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2K20/MC/44

Assignment 3

(1) Let $X_1, X_2, \ldots X_n$ be n, i.i.d exponential random variables each with parameter λ.
Let $W_n = X_1 + \cdots + X_n$
$\therefore$ The distribution function W_n is $P[W_n = t] = f^n(t)$
& $f(t) = \lambda e^{-\lambda t}$ (an exponential random variable)

- $f^n(t)$ is convolution of n random variable

$$\Rightarrow f^{n+1}(t) = \int_0^t f^n(t-x) f(x)\,dx \Rightarrow f^2(t) = \lambda(\lambda t)(e^{-\lambda t})$$

$$\Rightarrow f^n(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}$$

$\therefore$ The distribution function of sum of n i.i.d exponential random variable each with parameter λ is:

(*) $P[W_n = t] = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}(e^{-\lambda t})$ [Erlang distribution]

- Let no. of renewals be $N(t) = n$ until time t,

Hence, $P[N(t)=n] = P[W_n \le t < W_{n+1}] = P[W_n \le t] - P[W_{n+1} \le t]$
$= F^n(t) - F^{n+1}(t)$

$$F^n(t) = \int_0^t \frac{\lambda(\lambda t)^n}{(n-1)!} e^{-\lambda t}\,dt = 1 - \left[\sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}\right] e^{-\lambda t}$$

(*) - Hence, Probability distribution for no. of renewals in $(0,t]$:

$$P[N(t)=n] = F^n(t) - F^{n+1}(t)$$
$$= 1 - \left[\sum_{i=1}^{n-1} \frac{(\lambda t)^i}{i!}\right] e^{-\lambda t} - 1 + \left[\sum_{i=0}^{n} \frac{(\lambda t)^i}{i!}\right] e^{-\lambda t}$$

$$\Rightarrow P[N(t)=n] = \frac{(\lambda t)^n (e^{-\lambda t})}{n!}$$, which is poisson's distribution

(*) Hence, the Sum follows Erlang & no. of renewals follows poisson's distribution.

② ⓐ Uniform distribution with parameter a :

$$f(t) = \begin{cases} 1/a ; & 0 \le t \le a \\ 0 ; & \text{otherwise} \end{cases}$$

Now, we require $L_m(s)$:

ⓘ $L_m(s) = \dfrac{L_f(s)}{1 - L_f(s)} = \dfrac{1/as}{1 - 1/as} = \dfrac{1}{a[s - 1/a]}$

ⓘⓘ $m(t) = L^{-1}[L_m(s)] = e^{-t/a}/a$

$M(t) = E[N(t)] = \int_0^t m(t)\,dt = \dfrac{1}{a}\int_0^t e^{-t/a}\,dt = \underline{1 - e^{-t/a}}$

☆ Expected no. of renewals by time t :

$$\Rightarrow E[N(t)] = \begin{cases} 1 - e^{-t/a} ; & 0 \le t \le a \\ 0 ; & \text{otherwise} \end{cases}$$

ⓑ Sum of independent & normally distributed random variable is normally distri with parameters $n\mu$ & $n\sigma^2$

$$F^n(t) = P(W_n \le t) = \Phi\left[\frac{t - n\mu}{\sqrt{n}\,\sigma}\right]$$

☆ & $M(t) = E[N(t)] = \sum_{n=1}^{\infty} F^n(t) = \sum_{n=1}^{\infty} \Phi\left[\dfrac{t - n\mu}{\sqrt{n}\,\sigma}\right]$

ⓒ Erland distribution with parameters m & λ i.e. mn exponential random variable $(\lambda e^{-\lambda t})$

$$F^n(t) = P(W_n \le t) = 1 - \left[\sum_{i=0}^{mn-1} \frac{(\lambda t)^i}{(i)!}\right] e^{-\lambda t}$$

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1-e^{-t/a}

able is g

nential

(3) For renewal process continuation over interval $(0, t]$

$1-\alpha \leq P(N(t) \leq n)$

& $P(N(t) \leq n) = 1 - X^n(t)$

$1-\alpha \leq 1 - X^n(t)$

Let the variables be normally distributed

$\Rightarrow X^n(t) = \phi\left(\frac{t - n\mu}{\sqrt{n}\,\sigma}\right) \Rightarrow 1-\alpha \leq 1-\phi\left(\frac{t-n\mu}{\sqrt{n}\,\sigma}\right)$

⊙ let $t = 200$, $1-\alpha = 0.99$, $\mu = 8$ & $\sigma^2 = 25$

$1-\alpha \leq 1-\phi\left(\frac{200-8n}{\sqrt{n}(5)}\right)$

⊙ It is equivalent to $Z_{0.01} = 2.32 \leq \frac{8n-200}{5\sqrt{n}}$

$\Rightarrow$ At least 34 parts must be available to ensure the continuous renewal process in $(0, t]$ with 0.99 as probability of success.

