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Assignment - 1

MC-303 Stochastic Processes

2K20/MC/08

Abhishek Kumar

Q1 Ans. Non-homogenous Bernoulli process is a sequence of Bernoulli trials over time t_0, where the probability of success in each trial may or may not be same throughout the sequence.

For deriving its probability distribution:

Let the state ~~off~~ of after n trials be $S_n = X_1 + X_2 + X_3 - - - + X_n$.

The probability of success at i^{th} trial will be p_i.
So, the moment generating function of i^{th} trial is:

$$E(e^{X_i s}) = q_i e^{0 \cdot s} + p_i e^{1 \cdot s}$$ ($q \to$ probability of failure $= 1 - p_i$)

$$= q_i + p_i e^{s}$$

So, the moment generating function of n trials will be:

$$E(e^{S_n s}) = \prod_{i=1}^{n} (E_i(e^{X_i s}))$$

$$= \prod_{i=1}^{n} (q_i + p_i e^{s})$$

$$= \prod_{i=1}^{n} q_i + \cancel{e^{s}\left(\prod_{i=1}^{n} p_i \prod_{j\neq i}^{n} (q_j)\right)} e^{s}\left(\sum_{i=1}^{n}\left(p_i \prod_{j\neq i} q_j\right)\right)$$

$$+ e^{2s}\left(\sum_{i_1=1, i_2=2}^{n}\left(p_{i_1} p_{i_2} \prod_{\substack{j\neq i_1 \\ j\neq i_2}}^{n} q_j\right)\right) + - - -$$

$$+ e^{ns} \prod_{i=1}^{n} p_i$$

So, $P(S_n = k)$ is the coefficient of $e^{k\theta}$ in the m.g.f. of Bernoulli Process.

Hence, $$P(S_n = k) = \sum_{i_1 < i_2 < i_3 < i_4 \ldots < i_k}^{n} \left(p_{i_1} \cdot p_{i_2} \cdot p_{i_3} \, p_{i_4} \ldots p_{i_k} \cdot \prod_{\substack{j \neq i_1 \\ j \neq i_2 \\ j \neq i_k}}^{n} q_j \right)$$

This is the probability distribution of non-homogeneous Bernoulli process.

By the definition of Bernoulli process & Bernoulli trials; we have:

$$P(X_i = 1) = p_i \quad ; \quad P(X_i = 0) = q_i$$

So, $$P(X_i = 1 \mid X_{i-1}) = P(X_i = 1 \mid X_{i-1} X_{i-2} X_{i-3} \ldots X_1) = p_i$$

& $$P(X_i = 0 \mid X_{i-1}) = P(X_i = 0 \mid X_{i-1} X_{i-2} X_{i-3} \ldots X_1) = q_i$$

Hence, the non-homogeneous Bernoulli process is a Markov process.

For homogeneous Bernoulli process, $p_1 = p_2 = p_3 = \ldots = p_n = p$

$$P(S_n = k) = \sum_{i_1 < i_2 < i_3 \ldots < i_k}^{n} p^k \cdot q^{n-k}$$

$= p^k q^{n-k}$ (No. of ways of choosing k distinct elements from 1 to n)

$= p^k \cdot q^{n-k} \; {}^nC_k$

$= {}^nC_k \cdot q^{n-k} \, p^k$

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Q 2(b). Homogeneous Poisson process is a counting process in which the rate of arrival is independant of the instance in time, t, denoted by λ. It has the following characteristics:

i) $X(t=0) = 0$.

ii) No. of arrivals between t_j & t_i is independant of the arrivals till t_i.

iii) No. of arrivals is dependant only on the length of interval and not on the location of the interval.

The probability distribution of homogenous Poisson Process is:

$$P(X(t)=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

By the property of the Poisson process:

$$P(X(t_n)=k_n \mid X(t_{n-1})=k_{n-1}) = P(X(t_n - t_{n-1}) = k_n - k_{n-1})$$

$$= \frac{(\lambda(t_n - t_{n-1}))^{k_n - k_{n-1}} e^{\lambda(t_n - t_{n-1})}}{(k_n - k_{n-1})!}$$

$$P(X(t_n)=k_n \mid X(t_1)=k_1, X(t_2)=k_2, X(t_3)=k_3 \ldots X(t_{n-1})=k_{n-1})$$

$$= P(X(t_n) - X(t_{n-1}) = k_n - k_{n-1})$$

$$= P(X(t_n - t_{n-1}) = k_n - k_{n-1})$$

$$= \frac{(\lambda(t_n - t_{n-1}))^{k_n - k_{n-1}} e^{\lambda(t_n - t_{n-1})}}{(k_n - k_{n-1})!}$$

