + e^{-x/2}\left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right]$$

$$+ xe^x - \frac{1}{730}\left[\sin 3x - 27 \cos 3x\right] - 2.$$

9) Solve $y'' - 2y' + y = xe^x \sin x$.

Sol:- The A.E is $f(m) = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\Rightarrow (m-1) = 0$ & $(m-1) = 0$

$\Rightarrow m = 1, 1$

$\therefore$ The roots are real & Equal.

$y_c = [c_1 + c_2 x] e^x$.

$y_p = P.I = \frac{1}{D^2 - 2D + 1} [x e^x . \sin x]$

$= e^x . \frac{x \sin x}{[(D+1)-1]^2} = e^x . \frac{x \sin x}{[(D+1)-1]^2}$

$= e^x . \frac{x \sin x}{D^2} = e^x . \frac{1}{D^2} [x \sin x]$

$= e^x . \frac{1}{D} \left[\int x \sin x \right] = e^x . \frac{1}{D} \left[x \int \sin x \, dx - \int \left[\frac{d}{dx}(x) . \int \sin x \, dx \right] dx \right.$

$= e^x . \frac{1}{D} \left[x . (-\cos x) - \int 1 . (-\cos x) . 1 . dx \right]$

$= e^x . \frac{1}{D} [-x . \cos x + \sin x]$

$$= e^x \left[- \int x \cos x \, dx + \int \sin x \, dx \right]$$

$$= e^x \left[- \left\{ x \sin x - \int 1 . \sin x \right\} - \cos x \right]$$

$$= e^x . \left[- \left\{ x . \sin x + \cos x \right\} - \cos x \right]$$

$$= e^x . \left[-x \sin x - \cos x - \cos x \right]$$

$$= e^x \left[-x \sin x - 2 \cos x \right]$$

$$= -x e^x . \sin x - 2 . e^x . \cos x$$

$$y_p = -x e^x \sin x - 2 e^x \cos x .$$

$$\therefore \; y = y_c + y_p$$

$$\Rightarrow y = \left[c_1 + c_2 x \right] e^x - \left[x e^x \sin x + 2 e^x \cos x \right]$$

10) Solve $(D^2+4)y = x^2+1+\cos 2x$.

Sol:- The given D.E is

$$(D^2+4)y = x^2+1+\cos 2x$$

The A.E is $f(m)=0$

$$\Rightarrow m^2+4=0$$

$$\Rightarrow m^2=-4 \Rightarrow m=\pm 2i.$$

$$y_c = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$y_p = \frac{1}{D^2+4}\left[x^2+1+\cos 2x\right]$$

$$= \frac{1}{D^2+4}\cdot x^2 + \frac{1}{D^2+4}\cdot 1\cdot e^{0x} + \frac{1}{D^2+4}\cos 2x.$$

$$y_p = PI_1 + PI_2 + PI_3$$

$$\Rightarrow PI_1 = \frac{1}{D^2+4}(x^2) = \frac{1}{4\left[1+\frac{D^2}{4}\right]}(x^2)$$

$$= \frac{1}{4}\cdot\frac{1}{\left(1+\frac{D^2}{4}\right)}(x^2) = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}\cdot(x^2)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}\right](x^2) \qquad \left[\because (1+D)^{-1} = 1-D+D^2+\cdots\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{D^2}{4}(x^2)\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{2}{4}\right] = \frac{1}{4}\left[x^2 - \frac{1}{2}\right]$$

$$PI_2 = \frac{1}{D^2+4} e^{0x} = \frac{1}{0^2+4} = \frac{1}{4}.$$

$$PI_3 = \frac{1}{D^2+4}(\cos 2x)$$

$$\boxed{D^2 = -2^2 = -4}$$

$$= \frac{1}{-2^2+2^2}[\cos 2x] = \frac{1}{0}\cos 2x$$

(Case Failure).

$$= \frac{x}{2D}[\cos 2x]$$

$$= \frac{x}{2D}(\cos 2x)$$

$$= \frac{x}{2} \cdot \frac{1}{D}(\cos 2x)$$

$$= \frac{x}{2}\int \cos 2x.$$

$$= \frac{x}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{x}{4}\sin 2x$$

$$PI_3 = \frac{x}{4}(\sin 2x)$$

$$y_p = P.I = PI_1 + PI_2 + PI_3.$$

$$\Rightarrow y_p = \frac{1}{4}\left[x^2 - \frac{1}{2}\right] + \frac{1}{4} + \frac{x}{4}(\sin 2x).$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1\cos 2x + c_2 \sin 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{1}{4} + \frac{x}{4}(\sin 2x)$$

11) Find the general solution of

$$y'' + 4y' + 4y = 6 \cdot e^{-2x} \cdot \cos^2 x.$$

Sol: The given D.E is

$$y'' + 4y' + 4y = 6 \cdot e^{-2x} \cdot \cos^2 x.$$

$f(m) = 0$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow m = -2, -2$ .. The roots are real & Equal.

$\cos 2x = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x = \cos 2x + 1$

$\Rightarrow \cos^2 x = \frac{1}{2}[1+\cos 2x]$

$\therefore y_c = [c_1 + c_2 x] e^{-2x}.$

$$\therefore y_p = \frac{1}{D^2+4D+4}\left[6 e^{-2x} \cdot \cos^2 x\right]$$

$$= \frac{1}{D^2+4D+4}\left[6 \cdot e^{-2x} \cdot \frac{1}{2}(1+\cos 2x)\right]$$

$$= \frac{1}{D^2+4D+4}\left[3 \cdot e^{-2x} \cdot (1+\cos 2x)\right]$$

$\boxed{\text{Put } D = D-2}$

$$= 3 \cdot e^{-2x} \frac{1}{(D-2)^2+4(D-2)+4} \cdot [1+\cos 2x]$$

$$= 3e^{-2x} \cdot \frac{1}{D^2}[1+\cos 2x]$$

$$\Rightarrow y_p = 3e^{-2x} \cdot \frac{1}{D} \cdot \frac{1}{D}(1+\cos 2x)$$

$$= 3e^{-2x} \cdot \frac{1}{D} \int [1+\cos 2x]\, dx$$

$$= 3e^{-2x} \cdot \frac{1}{D}\left[x + \frac{\cos 2x}{2}\right]$$

$$= 3e^{-2x} \cdot \int \left[x + \frac{\cos 2x}{2}\right] dx$$

$$= 3e^{-2x} \cdot \left[\frac{x^2}{2} + \frac{\sin 2x}{2 \cdot 2}\right]$$

$$= 3e^{-2x}\left[\frac{x^2}{2} + \frac{\sin 2x}{4}\right]$$

$$\Rightarrow y_p = \frac{3}{2} e^{-2x}\left[x^2 + \frac{\sin 2x}{2}\right]$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2 x)e^{-2x} + \frac{3}{2} e^{-2x}\left[x^2 + \frac{\sin 2x}{2}\right]$$

12) Apply method of Variation of Parameters to solve $\frac{d^2y}{dx^2}+y=\sec x$.

Sol: Given that the D.E is

$$(D^2+1)y = \sec x \quad , \quad \& \; R=\sec x.$$

The A.E is $m^2+1=0$.

$$\Rightarrow m^2=-1 \Rightarrow m=\pm i.$$

$$\Rightarrow y_c = e^{0x}\left[c_1 \cos x + c_2 \sin x\right]$$

$$\Rightarrow y_c = c_1 \cdot \cos x + c_2 \cdot \sin x.$$

$$\Rightarrow y_c = c_1 \cdot u + c_2 \cdot v.$$

where $u=\cos x$ & $v=\sin x$.

$$u' = \frac{du}{dx} = -\sin x \quad \& \quad \frac{dv}{dx} = \cos x = v'$$

$$\therefore A = -\int \frac{v \cdot R}{uv'-u'v}\,dx = -\int \frac{\sin x \cdot \sec x}{\cos x \cdot \cos x - (-\sin x)(\sin x)}\,dx$$

$$A = -\int \frac{\sin x \cdot \frac{1}{\cos x}}{\cos^2 x + \sin^2 x}\,dx = -\int \frac{\frac{\sin x}{\cos x}}{1}\,dx$$

$$\Rightarrow A = -\int \frac{\sin x}{\cos x}\,dx = \underline{\log|\cos x|}.$$

$$B = \int \frac{u \cdot R}{uv' - u'v} \, dx.$$

$$\Rightarrow B = \int \frac{\cos x \cdot \sec x}{1} \, dx = \int \cos x \cdot \sec x \, dx.$$

$$B = \int \cos x \cdot \frac{1}{\cos x} \, dx = \int 1 \, dx = \underline{\underline{x + c}}$$

$$P.I = y_p = A \cdot u + B \cdot V$$

$$\Rightarrow y_p = \log|\cos x| \cos x + x \cdot \sin x.$$

$$\Rightarrow y_p = (\cos x)(\log|\cos x|) + x \cdot \sin x.$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x + (\cos x)(\log|\cos x|) + x \cdot \sin x.$$

