## 12.2 CLASSIFICATION OF STOCHASTIC PROCESSES

Stochastic processes are classified on the basis of the underlying *index set* $T$ and the *state space* $S$.

If $T = \{0, 1, 2, \ldots\}$, or $T = \{0, \pm 1, \pm 2, \ldots\}$, the stochastic process is said to be *discrete parameter process* and is usually indicated by $\{X_n\}$. If $T = \{t: t \geq 0\}$ or $T = \{-\infty < t < \infty\}$, the stochastic process is said to be a *continuous parameter process* and is indicated by $\{X(t), t \geq 0\}$, or $\{X(t), -\infty < t < \infty\}$ as the case is. The state space is classified as *discrete* if it is finite or countable and process is often referred to as *chain*. The state space is classified as *continuous* if it consists of an interval, finite or infinite of the real line and the process is referred to as continuous-state process. The different classifications can be summarized as follows.

| *State space* | *Index set* | |
|---|---|---|
| | *Discrete* | *Continuous* |
| Discrete | Discrete parameter stochastic chain | Continuous parameter stochastic chain |
| Continuous | Discrete parameter continuous-state process | Continuous parameter continuous-state process. |

For example, the waiting time of an arriving inequiry message until process is begun, given by $\{w(t), t \geq 0\}$ is *continuous parameter continuous-state process*. The number of inquiries that arrive in the interval $[0, t]$ given by $\{N(t), t \geq 0]$ is a *continuous parameter stochastic chain*. The average time to run a batch job at a system on the $n$th day of the week, denoted by $\{X_n, n = 1, 2, 3, 4, 5, 6, 7\}$ is a *discrete parameter continuous-state process*; and the number of batch jobs run at a system on the $n$th day of the week, denoted by $\{X_n, n = 1, 2, 3, 4, 5, 6, 7\}$ is a *discrete parameter stochastic chain*. Specific realizations of these classes of stochastic processes are shown respectively in Figs. 12.1, 12.2, 12.3, and 12.4.

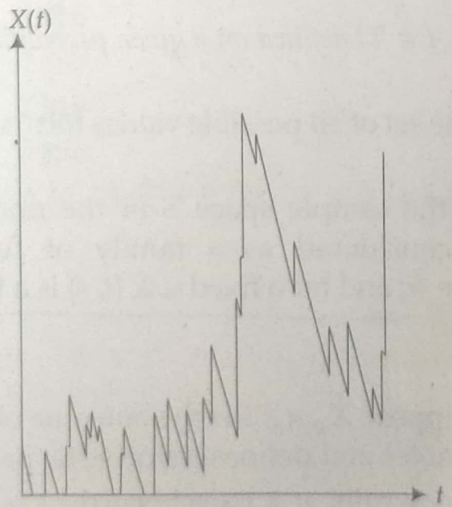



Fig. 12.1

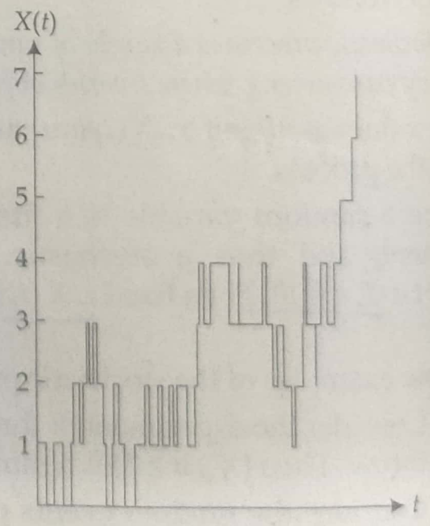



Fig. 12.2

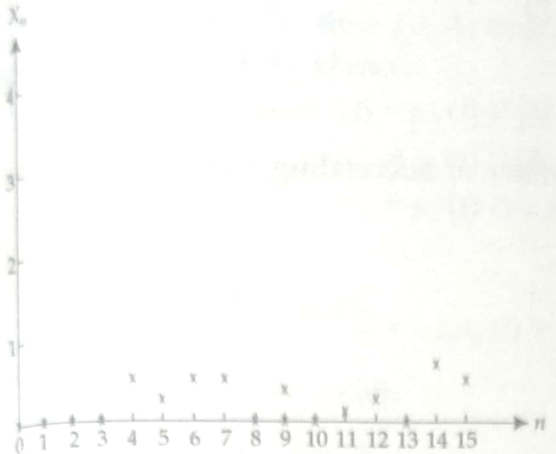



Fig. 12.3

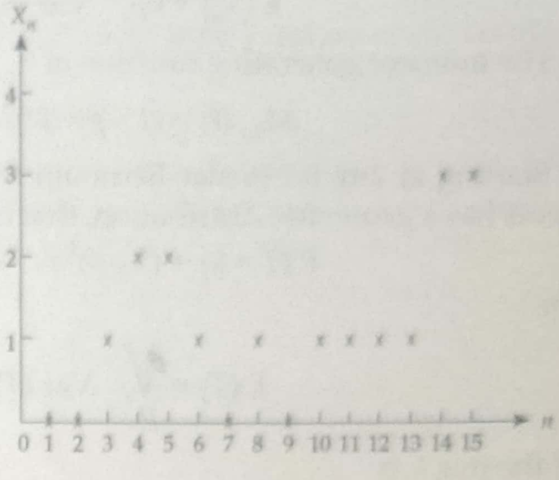



Fig. 12.4

A stochastic process $\{X(t), t \in T\}$ is said to be a *stationary process*, if the joint distribution of $X(t_1)$, and $X(t_2)$, $t_1, t_2 \in T$ depends only on $t_2 - t_1$. A process which is not stationary is called an *evolutionary* process.

A stochastic process $\{X(t), t \in T\}$ is said to be a *Markov process* if for any $t_0 < t_1 < t_2 < ... < t_n < t$, the conditional distribution of $X(t)$ for given values of $X(t_0)$, $X(t_1)$ ..., $X(t_n)$ depends only on $X(t_n)$, that is,

$$P[X(t) \le x \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, ..., X(t_0) = x_0]$$
$$= P[X(t) \le x \mid X(t_n) = x_n]$$

In case of discrete state space, the process is called a *Markov chain*.

A measure of dependence among the random variables of a stochastic process is given by its *autocorrelation function* R, defined by

$$R(t_1, t_2) = E[X(t_1) X(t_2)].$$

In the subsequent sections, we discuss certain specific stochastic processes, which are good models for many practical situations.

P[Xi≥1

## 12.3 THE BERNOULLI PROCESS

The Bernoulli process is an example of a discrete-parameter discrete-state process. Suppose $X_i$, $i = 1, 2, \ldots, n$ are independent and identically distributed Bernoulli random variables each with probability $p$ of success, that is, $P\{X_i = 1\} = p$, and probability $1 - p$ of failure, that is, $P\{X_i = 0\} = 1 - p$. Let $S_n = X_1 + \ldots + X_n$ be the number of successes in $n$ Bernoulli trials. Then $\{S_n, n = 1, 2, \ldots\}$ is called a *Bernoulli process* with state space $\{0, 1, 2, \ldots\}$, and so is a *discrete-parameter discrete-state process*.

Rewriting $S_n$ as $S_n = S_{n-1} + X_n$, we have

$$P\{S_n = k \mid S_{n-1} = k\} = P\{X_n = 0\} = 1 - p$$

and

$$P\{S_n = k \mid S_{n-1} = k - 1\} = P\{X_n = 1\} = p$$

Thus the process is a Markov process. For each $n$, $S_n$ has a *binomial distribution* with parameters $n$ and $p$, and

$$P\{S_n = k\} = {}^nC_k\, p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots n.$$

$$E(S_n) = np, \quad \text{Var}(S_n) = np(1-p)$$

The moment generating function of $S_n$ is

$$M_{S_n}(t) = (1 - p + pt)^n$$

Starting at any particular Bernoulli trial, the number of succeeding trials $T$ before the next success has a geometric distribution, that is,

$$P(T = k) = (1-p)^k\, p, \quad k = 0, 1, 2, \ldots$$

with

$$E(T) = \frac{q}{p}, \quad \text{Var}(T) = \frac{1-p}{p^2}$$

and the m.g.f. is

$$M_T(t) = \frac{tp}{1 - t(1-p)}.$$

In case each of the Bernoulli variate $X_i$ has the distinct parameter $p_i$, then the Bernoulli process is called a *non–homogeneous Bernoulli process*. Another generalization of the Bernoulli process is to assume that each trial has more than two possible outcomes.

## 12.4 THE POISSON PROCESS

The Poisson process is a continuous–parameter discrete-state stochastic process applicable in many practical situations, where the interest is in counting the number of events $N(t)$ occurring in the time interval $(0, t]$, for example, the number of incoming telephone calls to a switchboard, or the number of customers arriving at a service window.

