2

Forward and Futures Contracts

2.1 Introduction

A *forward contract* is probably the simplest of all derivative securities. It has a simple pricing mechanism and has wide applications, particularly in commodity and foreign exchange markets. We had already initiated some discussion on forward contracts in Chapter 1 where we had also explained the pricing methodology through a simple example.

Though *futures contracts* are very much in the spirit of forward contracts, they have specially been designed to standardize these contracts so as to eliminate the risk of default by the party suffering the loss. For this, a process called *marking to market* is conceptualized which requires an individual to open a *margin account* which is managed by an organized clearing house/exchange.

This chapter continues our earlier discussion on forward contracts and explains various concepts related with the working of futures contracts. We also describe very briefly another derivative security called *swap* which is again very popular in commodity and foreign exchange markets.

2.2 Forward Contract

Let us recall the definition of a *forward contract* from Chapter 1 and note that it is an agreement between two investors (also called parties) to buy or sell a risky asset at a specified future time (called the *delivery date*) for a price F (called the *forward price*) fixed at the present time (say $t = 0$). The party which agrees to buy the asset is said to enter into a *long forward contract* or to take a *long forward position*. The party which agrees to sell the asset is said to enter into a *short forward contract* or to take a *short forward position*. There is no payment of money by either party when the agreement for the forward contract is made. Some typical examples of forward contract could be a farmer wishing to fix the sale price of his/her crops in advance, an importer arranging to buy foreign currency at a fixed rate in future, a fund manager wishing to sell stock for a fixed price in future or a country wishing to import commodities (like wheat, sugar, oil etc.) from another country at a fixed rate at some specified future date. In these situations, forward contracts become very handy because they provide an opportunity to hedge against the unknown future price of the underlying risky asset.

Let $t = 0$ denote the time when the two parties enter into the specific forward contract agreement and $t = T$ be the delivery date. Let the agreed forward price be $F(0,T)$. Here the two arguments in F denote the time $t = 0$ and $t = T$ respectively. In case the context is clear and there is no ambiguity, we write $F(0,T)$ simply as F.

Let us now analyze the two scenarios, namely $F(0,T) < S(T)$ and $F(0,T) > S(T)$. Let us first consider the case $F(0,T) < S(T)$. In this case the party having the long forward contract will benefit because it can buy the asset for $F(0,T)$ and sell the same for the market price $S(T)$, making a profit of $S(T) - F(0,T)$. But the other party which has taken a short forward position will suffer loss of $S(T) - F(0,T)$ because it will have to sell below the market price. Fig. 2.1 clearly exhibits these two positions.

