

Renewal Process

Poisson process \Rightarrow times b/w successive events are i.i.d. random variables which are distributed exponentially.

Renewal process \Rightarrow times b/w successive events are i.i.d. random variables which are not necessarily exponentially distributed.

Renewal process is a discrete parameter process $\{X_n, n \geq 1\}$ where X_1, X_2, X_3, \dots are i.i.d non-negative r.v. where X_i is defined as the time elapsed b/w $(i-1)^{\text{st}}$ event and i^{th} event, removing the restriction of exponential distribution.

An example of a renewal process can be a duration for which the candle remains lit and durations after which it has to be lit again (renewed).

If $\{X_n, n \geq 1\}$ is the renewal process, the C.D.F (common distribution function) of the random variables X_i is

$$F(x) = P[X_n \leq x], n=1, 2, 3, \dots$$

$N(t)$ is the count of the number of events that have occurred till time 't'. Here, an event means a renewal happening (i.e. the candle being lit again).

The sum W_n ,

$$W_n = X_1 + X_2 + X_3 + \dots + X_n, n \geq 1$$

is called the waiting time until n^{th} renewal.

$$W_0 = 0$$

If $N(t)$ defines the number of renewals in time interval $[0, t]$, then,

$$M(t) = E(N(t)) \quad \left\{ \begin{array}{l} \text{renewal function (avg. no. of renewals} \\ \text{b/w } [0, t] \end{array} \right.$$

$m(t) = \frac{dM(t)}{dt}$ is called the renewal density. This is the probability of occurrence of a renewal in the interval $[t, t+dt]$.

[In case of a poisson process (λ), $M(t) = \lambda t$ and $m(t) = \lambda$]

Now let X_i 's be the i.i.d r.v. with $F(x)$ as the C.D.F and $W_n = \sum_{i=1}^n X_i$ be the waiting time till n^{th} renewal, with distribution function as $F^{(n)}(t)$.

For $F^{(n)}(t)$, we define,

$$F_x^{(n)}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

If $N(t)$ = number of renewals till t , then

$$P[N(t) = n] = \text{Prob}[n \text{ renewals till time } t]$$

(i) - $P[t \text{ is somewhere b/w the } n^{th} \text{ renewal and } n+1^{th} \text{ renewal}]$

$$= P[W_n \leq t \leq W_{n+1}]$$

$$= P[W_n \leq t] - P[W_{n+1} \leq t]$$

prob. that 't' is greater than the waiting time of n^{th} renewal,
cause it has already occurred

prob. that 't' is greater than waiting time of ' $n+1$ 'th renewal, which means $n+1$ th renewal has also occurred

$$= F^{(n)}(t) - F^{(n+1)}(t).$$

Now, $M(t) = E(N(t))$ (= expectation of 'n' renewals)

$$= \sum_{n=0}^{\infty} n \cdot P[N(t)=n]$$

$$= \sum_{n=0}^{\infty} n \cdot F^{(n)}(t) - \sum_{n=0}^{\infty} n \cdot F^{(n+1)}(t)$$

$$= F^{(1)}(t) + 2F^{(2)}(t) + \dots - [F^{(2)}(t) + 2F^{(3)}(t) + \dots]$$

$$= F^{(1)}(t) + F^{(2)}(t) + F^{(3)}(t) + \dots$$

$$= \boxed{\sum_{n=1}^{\infty} F^{(n)}(t)} - \textcircled{A}$$

$$= F^{(1)}(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t)$$

$F^{(n+1)}$ is the convolution of $F^{(n)}$ and F .

If 'f' is the density function, then

$$\frac{dF}{dt} \quad F^{(n+1)}(t) = \int_0^t F^{(n)}(t-x) \cdot f(x) dx.$$

$$\therefore F^{(1)}(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t)$$

$$= M(t) = F^{(1)}(t) + \sum_{n=1}^{\infty} \left[\int_0^t F^{(n)}(t-x) \cdot f(x) dx \right]$$

$$= F^{(1)}(t) + \int_0^t \left[\sum_{n=1}^{\infty} F^{(n)}(t-x) \right] \cdot f(x) dx$$

$$M(t) = F^{(1)}(t) + \int_0^t M(t-x) \cdot f(x) dx - \textcircled{B}$$

$$\sum_{n=1}^{\infty} F^{(n)}(t-x) = M(t-x) \text{ from Eq } \textcircled{A}$$

Fundamental

Renewal Equation.

Also from Eqn. A $\Rightarrow M(t) = \sum_{n=1}^{\infty} f^{(n)}(t)$

differentiating A term by term \Rightarrow

$$\frac{dM(t)}{dt} = m(t) = \sum_{n=1}^{\infty} f^{(n)}(t)$$

$f^{(n)}$ can be treated as the convolution of $f^{(n-1)}$ and f .

From Eqn. B \Rightarrow

$$m(t) = f''(t) + \int_0^t m(t-x) \cdot f(x) dx$$

Renewal Equation.

Solution of Renewal Equation is obtained by using Laplace Transform
Take Laplace transform on both sides \Rightarrow

$$L(m(t)) = L(f(t)) + L\left[\int_0^t m(t-x) \cdot f(x) dx\right]$$

$$L_m(s) = L_f(s) + \frac{L_m(s) \cdot L_f(s)}{s}$$

$$\text{where } \left\{ L_f(s) = \int_0^{\infty} e^{-st} f(t) dt \right\}$$

$$\text{This gives } L_m(s) = \frac{L_f(s)}{1 - L_f(s)}$$

$$L_f(s) = \frac{L_m(s)}{1 + L_m(s)}$$

Now using these 2 equations, $m(t)$ can be found using $f(t)$ or vice-versa.