13) Find the particular integral of

$$(D^2-6D+9)y = x^2+2x.$$

Sol: The given D.E is

$$(D^2-6D+9)y = x^2+2x.$$

& The A.E is

$$m^2-6m+9=0$$

$$\Rightarrow (m-3)^2=0 \Rightarrow m=3,3.$$

$$\therefore y_c = (c_1+c_2x)e^{3x}.$$

$$y_p = P.I = \frac{1}{[D^2-6D+9]}[x^2+2x]$$

$$= \frac{1}{9\left[1+\frac{D^2-6D}{9}\right]}(x^2+2x)$$

$$= \frac{1}{9}\left[1+\frac{D^2-6D}{9}\right]^{-1}[x^2+2x]$$

$$= \frac{1}{9}\left[1-\left(\frac{6D-D^2}{9}\right)\right]^{-1}(x^2+2x)$$

$$= \frac{1}{9}\left[1+\left[\frac{6D-D^2}{9}\right]+\left[\frac{6D-D^2}{9}\right]^2\right](x^2+2x)$$

$$= \frac{1}{9}\left[x^2+2x+\frac{6}{9}D(x^2+2x)-\frac{1}{9}D^2(x^2+2x)\right.$$
$$\left.+\frac{36}{81}D^2(x^2+2x)\right]$$

$$= \frac{1}{9}\left[x^2+2x+\frac{6}{9}(2x+2)-\frac{1}{9}(2)+\frac{36}{81}(2)\right]$$

$$= \frac{1}{9}\left[x^2+2x+\frac{4x}{3}+\frac{4}{3}-\frac{2}{9}+\frac{8}{9}\right]$$

$$= \frac{1}{9}\left[x^2+\frac{10}{3}x+2\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[x^2+\frac{10}{3}x+2\right]$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = [c_1+c_2x]e^{3x}+\frac{1}{9}\left[x^2+\frac{10}{3}x+2\right].$$

14) Solve $x^2 \cdot \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx} - 3y = x^2 \cdot \log x$.

Sol: The given D.E is

$$\left[x^2 D^2 - x D - 3\right] y = x^2 \cdot \log x.$$

Now, it is in the form of Cauchy's homogeneous Linear D.E.

Put $x = e^z \Rightarrow \log x = z$

$$x D = D, \quad x^2 D^2 = D(D-1)$$

$\Rightarrow$ [illegible]

$$\Rightarrow \left[D(D-1) - D - 3\right] y = e^{2z} \cdot z$$

$$\Rightarrow \left[D^2 - D - D - 3\right] y = e^{2z} \cdot z$$

$$\Rightarrow \left[D^2 - 2D - 3\right] y = e^{2z} \cdot z.$$

The A.E is $m^2 - 2m - 3 = 0$

$$\Rightarrow (m+1)(m-3) = 0$$

$$\Rightarrow m = -1, 3.$$

$$y_c = C_1 e^{-x} + C_2 e^{3x}.$$

$$P.I = y_p = \frac{1}{D^2 - 2D - 3}\left(z \cdot e^{2z}\right)$$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} \cdot z$$

Put $D = D+2$

$$= e^{2z} \frac{1}{D^2 + 2D - 3} \cdot z$$

$$= e^{2z} \cdot \frac{1}{-3\left[1 - \frac{D^2+2D}{3}\right]} \cdot z.$$

$$= -\frac{e^{2z}}{3} \cdot \frac{1}{\left[1 - \frac{2D+D^2}{3}\right]} \cdot z.$$

$$= -\frac{e^{2z}}{3} \left[1 - \frac{2D+D^2}{3}\right]^{-1} \cdot (z)$$

$$= -\frac{e^{2z}}{3}\left[z + \frac{2}{3}\right]$$

$$y_p = -\frac{x^2}{3} \cdot \left[\log x + \frac{2}{3}\right]$$

$\boxed{z = \log x}$

$\boxed{x = e^z}$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{3}\left[\log x + \frac{2}{3}\right]$$

---

15) Apply method of Variation of Parameters to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

Sol: Given D.E is $(D^2+1)y = \operatorname{cosec} x$.

& $R = \operatorname{cosec} x$

The A.E is $m^2+1 = 0$.

$$\Rightarrow m^2 = -1 \Rightarrow m = \pm i.$$

$$y_c = c_1 \cdot \cos x + c_2 \sin x.$$

By the method of Variation of parameters

$$y_c = c_1 \cdot u + c_2 \cdot v$$

where $u = \cos x$ & $v = \sin x$.

$u' = -\sin x$ & $v' = \cos x$.

$$A = -\int \frac{v \cdot R}{uv' - v \cdot u'} \cdot dx.$$

$$= -\int \frac{\sin x \cdot \csc x}{\cos^2 x + \sin^2 x}\, dx = -\int 1\, dx = -x$$

$$A = -x.$$

& $$B = \int \frac{u \cdot R}{uv' - vu'}\, dx$$

$$= \int \frac{\cos x \cdot \csc x}{\cos^2 x + \sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin x}\, dx = \log|\sin x|$$

$$\therefore \quad B = \log|\sin x|.$$

$$y_p = P.I = A \cdot u + B \cdot V$$

$$= (-x) \cdot \cos x + \log|\sin x| \cdot (\sin x)$$

$$\therefore \ y = y_c + y_p$$

$$\Rightarrow \ y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log|\sin x|$$

UNIT-IV SAQ Question Bank

Q1 Evaluate $\gamma\left(-\frac{5}{2}\right)$

CO	BTL
CO1	3

Sol: We know that $\gamma(n) = \frac{\gamma(n+1)}{n}$

$$\therefore \gamma\left(-\frac{5}{2}\right) = \frac{\gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\gamma\left(\frac{3}{2}\right)$$

$$= -\frac{2}{5}\frac{\gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \left(-\frac{2}{5}\right)\left(-\frac{2}{3}\right)\gamma\left(-\frac{1}{2}\right)$$

$$= \frac{4}{15}(-2)\frac{\gamma\left(-\frac{1}{2}+1\right)}{\cancel{-\frac{1}{2}}} = -\frac{8}{15}\gamma\left(\frac{1}{2}\right)$$

$$= -\frac{8}{15}\sqrt{\pi} \qquad \left\{\because \gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right\}$$

Q.2 State Beta function

CO1	BTL5

Sol. The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the "beta function" & is denoted by $B(m,n)$. $m>0, n>0$

i.e $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

Q.3 State Gamma function

Sol. The definite integral $\int_0^\infty e^{-x}x^{n-1}dx$ is called Gamma function & is denoted as $\gamma(n)$

$$\gamma n = \int_0^\infty e^{-x}x^{n-1}dx.$$

Q.4: State Rodrigues formula. $\boxed{CO5 | BTL1}$

Sol. The relation $P_n(x) = \frac{1}{2^n \, n!} \frac{d^n}{dx^n} (x^2-1)^n$

is known as "Rodrigues Formula".

Q.5. State the relation between Beta & Gamma function. $\boxed{CO5 | BTL1}$

Sol:- $B(m,n) = \frac{\gamma_m \gamma_n}{\gamma_{m+n}}$ or $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0, n>0$.

Q.6. Prove that $P_n(1) = 1$ $\boxed{CO4 | BTL5}$

Sol. We know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

If $x=1$

$$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$((1-t)^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$(1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\Rightarrow 1+t+t^2+ \cdots + t^n = \sum_{n=0}^{\infty} t^n P_n(1)$$

Comparing coefficients of t^n on b/s,

$$\Rightarrow P_n(1) = 1$$

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7Q. prove that $erf(x) + erf_c(x) = 1$.

Soln By the defn we have.

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du + \frac{2}{\pi} \int_x^{\infty} e^{-u^2} du.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du.$$

$= erf(\infty) = 1.$

$\therefore erf(x) + erf_c(x) = 1.$

Q.8 Find the value of $B(9/2, 7/2)$

Sol→ we know $B(m,n) = \frac{\gamma_m \gamma_n}{\gamma_{m+n}}$

$$B(9/2, 7/2) = \frac{\gamma(9/2)\gamma(7/2)}{\gamma(9/2 + 7/2)} = \frac{\frac{7}{2}\gamma(7/2)\ \frac{5}{2}\gamma(5/2)}{\gamma(16/2)}$$

$$= \frac{\frac{5}{2}\ 7/2 \left[\frac{5}{2}\gamma(5/2)\ \frac{3}{2}\gamma(3/2)\right]}{7!}$$

$$= \frac{3/2\,(5/2)^2\,(7/2)\left[3/2\ \gamma(3/2) \cdot \frac{1}{2}\gamma(1/2)\right]}{7!}$$

$$= \frac{\left(\frac{1}{2}\right)(3/2)^2(5/2)^2(7/2)\left[\frac{1}{2}\gamma(1/2) \cdot \sqrt{\pi}\right]}{7!}$$

$$= \frac{\left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)^2\left(\frac{5}{2}\right)^2\left(\frac{7}{2}\right)\pi}{7!}$$

