



LORDSINSTITUTEOFENGINEERINGANDTECHNOLOGY
(UGC Autonomous)

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B.E-I SEMESTER, QUESTION BANK

MATHEMATICS-I 2023-24

(COMMON FOR ALL BRANCHES)

SAQ UNIT-I			
S.NO		CO MAPPING	Bloom's Taxonomy Level
1	Define limit of Sequence and give one example.	CO2	BTI1
2	Define Convergent, Divergent and Oscillatory Sequences with examples.	CO2	BTI1
3	Define Convergent Series, Divergent Series and Oscillatory Series .	CO2	BTI1
4	Test for Convergence of the Series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$	CO2	BTI4
5	Write the statement of P-Series and Give one Example.	CO2	BTI1
6	Test for Convergence of $\sum \frac{1}{2^n}$.	CO2	BTI4
7	Write the n^{th} term of the series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$	CO2	BTI2
8	Test the Convergence of $\sum \frac{2^n}{n^3}$	CO2	BTI3
9	Define D' Alembert Ratio Test.	CO2	BTI 1
10	Test the convergence of $\sum \frac{n}{3^n(n^2+2)}$.	CO2	BTI4

11	Test the convergence of $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$	CO2	BTL4
12	Test the Convergence of the following Series $\frac{3}{1} + \frac{4}{8} + \frac{5}{27} + \frac{6}{64} + \dots$	CO2	BTL4
13	Test the Convergence of Series $\sum \frac{\cos n\pi}{(n^2+1)}$	CO2	BTL4
14	Test the convergence of the Series $\sum \frac{n!2^n}{n^n}$.	CO2	BTL4
15	Test for convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$	CO2	BTL4

LAQ UNIT - I			
1	Test for convergence of the series $\sum \sqrt{n^4 + 1} - n^2$.	CO2	BTL 4
2	Test for Convergence of the Series $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \dots$	CO2	BTL 4
3	Test for convergence of the series $\frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots$	CO2	BTL5
4	Test for Convergence of the Series $\sum \frac{4.7 \dots (3n+1)}{1.2.3 \dots n} x^n$	CO2	BTL5
5	Find whether the following Series is Convergent or Divergent. $\sum_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^{n+1}}$	CO2	BTL5
6	Test for Convergence of the Series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$	CO2	BTL 4
7	Test for Convergence of the following Series $\frac{3.6.9 \dots 3n}{7.10.13 \dots (3n+4)} x^n$	CO2	BTL 4

8	Examine the absolute or conditional convergence of $3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^4 + \dots$.	CO2	BTL5
9	Test for absolute or conditional Convergence of the Series $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \dots, 0 < x < 1$.	CO2	BTL 4
10	Discuss the Convergence of the Series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$	CO2	BTL 5
11	Test for convergence of the series $\sum \frac{1.2.3\dots\dots n}{3.5.7\dots\dots(2n+1)}$	CO2	BTL4
12	Test for convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n+1} - \sqrt{n})$	CO2	BTL4
13	Examine the following series for absolute or conditional convergence $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$	CO2	BTL4
14	Test for convergence of the series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \dots$	CO2	BTL4
15	Test for convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$	CO2	BTL4

unit 1

sequences & series.

19 Define limit of a sequence and give one example

Ans Let $\{u_n\}$ be a sequence and $k \in \mathbb{R}$, k is said to be limit of a sequence $\lim_{n \rightarrow \infty} u_n = k$.

$$\text{Ex. } u_n = \frac{n^2+1}{2n^2+3}, \quad u_n = \frac{n+1}{2n+4}$$

20 Define convergent, divergent and oscillatory sequence with example

Ans Convergent sequence If $\lim_{n \rightarrow \infty} u_n = k$ then we say $\{u_n\}$ converges to k

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 = \text{finite}$$

Divergent Sequence - A sequence is said to be divergent if its limit is infinite $\lim_{n \rightarrow \infty} u_n = \infty$, $\lim_{n \rightarrow \infty} n^2 = \infty$

Oscillatory Sequence - A sequence $\{u_n\}$ is said to be oscillatory if $\lim_{n \rightarrow \infty} u_n$ is not unique ex: $\lim_{n \rightarrow \infty} u_n = \{+1, -1\}$ { even odd }

3Q Define convergent series, divergent series, oscillatory series.

Ans Convergent series :- A series $\sum U_n$ is said to be convergent if

$\lim_{n \rightarrow \infty} S_n = \text{finite}$, where S_n is sum of first n terms of the series.

Divergent Series : A series $\sum U_n$ is said to be divergent if $\lim_{n \rightarrow \infty} S_n = +\infty$

Oscillatory Series : A series $\sum U_n$ is said to be oscillatory if $\lim_{n \rightarrow \infty} S_n$ or $\lim_{n \rightarrow \infty} U_n$ is finite but not unique.

4Q Test for convergence of the series. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Ans power of n in Nr & Dr is different \therefore Apply comparison test

$$\text{Step 1 assume } V_n = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}-1}}$$

$$\Rightarrow V_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{u_n}{v_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{\sqrt{n}}{n^2+1} \times \frac{n^{3/2}}{1}$$

$$= \lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{\sqrt{n}}{\sqrt{n}} \times \frac{n\sqrt{n}}{n\sqrt{n}}}{\frac{n^2+1}{n^2}}$$

$$= \frac{1}{1 + \frac{1}{n^2}} = \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{1+0} = 1 \\ = 1 = \text{finite}$$

P-test on $v_n = \frac{1}{n^{3/2}}$

$$\frac{3}{2} > 1 \Rightarrow v_n \text{ is convergent}$$

by comparison $\sum u_n$ & $\sum v_n$ are converge together or diverge together

$\therefore \Rightarrow u_n$ is also convergent.

5Q write the statement of p-series
and give example

Sol Let $\sum u_n = \sum \frac{1}{n^p}$ be a sequence
of +ve terms.

then $p > 1 \Rightarrow u_n$ is convergent

$\Rightarrow p = 1 \Rightarrow u_n$ is divergent

$$\sum \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3^2} + \dots$$

6Q Test for convergence of $\sum \frac{1}{2^n}$.

Sol $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

it is a geometric series.

$r = \frac{1}{2}$ = common ratio.

$r = \frac{1}{2} < 1 \Rightarrow u_n$ is convergent

7Q write the n^{th} term of the series

$$\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots$$

Sol n^{th} term = $\frac{x^n}{n(n+1)}$

$$a + (n-1)d = n^{\text{th}} \text{ term.}$$

89) Test the convergence of $\sum \frac{2^n}{n^3}$

Sol) $U_n = \frac{2^n}{n^3}, (U_n)^{\frac{1}{n}} = \left(\frac{2^n}{n^3}\right)^{\frac{1}{n}}$

$$= \frac{2}{n^{3/n}}$$

$$\lim_{n \rightarrow \infty} (U_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2}{n^{3/n}} = \frac{2}{\infty^0} = 2$$

$2 > 1 \Rightarrow$ divergent

90) Define D'Alembert Ratio Test

Sol) If $\sum U_n$ is a series of +ve terms

then $\lim_{n \rightarrow \infty} \frac{U_n}{U_{n+1}} = k,$

$k > 1 \Rightarrow$ convergent

$k < 1 \Rightarrow$ divergent

$k = 1 \Rightarrow$ test fails.

[109]

Test the convergence of $\sum \frac{n}{3^n(n^2+2)}$

[Solⁿ]

$$u_n = \frac{n}{3^n(n^2+2)}$$

$$u_{n+1} = \frac{n+1}{3^{n+1}(n^2+1+2n+2)}$$

$$= \frac{n+1}{3^n \cdot 3(n^2+2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n}{3^n(n^2+2)} \times \frac{3^n \cdot 3(n^2+2n+3)}{(n+1)}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{n/n \cdot 3(\frac{n^2+2n+3}{n^2})}{(\frac{n^2+2}{n^2})(\frac{n+1}{n})}$$

$$= \frac{3 \cdot (1+0+0)}{(1+0)(1+0)} = 3 > 1$$

$3 > 1 \Rightarrow$ convergent

by D'Alembert's ratio test

7-

119 Test the convergence of $\sum \left[1 + \frac{1}{\sqrt{n}} \right]^{-n^{3/2}}$

Solⁿ

$$u_n = \left[1 + \frac{1}{\sqrt{n}} \right]^{-n\sqrt{n}}$$

$$(u_n)^n = \left\{ \left[1 + \frac{1}{\sqrt{n}} \right]^{-n\sqrt{n}} \right\}^n$$

$$(u_n)^n = \left[1 + \frac{1}{\sqrt{n}} \right]^{-\sqrt{n}}$$

$$= \left\{ \left[1 + \frac{1}{\sqrt{n}} \right]^{\frac{1}{\sqrt{n}}} \right\}^n = \frac{1}{e} < 1$$

\Rightarrow convergent

$$\therefore \left[1 + \frac{1}{n} \right]^n = e$$

120 Test the convergence of the following series $\frac{3}{1} + \frac{4}{8} + \frac{5}{27} + \frac{6}{64} + \dots$

Solⁿ

$$u_n = \frac{n+2}{n^3}, \quad v_n = \frac{1}{n^{3-1}}$$

$$\lim_{n \rightarrow 0} \frac{u_n}{v_n} = \lim_{n \rightarrow 0} \frac{n+2}{n^3} \times \frac{n^2}{1}$$

$$= \lim_{n \rightarrow 0} \frac{\frac{n+2}{n}}{\frac{n^3}{n^2}} \times \frac{n^2}{n^2}$$

$$= (1+0) = 1.$$

p-test on $v_n = \frac{1}{n^2} \Rightarrow$ convergent

$\Rightarrow u_n$ is convergent.

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Test the convergence of the

series. $\sum \frac{\cos n\pi}{(n^2+1)}$

Sol

$$\frac{\cos n\pi}{n^2+1} = \frac{(-1)^n}{n^2+1}$$

It is an alternating series
To test the alternating series
we apply Leibnitz test.

$$\lim_{n \rightarrow 0} u_n = \frac{1}{n^2+1} = \lim_{n \rightarrow 0} \frac{\frac{1}{n^2}}{n^2+1} = 0$$

$$u_n > u_{n+1}$$

$$\Rightarrow u_n - u_{n+1} \neq 0$$

$$\frac{1}{n^2 + 1} - \frac{1}{(n+1)^2 + 1}$$

$$= \frac{[(n+1)^2 + 1] - n^2 + 1}{(n^2 + 1)[(n+1)^2 + 1]} \neq 0$$

$\Rightarrow u_n$ is convergent.

Test the convergence of the series $\sum \frac{n! 2^n}{n^n}$

So?

$$u_n = \frac{n! 2^n}{n^n}, \quad u_{n+1} = \frac{(n+1)! 2^{n+1}}{(n+1)^{n+1}}$$

$$= \frac{u_n}{u_{n+1}} = \frac{n! 2^n}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)! 2^{n+1}}$$

$$(n+1)! = (n+1)[n+1-1]!$$

\Rightarrow

$$\frac{n! 2^n}{n^n} \times \frac{(n+1)^n (n+1)}{(n+1) n! 2^n \cdot 2}$$

$$= \left(\frac{n+1}{n}\right)^5 = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{n}\right]^n = e$$

$e > 1 \Rightarrow$ Convergent

$$e = 2.7182$$

150 Test for convergence of
the series $\sum_{n=1}^{\infty} \left(\frac{2^n + 3}{3^n + 1} \right)^{1/2}$

Solⁿ $u_n = \left\{ \frac{n(2^n + 3)}{2^n} \times \left(\frac{3^n + 1}{3^n} \right)^{-1} \right\}^{1/2}$

$$u_n = \left(\frac{1 + \frac{3}{2^n}}{1 + \frac{1}{3^n}} \right)^{1/2} \left(\frac{2^n}{3^n} \right)^{1/2}$$

$$v_n = \left(\frac{2}{3} \right)^{n/2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \left(\frac{1+0}{1+0} \right)^{1/2} \times \left(\frac{2}{3} \right)^{n/2} \times \left(\frac{3}{2} \right)^{n/2} \\ = 1$$

$$v_n = \left(\frac{2}{3} \right)^{n/2}, \quad r = \sqrt{\frac{2}{3}} < 1$$

geometric series convergent

L A Q

19 Test for convergence of the series $\sum \sqrt{n^4+1} - n^2$

Ans

$$U_n = \frac{\sqrt{n^4+1} - n^2}{\sqrt{n^4+1} + n^2} \times \frac{\sqrt{n^4+1} + n^2}{\sqrt{n^4+1} + n^2}$$

$$= \frac{(\sqrt{n^4+1})^2 - (n^2)^2}{\sqrt{n^4+1} + n^2} = \frac{1}{\sqrt{n^4+1} + n^2}$$

$$V_n = \frac{1}{n^2}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{U_n}{V_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{\sqrt{n^4+1} + n^2} \times \frac{n^2}{1}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{n^2}{n^2}}{\sqrt{\frac{n^4+1}{n^4} + \frac{n^2}{n^2}}} = \frac{1}{\sqrt{1+0+1}} = \frac{1}{2}$$

p-test on $V_n = \frac{1}{n^2}$

$p=2 > 1 \Rightarrow$ convergent

\Rightarrow by comparison test

U_n is convergent

29 Test the convergence of the series

$$\frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16} + \dots$$

$$\text{So } u_n = \frac{1}{(3n+1)(3n+4)(3n+7)} \quad \left| \begin{array}{l} 9+(n-1)d \\ 4+(n-1)3 \\ 4+3n-3 \\ 3n+1 \end{array} \right.$$

Power of n in Nr & Dr is different, we apply

Comparison test

$$v_n = \frac{1}{n^{3-0}} = \frac{1}{n^3} \quad \left| \begin{array}{l} 3n+4 \\ 10, 13, 16, \dots \\ 10+(n-1)3 \end{array} \right.$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{(3n+1)(3n+4)(3n+7)} \times \frac{n^3}{1} \quad \left| \begin{array}{l} 10+3n-3 \\ 3n+7 \end{array} \right.$$

$$\lim_{n \rightarrow \infty} \frac{n^3 / n^3}{\left(\frac{3n+1}{n} \right) \left(\frac{3n+4}{n} \right) \left(\frac{3n+7}{n} \right)}$$

$$= \frac{1}{(3+0)(3+0)(3+0)} = \frac{1}{27}$$

$$p\text{-test on } v_n = \frac{1}{n^3}$$

$\Rightarrow p = 3 > 1 \Rightarrow$ Convergent
 u_n is convergent.

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Test for convergence of the series

$$\frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$$

so/n

$$U_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2 \cdot 4 \cdot 6 \cdots 2n}$$

$$U_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)[2(n+1)-1]x^{n+1}}{2 \cdot 4 \cdot 6 \cdots 2n \cdot 2(n+1)}$$

$$U_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)[2n+1]x^{n+1} \cdot x}{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2n+2)}$$

$$\frac{U_n}{U_{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2 \cdot 4 \cdots 2n} \times \frac{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2n+2)}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)x^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \left(\frac{2n+2}{2n+1} \right) \frac{1}{x}$$

$$\lim_{n \rightarrow 0} \frac{U_n}{U_{n+1}} = \lim_{n \rightarrow 0} \frac{1}{x} \left[\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}} \right]$$

$$= \lim_{n \rightarrow 0} \frac{1}{x} \left[\frac{2+\frac{2}{n}}{2+\frac{1}{n}} \right]$$

$$\lim_{n \rightarrow 0} \left[\frac{1}{x} \left[\frac{2+\frac{2}{n}}{2+\frac{1}{n}} \right]^n \right] = \frac{1}{x} \left[\frac{2+0}{2+0} \right]$$

$\frac{1}{x} > 1 \Rightarrow$ convergent

$\frac{1}{x} < 1 \Rightarrow$ divergent

$\frac{1}{x} = 1 \Rightarrow$ test fails.

To solve further put $x = 1$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{n \rightarrow 0} \frac{1}{1} \left(\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}} \right) \Rightarrow \lim_{n \rightarrow 0} \left[\frac{2 + \frac{2}{n}}{2 + \frac{1}{n}} \right] = 1.$$

test fails.

Apply Raabe's test.

$$\lim_{n \rightarrow 0} n \left[\frac{u_n}{u_{n+1}} - 1 \right]$$

$$\lim_{n \rightarrow 0} n \left[\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}} - 1 \right]$$

$$\lim_{n \rightarrow 0} n \left[\frac{(2n+2) - (2n+1)}{2n+1} \right]$$

$$\lim_{n \rightarrow 0} n \left[\frac{1}{2n+1} \right] \Rightarrow \lim_{n \rightarrow 0} \frac{n}{n} \left[\frac{1}{2n+1} \right]$$

$$= \frac{1}{2} < 1$$

\Rightarrow divergent

[4Q] Test for convergence of the series. $\sum \frac{4 \cdot 7 \cdots (3n+1)x^n}{1 \cdot 2 \cdots n}$

$$\boxed{\text{Sol'n}} \quad u_n = \frac{4 \cdot 7 \cdots (3n+1)x^n}{1 \cdot 2 \cdots n}$$

$$u_{n+1} = \frac{4 \cdot 7 \cdots (3n+1)[3(n+1)+1]x^{n+1}}{1 \cdot 2 \cdots n(n+1)}$$

$$u_{n+1} = \frac{4 \cdot 7 \cdots (3n+1)(3n+4)x^n \cdot x}{1 \cdot 2 \cdots n(n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4 \cdot 7 \cdots (3n+1)x^n}{1 \cdot 2 \cdots n} \times \frac{1 \cdot 2 \cdots n(n+1)}{4 \cdot 7 \cdots (3n+1)(3n+4)x^n \cdot x}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)}{(3n+4)x}$$

$$\lim_{n \rightarrow \infty} \frac{1}{x} \left[\frac{\frac{n+1}{n}}{\frac{3n+4}{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{x} \left[\frac{1 + \frac{1}{n}}{3 + \frac{4}{n}} \right]$$

$$= \frac{1}{3x} > 1 \Rightarrow \text{convergent}$$

$$= \frac{1}{3x} < 1 \Rightarrow \text{divergent}$$

$$= \frac{1}{3x} = 1 \Rightarrow \text{test fail}$$

To solve further

$$\frac{1}{3x} = 1 \Rightarrow 3x = 1$$

$$x = \frac{1}{3}$$

Put $x = \frac{1}{3}$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{\frac{1}{n} \rightarrow 0} \left(\frac{\frac{n+1}{n}}{\frac{3n+4}{n}} \right) \frac{1}{\sqrt[3]{3}} = 3 \lim_{\frac{1}{n} \rightarrow 0} \left[\frac{1 + \frac{1}{n}}{3 + \frac{4}{n}} \right]$$

$$= 3 \left[\frac{1+0}{3+0} \right] = 1. \text{ test fails.}$$

Apply Rabee's.

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{3(n+1)}{3n+4} - 1 \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{3n+3-3n-4}{3n+4} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{n}{n} \left[\frac{-1}{\frac{3n+4}{n}} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \left[\frac{-1}{3 + \frac{4}{n}} \right] = -\frac{1}{3} < 1$$

$\Rightarrow u_n$ is divergent

[50] find whether the following series is convergent or divergent $\sum_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^{n+1}}$

So $U_n = \frac{[(n+1)x]^n}{n^n}$

$$(U_n)^{\frac{1}{n}} = \left(\frac{[(n+1)x]^n}{n^{n+1}} \right)^{\frac{1}{n}}$$

$$(U_n)^{\frac{1}{n}} = \frac{(n+1)x}{n^{1+\frac{1}{n}}} = \frac{(n+1)x}{n \cdot n^{\frac{1}{n}}}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \left[\frac{n+1}{n} \right] x \cdot \left(\frac{1}{n} \right)^{\frac{1}{n}} \quad \lim_{\frac{1}{n} \rightarrow 0} \left(\frac{1}{n} \right)^{\frac{1}{n}} = 1$$

$$\lim_{\frac{1}{n} \rightarrow 0} \left[1 + \frac{1}{n} \right] x \cdot 1 \\ = x > 1 \Rightarrow \text{divergent} \\ x < 1 \Rightarrow \text{convergent}$$

To solve further put $x=1$ in

$$U_n,$$

$$U_n = \left[\frac{(n+1)1}{n^{n+1}} \right]^n$$

Power of n^2 in $N_n \in D_r$ is different we apply comparison test-

$$V_n = \frac{1}{\frac{n+1-n}{n}} = \frac{1}{n}$$

$$\begin{aligned} \lim_{\frac{1}{n} \rightarrow 0} \frac{U_n}{V_n} &= \lim_{\frac{1}{n} \rightarrow 0} \left[\frac{(n+1)^n}{n^n \cdot n} \right] \\ &= \lim_{\frac{1}{n} \rightarrow 0} \left[\frac{n+1}{n} \right]^n \\ &= \lim_{\frac{1}{n} \rightarrow 0} \left(1 + \frac{1}{n} \right)^n = e = \text{finite} \end{aligned}$$

P-test on $V_n = \frac{1}{n}$

$p=1 \Rightarrow V_n$ is convergent

\Rightarrow by comparison test U_n is also convergent.