Suppose that 'events' are occurring at random time points, and let $N(t)$ denote the number of events that occurs in the time interval $(0, t]$. These events are said to constitute a *Poisson process with rate* $\lambda$ $(> 0)$, if

*(i)* $N(0) = 0$

*(ii) The number of events that occur in disjoint time intervals are independent.*

*(iii) The distribution of the number of events that occur in a given interval depends only on the length of the interval and not on its location.*

**(iv)* $P\{N(\Delta t) = 1\} = \lambda \Delta t + o(\Delta t)$, *and* $P\{N(\Delta t) \geq 2\} = o(\Delta t)$.

The condition (i) states that the process begins at time 0. Condition (ii) states that the number of events $N(t)$ by time $t$ is independent of the number of events $N(t+s) - N(t)$ that occurs between $t$ and $t + s$, $(s > 0)$. Condition (iii) is the *stationary increment assumption* that states and probability distribution of $N(t+s) - N(t)$ is the same for all values of $t$. Condition (iv) ensures that in a small interval of length $\Delta t$, the probability of occurring one 'event' is approximately $\lambda \Delta t$, whereas of occurring two or more events is approximately 0.

* We say that the function $f$ is $o(\Delta t)$, if $\lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t} = 0$, that is, for small values of $\Delta t$, $f(\Delta t)$ is small even in comparison with $\Delta t$.

Next we obtain an expression for $p_n(t) = P[N(t) = n]$, $n = 0, 1, 2, \ldots$, the probability that the number of events occurring in the interval $(0, t]$ is $n$. First we consider for $n = 0$.

No event occurs by time $t + \Delta t$ only if no event occurs by time $t$ and no event occurs in the interval from $t$ to $t + \Delta t$. Hence,

$$\begin{aligned} p_0(t + \Delta t) &= p_0(t)\, P[N(t + \Delta t) - N(t) = 0] \\ &= p_0(t)\, P[N(\Delta t) = 0] \\ &= p_0(t)\,(1 - \lambda \Delta t + o(\Delta t)) \end{aligned}$$

This gives

$$\frac{p_0(t + \Delta t) - p_0(t)}{\Delta t} = -\lambda p_0(t) + \frac{o(\Delta t)}{\Delta t}$$

Making $\Delta t \to 0$, we obtain

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad \ldots(12.1)$$

Using the initial condition $p_0(0) = 1$, this gives

$$p_0(t) = e^{-\lambda t} \qquad \ldots(12.2)$$

Next, suppose $n > 0$. The $n$ events can occur by time $t + \Delta t$ in three mutually exclusive ways:

(i) $n$ events occur by time $t$ and no event occurs in the interval from $t$ to $t + \Delta t$.

(ii) $n - 1$ events occur by time $t$ and one event occurs in the interval from $t$ to $t + \Delta t$.

(iii) $n - m$ events occur by time $t$ for some $m \in \{2, 3, \ldots, n\}$ and exactly $m$ events occur in the interval from $t$ to $t + \Delta t$.

Summing up the probabilities of the events (i), (ii), and (iii), we obtain

$$p_n(t + \Delta t) = p_n(t)\,(1 - \lambda \Delta t + o(\Delta t)) + \lambda \Delta t\, p_{n-1}(t) + o(\Delta t)$$

This gives

$$\frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(\Delta t)}{\Delta t}$$

Making $\Delta t \to 0$, we obtain

$$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n \geq 1 \qquad \ldots(12.3)$$

The initial conditions are $p_n(0) = 0$, $n \geq 1$.

We solve for $p_n(t)$ by induction on $n$.

For $n = 1$ Eq. (12.3), using (12.2) becomes

$$\frac{dp_1(t)}{dt} + \lambda p_1(t) = \lambda e^{-\lambda t},$$

which is a linear differential equation in $p_1(t)$. This gives

$$p_1(t) = e^{-\lambda t} \lambda t,$$

using the condition $p_1(0) = 0$.

Next, let us suppose that

$$p_{n-1}(t) = \frac{e^{-\lambda t}}{(n-1)!} (\lambda t)^{n-1}$$

Substituting this in Eq. (12.3) and solving the linear differential equation obtained using the initial conditions $p_n(0) = 0, n \geq 1$, we obtain

$$p_n(t) = \frac{e^{-\lambda t}}{n!} (\lambda t)^n, \quad \text{...(12.4)}$$

and this proves the induction step.

It can be verified by direct substitution that (12.4) satisfies the Eq. (12.3). We note that (12.4) is the probability distribution of a Poisson variate with parameter $\lambda t$.

Thus, we have proved the following result:

**Theorem 12.1:** *Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Then the random variable $N(t)$ describing the number of events in any time interval of length $t > 0$ has a Poisson distribution with parameter $\lambda t$, that is,*

$$P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

*Thus, the average number of events occurring in any time interval of length $t$ is $\lambda t$.*

Next, let in the Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda$, $0 = t_0 < t_1 < t_2 < \ldots$ be the successive occurrence times of events, and let $\tau_n = t_n - t_{n-1}, n = 1, 2, \ldots,$ be the inter-arrival times. Consider the sequence $\{\tau_n\}$ of inter-arrivals times. Obviously, by the definition of a Poisson process $\tau_n$ are mutually independent and identically distributed random variables, and further for any $s \geq 0$ and any $n \geq 1$

$$\begin{aligned} P[\tau_n > s] &= P[N(t_{n-1} + s) - N(t_{n-1}) = 0] \\ &= P[N(s) = 0] = e^{-\lambda s} \end{aligned} \quad \text{...(12.5)}$$

using the *stationary increment assumption* of a Poisson process, and then using (12.4). Therefore,

$$P[\tau_n \leq s] = 1 - e^{-\lambda s}, \quad s \geq 0 \quad \text{...(12.6)}$$

which is the distribution function of an exponential variate with mean $1/\lambda$. Thus, we have the following result:

**Theorem 12.2** *Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Then the inter-arrival times $\{\tau_n\}$ of successive events are mutually independent, and identically distributed exponential random variables each with mean $1/\lambda$.*

**Remarks**

1. From (12.4), we observe that in a Poisson process, the number of events $N(t)$ in the interval $(0, t]$ has a Poisson distribution with parameter $\lambda t$. This implies that Poisson process is not *a stationary process*.
2. Since $E(N(t)) = Var(N(t)) = \lambda t$, thus as $t$ approaches $\infty$, $E(N(t)/t)$ approaches $\lambda$ and $Var(N(t)/t)$ approaches zero, that is, $N(t)/t$ the number of events per unit time in a Poisson process converges to $\lambda$ as $t$ approaches infinity, and as such, the parameter $\lambda$ is called the *arrival rate* of the Poisson process.
3. The pooled effect of two independent Poisson processes with rates $\lambda_1$ and $\lambda_2$ is also a Poisson process with rate $\lambda_1 + \lambda_2$. But the difference of two Poisson processes is not a Poisson process.
4. Poisson process is widely used in reliability theory and queueing theory. It is a special case of a general type of stochastic process called a *birth–and–death process*. Further the *Poisson process is Markovian*, since the future state depends on the present only and not on the past state.

5. In case the parameter $\lambda$ is not a constant but a function of $t$, then the process is called a *non-homogeneous Poisson* process.

**Example 12.1:** Suppose that the customers are arriving at a ticket counter according to a Poisson process with a mean rate of 2 per minute. Find the probability that in an interval of five minutes, the number of customers arriving is (a) exactly 3, (b) less than 3 (c) greater than 3.

**Solution:** If $N(t)$ denotes the number of customers arriving in an interval $(0, t]$, then $N(t)$ follows Poisson distribution with mean $2t$.

(a) The probability the number of customers arriving is exactly 3 in an interval of five minutes is

$$P[N(5) = 3] = \frac{e^{-10}(10)^3}{3!} = 0.0076$$

(b) $$P[N(5) < 3] = \sum_{k=0}^{2} \frac{e^{-10}(10)^k}{k!}$$

$$= e^{-10}\left[1 + 10 + \frac{100}{2}\right] = 0.0028$$

(c) $$P[N(5) > 3] = 1 - P(N(5) \leq 3)$$
$$= 1 - 0.0104 = 0.9896$$

**Example 12.2:** Suppose that the passengers arrive at a train terminal according to a Poisson process with rate $\lambda$. If the train is dispatched at time $t$, find the expected sum of the waiting times of all those that enter the train.