Q.9. Prove That $B'(m,n) = B(n,m)$ $\boxed{CO4 \mid BTL4}$

Sol: we know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

let $(1-x) = y$ $\mid x \to 0, y \to 1$

$\Rightarrow dx = -dy$ $\quad x \to 1, y \to 0$

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$\boxed{B(m,n) = B(n,m)}$$

Q.10 Express $f(x) = 2x^3 - 6x^2 + 5x - 3$ in terms of Legendre Polynomial $P_n(x)$. CO4 | BTL

Sol. By Rodrigues Formula $P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}}\cancel{2}x$$

$$P_1(x) = x \quad \text{or} \quad \boxed{x = P_1(x)}$$

$$P_2(x) = \frac{1}{2!2^2}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8_2}\frac{d}{dx}\cancel{2}(x^2-1)\cdot\cancel{2}x$$

$$P_2(x) = \frac{1}{2}\frac{d}{dx}(x^3 - x) = \frac{1}{2}(3x^2-1)$$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\Rightarrow 3x^2-1 = 2P_2(x)$

$3x^2 = 2P_2(x)+1$

$$\boxed{x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)}$$

$P_3(x) = \frac{1}{3!\,2^3}\frac{d^3}{dx^3}(x^2-1)^3$

$= \frac{1}{6\times 8}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$

$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5-12x^3+6x)$

$= \frac{1}{48}\frac{d}{dx}(30x^4-36x^2+6)$

$= \frac{1}{48}(120x^3-72x)$

$= \frac{\cancel{24}\,1}{\cancel{48}\,2}(5x^3-3x)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$\Rightarrow 5x^3-3x = 2P_3(x)$

$5x^3 = 2P_3(x)+3x$

$$\boxed{x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)}$$

Substituting x, x^2, x^3 in the given Polynomial, we get

$f(x) = 2\left[\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\right] - 6\left[\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right] + 5P_1(x) - 3P_0(x)$

$\Rightarrow f(x) = \frac{4}{5}P_3(x) + \frac{6}{5}P_1(x) - 4P_2(x) - 2P_0(x) + 5P_1(x) - 3P_0(x)$

$\therefore f(x) = \frac{4}{5}P_3(x) + \frac{31}{5}P_1(x) - 4P_2(x) - 5P_0(x)$ //

Q.11 Prove that $\Gamma(n+1) = n!$ $\boxed{CO4 \mid BTL2}$

Sol: we know $\Gamma(n+1) = n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$= n(n-1)(n-2)(n-3)\Gamma(n-3)$

$\vdots$

$\Gamma(n+1) = n(n-1)(n-2)(n-3) \cdots 1$

$\boxed{\Gamma(n+1) = n!}$, $(n = 0, 1, 2, \cdots)$

Q.12 Evaluate $\int_{-1}^{0} (1-x^2)^n \, dx$.

Sol: Let $x^2 = t$ | $x \to -1, t \to 1$

$2x\,dx = dt$ | $x \to 0, t \to 0$

$\int_1^0 (1-t)^n \frac{dt}{2x} = \int_1^0 (1-t)^n \frac{dt}{2\sqrt{t}}$

$= \frac{1}{2} \int_1^0 (1-t)^n t^{-1/2} \, dt$

$= \frac{-1}{2} \int_0^1 t^{1/2-1} (1-t)^{n+1-1} \, dt$

$= -\frac{1}{2} B(\frac{1}{2}, n) = -\frac{1}{2} \frac{\Gamma(1/2)\Gamma(n)}{\Gamma(n+1/2)}$ //

Q.13 Express as Polynomial in x,
$3P_3(x)+2P_2(x)+4P_1(x)+5P_0(x)$ $\boxed{CO4|BTL3}$

Sol. We know $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

$P_3(x)=\frac{1}{2}(5x^3-3x)$

substituting in given eqn.

$\frac{3}{2}(5x^3-3x)+(3x^2-1)+4x+5$

$\Rightarrow \frac{15}{2}x^3-\frac{9}{2}x+3x^2-1+4x+5$

$\Rightarrow \frac{15}{2}x^3+3x^2+\frac{1}{2}x+4$

Q.14. Determine the nature of point $x=0$
for eqn. $xy''+y\sin x=0$ $\boxed{CO4|BTL2}$

Sol: Given $xy''+y\sin x=0$

$\Rightarrow y''+\frac{\sin x}{x}y=0$

$y''+P(x)y'+Q(x)y=0$

$P(x)=0$, $Q(x)=\frac{\sin x}{x}$

for $x=0$, $Q(x)=1$

$\therefore x=0$ is a regular singular point.

Q.15. Evaluate $\Gamma(9/2)$

Sol $\Gamma(9/2) = \frac{7}{2}\Gamma(7/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma(5/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma(3/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

L.A.Q UNIT – IV

Q1 Evaluate $\frac{d}{dx}\left(erf(ax)\right)$ $\boxed{CO4 | BTL4}$

Sol. By definition, $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$= \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$

We know $e^{-u^2} = 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots$

$\left\{ \because e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right\}$.

$$\therefore erf(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right) du$$

$$\Rightarrow erf(ax) = \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{10} - \frac{u^7}{42} + \cdots\right]_0^{ax}$$

$$\Rightarrow erf(ax) = \frac{2}{\sqrt{\pi}} \left[ax - \frac{a^3x^3}{3} + \frac{a^5x^5}{10} - \frac{a^7x^7}{42} + \cdots\right]$$

Now $$\frac{d}{dx}\left[erf(ax)\right] = \frac{2}{\sqrt{\pi}} \cdot \frac{d}{dx}\left[ax - \frac{a^3x^3}{3} + \frac{a^5x^5}{10} - \frac{a^7x^7}{42} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[a - \frac{\cancel{3}a^3x^2}{\cancel{3}_1} + \frac{\cancel{5}a^5x^4}{\cancel{10}_2} - \frac{\cancel{7}a^7x^6}{\cancel{42}_6} + \cdots\right]$$

$$= \frac{2a}{\sqrt{\pi}} \left[1 - a^2x^2 + \frac{a^4x^4}{2} - \frac{a^6x^6}{3!} + \cdots\right]$$

$$= \frac{2a}{\sqrt{\pi}} \left[1 - \frac{(a^2x^2)}{1!} + \frac{(a^2x^2)^2}{2!} - \frac{(a^2x^2)^3}{3!} + \cdots\right]$$

$$\therefore \frac{d}{dx}\left[erf(ax)\right] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

Q.2 Show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$. CO4 | BTLS

Sol: By def^n. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $x = \frac{1}{1+y}$ | $x \to 0, y \to \infty$; $x \to 1, y \to 0$

$dx = -\frac{1}{(1+y)^2} dy$

$\therefore B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \left(-\frac{1}{(1+y)^2}\right) dy$

$= \int_0^\infty \left(\frac{1}{1+y}\right)^{m+1} \left(\frac{\cancel{1}+y-\cancel{1}}{1+y}\right)^{n-1} dy$

$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

{$\because$ ~~Bd~~ is symmetric}

$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Q.3. Prove that $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ CO4 | BTLS

Sol:- we know that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \to (1)$

now, $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$, let $x = \frac{1}{y}$, $dx = -\frac{1}{y^2} dy$ | $x \to 1, y \to 1$; $x \to \infty, y \to 0$

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$$= \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{1}{y^{m-1}\left(\frac{y+1}{y}\right)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^n \cdot \cancel{y^m}}{\cancel{y^m} \cdot y^{-1} (1+y)^{m+n}} \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\therefore B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\boxed{B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.}$$

Q.4 Define Gamma function & show that $\Gamma(1/2) = \sqrt{\pi}$. [CO4|BTL4]

Sol.- The definite Integral $\int_0^\infty e^{-x} x^{n-1} dx$ is defined as gamma function & is denoted as

$$\gamma_n \text{ or } \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

To show that $\Gamma(1/2) = \sqrt{\pi}$.

we know that $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Let $m = n = 1/2$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1/2+1/2)} = \left(\Gamma(1/2)\right)^2 \rightarrow (A)$$

$$\cancel{= \frac{\left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right)}{\Gamma(2/2)}}$$

$$\cancel{= \frac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)}}$$

we know that $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\left[\theta\right]_0^{\pi/2}$$

$$= \cancel{2}\left[\pi/\cancel{2} - 0\right]$$

$$\left[\Gamma(1/2)\right]^2 = \pi \quad \{\text{using } (A)\}$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Q.5. Evaluate $\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}\,dx$ using Beta & gamma function.