(4) (i) Queuing Disciplines are:

1) LIFO (last in first out). eg: Stack structure
2) FIFO (First in First out). eg: Queue
3) Priority Scheduling. eg: scholarship on basis of score.
4) Service In Random Order (SIRO).

(ii) Custom Behaviours are:

1) Balking: At each arrival customer join queue or not depending

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(ii) [illegible] Behaviours are :

[illegible]ing : At each arrival customer chooses whether to join queue or not depending on the queue length.

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2) <u>Reneging</u>: After waiting for some time the customer might get impatient & leave the queue.

3) <u>Collusion</u>: Several customers may cooperate & only on[e of] them stands in the queue.

4) <u>Jockeying</u>: In case of multiple queues customer might [keep] changing it's queue to be in the shortest one.

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(5) $\lambda = 2$ (arrival rate), $\mu = 3$ (departure rate)
$C = 7$ lines, $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$

(a) Probability of customer loss $= \frac{(\rho)^c}{c!} \times \frac{1}{1 + \frac{\rho}{1!} + \dots + \frac{\rho^c}{c!}}$

$$= \frac{(2/3)^7}{7!} \times \left[\frac{1}{1 + \left(\frac{2}{3}\right) + \dots + \frac{(2/3)^7}{7!}} \right].$$

(b) Mean no. of busy channels $= \sum_{n=0}^{7} n P_n = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6 + 7P_7$

$$\& \; P_n = \frac{\rho^n / n!}{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \dots + \frac{\rho^c}{c!}}$$

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(6) Distribution of waiting time in the queue in M/M/1 model:

- Let P_n be the probability of having customers in system at time t.

$\Rightarrow P_n(t+\Delta t) = P_n(t)P + P_{n-1}(t).P + P_{n+1}(t).P + P_n(t)P$

(No arrival & no service in Δt, 1 arrival in Δt, 1 Service in Δt, 1 arrival & 1 service in Δt)

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t + o(\Delta t)] + P_{n-1}(t)[\lambda(\Delta t) + o(\Delta t)] + P_{n+1}(t)[\mu\Delta t + o(\Delta t)] + P_n(t)[o(\Delta t)]$$

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t] + P_{n-1}(t)[\lambda\Delta t] + P_{n+1}(t)[\mu\Delta t] \quad \text{—(1)}$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \frac{-P_n[(\lambda+\mu)\Delta t] + P_{n-1}(\lambda\Delta t) + P_{n+1}(\mu\Delta t)}{\Delta t}$$

$$\therefore \frac{d P_n(t)}{dt} = -P_n[\lambda+\mu] + P_{n-1}\lambda + P_{n+1}\mu \quad \text{—(2)}$$

for $n=0$: $P_0(t+\Delta t) = P_0[-\lambda \ +1]\Delta t + P_1[\mu\Delta t]$ —(3)

$$\& \ \frac{d P_n(t)}{dt} = -P_0\lambda + P_1\mu \quad \text{—(4)}$$

- After some time, steady state is achieved i.e. probability distribution of each state is independent of time.

$$\Rightarrow \frac{d P_n(t)}{dt} = 0; \ n = 0, 1, \ldots$$

$\Rightarrow$ Using eqn. (2) & (4)

$$0 = -P_n(\lambda+\mu) + P_{n-1}(\lambda) + P_{n+1}(\mu) \quad \text{—(5)}$$

& $P_0\lambda + P_1\mu = 0$ — ⑥ (for $n=0$)

⇒ Using ③ & ⑥:

$$P_1 = \frac{\lambda}{\mu} P_0, \quad P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$\Rightarrow \boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n P_0} \Rightarrow \boxed{P_n = \rho^n P_0} \quad (\rho = \lambda/\mu)$$

⊙ At any given time, total probability over entire distribution is 1.