69 Test for convergence of the

series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n 3^n}$

(Sol)

$$U_n = \frac{x^{n-1}}{n 3^n}, \quad U_{n+1} = \frac{x^{n+1-1}}{(n+1) 3^{n+1}}$$

$$U_{n+1} = \frac{x^n}{(n+1) 3^n \cdot 3}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{3^n n} \times \frac{(n+1)3^n \cdot 3}{x^n}$$

$$= \frac{x^{n-1-n}}{n} \underbrace{3(n+1)}_n$$

$$= \frac{3}{x} \left[\frac{n+1}{n} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{3}{x} \left[1 + \frac{1}{n} \right] = \frac{3}{x} [1+0].$$

$\frac{3}{x} > 1$ convergent

$\frac{3}{x} < 1 \Rightarrow$ divergent

$\frac{3}{x} = 1 \Rightarrow$ test fails

To solve further $\frac{3}{x} = 1$

$x = 3 \cdot$ put in u_n .

$$u_n = \frac{x^{n-1}}{3^n \cdot n} = \frac{3^{n-1}}{3^n \cdot n} = \frac{3^{-1}}{3^n \cdot n}$$

$$= \frac{1}{3n}, \text{ assume } v_n = \frac{1}{n}$$

$$\lim_{n \rightarrow 0} \frac{u_n}{v_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{3n} \times \frac{n}{1} = \frac{1}{3}.$$

P-test on v_n .

$$v_n = \frac{1}{n}, p=1 \Rightarrow \text{divergent}$$

\therefore By comparison test
 u_n is convergent

Test the convergence of the following

70

$$\text{series } \frac{3 \cdot 6 \cdot 9 \dots 3n}{7 \cdot 10 \dots (3n+4)} x^n$$

Solⁿ

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n \cdot (3n+3) x^{n+1}}{7 \cdot 10 \dots (3n+4)(3(n+1)+4)}$$

$$u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots 3n(3n+3) x^n \cdot x}{7 \cdot 10 \dots (3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{3 \cdot 6 \cdot 9 \dots 3n x^n}{7 \cdot 10 \dots (3n+4)} \times \frac{7 \cdot 10 \dots (3n+4)(3n+7)}{3 \cdot 6 \cdot 10 \dots 3n(3n+3) x^n \cdot x}$$

$$= \frac{(3n+7)}{(3n+3)x} \Rightarrow \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{x} \left(\frac{\frac{3n+7}{n}}{\frac{3n+3}{n}} \right)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{3 + \frac{7}{n}}{3 + \frac{3}{n}} \right] = \frac{1}{n} \left[\frac{3+0}{3+0} \right] = \frac{1}{n}$$

$\frac{1}{n} > 1 \Rightarrow$ convergent

$\frac{1}{n} < 1 \Rightarrow$ divergent

$\frac{1}{n} = 1 \Rightarrow$ test fails

put $n=1$ in $\frac{u_n}{u_{n+1}}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{3 + \frac{7}{n}}{3 + \frac{3}{n}} \right] = 1. \text{ test fails}$$

Apply Rabee's test

$$\lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right]$$

$$\lim_{n \rightarrow \infty} n \left[\frac{3n+7}{3n+3} - 1 \right]$$

$$\lim_{n \rightarrow \infty} n \left[\frac{(3n+7) - (3n+3)}{3n+3} \right]$$

$$\lim_{n \rightarrow \infty} n \left[\frac{4}{3n+3} \right] \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{n} \left[\frac{\frac{4}{n}}{\frac{3n+3}{n}} \right]$$

$$\frac{4}{3+0} = \frac{4}{3} > 1 \Rightarrow \text{convergent}$$

89 Examine the absolute or conditional convergence of

$$3\frac{1}{2} - 4\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^4 + \dots$$

Sol

$$U_n = \frac{n+2}{2^n} \quad \frac{a+(n-1)d}{3+(n-1)i}$$

it is an alternating series.

$$U_{n+1} = \frac{n+3}{2^{n+1}}$$

$$\lim_{\frac{1}{n} \rightarrow 0} (-1)^{n-1} \frac{(n+2)}{2^n} = 0$$

$$U_n > U_{n+1}$$

$$\Rightarrow U_n - U_{n+1} \neq 0$$

$$\frac{n+2}{2^n} - \frac{n+3}{2^{n+1}} \neq 0$$

\Rightarrow convergent

$$\left| \frac{u_n}{u_{n+1}} \right| = \frac{n+2}{2^n} \times \frac{2^n \cdot 2}{(n+3)}$$

$$\lim_{\frac{1}{n} \rightarrow 0} 2 \left[\frac{n+2}{n+3} \right] = \lim_{\frac{1}{n} \rightarrow 0} 2 \left[\frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{3}{n}} \right]$$

$$= \lim_{\frac{1}{n} \rightarrow 0} 2 \left[\frac{1 + \frac{2}{n}}{1 + \frac{3}{n}} \right] = 2$$

$2 > 1 \Rightarrow$ convergent

u_n is convergent

$|u_n|$ is also convergent

absolute convergent.

Q8 Test the convergence of
the series $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \dots$

Sol It is an alternating series

$$u_n = (-1)^n \frac{x^n}{1+x^n}, \quad u_n = \frac{x^n}{1+x^n}$$

$$u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\begin{aligned}
 U_n - U_{n+1} &= \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}} \\
 &= \frac{x^n(1+x^{n+1}) - (1+x^n)(x^{n+1})}{(1+x^n)(1+x^{n+1})} \\
 &= \frac{x^n + x^{2n+n} - x^{n+1} - x^{2n+1}}{(1+x^n)(1+x^{n+1})} \neq 0
 \end{aligned}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{x^n}{1+x^n} = 0$$

—

\Rightarrow convergent

Ques. Discuss the convergence of

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$$

$$\begin{aligned}
 U_n &= \frac{x^{2n-2}}{(n+1)\sqrt{n}}, \quad \frac{a+(n-1)d}{0+(n-1)2} \\
 &\quad = 2n-2
 \end{aligned}$$

$$U_{n+1} = \frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{1}{x^2} \left[\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{x^2} \left[\frac{\frac{n+2}{n} \sqrt{\frac{n+1}{n}}}{\left(\frac{n+1}{n}\right) \sqrt{\frac{n}{n}}} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{x^2} \left[\frac{\left(1 + \frac{2}{n}\right) \sqrt{1 + \frac{1}{n}}}{\left(1 + \frac{1}{n}\right) (1)} \right]$$

$$= \frac{1}{x^2} \left(\frac{1+0}{1+0} \right) \frac{\sqrt{1+0}}{(1+0)(1)} = \frac{1}{x^2}$$

$\frac{1}{x^2} > 1 \Rightarrow$ Convergent

$\frac{1}{x^2} < 1 \Rightarrow$ Divergent

$\frac{1}{x^2} = 1 \Rightarrow$ Test fails

$\frac{1}{x^2} = 1 \Rightarrow$ To solve further

put $x = 1$ in $\frac{u_n}{u_{n+1}}$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{1} \left[\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \right]$$
$$= \frac{\left(1 + \frac{2}{n}\right) \cancel{\sqrt{1 + \frac{1}{n}}}}{\cancel{\left(1 + \frac{1}{n}\right) \sqrt{n}}} =$$

put $x=1$ in U_n

$$U_n = \frac{1}{1^2} \frac{1}{(n+1)\sqrt{n}},$$

$$\text{assume } V_n = \frac{1}{n\sqrt{n}}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{U_n}{V_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{(n+1)\sqrt{n}} \times n\sqrt{n}$$

$$= \lim_{\frac{1}{n} \rightarrow 0} \left[\frac{\frac{n\sqrt{n}}{n\sqrt{n}}}{\frac{(n+1)\sqrt{n}}{n\sqrt{n}}} \right] = 1.$$

$$V_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

$$\frac{3}{2} > 1 \Rightarrow \text{convergent}$$

\Rightarrow by comparison U_n is also
convergent

[119] Test for convergence of the

$$\text{series } \sum \frac{1 \cdot 2 \cdot 3 \cdots n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

Sol: $U_{n+1} = \frac{1 \cdot 2 \cdot 3 \cdots n(n+1)}{3 \cdot 5 \cdot 7 \cdots (2n+1)[2(n+1)+1]}$

$$\frac{U_n}{U_{n+1}} = \frac{1 \cdot 3 \cdots n}{3 \cdot 5 \cdots (2n+1)} \times \frac{3 \cdot 5 \cdots (2n+1)(2n+3)}{1 \cdot 2 \cdots n(n+1)}$$

$$= \frac{2n+3}{n+1}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{2n+3}{n}}{\frac{n+1}{n}} = \frac{2 + \frac{3}{n}}{1 + \frac{1}{n}} = 2$$

$$\frac{U_n}{U_{n+1}} = 2 > 1 \Rightarrow \text{convergent}$$

[120] Test the convergence of the series. $\sum_{n=1}^{\infty} (-1)^{n+1} [\sqrt{n+1} - \sqrt{n}]$

$$U_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$U_n = \frac{(\sqrt{n+1} - \sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}}.$$

$$U_n = \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$V_n = \frac{1}{\sqrt{n}}.$$

$$\frac{U_n}{V_n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \times \frac{\sqrt{n}}{1}$$

$$\lim_{n \rightarrow \infty} \left[\frac{\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{\frac{n+1}{n}} + \sqrt{\frac{n}{n}}} \right] \\ = \frac{1}{\sqrt{1+0} + \sqrt{1}} = \frac{1}{2}$$

$$V_n = \frac{1}{\sqrt{n}} \quad P = \frac{1}{2}$$

divergent
by C.T

$$|u_n| = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

is convergent

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

$$u_n - u_{n+1} \neq 0$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{\sqrt{n+2} + \sqrt{n+1}} \neq 0$$

$$\frac{\sqrt{n+2} + \sqrt{n+1} - \sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+2} + \sqrt{n+1})} \neq 0$$

$\Rightarrow u_n$ is absolute convergent

130 Examine the following series
for absolute or conditional convergence

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

$$U_n = \frac{n}{n^2 + 1}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{n}{n^2}}{\frac{n^2 + 1}{n^2}} = \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = 0$$

$$U_n - U_{n+1} \neq 0$$

$$\frac{n}{n^2 + 1} - \frac{n+1}{(n+1)^2 + 1} \neq 0$$

$$\frac{n(n^2 + 2n + 2) - (n+1)(n^2 + 1)}{(n^2 + 1)(n^2 + 2n + 2)} \neq 0$$

U_n is convergent

$$|U_n| = \frac{n}{n^2 + 1}, \quad V_n = \frac{1}{n}$$

$$\frac{U_n}{V_n} = \frac{U_n}{V_n} = \frac{n}{n^2 + 1} \times \frac{n}{1}$$

$$\begin{aligned} \lim_{\frac{1}{n} \rightarrow 0} \frac{n^2}{n^2 + 1} &= \lim_{\frac{1}{n} \rightarrow 0} \frac{\frac{n^2}{n^2}}{\frac{n^2 + 1}{n^2}} \\ &= \frac{1}{1 + \frac{1}{n^2}} = 1 \end{aligned}$$

$$V_n = \frac{1}{n}, \quad p=1$$

$\Rightarrow V_n$ is divergent

$\Rightarrow U_n$ is divergent

$\Rightarrow U_n$ is conditional
convergent

140 Test for convergence of
the series.

$$\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \dots$$

$$U_n = \frac{x^n}{(2n-1)2n}$$

$$U_{n+1} = \frac{x^{n+1}}{[2(n+1)-1][2(n+1)]}$$

$$U_{n+1} = \frac{x^n \cdot x}{(2n+1)(2n+2)}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^n}{(2n-1)(2n)} \times \frac{(2n+1)(2n+2)}{x^n \cdot x}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{\pi} \left[\frac{(2n+1)(2n+2)}{(2n-1)(2n)} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{\pi} \left[\frac{\left(2 + \frac{1}{n}\right) \left(2 + \frac{2}{n}\right)}{\left(2 - \frac{1}{n}\right) \left(\frac{2n}{n}\right)} \right] = \frac{1}{\pi}.$$

test $\frac{1}{\pi} > 1 \Rightarrow$ convergent

$\frac{1}{\pi} < 1 \Rightarrow$ divergent

$\frac{1}{\pi} = 1 \Rightarrow$ test fails

put $\frac{1}{\pi} = 1 \Rightarrow \pi = 1$.

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{1}{1} \left[\frac{2 + \frac{1}{n}}{2 - \frac{1}{n}} \times \frac{2 + \frac{2}{n}}{\frac{2n}{n}} \right] = 1$$

Apply Rabee's

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{u_n}{l u_{n+1}} - 1 \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{(2n+1)(2n+2)}{(2n-1)(2n)} - 1 \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{2n+1(2n+2) - (2n-1)(2n)}{2n-1(2n)} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{4n^2 + 4n + 2n + 2 - 4n^2 + 2n}{4n^2 - 2n} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} n \left[\frac{8n + 2}{4n^2 - 2n} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \frac{n}{n} \left[\frac{\frac{8n+2}{n}}{\frac{4n^2-2n}{n^2}} \right]$$

$$= \frac{8}{4} = 2 > 1.$$

\Rightarrow convergent

150

Test for Convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1} \right)^n$

Sol?

$$\lim_{n \rightarrow \infty} \left[\frac{n}{3n+1} \right]^n$$

$$(u_n)^{1/n} = \left[\frac{n}{3n+1} \right]^{n/n}$$

$$\lim_{\frac{1}{n} \rightarrow 0} \left[\frac{\frac{n}{n}}{\frac{3n+1}{n}} \right]$$

$$\lim_{\frac{1}{n} \rightarrow 0} \left[\frac{1}{3 + \frac{1}{n}} \right] = \frac{1}{3}$$

$\frac{1}{3} < 1 \Rightarrow$ convergent

unit II

SQ

- 19 check whether Rolle's theorem can be applicable for $f(x) = |x|$ in $[-1, 1]$

Soln

$$f(x) = |x|$$

$$f(x) = x, \quad x \geq 0$$

$$\therefore = -x, \quad x < 0$$

$f(x)$ exists $[-1, 1]$. $\lim_{x \rightarrow 0^+} \frac{(-x)}{x} = 0$, $\lim_{x \rightarrow 0^-} \frac{(x)}{x} = 0$
 $\therefore f(x)$ is continuous in $[-1, 1]$

$$f'(x) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{x - 0}{x} = 1$$

$$f'(x) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{-x - 0}{x} = -1$$

$\therefore f'(x)$ not unique

$f(x)$ is not derivable in $(-1, 1)$

Hence the theorem cannot be verified.

20 find a point at which the tangent to the curve $y = \ln x$ is parallel to the chord joining the point $(1, 0)$ and $(e, 1)$.

Soln by Lagrange's mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = \log x, \quad f'(x) = \frac{1}{x}, \quad f'(c) = \frac{1}{c}$$

$$a = 1, \quad b = e$$

$$\frac{1}{c} = \frac{\log(e) - \log(1)}{e - 1} =$$

$$\frac{1}{c} = \frac{1 - 0}{e - 1} \Rightarrow \frac{1}{c} = \frac{1}{e - 1}$$

$$c = e - 1, \quad 1 < (e - 1) < e$$

the point at which the tangent is parallel to the chord is $(e - 1)$

39 find the radius of curvature
for the curve $y = x^2 - 6x + 10$
at (3,1)

Soln

$$y = x^2 - 6x + 10$$

$$\frac{dy}{dx} = 2x - 6 = y'$$

$$= 2(3) - 6 = 0$$

$$y'' = \frac{d^2y}{dx^2} = 2$$

$$\text{radius} = \frac{\left[1 + (y')^2\right]^{3/2}}{y''}$$

$$= \frac{[1+0]^{3/2}}{2} = \frac{1}{2}$$

40 find the point on the curve
 $f(x) = x^2 - 2x$ in $[0, 2]$ which
 the tangent is parallel to x-axis

Soln $f(x) = x^2 - 2x, f'(x) = 2x - 2$

$$f'(c) = 2c - 2$$

$$2c - 2 = 0 \Rightarrow c = 1 \in (0, 2)$$

\therefore The point on the curve at which the tangent is parallel to x-axis

[5g] obtain Taylor series expansion

$$f(x) = \sin x \text{ at } x = \pi/4$$

[Soln] $f(x) = \sin x, f'(x) = \cos x$

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4}, \quad f'\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4}$$
$$= \frac{1}{2}, \quad = \frac{1}{2}$$

$$f''(x) = -\sin x, \quad f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4}$$
$$= -\frac{1}{2}$$

Taylor series

$$f(x) = f\left(\frac{\pi}{4}\right) + (x - \pi/4)f'\left(\frac{\pi}{4}\right) + \frac{(x - \pi/4)^2}{2!} f''\left(\frac{\pi}{4}\right) + \dots$$

$$f(x) = \sin x = \frac{1}{\sqrt{2}} + (x - \pi/4) \frac{1}{\sqrt{2}} + \frac{(x - \pi/4)^2}{2!} (-\frac{1}{\sqrt{2}}) + \dots$$

69 find c of Cauchy's mean value theorem

$f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$
in $[a, b]$, where $0 < a < b$

Soln

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2}c^{-\frac{3}{2}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$-\frac{\frac{-1/2}{c}}{\frac{-3/2}{c}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}}$$

$$-C^{-\frac{1}{2} + \frac{3}{2}} = -\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} \times \sqrt{ab}$$

$$C^{\frac{1}{2}} = \sqrt{ab}$$

$$\sqrt{c} = \sqrt{ab}$$

$$a < \underline{c} < b$$

⑦ verify Cauchy's mean value theorem
 for $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $(1, e)$

$$\underline{\text{Soln}} \quad f'(x) = \frac{1}{x}, \quad g'(x) = -\frac{1}{x^2}$$

$$f'(c) = \frac{1}{c}, \quad g'(c) = -\frac{1}{c^2}$$

$$f(x) = \log x, \quad f(e) = \log e = 1$$

$$f(1) = \log 1 = 0,$$

$$g(x) = \frac{1}{x}, \quad g(e) = \frac{1}{e}$$

$$g(1) = \frac{1}{1} = 1, \quad g(e) = \frac{1}{e}$$

$f(x)$ & $g(x)$ are continuous

and derivable

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{c}}{-\frac{1}{c^2}} = \frac{\frac{1}{e} - 1}{\frac{1}{e}}$$

$$-c = \frac{1-e}{e} = \frac{e}{e-1} = \frac{e}{1-e}$$

$$c = \frac{e}{e-1} \in (1, e)$$

(88) calculate the radius of curvature of $x^2 + y^2 = 16$

Sol)

$$2x + 2yy' = 0$$

$$2(x + yy') = 0$$

$$x = -yy'$$

$$y' = -\frac{y}{x}$$

$$y'' = - \left[\frac{x(y') - y(1)}{x^2} \right]$$

$$y'' = - \left[\frac{x(-\frac{y}{x}) - y}{x^2} \right]$$

$$y'' = - \left[-\frac{2y}{x^2} \right] = \frac{2y}{x^2}$$

$$\rho = \frac{\left[1 + (y')^2 \right]^{3/2}}{y''}$$

$$\rho = \frac{\left[1 + (-\frac{y}{x})^2 \right]^{3/2}}{\frac{2y}{x^2}}$$

$$\rho = \frac{\left[(x^2 + y^2) / x^2 \right]^{3/2}}{2y}$$

$$\rho = \frac{(x^2 + y^2)^{3/2}}{(x^2)^{3/2}} \times \frac{x^2}{2y}$$

$$\rho = \frac{(x^2 + y^2)^{3/2}}{x^3} \cdot \left(\frac{x^2}{2y} \right)$$

$$\rho = \frac{(x^2 + y^2)^{3/2}}{2xy}$$

90 use Taylor series to express
the polynomial $2x^3 + 7x^2 + x - 6$ in
powers of $(x-2)$.

Soln $f(x) = 2x^3 + 7x^2 + x - 6$

$$f(2) = 2(2^3) + 7(2^2) + 2 - 6 \\ = 16 + 28 + 2 - 6 = 40$$

$$f'(x) = 6x^2 + 14x + 1$$

$$f'(2) = 6(2^2) + 14(2) + 1$$

$$f'(2) = 24 + 28 + 1 = 53$$

$$f''(x) = 12x + 14$$

$$f''(2) = 12(2) + 14 = 24 + 14 = 38$$

$$f'''(x) = 12$$

Taylor series

$$\begin{aligned} f(x) &= f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!} + \dots \\ &= 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12) + \dots \end{aligned}$$

10] find the value of \bar{c} by using
generalized mean value theorem

for $f(x) = e^x$, $g(x) = \bar{e}^x$ in $[3, 7]$.

$$\boxed{\text{Soln}} \quad \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{\bar{e}^c} = \frac{e^7 - e^3}{\bar{e}^7 - \bar{e}^3}$$

$$-\frac{c+c}{e} = \frac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}}$$

$$-\frac{2c}{e} = \frac{e^7 - e^3}{\frac{e^3 - e^7}{e^{7+3}}}$$

$$-\frac{2c}{e} = -\left[\frac{e^7 - e^3}{e^7 - e^3} \right] \cdot e^{10}$$

$$2c = 10$$

$$c = 5$$

11Q State Cauchy's mean value theorem

Statement:- Let $f: [a, b] \rightarrow \mathbb{R}$, $g: [a, b] \rightarrow \mathbb{R}$

(i) $f(x)$ and $g(x)$ are continuous $[a, b]$

(ii) $f(x)$ and $g(x)$ are derivable (a, b)

then \exists a point $c \in (a, b)$ \Rightarrow

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

12Q write the expression for Taylor's series and Maclaurin series.

Statement $f: [a, b] \rightarrow \mathbb{R}$ having partial derivatives

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

[13Q] using Rolle's mean value theorem of
 $g(x) = 8x^3 - 6x^2 - 2x + 1$, find the
 value of c defined over $(0, 1)$

Soln $g'(c) = 24c^2 - 12c - 2$

$$g'(c) = 0$$

$$24c^2 - 12c - 2 = 0$$

$$c = \frac{3 \pm \sqrt{21}}{12} = 0.6318 \in (0, 1)$$

$$g(0) = g(1)$$

$$= 8 - 6 - 2 + 1 = 1$$

$g(x)$ is continuous and $g(x)$ is
 derivable.

[14Q] verify Lagrange's mean value theorem
 for $f(x) = (x-1)(x-2)(x-3)$ in $[0, 4]$

$$\text{for } f(x) = (x-1)(x-2)(x-3) \text{ in } [0, 4]$$

Soln $f(0) = (0-1)(0-2)(0-3) = -6$

$$f(4) = (4-1)(4-2)(4-3) = 6$$

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(c) = 3c^2 - 12c + 11$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$3c^2 - 12c + 11 = \frac{6+6}{4-0} = \frac{12}{4} = 3$$

$$3c^2 - 12c + 11 = 3$$

$$3c^2 - 12c + 8 = 0$$

$$c = \frac{12 \pm \sqrt{144-96}}{6} = \frac{6 \pm 2\sqrt{3}}{3} \in (0, 4)$$

====

(150) find the coordinates of the centre of curvature at any point of the rectangular hyperbola $xy = c^2$

Soln $xy = c^2$, diff $y = w.r.t x$
 $xy' + y(1) = 0 \Rightarrow y' = -\frac{y}{x}$

$$y'' = -\left[\frac{xy' - y(1)}{x^2}\right] = -\left[\frac{x(-\frac{y}{x}) - y}{x^2}\right]$$

$$y'' = -\frac{[-2y]}{x^2} = \frac{2y}{x^2}$$

$$X = x - \frac{y'[1 + (y')^2]}{y''},$$

$$X = x - \frac{-\frac{y}{x}[1 + \frac{y^2}{x^2}]}{\frac{2y}{x^2}}$$

$$X = x + \frac{y}{x} \frac{\cancel{[x^2 + y^2/x^2]}}{\cancel{2y/x^2}}$$

$$X = x + \frac{(x^2 + y^2)}{2y/x} \Rightarrow X = \frac{3x^2 + y^2}{2x}$$

$$Y = y + \frac{[1 + (\underline{y'})^2]}{y''}$$

$$Y = y + \frac{[1 + (-\frac{y}{x})^2]}{\frac{2y}{x^2}}$$

$$Y = y + \frac{[x^2 + y^2]/x^2}{2y/x^2}$$

$$Y = \frac{3y^2 + x^2}{2y}$$

(x, Y) is coordinates of
Centre of curvature.