**Solution:** If $t_i$ is the time of the $i$th arrival, $i \geq 1$ and $N(t)$ is the number of arrivals by time $t$, then the total waiting time $W(t)$ is given by

$$W(t) = \sum_{i=1}^{N(t)} (t - t_i)$$

The total waiting time by time $t + \Delta t$ is equal to the total waiting time by $t$, plus $\Delta t$ time the number present at $t$, plus the sum of the waiting times of all arrivals between $t$ and $t + \Delta t$. Thus,

$$W(t + \Delta t) = W(t) + \Delta t\, N(t) + W(t, t + \Delta t) \quad ...(12.7)$$

Let $E[W(t)] = w(t)$, and obviously $E[W(t, t + \Delta t)] = o(\Delta t)$

Taking expectations on both sides of Eq. (12.7), we obtain

$$w(t + \Delta t) = w(t) + \lambda t\, \Delta t + o(\Delta t)$$

or, $$\frac{w(t + \Delta t) - w(t)}{\Delta t} = \lambda t + \frac{o(\Delta t)}{\Delta t}$$

Making $\Delta t \to 0$, we obtain

$$\frac{dw(t)}{dt} = \lambda t$$

and thus, $$w(t) = \frac{\lambda t^2}{2} + c.$$

At $t = 0$, $w(t) = 0$. This gives $c = 0$ and hence
$$E[W(t)] = \lambda t^2/2.$$

## 12.5 BIRTH-AND-DEATH PROCESS

In the preceding section, we have studied the Poisson process $\{N(t), t \geq 0\}$ in which the occurrence of an 'event' can be considered as an arrival of some entity at an average arrival rate $\lambda$. We can interpret such an arrival as 'birth' with birth rate $\lambda$. For some systems, such as a biological species or a queueing system, the birth rate may reasonably be considered to depend on the population size $n$, say $\lambda_n$ at that instant. Similarly, it is reasonable to assume all deaths or decreases in the population with some rate, say $\mu_n$. Population may be of 'customers' joining a queue with some arrival rate before a service window and then leaving with some departure rate after receiving the service. Here arrival of a customer corresponds to a 'birth' and departure after being served corresponds to a 'death'.

Consider a continuous parameter stochastic process $\{X(t), t \geq 0\}$ with the discrete state space. We say that the process is in state $E_n$ at time $t$, if $X(t) = n$, $n = 0, 1, 2, \ldots$ The process is described as a *birth-and-death process* if there exist non-negative birth rates $\lambda_n$, $n = 0, 1, 2, \ldots$ and non-negative death rates $\mu_n$, $n = 0, 1, 2, \ldots$ such that the following assumptions are satisfied.

1. *State changes are allowed only from $E_n$ to $E_{n+1}$, or $E_n$ to $E_{n-1}$, if $n \geq 1$ but from $E_0$ to $E_1$ only.*
2. *If at time $t$ the system is in state $E_n$, the probability between time $t$ and time $t + \Delta t$ of state changes from $E_n$ to $E_{n+1}$ equals $\lambda_n \Delta t + o(\Delta t)$, and from $E_n$ to $E_{n-1}$ equals $\mu_n \Delta t + o(\Delta t)$. The probability of more than one transition in the time interval $t$ to $t + \Delta t$ is $o(\Delta t)$.*

Let $p_n(t) = P[X(t) = n]$ be the probability that the system is in state $E_n$ at time $t$. We derive the differential–difference equation for $p_n(t)$.

If $p_n(t + \Delta t)$, $n \geq 1$ is the probability that at time $t$, system is in state $E_n$, then this can happen in four mutually exclusive ways:

(a) The system was in state $E_n$ at time $t$ and no birth or death occurred in the interval from $t$ to $t + \Delta t$.

(b) The system was in state $E_{n-1}$ at time $t$ and one birth occurred in the interval from $t$ to $t + \Delta t$.

(c) The system was in state $E_{n+1}$ at time $t$ and one death occurred in the interval from $t$ to $t + \Delta t$.

(d) The system was in state $E_n$ at time $t$ and two or more transitions occurred in the interval from $t + \Delta t$; with associated probability zero.

Summing up the probabilities of the events at (a), (b), (c) and (d), we obtain
$$p_n(t + \Delta t) = p_n(t)\,[1 - \lambda_n \Delta t - \mu_n \Delta t] + p_{n-1}(t)\,\lambda_{n-1} \Delta t + p_{n+1}(t)\,\mu_{n+1} \Delta t + o(\Delta(t))$$
Rearranging, dividing by $\Delta t$, and making $\Delta t \to 0$, we obtain
$$\frac{dp_n(t)}{dt} = -(\lambda_n + \mu_n)\,p_n(t) + \lambda_{n-1}\,p_{n-1}(t) + \mu_{n+1}\,p_{n+1}(t),\ n \geq 1. \qquad \ldots(12.8a)$$
By similar reasoning for $n = 0$, we obtain
$$\frac{dp_0(t)}{dt} = -\lambda_0\,p_0(t) + \mu_1\,p_1(t). \qquad \ldots(12.8b)$$

If initially the process is in state $E_i$, then initial conditions are

$$p_i(0) = 1 \quad \text{and} \quad p_j(0) = 0 \quad \text{for } j \neq i. \qquad \ldots(12.9)$$

The transient solution of this system of differential-difference equations is a difficult task except for some very special cases. As special cases, first we discuss the *pure birth process* in which births (arrivals) only are allowed, and then the *pure death process*, in which deaths (departures) only are permitted. An example of the pure birth process is the creation of birth certificates for newly born babies and the pure death process, may be explained by the random withdrawl of stocked items in a store. Also we derive the steady state solutions of the differential difference Eqs (12.8).

### 12.5.1 Pure Birth Process

Consider a pure birth (arrival) process with $\lambda_n = \lambda > 0$ and $\mu_n = 0$ for all $n$, the system of Eqs (12.8) becomes

$$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n \geq 1$$

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

The solution of this system of equations satisfying the given initial conditions (12.9) as observed in *Section 12.4*, is

$$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \geq 0, \quad t \geq 0.$$

*Thus the pure–birth process with constant birth rate $\lambda$ is a Poisson process with average arrival rate equal to the birth rate $\lambda$.*

**Example 12.3:** Babies are born in a state at the rate of one birth every 20 minutes. The inter-arrival time follows an exponential distribution. Find

(a) The expected number of births per year

(b) The probability that no births will occur in any one day.

(c) The probability of occurring 20 births in 4 hours given that 15 births have already occurred in the first 3 hours of the 4–hour period.

**Solution:** Let $\lambda$ be the birth rate per day, then

$$\lambda = \frac{24 \times 60}{20} = 72 \text{ births per day.}$$

(a) The expected number of birth per year in the state is

$$\lambda t = 72 \times 365 = 26280 \text{ births per year.}$$

(b) The probability of no birth in any one day, $p_0(t)$ at $t = 1$, is

$$p_0(1) = \frac{(72 \times 1)^0 e^{-72 \times 1}}{0!} \approx 0.$$

(c) The probability of occurring 20 births in a 4-hour period given that 15 births have already occurred in the first 3 hours of that period is simply equal to the probability of $(20 - 15) = 5$ births in $(4 - 3) = 1$ hour. The birth rate per hour is $60/20 = 3$, thus

$$p_5(1) = \frac{(3 \times 1)^5 e^{-3 \times 1}}{5!} = 0.0112.$$

### 12.5.2 Pure Death Process

In the pure death (departure) process, we assume that system starts with $N$ inventories in a store at time $t = 0$ and no other arrivals are allowed. The departures occur at rate of $\mu$ items per unit time, that is, $\lambda_n = 0$ and $\mu_n = \mu$ for all $n \le N$. The differential–difference Eqs. (12.8) are modified to

$$\frac{dp_N(t)}{dt} = -\mu p_N(t)$$

$$\frac{dp_n(t)}{dt} = -\mu p_n(t) + \mu p_{n+1}(t), \quad 0 < n < N$$

$$\frac{dp_0(t)}{dt} = \mu p_1(t)$$

with initial conditions $p_N(0) = 1$ and $p_k(0) = 0 < k < N$. Proceeding on the same lines as in Section 12.4, the solution of the system of equations is the *truncated Poisson distribution* with rate $\mu$ given by

$$p_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}, \quad n = 1, 2, \ldots, N$$

$$p_0(t) = 1 - \sum_{n=1}^{N} p_n(t).$$

### 12.5.3 Birth and Death Process: The Steady-State Solution

The *steady–state solution* of the differential–difference equation (12.8), are obtained by setting $dp_n/dt$ equal to zero and setting $p_n = \lim_{t \to \infty} p_n(t)$ (assuming it exists) for all $n$. We obtain the set of difference equations

$$0 = -(\lambda_n + \mu_n)\, p_n + \lambda_{n-1}\, p_{n-1} + \mu_{n+1}\, p_{n+1}, \quad n \ge 1 \qquad \ldots(12.10)$$

$$0 = -\lambda_0\, p_0 + \mu_1 p_1, \quad n = 0 \qquad \ldots(12.11)$$

By rearranging Eq. (12.10), we obtain

$$\begin{aligned} \lambda_n p_n - \mu_{n+1} p_{n+1} &= \lambda_{n-1} p_{n-1} - \mu_n p_n \\ &= \lambda_{n-2} p_{n-2} - \mu_{n-1} p_{n-1} \\ &= \lambda_{n-3} p_{n-3} - \mu_{n-2} p_{n-2} \\ &\;\;\vdots \\ &= \lambda_0 p_0 - \mu_1 p_1 \\ &= 0, \quad \text{using (12.11)}. \end{aligned}$$