CO5	BTL5

Sol: $\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}\,dx.$

$$= \int_{x=0}^{\infty} \frac{x^{3/2}}{a\sqrt{1+x^2/a^2}}\,dx \rightarrow (1)$$

Put $\frac{x^2}{a^2} = y \Rightarrow x^2 = a^2 y$

$x = a\sqrt{y}$

Diff. b/s w.r.t x

$2x\,dx = a^2 dy$ If $x=0, y=0$

If $x=\infty, y=\infty$

$x\,dx = \frac{a^2}{2} dy$

$dx = \frac{a^2}{2}\frac{dy}{x}$

$dx = \frac{a^2}{2}\frac{dy}{a\sqrt{y}} = \frac{a}{2} y^{-1/2}$

substitute in (1)

$$\int_0^\infty \frac{x^{3/2}}{\sqrt{a^2+x^2}}\,dx = \int_{y=0}^{\infty} \frac{(a\sqrt{y})^{3/2}}{a\sqrt{1+y}} \frac{a}{2} y^{-1/2} dy$$

$$= \int_0^\infty \frac{a^{3/2}}{2} \frac{y^{3/4} \cdot y^{-1/2}}{(1+y)^{1/2}} dy$$

$$= \frac{a^{3/2}}{2} \int_0^\infty \frac{y^{1/4}}{(1+y)^{1/2}} dy$$

$$= \frac{a^{3/2}}{2}\int_0^\infty \frac{y^{5/4-1}}{(1+y)^{5/4+(-3/4)}}\,dy$$

$$= \frac{a^{3/2}}{2} B\left(5/4, -3/4\right) \quad \left\{\because B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\right.$$

$$= \frac{a^{3/2}}{2} \frac{\Gamma(5/4)\,\Gamma(-3/4)}{\Gamma(5/4-3/4)}$$

$$= \frac{a^{3/2}}{2} \frac{\frac{1}{4}\Gamma(1/4)\,\Gamma(-3/4)}{\Gamma(1/2)} \quad \left\{\because \Gamma(n)=\frac{\Gamma(n+1)}{n}\right\}$$

$$= \frac{a^{3/2}}{8} \frac{\Gamma(1/4)\,\Gamma(-3/4)}{\sqrt{\pi}} \quad \because \Gamma_{1/2}=\sqrt{\pi}.$$

$$= \frac{a^{3/2}}{8} \frac{\Gamma(1/4)\,\Gamma(-3/4)}{\sqrt{\pi}}.$$

Q.6 Express $2x^3+3x^2-x+1$ in terms of Legendre Polynomial. |CO4|BTL5|

Sol: We know that By Rodrigues Formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\boxed{P_0(x)=1}$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2}\times 2x$$

$$\boxed{P_1(x)=x}$$

$$P_2(x) = \frac{1}{2!\,4}\frac{d^2}{dx^2}(x^2-1)^2$$

$$P_2(x) = \frac{1}{8_2}\frac{d}{dx}\cancel{2}(x^2-1)\times\cancel{2}x$$

$$P_2(x) = \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$P_2(x) = \frac{1}{2}(3x^2-1)^2$$

$$2P_2(x) = 3x^2-1$$

$$\Rightarrow 3x^2 = 2P_1(x)+1 = 2P_2(x)+P_0(x)$$

$$\Rightarrow x^2 = \frac{1}{3}\left[2P_2(x)+P_0(x)\right]$$

$$P_3(x) = \frac{1}{3!\,2^3}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5-12x^3+6x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4-36x^2+6)$$

$$= \frac{1}{48}\cancel{\frac{d}{dx}}(120x^3-72x)$$

$$= \frac{\cancel{24}}{\cancel{48}_2}(5x^3-3x)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\Rightarrow 2P_3(x) = 5x^3-3x$$

$$\Rightarrow 5x^3 = 2P_3(x)+3x$$

$$\Rightarrow \cancel{5}x^3 = \frac{\cancel{2}}{5}(2P_3(x)+3x)$$

$$\Rightarrow x^3 = \frac{1}{5}\left[2P_3(x)+3x\right]$$

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Substituting x, x^2, x^3, we get

$$2x^3+3x^2-x+1 = 2\left[\frac{1}{5}\left(2P_3(x)+3P_1(x)\right)\right] + 3\left[\frac{1}{3}\left(2P_2(x)+P_0(x)\right)\right]$$
$$- P_1(x) + P_0(x)$$

$$\boxed{2x^3+3x^2-x+1 = \frac{4}{5}P_3(x) + 2P_2(x) + \frac{1}{5}P_1(x) + 2P_0(x)}$$

Q.7 First Recurrence Relation for $P_n(x)$.
Prove that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

CO5 | BTL3

Sol:- We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n . P_n(x)$$

Differentiating Partially w.r.t 't' on b/s

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(1-2xt+t^2)^{-1}(x-t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} x t^n P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - \sum_{n=0}^{\infty} 2xn t^n P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x)$$

Equating coeff of t^n on b/s.

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = 2xnP_n(x) + xP_n(x) - (n-1)P_{n-1}(x) - P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Q.8 Find the Power Series solution of the differential eqn $(1-x)^2 y'' - 2xy' + 2y = 0$ about $x = 0$ $\boxed{\text{CO4} | \text{BTL5}}$

Sol:- $(1-x^2)y'' - 2xy' + 2y = 0 \to (1)$

$P_0(x) = 1 - x^2$, At $x = 0 \Rightarrow 1 - x^2 \neq 0$

$\therefore x = 0$ is ordinary Point of (1).

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \to (2)$ be a solution of (1)

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \cdots$$

Substitute in (1), we get

$$(1-x^2)(2a_2 + 6a_3x + 12a_4x^2 + \cdots) - 2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots) + 2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0$$

Comparing the coefficients

x^0 (constants) $\Rightarrow 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

$x^1 \Rightarrow 6a_3 - 2a_1 + 2a_1 = 0 \Rightarrow 6a_3 = 0 \Rightarrow a_3 = 0$

$x^2 \Rightarrow -\cancel{2a_2} + 12a_4 - 4a_2 + \cancel{2a_2} = 0$

$\Rightarrow {}_{3}\cancel{12}a_4 = \cancel{4}a_2$

$a_4 = \frac{1}{3}a_2 = \frac{1}{3}(-a_0)$

Substitute in (3), we get

$$y = a_0 + a_1 x + (-a_0)x^2 + 0x^3 + \left(-\frac{a_0}{3}\right)x^4 + \cdots$$

$$y = a_0\left(1 - x^2 - \frac{x^4}{3} + \cdots\right) + a_1 x$$

where a_0 & a_1 are arbitrary constant.

Q.9 Express the following sum of the Legendre Polynomial in terms of x, $8P_4(x) + 2P_2(x) + P_0(x)$. $\boxed{CO4 \mid BTL4}$

Sol: we know By Rodrigues formula,

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\rightarrow P_0(x) = \frac{1}{2^0 \cdot 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1 \Rightarrow \underline{P_0(x) = 1}$$

$$\rightarrow P_1(x) = \frac{1}{2^1 \cdot 1!}\frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}}\cancel{2}x = x \Rightarrow \underline{P_1(x) = x.}$$

$$\rightarrow P_2(x) = \frac{1}{2^2 \cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{\cancel{8}_2}\frac{d}{dx}\cancel{2}(x^2-1)\cdot\cancel{2}x$$

$$P_2(x) = \frac{1}{2}\frac{d}{dx}(x^3 - x) = \underline{\frac{1}{2}(3x^2 - 1)}$$

$$\rightarrow P_3(x) = \frac{1}{2^3 \cdot 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{8\times 6}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5 - 12x^3 + 9x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4 - 36x^2 + 9)$$

$$= \frac{1}{48}(120x^3 - 72x) = \frac{24}{48_2}(5x^3 - 3x)$$

$$\boxed{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

$$\rightarrow P_4(x) = \frac{1}{2^4\,4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{\cancel{16}_{2}\times 24}\frac{d^3}{dx^3}\cancel{4}(x^2-1)^3\times \cancel{2}x$$

$$= \frac{1}{48}\frac{d^3}{dx^3}x(x^6-3x^4+3x^2-1)$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^7-3x^5+3x^3-x)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(7x^6-15x^4+9x^2-1)$$

$$= \frac{1}{48}\frac{d}{dx}(42x^5-60x^3+18x)$$

$$= \frac{6}{48}\frac{d}{dx}(7x^5-10x^3+3x)$$

$$= \frac{6}{\cancel{48}_{8}}(35x^4-30x^2+3)$$

$$\boxed{P_4(x) = \frac{1}{8}(35x^4-30x^2+3)}$$

$$\therefore 8P_4(x)+2P_2(x)+P_0(x) = \frac{\cancel{8}}{\cancel{8}}(35x^4-30x^2+3) + \frac{\cancel{2}}{\cancel{2}}(3x^2-1) + 1$$

$$= 35x^4-30x^2+3+3x^2\cancel{-1}\cancel{+1}$$

$$= 35x^4-27x^2+3 \;//\; \underline{\text{Ans}}.$$

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Q.10 Evaluate the Improper Integral,

$$\int_0^\infty \sqrt{x}\cdot e^{-x^2}\,dx.$$ $\boxed{CO4/BTL5}$

Sol: Given $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx$

Let $x^2 = t \Rightarrow x = \sqrt{t}$ | $x\to 0, t\to 0$; $x\to\infty, t\to\infty$

$2x\,dx = dt$

$dx = \dfrac{dt}{2\sqrt{t}}$

$$\therefore \int_0^\infty e^{-t}\cdot t^{1/4}\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/4}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{3/4-1}\,dt$$

$$= \frac{1}{2}\,\Gamma(3/4)$$

Q.11 Prove that $B(m,n) = 2\int_0 \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

Sol. By Beta function def.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$dx = \sin 2\theta\, d\theta$

If $x=0 \Rightarrow \sin^2\theta = 0 \Rightarrow \theta = 0$

If $x = 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \pi/2$.