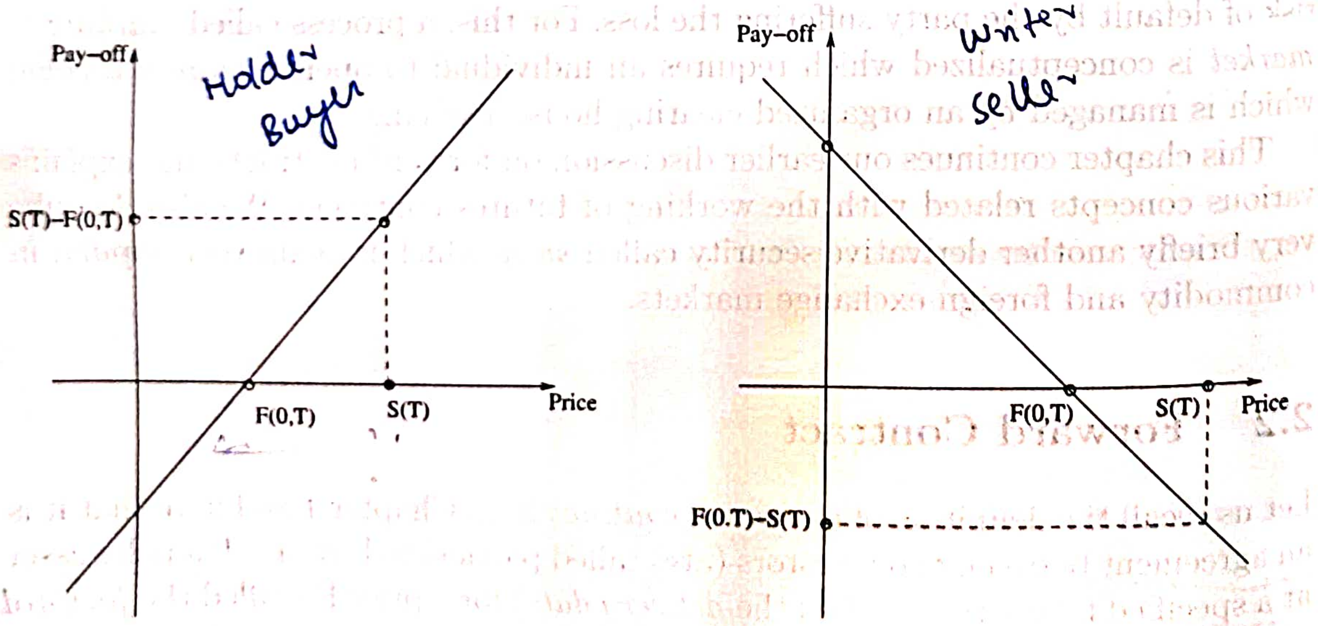

Fig. 2.1. Pay-off of a forward contract

If $F(0,T) > S(T)$ then these arguments will simply get reversed. In this case the party taking a short forward position will benefit and make a profit of $F(0,T)-S(T)$ whereas the other party (which has taken a long forward position) will make a loss of $F(0,T)-S(T)$ and analogue of Fig. 2.1 can be drawn for this case as well.

Therefore in either case the pay-off of a long forward contract at delivery is $S(T)-F(0,T)$ and that of a short forward contract is $F(0,T)-S(T)$. Sometimes the contract may be initiated at time $t<T$ rather than $t=0$. In this scenario the pay-offs of a long forward position and a short forward position are $S(T)-F(t,T)$ and $F(t,T)-S(T)$ respectively where $F(t,T)$ denotes the forward price.

2.3 Forward Price Formula

We now look into the problem of determining the price of the forward contract, i.e. the forward price $F(0,T)$. We shall make use of the *no arbitrage principle* and follow the methodology indicated in Chapter 1 to get the next theorem.

Theorem 2.3.1 (Forward Price Formula) *Let the price of an asset at $t=0$ be $S(0)$. Then the forward price $F(0,T)$ is given by*

$$F(0,T)=\frac{S(0)}{d(0,T)} \tag{2.1}$$

where $d(0,T)$ is the discount factor between $t=0$ and $t=T$.

Proof. If possible let $F(0,T)>(S(0)/d(0,T))$. Then at $t=0$ we construct the portfolio P as follows

(i) borrow $S(0)$ amount of cash until $t=T$,
(ii) buy one unit of the underlying asset on the spot market at price $S(0)$ (if the underlying is the stock then it will mean buying one share of stock for amount $S(0)$),
(iii) take a short forward position with delivery at $t=T$ and forward price as $F(0,T)$.

This gives the value of the portfolio P at $t=0$ as

$$V_P(0)=-S(0)+S(0)=0.$$

Now at $t=T$, we sell the asset for $F(0,T)$ and pay the amount $(S(0)/d(0,T))$ to clear the loan with interest and thereby close the short position. This gives

$$V_P(1)=F(0,T)-(S(0)/d(0,T))\,,$$

which is strictly positive under the stated assumption. Thus there is a strictly positive amount of risk free profit with zero net investment. This is contrary to the no arbitrage principle.

Next suppose that $F(0,T) < (S(0)/d(0,T))$. In this case, at $t = 0$, we construct the portfolio Q as follows

(i) short sell one unit of the underlying asset (thus if the underlying asset is stock, then we borrow one share of stock and sell the same in the market with all conditions and procedure as specified in short selling),
(ii) invest the proceeds (i.e. the amount $S(0)$ received by selling one unit of the asset) at the risk free rate up to $t = T$,
(iii) enter into a long forward contract with delivery as $t = T$ and forward price $F(0,T)$.

This gives

$$V_Q(0) = S(0) - S(0) = 0.$$

Now at $t = 1$, we

(i) cash the risk free investment with interest, i.e. collect cash $(S(0)/d(0,T))$,
(ii) buy the underlying asset for $F(0,T)$ using the long forward position,
(iii) close the short position on the underlying asset by returning the borrowed unit of asset to the owner.