So, $P(X(t_n)=k_n \mid X(t_{n-1})=k_{n-1})$

$$= P(X(t_n)=k_n \mid X(t_1)=k_1, X(t_2)=k_2, X(t_3)=k_3 \ldots X(t_{n-1})=k_{n-1})$$

$$= P(X(t_n) - X(t_{n-1}) = k_n - k_{n-1})$$

So, the state $X(t)$ depends only on the present i.e the moment of start of the interval.

Hence, the homogeneous Poisson process is a Markov process.

In case of non-homogeneous Poisson process, the ~~p.d~~ probability distribution is given by:

$$P(X(t)=n) = \frac{(m(t))^n e^{-m(t)}}{n!}$$

where $$m(t) = \int_0^t \lambda_s(u)\, du$$

where $\lambda(t)$ is the intensity function of arrival rate.

Non-homogeneous Markov process is also a Markov Process as:

$$P(X(t_n)=k_n \mid X(t_1)=k_1, X(t_2)=k_2 \dots X(t_{n-1})=k_{n-1}) = P(X(t_n)=k_n \mid X(t_{n-1})=k_{n-1})$$
$$= P(X(t_n - t_{n-1}) = k_n - k_{n-1})$$

(as the process is independant of the position of interval)

In case of homogeneous ~~Markov~~ Poisson process, $\lambda_s(t) = \lambda$

So, $$m(t) = \int_0^t \lambda\, du = \lambda t.$$

Hence, $$P(X(t)=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

which is the PDF of homogeneous Poisson process.

Q3 Ans. Pure death process is a counting process in which the rate of arrival, λ is 0. and the rate of departure, μ may not be equal to zero. It is a stationary process.

$$\lambda_n = 0, \quad \mu_n = \mu > 0.$$

At time $t = 0$, there are N momentaries in the list.

$$\left.\begin{aligned} P_N(0) &= 1. \\ P_n(0) &= 0 \qquad 0 < n \leq N-1 \end{aligned}\right\} \text{Base condition}$$

$$\frac{dP_N(t)}{dt} = -\mu P_N(t), \quad \text{——— (i)}$$

$$\frac{dp_n(t)}{dt} = -\mu p_n(t) + \mu p_{n+1}(t), \quad 0 < n < N. \quad \text{——— (ii)}$$

This is the differential - difference equation of pure death process.

The probability distribution of the number of departures at time t is the solution of above equation.

$$\frac{dp_0(t)}{dt} = \mu p_1(t). \quad \text{——— (iii)}$$

Solving eq^n (i), we get: $P_N(t) = e^{-\mu t}$.

So, $$\frac{dp_{N-1}(t)}{dt} = -\mu p_{N-1}(t) + \mu e^{-\mu t}$$

$$\Rightarrow p_{N-1}(t) = (\mu t)e^{-\mu t}$$

Similarly, $P_{N-2}(t) = \frac{(\mu t)^2 e^{-\mu t}}{2!}$

And therefore, by induction we get:

$$P_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}$$

This is the probability distribution of Pure Death process.

Q 4A) We know that the interarrival time of Poisson process ~~is~~ follows exponential distribution.

So, sum of n ~~tsts~~ exponential random variables
= sum of n interarrival time of Poisson process.
= Arrival of n^{th} events at the time t, where t = sum of n interarrival times

So, $T_n = S_1 + S_2 + S_3 + S_4 + \cdots + S_n$.

where S_i is exponentially distributed with factor λ

$$P(X(T_n) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{(n)!}$$

is the distribution of arrival of n events ~~process~~ by time T_n, i.e. the sum of n interarrival times of events in addition to some residual time.

