Aufgabe 35: Likelihood-Quotienten-Test

a) $H_0: \mu = \mu_0 \quad \Rightarrow \Theta_0 = \{(\mu, \sigma^2): \mu = \mu_0, \sigma^2 > 0\}$

$H_1: \mu \neq \mu_0 \quad \Rightarrow \Theta_1 = \{(\mu, \sigma^2): \mu \neq \mu_0, \sigma^2 > 0\}$

$$\Rightarrow \mathcal{L}((\mu, \sigma^2)|x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right)$$

Teststatistik: $$\Gamma(x) = \frac{\sup_{\theta\in\Theta_0} \mathcal{L}(\theta|x)}{\sup_{\theta\in\Theta_1} \mathcal{L}(\theta|x)}$$

b) $$\frac{\partial}{\partial\sigma} \log(\mathcal{L}) = \frac{\partial}{\partial\sigma}\left[\log\left(\frac{1}{(2\pi)^{n/2}} \cdot \sigma^{-n}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] \overset{!}{=} 0$$

$$0 = \cancel{(2\pi)^{n/2}} \cdot \cancel{\sigma^{n}} \cdot (-n) \cdot \cancel{(2\pi)^{-n/2}} \cdot \sigma^{-n-1} + \frac{1}{\sigma^3} \cdot \sum_{i=1}^{n}(x_i-\mu)^2$$

$$0 = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial\mu} \log(\mathcal{L}) = \frac{\partial}{\partial\mu}\left[\log\left(\frac{1}{(2\pi)^{n/2}} \sigma^{-n}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] \overset{!}{=} 0$$

$$0 = \frac{-1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu)\cdot(-1) = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - n\mu$$

$$\Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$H_0: \mu = \mu_0$

$H_1: \mu = \frac{1}{n}\sum_{i=1}^{n} x_i$

c) $$\sup_{\theta\in\Theta_0} \mathcal{L}(\theta|x) = \frac{1}{(2\pi\hat{\sigma}^2)^{n/2}} \exp\left(-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)$$

$$= \frac{1}{\left(\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)^{n/2}} \underbrace{\exp\left(\frac{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2}\right)}_{\exp(-\frac{n}{2})}$$

$$\sup_{\theta\in\Theta_1} \mathcal{L}(\theta|x) = \left(\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-n/2} \exp\left(-\frac{n}{2}\right)$$

$$\Rightarrow \Gamma(x) = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\mu_0)^2}\right)^{n/2}$$

Für eine Stichprobenvarianz gilt $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$\Rightarrow \quad \Gamma(x) = \left(\frac{s^2(n-1)}{\sum_{i=1}^{n} (x_i - \mu_0)^2} \right)^{n/2} = \left(\frac{s^2(n-1)}{n(\bar{x} - \mu_0)^2} \right)^{n/2} < C$$

d) $\sqrt{N} = \sqrt{25} = 5 \qquad \mu_0 = 200ml \qquad \bar{x} = 205ml \qquad s = 10ml$

$$\Rightarrow T = 5 \cdot \frac{(205ml - 200ml)}{10ml} = 2{,}5$$