LAQ

Q State and prove Rolle's Theorem.

Statement:- Let $f: [a, b] \rightarrow \mathbb{R}$.

i) $f(x)$ is continuous over $[a, b]$

ii) $f(x)$ is derivable over (a, b)

iii) $f(a) = f(b)$ then \exists a point

$c \in (a, b) \ni f'(c) = 0$

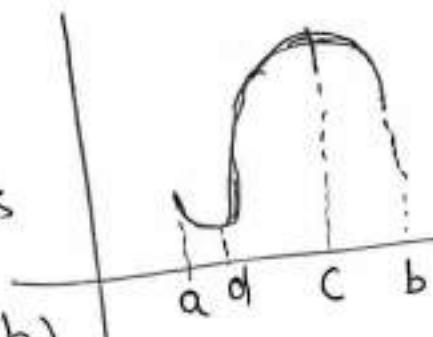
So $f(x)$ is continuous over $[a, b]$

$\Rightarrow f(x)$ is bounded.

\therefore Every continuous function
is bounded.

It attains its

bounds (gLB, LUb)



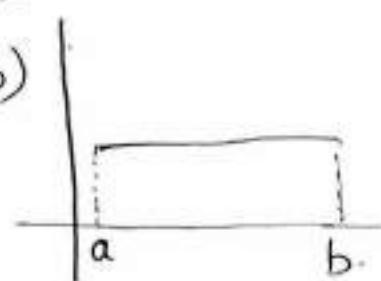
gLB = greatest Lower bound.

$L Ub$ = least upper bound.

case i) if $f(x)$ is constant

\exists a point $c \in (a, b)$

$$f'(c) = 0$$



Case (ii) $f(x)$ is not constant

It has lower bound and upper bound.

Let $f(c) = \text{maximum value} = M$.

$$f(c+h) \leq M \text{ for } h > 0$$

$$f(c+h) - f(c) \leq 0$$

$$M - M \leq 0$$

If $h > 0$

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \leq 0$$

If $h < 0$

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \geq 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = 0$$

$$f'(c) = 0$$

29 If $a < b$ prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

hence deduce

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Sol by Lagrange's mean value theorem we have.

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{--- I}$$

$$\text{let } f(x) = \tan^{-1} x.$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(c) = \frac{1}{1+c^2}$$

Sub in --- I

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

given $a < c < b$.

$$a^2 < c^2 < b^2$$

$$(1+a^2) < (1+c^2) < (1+b^2)$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} .$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+a^2}$$

$$\text{let } a = \frac{\pi}{4}, \quad b = \frac{4}{3}, \quad a=1$$

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} > \tan^{-1} \frac{4}{3} - \tan^{-1} 1 > \frac{\frac{4}{3}-1}{1+1^2}$$

$$\frac{1}{3} \times \frac{9}{25} > \tan^{-1} \frac{4}{3} - \frac{\pi}{4} > \frac{1}{3(2)}$$

$$\frac{3}{25} + \frac{\pi}{4} > \tan^{-1} \frac{4}{3} > \frac{\pi}{4} + \frac{1}{6}$$

30 State and prove Lagrange's Mean Value Theorem

Soln Statement: Let $f: [a, b] \rightarrow \mathbb{R}$

(i) $f(x)$ is continuous over $[a, b]$

(ii) $f(x)$ is derivable over (a, b)

\exists a point $c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof Assume $g(x) = f(x) + Ax$

To check $g(x)$ satisfies all conditions of Rolle's theorem.

(i) $g(x) = f(x) + Ax$

is a polynomial, every polynomial is continuous.

$\therefore g(x)$ is continuous over $[a, b]$.

(ii) $g'(x) = f'(x) + A(1)$

$g'(x)$ exist

$\therefore g(x)$ is derivable over (a, b) .

(iii) $g(a) = g(b)$

$$f(a) + Aa = f(b) + Ab.$$

$$f(b) - f(a) = Aa - Ab$$

$$f(b) - f(a) = -A(b-a)$$

$$\frac{f(b)-f(a)}{b-a} = -A$$

$g(x)$ satisfied all conditions
of Rolle's theorem
 $\therefore \exists$ a point $c \in (a, b) \Rightarrow g'(c) = 0$

$$g'(c) = f'(c) + A = 0$$

$$f'(c) = -A$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

—

40 verify Rolle's theorem for
 $f(x) = e^x \sin x$ in $[0, \pi]$

Sol) $f(x)$ is continuous. $[0, \pi]$

e^x is continuous.

$\sin x$ is continuous.

$$f'(x) = e^x \sin x + e^x \cos x$$

$f'(x)$ exist

$\therefore f'(x)$ is derivable $(0, \pi)$

$$f(0) = f(\pi)$$

$$e^0 \sin 0 = e^\pi \sin \pi \quad \because \sin \pi = 0$$
$$0 = 0$$

$f(x)$ satisfied all conditions
of Rolle's theorem \exists a point

$$c \in (0, \pi) \Rightarrow f'(c) = 0$$

$$e^c \sin c + e^c \cos c = 0$$

$$e^c [\sin c + \cos c] = 0$$

$$\cos c = -\sin c$$

$$\left| \frac{\sin c}{\cos c} \right| = |-\tan c| = 1$$

$$= \tan \frac{\pi}{4} = 1.$$

$$0 < c < \pi \Rightarrow 0 < \frac{\pi}{4} < \pi$$

59 If $f(x) = \log x$ $g(x) = x^2$ in $[a, b]$
 with $b > a > 1$ using Cauchy's
 theorem prove that $\frac{\log b - \log a}{b - a} = \frac{a + b}{2c^2}$

Sol: by Cauchy's we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

$$f'(x) = \frac{1}{x}, \quad g'(x) = 2x$$

$$f'(c) = \frac{1}{c}, \quad g'(c) = 2c$$

$$\frac{\frac{1}{c}}{2c} = \frac{\log b - \log a}{b^2 - a^2}$$

$$\frac{1}{2c^2} = \frac{\log b - \log a}{(b+a)(b-a)}$$

$$\frac{a+b}{2c^2} = \frac{\log b - \log a}{b-a}$$

69 State and prove Cauchy's mean value theorem.

Proof assume $h(x) = f(x) + A g(x)$, $[a, b]$
check whether $h(x)$ satisfy all
conditions of Rolle's theorem.

$h(x)$ is continuous over $[a, b]$

$\therefore h(x)$ is sum of two continuous functions.

$$\Rightarrow h'(x) = f'(x) + A g'(x)$$

$\Rightarrow h(x)$ is derivable.

$$h(a) = h(b)$$

$$f(a) + A g(a) = f(b) + A g(b)$$

$$-A g(b) + A g(a) = f(b) - f(a)$$

$$-A [g(b) - g(a)] = f(b) - f(a)$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = -A$$

$h(x)$ satisfies all conditions of
Rolle's theorem

$\therefore \exists$ a point $c \in (a, b) \ni$
 $h'(c) = 0$

$$h'(c) = f'(c) + A g'(c) = 0$$

$$f'(c) = -A g'(c)$$

$$\frac{f'(c)}{g'(c)} = -A$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Statement: of Cauchy's.

Let $f: [a, b] \rightarrow \mathbb{R}$, $g: [a, b] \rightarrow \mathbb{R}$.

(i) $f(x)$ and $g(x)$ are continuous

over $[a, b]$

(ii) $f(x)$ and $g(x)$ are derivable over (a, b)
then \exists a point $c \in (a, b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

79 Verify Rolle's Theorem for $f(x)$

$$f(x) = \frac{\sin x}{e^x} \text{ in } [0, \pi].$$

Soln (i) $\sin x$ is continuous function $[0, \pi]$
 e^x is continuous function $[0, \pi]$

(ii) $\sin x$ and e^x are differentiable $(0, \pi)$

$$f(0) = \frac{\sin 0}{e^0} = 0, \quad f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$$f'(x) = \frac{e^x \cos x - \sin x e^x}{(e^x)^2}$$

$$= \frac{(\cos x - \sin x)e^x}{(e^x)e^x} = \frac{\cos x - \sin x}{e^x}$$

$$f'(c) = 0 \Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

$$\cos c = \sin c \Rightarrow \frac{\sin c}{\cos c} = 1$$

$$\tan c = 1 = \tan \frac{\pi}{4}$$

$$0 < \frac{\pi}{4} < \pi$$

89 S.T $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \dots$

and hence deduce that

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{2} - \frac{x^3}{48} \dots$$

Soln $\log(1+e^x) = f(x)$

$$\log(1+e^0) = f(0) =$$

$$\log(1+1) = \log 2$$

$$f'(x) = \frac{e^x}{1+e^x}$$

$$f'(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \quad d(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$$

$$f''(0) = \frac{(1+e^0)e^0 - e^0 \cdot e^0}{(1+e^0)^2} = \frac{2-1}{2^2} = \frac{1}{4}$$

$$f''(x) = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x)e^x \cdot e^x}{(1+e^x)^4}$$

$$f'''(0) = \frac{(1+e^0)^2 e^0 - 2(1+e^0)e^0 \cdot e^0}{(1+e^0)^4}$$

$$= \frac{(2)^2 - 2(2)}{(1+1)^4} = 0$$

Mac laurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\log(1+e^x) = \log 2 + x \left(\frac{1}{2} + \frac{x^2}{2} C_4 + \frac{x^3}{3!} C_6 + \dots\right)$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} + \dots$$

Q9 find the eqn of circle of curvature
at the point (x_1, y_1) of the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Sol" given $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0$$

$$\frac{1}{2} \left[\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} y' \right] = 0$$

$$\frac{1}{\sqrt{x}} = -\frac{1}{\sqrt{y}} y'$$

$$y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{a_1}{a_1 u}} = -1$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}, \quad y'' = - \left[\frac{\sqrt{x} \frac{1}{2\sqrt{y}} y' - \frac{1}{2\sqrt{x}} \sqrt{y}}{(\sqrt{x})^2} \right]$$

$$y'' = - \frac{\left[\frac{1}{2} \sqrt{\frac{a_1}{a_1 u}} (-1) - \frac{1}{2} \sqrt{\frac{a_1}{a_1 u}} \right]}{a_1 u}$$

$$= -\left(-\frac{1}{2} - \frac{1}{2}\right) \times \frac{4}{a}$$

$$y'' = \frac{4}{a}$$

Centre of Curvature (x, y)

$$x = x - \frac{y' [1 + (y')^2]}{y''}$$

$$x = \frac{a}{4} - \frac{(-1) [1 + (-1)]^2}{4/a}$$

$$x = \frac{a}{4} + \frac{a}{4}(2) = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$y = y + \frac{[1 + (y')^2]}{y''}$$

$$y = \frac{a}{4} + \frac{[1 + (-1)^2]}{4/a}$$

$$y = \frac{a}{4} + \frac{a}{4}(2) = \frac{3a}{4}$$

$$\rho = \frac{[1 + (y')^2]^{3/2}}{y''} = \text{radius.}$$

$$= \frac{[1 + (-1)^2]^{3/2}}{4/a} = \frac{\frac{a}{4} 2\sqrt{2}}{\frac{a}{2}} = \frac{a}{\sqrt{2}}$$

Eqn of circle of curvature

$$(x - x)^2 + (y - y)^2 = r^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2$$

—

[10Q] find the radius of curvature at
the point $(\frac{3a}{2}, \frac{3a}{2})$ of the curve

$$x^3 + y^3 = 3axy$$

$$\text{Soh given } x^3 + y^3 = 3axy$$

$$3x^2 + 3y^2 y' = 3a(xy' + y(1))$$

$$3[x^2 + y^2 y'] = 3a[xy' + y]$$

$$y^2 y' - a x y' = a y - x^2$$

$$y' = \frac{ay - x^2}{y^2 - ax}$$

$$y' = -\left(\frac{x^2 - ay}{y^2 - ax}\right)$$

$$y' = + \frac{\left[\frac{9a^2}{4} - a\left(\frac{3a}{2}\right) \right]}{a\left(\frac{3a}{2}\right) - \left(\frac{9a^2}{4}\right)}$$

$$y' = - \frac{\left[\frac{9a^2}{4} - a\left(\frac{3a}{2}\right) \right]}{\left[\frac{9a^4}{16} - a\left(\frac{3a}{2}\right) \right]} = -1.$$

$$y'' = - \frac{\left[\underbrace{(y^2 - ax)(2x - ay')}_{} - (x^2 - ay)(2yy' - a) \right]}{(y^2 - ax)^2}$$

$$y'' = - \frac{\left[\cancel{\left(\frac{9a^2}{4} - a\left(\frac{3a}{2}\right) \right)} \left(2\left(\frac{3a}{2}\right) - a(-1) \right) - \cancel{\left(\frac{9a^2}{16} - a\left(\frac{3a}{2}\right) \right)} \left(2\left(\frac{3a}{2}\right)(-1) - a \right) \right]}{\left[\cancel{\left(\frac{9a^2}{4} - a\left(\frac{3a}{2}\right) \right)} \right]^2}$$

$$y'' = - \frac{\left[4a + 4a \right]}{\frac{9a^2}{4} - \frac{3a^2}{2}} = \frac{8a}{\frac{9a^2 - 6a^2}{4}}$$

$$= \frac{32a}{3a^2} = \frac{32}{3a}$$

$$\text{radius } \rho = \frac{[1 + (y')^2]^{3/2}}{y''}$$

$$= \frac{[1 + (-1)^2]^{3/2}}{\frac{32}{3a}}$$

$$= \frac{3a(2\sqrt{2})}{32} = \frac{3a(\sqrt{2})}{16}$$

$$\frac{3a\sqrt{2}}{8\sqrt{2}\sqrt{2}} = \frac{3a}{8\sqrt{2}}$$

≡

[119] S.T the radius of curvature
of the cycloid $x=a(t+\sin t)$

$$y = a(1 - \cos t) \text{ is } 4a \cdot \cos(t/2)$$

solⁿ $x = a(t + \sin t)$, $y = a(1 - \cos t)$

$$\frac{dx}{dt} = a(1 + \cos t) \quad \frac{dy}{dt} = a \sin t$$

$$y' = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{a \sin t}{a(1 + \cos t)}$$

$$y' \frac{a \sin t}{a(1+\cos t)} = \frac{2 \sin t/2 \cos t/2}{2 \cos^2 t/2} = \tan t/2$$

$$y'' = \sec^2 \frac{t}{2} \left(\frac{1}{2}\right) \frac{dt}{dx}$$

$$= \sec^2 \frac{t}{2} \left(\frac{1}{2}\right) \cdot \frac{1}{a(1+\cos t)}$$

$$= \frac{1}{2a} \sec^2 \frac{t}{2} \cdot \frac{1}{2 \cos^2(t/2)}$$

$$\frac{1}{2 \cdot 2a} \sec^4 \frac{t}{2} = \frac{1}{4a} \sec^4 \frac{t}{2}.$$

$$C = \frac{\left[1 + (y')^2\right]^{3/2}}{y''} = \frac{\left[1 + \tan^2 \frac{t}{2}\right]^{3/2}}{\frac{1}{4a} \sec^4 \frac{t}{2}}$$

$$4a \cdot \frac{\left[\sec^2 \frac{t}{2}\right]^{3/2}}{\sec^4 \frac{t}{2}} = 4a \cdot \frac{\sec^3 \frac{t}{2}}{\sec^4 \frac{t}{2}}$$

$$4a \cdot \frac{1}{\sec^2 \frac{t}{2}} = 4a \underline{\underline{(\cos^2 \frac{t}{2})}}$$

129 find the radius of curvature

of $x = at^2, y = 2at$

Soln $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

$$y' = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y'' = -\frac{1}{t^2} \frac{dt}{dx} = -\frac{1}{t^2} \times \frac{1}{2at}$$

$$y'' = \frac{-1}{2at^3}$$

$$r = \left[1 + (y')^2 \right]^{3/2} / y''$$

$$r = \left| \frac{\left[1 + \frac{1}{t^2} \right]^{3/2}}{-\frac{1}{2at^3}} \right|$$

$$r = \left[\left(t^2 + 1 \right) / t^2 \right]^{3/2} \times 2at^3$$

$$r = (t^2 + 1)^{3/2} \cdot 2a$$

[30] S.O.T the radius of curvature
for the rectangular parabola $xy = c^2$
is $\frac{(x^2 + y^2)^{3/2}}{2c^2}$

Soln $xy = c^2$ (given)

$$xy' + y(1) = 0$$

$$y' = -\frac{y}{x}$$

$$y'' = -\left[\frac{xy' - y(1)}{x^2}\right]$$

$$y'' = -\left[\frac{x(-\frac{y}{x}) - y}{x^2}\right]$$

$$y'' = \frac{2y}{x^2}$$

$$\rho = \frac{\left[1 + (y')^2\right]^{3/2}}{y''}$$

$$= \frac{\left[1 + (-\frac{y}{x})^2\right]^{3/2}}{2y/x^2}$$

$$\rho = \frac{[x^2 + y^2]^{3/2}}{(x^2)^{3/2}} \times \frac{x^2}{2y}$$

$$= \frac{[x^2 + y^2]^{3/2}}{x^3} \times \frac{x^2}{2y}$$

$$= \frac{[x^2 + y^2]^{3/2}}{2xy} = \frac{[x^2 + y^2]^{3/2}}{2c^2}$$

(140) find the radius of curvature
of $\gamma = a(1 - \cos\theta)$.

so $\gamma' = a(0 - (-\sin\theta)) = a\sin\theta$

$$\gamma'' = a\cos\theta$$

$$\rho = \frac{[\gamma^2 + (\gamma')^2]^{3/2}}{\gamma^2 + 2\gamma'\gamma'' - \gamma\gamma''}$$

$$\begin{aligned}
 & \left[\gamma^2 + (\gamma')^2 \right]^{3/2} = \left[a^2(1-\cos\theta)^2 + a^2 \sin^2 \theta \right]^{3/2} \\
 &= (a^2)^{3/2} \left[1 + \cos^2 \theta - 2\cos\theta + \sin^2 \theta \right]^{3/2} \\
 &= a^3 [2 - 2\cos\theta]^{3/2} \\
 &= a^3 (2)^{3/2} [1 - \cos\theta]^{3/2} \\
 &= a^3 2\sqrt{2} [2\sin^2 \theta/2]^{3/2} \\
 &= 2\sqrt{2} a^3 2\sqrt{2} \sin^3 \theta/2 \\
 &= 8a^3 \sin^3 \theta/2 \\
 \\
 & \gamma^2 + 2\gamma_1 - \gamma\gamma_2 \\
 &= a^2(1-\cos\theta)^2 + 2(a\sin\theta)^2 - a(1-\cos\theta)(a\cos\theta) \\
 &= a^2 [1 + \cos^2 \theta - 2\cos\theta + 2\sin^2 \theta - \cos\theta + \cos^2 \theta] \\
 &= a^2 [3 - 3\cos\theta] = a^2(3)[1 - \cos\theta] \\
 &= a^2(3)(2\sin^2 \theta/2) \\
 &= 6a^2 \sin^2 \theta/2 \\
 \ell &= \frac{8a^3 \sin^3 \theta/2}{6a^2 \sin^2 \theta/2} = \frac{4}{3}a \sin \theta/2
 \end{aligned}$$

159 find the radius of curvature

for $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$

$$y^2 = \frac{a^2(a-x)}{x}$$

$$xy^2 = a^2(a-x)$$

$$xy^2 = a^3 - a^2x$$

$$x^2yy' + y^2 \cdot 1 = -a^2$$

$$y' = -\frac{a^2 - y^2}{2xy}$$

$$\text{at } (a, 0) = y' = \infty$$

find $\frac{dy}{dx} = \frac{2xy}{-a^2 - y^2} = 0$

$$x'' = -2 \left[\frac{(a^2 + y^2)[x \cdot 1 + y(x') - xy(2y)]}{(a^2 + y^2)^2} \right]$$

$$x'' = -2 \left[\frac{(a^2 + 0)[0 + 0] - 0}{(a^2 + 0)^2} \right]$$

$$x'' = -2 \left[\frac{a^3/a^4}{a^2} \right] = -\frac{2}{a}$$

$$P = \frac{[1 + (x')^2]^{3/2}}{x''} = \left| \frac{(1+0)^{3/2}}{-2/a} \right| = \frac{a}{2}$$

(1)

UNIT-3

① Evaluate the $\lim_{(x,y) \rightarrow (1,1)} \frac{x(y-1)}{y(x-1)}$, if it exists?

$$\text{Ans: } \lim_{x \rightarrow 1} \left[\lim_{y \rightarrow 1} \frac{x(y-1)}{y(x-1)} \right]$$

$$\lim_{x \rightarrow 1} \left[\frac{x(1-1)}{1(x-1)} \right] = 0 \quad -\textcircled{1}$$

$$\begin{aligned} \lim_{y \rightarrow 1} \left[\lim_{x \rightarrow 1} \frac{x(y-1)}{y(x-1)} \right] &= \lim_{y \rightarrow 1} \left[\frac{y-1}{y(1-1)} \right] \\ &= \lim_{y \rightarrow 1} \frac{y-1}{0} \\ &= \infty \quad -\textcircled{2} \end{aligned}$$

$$\therefore \textcircled{1} \neq \textcircled{2}$$

limit does not exist.

② S.T $\lim_{(x,y) \rightarrow (0,0)} \left[\frac{x^3y}{x^6+y^2} \right]$ does not exist?

$$\text{Ans: } \lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} \frac{x^3y}{x^6+y^2} \right]$$

$$\lim_{x \rightarrow 0} \left[\frac{x^3(0)}{x^6+0} \right] = 0 \quad -\textcircled{1}$$

$$\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow 0} \frac{x^3y}{x^6+y^2} \right] = 0 \quad -\textcircled{2}$$

$$\text{Let } y = mx \Rightarrow \lim_{y \rightarrow 0} \left[\lim_{x \rightarrow m0} \frac{x^3y}{x^6+y^2} \right]$$

(2)

$$= \lim_{x \rightarrow 0} \left[\frac{x^3 \cdot m(x)}{x^6 + (mx)^2} \right] = 0 \quad \text{--- (3)}$$

Let $y = mx^2$

$$= \lim_{x \rightarrow 0} \left[\frac{x^3 \cdot mx^2}{x^6 + (mx^2)^2} \right] = 0 \quad \text{--- (4)}$$

Let $y = mx^3$

$$= \lim_{x \rightarrow 0} \left[\frac{x^3 \cdot mx^3}{x^6 + (mx^3)^2} \right] = \lim_{x \rightarrow 0} \frac{m}{x^6 + m^2 x^6}$$

$$= \frac{m}{1+m^2} \quad \text{--- (5)}$$

$$\therefore ① = ② = ③ = ④ + ⑤.$$

\therefore limit does not exist.

3. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ then P.T $xU_x + yU_y = 1$

Sol:

$$u = \log_e\left(\frac{x^2+y^2}{x+y}\right)$$

$$\begin{cases} U_x = \frac{\partial u}{\partial x} \\ U_y = \frac{\partial u}{\partial y} \end{cases}$$

$$e^u = \frac{x^2+y^2}{x+y}$$

which is homogeneous fn of degree "1" in x, y .

By Euler we have.

$$x \frac{\partial}{\partial x}(e^u) + y \frac{\partial}{\partial y}(e^u) = 1 \cdot e^u.$$

$$x e^u \cdot \frac{\partial u}{\partial x} + y e^u \cdot \frac{\partial u}{\partial y} = e^u.$$

$$\cancel{e^u} \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = \cancel{e^u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$$

$$xU_x + yU_y = 1 //.$$

4. Discuss the continuity of the fn $f(x,y) = \frac{(x-y)^2}{x^2+y^2}$,
 $(x,y) \neq (0,0); \quad 0, \quad (x,y) = (0,0).$?

Sol: Given $f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} \frac{(x-y)^2}{x^2+y^2} \right] = \lim_{x \rightarrow 0} \left[\frac{(x-0)^2}{x^2+0} \right] = \lim_{x \rightarrow 0} \left[\frac{x^2}{x^2} \right] = 1$$

$$\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow 0} \frac{(x-y)^2}{x^2+y^2} \right] = \lim_{y \rightarrow 0} \left[\frac{(0-y)^2}{0^2+y^2} \right] = \lim_{y \rightarrow 0} \frac{y^2}{y^2} = 1.$$

\therefore limit of the fn exist & value is unique = 1
 \Rightarrow It is continuous

5. Define homogeneous fn with example?

Sol: If "u" is a H.F of degree 'n' in $x^a y^b$ then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u.$$

e.g. If $u = \tan^{-1} \left(\frac{x^3+y^3}{x-y} \right)$ P.T. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u.$

$$\begin{aligned} \tan u &= \frac{x^3+y^3}{x-y} \\ &= \frac{x^3 \left[1 + \frac{y^3}{x^3} \right]}{x \left[1 - \frac{y}{x} \right]} = x^2 f \left(\frac{y}{x} \right) \end{aligned}$$

is a H.F of degree '2' in $x^a y^b$.

$$\therefore \text{By Euler } x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u.$$

$$x^2 \sec^2 u \frac{\partial u}{\partial x} + y^2 \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u.$$

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If $x+y+z = u$, $y+z = uv$, $x = uvw$. Then
evaluate $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

SOL:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{aligned} x+y+z &= u & y+z &= uv & z &= uvw \\ x+uv &= u & y+uvw &= uv \\ x = u-uv & & y = uv-uvw & \end{aligned}$$

$$\frac{\partial x}{\partial u} = 1-v, \quad \frac{\partial x}{\partial v} = -u, \quad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v-vw, \quad \frac{\partial y}{\partial v} = u-uw, \quad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw, \quad \frac{\partial z}{\partial v} = uw, \quad \frac{\partial z}{\partial w} = uv$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[uv - uv^2vw + uv^2vw] + u[uv^2 - uv^2vw + uv^2vw] + 0$$

$$= (1-v)(uv) + u(uv^2)$$

$$= uv^2 - uv^2 + u^2v^2$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = uv^2,$$

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If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$. Find
 $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Sol:-

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$u = x^2 - 2y$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2, \quad \frac{\partial u}{\partial z} = 0$$

$$v = x + y + z$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial v}{\partial z} = 1$$

$$w = x - 2y + 3z$$

$$\frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = -2, \quad \frac{\partial w}{\partial z} = 3$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) + 0$$

$$= 2x(5) + 2(2)$$

$$= 10x + 4.$$

9 Obtain Taylor's series for $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ at the point $(-1,1)$.

$$f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$$

$$x_0 = -1, \quad y_0 = 1$$

$$x = x_0 + h \quad y = y_0 + k$$

$$x = -1 + h \quad y = 1 + k$$

$$h = x + 1 \quad k = y - 1$$

Taylor's Series:-

$$f(x,y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) + \frac{1}{2!} \left(h \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \dots$$

$$\begin{aligned} f(x_0, y_0) &= 2(-1)^2 - (-1)(1) + (1)^2 + 3(-1) - 4(1) + 1 \\ &= 2 + 1 + 1 - 3 - 4 + 1 \\ &= -2. \end{aligned}$$

$$\frac{\partial f}{\partial x} = 4x - y + 3 \Rightarrow \left(\frac{\partial f}{\partial x} \right)_{(-1,1)} = 4(-1) - (1) + 3 = -4 - 1 + 3 = -2$$

$$\frac{\partial f}{\partial y} = -x + 2y - 4 \Rightarrow \left(\frac{\partial f}{\partial y} \right)_{(-1,1)} = -(-1) + 2(1) - 4 = 1 + 2 - 4 = -1$$

$$\frac{\partial^2 f}{\partial x^2} = 4; \quad \frac{\partial^2 f}{\partial y^2} = 2; \quad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\begin{aligned} f(x,y) &= -2 + ((x+1)(-2) + (y-1)(-1)) + \frac{1}{2!} \left((x+1)^2(4) + 2(x+1) \right. \\ &\quad \left. (y-1)(-1) + (y-1)^2(2) \right) \dots \end{aligned}$$

$$\begin{aligned}
 \Rightarrow f(x,y) &= -2 + [-2x-2-y+1] + \frac{1}{2!} [4(x+1)^2 - 2(x+1)(y+1) + 2(y+1)^2] + \dots \\
 &= -2 + [-2x-y-1] + \frac{1}{2!} [4x^2 + 8x + 4 - 2xy + 2x - 2y + 2 \\
 &\quad + 2y^2 - 4y + 2] + \dots \\
 &\text{---} \\
 &= -2 - [2x+y+1] + \frac{1}{2!} [4x^2 + 2y^2 + 10x - 6y - 2xy + 8] + \dots \\
 &\text{---}
 \end{aligned}$$

10. Write a working rule to find the maximum and minimum values of $f(x,y)$

Sol ① Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate it to zero to get two equations

$\frac{\partial f}{\partial x} = 0 \quad \text{--- } ① \quad \frac{\partial f}{\partial y} = 0 \quad \text{--- } ②$ solve equation ① + ② and obtain the values of x and y as $(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots$ called as stationary points

② Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$, find r, s, t at the stationary points obtained in the step-1

③ If $rt - s^2 > 0, r > 0$ at (a_1, b_1) then $f(x,y)$ is said to have minimum value at (a_1, b_1) and minimum value is $f(a_1, b_1)$

(6)

- i) If $\lambda t - s^2 > 0$, $t < 0$ at (a_1, b_1) then $f(x, y)$ is said to have maximum value at (a_1, b_1) and maximum value is $f(a_1, b_1)$
- ii) If $\lambda t - s^2 < 0$, no maxima and minima exist and the point is called saddle point
- iii) If $\lambda t - s^2 = 0$, no conclusion can be drawn further investigation is ~~and~~ required.

11 Write about Lagrange's method of multipliers.

Working Rule:-

To find the extreme values for the function $f(x, y, z)$ where x, y, z are not all independent and are connected by a relation $\phi(x, y, z) = 0$

① Form the Langranges function $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$ where λ is called Langrange's multiplier.

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z) \quad \text{①}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \text{③}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \text{④}$$

③ Solving equations ②, ③, ④ and the given relation we get the values of x, y, z . Hence the stationary points

(12) find the maximum and minimum of the function $f(x) = x^5 - 3x^4 + 5$

Sol. $f(x) = x^5 - 3x^4 + 5$

diff w.r.t x

$$f'(x) = 5x^4 - 12x^3$$

$$f'(x) = 0$$

$$5x^4 - 12x^3 = 0$$

$$x^3(5x - 12) = 0$$

$$x^3 = 0, \quad 5x - 12 = 0$$

$$x = 0, \quad x = 12/5$$

$$f''(x) = 20x^3 - 36x^2$$

for $x = 0, f''(x) = 0$

$\therefore x = 0$ no maxima & minima exist

for $x = 12/5$

$$f''(x) = 20 \left(\frac{12}{5}\right)^3 - 36 \left(\frac{12}{5}\right)^2$$

$$f''(x) = 20 \times \frac{12^3}{5^3} - 36 \times \frac{12^2}{5^2}$$

$$f''(x) = 276.48 - 207.36 \Rightarrow 69.12 > 0$$

$\therefore f''(x)$ is minimum at $x = 12/5$, & minimum value is

$$f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^5 - 3\left(\frac{12}{5}\right)^4 + 5$$

$$= -14.9 //.$$

(3) without the maximum and minimum of $x^2 + y^2 + 6x + 12$

Given function $f(x, y) = x^2 + y^2 + 6x + 12$

diff w.r.t x, y

$$\frac{\partial f}{\partial x} = 2x + 6 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 2y \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$2x + 6 = 0 \quad 2y = 0$$

$$x = -\frac{6}{2} \quad 2y = 0$$

$$x = -3 \quad y = 0, \quad \therefore$$

$(-3, 0)$ as the stationary points

$$D = \frac{\partial^2 f}{\partial x^2} = 2$$

$$L = \frac{\partial^2 f}{\partial y^2} = 2$$

$$R = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$D(L-R)^2 = (2)(2) - (0)^2 = 4, \quad r=2 > 0.$$

$D(L-R)^2 > 0$, \therefore the function has minimum value at $(-3, 0)$

$$f(-3, 0) = (-3)^2 + (0)^2 + 6(-3) + 12$$

$$= 9 - 18 + 12$$

$$= 3.$$

\therefore minimum value is 3

(14) Examining for max & min value of the fn

$$f(x, y) = x^2 - 3xy + y^2 + 2x.$$

Sol: $f(x, y) = x^2 - 3xy + y^2 + 2x$

Find x, y.

$$\frac{\partial f}{\partial x} = 2x - 3y + 2 = 0 \quad ; \quad \frac{\partial f}{\partial y} = -3x + 2y = 0$$

$$\Rightarrow 2x - 3y = -2 \quad \textcircled{1} \quad \Rightarrow -3x + 2y = 0 \quad \textcircled{2}$$

solving \textcircled{1} \times 3 & \textcircled{2} \times 2.

$$\begin{array}{r} 6x - 9y = -6 \\ -6x + 4y = 0 \\ \hline -5y = -6 \Rightarrow y = \frac{6}{5} \end{array}$$

$$\text{from } \textcircled{1} \Rightarrow 2x - 3\left(\frac{6}{5}\right) = -2$$

$$\begin{aligned} 2x &= -2 + \frac{18}{5} \\ 2x &= \frac{8}{5} \\ x &= 4/5 \end{aligned}$$

\therefore The stationary pt is $\left(\frac{4}{5}, \frac{6}{5}\right)$

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = -3, \quad t = \frac{\partial^2 f}{\partial y^2} = 2.$$

$$rt - s^2 = (2)(2) - (-3)^2$$

$$= 4 - 9$$

$$= -5 < 0$$

No max & min exist & the point is called saddle pt

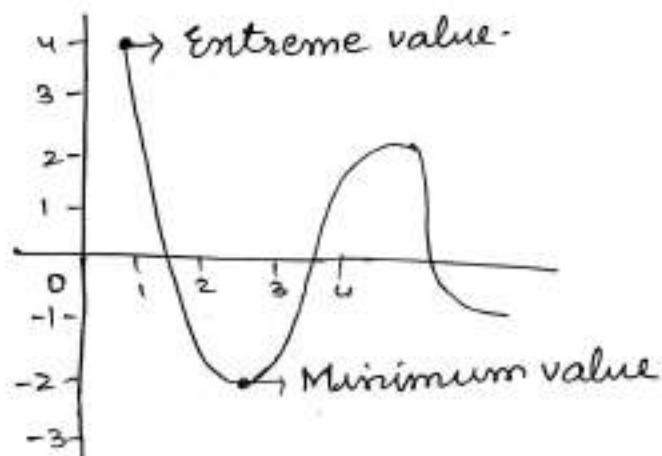
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Define the terms

- (i) Extremum (ii) Saddle point

Ans (i) Extremum:-

An Extremum (or extreme value) of a fn is a point at which a maximum or minimum value of the fn is obtained at some interval.



(ii) Saddle point:- A point where the derivative of the fn is zero but the derivatives does not change sign is known as saddle point.

* Obtain the total derivative of $z = \tan^{-1}\left(\frac{x}{y}\right)$:
 $(x, y) \neq (0, 0)$ where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

Sol:-

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$z = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{\frac{1}{x}}{\frac{y^2+x^2}{y^2}} = \frac{y}{y^2+x^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \left(\frac{-x}{y^2} \right) = \frac{-x}{y^2+x^2} = \frac{-x}{x^2+y^2}$$

$$x = e^t - e^{-t}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}} \cdot x \left(\frac{-1}{y^2} \right)$$

$$\frac{dx}{dt} = e^t + e^{-t} = y$$

$$y = e^t + e^{-t} \Rightarrow \frac{dy}{dt} = e^t - e^{-t} = x$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y^2+x^2} = \frac{-x}{x^2+y^2}$$

$$\frac{dz}{dt} = \frac{y}{x^2+y^2} = \frac{y}{x^2+y^2}(y) + \frac{-x}{x^2+y^2}(x)$$

$$\frac{dz}{dt} = \frac{y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

Properties of Jacobians :-

D) If $J = \frac{\partial(u, v)}{\partial(x, y)}$, $J' = \frac{\partial(x, y)}{\partial(u, v)}$ then $JJ' = 1$

E) If $u = u(x, y)$, $v = v(x, y)$ & $x = f(r, t)$,
 $y = g(r, t)$ then

$$\frac{\partial(u, v)}{\partial(r, t)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, t)}$$

Q** If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, S.T. $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

Sol:-
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$u = \frac{yz}{x}$$

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x}$$

$$v = \frac{zx}{y}$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}$$

$$w = \frac{xy}{z}$$

$$\frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$\begin{aligned} \omega(u, v, w) \\ \omega(x, y, z) &= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= -\frac{yz}{x^2} \left[\frac{x^2}{yz} - \frac{x^2}{yz} \right] - \frac{z}{x} \left[-\frac{x}{z} - \frac{x}{z} \right] + \frac{y}{x} \left[\frac{x}{y} + \frac{x}{y} \right] \\ &= 0 - \frac{z}{x} \left(-\frac{2x}{z} \right) + \frac{y}{x} \left(\frac{2x}{y} \right) \\ &= 2+2 = 4 \end{aligned}$$

3Q) If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = ut^3$.

$$\text{ST } \frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$$

Sof: $\frac{dy}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

$$u = \sin^{-1}(x-y)$$

$$\frac{\partial}{\partial x} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-(x-y)^2}} (+1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}} (-1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1-(x-y)^2} (-1)$$

$$x = 3t, \quad y = 4t^3$$

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 12t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} - \frac{1}{\sqrt{1-(x-y)^2}} 12t^2$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}} = \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3(1-4t^2)}{\sqrt{(1-4t^2)(1-t^2)}}$$

$$= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}} = \frac{3}{\sqrt{1-t^2}}$$

Q4. If $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: If $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}, J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = JJ' = 1$

$$J' = \left| \begin{array}{ccc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{array} \right| ; \quad u = \frac{2yz}{x}$$

$$\frac{\partial u}{\partial x} = -\frac{2yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{2z}{x}, \quad \frac{\partial u}{\partial z} = \frac{2y}{x}$$

$$v = \frac{3zx}{y}$$

$$\frac{\partial v}{\partial x} = \frac{3z}{y}, \quad \frac{\partial v}{\partial y} = -\frac{3zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{3x}{y}$$

$$\omega = \frac{uxy}{z}$$

$$\frac{\partial \omega}{\partial x} = \frac{uy}{z}, \quad \frac{\partial \omega}{\partial y} = \frac{ux}{z}, \quad \frac{\partial \omega}{\partial z} = -\frac{uxy}{z^2}$$

$$\begin{aligned}\frac{\partial(u, v, \omega)}{\partial(x, y, z)} &= \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{z} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix} \\ &= -\frac{2yz}{x^2} \left[\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz} \right] - \frac{2z}{x} \left[-\frac{12xyz}{yz^2} - \frac{12xy}{yz} \right] \\ &\quad + \frac{2y}{z} \left[\frac{12x^2}{yz^2} + \frac{12xyz}{y^2z} \right] \\ &= -\frac{2yz}{x^2} \left(\frac{12x^2}{yz} - \frac{12x^2}{yz} \right) - \frac{2z}{x} \left(-\frac{12x}{z} - \frac{12x}{z} \right) + \frac{2y}{z} \left(\frac{12x}{y} + \frac{12x}{y} \right) \\ &= -\frac{2yz}{x^2}(0) - \frac{2z}{x}(-24x) + \frac{2y}{z}(24x)\end{aligned}$$

$$J' = 2(2u) + 2(2u) = 96$$

$$JJ' = 1 \Rightarrow J = \frac{1}{J'} = \frac{1}{96}$$

$$J = \underline{\underline{\frac{1}{96}}}$$

5
Examine whether the fns are functionally dependent $u = \frac{x+y}{1-xy}$, $v = \frac{\tan^{-1}x + \tan^{-1}y}{x+y}$. If so find the relation between them.

Ans:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

\therefore If $J(u,v) = 0$, then
 u, v is F.D.

If $J(u,v) \neq 0$, then
 u, v is F.I.

$$u = \frac{x+y}{1-xy}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} = \frac{1-xy + xy + y^2}{(1-xy)^2},$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2} = \frac{1-xy + x^2 + xy}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$v = \tan^{-1}x + \tan^{-1}y.$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

\therefore The given fn's are functionally dependent
where $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$v = \tan^{-1}y \text{ or } \underline{\tan v = y}.$$

*6

If $u = x + 3y^2 + z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$
 evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1, -1, 0)$.

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$u = x + 3y^2 + z^3$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 6y, \quad \frac{\partial u}{\partial z} = 3z^2.$$

$$v = ux^2yz$$

$$\frac{\partial v}{\partial x} = 8xyz, \quad \frac{\partial v}{\partial y} = ux^2z, \quad \frac{\partial v}{\partial z} = ux^2y$$

$$w = 2z^2 - xy$$

$$\frac{\partial w}{\partial x} = -y, \quad \frac{\partial w}{\partial y} = -x, \quad \frac{\partial w}{\partial z} = 4z$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & ux^2z & ux^2y \\ -y & -x & 4z \end{vmatrix}$$

$$\left. \frac{\partial(u,v,w)}{\partial(x,y,z)} \right|_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(0 - 0) + 6(0 + 0) + 0$$

$$= -4 + 2(0)$$

$$= \underline{\underline{20}}$$

8(i) Find the first 3 terms of the Taylor's series of $f(x, y) = e^x \cos y$ around $(0, 0)$.

Sol: $f(x, y) = e^x \cos y$, $f(0, 0) = e^0 \cos 0 = 1$

$$f_x = e^x \cos y = e^0 \cos 0 = 1$$

$$f_{xx} = e^x \cos y = e^0 \cos 0 = 1$$

$$f_{xy} = e^x [-\sin y] = e^0 (-\sin 0) = 0$$

$$f_{yy} = -e^x (\cos y) = -1$$

$$f_{xxy} = -e^x [-\sin y] = e^0 (\sin 0) = 0$$

$$f_{xyy} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = \frac{\partial}{\partial x} (-e^x \sin y) \\ = -e^0 \sin y = 0$$

$$f_{xxy} = \frac{\partial}{\partial x} (f_{xy}) = -\frac{\partial}{\partial x} (e^x \sin y) \\ = -(e^x \sin y) = 0$$

$$f_{xxyy} = \frac{\partial}{\partial x} (f_{yy}) = \frac{\partial}{\partial x} (-e^x \cos y) = -1$$

$$e^x \cos y = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) \\ + y^2 f_{yy}(0, 0) + 2xy f_{xy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + \\ 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots$$

$$= 1 + [x(1) + y(0)] + \frac{1}{2!} [x^2(1) + y^2(-1) + 2xy(0)] \\ + \frac{1}{3!} [x^3(1) + y^3(1) + 3x^2y(0) + 3xy^2(-1)] + \dots$$

* * 9

Evaluate the first three terms of the Taylor's series of the fn $f(x, y) = e^x \sin y$ around (0, 0).