Thus,

$$p_{n+1} = \frac{\lambda_n}{\mu_{n+1}}\, p_n, \quad n \ge 0.$$

$$p_n = \frac{\lambda_{n+1}}{\mu_n} p_{n-1}, \quad n \geq 1.$$

or

This gives

$$p_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} p_0 = p_0 \prod_{i=0}^{n-1} \left( \frac{\lambda_i}{\mu_{i+1}} \right), \quad n \geq 1. \quad \dots(12.12)$$

Since $\sum_{n=0} p_n = 1$, thus we have

$$p_0 = \frac{1}{1 + \sum_{n \geq 1}^{n} \prod_{i=0}^{n-1} \left( \frac{\lambda_i}{\mu_{i+1}} \right)}, \quad \dots(12.13)$$

the probability of being the system in $E_0$ state. Thus, the limiting distribution ($p_0, p_1, \dots$) is now completely determined. Also the probabilities are non-zero if $p_0$ is non-zero, that is, if the series $\sum_{n \geq 1}^{n} \prod_{i=0}^{n-1} \left( \frac{\lambda_i}{\mu_{i+1}} \right)$ is convergent.

The Eqs. (12.12) and (12.13) are extremely useful in generating a variety of queuing models.

**Example 12.4:** The number of counters in operation in a retail store depends on the number of customers in the store as per the schedule given below:

| | | | |
|---|---|---|---|
| No. of customers in the store: | 1–3 | 4–6 | 7 and more |
| No. of counters in operation: | 1 | 2 | 3 |

Customers arrive in the counter area according to a Poissson distribution with a mean rate of 10 customers per hour. The average checkout time per customers is exponential with mean 12 minutes. Determine the steady-state probability $p_n$ of $n$ costomers in the checkout area. Also find the probability that three counters will be in operation.

**Solution:** Here we have

$$\lambda_n = \lambda = 10 \text{ customers per hour, } n = 0, 1, 2, \dots$$

$$\mu_n = \begin{cases} \frac{60}{12} = 5 & \text{customers per hour,} \quad n = 1, 2, 3 \\ 2 \times 5 = 10 & \text{customers per hour,} \quad n = 4, 5, 6 \\ 3 \times 5 = 15 & \text{customers per hour,} \quad n = 7, 8, \dots \end{cases}$$

Thus,

$$p_1 = \left( \frac{10}{5} \right) p_0 = 2\, p_0$$

$$p_2 = \left( \frac{10}{5} \right)^2 p_0 = 4\, p_0$$

$$p_3 = \left(\frac{10}{5}\right)^3 p_0 = 8\, p_0$$

$$p_4 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right) p_0 = 8\, p_0$$

$$p_5 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^2 p_0 = 8\, p_0$$

$$p_6 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 p_0 = 8\, p_0$$

$$p_n = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{10}{15}\right)^{n-6} p_0 = 8\left(\frac{2}{3}\right)^{n-6} p_0, \quad n = 7, 8, \ldots$$

The value of $p_0$ is determined from the fact that

$$p_0 + \sum_{n=1}^{\infty} p_n = 1$$

This gives $p_0 + p_0\,(2 + 4 + 8 + 8 + 8 + 8 + 8(2/3) + 8(2/3)^2 + \ldots] = 1$

or, $$p_0\left[31 + 8\left(\frac{1}{1-(2/3)}\right)\right] = 1$$

or, $$55p_0 = 1 \quad \text{or} \quad p_0 = 1/55.$$

Thus,

$$p_n = \frac{8}{55}\left(\frac{2}{3}\right)^{n-6}$$

The probability that 3 counters will be in operation is equivalent to the probability that number of customers will be 7 or more.

This is given by

$$P[n \geq 7] = \sum_{n=7}^{\infty} p_n = \frac{8}{55}\sum_{n=7}^{\infty}\left(\frac{2}{3}\right)^{n-6}$$

$$= \frac{8}{55}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\right]$$

$$= \frac{8}{55}\,\frac{2/3}{1-(2/3)} = \frac{16}{55}$$

## 12.6 RENEWAL PROCESS

We have observed that time between successive events in a Poisson process are independent, and identically distributed exponential random variables. A *renewal process* (or, *recurrent process*) is a generalization of the Poisson process obtained by removing the restriction of the exponential inter-arrival time between two successive events.

*A renewal process is defined to be a discrete-parameter independent process $\{X_n, n \geq 1\}$, where $X_1, X_2, \ldots$ are independent and identically distributed, non-negative random variables.* Here $X_i$ may be interpreted as the time between the $(i-1)$st and the $i$th events, removing the restriction of exponential distribution.

As an example of a renewal process, consider a system in which the replacement after a failure is performed, requiring negligible time. The times between successive failures might well be independent and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$.

If $\{X_n, n \geq 1\}$ is the renewal process, then the common distribution function of the random variables $X_i$ is

$$F(x) = P[X_n \leq x], \quad n = 1, 2, 3, \ldots$$

Normally, we use $\mu$ for the common mean and $\sigma^2$ for the common variance of the sequence $\{X_n, n \geq 1\}$. Further when an event counted by $N(t)$ occurs, we say that a *renewal* has taken place. The sum

$$W_n = X_1 + X_2 + \ldots + X_n, \quad n \geq 1, \text{ with } W_0 = 0$$

is called the *waiting time until the nth renewal.*

The process $\{N(t), t \geq 0\}$ is called the *renewal counting process*, and $N(t)$ is called the *renewal random variable.*

A common example of a renewal process is the light bulb example. A light bulb is installed at time $W_0 = 0$. When it burns out at time $W_1 = X_1$, it is replaced by a new bulb, that burns out at time $W_2 = X_1 + X_2$; and the process continues indefinitely, as each bulb burns out it is replaced with a fresh one. It is assumed that sucessive bulb lifetimes $\{X_n, n \geq 1\}$ are independent and identically distributed random variables. Here $N(t)$ is the number of light bulb replacements that occured in the interval $(0, t]$.

The function $M(t)$, defined for all $t > 0$, given by

$$M(t) = E[N(t)] \quad \ldots(12.14)$$

is called the *renewal function* of the renewal process and gives the average number of renewals in the interval $(0, t]$. The derivative of the renewal function $M(t)$, given by

$$m(t) = \frac{dM(t)}{dt} \quad \ldots(12.15)$$

is called the *renewal density*. This can be interpreted as the probability of occurrence of a renewal in the interval $[t, t + \Delta t]$.

For example, in case of a Poisson process with rate $\lambda$, $M(t) = \lambda t$ and $m(t) = \lambda$.

### 12.6.1 Fundamental Renewal Equation

Consider the renewal process $\{X_n, n \geq 1\}$, where $X_i$'s are independent and identically distributed random variables with the underlying distribution function $F(x)$. Let $W_n = \sum_{i=1}^{n} X_i$ be the waiting time until the $n$th renewal, and $F^{(n)}(t)$ be its distribution function. Obviously, $F^{(n)}$ is the $n$-fold convolution of $F$ with itself, where we define

$$F^{(0)}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

If $N(t)$ is number of renewals until $t$, then

$$\begin{aligned} P[N(t) = n] &= P[W_n \leq t < W_{n+1}] \\ &= P[W_n \leq t] - P[W_{n+1} \leq t]. \\ &= F^{(n)}(t) - F^{(n+1)}(t) \end{aligned} \quad \text{...(12.16)}$$

The renewal function $M(t)$ of the process is

$$M(t) = E[N(t)]$$

$$= \sum_{n=0}^{\infty} n\,P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} n\,F^{(n)}(t) - \sum_{n=0}^{\infty} n\,F^{(n+1)}(t)$$

$$= \sum_{n=1}^{\infty} F^{(n)}(t) \quad \text{...(12.17)}$$

$$= F(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t) \quad \text{...(12.18)}$$

Since $F^{(n+1)}$ is the convolution of $F^{(n)}$ and $F$, in case $f$ is the density function, then we can write

$$F^{(n+1)}(t) = \int_0^t F^{(n)}(t-x)\,f(x)\,dx,$$

Therefore (12.18) becomes

$$M(t) = F(t) + \sum_{n=1}^{\infty}\left[\int_0^t F^{(n)}(t-x)\,f(x)\,dx\right]$$

$$= F(t) + \int_0^t \left(\sum_{n=1}^{\infty} F^{(n)}(t-x)\right) f(x)\,dx$$

$$= F(t) + \int_0^t M(t-x)\,f(x)\,dx, \quad \text{...(12.19)}$$

using (12.17).