$$\therefore B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \cdot \sin 2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \, (\cos^2\theta)^{n-1} \cdot 2\sin\theta \cdot \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \; \cos^{2n-2+1} d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1} d\theta$$

Q.12 Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$. $\boxed{CO4|BTL5}$

Sol: $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 x^2 (1-x^5)^{-1/2} dx$

$$= \int_0^1 \frac{x^2}{x^4} (1-x^5)^{-1/2} x^4 dx$$

$$= \int_0^1 x^{-2} (1-x^5)^{-1/2} x^4 dx.$$

Put $x^5 = y \Rightarrow 5x^4 dx = dy$

$x = y^{1/5}$ or $x^4 dx = \frac{dy}{5}$

when $x=1$, $y=1$

when $x=0$, $y=0$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 y^{-2/5} (1-y)^{-1/2} \frac{dy}{5}$$

$$= \frac{1}{5}\int_0^1 y^{3/5 - 1} (1-y)^{\frac{1}{2}-1} dy$$

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{1-x^5}} = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)}$$

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"OR"

Let $x^5 = y \Rightarrow x = y^{1/5}$

Diff. b/s ~~w.r.t.~~

~~$5x^4 dx = dy$~~ $dx = \frac{1}{5} y^{-4/5} dy$

~~or x~~ when $x=0,\ y=0$

when $x=1,\ y=1$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 \frac{y^{2/5}}{(1-y)^{1/2}} \cdot \frac{1}{5} y^{-4/5} dy$$

$$= \frac{1}{5} \int_0^1 \frac{y^{2/5 - 4/5}}{(1-y)^{1/2}} dy$$

$$= \frac{1}{5} \int_0^1 y^{-2/5} (1-y)^{-1/2} dy$$

$$= \frac{1}{5} \int_0^1 y^{3/5 - 1} (1-y)^{1/2 - 1} dy$$

comparing with $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)}$$

Q.13. Prove that $2^{2n-1} \Gamma(n) \Gamma(n + \frac{1}{2}) = \Gamma(2n) \sqrt{\pi}$

CO4 | BTL4

Sol. By def, we have

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

or

$$\int_0^1 x^{n-1} (1-x)^{m-1} dx = B(m,n) = \frac{\Gamma(n) \cdot \Gamma(m)}{\Gamma(n+m)} \rightarrow (1)$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta \cos\theta \, d\theta$

when $x \to 0,\ \theta \to 0$

$x \to 1,\ \theta \to \pi/2$

From eqn (1)

$$\int_0^{\pi/2} \sin^{2n-2}\theta \cos^{2m-2}\theta\,(2\sin\theta\cos\theta)\,d\theta = \frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m)}$$

or $$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\,d\theta = \frac{\Gamma(n)\cdot\Gamma(m)}{2\Gamma(n+m)} \rightarrow (2)$$

Put $m=\frac{1}{2}$ in eqn (2), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta\,d\theta = \frac{\Gamma(n)\cdot\Gamma(\frac{1}{2})}{2\Gamma(n+\frac{1}{2})} = \frac{\sqrt{\pi}\cdot\Gamma(n)}{2\,\Gamma(n+\frac{1}{2})} \rightarrow (3)$$

now Put $m=n$ in eqn (2), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{[\Gamma(n)]^2}{2\Gamma(2n)}$$

or $$\frac{[\Gamma(n)]^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1}\,d\theta$$

$$= \frac{1}{2^{2n-1}}\int_0^{\pi/2} \sin^{2n-1}2\theta\,d\theta$$

$$= \frac{1}{2^{2n-1}}\cdot\frac{1}{2}\int_0^{\pi/2} \sin^{2n-1}\phi\,d\phi \quad (\text{Put } 2\theta=\phi)$$

$$= \frac{1}{2^{2n}}\int_0^{\pi/2} \sin^{2n-1}\phi\,d\phi$$

$$\Rightarrow \frac{[\Gamma(n)]^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}}\,\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \quad \{\text{using } (3)\}$$

$$\Rightarrow \boxed{2^{2n-1}\,\Gamma(n)\cdot\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\cdot\Gamma(2n)}$$

Q.14 Evaluate $\int_0^{\infty} 3^{-4x^2} dx$.

Let $I = \int_0^{\infty} 3^{-4x^2} dx$

Put $3^{-4x^2} = e^{-t}$

Taking log on b/s, we get

$\log 3^{-4x^2} = \log e^{-t}$

$\cancel{-}4x^2 \log 3 = \cancel{-}t \Rightarrow 4x^2 \log 3 = t$

$\Rightarrow x^2 = \frac{t}{4\log 3} \Rightarrow x = \frac{\sqrt{t}}{2\sqrt{\log 3}} = \frac{(t)^{1/2}}{2(\log 3)^{1/2}}$

Diff. b/s w.r.t x,

$dx = \frac{t^{-1/2}}{4\sqrt{\log 3}} dt$

$\therefore I = \int_0^{\infty} \frac{e^{-t} \cdot t^{-1/2}}{4\sqrt{\log 3}} dt$

$\Rightarrow I = \frac{1}{4\sqrt{\log 3}} \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt$

$\Rightarrow \boxed{I = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}}$ $\left\{ \because \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt = \sqrt{\pi} \right\}$

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LQ. Evaluate $\int_0^1 x^m (1-x^2)^n$

Soln: Let $x^2 = t \Rightarrow x = t^{1/2}$.

$$2x\,dx = dt$$

$$dx = \frac{dt}{2x} = \frac{dt}{2t^{1/2}}$$

$$U.L = 0 \Rightarrow t = 0$$

$$L.L = 1 \Rightarrow t = 1$$

Subst in Above eqn.

$$\int_0^1 x^m (1-x^2)^n = \int_0^1 \left(t^{1/2}\right)^m (1-t)^n \frac{dt}{2t^{1/2}}$$

$$= \frac{1}{2}\int_0^1 t^{m/2}\, t^{-1/2} (1-t)^n\, dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{m-1}{2}} (1-t)^n\, dt$$

$$\because \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$m = 2m - 1$

Hence in

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1} (1-x)^{n+1-1}\, dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, n+1\right).$$

M-II - UNIT-5 - Q.B.

SAQ & LAQ's

1. find $L\{e^{-t} \cdot \sin 2t\}$.

Sol:- $$L\{\sin at\} = \frac{a}{s^2+a^2}$$

Here $a=2$.

$$L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$$

Now, $$L\{e^{-t} \cdot \sin 2t\} = \left[\frac{2}{s^2+4}\right]_{s \to s+1}$$

$$= \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+1+4} = \frac{2}{s^2+2s+5}.$$

$$\therefore L\{e^{-t} \cdot \sin 2t\} = \frac{2}{s^2+2s+5}.$$

---

2. State convolution Theorem of Laplace Transforms.

Sol:- If $f(s)$ & $g(s)$ are Laplace transform of $F(t)$ & $G(t)$ respectively,

i.e., if $L^{-1}[f(s)] = F(t)$ & $L^{-1}[g(s)] = G(t)$,

then $$L^{-1}\{f(s) * g(s)\} = \int_0^t F(u) \cdot G(t-u)\, du.$$

---

3). Find $L\{t \cdot \cos 2t\}$

Sol:- $L\{t^n \cdot f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n}[f(t)]$.

$\Rightarrow L\{t^1 \cdot \cos 2t\} = (-1)^1 \cdot \frac{d^1}{ds^1}[\cos 2t]$

$L[\cos 2t] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4} = \frac{s}{s^2+a^2}$.

$\Rightarrow L[t \cdot \cos 2t] = (-1) \cdot \frac{d}{ds} \cdot \left[\frac{s}{s^2+4}\right]$

$= (-1) \frac{-s^2+4}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$.

$$\boxed{\frac{u'v - uv'}{v^2}}$$

4) Evaluate

$L^{-1}\left[\frac{2}{s^3} + \frac{1}{s^2}\right]$

Sol:- $2 \cdot L^{-1}\left[\frac{1}{s^3}\right] + L^{-1} \cdot \left[\frac{1}{s^2}\right]$

$= 2 \cdot \frac{t^2}{2!} + t \cdot = t^2 + t$.

$[\because L^{-1}\left[\frac{1}{s^3}\right] =$

$\bar{f}(s) = \frac{1}{s^{n+1}} = \frac{1}{s^{2+1}} = \bar{f}(s)$

$L^{-1}[\bar{f}(s)] = f(t)$

$= \frac{t^n}{n!}$.

(5) find $L\{t \cdot e^{-t}\}$

Sol:- $L\{e^{-t}\} = \frac{1}{s+1} = \bar{f}(s)$

$L\{t^1 \cdot e^{-t}\} = (-1)^n \cdot \frac{d^n}{ds^n}[\bar{f}(s)]$

$= (-1)^1 \frac{d^1}{ds^1}\left[\frac{1}{s+1}\right]$

$= (-1) \cdot \frac{d}{ds}(s+1)^{-1}$

$= (-1) \cdot (-1) \cdot (s+1)^{-1-1}$

$= (-1) \cdot (-1) \cdot (s+1)^{-2}(1) = (s+1)^{-2} = \frac{1}{(s+1)^2}$

(6) find ~~inverse~~ Laplace transform of

$f(t) = \cancel{\ldots} \cdot t \cdot e^{3t} \cdot \sin 2t$

Sol:- we know that

$L\{\sin 2t\} = \frac{2}{s^2+4}$

$\therefore$ By first shifting Th.