Sol:- $f(x, y) = e^x \sin y$ at (0, 0)

$$x_0 = 0, \quad y_0 = 0$$

$$h = x, \quad k = y$$

We have Taylor's Series.

$$f(x, y) = f(0, 0) + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) + \frac{1}{2!} \left(\frac{x^2 \partial^2 f}{2x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + \right)$$

$$+ y^2 \frac{\partial^2 f}{\partial y^2} \Big) + \frac{1}{3!} \left(x^2 \frac{\partial^3 f}{\partial x^3} + 3x^2 y \frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3 f}{\partial x \partial y^2} + y^3 \frac{\partial^3 f}{\partial y^3} \right) + \dots$$

$$f(x, y) = e^x \sin y \Rightarrow f(0, 0) = 0$$

$$\frac{\partial f}{\partial x} = e^x \sin y \Rightarrow \left(\frac{\partial f}{\partial x} \right)_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = e^x \cos y \Rightarrow \left(\frac{\partial f}{\partial y} \right)_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \sin y \Rightarrow \left(\frac{\partial^2 f}{\partial x^2} \right)_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^x \cos y \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \sin y \Rightarrow \left(\frac{\partial^2 f}{\partial y^2} \right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = e^x \sin y \Rightarrow \left(\frac{\partial^3 f}{\partial x^3} \right)_{(0,0)} = 0$$

$$\left(\frac{\partial^3 f}{\partial x^2 \partial y} \right) = e^x \cos y \Rightarrow \left(\frac{\partial^3 f}{\partial x^2 \partial y} \right)_{(0,0)} = 1$$

$$\left(\frac{\partial^3 f}{\partial x \partial y^2} \right) = -e^x \sin y \Rightarrow \left(\frac{\partial^3 f}{\partial x \partial y^2} \right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = -e^x \cos y \Rightarrow \left(\frac{\partial^3 f}{\partial y^3} \right)_{(0,0)} = -1$$

$$\therefore f(x, y) = 0 + [x(0) + y(1)] + \frac{1}{2!} [x^2(0) + 2xy(1) + y^2(0)] \\ + \frac{1}{3!} [x^2(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)] + \dots$$

$$f(x, y) = y + xy + \frac{1}{6} (3x^2y - y^3) + \dots$$

=====

10

- ** Find the stationary point of $u(x,y) = \sin x \sin y \sin(x+y)$ where $0 < x < \pi$, $0 < y < \pi$ & find the max. u.

$$\text{Ans} \quad u(x,y) = \sin x \sin y \sin(x+y)$$

Distr. 'x' & 'y'

$$\frac{\partial u}{\partial x} = \sin y [\sin x \sin(x+y)]$$

$$= \sin y [\sin x \cos(x+y) + \cos x \sin(x+y)]$$

$$= \sin y [\sin(2x+y)]$$

$$\frac{\partial u}{\partial y} = \sin x [\sin y \sin(x+y)]$$

$$= \sin x [\sin y \cos(x+y) + \cos y \sin(x+y)]$$

$$= \sin x \sin(2x+2y)$$

$$\frac{\partial u}{\partial x} = \sin y [\sin(2x+y)] = 0 ; \quad \frac{\partial u}{\partial y} = 0$$

$$\sin y [\sin(2x+y)] = 0 , \quad \sin x \sin(2x+2y) = 0$$

$$y = 0, \pm \pi$$

$$x = 0, \pm \pi$$

$$2x+y = 0$$

$$x+2y = 0$$

$$2x+y = \pm \pi - \textcircled{1}$$

$$x+2y = \pm \pi - \textcircled{2}$$

solving ① $\frac{\partial^2 u}{\partial x^2}$ ②.

$$4x + 2y = 2\pi$$

$$\underline{2x + 2y = \pi}$$

$$3x = \pi$$

$$x = \pi/3$$

$$\text{from } ① \Rightarrow 2(\frac{\pi}{3}) + y = \pi$$

$$y = \pi - \frac{2\pi}{3}$$

$$y = \pi/3$$

$(\frac{\pi}{3}, \frac{\pi}{3})$ $(-\frac{\pi}{3}, -\frac{\pi}{3})$ are the stationary points

$$r = \frac{\partial^2 f}{\partial x^2} = \sin y \cos(2x+y)(2) = \sin(\frac{\pi}{3})[\cos(\frac{2\pi}{3} + \frac{\pi}{3})(2)] \\ = \frac{\sqrt{3}}{2}(-1)(2) = -\sqrt{3}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \sin y \cos(2x+y) + \cos y \sin(2x+y) \\ = \sin(2x+2y) = \sin(\frac{2\pi}{3} + \frac{2\pi}{3}) = -\frac{\sqrt{3}}{2}$$

$$t = \frac{\partial^2 f}{\partial y^2} = \sin y \cos(2x+y)(2) \\ = \sin(\frac{\pi}{3}) \cos(\frac{\pi}{3} + \frac{2\pi}{3})(2) \\ = \frac{\sqrt{3}}{2}(-1)(2) = -\sqrt{3}.$$

$$\therefore rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$$r = -\sqrt{3} < 0$$

$u(x, y)$ is maximum & the max value is

$$u(\frac{\pi}{3}, \frac{\pi}{3}) = \sin \frac{\pi}{3} \cdot \sin \frac{\pi}{3} \sin(\frac{\pi}{3} + \frac{\pi}{3}) \\ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

11Q.

Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Sol: Let (x, y, z) be a point on the surface $xyz^2 = 2$.
 distance from origin $d = \sqrt{x^2 + y^2 + z^2}$ — (1)
 $f = d^2 = x^2 + y^2 + z^2$; $\phi = xyz^2 - 2$

Lagrange's fn:

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

$$F = x^2 + y^2 + z^2 + \lambda(xyz^2 - 2) — (2)$$

Point x, y, z :

$$\frac{\partial F}{\partial x} = 2x + \lambda yz^2, \frac{\partial F}{\partial y} = 2y + \lambda xz^2, \frac{\partial F}{\partial z} = 2z + 2\lambda xy^2$$

$$2x + \lambda yz^2 = 0 — (3), 2y + \lambda xz^2 = 0 — (4), 2z + 2\lambda xy^2 = 0 — (5)$$

$$\text{from (3)} \quad \lambda = -\frac{2x}{yz^2} — (i)$$

$$\text{from (4)} \quad \lambda = -\frac{2y}{xz^2} — (ii)$$

$$\text{from (5)} \quad \lambda = -\frac{2z}{xy^2} \Rightarrow \lambda = -\frac{1}{xy}$$

$$\text{from (i), (ii)} \Rightarrow -\frac{2x}{yz^2} = -\frac{2y}{xz^2} \Rightarrow \frac{x}{y} = \frac{y}{x} \Rightarrow x^2 = y^2$$

$$\text{from (i) & (iii)} \Rightarrow -\frac{2x}{yz^2} = -\frac{1}{xy} \Rightarrow 2y^2 = z^2 \Rightarrow y^2 = \frac{z^2}{2}$$

$$\Rightarrow z^2 = 2y^2$$

$$\Rightarrow z = 2y$$

$$\text{from (1)} \Rightarrow xyz^2 = 2$$

$$(x)(y)(2y^2) = 2$$

$$\cancel{\lambda} x^4 = 2 \Rightarrow x^4 = 1$$

$$\therefore x = 1, y = 1, z = \sqrt{2}$$

\therefore The pt on the surface is $(1, 1, \sqrt{2})$ at the
 shortest distance from origin is $d = \sqrt{1^2 + 1^2 + 2}$
 $= \sqrt{4} = 2$

12
**

Dinars the max or min $f(x, y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$?

Note:

$$f(x, y) = 4x^2 + 2y^2 + 4xy - 16x - 2y - 3$$

Want $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 8x + 4y - 10 = 0, \quad \frac{\partial f}{\partial y} = 4y + 4x - 2 = 0$$
$$4x + 4y = 10, \quad 4x + 4y = 2.$$

solving ① & ②

$$\begin{array}{r} 8x + 4y = 10 \\ 4x + 4y = 2 \\ \hline 4x = 8 \end{array}$$

$$x = 2 \text{ sub in } ①.$$

$$8(2) + 4(y) = 10$$

$$\begin{array}{l} 4y = 10 - 16 \\ 4y = -6 \end{array}$$

$$y = -\frac{3}{2}$$

The stationary pt is $(2, -\frac{3}{2})$

$$\gamma = \frac{\partial^2 f}{\partial x^2} = 8, \quad m = \frac{\partial^2 f}{\partial x \partial y} = 4, \quad t = \frac{\partial^2 f}{\partial y^2} = 4$$

$$\gamma t - m^2 = (8)(4) - 4^2 = 32 - 16 = 16 > 0$$

$$\gamma = 8 > 0$$

$\therefore f(x, y)$ has minimum value at $(2, -\frac{3}{2})$

$$\begin{aligned} \text{min value is } f(2, -\frac{3}{2}) &= 4x^2 + 2y^2 + 4xy - 10x - 2y - 3 \\ &= 4(2)^2 + 2\left(-\frac{3}{2}\right)^2 + 4(2)\left(-\frac{3}{2}\right) - \\ &\quad 10(2) - 2\left(-\frac{3}{2}\right) - 3 \\ &= 4(4) + 2\left(\frac{9}{4}\right) + 4\left(-\frac{6}{2}\right) - 20 + 3 - 3 \\ &= 16 + \frac{9}{2} - 12 - 20 \\ &= 16 + \frac{9}{2} - 32 \\ &= \frac{32 + 9 - 64}{2} = -\frac{23}{2} \end{aligned}$$

$$f(2, -\frac{3}{2}) = -\frac{23}{2} //$$

where λ and μ are the lagrange multipliers
and proceed as above.

13*

Find the maximum and minimum values
of the $x+y+z$ subject to condition

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Ans:

$$f(x, y, z), \quad \phi(x) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

Lagrange's function

$$\begin{aligned} F(x, y, z) &= f(x, y, z) + \lambda \phi(x, y, z) \\ &= x + y + z + \lambda \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \right) \quad \text{--- (1)} \end{aligned}$$

Our x, y, z .

$$\frac{\partial F}{\partial x} = 1 - \frac{\lambda}{x^2} = 0 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial y} = 1 - \frac{\lambda}{y^2} = 0 \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2} = 0 \quad \text{--- (4)}$$

$$\begin{aligned} \text{From (2)} \Rightarrow \frac{\lambda}{x^2} &= 1 \Rightarrow x^2 = \lambda \\ x &= \pm \sqrt{\lambda} \end{aligned}$$

$$\text{From (3)} \Rightarrow \frac{\lambda}{y^2} = 1 \Rightarrow y^2 = \lambda \Rightarrow y = \pm \sqrt{\lambda}$$

$$\text{From (4)} \Rightarrow \frac{\lambda}{z^2} = 1 \Rightarrow z^2 = \lambda \Rightarrow z = \pm \sqrt{\lambda}.$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} = 1$$

$$\frac{3}{\sqrt{\lambda}} = 1 \Rightarrow \lambda = 9.$$

$$x = \pm\sqrt{9}, y = \pm\sqrt{9}, z = \pm\sqrt{9}$$

$$x = \pm 3, y = \pm 3, z = \pm 3.$$

$(3, 3, 3), (-3, -3, -3)$ are the stationary pts

$$\text{at } (3, 3, 3), f(x, y, z) = 3+3+3=9$$

$$\text{at } (-3, -3, -3), f(x, y, z) = -3-3-3=-9$$

$$\text{max value at } (3, 3, 3) = 9$$

$$\text{min value at } (-3, -3, -3) = -9.$$

* Find the min value at $x^2+y^2+z^2$ with the constraint $x+y+z = 3a$.

$$f = x^2+y^2+z^2, \phi = x+y+z - 3a$$

Lagrange's Function

$$\begin{aligned} F(x, y, z) &= f(x, y, z) + \lambda\phi(x, y, z) \\ &= x^2+y^2+z^2 + \lambda(x+y+z-3a) \quad \text{--- (1)} \end{aligned}$$

Diff w.r.t x, y, z .

$$\frac{\partial F}{\partial x} = 2x + \lambda = 0 \Rightarrow 2x = -\lambda \Rightarrow x = -\lambda/2$$

$$\frac{\partial F}{\partial y} = 2y + \lambda = 0 \Rightarrow 2y = -\lambda \Rightarrow y = -\lambda/2$$

$$\frac{\partial F}{\partial z} = 2z + \lambda = 0 \Rightarrow 2z = -\lambda \Rightarrow z = -\lambda/2$$

$$x+y+z = 3a.$$

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a.$$

$$-\frac{3\lambda}{2} = 3a$$

$$\lambda = -2a.$$

$$x = \frac{2a}{2}, \quad y = \frac{2a}{2}, \quad z = \frac{2a}{2}$$

$$x = a, \quad y = a, \quad z = a$$

(a, a, a) are the stationary points

$$\text{at } (a, a, a) = a^2 + a^2 + a^2 = 3a^2$$

∴ Hence the min value is $3a^2$

- 15
** Find the volume of the greatest rectangular parallelopipede that can be inscribed in the ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

- Ans: Let $2x, 2y, 2z$ be the length, breadth & height of the rectangular parallelopipede that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — (1)
then the centroid of the parallelopipede coincide with the centre of the ellipsoid $(0, 0, 0)$ & the corners of the parallelopipede lie on the surface of the ellipsoid

$$\text{volume} = l \times b \times h$$

$$= 8xyz$$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Lagrange's Function

$$F(x, y, z) = f(x, y, z) + \lambda \{\phi(x, y, z)\}$$

$$F = 8xyz + \lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right] \quad \text{--- (2)}$$

Diff w.r.t x, y, z .

$$\frac{\partial F}{\partial x} = 8xyz + \frac{2x\lambda}{a^2} = 0 \Rightarrow 8yz = -\frac{2x\lambda}{a^2} \Rightarrow \lambda = -\frac{4a^2yz}{x}$$

$$\frac{\partial F}{\partial y} = 8xz + \frac{2y\lambda}{b^2} = 0 \Rightarrow \lambda = -\frac{4b^2xz}{y}$$

$$\frac{\partial F}{\partial z} = 8xy + \frac{2z\lambda}{c^2} = 0 \Rightarrow \lambda = -\frac{4c^2xy}{z}$$

Equating

$$-\frac{4a^2yz}{x} = -\frac{4b^2xz}{y}, \quad -\frac{4b^2xz}{y} = -\frac{4c^2xy}{z}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\frac{y^2}{b^2} = \frac{z^2}{c^2}$$

sub in ①

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$\text{Hence } y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$\therefore \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is the stationary point

$$\therefore \text{The max volume } V = 8 \left(\frac{a}{\sqrt{3}}\right) \left(\frac{b}{\sqrt{3}}\right) \left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}}$$

1. Evaluate $\int_0^1 \int_0^3 (x^2 + y^2) dy dx$

Sol: $\int_0^1 \left(\int_0^3 (x^2 + y^2) dy \right) dx$

$$= \int_0^1 \left(\int_0^3 x^2 dy + \int_0^3 y^2 dy \right) dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \Big|_0^3 \right) dx = \int_0^1 \left(3x^2 y + \frac{27}{3} - 0 \right) dx = \int_0^1 (3x^2 + 9) dx$$

$$= \left(3 \frac{x^3}{3} + 9x \right)_0^1 = 1 + 9 = 10 //$$

2. Evaluate $\int_0^{\pi} \int_0^1 x \cos(xy) dy dx$

Sol: $\int_0^{\pi} \int_0^1 x \cos(xy) dy dx$

$$= \int_0^{\pi} \left(\int_0^1 x \cos(xy) dy \right) dx$$

$$\therefore \int \cos ax dx = \frac{\sin ax}{a}$$

$$= \int_0^{\pi} \left(\left(x \times \frac{\sin xy}{x} \right)_0^1 \right) dx$$

$$= \int_0^{\pi} (\sin x - 0) dx = \int_0^{\pi} \sin x dx = (-\cos x)_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= -(-1) + 1$$

$$= 1 + 1$$

$$= 2 //$$

3. Evaluate $\iint x^2 dx dy$ over the region bounded by hyperbola $xy = 4, y = 0,$
 $x = 1, x = 4$

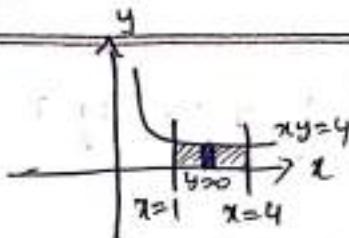
$$\int \int z^2 dx dy$$

$$z \rightarrow 1-4$$

$$y \rightarrow 0-4/x$$

$$= \int_1^4 \int_0^{4/x} z^2 dy dx = \int_1^4 z^2 (y)_0^{4/x} dx = \int_1^4 z^2 \left(\frac{4}{x}\right) dx = 4 \int_1^4 z dx$$

$$= 4 \left(\frac{x^3}{3} \right)_1^4 = 4 \left(\frac{16}{3} - \frac{1}{3} \right) = 4 \left(\frac{15}{3} \right) = 30 //$$



4. Evaluate $\int_0^2 \int_0^x e^{(x+y)} dx dy$

$$\text{Sol: } \int_0^2 \int_0^x e^{(x+y)} dx dy = \int_0^2 \int_0^x e^{x+y} dy dx$$

$$= \int_0^2 e^x (e^y)_0^x dx = \int_0^2 e^x (e^x - e^0) dx = \int_0^2 e^x (e^x - 1) dx$$

$$= \int_0^2 e^{2x} - e^x dx = \left(\frac{e^{2x}}{2} - e^x \right)_0^2 = \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$$

$$= \frac{e^4}{2} - e^2 + \frac{1}{2}$$

$$= \frac{e^4 - 2e^2 + 1}{2}$$

5. Evaluate $\int_0^2 \int_0^1 y dy dx$

$$\begin{aligned} \int_0^2 \int_0^1 y dy dx &= \int_0^2 \left(\frac{y^2}{2} \right)_0^1 dx = \int_0^2 \left(\frac{x^2}{2} \right) dx = \left(\frac{x^3}{6} \right)_0^2 = \frac{2^3}{6} = \frac{8}{6} \\ &= \frac{4}{3} \end{aligned}$$

6. Evaluate $\int_0^2 \int_{\sqrt{x}}^{\sqrt{2x}} (x^2 + y^2) dy dx$

$$\begin{aligned} \text{Soln} &= \int_0^1 \int_{\sqrt{x}}^{\sqrt{2x}} (x^2 + y^2) dy dx \\ &= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_{\sqrt{x}}^{\sqrt{2x}} dx \\ &= \int_0^1 x^2 \sqrt{x} + \frac{x^3/2}{3} - x^3 - \frac{x^3}{3} dx \\ &= \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} dx \\ &= \left(\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{(5/2)3} - \frac{x^4}{12} \right)_0^1 = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{2}{7} + \frac{2-5}{15} \\ &= \frac{2}{7} - \frac{1}{5} = \frac{10-7}{35} = \frac{3}{35} \end{aligned}$$

7. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

$$\text{Soln} \quad \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

$$= \int_0^\infty e^{-y^2} \left\{ \int_0^\infty e^{-x^2} dx \right\} dy.$$

We know $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, using gamma function.

$$\begin{aligned}
 & \Rightarrow \int_0^\infty e^{-y^2} \frac{\sqrt{\pi}}{2} dy = \\
 & = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2} dy \\
 & = \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} \quad \left(\text{If } \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \right) \\
 & = \frac{\pi}{4}.
 \end{aligned}$$

8. Evaluate $\int_0^2 \int_0^{1-x} \int_0^{1-x-y} 6 dz dy dx$

$$\begin{aligned}
 \text{Solt:} \quad & 6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6 dz dy dx \\
 & = 6 \int_0^1 \int_0^{1-x} (z) \Big|_0^{1-x-y} dy dx \\
 & = 6 \int_0^1 \int_0^{1-x} (1-x-y) dy dx \\
 & = 6 \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Big|_0^{1-x} dx = 6 \int_0^1 1-x-y(1-x) - \frac{(1-x)^2}{2} dx \\
 & = 6 \int_0^1 1-x-y + y^2 - \frac{1-2x+x^2}{2} dx = 6 \int_0^1 1-2x+x^2 - \frac{1-2x+x^2}{2} dx \\
 & = 6 \int_0^1 \frac{2-4x+2x^2-1+2x-x^2}{2} dx = \frac{3}{2} \int_0^1 x^2-2x+1 dx = \frac{3}{2} \left(\frac{x^3}{3} - \frac{2x^2}{2} + x \right)_0^1 \\
 & = 3 \left(\frac{1}{3} - 2 + 1 \right) = 1 \Rightarrow \text{Ans}
 \end{aligned}$$

9. Evaluate $\int_0^{\frac{\pi}{6}} \int_0^1 \int_{-2}^3 y \sin z \, dz \, dy \, dx$

Sol: $\int_0^{\frac{\pi}{6}} \int_0^1 \int_{-2}^3 y \sin z \, dz \, dy \, dx$

$$\int_0^{\frac{\pi}{6}} \int_0^1 y \sin z \, (z) \Big|_{-2}^3 \, dy \, dx$$

$$\int_0^{\frac{\pi}{6}} \int_0^1 y \sin z (3+2) \, dy \, dz = 5 \cdot \int_0^{\frac{\pi}{6}} \sin z \left(\frac{y^2}{2} \right) \Big|_0^1 \, dz$$

$$= 5 \int_0^{\frac{\pi}{6}} \sin z \left(\frac{1}{2} \right) \, dz = \frac{5}{2} \int_0^{\frac{\pi}{6}} \sin z \, dz = \frac{5}{2} (-\cos z) \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{5}{2} \left(-\cos \frac{\pi}{6} + \cos 0 \right) = \frac{5}{2} \left(-\frac{\sqrt{3}}{2} + 1 \right) = \frac{5}{2} \left(\frac{-\sqrt{3} + 2}{2} \right)$$

$$= \frac{5}{2} \left(\frac{2-\sqrt{3}}{2} \right) = \frac{5}{4} (2-\sqrt{3})$$

Evaluate $\iint xy \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$

10.

Sol:

$$x^2 + y^2 = a^2$$

$$C(0,0) \quad r = a$$

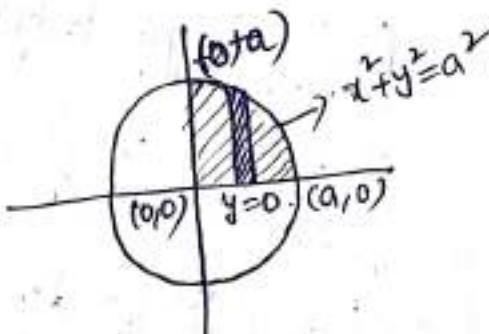
$$x \rightarrow 0-a$$

$$y \rightarrow 0 - \sqrt{a^2 - x^2}$$

area: $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dy \, dx$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} my \, dy \, dx = \int_0^a x \left(\frac{y^2}{2} \right) \Big|_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \int_0^a x \left(\frac{a^2-x^2}{2} \right) \, dx = \int_0^a x \left(\frac{a^2-x^2}{2} \right) \, dx$$



$$\begin{aligned}
 &= \int_0^a \frac{2x^2 - x^3}{2} dx \\
 &= \frac{1}{2} \int_0^a x^2 - x^3 dx \\
 &= \frac{1}{2} \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^a \\
 &= \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) \\
 &= \frac{1}{2} \left(\frac{a^4}{4} \right) \\
 &= \frac{a^4}{8}.
 \end{aligned}$$

ii) Evaluate $I = \int_0^1 \int_0^2 \int_0^3 xyz dz dy dx$.