This is known as the *fundamental renewal equation*.

Differentiating (12.17) term by term, we have renewal density $m(t)$ as

$$m(t) = \sum_{n=1}^{\infty} f^{(n)}(t) \qquad \ldots(12.20)$$

where $f^{(n)}$ can be considered as the convolution of $f^{(n-1)}$ with $f$. Further from (12.19), we have

$$m(t) = f(t) + \int_0^t m(t-x)\, f(x) dx \qquad \ldots(12.21)$$

This is known as the *renewal equation*.

Under appropriate conditions, the *asymptotic renewal rate*, given by $\lim_{t\to\infty} m(t)$, can be shown equal to $1/E(X)$.

Solution of the renewal equation, (12.21) is obtained using Laplace transform.

Taking Laplace transform on both sides of (12.21), we have

$$L[m(t)] = L[f(t)] + L\left[\int_0^t m(t-x)\, f(x)\, dx\right]$$

or,

$$L_m(s) = L_f(s) + L_m(s)\, L_f(s),$$

where $L_m(s) = \int_0^\infty e^{-st}\, m(t)$, $s > 0$ etc.

This gives

$$L_m(s) = \frac{L_f(s)}{1 - L_f(s)} \qquad \ldots(12.22)$$

or,

$$L_f(s) = \frac{L_m(s)}{1 + L_m(s)} \qquad \ldots(12.23)$$

Using these two equations $m(t)$ can be determined by $f(t)$ or vice versa.

In case of a process with inter-arrival times exponentially distributed (that is, the Poisson process) with parameter $\lambda$, we have

$$f(t) = \lambda e^{-\lambda t}, \quad \text{and so,} \quad L_f(s) = L[\lambda e^{-\lambda t}] = \frac{\lambda}{s+\lambda}$$

Thus, $$L_m(s) = \frac{\lambda/(s+\lambda)}{1-\lambda/(s+\lambda)} = \frac{\lambda}{s}$$

Taking inverse transform

$$m(t) = L^{-1}\left[\frac{\lambda}{s}\right] = \lambda,$$

and, therefore,

$$M(t) = \lambda t, \quad t \geq 0.$$

Also $F^{(n)}(t)$ is the convolution of $n$ identical exponential distributions, that is,

$$F^{(n)}(t) = 1 - \left(\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right) e^{-\lambda t}$$

Thus,

$$P[N(t) = n] = F^{(n)}(t) - F^{(n+1)}(t), \quad \text{refer to (12.16)}$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

a result in confirmation with the result obtained in case of a Poisson process.

**Example 12.5:** Suppose $\{N(t),\ t \geq 0\}$ is a renewal counting process with renewal function $M(t) = 2t$. Find the probability distribution of the number of renewals by time 10.

**Solution:** Let $\{X_n,\ n \geq 1\}$ be the renewal process with $F(t)$ and $f(t)$ respectively as the underlying probability distribution and density functions.

Here the renewal function is $M(t) = 2t$. Hence, the renewal density is

$$m(t) = \frac{d}{dt}M(t) = 2$$

If $L_m(s)$ and $L_f(s)$ are the Laplace transforms of the renewal density and probability density function, then

$$L_f(s) = \frac{L_m(s)}{1 + L_m(s)} = \frac{2/s}{1 + 2/s} = \frac{2}{s+2}$$

Taking inverse Laplace transform, the p.d.f. is

$$f(t) = 2e^{-2t},$$

The inter–arrival time is exponential with parameter 2, and hence the renewal process is Poisson with rate 2. Thus the probability distribution of the number of renewals $N(t)$ in the interval (0, 10] is

$$P[N(t) = n] = \frac{(20)^n e^{-20}}{n!}, \quad n \geq 0.$$

## 12.7 MARKOV CHAINS

Consider a discrete parameter discrete-state space stochastic process $X_n$, $n = 0, 1, 2, \ldots$ that takes on a finite or countable number of possible values. If $X_n = i$, then the process is said to be in state $i$ at time $n$, and let whenever the process is in state $i$, then there is a probability $p_{ij}$ that it will next be in state $j$. Such a stochastic process is said to be a *Markov chain*, if

$$P[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_o] = P[X_{n+1} = j \mid X_n = i] = p_{ij} \quad \ldots(12.24)$$

The property (12.24) may be interpreted that *any future state of the process depends upon its present state and is independent of its past.*

Normally, *the one-step transition probabilities*

$$P[X_{n+1} = j \mid X_n = i] = p_{ij}$$

depend upon the index $n$. However, we are interested primarily in Markov chains for which these probabilities are independent of $n$. Such a Markov chain is said to be *homogeneous in time*. Here we shall assume the Markov chains to be *time-homogeneous*.

The one-step transition probabilities are completely specified in the form of a *transition probability matrix* **P** given by

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad \ldots(12.25)$$

with $p_{ij} \geq 0;\ i, j = 0, 1, 2, \ldots,$ and $\sum_j p_{ij} = 1$, for each $i$.

It is a square matrix. If the number of states is finite, say $n$, then it will be an $n \times n$ square matrix, otherwise, the matrix will be infinite.

Any such square matrix that has non-negative entries with row sums all equal to unity is called a *stochastic matrix*. Further a chain is said to be *regular* if all the entries of $\mathbf{P}^m$ are positive for some $m \geq 1$.

Consider a communication system that transmits the digits 0 and 1. Each digit must pass through several stages and at each stage there is probability $p$ that digit entered will leave unchanged. If $X_n$ denotes the digit leaving the $n$th stage of the system and $X_0$ denotes the digit entering the first stage, then $\{X_n, n \geq 0\}$ is a two-state Markov chain with transition probability matrix.

$$P = \text{State of } X_{n-1} \begin{array}{c|cc} & \multicolumn{2}{c}{\text{State of } X_n} \\ & 0 & 1 \\ \hline 0 & p & 1-p \\ 1 & 1-p & p \end{array}$$

This matrix *P* can be easily obtained from the *channel diagram*

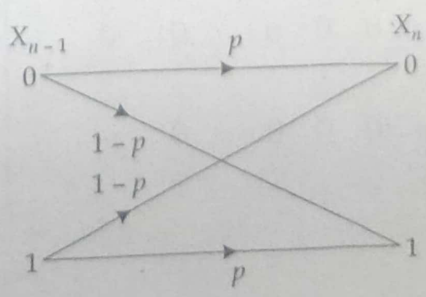



**Fig. 12.5**

A graphical representation of this Markov chain is provided by the state diagram as shown in Fig. 12.6.

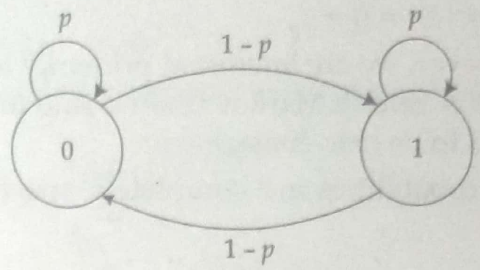



**Fig. 12.6**

The *joint probability* $[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0]$ in case of a Markov chain is given by

$$P\,[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0]$$

$$= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_0 = i]\; P\,[X_{n-1} = i_{n-1}, \ldots, X_0 = i_1]$$

$$= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}]\; P[X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] \qquad \text{(Markovian property)}$$

$$= P\,[X_n = i_n \mid X_{n-1} = i_{n-1}]\; P\,[X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}] \ldots P[X_1 = i_1 \mid X_0 = i_0]\; P[X_0 = i_0] \quad \text{...(12.26)}$$

*Thus the joint probability is the product of the successive one-step transition probabilities and the initial probability.*

**Example 12.6:** Consider a sequence of Bernoulli trials with probability $p$ of success. Let $X_n$ the outcome of the $n$th trial be $k$, where $k = 0, 1, 2, \ldots, n$ denotes that there is a run (uninterrupted block) of $k$ successes. Find the transition probability matrix of the process.

**Solution:** Obviously $\{X_n; n \geq 0\}$ constitutes a Markov chain with one-step transition probabilities

$$p_{jk} = P\,[X_n = k \mid X_{n-1} = j],$$

given by

$$p_{jk} = \begin{cases} p, & k = j+1 \\ q, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus the transition probability matrix $P$ is an infinite matrix given as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & k & k+1 & \cdots \end{array} \\ \begin{bmatrix} q & p & 0 & 0 & & 0 & 0 & \\ q & 0 & p & 0 & & 0 & 0 & \\ q & 0 & 0 & p & & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ q & 0 & 0 & 0 & & 0 & p & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix} \end{array}$$

**Example 12.7:** Let $\{X_n, n \geq 0\}$ be a three state 0, 1, 2 Markov chain with transition probability matrix

$$\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.50 & 0.25 \\ 0 & 0.75 & 0.25 \end{bmatrix}$$

with initial distribution $p_i = P[X_0 = i] = 1/3, \quad i = 0, 1, 2.$

Find $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$.