$L\{e^{3t} \cdot \sin 2t\} = \frac{2}{(s-3)^2+4} = \frac{2}{s^2-6s+13}$

$\Rightarrow L\{t \cdot e^{3t} \cdot \sin 2t\} = (-1) \cdot \frac{d}{ds}\left[\frac{2}{s^2-6s+13}\right]$

$$= (-2) \cdot \frac{(-1)}{(s^2-6s+13)^2} (2s-6)$$

$$= \frac{4(s-3)}{(s^2-6s+13)^2}.$$

⑦ Find $L\{t^3 \cdot e^{-4t}\}$

Sol:— $L\{e^{-4t}\} = \frac{1}{s+4}$

$$L\{t^3 \cdot e^{-4t}\} = (-1)^3 \cdot \frac{d^3}{ds^3}\left[\frac{1}{s+4}\right]$$

$$= (-1) \cdot \frac{d^3}{ds^3}[s+4]^{-1} = (-1) \cdot \frac{d^2}{ds^2} \cdot \frac{d}{ds}(s+4)^{-1}$$

$$= (-1) \cdot \frac{d^2}{ds^2} (-1) \cdot [s+4]^{-1-1} (1)$$

$$= \frac{d^2}{ds^2}[s+4]^{-2} = \frac{d}{ds} \cdot \frac{d}{ds}[s+4]^{-2}$$

$$= \frac{d}{ds} \cdot (-2)(s+4)^{-2-1} = \frac{d}{ds}\left[(-2)(s-4)^{-3}\right]$$

$$= (-2)(-3)(s-4)^{-3-1}$$

$$= 6 \cdot (s-4)^{-4} = \frac{6}{(s-4)^4}$$

3

(8) Find $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right]$

Sol:- $\frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$. — (1)

$1 = A(s+3) + B(s+2) = As + 3A + Bs + 2B$

$\Rightarrow 1 = (A+B)s + 3A + 2B$

$A + B = 0$

$3A + 2B = 1$

$\Rightarrow 3A + 3B = 0$

& $3A + 2B = 1$

solving the above two eq's, we get

$B = -1$ & $A = 1$

$= \frac{1}{s+2} - \frac{1}{s+3}$.

$\Rightarrow L^{-1}\left[\frac{1}{(s+2)(s+3)}\right] = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right]$

$\Rightarrow e^{-2t} - e^{-3t}$.

(9) Find the Laplace transform of

$f(t) = \sin^2 t$.

Sol:- $L[\sin^2 t]$

$$\Rightarrow L\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2} \cdot L[1-\cos 2t]$$

$$= \frac{1}{2} \cdot \left[L[1] - L[\cos 2t]\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{1}{2} \cdot \frac{4}{s(s^2+4)} = \frac{2}{s(s^2+4)}$$

$\cos 2A = 1 - 2\sin^2 A$.

$\cos 2A - 1 = -2\sin^2 A$.

$\Rightarrow 2\sin^2 A = 1 - \cos 2A$

$\Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$

$$\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$$

(10) Explain Unit Impulse function. (Dirac Delta Function).

Sol:- The unit impulse function is considered as the limiting form of the function.

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a. \\ 0 & \text{for } t \neq a. \end{cases}$$

Such that $\int_0^\infty \delta(t-a)\,dt = 1$. $(a \geq 0)$

(11) Evaluate $L\{\cos^2 t\}$.

Sol. → $\cos^2 t = \frac{1+\cos 2t}{2}$

$$L[\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right] = \frac{1}{2}\cdot L[1+\cos 2t]$$

$$= \frac{1}{2}\cdot L[1] + L[\cos 2t]$$

$$= \frac{1}{2}\cdot\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right] = \frac{1}{2}\left[\frac{2s^2+4}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{2(s^2+2)}{s(s^2+4)}\right]$$

$$= \frac{s^2+2}{s(s^2+4)}.$$

12. Find $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$

Sol:- By partial fraction

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$\cancel{\frac{1}{(s+1)(s+2)}} = \frac{A(s+2)+B(s+1)}{\cancel{(s+1)(s+2)}}$$

$$1 = A(s+2)+B(s+1)$$

put $s=-1$

$$1 = A(-1+2)+B(-1+1)$$

$$1 = A(1)+0$$

$$\boxed{A=1}$$

put $s=-2$

$$1 = A(-2+2)+B(-2+1)$$

$$1 = 0+B(-1)$$

$$\boxed{B=-1}$$

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

13. Find $L\{t^3 e^{-4t}\}$

Sol: $L\{e^{-4t}\} = \frac{1}{s+4}$

By multiplication property

$$L\{t^3 e^{-4t}\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+4}\right) \quad \left[\because L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[f(s)]\right]$$

$$= -1 \frac{d^2}{ds^2}\left(\frac{d}{ds}(s+4)^{-1}\right)$$

$$= -1\cdot(-1)\frac{d^2}{ds^2}(s+4)^{-2}$$

$$= 1\frac{d}{ds}\left(\frac{d}{ds}(s+4)^{-2}\right) = 1\cdot(-2)\frac{d}{ds}(s+4)^{-3}$$

$$= -2(-3)(s+4)^{-4}$$

$$= \frac{6}{(s+4)^4}$$

14. Find $L^{-1}\left\{\frac{1}{s(s+1)}\right\}$

sol:- $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

By division Rule

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^t = \left[\frac{e^{-t} - e^{0}}{-1}\right]$$

$$= \frac{e^{-t} - 1}{-1} = 1 - e^{-t}$$

15. Find $L\{t^3\}$

sol:- $L\{t^3\} = \frac{3!}{s^4}$

$$= \frac{3 \times 2 \times 1}{s^4}$$

$$= \frac{6}{s^4}$$

*

$s^2+6s+34$

1. Find the Laplace transform of

$$f(t) = \frac{e^{-2t}\sin 3t}{t}$$

We know that

$$L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\sin 3t}{t}\right\} = 3\int_s^\infty \frac{1}{s^2+9}\,ds \text{ (By division property)}$$

$$= 3\int_s^\infty \frac{1}{s^2+3^2}\,ds$$

$$= \left[3 \cdot \frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

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$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) \qquad \left[\tan^{-1}(x) + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$= \cot^{-1}\left(\frac{s}{3}\right)$$

$$L\left\{\frac{\sin 3t}{t}\right\} = \cot^{-1}\left(\frac{s}{3}\right)$$

$$L\left\{e^{-2t}\,\frac{\sin 3t}{t}\right\} = \cot^{-1}\left(\frac{s+2}{3}\right)$$

2 Using Laplace Transform solve the initial value problem

$y'' + y = e^t \sin t$, $y(0) = 0 = y'(0)$

Sol: Given

$y'' + y = e^t \sin t$

Taking L.T on both sides

$L\{y''\} + L\{y\} = L\{e^t \sin t\}$

$[\because L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0)]$

$s^2 L\{y\} - s y(0) - y'(0) + L\{y\} = \dfrac{1}{(s-1)^2+1^2}$

using initial condition

$y(0)=0,\ y'(0)=0$

$s^2 L\{y\} - 0 - 0 + L\{y\} = \dfrac{1}{(s-1)^2+1^2}$

$$L\{y\}\left[s^2+1\right] = \frac{1}{s^2+1-2s+1}$$

$$L\{y\} = \frac{1}{s^2-2s+2} \cdot \frac{1}{s^2+1}$$

$$\frac{1}{(s^2-2s+2)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

$$\cancel{\frac{1}{(s^2-2s+2)(s^2+1)}} = \frac{(As+B)(s^2-2s+2)+(Cs+D)(s^2+1)}{\cancel{(s^2-2s+2)(s^2+1)}}$$

$$1 = (As+B)(s^2-2s+2) + (Cs+D)(s^2+1)$$

$$1 = As^3 - 2As^2 + 2As + Bs^2 - 2Bs + 2B$$
$$+ Cs^3 + Cs + Ds^2 + D$$

equating the co-efficient

s^3 co-efficient $\quad A + C = 0$

$A = -C \rightarrow (1)$

s^2 " $\quad -2A + B + D = 0 \rightarrow (2)$

$2A - 2B + C = 0 \rightarrow (3)$

s " $\quad 2B + D = 1 \rightarrow (4)$

constants

$2(-C) - 2B + C = 0$

$-2C - 2B + C = 0$

$-2B - C = 0$

$-2B = C \quad B = -\frac{C}{2}$

$-2(C) + \cancel{2B}\, C + C = 0$

$\cancel{2B}+$

$B + \cancel{D} = -2C$

$2B + \cancel{D} = -1$

$-B = -2C - 1$

$2A - 2B + C = 0$

$2\cancel{C} + B + D = 0$

$-2\cancel{C} - 2B + C = 0$

$-B + C + D = 0$

$-4B + B + D = 0$

$-3B + D = 0$

$2B + D = 1$

$-5B = -1$

$B = 1/5$

$A = 2/5$

$C = -2/5$

$-2C - 2B + C = 0$

$-C - 2B = 0$

$-2B = C$

$-2B = -A$

$A = 2B$

$A = 2/5,\ B = 1/5\ \ C = -2/5\ \ D = 3/5$

sub in eqn (1)