Sol:

$$\begin{aligned}
 &\int_0^1 \int_0^2 \int_0^3 xyz dz dy dx \\
 &\int_0^1 \int_0^2 yz \left(\frac{x^2}{2} \right)_0^3 dy dz = \int_0^1 \int_0^2 yz \left(\frac{9}{2} \right) dy dz \\
 &\frac{9}{2} \int_0^1 z \left(\frac{y^2}{2} \right)_0^3 dy dz \\
 &= \frac{9}{2} \int_0^1 z \left(\frac{9}{2} \right) dz \\
 &= \frac{9}{2} \times z \int_0^1 \frac{9}{2} dz = 9 \left(\frac{z^2}{2} \right)_0^1 = 9 \left(\frac{1}{2} \right) = \frac{9}{2}.
 \end{aligned}$$

12. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2) dy dx$

Sol: $= \int_0^5 \left(\int_0^{x^2} x(x^2+y^2) dy \right) dx$

$$= \int_0^5 \left(\cdot \int_0^{x^2} x^3 + xy^2 dy \right) dx$$

$$= \int_0^5 \left(x^3 y + \frac{xy^3}{3} \right)_0^{x^2} dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3} - 0 \right) dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^5 = \frac{5^6}{6} + \frac{5^8}{24} = \frac{5^6 \times 4 + 5^8}{24} = \frac{5^6(4+25)}{24}$$

$$= 5^6 \left(\frac{29}{24} \right)$$

13. Evaluate $\int_0^4 \int_{y^2/4}^y \frac{y}{(x^2+y^2)} dx dy$

Sol: $= \int_0^4 \left(\int_{y^2/4}^y \left(\tan^{-1} \frac{x}{y} \right)_{y^2/4}^y \right) dy$

$$= \int_0^4 \left(\tan^{-1}(a) - \tan^{-1}\left(\frac{y^2/4}{y}\right) \right) dy$$

$$= \int_0^4 \left(\frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right) \right) dy \rightarrow ①$$

Consider $\int \tan^{-1}\left(\frac{y}{4}\right) dy$

$$\tan^{-1}\left(\frac{y}{4}\right) \int 1 dy - \int \left(\frac{d}{dy} \tan^{-1}\left(\frac{y}{4}\right) \int 1 dy \right) dy$$

$$= \tan^{-1}\left(\frac{y}{4}\right) y - \int \frac{1}{1 + (\frac{y}{4})^2} \frac{y}{4} dy$$

$$= y \tan^{-1}\left(\frac{y}{4}\right) - \int \frac{y}{16+y^2} dy$$

$$= y \tan^{-1}\left(\frac{y}{4}\right) - 2 \int \frac{dy}{16+y^2}$$

$$= y \tan^{-1}\left(\frac{y}{4}\right) - 2 \log(16+y^2)$$

Eq (1) becomes

$$= \int_0^4 \frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right) dy$$

$$= \left[\frac{\pi}{4} y - y \tan^{-1}\left(\frac{y}{4}\right) + 2 \log(16+y^2) \right]_0^4$$

$$= \frac{\pi}{4} (4) - 4 \tan^{-1}(1) + 2 \log 32 - 2 \log 16$$

$$= \cancel{\frac{\pi}{4}} - \cancel{\pi} + 2 \log 32 - 2 \log 16$$

$$= 2 \left(\log \frac{32}{16} \right) = 2 \log 2 = \log 4 //$$

14. Evaluate $\iint (x^2+y^2) dx dy$ in the positive quadrant for which $x+y \leq 1$

Soln

$$x \rightarrow 0 \cdot (1-y)$$

$$y \rightarrow 0 - 1$$

$$= \int_0^1 \int_0^{(1-y)} (x^2+y^2) dx dy$$

$$= \int_0^1 \int_0^{(1-y)} (x^2+y^2) dx dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + y^2 x \right) \Big|_0^{(1-y)} dy$$

$$= \int_0^1 \left(\frac{(1-y)^3}{3} + y^2 (1-y) \right) dy$$

$$= \int_0^1 \frac{1 - 3(1)^2(y) + 3(1)(y^2) - y^3}{3} + y^2 - y^3 dy$$

$$= \int_0^1 \frac{1 - 3y + 3y^2 - y^3}{3} + y^2 - y^3 dy$$

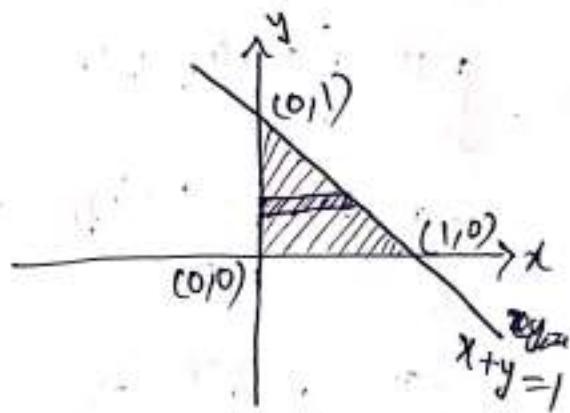
$$= \int_0^1 \frac{(1 - 3y + 3y^2 - y^3)}{3} + y^2 - y^3 dy$$

$$= \int_0^1 \frac{(1 - 3y + 3y^2 - y^3 + 3y^2 - 3y^3)}{3} dy$$

$$= \frac{1}{3} \int_0^1 (1 - 3y + 6y^2 - 4y^3) dy$$

$$= \frac{1}{3} \left(y - \frac{3y^2}{2} + \frac{6y^3}{3} + \frac{0y^4}{4} - \frac{4y^4}{4} \right) \Big|_0^1$$

$$= \frac{1}{3} \left(1 - \frac{3}{2} + 2 - 1 \right) = \frac{1}{3} \left(\frac{1}{2} \right) = \frac{1}{6}$$



$$15. \text{ Evaluate } \int_0^{\pi} \int_0^{a\sin\theta} r dr d\theta$$

$$= \int_0^{\pi} \int_0^{a\sin\theta} r dr d\theta$$

$$= \int_0^{\pi} \left(\frac{r^2}{2} \right)_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{a^2 \sin^2 \theta}{2} d\theta$$

$$\frac{a^2}{2} \int_0^{\pi} \sin^2 \theta d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \sin^2 \theta d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{4} \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{\pi}$$

$$= \frac{a^2}{4} (\pi - 0)$$

$$= \frac{\pi a^2}{4}$$

LAQ's

1. Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta dr d\theta$

Soln $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta dr d\theta$

$$\int_0^{\pi} \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} \cos\theta d\theta$$

$$= \int_0^{\pi} \frac{a^3(1+\cos\theta)^3}{3} \cos\theta d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} (1+3\cos\theta + 3\cos^2\theta + \cos^3\theta) \cos\theta d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \cos\theta + 3\left(\frac{1+\cos 2\theta}{2}\right) + \frac{3}{4} (\cos 3\theta + 3\cos\theta) + \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta + \frac{3}{4}\cos 3\theta + \frac{9}{4}\cos\theta + \frac{1}{4}(1+2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \frac{13}{4}\cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta + \frac{3}{4}\cos 3\theta + \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1+\cos 4\theta}{8} d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \left(\frac{13}{4}\cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta + \frac{3}{4}\cos 3\theta + \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta \right) d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \left(\frac{13}{4} \cos\theta + \frac{15}{8} + 2\cos 2\theta + \frac{3}{4} \cos 3\theta + \frac{1}{8} \cos 4\theta \right) d\theta$$

$$= \frac{a^3}{3} \left[\frac{13}{4} \sin\theta + \frac{15}{8}\theta + 2 \frac{\sin 2\theta}{2} + \frac{3}{4} \frac{\sin 3\theta}{3} + \frac{1}{8} \frac{\sin 4\theta}{4} \right]_0^{\pi}$$

$$= \frac{a^3}{3} \left[\frac{15}{8}\pi \right]$$

$$= \frac{5\pi a^3}{8}$$

2. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x} dy dx$

Sol

$$\int_0^4 \left(\frac{e^{y/x}}{1/x} \right)_{0^+}^{x^2} dx \quad \left\{ \because \int e^{ax} dx = \frac{e^{ax}}{a} \right\}$$

$$\int_0^4 x e^x - x dx$$

$$= \left[(x e^x - \int 1 e^x dx) - \frac{x^2}{2} \right]_0^4$$

$$= \left[x e^x - e^x - \frac{x^2}{2} \right]_0^4$$

$$= 4e^4 - e^4 - \frac{16}{2} + 1$$

$$= 4e^4 - e^4 - 8 + 1$$

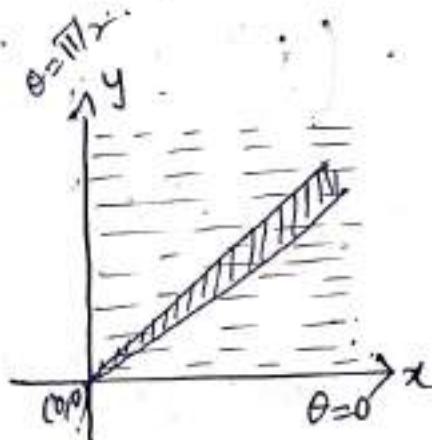
$$= 3e^4 - 7$$

3. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ by changing to polar coordinates

$$x = r \cos \theta, y = r \sin \theta, dx dy = r dr d\theta$$

$$\begin{aligned} & x=0, y=0 \\ & x=\infty, y=\infty \end{aligned}$$

$$\begin{aligned} r &\rightarrow 0 \rightarrow \infty \\ \theta &\rightarrow 0 \rightarrow \frac{\pi}{2} \end{aligned}$$



$$\frac{\pi}{2} \int_0^\infty e^{-r^2} r dr d\theta$$

$$\text{multiply \& divide by } -2$$

$$= -\frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-r^2} (-2r dr d\theta) = -\frac{1}{2} \int_0^{\pi/2} (e^{-r^2})_0^\infty d\theta$$

$$\begin{aligned}
 &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (e^{-\theta} - e^0) d\theta \\
 &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0-1) d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (d\theta) = \frac{1}{2} (\theta)_0^{\frac{\pi}{2}} = \frac{1}{2} \left(\frac{\pi}{2}\right) = \frac{\pi}{4} //
 \end{aligned}$$

Evaluate $\int_0^2 \int_0^x e^{(x+y)} dy dx$.

$$\int_0^2 \int_0^x e^{(x+y)} dy dx$$

$$\int_0^2 \int_0^x e^x \cdot ey dy dx$$

$$\int_0^2 e^x (ey)_0^x dx$$

$$\begin{aligned}
 &\pm \int_0^2 e^x (e^x - e^0) dx = \int_0^2 e^x (e^x - 1) dx = \int_0^2 e^{2x} - e^x dx \\
 &= \left(\frac{e^{2x}}{2} - e^x \right)_0^2 = \frac{e^4 - e^2}{2} - \frac{1}{2} + 1
 \end{aligned}$$

$$= \frac{e^4 - e^2 + 1}{2} = \frac{e^4 - 2e^2 + 1}{2} //$$

Evaluate $\int_0^2 \int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2+y^2)} dy dx$

$$\int_0^2 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2} dy dx$$

$$\begin{aligned}
 &= \int_0^2 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1} \left(\frac{y}{\sqrt{1+x^2}} \right) \right)_0^{\sqrt{1+x^2}} dx
 \end{aligned}$$

$$\boxed{\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}}$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} \left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \right) - 0 \, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} 1 \, dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} \, dx$$

$$\therefore \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx = \frac{\pi}{4} (\sinh^{-1} x)_0^1 = \frac{\pi}{4} (\sinh^{-1} 1)$$

68. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2) \, dx \, dy$ by changing to polar coordinates.

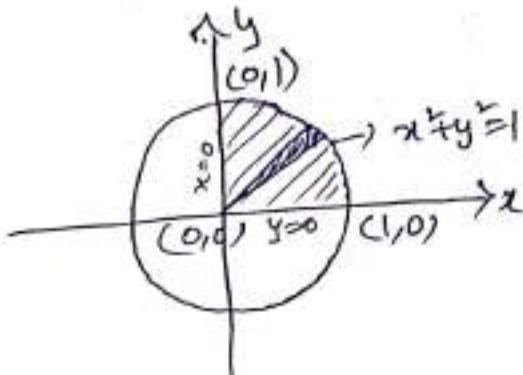
Sol: $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2) \, dx \, dy$.

$$x = r \cos \theta \quad y = r \sin \theta$$

$$dx \, dy = r \, dr \, d\theta$$

$$x=0 \quad x=\sqrt{1-y^2}$$

$$y=0 \quad y=1$$



$$x^2 + y^2 = 1$$

$$r^2 = 1$$

$$r = 1$$

$$r \rightarrow 0 \rightarrow 1$$

$$\theta \rightarrow 0 \rightarrow \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \int_0^1 r^2 r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^1 \, d\theta \quad \int_0^{\pi/2} \frac{1}{4} \, d\theta = \frac{1}{4} \left[\theta \right]_0^{\pi/2}$$

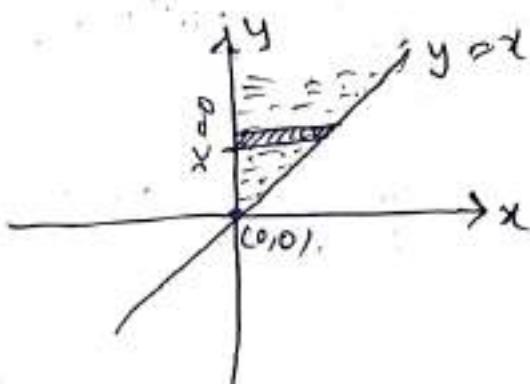
$$= \frac{1}{4} \left[\frac{\pi}{2} \right] = \frac{\pi}{8}.$$

18. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ by changing the order of integration.

Sol: $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$.

$$x=0 \quad x=\infty$$

$$y=x \quad y=\infty$$



$x \rightarrow 0 - y$ (By considering horizontal strip)
 $y \rightarrow 0 - \infty$.

$$\begin{aligned} \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy &= \int_0^\infty \left[\frac{e^{-y}}{y} \right]_0^y dy = \int_0^\infty \frac{e^{-y}}{y} (y) dy \\ &= \int_0^\infty e^{-y} dy = (e^{-y}) \Big|_0^\infty \\ &= 0 + 1 = 1 \end{aligned}$$

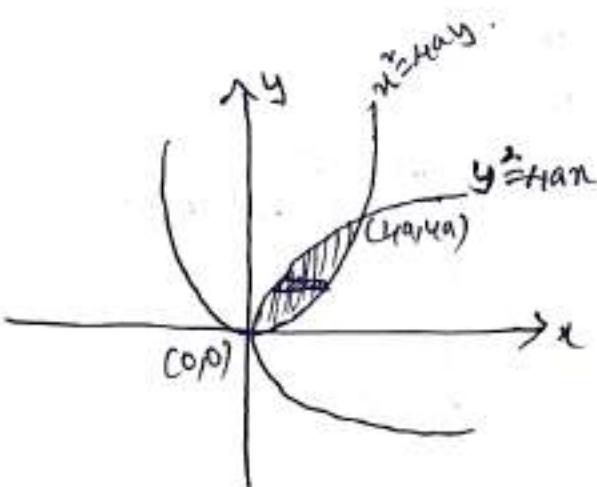
19. Evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx$ by changing the order of integration.

Sol: $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx$.

$$x=0 \quad x=4a$$

$$y = \frac{x^2}{4a} \quad y = 2\sqrt{ax}.$$

$$\Rightarrow x^2 = 4ay \quad y^2 = 4ax.$$



$$x \rightarrow y^2/4a - 2\sqrt{ax}$$

$$y \rightarrow 0 - 4a$$

(By considering horizontal strip)

$$\begin{aligned}
 & \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx dy = \int_0^{4a} (x)_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy \\
 &= \int_0^{4a} 2\sqrt{a} \sqrt{y} - \frac{1}{4a} y^2 dy = \left(2\sqrt{a} \frac{y^{3/2}}{3/2} - \frac{1}{4a} \frac{y^3}{3} \right) \Big|_0^{4a} \\
 &= \frac{4}{3} \sqrt{a} \cdot (4a)^{3/2} - \frac{1}{4a} \frac{(4a)^3}{3} \\
 &= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{a^2}{3} (32 - 16) = \underline{\underline{\frac{16a^2}{2}}}.
 \end{aligned}$$

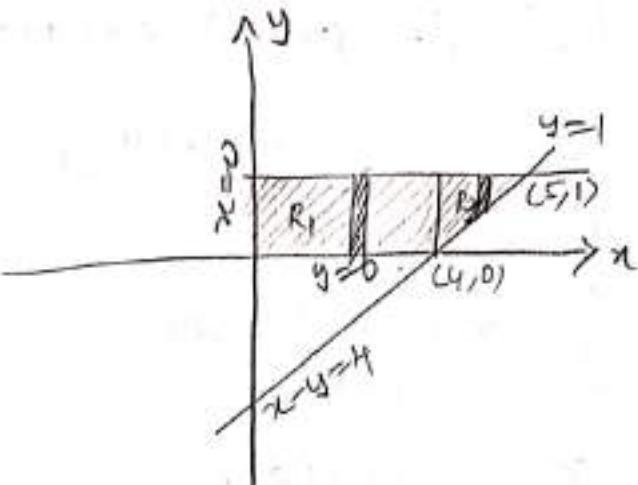
9. Evaluate $\int_0^1 \int_0^{y+4} \frac{2y+1}{x+1} dx dy$ by changing the order of integration.

Sol:

$$\int_0^1 \int_0^{y+4} \frac{2y+1}{x+1} dx dy$$

$$\begin{aligned} x &= 0 & x &= y+4 \\ y &= 0 & x-y &= 4 \\ y &= 1 & y &= 5 \end{aligned}$$

$$\iint_R = \iint_{R_1} + \iint_{R_2}$$



Region 1:

$$x \rightarrow 0 - 4$$

$$y \rightarrow 0 - 1$$

$$\int_0^4 \int_0^{x+1} \frac{2y+1}{x+1} dy dx$$

$$\int_0^4 \frac{1}{x+1} \left(\frac{2y^2}{2} + y \right)_0^1 dx$$

$$\int_0^4 \frac{1}{x+1} (2) dx$$

$$2 \left(\log(x+1) \right)_0^4$$

$$2 \log 5 - 0 = 2 \log 5$$

Region 2:

$$x \rightarrow 4 - 5$$

$$y \rightarrow x - 4 - 1$$

$$\int_4^5 \int_{x-4}^1 \frac{2y+1}{x+1} dy dx$$

$$\int \frac{1}{x+1} \left[\frac{x^2y^2}{2} + y \right] dx$$

$$\int_4^5 \frac{1}{x+1} (2 - (x-4)^2 - (x-4)) dx$$

$$\int_4^5 \frac{2-x^2+8x-16-x+4}{x+1} dx$$

$$\int_4^5 \frac{-x^2+7x-10}{x+1} dx$$

$$-\int_4^5 \frac{x^2-7x+10}{x+1} dx$$

By division method

$$\begin{array}{r} x+1) \overline{x^2-7x+10} (x-8 \\ \underline{x^2+x} \\ \hline -8x+10 \\ \underline{-8x-8} \\ \hline 18 \end{array}$$

$$= \int_4^5 x-8 + \frac{18}{x+1} dx$$

$$= - \left[\frac{x^2}{2} - 8x + 18 \log(x+1) \right]_4^5$$

$$= - \left[\frac{25}{2} - 40 + 18 \log 6 - \left[\frac{16}{2} + 32 - 18 \log 5 \right] \right]$$

$$= - \left[\frac{25}{2} - 16 + 18 \log 6 - 18 \log 5 \right]$$

$$\iint_R = \iint_R + \iint_{R_2} = 2 \log 5 - \frac{25}{2} + 16 - 18 \log 6 + 18 \log 5$$

$$= 20 \log 5 + \frac{7}{2} - 18 \log 6 //$$

10Q. Evaluate $\iiint (xy + yz + zx) dx dy dz$ where R is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$.

Sol: $\iiint (xy + yz + zx) dx dy dz$

$$x \rightarrow 0-1$$

$$y \rightarrow 0-2$$

$$z \rightarrow 0-3.$$

$$\begin{aligned} \int_0^3 \int_0^2 \int_0^1 (xy + yz + zx) dx dy dz &= \int_0^3 \int_0^2 \left(\frac{x^2}{2} y + xy z + x \frac{z^2}{2} \right)_0^1 dy dz \\ &= \int_0^3 \int_0^2 \left(\frac{1}{2} y + yz + \frac{1}{2} z \right) dy dz = \int_0^3 \left(\frac{1}{2} \frac{y^2}{2} + z \frac{y^2}{2} + \frac{1}{2} z y \right)_0^2 dz \\ &= \int_0^3 \left(\frac{1}{2} \times 2 + z \times 2 + \frac{1}{2} z^2 \right) dz = \int_0^3 (1 + 2z + z^2) dz \\ &= \int_0^3 1 + 3z + z^2 dz = \left(z + 3 \frac{z^2}{2} \right)_0^3 = 3 + 3 \left(\frac{9}{2} \right) \\ &= 3 + \frac{27}{2} = \frac{6+27}{2} \\ &= \frac{33}{2} \end{aligned}$$

11Q. Evaluate $\int_0^{\ln c} \int_0^{\ln b} \int_0^{\ln a} e^{x+y+z} dx dy dz$.

Sol: $\int_0^{\ln c} \int_0^{\ln b} \int_0^{\ln a} e^{x+y+z} dx dy dz$.

$$= \int_0^{\ln c} \int_0^{\ln b} e^{y+z} \left(e^x \right)_0^{\ln a} dy dz$$

$$= \int_0^{\ln c} \int_0^{\ln b} e^{y+z} \left(e^{\ln a} - e^0 \right) dy dz$$

$$= \int_0^{\ln c} \int_0^{\ln b} e^{y+z} (a-1) dy dz = (a-1) \int_0^{\ln c} e^z (e^y)_0^{\ln b} dz.$$

$$= (a-1) \int_0^{\ln c} e^z (e^{\ln b} - e^0) dz = (a-1) \int_0^{\ln c} e^z (b-1) dz$$

$$= (a-1)(b-1) \int_0^{\ln c} e^z dz = (a-1)(b-1) (e^z)_0^{\ln c}$$

$$= (a-1)(b-1) (e^{\ln c} - e^0)$$

$$= (a-1)(b-1)(c-1)$$

—

128. Evaluate $\iiint (x^2 + y^2 + z^2) dx dy dz$ taken over the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$; by transforming to spherical coordinates.