**Solution:** Since $\{X_n, n \geq 0\}$ is a Markov chain, thus the joint probability

$$P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$$
$$= P[X_3 = 1 | X_2 = 2]\, P[X_2 = 2 | X_1 = 1]\, P[X_1 = 1 | X_0 = 2]\, P[X_0 = 2]$$
$$= p_{21}\, p_{12}\, p_{21}\, p_2$$
$$= (0.75)(0.25)(0.75)\left(\frac{1}{3}\right) = 0.047$$

### 12.7.1 The $n$-step Transition Probabilities

*The n-step transition probability $p_{ij}^{(n)}$ of a Markov chain is the conditional probability given that chain is currently in state i, that it will be in state j after n additional transitions (steps),* that is

$$p_{ij}^{(n)} = P[X_{n+m} = j | X_m = i], \quad n \geq 0, i, j \geq 0 \qquad \text{...(12.27)}$$

Obviously, $p_{ij}^{(1)} = p_{ij}$ and we define

$$p_{ij}^{(0)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

The $n$-step transition probabilities (12.27) can be computed using the *Chapman-Kolmogorov equations* given by

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)}\, p_{kj}^{(m)}, \quad n, m \geq 0, i, j \geq 0 \qquad \text{...(12.28)}$$

A justification for Eqs. (12.28) can be obtained as follows.

We have

$$p_{ij}^{(n+m)} = P[X_{n+m} = j \mid X_0 = i]$$
$$= \sum_k P[X_{n+m} = j, X_n = k \mid X_0 = i]$$
$$= \sum_k P[X_{n+m} = j \mid X_n = k, X_0 = i]\, P[X_n = k \mid X_0 = i]$$
$$= \sum_k P[X_n = k | X_0 = i]\, P[X_{n+m} = j \mid X_n = k]$$

$$= \sum_k p_{ij}^{(n)} p_{kj}^{(m)}.$$

In particular, when $m = 0$

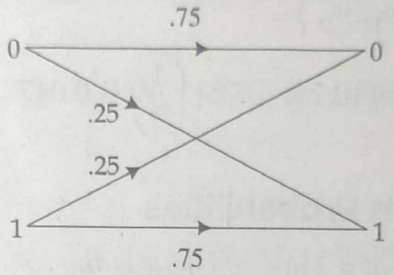
$$p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)} p_{kj}, \quad n = 2, 3, \ldots, \text{and } i, j \geq 0 \qquad \ldots(12.29)$$

If we denote the stochastic matrix of $n$-step transition probabilities $p_{ij}^{(n)}$ by $P^{(n)}$ then from (12.29)

$$P^{(n)} = P^{(n-1)} P$$

that is, $P^{(n)}$ can be obtained as the matrix product of $P^{(n-1)}$ by $P$. Hence $P^{(n)}$ can be calculated as $P^n$, the $n$th power of the stochastic matrix $P$.

**Example 12.8:** Consider the communication system given by the channel diagram




What is the probability that a 0 entered at the first stage is received as 0 by the fifth stage?

**Solution:** The transition probability matrix is

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

We want to find the $p_{00}^{(5)}$, we have

$$P^{(2)} = P^2 = PP = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix}, \quad P^{(4)} = P^4 = P^2 P^2 = \begin{bmatrix} 0.531 & 0.469 \\ 0.469 & 0.531 \end{bmatrix}$$

and,

$$P^{(5)} = P^5 = PP^4 = \begin{bmatrix} 0.516 & 0.484 \\ 0.484 & 0.516 \end{bmatrix}$$

Thus, $p_{00}^{(5)} = 0.516$.

**Remark:** We must note that $p_{ij}^{(n)}$ is the probability that the state after $n$ additional stages is $j$ subject to the condition that the current state is $i$. In case we need to calculate the unconditional distribution of the state after $n$ stages (time), that is $P[X_n = j]$, $j = 1, 2, \ldots$ we must have the initial probability distribution of the states 0, 1, 2, … If $P[X_0 = i] = p_i$, $i \geq 0$, $\sum_i p_i = 1$ is the initial probability distribution, then for any $j$,

$$p_j^{(n)} = P[X_n = j] = \sum_i P[X_n = j \mid X_0 = i]\, P[X_0 = i] = \sum_i p_{ij}^{(n)}\, p_i, \quad j = 1, 2, \ldots, \qquad \text{...(12.30)}$$

This can be obtained by multiplying the row vector $p = (p_0, p_1, p_2, \ldots)$ by the $j$th column of the stochastic matrix $P$ using matrix multiplication, and this defines the probability distribution of $X_n$.

**Example 12.9:** Suppose that whether it rains today depends on previous weather conditions only from the last two days and let

$P$ [If it has rained for the past two days, then it will rain tomorrow] = 0.7

$P$ [If it has rained today but not yesterday, then it will rain tomorrow] = 0.5

$P$ [If it rained yesterday but not today, then it will rain tomorrow] = 0.4

$P$ [If it has not rained past two days, then it will rain tomorrow] = 0.2

Assuming the system to be homogeneous, write it as Markov chain. Let it rained on both Monday and Tuesday. What is the probability that it will rain on Thursday?

**Solution:** Define

State 0: If it rained both today and yesterday

State 1: If it rained today but not yesterday

State 2: If it rained yesterday but not today

State 3: If it rained neither today nor yesterday

On the basis of the probabilities given for different possibilities, the situation can be represented by a four-state Markov chain with transition probability matrix given as

$$P = \begin{bmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.0 & 0.8 \end{bmatrix}$$

The two-step transition probability matrix is

$$P^2 = \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix}$$

Now the chain is in state 0 on Tuesday and it will rain on Thursday if the chain is in either state 0 or state 1 on that day. Hence, the desired probability is

$$p_{00}^{(2)} + p_{01}^{(2)} = 0.49 + 0.12 = 0.61.$$

**Example 12.10:** Given a two state Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, 0 \le p, q \le 1, \; |1-p-q| < 1$$

Show that the $n$-step transition probability matrix $P^{(n)}$ is given by

$$P^{(n)} = \begin{bmatrix} \dfrac{q+p(1-p-q)^n}{p+q} & \dfrac{p-p(1-p-q)^n}{p+q} \\ \dfrac{q-q(1-p-q)^n}{p+q} & \dfrac{p+q(1-p-q)^n}{p+q} \end{bmatrix}$$

**Solution:** We prove it by induction. Obviously the result is true for $n = 1$.

Assume the result is true for $n = k$, then for $n = k + 1$

$$P^{(k+1)} = P^{(k)}\,P = \begin{bmatrix} \dfrac{q+p(1-p-q)^k}{p+q} & \dfrac{p-p(1-p-q)^k}{p+q} \\ \dfrac{q-q(1-p-q)^k}{p+q} & \dfrac{p+q(1-p-q)^k}{p+q} \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

The element at the (1, 1) place of the matrix $P^{(k+1)}$ is

$$\frac{[q+p(1-p-q)]^k(1-p)+[p-p(1-p-q)^k]q}{p+q} = \frac{q(1-p)+pq+(1-p-q)^k(p-p^2-pq)}{p+q}$$

$$= \frac{q+p(1-p-q)^{k+1}}{p+q}$$

Similarly the elements at the places (1, 2), (2, 1) and (2, 2) can be obtained, and we have

$$P^{(k+1)} = \begin{bmatrix} \dfrac{q+p(1-p-q)^{k+1}}{p+q} & \dfrac{p-p(1-p-q)^{k+1}}{p+q} \\ \dfrac{q-q(1-p-q)^{k+1}}{p+q} & \dfrac{p+q(1-p-q)^{k+1}}{p+q} \end{bmatrix}$$

This proves the induction step.

Hence, the result is true for all integral values of $n$ provided $|1-p-q| < 1$.

We note here that as $n$ tends to large, $P^{(n)}$ tends to

$$\begin{bmatrix} \dfrac{q}{p+q} & \dfrac{p}{p+q} \\ \dfrac{q}{p+q} & \dfrac{p}{p+q} \end{bmatrix}$$

Thus, the $n$ step probabilities become independent of $n$.

**Example 12.11:** A gambler has a fortune of Rs. 2. He bets Re 1 at a time and wins Re 1 with probability 1/2. He stops playing if he loses all his fortune or doubles it. Write the transition probability matrix. What is the probability that he loses his fortune at the end of three plays?