This gives

$$V_Q(T) = (S(0)/d(0,T)) - F(0,T),$$

which is strictly positive. This again allows a strictly positive risk free profit with zero net investment contradicting the no arbitrage principle. Thus $F(0,T) = S(0)/d(0,T)$.

$\square$

Remark 2.3.1 *For the case of constant interest rate r being compounded continuously, we have $d(0,T) = e^{-rT}$ and hence*

$$F(0,T) = S(0)\, e^{rT}. \tag{2.2}$$

Remark 2.3.2 *If the contract is initiated at some intermediate time t, $0 < t < T$, then $d(t,T) = e^{-r(T-t)}$ and*

$$F(t,T) = S(t)\, e^{r(T-t)}, \tag{2.3}$$

where r is same as in Remark 2.3.1.

Remark 2.3.3 *In terms of zero coupon bond prices, the forward price formula 2.1 becomes*

$$F(0,T) = \frac{S(0)}{B(0,T)}, \tag{2.4}$$

which is more convenient to use as it does not require the assumption that the interest rate r is constant.

It is a matter of common knowledge that holding of physical assets like gold, sugar, oil, wheat etc. entail inventory carrying costs, such as rental for storage and insurance fees etc.. These costs affect the theoretical forward price formula as given at (2.1). The standard forward price formula (2.1) can be generalized in several ways to take into consideration the costs of carry and also to include dividends. We have the below given result in this regard.

Theorem 2.3.2 (Forward Price Formula with Carrying Costs) *Let an asset carry a holding cost of $c(i)$ per unit in period i $(i = 0, 1, 2, \ldots (n-1))$. Also let at $t = 0$, the price of this asset be $S(0)$ and short selling be allowed. Then*

$$F(0,T) = \frac{S(0)}{d(0,T)} + \sum_{i=0}^{n-1} \frac{c(i)}{d(i,n)}, \tag{2.5}$$

where delivery date is $t = T$ and between $t = 0$ and $t = T$ there are n periods which have appropriately been identified as per the given context.

Next we consider the case when the underlying asset (e.g. stock) is dividend paying.

Theorem 2.3.3 (Forward Price Formula with Dividend) *Let an asset be stored at zero cost and also sold short. Let the price of this asset at $t = 0$ be $S(0)$ and a dividend of Rs div be paid at time τ, $0 < \tau < T$. Then*

$$F(0,T) = \frac{[S(0) - (div)\,(d(0,\tau))]}{d(0,T)}. \tag{2.6}$$

For the case of constant interest rate r being compounded continuously, the above formula becomes

$$F(0,T) = [S(0) - (div)\ e^{-r\tau}]\,e^{rT}. \tag{2.7}$$

Sometimes the asset (stock) may pay dividends continuously at a rate of $r_{div} > 0$. This is called *continuous dividend yield.* If the dividends are reinvested in the stock, then an investment in one share held at $t = 0$ will become $e^{(r_{div})T}$ shares at $t = T$. This is very similar to continuous compounding and therefore $e^{-(r_{div})T}$ shares at $t = 0$ will give one share at $t = T$. This observation leads to Theorem 2.3.4.

Theorem 2.3.4 (Forward Price Formula with Continuous Dividend) *Let an asset be stored at zero cost and also sold short. Let the price of the asset at $t = 0$ be $S(0)$ and the asset pay dividends continuously at a rate of r_{div}. Then*

$$F(0,T) = S(0)\, e^{(r-r_{div})T}. \tag{2.8}$$

Theorems 2.3.2, 2.3.3 and 2.3.4 are not proved here as their proofs can be constructed analogous the proof of Theorem 2.3.1. Interested readers may refer to Luenberger [85] and Capinski and Zastawniak [25] in this regard.

Forward contracts in foreign currency market are very common and we wish to derive a formula for determining the price of the same. To be specific, let us consider the two currencies as British pound and US dollars with the latter as the underlying. The readers may imagine a British importer of US goods requiring US dollars after $t = T$. So this British importer may think of taking a forward contract on US dollars with delivery as $t = T$ to hedge against the fluctuating exchange rate of British pound versus US dollars.