Sol: $\iiint (x^2 + y^2 + z^2) dx dy dz$.

spherical coordinates.

$$x = \rho \sin \theta \cos \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \theta$$

$$dx dy dz = \rho^2 \sin \theta d\rho d\theta d\phi$$

$$\rho \rightarrow 0 \rightarrow 1$$

$$\theta \rightarrow 0 \rightarrow \pi$$

$$\phi \rightarrow 0 \rightarrow 2\pi$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin \theta d\rho d\theta d\phi = \int_0^{\pi} \int_0^{\pi} \left(\frac{\rho^3}{3} \right)_0^1 \sin \theta d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} \sin \theta d\theta d\phi = \frac{1}{3} \int_0^{\pi} (-\cos \theta)_0^{\pi} d\phi$$

$$= \frac{1}{3} \int_0^{2\pi} (-\cos\theta + \cos 0) d\phi$$

$$= \frac{1}{3} \int_0^{2\pi} (1+1) d\phi = \frac{2}{3} (\phi)_0^{2\pi} = \frac{2}{3} (2\pi) = \frac{4\pi}{3}$$

13Q. find the volume of the unit sphere $x^2+y^2+z^2=1$

Sol: Volume = $\iiint dxdydz$.

in spherical coordinates.

$$dxdydz = r^2 \sin\theta dr d\theta d\phi$$

$$r \rightarrow 0 \rightarrow 1$$

$$\theta \rightarrow 0 \rightarrow \pi$$

$$\phi \rightarrow 0 \rightarrow 2\pi$$

$$V = \int_0^{2\pi} \int_0^\pi \int_0^1 r^2 \sin\theta dr d\theta d\phi = \int_0^{2\pi} \int_0^\pi \left(\frac{r^3}{3}\right)_0^1 \sin\theta d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \frac{1}{3} \sin\theta d\theta d\phi = \frac{1}{3} \int_0^{2\pi} (-\cos\theta)_0^\pi d\phi$$

$$= \frac{1}{3} \int_0^{2\pi} (-\cos\pi + \cos 0) d\phi = \frac{1}{3} \int_0^{2\pi} (1+1) d\phi$$

$$= \frac{2}{3} (\phi)_0^{2\pi} = \frac{2}{3} (2\pi) = \frac{4\pi}{3}$$

\therefore Volume of unit sphere = $\frac{4\pi}{3}$

14Q. Transform the following to Cartesian form and hence evaluate. $\int_0^\pi \int_0^a r^3 \sin\theta \cos\theta dr d\theta$

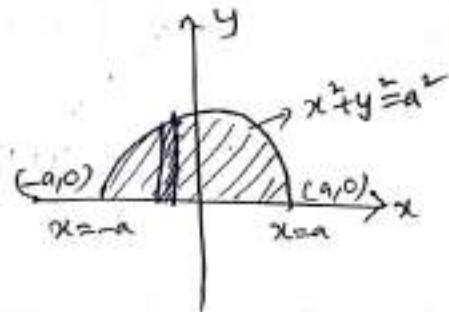
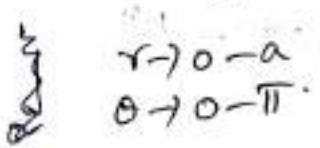
Sol: $\int_0^\pi \int_0^a r^3 \sin\theta \cos\theta dr d\theta$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$dr dy = r dr d\theta$$

$$r \rightarrow 0 \text{ to } a$$

$$\theta \rightarrow 0 \text{ to } \pi$$



By considering the vertical strip.

$$x \rightarrow -a \text{ to } a$$

$$y \rightarrow 0 \text{ to } \sqrt{a^2 - x^2}$$

$$\int_0^{\pi} \int_0^a r^3 \sin \theta \cos \theta dr d\theta = \int_0^{\pi} \int_0^a (r \cos \theta)(r \sin \theta)(r dr d\theta)$$

$$= \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} xy dy dx.$$

$$= \int_{-a}^a x \left(\frac{y^2}{2} \right)_0^{\sqrt{a^2 - x^2}} dx = \int_{-a}^a x \left(\frac{a^2 - x^2}{2} \right) dx.$$

$$= \frac{1}{2} \int_{-a}^a a^2 x - x^3 dx = \frac{1}{2} \left\{ a^2 \frac{x^2}{2} - \frac{x^4}{4} \right\}_a^{-a}$$

$$= \frac{1}{2} \left\{ \frac{a^4}{2} - \frac{a^4}{4} - \frac{a^4}{2} + \frac{a^4}{4} \right\} = 0.$$

158. Evaluate $\int_0^{\pi/2} \int_0^a r^2 dr d\theta$.

Sol: $\int_0^{\pi/2} \int_0^a r^2 dr d\theta = \int_0^{\pi/2} \left(\frac{r^3}{3} \right)_0^a d\theta$

$$= \int_0^{\pi/2} \left(\frac{a^3}{3} - \frac{a^3(1-\cos \theta)^3}{3} \right) d\theta.$$

$$\begin{aligned}
 &= \frac{a^3}{3} \int_0^{\pi/2} 1 - (1 - \cos\theta)^3 d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} 1 - (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta) d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} 1 - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - 3\cos^2\theta + \cos^3\theta d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - 3\left(1 + \frac{\cos 2\theta}{2}\right) + \frac{1}{4}(3\cos 3\theta + 3\cos\theta) d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta d\theta \\
 &= \frac{a^3}{3} \int_0^{\pi/2} \frac{15}{4}\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta d\theta \\
 &= \frac{a^3}{3} \left[\frac{15}{4}\sin\theta - \frac{3}{2}\theta - \frac{3}{2} \frac{\sin 2\theta}{2} + \frac{1}{4} \frac{\sin 3\theta}{3} \right]_0^{\pi/2} \\
 &= \frac{a^3}{3} \left[\frac{15}{4}(1) - \frac{3}{2}\left(\frac{\pi}{2}\right) - 0 + \frac{1}{12}(-1) \right] \\
 &= \frac{a^3}{3} \left\{ \frac{15}{4} - \frac{1}{2} - \frac{3\pi}{4} \right\} = \frac{a^3}{3} \left\{ \frac{45 - 1 - 9\pi}{12} \right\} \\
 &= \frac{a^3}{3} \left\{ \frac{44 - 9\pi}{12} \right\} = \frac{a^3}{36} (44 - 9\pi)
 \end{aligned}$$

UNIT-05

SAQ :-

1. If \vec{a} is a constant vector and $\vec{r} = xi\hat{i} + y\hat{j} + zk\hat{k}$ then evaluate $\operatorname{div}(\vec{a} \times \vec{r})$

Sol:-

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\vec{a} \times \vec{r} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

$$\operatorname{div}(\vec{a} \times \vec{r}) = \nabla \cdot (\vec{a} \times \vec{r})$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot ((a_2z - a_3y)\hat{i} - (a_1z - a_3x)\hat{j} + (a_1y - a_2x)\hat{k})$$

$$= \frac{\partial}{\partial x}(a_2z - a_3y) - \frac{\partial}{\partial y}(a_1z - a_3x) + \frac{\partial}{\partial z}(a_1y - a_2x)$$

$$= 0.$$

2. Prove that $\operatorname{curl}(\operatorname{grad}f) = 0$ where f is a differentiable scalar field.

Sol:- $\operatorname{grad}f = \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$

$$\operatorname{curl}(\operatorname{grad}f) = \nabla \times \operatorname{grad}f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) - j \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) + k \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= 0$$

3. Find ∇f if $f = \log_e(x^2 + y^2 + z^2)$.

Sol: $\nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$

$$= \bar{i} \frac{2x}{x^2 + y^2 + z^2} + \bar{j} \frac{2y}{x^2 + y^2 + z^2} + \bar{k} \frac{2z}{x^2 + y^2 + z^2}$$

$$= \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{x^2 + y^2 + z^2}$$

4. Evaluate $\int \vec{V} \cdot d\vec{r}$ when $\vec{V} = x\bar{i} + y\bar{j} + z\bar{k}$ and c is the line segment from $A(1, 2, 2)$ to $B(3, 6, 6)$.

Sol: $\int \vec{V} \cdot d\vec{r} = \int_A^B (x\bar{i} + y\bar{j} + z\bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$

$$= \int_{(1,2,2)}^{(3,6,6)} x dx + y dy + z dz$$

Let $x = t$ $y = 2t$ $z = 2t$
 $dx = dt$ $dy = 2dt$ $dz = 2dt$.

$$t \rightarrow 1-3$$

$$\begin{aligned}
 &= \int_1^3 t dt + 4t dt + 4t dt = \int_1^3 9t dt = 9 \left(\frac{t^2}{2} \right)_1^3 \\
 &= 9 \left(\frac{9}{2} - \frac{1}{2} \right) \\
 &= 9 \left(\frac{8}{2} \right) = 36
 \end{aligned}$$

5. Find the unit normal vector to the surface $f(x,y,z) = x^2y - y^2z - xy^2$ at $P(1, -1, 0)$.

Sol.

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\nabla f = i(2xy - y^2) + j(x^2 - 2yz - xz) + k(-y^2 - xy)$$

$$(\nabla f)_{(1, -1, 0)} = -2i + j$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{-2i + j}{\sqrt{4+1}} = \frac{-2i + j}{\sqrt{5}}$$

6. Find $\operatorname{div} \vec{f}$ where $\vec{f} = \operatorname{grad}(x^3 + y^3 + z^3 + 3xyz)$.

Sol.

$$\operatorname{grad}(x^3 + y^3 + z^3 + 3xyz)$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^3 + y^3 + z^3 + 3xyz)$$

$$= \vec{i}(3x^2 + 3yz) + \vec{j}(3y^2 + 3xz) + \vec{k}(3z^2 + 3xy)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2 + 3yz) + \frac{\partial}{\partial y}(3y^2 + 3xz) + \frac{\partial}{\partial z}(3z^2 + 3xy)$$

$$= 6x + 6y + 6z$$

$$\text{div } \vec{F} = 6(x+y+z).$$

7. State Green's theorem on a plane.

Ans Green's Theorem:-

If S is a closed region in xy -plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivative in the region R then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where

' C ' is traversed in positive direction
(anticlock wise).

8. Show that $\vec{V} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Sol:

$$\operatorname{div} \vec{V} = \nabla \cdot \vec{V}$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (3y^4z^2\vec{i} + 4x^3z^2\vec{j} + 3x^2y^2\vec{k})$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

$$\operatorname{div} \vec{V} = 0$$

$\therefore \vec{V}$ is solenoidal.

9. Find the directional derivative of the function $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\vec{i} - \vec{j} + \vec{k}$.

Sol: Directional Derivative = $\nabla \phi \cdot \hat{e}$

$$\phi = x^2yz + 4xz^2$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$(\nabla \phi) = \vec{i}(+4+4) + \vec{j}(-1) + \vec{k}(-2+8)$$

$$(1, -2, -1) = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\nabla \phi = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\hat{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i} - \bar{j} - \bar{k}}{\sqrt{4+1+1}} = \frac{2\bar{i} - \bar{j} - \bar{k}}{\sqrt{6}}$$

Directional derivative = $\nabla \phi \cdot \hat{e}$

$$= (8\bar{i} - \bar{j} - 10\bar{k}) \cdot \frac{(2\bar{i} - \bar{j} - \bar{k})}{\sqrt{6}} = \frac{16 + 1 + 10}{\sqrt{6}}$$

$$\frac{\partial \phi}{\partial \bar{a}} = \cancel{\frac{\partial \phi}{\partial \bar{i}}} = \frac{\partial \phi}{\cancel{\partial \bar{i}}} = \frac{27}{\sqrt{6}}$$

10. State Gauss divergence theorem.

Gauss divergence theorem:-

Let S be a closed surface enclosing a volume V if \bar{F} is continuously differentiable vector point function then

$$\int_V \operatorname{div} \bar{F} dV = \int_S \bar{F} \cdot \bar{n} dS$$

11. If \bar{a} is a constant vector and $\bar{r} = xi + yj + zk$ then evaluate $\operatorname{curl}(\bar{a} \times \bar{r})$.

Sol:

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$$\bar{r} = xi + yj + zk$$

$$\bar{a} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \bar{i}(a_2z - a_3y) - \bar{j}(a_1z - a_3x) + \bar{k}(a_1y - a_2x)$$

$$\text{curl}(\bar{a} \times \bar{r}) = \nabla \times (\bar{a} \times \bar{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_2 x \end{vmatrix}$$

$$= \hat{i}(a_1 + a_1) - \hat{j}(a_2 - a_2) + \hat{k}(a_3 + a_3)$$

$$= 2a_1 \hat{i} + 2a_2 \hat{j} + 2a_3 \hat{k}$$

$$= 2(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$$

$$= 2\bar{a}$$

12. Prove that $\text{curl}(\text{grad } f) = 0$, where f is a differentiable scalar field.

Sol:-

$$\text{grad } f = \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$$

$$\text{curl}(\text{grad } f) = \nabla \times \text{grad } f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z} \right) - \hat{j} \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z} \right) + \hat{k} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} \right)$$

$$= 0$$

13. State Green's theorem.

Ans Green's theorem:-

If S is a closed region in xy -plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivative in the region R then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where

' C ' is transversed in positive direction
(anticlockwise).

14. State Stoke's theorem.

Stoke's theorem:-

Let S be a open surface bounded by a closed non intersecting curve C , if \vec{F} is any differentiable vector point function then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds.$$

15. Show that $\vec{V} = 12x\hat{i} - 15y\hat{j} + \hat{k}$ is irrotational.

Sol:- Given $\vec{V} = 12x\hat{i} - 15y\hat{j} + \hat{k}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 12x & -15y & 1 \end{vmatrix}$$
$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$
$$= 0$$

$\therefore \vec{V}$ is irrotational.

UNIT-05 (LAQ)

1. Evaluate $\oint_C x dy - y dx$ where C is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$ using Greens theorem.

Sol.: By Greens theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

$$M = -y$$

$$N = x$$

$$\frac{\partial M}{\partial y} = -1$$

$$\frac{\partial N}{\partial x} = 1$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (1+1) dx dy = \iint_R 2 dx dy$$

$$x \rightarrow 0-2$$

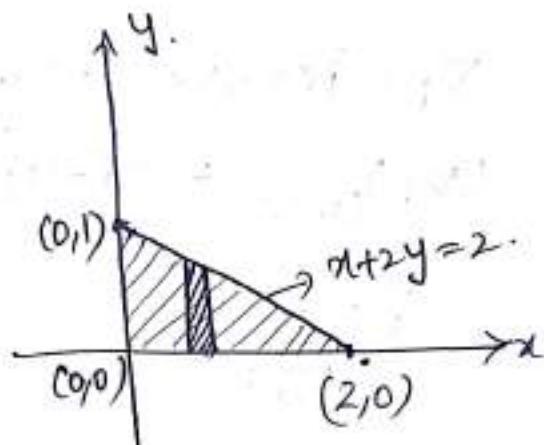
$$y \rightarrow 0 - \frac{2-x}{2}$$

$$= \iint_0^2 2 dy dx$$

$$= 2 \int_0^2 (y)_{0.}^{\frac{2-x}{2}} dx$$

$$= 2 \int_0^2 \frac{2-x}{2} dx = \left(2x - \frac{x^2}{2} \right)_0^2 = 4 - 2 = 2$$

$$\therefore \oint_C x dy - y dx = 2$$



2. Find the Directional Derivative of $f(x, y, z) = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line \overline{PQ} where Q is the point $(5, 0, 4)$.

Sol: $P(1, 2, 3)$ $Q(5, 0, 4)$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$

$$\overline{OP} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\overline{OQ} = 5\vec{i} + 4\vec{k}$$

$$\vec{\alpha} = \overline{PQ} = (5\vec{i} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k})$$

$$\vec{\alpha} = 4\vec{i} - 2\vec{j} + \vec{k}$$

$$\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\nabla f = \vec{i}(2x) + \vec{j}(-2y) + \vec{k}(4z)$$

$$(\nabla f)_{(1, 2, 3)} = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\hat{e} = \frac{\vec{\alpha}}{|\vec{\alpha}|}$$

$$= \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{16+4+1}}$$

$$= \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}}$$

Direction Derivative = $\nabla f \cdot \hat{e}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \cdot \frac{(4\vec{i} - 2\vec{j} + \vec{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} \Rightarrow \frac{28}{\sqrt{21}}$$

3. Find the angle between the surfaces $x^2+y^2+z^2=9$,
 $z+3=x^2+y^2$ at $(-2,1,2)$.

$$\phi_1 = x^2 + y^2 + z^2 - 9$$

$$\phi_2 = x^2 + y^2 - z - 3$$

$$\vec{n}_1 = \nabla \phi_1 = i \frac{\partial \phi_1}{\partial x} + j \frac{\partial \phi_1}{\partial y} + k \frac{\partial \phi_1}{\partial z}$$

$$\vec{n}_1 = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$(\vec{n}_1)_{(-2,1,2)} = -4\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\vec{n}_2 = \nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$$

$$(\vec{n}_2)_{(-2,1,2)} = -4\vec{i} + 2\vec{j} - \vec{k}$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{(-4\vec{i} + 2\vec{j} + 4\vec{k}) \cdot (-4\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{16+4+16} \sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

4. Find the constants a, b, c such that $\vec{F} = (ax+3y+cz)\vec{i} + (bx+cy+3z)\vec{j} + (2x+cy+3z)\vec{k}$ is irrotational, find the scalar function f such that $\vec{F} = \nabla f$.

Sol:- Given $\operatorname{curl} \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \vec{k} \\ ax+3y+cz & bx+cy+3z & 2x+cy+3z \end{vmatrix} = 0$$

$$\Rightarrow i(c-3) - j(2-a) + k(b-3) = 0$$

$$c-3=0, \quad -2+a=a, \quad b-3=0$$

$$c=3, \quad a=2, \quad b=3$$

Since \vec{F} is irrotational \exists a scalar potential f such that $\vec{F} = \nabla f$

$$(2x+3y+2z)\vec{i} + (3x+2y+3z)\vec{j} + (2x+3y+3z)\vec{k}$$

$$= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2x+3y+2z \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = 3x+2y+3z \longrightarrow (2)$$

$$\frac{\partial f}{\partial z} = 2x+3y+3z \longrightarrow (3)$$

on Integrating (1), (2) and (3)

$$f = \frac{2x^3}{3} + 3xy + 2xz + c_1(y, z)$$

$$f = 3xy + \frac{2y^2}{2} + 3yz + c_2(x, z)$$

$$f = 2xz + 3yz + \frac{3z^2}{2} + c_3(x, y)$$

$$\therefore f = x^2 + y^2 + \frac{3z^2}{2} + 3xy + 2xz + 3yz + C.$$

5. Find constant a and b so that the surface $ax^2 - 2bxy - (a+4)x$ will be orthogonal to the surface $4x^2y + z^2 = 4$ at $(1, -1, 2)$.

Sol:-

$$\phi_1 = ax^2 - 2bxy - (a+4)x$$

$$\phi_2 = 4x^2y + z^2 - 4$$

$$\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = 0 \quad [\theta = 90^\circ] \\ \cos 90^\circ = 0$$

$$\vec{n}_1 \cdot \vec{n}_2 = 0 \rightarrow (1)$$

$$\vec{n}_1 = \nabla \phi_1 = i \frac{\partial \phi_1}{\partial x} + j \frac{\partial \phi_1}{\partial y} + k \frac{\partial \phi_1}{\partial z}$$

$$\vec{n}_1 = i(2ax - a - 4) + j(-2bx) + k(-2by)$$

$$(\vec{n}_1)_{(1, -1, 2)} = (a-4)\vec{i} - 4b\vec{j} + 2b\vec{k}$$

$$\vec{n}_2 = \nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$$

$$\vec{n}_2 = i(8xy) + j(4x^2) + k(2z)$$

$$(\vec{n}_2)_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 4\vec{k}$$

from equation (1)

$$((a-4)\vec{i} - 4b\vec{j} + 2b\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 4\vec{k})$$

$$-8(a-4) - 16b + 8b = 0$$

$$-a + 4 - 2b + b = 0$$

$$\Rightarrow a + b = 4 \quad \text{--- (2)}$$

Sub the point (1, -1, 2) in $\phi_1 = ax^2 - 2byz - (a+4)x$

$$\Rightarrow a(1)^2 - 2b(-1)(2) = (a+4)(1)$$

$$\Rightarrow a + 4b = a + 4$$

$$\Rightarrow 4b = a + 4 - a \Rightarrow 4b = 4$$

$$\boxed{b=1}$$

sub $b=1$ in equation (2)

$$\Rightarrow a + 1 = 4$$

$$\Rightarrow \boxed{a=3}$$

$$\therefore a=3, b=1$$

6. Find angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at point $(2, -1, 2)$.

Sol. $\phi_1 = x^2 + y^2 + z^2 - 9$

$$\phi_2 = x^2 + y^2 - z - 3$$

$$\vec{n}_1 = \nabla \phi_1 = i \frac{\partial \phi_1}{\partial x} + j \frac{\partial \phi_1}{\partial y} + k \frac{\partial \phi_1}{\partial z}$$

$$\vec{n}_1 = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$(\vec{n}_1)_{(2, -1, 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\vec{n}_2 = \nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$$

$$(\vec{n}_2)_{(2, -1, 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+16} \sqrt{16+4+1}}$$

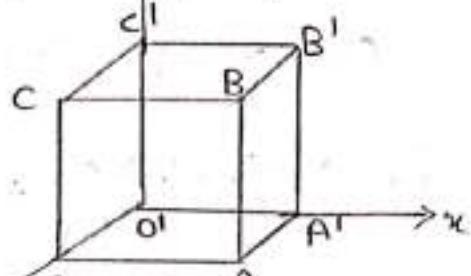
$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

7. Verify Gauss divergence theorem for
 $F = (x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}$ taken over the surface of the cube bounded by the planes $x=y=z=a$ and coordinate planes.