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**Solution:** If $X_n$ denotes the fortune of the gambler, then state space of $X_n$ is 0, 1, 2, 3, 4. The transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We want $p_{20}^{(3)} = P[X_3 = 0 \mid X_0 = 2]$

Using Chapman-Kolmogorov equation (12.29) for $n = 3$, we have

$$p_{20}^{(3)} = \sum_{k=0}^{4} p_{2k}^{(2)} p_{k0}$$

$$= p_{20}^{(2)} p_{00} + p_{21}^{(2)} p_{10} + p_{22}^{(2)} p_{20} + p_{23}^{(2)} p_{30} + p_{24}^{(2)} p_{40}$$

$$= p_{20}^{(2)} + \frac{1}{2} p_{21}^{(2)}, \quad \text{since } p_{00} = 1, p_{10} = 1/2, p_{20} = p_{30} = p_{40} = 0$$

Also, $$p_{20}^{(2)} = \sum_{k=0}^{4} p_{2k} p_{k0}$$

$$= p_{20} p_{00} + p_{21} p_{10} + p_{22} p_{20} + p_{23} p_{30} + p_{24} p_{40}$$

$$= p_{20} + \frac{1}{2} p_{21} = 0 + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

and, $$p_{21}^{(2)} = \sum_{k=0}^{4} p_{2k} p_{k1}$$

$$= p_{20} p_{01} + p_{21} p_{11} + p_{22} p_{21} + p_{23} p_{31} + p_{24} p_{41}$$

$$= 0 + \frac{1}{2}(0) + 0 + \frac{1}{2} + (0) + 0$$

$$= 0$$

Thus, $$p_{20}^{(3)} = \frac{1}{4}$$

### 12.7.2 Classification of States

First we define *periodic* and *aperiodic* states.

The *period* of a state $i$ is the greatest common divisor (g.c.d.) of the set of all positive integers $n$ such that $p_i^{(n)} > 0$. If the g.c.d. is greater than 1, the state $i$ is said to be *periodic*, otherwise *aperiodic*. A Markov chain is said to be *aperiodic* if every state has period 1.

For example, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is aperiodic since every power of $P$ is the identity matrix. On the other hand, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is periodic with period 2, since every even power of $P$ is the identity matrix and every odd power of $P$ is the matrix itself.

Next, a state $j$ of a Markov chain $\{X_n, n \geq 0\}$ is said to be *accessible* from state $i$ if $p_{ij}^{(n)} > 0$ for some $n \geq 0$. If state $j$ is accessible from state $i$, and state $i$ is accessible from state $j$, then we say that the two states $i$ and $j$ *communicate*, and symbolically it is expressed as $i \leftrightarrow j$.

We can very easily check that the relation *communication* satisfies the following three properties:

1. $i \leftrightarrow i$ (*reflexivity*)
2. If $i \leftrightarrow j$, then $j \leftrightarrow i$ (*symmetry*)
3. If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (*transitivity*)

Thus, 'communication' is an *equivalence relation* and hence partitions the states of a Markov chain in *equivalence classes* (mutually exclusive and exhaustive) of states such that the two states $i$ and $j$ are in the same *class* if, and only if $i \leftrightarrow j$.

A Markov chain is said to be *irreducible* if and only if there is only one equivalence class. In this case every state is accessible from every other state, that is, all states communicate with each other. Further, if $i \leftrightarrow j$, then $i$ and $j$ have the same period.

For example, the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

is irreducible since every state is accessible from every other state.

Consider a Markov chain with four states 0, 1, 2, 3 with transition probability matrix.

$$P = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

The classes of this Markov chain are $\{0, 1\}$, $\{2\}$, and $\{3\}$. We observe that

1. States 0 and 1 communicate.
2. States 0 and 1 are accessible from the state 2 but converse is not true.
3. State 3 is an absorbing state that is, $p_{33} = 1$, no other state is accessible from it.

Next, we define *transient* and *recurrent* states.

Let for each state $i$ of a Markov chain $\{X_n, n \geq 0\}$, $f_i^{(n)}$ be the probability that the first return to the state $i$ occurs in $n$ transitions after leaving $i$. That is

$$f_i^{(n)} = P[X_n = i, X_k \neq i \text{ for } k = 1, 2, \ldots, n-1 \mid X_0 = i],$$

where we define $f_i^{(0)} = 1$, for all $i$.

The probability of ever returning to the state $i$ is given by

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)} \qquad \ldots(12.31)$$

The state $i$ is said to be *transient* (or *non-recurrent*) state, if $f_i < 1$, and it is said to be *recurrent* state, if $f_i = 1$. Thus, in case state $i$ is transient, then there is a positive probability $1 - f_i$ that the process will never visit the state $i$ again.

Further if the state $i$ is a recurrent state, then

$$m_i = \sum_{n=1}^{\infty} n f_i^{(n)} \qquad \ldots(12.32)$$

is defined as the *mean recurrence time of the state* $i$. If $m_i$ is infinite, then state $i$ is said to be *recurrent null*; in case of finite $m_i$, the state $i$ is said to be *positive recurrent*, or *recurrent non-null*.

In a Markov chain if a state $i$ is recurrent, then starting in state $i$, the process will re-enter the state $i$ again and again and in fact infinitely often. The expected number of time periods that the process is in state $i$ is infinite. In case, we define

$$I_n = \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}$$

then $\sum_{n=0}^{\infty} I_n$ represents the number of periods that the process is in state $i$. Consider

$$E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P[X_n = i \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

Thus, we have the following result:

*A state i is recurrent, if* $\sum_{n=0}^{\infty} p_{ij}^{(n)} = \infty$. *In case the state i is transient, then* $\sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty$.

We have observed that while a recurrent state will be visited infinitely many times but transient state will be visited only a finite number of times. *Thus, in case a Markov chain has finite number of states, then not all the states can be transient; at least one of the states must be recurrent*. Also, recurrence is a class property, that is, if state is recurrent and communicates with state $j$, then state $j$ is also recurrent. Similarly, transient is also a class property.

Further in a finite-state Markov chain all recurrent states are positive recurrent and positive recurrent aperiodic states are called *ergodic*.

Another important result is that *a finite-state Markov chain that is irreducible and aperiodic is ergodic and the n-step transition probabilities* $p_{ij}^{(n)}$ *for an irreducible, ergodic Markov chain become independent of i as* $n \to \infty$*, and this is called steady-state probability distribution*.

For example, the finite-state Markov chain considered in *Example* 12.10 is irreducible and aperiodic for $0 < p, q < 1$ and hence the steady-state distribution exists, and is given by $\left[\frac{q}{p+q}, \frac{p}{p+q}\right]$.

However, the steady-state distribution $v = [v_0 \ v_1]$ in case it exists, can be obtained directly also as the solution to the system of equations.

$$v = vP \qquad \text{...(12.33)}$$

For example, in case of the transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \quad 0 < p, q < 1, \ |1-p-q| < 1$$

The system of Eq. (12.33) is given by

$$[v_0 \ v_1] = [v_0 \ v_1] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

or,

$$v_0 = (1-p)\, v_0 + q v_1$$
$$v_1 = p v_0 + (1-q)\, v_1$$

Rearranging these as

$$p v_0 - q v_1 = 0$$
$$-p v_0 + q v_1 = 0$$

which is a set of two linearly dependent equations. Taking one of these and solving it along with the constraint $v_0 + v_1 = 1$, we obtain

$$v_0 = \frac{q}{p+q} \quad \text{and} \quad v_1 = \frac{p}{p+q},$$

the steady-state solution as obtained earlier.

**Example 12.12:** Consider a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.0 & 0.8 \end{bmatrix}$$

Determine whether or not this Markov chain is irreducible.

**Solution:** The answer is not obvious from looking at $P$. Consider $P^{(2)}$, we have

$$P^{(2)} = P^2 = \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix}$$

Since each element of $P^{(2)}$ is positive, thus every state is accessible from any other state in two steps, hence the Markov chain is irreducible.

**Example 12.13:** A salesman's territory consists of three cities $A$, $B$ and $C$. He never visit in the same city on two consecutive days. If he visits city $A$, then next day he visits city $B$. However, if he visits either $B$ or $C$, then next day he is twice as likely to visit city $A$ as other city. Find how often does he visit each of the cities in the long run.

**Solution:** The state space of the Markov chain $\{X_n\}$ is $A$, $B$, and $C$. The transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

It is easy to see that this is a finite state irreducible aperiodic Markov chain. Hence, the steady probabilities exist. Let $v = [v_A \; v_B \; v_C]$ be the steady state probability distribution. It is given by probability distribution

$$[v_A \; v_B \; v_C] = [v_A \; v_B \; v_C] \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

This gives

$$v_A = 2/3\, v_B + 2/3\, v_C$$
$$v_B = v_A + 1/3\, v_C$$
$$v_C = 1/3\, v_B$$

From the first two equations, $8v_C = 3v_A$

From the third equation, $3v_C = v_B$.