$$L\{y\} = \frac{\frac{2}{5}s + \frac{1}{5}}{s^2+1} + \frac{-\frac{2}{5}s + \frac{3}{5}}{s^2-2s+2}$$

$$y = L^{-1}\left[\frac{\frac{2}{5}s + \frac{1}{5}}{s^2+1}\right] + L^{-1}\left[\frac{-\frac{2}{5}s + \frac{3}{5}}{s^2-2s+2}\right]$$

$$= \frac{2}{5}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right] +$$

$$\left(-\frac{1}{5}L^{-1}\left(\frac{2(s-1)-1}{s}\right)\right.$$

$$-\frac{1}{5}\left(\frac{L^{-1}[2s-3]}{(s-1)^2+1}\right)$$

$$-\frac{1}{5}\left[L^{-1}\left(\frac{2(s-1)-1}{(s-1)^2+1}\right)\right)$$

$$= \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{1}{5}e^t(\cos t - \sin t)$$

5) Evaluate $L^{-1}\left\{\log\left(\frac{s-3}{s+3}\right)\right\}$

Let $\bar{f}(s) = \log\left(\frac{s-3}{s+3}\right)$

$= \log(s-3) - \log(s+3)$

diff w.r.to "s" on both side

$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\log(s-3) - \frac{d}{ds}\log(s+3)$

$= \frac{1}{s-3} - \frac{1}{s+3}$

Taking I. L. Transform on both sides

$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}$

$= -t.f(t) = e^{3t} - e^{-3t}$

$f(t) = \frac{e^{-3t} - e^{3t}}{t}$

convolation theorem,

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5. Find the Laplace transform of $f(t) = \frac{2\sin^2 t}{t}$

Sol: Given

$$f(t) = \frac{2\sin^2 t}{t}$$

$$\left[\begin{array}{l} \cos 2\theta = 1 - 2\sin^2\theta \\ 2\sin^2\theta = 1 - \cos 2\theta \end{array} \right.$$

W.K.T

$$L\{1 - \cos 2t\}$$

$$= L\{1\} - L\{\cos 2t\}$$

$$= \frac{1}{s} - \frac{s}{s^2+2^2}$$

By division property

$$L\{f(t)\} = L\left\{\frac{2\sin^2 t}{t}\right\} = L\left\{\frac{1-\cos 2t}{t}\right\}$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{2s}{2(s^2+2^2)}\right) ds$$

$$= \left| \log s - \log(s^2+4) \right|_s^\infty$$

$$= \left[\log s - \log(s^2+4)^{1/2} \right]_s^\infty$$

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$$= \left[\log \frac{s}{(s^2+4)^{1/2}}\right]_s^\infty$$

$$= \lim_{s\to\infty} \log \frac{\cancel{s}}{\cancel{s}\left(1+\frac{4}{s^2}\right)^{1/2}} - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= \log 1 - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= 0 - \log \frac{s}{(s^2+4)^{1/2}}$$

$$= \log \frac{(s^2+4)^{1/2}}{s}$$

$$= \log \left(\frac{s^2+4}{s^2}\right)^{1/2}$$

$$= \frac{1}{2} \log \left(\frac{s^2}{s^2} + \frac{4}{s^2}\right)$$

$$= \frac{1}{2} \log \left(1+\frac{4}{s^2}\right)$$

$$= \frac{1}{2} \log (1+4s^{-2})$$

6. Find the inverse L.T of $\frac{s}{s^4+s^2+1}$

Sol:- Let $\bar{f}(s) = \frac{s}{s^4+s^2+1} =$

$$= \frac{s}{(s^4+2s^2+1)-s^2} = \frac{s}{(s^2+1)^2-s^2}$$ $\left[a^2-b^2 = (a+b)(a-b)\right.$

$$= \frac{s}{(s^2+s+1)(s^2+1-s)}$$

$$= \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1} \longrightarrow (A)$$

Using partial fraction

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

$$s = As^3 - As^2 + As + Bs^2 - Bs + B + Cs^3 + Cs^2 + Cs + Ds^2 + Ds + D$$

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On equating the co-efficient of

i) $s^3 \Rightarrow 0 = A + C \rightarrow (1)$

ii) $s^2 \Rightarrow 0 = -A + B + C + D \rightarrow (2)$

iii) $s \Rightarrow 1 = A - B + C + D \rightarrow (3)$

$1 = -B + D \quad (A + C = 0) \, (\text{from } (1))$

iv) Constants $0 = B + D \Rightarrow (4)$

from (2)

$-A + C = 0 \rightarrow (5)$

from (1), (5)

$$\begin{array}{r} \cancel{A} + C = 0 \\ -\cancel{A} + C = 0 \\ \hline 2C = 0 \\ C = 0 \end{array}$$

from (1) $A + 0 = 0$

Also,

$$\begin{array}{r} \cancel{B} + D = 0 \\ -\cancel{B} + D = 1 \\ \hline 2D = 1 \\ D = 1/2 \end{array} \qquad \begin{array}{l} B + D = 0 \\ B = -D \\ B = -1/2 \end{array}$$

from (A)

$$\frac{s}{s^4 + s^2 + 1} = -\frac{1}{2} \cdot \frac{1}{s^2 + s + 1} + \frac{1}{2} \cdot \frac{1}{s^2 - s + 1}$$

Taking I.L.T on both sides

$$L^{-1}\{\bar{f}(s)\} = -\frac{1}{2} L^{-1}\left\{\frac{1}{s^2 + s + 1}\right\} + \frac{1}{2}\left\{\frac{1}{s^2 - s + 1}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s^2 + s + \frac{1}{4}) + (1 - 1/4)}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s^2 - s + 1/4) + (1 - 1/4)}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\}$$

By shifting property

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = -\frac{1}{2} e^{-t/2} L^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\} + \frac{1}{2} e^{t/2} L^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\}$$

$$= -\frac{1}{2} \cdot e^{-t/2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2} t + \frac{1}{2} e^{t/2} \frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2} t$$

$$= -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\, e^{-t/2}\cdot \sin\frac{\sqrt{3}}{2}t + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\, e^{t/2}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,\left(e^{t/2}-e^{-t/2}\right)\times\frac{2}{2}$$

$$\left[\frac{e^{t}-e^{-t}}{2} = \sinh t\right]$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left(\frac{e^{t/2}-e^{-t/2}}{2}\right)$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\cdot\sinh\frac{t}{2}$$

(7) To prove

$$L\{e^{at}\} = \frac{1}{s-a}$$

We know that

$$L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot f(t)\,dt$$

Here $f(t) = e^{at}$

$$\therefore L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at}\,dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\,dt \quad \left[\begin{matrix} -st+at \\ -(s-a)t \end{matrix}\right]$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= \frac{1}{-(s-a)}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{1}{-(s-a)}(0-1) = \frac{1}{s-a}$$

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8) Find the LT of the function

$$f(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases}$$ and

the period of $f(t)$ is 2π

$$L\{f(t)\} = \frac{\int_0^t e^{-st} f(t)\,dt}{1 - e^{-st}}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} (0)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt \right]$$

$$\left[\because \int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt) \right]$$

$a = -s,\ b = 1$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{(-s)^2+1^2} (-s \sin t - 1 \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{e^{-s\pi}}{s^2+1} (s \sin \pi + \cos \pi) - (0 + \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{e^{-s\pi}}{s^2+1} (0+1) + \frac{1}{s^2+1} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1^2} \right)$$

$$= \frac{1}{1^2-(e^{-\pi s})^2} \left(\frac{e^{-\pi s}+1}{s^2+1} \right)$$

$$= \frac{1}{\cancel{(1+e^{-\pi s})}\,(1-e^{-\pi s})} \cdot \left(\frac{\cancel{e^{-\pi s}+1}}{s^2+1}\right)$$

$$= \frac{1}{(1-e^{-\pi s})\,(s^2+1)}$$

Q9) show that $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

Sol:- we know that

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

here $f(t) = \sin at$

$$L\{\sin at\} = \int_0^{\infty} e^{-st} \sin at \, dt$$

$$\left[\because \int_0^{\infty} e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]\right.$$

$$= \frac{e^{-st}}{s^2+a^2}\left[-s\sin at - a\cos at\right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{s^2+a^2}\left(-s\sin\infty - a\cos\infty\right) - \frac{e^{-0}}{s^2+a^2}\left[-s\sin 0 - a\cos 0\right]$$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

$$= \frac{a}{s^2+a^2}$$

* ——— *

LAQ:-

1.Q) Find the Laplace transform of

$f(t) = \bar{e}^{t}(2\cos 5t - 3\sin 5t)$

$\therefore L\{2\cos 5t - 3\sin 5t\}$

$= 2L\{\cos 5t\} - 3L\{\sin 5t\}$ [By using Laplace Transform]

