Aufgabe 23:

a) Punkte der Geraden: $\begin{pmatrix}x_1\\z_1\end{pmatrix}$ und $\begin{pmatrix}x_2\\z_2\end{pmatrix}$

$$x_1 = a z_1 + b \qquad \Rightarrow a = \frac{x_2 - x_1}{z_2 - z_1}$$
$$x_2 = a z_2 + b \qquad \Rightarrow b = x_1 - \frac{x_2 - x_1}{z_2 - z_1} \cdot z_1$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}z_1 & 1\\ z_2 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \underbrace{\frac{1}{z_2 - z_1}\begin{pmatrix}1 & -1\\ -z_2 & z_1\end{pmatrix}}_{B \,\hat{=}\, \text{Jacobimatrix}}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$V\begin{bmatrix}a\\b\end{bmatrix} = \frac{1}{(z_2-z_1)^2}\begin{pmatrix}1 & -1\\ -z_2 & z_1\end{pmatrix}\cdot\begin{pmatrix}\sigma_{x_1}^2 & 0\\ 0 & \sigma_{x_2}^2\end{pmatrix}\cdot\begin{pmatrix}1 & -z_2\\ -1 & z_1\end{pmatrix} \rightarrow \text{BVB-Formel}$$

$$= \frac{1}{(z_2-z_1)^2}\begin{pmatrix}\sigma_{x_1}^2 & -\sigma_{x_2}^2\\ -\sigma_{x_1}^2 z_2 & \sigma_{x_2}^2 z_1\end{pmatrix}\begin{pmatrix}1 & -z_2\\ -1 & z_1\end{pmatrix}$$

$$= \frac{1}{(z_2-z_1)^2}\begin{pmatrix}\sigma_{x_1}^2 + \sigma_{x_2}^2 & -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1)\\ -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1) & \sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2\end{pmatrix} \quad \text{Kovarianzmatrix}$$

$$\left.\begin{aligned}\sigma_a &= \frac{1}{(z_2-z_1)}\cdot\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}\\ \sigma_b &= \frac{1}{z_2-z_1}\cdot\sqrt{\sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2}\end{aligned}\right\} \text{Fehler}$$

$$p(a,b) = \frac{1}{\sigma_a \sigma_b}\,\mathrm{Cov}(a,b) = -\frac{(z_2\sigma_{x_1}^2 + z_1\sigma_{x_2}^2)}{\sqrt{(\sigma_{x_1}^2+\sigma_{x_2}^2)\,(\sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2)}} \quad \text{Korelations-koeffizient}$$

b) $$x_3 = a\cdot z_3 + b = \frac{x_2-x_1}{z_2-z_1}\cdot z_3 + x_1 - \frac{x_2-x_1}{z_2-z_1}\cdot z_1$$

$$V[x_3] = \underbrace{\begin{pmatrix}\frac{\partial x_3}{\partial x_1} & \frac{\partial x_3}{\partial x_2}\end{pmatrix}}_{\text{Jacobi}}\cdot\frac{1}{(z_2-z_1)^2}\begin{pmatrix}\sigma_{x_1}^2 + \sigma_{x_2}^2 & -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1)\\ -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1) & \sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial x_3}{\partial x_1}\\ \frac{\partial x_3}{\partial x_2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{z_2-z_3}{z_2-z_1} & \frac{z_3-z_1}{z_2-z_1}\end{pmatrix}\cdot\begin{pmatrix}\sigma_{x_1}^2 + \sigma_{x_2}^2 & -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1)\\ -(\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1) & \sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2\end{pmatrix}\frac{1}{(z_2-z_1)^2}\cdot\begin{pmatrix}\frac{z_2-z_3}{z_2-z_1}\\ \frac{z_3-z_1}{z_2-z_1}\end{pmatrix}$$

$$= \frac{1}{(z_2-z_1)^4}\left[(\sigma_{x_1}^2+\sigma_{x_2}^2)(z_2-z_3)^2 - (\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1)(z_3-z_1)^2\right]$$

$$\sigma_{x_3} = \sqrt{V[x_3]} = \frac{1}{(z_2-z_1)^2}\cdot\sqrt{(\sigma_{x_1}^2+\sigma_{x_2}^2)(z_2-z_3)^2 - (\sigma_{x_1}^2 z_2 + \sigma_{x_2}^2 z_1)(z_3-z_1)^2}$$

c) $$V[x_3] = \frac{1}{(z_2-z_1)^4}\begin{pmatrix}z_2-z_3 & z_3-z_1\end{pmatrix}\cdot\begin{pmatrix}\sigma_{x_1}^2+\sigma_{x_2}^2 & 0\\ 0 & \sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2\end{pmatrix}\begin{pmatrix}z_2-z_3\\ z_3-z_1\end{pmatrix}$$

ohne Korellation $$= \frac{1}{(z_2-z_1)^4}\cdot\left[(\sigma_{x_1}^2+\sigma_{x_2}^2)(z_2-z_3)^2 + (\sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2)(z_3-z_1)^2\right]$$

$$\sigma_{x_3\,\text{ohne}} = \frac{1}{(z_2-z_1)^2}\sqrt{(\sigma_{x_1}^2+\sigma_{x_2}^2)(z_2-z_3)^2 + (\sigma_{x_1}^2 z_2^2 + \sigma_{x_2}^2 z_1^2)(z_3-z_1)^2}$$