The residual time can be removed by:

differentiating both sides by time t, so as to get the probability density function of T_n:

So, $\frac{dP(X(T_n)=n)}{dt} = \frac{\lambda^n . n t^{n-1} e^{-\lambda t}}{n!} - \frac{\lambda^n t^n e^{-\lambda t}}{n!}$

$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} - \frac{\lambda^n t^n e^{-\lambda t}}{n!}$

$P(X(T_n)=n \mid X(T_n-\Delta t)=n-1) \cap \epsilon$

$= P(X(\Delta t)=1) . P(X(T_n-\Delta t)=n-1)$

$= \lambda(\Delta t) . (\lambda(T_n-\Delta t))^n e^{-\lambda(T+\Delta t)}$

We know $f_1(t) = \lambda e^{-\lambda t}$ i.e. probability density function for one interarrival time.

So, $f_2(t) = \int_0^t f_1(t-s) . f(s)\, ds = \int_0^t \lambda e^{-\lambda(t-s)} . \lambda e^{-\lambda s} ds =$

$= \lambda e^{-\lambda t} \int \lambda ds = \lambda^2 t e^{-\lambda t}$.

similarly by induction we get

$f_n(t) = \int_0^t f_{n-1}(t-s) f_1(s)\, ds$

$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

So, $f_n(t)$ is a Gamma distribution.

So, the probability density function of n random vars. with exponential distribution is

$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Q5 Ans a) Gaussian Process.

Gaussian Processes are processes where random variables follow Gaussian distribution, i.e. Normal distribution. ~~which is the central tendency~~

An example of Gaussian process is the stock prices of a company over time t.

b) Brownian Process

An example of Brownian Process is the motion of an electron in a conductor.

c) Stationary Process

An example of stationary process is bacteria cultivation in a culture. In this case, the growth of bacteria is only dependant on the time ~~rate~~ length of the interval and not on the position of the interval.

Q6 Ans. In case of unrestricted random walk, the random variable of the position follows ~~a~~ normal distribution as a virtue of the Central Limit theorem ~~as~~ when the no. of steps becomes very large.

In that ~~scenario~~ ~~scene~~ scenario,

$$\left.\begin{aligned} \text{Mean, } \mu &= E(X) = (p-q) \\ \text{Variance, } \sigma^2 &= E(X^2) - (E(X))^2 \\ &= (p+q) - (p-q)^2 \end{aligned}\right\} \text{Single step}$$

For n steps: $\mu_N = n\mu = n(p-q)$

$\& \ \sigma_N^2 = n\sigma^2$

~~the weather~~

And the probability to exist in range between $x = a$ & $x = b$ is :

$$P(a < x < b) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu_N}{\sigma_N}\right)^2} dx .$$

The property of Normal distribution shows that most chance of finding a random variable is close to mean, μ_N.

And $P(\mu_N - 3\sigma_N < x < \mu_N + 3\sigma_N) = 99.7\%$

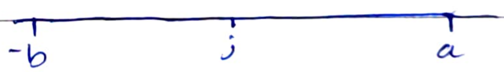
So, the position is very much probable to be around mean.

We have, $p < q \Rightarrow \mu_N = n(p-q) < 0$.

as $n \to \infty$, $\mu_N \to -\infty$.

~~So~~. Thus, as n increases, the element will drift towards negative infinity.

Q 7a) Let us configure a starting point to be $x_0 = j$ and the ~~barrier~~ barriers at $x = a$ & $x = -b$. So, $x_0 = j$ lies between $x = -b$ & $x = a$.



And let $f_{ja}^{(n)}$ be the distribution of the probability of absorption at 'a' when starting at j.

So, $f_{ja}^{(n)}$ is the probability that starting from $x = j$, the particle first visits 'a' at n^{th} step, and not visited $x = -b$ prior to that.

$$f_{ja}^{(n)} = P\{-b < x_1 < a, \ldots\ldots -b < x_{n-1} < a,\ x_n = a \mid x_0 = j\}$$

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For $n=0$, we have $f_{ja}^{(n)} = \begin{cases} 1 & , j=a \\ 0 & , j\neq a \end{cases}$

Let A_n be the event 'absorption at 'a' at time n'.

$f_{ja}^{(n)} = \text{Probab}\{A_n \mid \text{start at } j\}$

$\Rightarrow f_{ja}^{(n)} = p \cdot \text{Probab}\{A_{n-1} \mid \text{start at } j+1\} + (1-p-q)\,\text{Probab}\{A_{n-1}$
$+ q\,\text{Probab}\{A_{n-1} \mid \text{start at } j-1\}.$

$$= p\, f_{(j+1)a}^{n-1} + (1-p-q)\, f_{ja}^{(n-1)} + q\, f_{(j-1)a}^{(n-1)}$$

This is the difference equation of order 1 in n & 2 in j, for $j \in (-b+1, a-1)$

The boundary condition are: $f_{-ba}^{(n)} = f_{aa}^{(n)} = 0$.