$= 2 \cdot \frac{s}{s^2+5^2} - 3\frac{5}{s^2+5^2}$

$= \frac{2s}{s^2+5^2} - \frac{15}{s^2+5^2}$

$= \frac{2s-15}{s^2+5^2}$

By shifting property

$L\{\bar{e}^{3t}(2\cos 5t - 3\sin 5t)\}$

$= \frac{2(s+3)-15}{(s+3)^2+5^2}$

$= \frac{2s+6-15}{s^2+6s+9+25} = \frac{2s-9}{s^2+6s+34}$

$= \frac{2s-9}{s^2+6s+34}$

Convolution theorem :-

Statement :- If $L^{-1}\{\bar{f}(s)\} = f(t)$, $L^{-1}\{\bar{g}(s)\} = g(t)$ then

$$L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = \int_0^t f(u) \cdot g(t-u)\,du$$

(1) Applying convolution theorem, solve $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol: $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right\}$$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+a^2}$

By I. L. T on both sides

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$$

Also, $$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \cdot \sin at = g(t)$$

Using convolution theorem,

$$L^{-1}\left\{\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right\} = \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u)\,du$$

$$\left[\because\ 2\cos A \sin B = \sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2a}\int_0^t 2\cos au \cdot \sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin a(\cancel{u}+t-\cancel{u}) - \sin a(u-t+u)\}\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin at - \sin a(2u-t)\}\,du$$

$$= \frac{1}{2a}\left[\sin at \int_0^t 1\,du - \int_0^t \sin a(2u-t)\,du\right]$$

$$= \frac{1}{2a}\left[\sin at\, |u|_0^t - \left(-\frac{\cos a(2u-t)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] \qquad \left[\because \begin{array}{l}\cos a(2t-t) \\ \cos at \\ \cos a(0-t) \\ \cos a(-t) \\ = \cos at\end{array}\right]$$

$$= \frac{t \sin at}{2a}$$

12). Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ with $y = \frac{dy}{dt} = 0$ when $t=0$ using Laplace transform.

Sol:- Given equation can be written as

$y'' + 2y' - 3y = \sin t$ with $y(0) = 0$ & $y'(0) = 0$

Taking Laplace Transform on both sides

$L\{y''\} + 2L\{y'\} - 3L\{y\} = L\{\sin t\}$

$s^2 L\{y\} - s y(0) - y'(0) + 2[sL\{y\} - y(0)] - 3L\{y\} = \frac{1}{s^2+1}$

using initial conditions, we get

$y(0) = 0,\ y'(0) = 0$

$\Rightarrow s^2 L\{y\} - 0 - 0 + 2[sL\{y\} - 0] - 3L\{y\} = \frac{1}{s^2+1}$

$L\{y\}[s^2 + 2s - 3] = \frac{1}{s^2+1}$

$\Rightarrow L\{y\}[(s+3)(s-1)] = \frac{1}{s^2+1}$

$L\{y\} = \frac{1}{(s^2+1)(s+3)(s-1)} \rightarrow (1)$

$\frac{1}{(s^2+1)(s+3)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s+3} + \frac{D}{s-1} \rightarrow (2)$

$\Rightarrow 1 = (As+B)(s+3)(s-1) + C(s-1)(s^2+1) + D(s+3)(s^2+1) \rightarrow (3)$

$\Rightarrow 1 = As(s^2+2s-3) + B(s^2+2s-3) + C(s^3+s-s^2-1) + D(s^3+s+3s^2+3)$

$\Rightarrow 1 = A(s^3+2s^2-3s) + B(s^2+2s-3) + C(s^3-s^2+s-1) + D(s^3+3s^2+s+3) \rightarrow (4)$

for $s=-3$ in eqn (3)

$1 = c(-3-1)(9+1)$

$\Rightarrow 1 = -40c$

$\boxed{c = -1/40}$

for $s=1$, from (3) $\Rightarrow 1 = D(1+3)(1+1)$

$1 = 8D \Rightarrow \boxed{D = 1/8}$

Equating the co-efficients of s^3 & s^2, we get

$A + C + D = 0$ & $2A + B - C + 3D = 0$

$\Rightarrow A - \frac{1}{40} + \frac{1}{8} = 0$

$\Rightarrow A = \frac{1}{40} - \frac{1}{8} = \frac{1-5}{40} = -\frac{1}{40}$

$\boxed{A = -\frac{1}{10}}$

$$\Rightarrow 2\left(-\frac{1}{10}\right)+B+\frac{1}{40}+\frac{3}{8}=0$$

$$\Rightarrow B=\frac{1}{5}-\frac{1}{40}-\frac{3}{8}$$

$$=\frac{8-1-15}{40}=-\frac{1}{5}$$

$$\boxed{B=-\frac{1}{5}}$$

Sub the values of A, B, C & D in eqn (2) & using in eqn (1) we get

$$L\{y\}=\frac{-\frac{1}{10}s-\frac{1}{5}}{s^2+1}+\frac{-\frac{1}{40}}{s+3}+\frac{\frac{1}{8}}{s-1}$$

$$L\{y\}=-\frac{1}{10}\left(\frac{s}{s^2+1}\right)-\frac{1}{5}\left(\frac{1}{s^2+1}\right)-\frac{1}{40}\left(\frac{1}{s+3}\right)+\frac{1}{8}\left(\frac{1}{s-1}\right)$$

$$y=L^{-1}\left[-\frac{1}{10}\left(\frac{s}{s^2+1}\right)-\frac{1}{5}\left(\frac{1}{s^2+1}\right)-\frac{1}{40}\left(\frac{1}{s+3}\right)+\frac{1}{8}\left(\frac{1}{s-1}\right)\right]$$

$$=-\frac{1}{10}L^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{1}{40}L^{-1}\left\{\frac{1}{s+3}\right\}+\frac{1}{8}L^{-1}\left\{\frac{1}{s-1}\right\}$$

(3) $$\Rightarrow y=-\frac{1}{10}\cos t-\frac{1}{5}\sin t-\frac{1}{40}e^{-3t}+\frac{1}{8}e^{t}$$

$$f(t) = \frac{e^{-3t} - e^{3t}}{t}$$

13 Q) Using convolation theorem, find $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$

Sol: Given

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\}$$

Let $\bar{f}(s) = \frac{1}{s+1}$ and $\bar{g}(s) = \frac{1}{s+2}$

Taking I.L.T on both sides

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} = g(t)$$

Using convolution theorem

$$L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} = \int_0^t f(u).g(t-u)\,du$$

$$= \int_0^t e^{-u} e^{-2(t-u)}\, du$$

$$= \int_0^t e^{-u} \cdot e^{-2t} \cdot e^{2u}\, du$$

$$= e^{-2t} \int_0^t e^{u}\, du \qquad = el$$

$$= e^{-2t} \left(e^{u} \right)_0^t$$

$$= e^{-2t} \left(e^{t} - e^{0} \right)$$

$$= e^{-2t} \left(e^{t} - 1 \right)$$

$$= e^{-2t} e^{t} - e^{-2t}$$

$$= e^{-t} - e^{-2t}$$

2 Using Laplace transform, solve

$\ldots t \sin(t)$, y

14) Find $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

Sol: Given

$L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$ $\left[\because L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t L^{-1}\{\bar{f}(s)\}\,dt\right]$

$= \int_0^t L^{-1}\left\{\frac{1}{s^2+3^2}\right\} dt$

$= \frac{1}{3}\int_0^t \sin 3t \, dt$

$= \frac{1}{3}\left(-\frac{\cos 3t}{3}\right)_0^t$

$= -\frac{1}{9}\left[\cos 3t - \cos(0)\right]$

$= -\frac{1}{9}\left[\cos 3t - 1\right]$

$= -\frac{\cos 3t}{9} + \frac{1}{9}$

$= \frac{1}{9} - \frac{\cos 3t}{9} = \frac{1-\cos 3t}{9}$

15. Find $L^{-1}\left\{\frac{1}{s(s+2)}\right\}$

Sol: Given

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \int_0^t e^{-2t}\,dt$$

$$= \left[\frac{e^{-2t}}{-2}\right]_0^t = \left[\frac{e^{-2t}-e^0}{-2}\right]$$

$$= \frac{e^{-2t}-1}{-2}$$

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\} = \frac{1-e^{-2t}}{2}$$

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6. Find the L.T of $f(t) = t^2 \sinh t$

Sol:- $L\{t^2 \sinh t\} = L\{t^2 \left(\frac{e^t - e^{-t}}{2}\right)\}$

$$= \frac{1}{2}\{L(e^t t^2) - L(e^{-t} t^2)\}$$

$$L\{t^2\} = \frac{n!}{s^{n+1}} = \frac{2}{s^3}$$

By shifting property

$$L\{e^t \cdot t^2\} = \frac{2}{(s-1)^3}$$

$$L\{e^{-t} \cdot t^2\} = \frac{2}{(s+1)^3}$$

$$\therefore L\{t^2 \sinh t\} = \frac{1}{2}\left\{\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\right\}$$

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