Sol:-

$$F = (x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}, \\ x=y=z=a$$



By Gauss Divergence theorem then we have

$$\int_S \bar{F} \cdot \bar{n} dS = \int_V \nabla \cdot \bar{F} dv$$

$$\int_S \bar{F} \cdot \bar{n} dS = \int_{OABC} + \int_{OAA'B'} + \int_{ODC'C'} + \int_{C'B'C'D'} + \int_{O'A'B'C'} + \int_{A'ABB'}$$

$S_1: O'A'B'C'$: its in xy plane

$$\iint_R \bar{F} \cdot \bar{n} \frac{dx dy}{|\bar{n} \cdot \bar{k}|}$$

$$\bar{n} = -\bar{k} \\ \bar{F} \cdot \bar{n} = -z$$

$$\int_{S_1} z dx dy = 0 \quad (\because z=0 \text{ in } xy \text{ plane})$$

$S_2: OAA'B'$: its in xz plane.

$$\iint_R \bar{F} \cdot \bar{n} \frac{dx dz}{|\bar{n} \cdot \bar{j}|} \quad \bar{n} = -\bar{j}$$

$$\bar{F} \cdot \bar{n} = 2x^2y$$

$$\int_{S_2} 2x^2y dx dz = 0 \quad (\because y=0 \text{ in } xz \text{ plane})$$

S₃ = OOLC Its in yx plane.

$$\iint_{S_3} \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{n} \cdot \vec{i}|} \quad \vec{n} = -\vec{i}$$

$$\vec{F} \cdot \vec{n} = -(x^3 - yz)$$

$$-\int_{S_3} x^3 - yz dy dz = \int_{S_3} yz dy dz = \iint_R yz dy dx$$

$$= \int_0^a \int_0^a yz dy dz$$

$$= \int_0^a y \left(\frac{z^2}{2} \right)_0^a dy \Rightarrow \frac{a^2}{2} \left(\frac{y^2}{2} \right)_0^a = \frac{a^4}{4}$$

S₄ : C' B' BC : Let the surface be projected in xy plane.

$$\int_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F} \cdot \vec{n} \frac{dx dz}{|\vec{n} \cdot \vec{j}|}$$

$$\vec{n} = \vec{j}$$

$$\vec{F} \cdot \vec{n} = -2x^2 y$$

$$= -2 \iint x^2 y dx dz$$

$$= -2a \int_0^a \int_0^a x^2 dx dz$$

$$= -2a \int_0^a x^2 (z) dx$$

$$= -2a^2 \int_0^a x^2 dx$$

$$= -2a^2 \left(\frac{x^3}{3}\right)_0^a$$

$$= -2a^2 \left(\frac{a^3}{3}\right) \Rightarrow -\frac{2a^5}{3}$$

S₅: OABC :- Let the surface be projected in xy plane.

$$\int_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$

$$\vec{n} = \vec{k}$$

$$\vec{F} \cdot \vec{n} = z = a$$

$$\iint_0^a \int_0^a adxdy = a \int_0^a (y)^a dx$$

$$= a^2 \times \int_0^a x^a \Rightarrow a^3$$

S₆: AA'BB' :- Let the surface be projected in yx plane.

$$\int_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dydz}{|\vec{n} \cdot \vec{i}|}$$

$$\vec{n} = \vec{i}$$

$$\vec{F} \cdot \vec{n} = x^3 - yz = a^3 - yz$$

$$\iint_0^a \int_0^a (a^3 - yz) dy dz$$

$$= \int_0^a (a^3 z - y \frac{z^2}{2})_0^a dy \Rightarrow \int_0^a a^4 - \frac{a^2 y}{2} dy = (a^4 y - \frac{a^2 y^2}{2})_0^a$$

$$= a^5 - \frac{a^4}{4}$$

$$\therefore \int_S \bar{F} \cdot \bar{n} dS = 0 + 0 + \frac{a^4}{4} - \frac{2a^5}{3} + a^3 + a^5 - \frac{a^4}{4}$$
$$= \frac{a^5}{3} + a^3$$

$$\underline{\text{R.H.S}} = \int_V \nabla \cdot \bar{F} dV$$

$$\nabla \cdot F = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot ((x^3 - yz)i - 2x^2yj + zk)$$
$$= 3x^2 - 2x^2 + 1 \Rightarrow x^2 + 1$$

$$\int_V \nabla \cdot F dV = \int_0^a \int_0^a \int_0^a (x^2 + 1) dx dy dz$$

$$= \int_0^a \int_0^a \left(\frac{x^3}{3} + x \right)_0^a dy dz$$

$$= \int_0^a \int_0^a \left(\frac{a^3}{3} + a \right) dy dz = \left(\frac{a^3}{3} + a \right) \int_0^a (y)_0^a dz$$

$$= \left(\frac{a^3}{3} + a \right) a \int_0^a dz$$

$$= a \left(\frac{a^3}{3} + a \right) (z)_0^a \Rightarrow a^2 \left(\frac{a^3}{3} + a \right)$$

$$\Rightarrow \frac{a^5}{3} + a^3$$

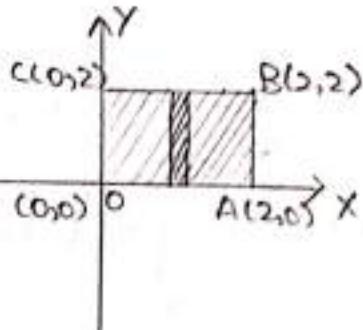
$\therefore \text{LHS} = \text{RHS}$; Hence Theorem is verified

8. Verify Green's theorem in the plane for

$\int (x^2 - xy^3) dx + (y^2 - 2xy) dy$ where C is a square with vertices $(0,0), (2,0), (2,2)$ and $(0,2)$.

Sol:- By Green's theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$



L.H.S

$$\oint_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $(0,0)(2,0)$

$$x \rightarrow 0-2 \quad y=0 \Rightarrow dy=0$$

$$\int_{OA} (x^2 - 0) dx = \int_0^2 x^2 dx = \left(\frac{x^3}{3} \right)_0^2 = \frac{8}{3}$$

Along AB $(2,0)(2,2)$

$$x=2 \Rightarrow dx=0, \quad y \rightarrow 0-2$$

$$\begin{aligned} \int_{AB} (0 + y^2 - 4y) dy &= \int_0^2 y^2 - 4y dy \\ &= \left(\frac{y^3}{3} - 4 \frac{y^2}{2} \right)_0^2 = \frac{8}{3} - 8 \Rightarrow -\frac{16}{3} \end{aligned}$$

Along BC (2,2) (0,2)

$$x \rightarrow 2-0, y \Rightarrow dy=0$$

$$\int_{BC} x^3 - 8x \, dx = \int_2^0 x^3 - 8x \, dx = \left(\frac{x^3}{3} - \frac{8x^2}{2} \right)_0^2$$

$$= 0 - \frac{8}{3} + 16 \Rightarrow \frac{40}{3}$$

Along CO (0,2) (0,0)

$$x=0 \Rightarrow dx=0 \quad y \rightarrow 2-0$$

$$\int_{CO} y^2 \, dy = \int_2^0 y^2 \, dy = \left(\frac{y^3}{3} \right)_2^0 = 0 - \frac{8}{3} = -\frac{8}{3}$$

L.H.S. $\oint_C M \, dx + N \, dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8$

R.H.S. $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$

$$M = x^2 - xy^3 \quad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2 \quad \frac{\partial N}{\partial x} = -2y$$

$$\iint_R (-2y + 3xy^2) \, dx \, dy$$

$x \rightarrow 0-2 \quad y \rightarrow 0-2$

$$\int_0^2 \int_0^2 -2y + 3xy^2 \, dx \, dy$$

$$\begin{aligned}
 &= \int_0^2 \left(-2y^2 + \frac{3y^2}{2} y^2 \right) dy \\
 &= \int -4y + 6y^3 dy \Rightarrow \left(-4\frac{y^2}{2} + 6\frac{y^4}{3} \right)_0^2 \\
 &\Rightarrow -8 + 16 = 8
 \end{aligned}$$

$$\therefore L.H.S = R.H.S$$

Hence the theorem is verified.

9. prove that $\nabla^2(\gamma^n) = n(n+1)\gamma^{n-2}$

$$\text{Soln- } \bar{\gamma} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\gamma = |\bar{\gamma}| = \sqrt{x^2 + y^2 + z^2}$$

$$\gamma^2 = x^2 + y^2 + z^2$$

diff p.w.r.t x, y and z

$$2x \frac{\partial \gamma}{\partial x} = 2x \Rightarrow \frac{\partial \gamma}{\partial x} = \frac{x}{\gamma}, \quad \frac{\partial \gamma}{\partial y} = \frac{y}{\gamma}, \quad \frac{\partial \gamma}{\partial z} = \frac{z}{\gamma}$$

$$\nabla^2 \gamma^n = \nabla(\nabla \gamma^n)$$

$$= \nabla \left(i \frac{\partial \gamma^n}{\partial x} + j \frac{\partial \gamma^n}{\partial y} + k \frac{\partial \gamma^n}{\partial z} \right)$$

$$= \nabla \sum i \frac{\partial}{\partial x} \gamma^n = \nabla \sum i n \gamma^{n-1} \frac{\partial \gamma}{\partial x}$$

$$= \nabla \sum i n \gamma^{n-1} \frac{x}{\gamma}$$

$$= \nabla \sum i n \gamma^{n-2} x$$

$$\begin{aligned}
 &= \nabla \sum m \gamma^{n-2} \mathbf{x}^{\circ} = \nabla (\gamma \gamma^{n-2} (x_i^{\circ} + y_j^{\circ} + z_k^{\circ})) \\
 &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) (\gamma \gamma^{n-2} (x_i^{\circ} + y_j^{\circ} + z_k^{\circ})) \\
 &= n \sum \frac{\partial}{\partial x} \gamma^{n-2} \cdot \mathbf{x} \\
 &= n \sum (n-2) \gamma^{n-3} \frac{\partial \gamma}{\partial x} + \gamma^{n-2} \\
 &= n \sum (n-2) \gamma^{n-3} \frac{x}{\gamma} \cdot \mathbf{x} + \gamma^{n-2} \\
 &= n \sum (n-2) \gamma^{n-4} \cdot \mathbf{x}^2 + \gamma^{n-2} \\
 &= n [(n-2) \gamma^{n-4} (x^2 + y^2 + z^2) + 3 \gamma^{n-2}] \\
 &= n [(n-2) \gamma^{n-4} \gamma^2 + 3 \gamma^{n-2}] \\
 &= n \gamma^{n-2} [n-2+3] \\
 &= n \gamma^{n-2} (n+1) \\
 &= n(n+1) \gamma^{n-2}
 \end{aligned}$$

10. (a) prove that $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$.

Sol:

$$\begin{aligned}
 \bar{a} \times \text{curl}(\bar{b}) &= \bar{a} \times (\nabla \times \bar{b}) = \bar{a} \times \sum \bar{i} \times \frac{\partial \bar{b}}{\partial x} = \sum \bar{a} \times \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right) \\
 &= \sum \left\{ \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) \bar{i} - (\bar{a} \cdot \bar{i}) \frac{\partial \bar{b}}{\partial x} \right\} = \sum \bar{i} \left\{ \bar{a} \frac{\partial \bar{b}}{\partial x} \right\} - \left\{ \bar{a} \cdot \sum \bar{i} \frac{\partial}{\partial x} \right\} \bar{b}
 \end{aligned}$$

$$\therefore \bar{a} \times \operatorname{curl} \bar{b} = \sum \vec{i} \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x_i} \right) - (\bar{a} \cdot \nabla) \bar{b} \rightarrow (1)$$

$$\text{Similarly, } \bar{b} \times \operatorname{curl} \bar{a} = \sum \vec{i} \left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x_i} \right) - (\bar{b} \cdot \nabla) \bar{a} \rightarrow (2)$$

(1) + (2) gives

$$\begin{aligned} \bar{a} \times \operatorname{curl} \bar{b} + \bar{b} \times \operatorname{curl} \bar{a} &= \sum \vec{i} \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x_i} \right) - (\bar{a} \cdot \nabla) \bar{b} + \\ &\quad \sum \vec{i} \left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x_i} \right) - (\bar{b} \cdot \nabla) \bar{a} \\ \Rightarrow \bar{a} + \operatorname{curl} \bar{b} + \bar{b} \times \operatorname{curl} \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + (\bar{b} \cdot \nabla) \bar{a} &= \\ = \sum \vec{i} \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x_i} + \bar{b} \cdot \frac{\partial \bar{a}}{\partial x_i} \right) &\Rightarrow \sum \vec{i} \frac{\partial}{\partial x_i} (\bar{a} \cdot \bar{b}) \\ = \nabla(\bar{a} \cdot \bar{b}) &= \operatorname{grad}(\bar{a} \cdot \bar{b}). \end{aligned}$$

b. Prove that $\operatorname{div}(\bar{a} \times \bar{b}) = \bar{b} \cdot \operatorname{curl} \bar{a} - \bar{a} \cdot \operatorname{curl} \bar{b}$.

$$\begin{aligned} \operatorname{div}(\bar{a} \times \bar{b}) &= \sum \vec{i} \frac{\partial}{\partial x_i} (\bar{a} \times \bar{b}) = \sum \vec{i} \left(\frac{\partial \bar{a}}{\partial x_i} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x_i} \right) \\ &= \sum \vec{i} \left(\frac{\partial \bar{a}}{\partial x_i} \times \bar{b} \right) + \sum \vec{i} \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x_i} \right) = \sum \left(\vec{i} \times \frac{\partial \bar{a}}{\partial x_i} \right) \cdot \bar{b} - \\ &\quad \sum \left(\vec{i} \times \frac{\partial \bar{b}}{\partial x_i} \right) \bar{a} \\ &= (\nabla \times \bar{a}) \cdot \bar{b} - (\nabla \times \bar{b}) \bar{a} = \bar{b} \operatorname{curl} \bar{a} - \bar{a} \operatorname{curl} \bar{b} \end{aligned}$$

Hence the theorem.

11. Evaluate $\int \bar{F} \cdot d\bar{r}$ where $\bar{F} = x^2 i + y^2 j$ and curve $y = x^2$ in the xy -plane from $(0,0)$ to $(1,1)$.

Sol:-

$$\int \bar{F} \cdot d\bar{r} = \int (x^2 i + y^2 j) \cdot (dx i + dy j)$$

$$= \int x^2 dx + y^2 dy$$

given $y = x^2$

$$dy = 2x dx \quad x \rightarrow 0-1$$

$$= \int x^2 dx + x^4 (2x dx)$$

$$= \int_0^1 (x^2 + 2x^5) dx \Rightarrow \left(\frac{x^3}{3} + \frac{2x^6}{6} \right)_0^1$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

12. Find the work done by the force $\vec{F} = (3x^2 - 6xy)\vec{i} + (2y + 3xz)\vec{j} + (1 - 4xyz^2)\vec{k}$ in moving article from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve C ; $x=t$, $y=t^2$, $z=t^3$.

Soln. Work done = $\int_C \vec{F} \cdot d\vec{r}$

$$= \int_C ((3x^2 - 6xy)\vec{i} + (2y + 3xz)\vec{j} + (1 - 4xyz^2)\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}).$$

$$= \int_C (3x^2 - 6xy)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz$$

$$\begin{array}{lll} x=t & y=t^2 & z=t^3 \\ dx=dt & dy=2t\,dt & dz=3t^2\,dt \end{array}$$

$$t \rightarrow 0 \rightarrow 1$$

$$= \int_0^1 (3t^2 - 6t^3)dt + (2t^2 + 3t^5)2t\,dt + (1 - 4t^9)3t^2\,dt$$

$$= \int_0^1 3t^2 - 6t^3 + 4t^3 + 6t^5 + 3t^2 - 12t^9\,dt$$

$$= \int_0^1 6t^2 - 2t^3 + 6t^5 - 12t^9\,dt$$

$$= \left(\frac{6t^3}{3} - \frac{2t^4}{4} + \frac{6t^6}{6} - \frac{12t^{10}}{10} \right)_0^1 = 2 - \frac{1}{2} + 1 - 1$$

$$= \frac{3}{2}.$$

=

13. Use Gauss divergence theorem to evaluate,
 $\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}) \cdot \vec{n} dS$, where S is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2 + y^2 + z^2 = 1$ above this plane.

Sol: Given $\iiint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}) \cdot \vec{n} dS$
 $x^2 + y^2 + z^2 = 1$

By Gauss divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

$$\vec{F} = yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}) \\ &= \frac{\partial}{\partial x} yz^2 + \frac{\partial}{\partial y} zx^2 + \frac{\partial}{\partial z} 2z^2 \\ &= 4z\end{aligned}$$

$$\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}) \cdot \vec{n} dS = \iiint_V 4z dx dy dz$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$dx dy dz = r^2 \sin \theta dr d\theta d\phi$$

$$r \rightarrow 0^+$$

$$\theta \rightarrow 0 - \pi$$

$$\phi \rightarrow 0 - 2\pi$$

$$= 4 \int_0^1 \int_0^\pi \int_0^{2\pi} r \cos^2 \theta \sin \theta dr d\theta d\phi$$

$$= 4 \int_0^1 \int_0^\pi \int_0^{2\pi} r^3 \sin^3 \theta \cos \theta dr d\theta d\phi$$

$$= 4 \int_0^1 \int_0^\pi \frac{r^4}{4} \sin^4 \theta \cos \theta d\theta d\phi$$

$$= \frac{2}{4} \int_0^1 \int_0^\pi (r^4)_0^1 \sin^4 \theta d\theta d\phi$$

$$= \frac{1}{2} \int_0^1 \int_0^\pi \sin^4 \theta d\theta d\phi$$

$$= \frac{1}{2} \int_0^1 \left(-\frac{\cos 2\theta}{2} \right)_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^1 \left(-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2} \right) d\phi$$

$$= \frac{1}{2} \int_0^1 \left(-\frac{1}{2} + \frac{1}{2} \right) d\phi$$

$$= 0$$

14. Verify Green's theorem in plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by $y = \sqrt{x}$ and $y = x^2$.

Sol:- By Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Given

$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$M = 3x^2 - 8y^2 \quad N = 4y - 6xy$$

$$\text{L.H.S} \Rightarrow \oint_C Mdx + Ndy = \oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$\oint_C = \int_{OA} + \int_{AO}$$

Along OA

$$(0,0) \rightarrow (1,1)$$

$$\int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$x^2 = y \\ 2x dx = dy$$

$$\int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)(2x dx).$$

$$\int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) dx$$

$$\int_0^1 3x^2 + 8x^4 - 20x^4 dx = \left(\frac{3x^3}{3} + \frac{8x^5}{5} - \frac{20x^5}{5} \right)_0^1 \\ = 1 + 2 - 4 = -1$$

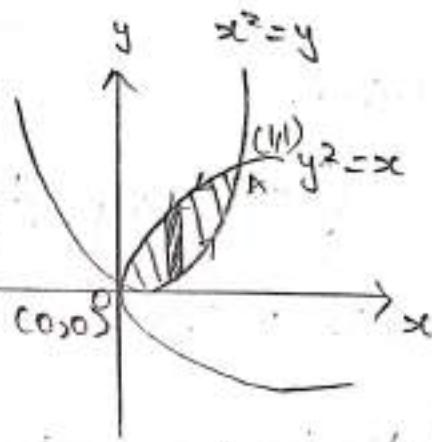
Along A0

$$(1,1) (0,0)$$

$$\int_{A0} (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

$$x = y^2$$

$$dx = 2y dy$$



$$\int_0^1 (3y^4 - 8y^2)(2y dy) + (4y - 6y^3) dy$$

$$\int_1^0 (6y^5 - 16y^3 + 4y - 6y^3) dy$$

$$\int_1^0 4y - 22y^3 + 6y^5 dy$$

$$\left[\frac{4y^2}{2} - \frac{22y^4}{4} + \frac{6y^6}{6} \right]_1^0$$

$$= 0 - \left(2 - \frac{11}{2} + 1 \right) = - \left(3 - \frac{11}{2} \right) = \frac{5}{2}$$

$$\therefore \text{L.H.S} \oint_M dx + N dy = -1 + \frac{5}{2} = \frac{3}{2}$$

R.H.S

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$M = 3x^2 - 8y^2$$

$$N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\iint_R -6y + 16y = \iint_R 10y dx dy$$

$$= 10 \iint y dx dy$$

$$x \rightarrow 0 \rightarrow y \rightarrow x^2 - \sqrt{x}$$

$$= 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y dy dx$$

$$= 10 \int_0^1 \left(\frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx \Rightarrow 10 \int_0^1 \left(\frac{x}{2} - \frac{x^4}{2} \right) dx$$

$$= 10 \left(\frac{x^2}{4} - \frac{x^5}{10} \right) = 10 \left(\frac{1}{4} - \frac{1}{10} \right)$$

$$= 10 \left(\frac{5-2}{20} \right) = \frac{3}{2} \quad \therefore \text{L.H.S} = \text{R.H.S.}$$

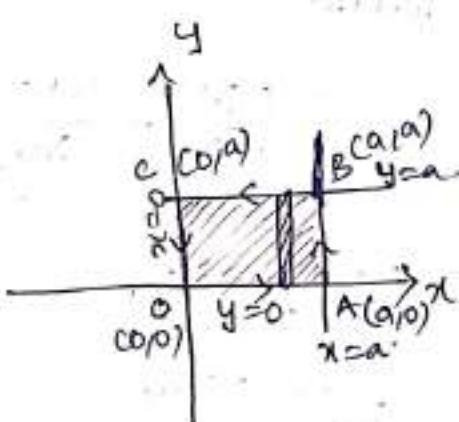
Hence theorem is verified.

15. Verify Stokes theorem for the function $F = x^2 i + xy j$
 Integrated round the square in the plane $z=0$
 whose sides are along the line $x=0, y=0, x=a, y=a$.

Sol:- $\vec{F} = x^2 i + xy j$

By Stokes theorem, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS$$



L.H.S:- $\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

$$= \int_{OA} (x^2 i + xy j) \cdot (dx i + dy j) - \int_{CO} x^2 dx + xy dy$$

Along OA:- (0,0)(a,0)

$$x \rightarrow 0-a \quad y=0 \\ dy=0$$

$$\int_{OA} x^2 dx = \int_0^a x^2 dx = \left(\frac{x^3}{3} \right)_0^a = \frac{a^3}{3}$$

Along AB:-

$$(a,0)(a,a)$$

$$x=a \quad y \rightarrow a-0 \\ dx=0$$

$$\int_A^B ay dy = a \int_0^a y dy = a \left(\frac{y^2}{2} \right)_0^a = \frac{a^3}{2}$$

Along BC :- (a, a) (0, a)

$$x \rightarrow a-0, y=a$$

$$dy=0$$

$$\int_{BC} x^2 dx = \int_a^0 x^2 dx = \left(\frac{x^3}{3} \right)_a^0 = 0 - \frac{a^3}{3} = -\frac{a^3}{3}$$

Along CO :- (0, a) (0, 0)

$$x=0 \quad y \rightarrow a-0$$

$$dx=0$$

$$\int_C \theta = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 \Rightarrow \frac{a^3}{2}$$

$$R.H.S = \int_S \text{curl } \vec{F} \cdot \vec{n} ds$$

$$\Rightarrow \text{curl } \vec{F} = \nabla \times \vec{F}$$

$$\begin{aligned}
 &= \begin{vmatrix} 0 & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} \\
 &= p(0) - j(0) + k(y)
 \end{aligned}$$

$$\operatorname{curl} \vec{F} = y \vec{k} \cdot \vec{n} = \vec{k}$$

$$\int_S \operatorname{curl} \vec{F} \cdot \vec{n} dS = \iint \operatorname{curl} \vec{F} \cdot \vec{n} \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$

$$= \int_0^a \int_0^a y dxdy$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^a dx$$

$$= \int_0^a \frac{a^2}{2} dx$$

$$= \frac{a^2}{2} (x)_0^a = \frac{a^3}{2}$$

$$\therefore L.H.S = R.H.S$$

Hence theorem is Verified.