To solve the above eq^n, we use generating function:

$$F_j(s) = \sum_{n=0}^{\infty} f_{ja}^{(n)} s^n$$

So, $F_j(s) = s\{p F_{j+1}(s) + (1-p-q) F_j(s) + q\, F_{j-1}(s)\}$.

We also know that $F_a(s) = 1$ & $F_{-b}(s) = 0$.

To solve the above difference equation, we let
$F_j(s) = \lambda^j$

$$\Rightarrow \lambda^j = s\{p\lambda^{j+1} + (1-p-q)\lambda^j + q\,\lambda^{j-1}\}$$

$$\Rightarrow \lambda^2 ps + \lambda((1-p-q)s-1) + qs = 0$$

The solution to above eq^n is:

$$\lambda_1(s) \,\&\, \lambda_2(s) = \frac{1-s(1-p-q) \pm \{(1-s(1-p-q))^2 - 4pqs^2}{2sp}$$

For real roots $\{1-s(1-p-q)\}^2 \geq 4pqs^2$

$$\Rightarrow 0 < s < \frac{1}{1-(\sqrt{p}-\sqrt{q})^2}$$

We take $\lambda_1(s) > \lambda_2(s)$

So, $F_j(s) = A[\lambda_1(s)]^j + B[\lambda_2(s)]^j$

Using boundary conditions, we get:

$$F_j(s) = \frac{\{\lambda_1(s)\}^{j+b} - \{\lambda_2(s)\}^{j+b}}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}}.$$

And for $p = q$: $F_j(s) = \dfrac{a}{a+b}$

This is the required probability of absorption at a specific barrier.

Q8 Ans. Let 0 & a be two reflecting barriers. So, from 0, the particle can stay at 0 or go to 1 & from a, to a & $a-1$.

$$X_n = \begin{cases} X_{n-1} + Z_n, & 0 \leq X_{n-1} + Z_n \leq a \\ a, & X_{n-1} + Z_n > a \\ 0, & X_{n-1} + Z_n < 0. \end{cases}$$

As $n \to \infty$, the moment of the particle settles down to a position, i.e. we do not observe any change in the probability distribution

So, $P_{jk}^{(n)}$ = probability of being at 'k' in n time starting from j

$$\Rightarrow p_{jk}^{(n)} = p\, P_{j(k-1)}^{(n-1)} + (1-p-q)\, P_{jk}^{(n-1)} + q\, P_{j(k+1)}^{(n-1)}$$

And for $k = a$: $p_{ja}^{(n)} = (1-q)\, p_{ja}^{(n-1)} + p\, P_{j(a-1)}^{(n-1)}$

& $k = 0$: $p_{j0}^{(n)} = (1-p)\, P_{j0}^{(n-1)} + (q)\, P_{j1}^{(n-1)}$

As $n \to \infty$, $p_{jk}^{(n)} \to \pi_k$

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$\pi_k = p\,\pi_{k-1} + (1-p-q)\,\pi_k + q\,\pi_{k+1}$

$\pi_a = \pi_a(1-q) + p\,\pi_{a-1}$

$\pi_0 = (1-p)\,\pi_0 + q\,\pi_1. \Rightarrow \pi_1 = \left(\frac{p}{q}\right)\pi_0$

$\pi_2 = \left(\frac{p}{q}\right)^2 \pi_0$

$\vdots$

$\pi_k = \left(\frac{p}{q}\right)^k \pi_0$

$\Rightarrow \left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^a\right)\pi_0 = 1$

$\Rightarrow \pi_0 = \dfrac{1 - p/q}{1 - (p/q)^{a+1}}$

$\Rightarrow \pi_k = \left[\dfrac{1 - p/q}{1 - (p/q)^{a+1}}\right]\left(\frac{p}{q}\right)^k$

Cases :

$p > q \Rightarrow$ Drift is towards 'a' and π_k is in increasing order.

$p < q \Rightarrow$ Drift is towards '0' and π_k is in decreasing order.

& $p = q \Rightarrow \pi_k = \pi_0 = \dfrac{1}{a+1}$