Solving these two along with the constraint, $v_A + v_B + v_C = 1$, we get

$$v_A = 2/5, \quad v_B = 9/20, \quad \text{and} \quad v_C = 3/20$$

Thus, the salesman visits the cities $A$, $B$, and $C$ respectively 40%, 45% and 15% times.

**Example 12.14:** A fair coin is tossed until 3 heads occur in a row. Let $X_n$ denote the longest string of heads ending at the $n$th trial. Find the transition probability matrix and classify the states.

**Solution:** The state space for $X_n$ is {0, 1, 2, 3}.

The transition probability matrix is

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The chain is not irreducible, here the state 3 is absorbing state and all other states are aperiodic.

**Example 12.15:** Describe the nature of the states of the Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

**Solution:** Let the states be described by $i = 0, 1, 2$.

We have

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad P^3 = P^2P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = P$$

Thus, $P^4 = P^2$ and in general $P^{2n} = P^2$ and $P^{2n+1} = P$.

From $P$ and $P^2$, we observe that

$$p_{00}^{(2)}, p_{01}^{(1)}, p_{02}^{(2)} > 0$$
$$p_{10}^{(1)}, p_{11}^{(2)}, p_{12}^{(1)} > 0$$
$$p_{20}^{(2)}, p_{21}^{(1)}, p_{22}^{(2)} > 0$$

Thus, the chain is irreducible.

Also $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \ldots > 0$, for all $i$, all the states are periodic with period 2. Further, since the chain is finite and irreducible, all its states are non-null persistent, hence all its states are non-ergodic.

## REVIEW EXERCISES

1. Differentiate between a random variable and a random process. Give examples.
2. Explain the terms 'state space' and 'parameter set' associated with a stochastic process. Give the classification based on state and parameter of a process. Give an example of each type.
3. What is a Markov process? Give examples.
4. Explain the Bernoulli process. Give example. When the process is said to be non-homogeneous?
5. Show that in a Bernoulli process, the number of succeeding trials before the next success has geometric distribution.
6. What is a Poisson process? Give examples. State properties of a Poisson process. Show that Poisson process is a Markov process.
7. Prove that sum of two independent Poisson processes is a Poisson process while difference is not a Poisson process.
8. If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, then show that the random variable $N(t)$, describing the number of events in any time interval $(0, t]$ has a Poisson distribution with parameter $\lambda t$.
9. Show that in a Poisson process with rate $\lambda > 0$, the inter-arrival times $\{\tau_n\}$ of successive events are mutually independent and identically distributed exponential random variables each with mean $1/\lambda$.
10. Explain birth and death process. Write the differential-difference equation for a general birth and death process and find the steady-state solution.
11. What is a renewal process? Give example. How this can be viewed as a generalization of a Poisson process?
12. Explain 'renewal function' and 'renewal density' in context with a renewal process.
13. Derive renewal equation and find its solution.
14. Define a Markov chain. Give example. When a Markov chain is called homogeneous?
15. Define $n$-step transition probability matrix of a Markov chain. What are Chapman-Kolmogorov equations?
16. How do you derive the probability distribution of the $n$th state $X_n$ in a Markov chain?
17. Define and explain the following:
    (1) Periodic and aperiodic states
    (2) Communicating states
    (3) Irreducible chain
    (4) Transient and recurrent states
    (5) Ergodic chains
18. What is meant by steady-state distribution of a Markov chain and how is it obtained

## PROBLEM SET

1. Show that the process $\{X(t)\}$ with probability distribution given by

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, \quad n = 0$$

is not stationary.

2. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 minutes (i) exactly 4 customers arrive, and (ii) more than 4 customers arrive.
3. Assume that a circuit has an IC whose time to failure is an exponentially distributed r.v. with expected lifetime of 3 months. If there are 10 spare IC's and time from failure to replacement is zero. What is the probability that the circuit can be kept operational for at least 1 year ?
4. A radioactive source emits particles at a rate of 6 per minute in accordance with Poisson process. Each particle emitted has a probability of 1/3 of being recorded. Find the probability that at least 5 particles are recorded in a 5-minute period.
5. Consider a computer system with Poisson job-arrival stream at an average rate of 60 per hour. Determine the probability that the time interval between successive job arrivals is
   (a) Longer than four minutes
   (b) Shorter than eight minutes
   (c) Between two and six minutes.
6. A service centre opens at 9 A.M. From 9 A.M. until 3 P.M. customer arrive at a Poission rate of four per hour and from 3 P.M. until 9 P.M. customers arrive at a Poisson rate of 6 per hour. Determine the probability distribution of the number of customers who enter the store on a given day.
7. People arrive at a bus stop according to a Poisson process with rate $\lambda_1$. Buses arrive at the stop according to a Poisson process with rate $\lambda_2$. A bus when arrives picks up everybody who is waiting. Find the expected value and the variance of the number of people who get on a bus.
8. Earthquakes occur in a given region in accordance with a Poisson process with a rate of 5 per year.
   (a) What is the probability there will be at least two earthquakes in the first half of 2010?
   (b) Assuming that the event in Part (a) occurs, what is the probability that there will be no earthquakes during the first 9 months of 2011?
9. Consider a computer system during a peak load where the CPU is saturated. Assume that the processing requirement of a job is exponentially distributed with mean $\mu$. Further assume that a fixed time $t_{sys}$ per job is spent performing overhead functions in the operating system. Let $N(t)$ be the number of jobs completed in the interval $(0, t]$. Show that

$$P[N(t) < n] = \sum_{k=0}^{n-1} e^{-\mu(t-nt_{sys})} \frac{[\mu(t-nt_{sys})]^k}{k!}, \text{ if } t \geq nt_{sys}$$

10. A particle performs a random walk with absorbing barriers at 0 and 4. That is, whenever it is at any position $r$, $(0 < r < 4)$, it moves to $r+1$ with probability $p$ or to $r-1$ with probability $q$, such that $p+q=1$. But as soon as it reaches to 0 or 4 it remains there itself. Write the transition probability of the process and classify the various states of the process.
11. A man drives a car or catches a train to go office each day. He never goes 2 days in a row by a train but if he drives on a day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week the man tossed a fair coin and drive to work if head appeared. Write the transition probability matrix and find the probability that he takes a train on the second day. Also find the probability that he drives to work in the long run.
12. The transition probability matrix of a Markov chain $\{X_n, n \geq 1\}$ with the three states 1, 2 and 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and the initial distribution (0.7, 0.2, 0.1). Find (i) $P\{X_2 = 3\}$, (ii) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$.
13. A communication source can generate one of the three possible messages 0, 1 and 2. Assume that the transmission can be described by a homogeneous Markov chain with the transition probability matrix

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

and the initial state probability distribution $p^{(0)} = (0.3, 0.3, 0.4)$. Find $p^{(3)}$ and limiting probability distribution.
14. The three state Markov chain is given by the transition probability matrix

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Prove that the chain is irreducible and all the states are aperiodic and non-null persiste
15. A transition probability matrix $P$ is said to be doubly stochastic if each column sum is a 1, that is, if $\sum_i p_{ij} = 1$, for all $j$. If such a chain is irreducible and aperiodic and has fini states 1, 2, ..., $m$, find its stationary probabilities.

16. At a system the job arrival rate is 2 per minute. Determine the following:
    (a) The average number of arrivals during 5 minutes
    (b) The probability that no arrival will occur during the next 0.5 minutes.
    (c) The probability of at least one arrival in the next 0.5 minutes.
    (d) The probability that the time between two successive arrivals is at least 3 minute
17. A barber shop serves one customer at a time and provides three seats for waiting customers. If the place is full, customer go elsewhere. Arrivals occur according to a Poisson distribution with a mean of 4 per hour. The time to get a haircut is exponential with mean 15 minutes. Determine the following
    (a) The steady-state probabilities
    (b) The expected number of customers in the shop
    (c) The probability customers will go elsewhere because the shop is full.

## ANSWERS

1. $E[X(t)] = 1$, $\text{Var}[X(t)] = 2at$; process is not stationary
2. (a) 0.133 (b) 0.715
3. 0.9972.
4. 0.9707

10. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

11. $$\begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & \quad C \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$ T: Train, C: Car; $\frac{1}{4}, \frac{2}{3}$

12. 0.279, 0.0048
13. (0.4083, 0.2727, 0.3190), (0.3409, 0.2727, 0.3864)
17. (a) $p_j = 0.2$, $j = 0, 1, 2, 3, 4$ (b) 2 (c) $p_4 = 0.2$.