$$\sum_{n=1}^{\infty} P_n = 1 \Rightarrow P_0 + \rho P_0 + \rho^2 P_0 + \cdots\cdots = 1$$

$$P_0(1+\rho+\rho^2+\cdots\cdots) = 1 \Rightarrow P_0\left[\frac{1}{1-\rho}\right] = 1$$

☆ $P_0 = 1-\rho$

$$\Rightarrow ☆ \quad \boxed{P_n = (1-\rho)\rho^n}$$

⊙ The mean number of members in the queue is given by:

$$L_q = E(N-1) = \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \frac{\rho^2}{1-\rho} = \boxed{\frac{\lambda^2}{\mu(\mu-\lambda)}}$$

⊙ For finding waiting time assume that a customer carries the system & has to wait W_q time. Expected length of this queue when the customer is at the head of queue is the expected number of arrivals during his waiting time.

$$L_q = \frac{W_q}{(1/\lambda)} \Rightarrow W_q = \frac{L_q}{\lambda} \Rightarrow \boxed{W_q = \frac{\lambda}{\mu(\mu-\lambda)}}$$

⊙ ⇒ To find distribution of time spent in queue, consider that a customer spends T_q time in queue.

For t_q to lie $(t, t+\Delta t)$ all n customers get serviced by time $t+\Delta t$.

$$f_q(t)\,dt = \sum_{n=1}^{\infty} P_n \cdot P[n-1 \text{ service in } t]\, P[1 \text{ service in } \Delta t]$$

$$= \sum_{n=1}^{\infty} \frac{(1-\rho)\rho^n (\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \mu\, dt$$

$$= [\rho\mu(1-\rho)][e^{-(1-\rho)\mu t}]dt$$

When arrival rate = service rate :

$\Rightarrow E(t_q) = \dfrac{\lambda}{\mu(\lambda-\mu)}$ & $\lambda=\mu$

$\Rightarrow E(t_q) = \infty \Rightarrow$ Both mean & Variance of waiting time approaches ∞ when $\lambda = \mu$.

(7) $\lambda = 2,\ \mu = 2,\ N = 5$

(i) Probability that customer is not allowed to join the system $= 1 - P_N = 1 - P_5 = 1 - \left(\dfrac{1}{5+1}\right) = \dfrac{5}{6}$

(ii) Mean no. of Customers in the System $= \sum_{n=0} n(P_n) = \sum_{n=0} \dfrac{n}{N+1}$

$= \dfrac{15}{6} = 2.5$

(iii) $\mu = 0.5$ $\qquad \rho = \lambda/\mu = 2/0.5 = 4$

$P_n = \dfrac{(1-\rho)\rho^n}{(1-\rho^{N+1})} \Rightarrow P_n = \dfrac{(1-4)(4^n)}{(1-4^6)} = \boxed{\dfrac{4^n}{1365}}$

(8) Let there be a M/M/1 queuing model where the arrival rate is 3 customers per hour ($\lambda = 3$) & service rate is 6 customers per hour ($\mu = 6$)

(·) $\lambda < \mu \Rightarrow$ steady state exists :

$$P_n = \rho^n (1-\rho) \quad , \rho = \frac{\lambda}{\mu} = 3/6 = 0.5$$

(i) In long run, no. of idle & busy periods become equal. Since arrival rate follows markovian property, mean time until next arrival, or end of idle period is $1/\lambda$

(·) The probability of server being idle is $P_0 = 1 - \rho$. So, over time T server sets idle for $T.(1-\rho)$ time & there are $\frac{T(1-\rho)}{(1/\lambda)} = T(\lambda)(1-\rho)$ idle periods.

From 9 AM to 5 PM i.e. 8 hours :

(a) Expected no. of idle periods $= (T)(\lambda)(1-\rho) = 8(3)(1-1/2)$
$= 24(1/2) = \underline{12}$

(b) Expected duration of idle periods in time $T = T(1-\rho)$
$= 8(1-1/2) = \underline{4}$

Expected duration of idle period $= 4/12$
$= 1/3$ hrs $= \underline{20 \text{ min}}$

(ii) Expected no. of customer served $= \dfrac{\text{expected busy period duration in } T}{\text{expected time to serve one customer}}$

$$= \frac{T(\rho)}{(1/\mu)} = 8(6)\left[\frac{1}{2}\right] = \underline{24}$$