Let at $t = \tau$, the buying and selling exchange rate be: 1 British pound = $P(t)$ US dollars. Also let the risk free interest rates for investments in British pounds and US dollars be r_{GBP} and r_{USD} respectively. Then the forward price is given by

$$F(0,T) = P(0)\, e^{(r_{USD}-r_{GBP})T}. \tag{2.9}$$

The above formula gives the agreed exchange rate at $t = T$, i.e. at $t = T$, the British importer will be able to buy $F(0,T)$ US dollars for one British pound.

To prove formula (2.9), let us consider two strategies as described below.

Strategy A
Invest $P(0)$ US dollars at the rate of r_{USD} until $t = T$.

Strategy B
Buy one British pound for $P(0)$ US dollars, invest it until $t = T$ at the rate of r_{GBP}, and take a short position in $\exp(r_{GBP} \cdot T)$ British pound forward contract with delivery as $t = T$ and forward price as $F(0,T)$. This gives

$$P(0)\, e^{(r_{USD})T} = F(0,T)\, e^{(r_{GBP})T},$$

and hence the result.

Example 2.3.1 *The current price of gold is Rs 18,000 per 10gm. Assuming a constant interest rate of 8% per year compounded continuously, find the theoretical price of gold for delivery after 9 months.*

Solution Here $S(0) = 18,000$, $r = 8\%$ per year and $T = 9$ months $= 3/4$ year. Hence by the forward price formula

$$
\begin{aligned}
F &= S(0)\, e^{rT} \\
&= (18000)\, e^{(0.08\times 3/4)} \\
&= (18000)\, e^{0.06} \\
&= \text{Rs } 19113.06.
\end{aligned}
$$

□

Example 2.3.2 *Find the forward price of a non-dividend paying stock traded today at Rs 100, with the continuously compounded interest rate of 8% per year, for a contract expiring seven months from today.*

Solution We have $S(0) = 100$, $r = 8\%$ per year and $T = 7/12$ year. This gives

$$F = 100\, e^{(0.08)(7/12)} = \text{Rs } 104.78.$$

□

Example 2.3.3 *The current price of sugar is Rs 60 per Kg and its carrying cost is 10 paisa per Kg per month to be paid at the beginning of each month. Let the constant interest rate r be 9% per annum. Find the forward price of the sugar (Rupees per Kg) to be delivered in 5 months.*

Solution The interest rate is $(.09)/12 = .0075$ per month. Therefore the reciprocal of one month discount rate for any month is (1.0075). Therefore we have

$(60) e^{rT}$

$(60) e^{(0.0075)\times 5}$

$$
\begin{aligned}
F(0,5) &= (1.0075)^5(60) + \Big(\sum_{i=1}^{5} (1.0075)^i\Big)(.1) \\
&= \text{Rs } 62.79.
\end{aligned}
$$

$(60)(1.0075)^5 + \sum_{i=0}^{4} 0.1\,(1.0075)^i$

□

Example 2.3.4 *Let the price of a stock on 1st April 2010 be 10% lower than it was on 1st January 2010. Let the risk free rate be constant at $r = 6\%$. Find the percentage drop of the forward price on 1st April 2010 as compared to the one on 1st January 2010 for a forward contract with delivery on 1st October 2010.*

Solution It is convenient to take 1st January 2010 as $t = 0$. Then 1st October 2010 is 9 months, i.e. 3/4 year. Thus $t = 3/4$. Also $t = \tau$ is 1st April 2010, i.e. $3/12 = 1/4$. Therefore using Theorem 2.3.1 we get

$$F(0,3/4) = S(0)\, e^{(0.06\times 3/4)}$$
$$F(1/4,3/4) = S(0)(0.9)\, e^{(0.06\times 2/4)}.$$

Here it may be noted that on 1*st* April 2010 the price of the stock is 10% lower than that on 1*st* January 2010, i.e. $(0.9)S(0)$. Now

$$\frac{F(0,3/4) - F(1/4,3/4)}{F(0,3/4)} = \frac{e^{(0.06\times 3/4)} - (0.9)\, e^{(0.06\times 1/2)}}{e^{(0.06\times 3/4)}}$$
$$= 0.1134.$$

Therefore the percentage drop in the forward price is 11.34%. □

Example 2.3.5 *An Indian importer wants to arrange a forward contract to buy US dollars in half a year. The interest rates for investment in Indian rupees and US dollars are* $r_{INR} = 6\%$ *and* $r_{USD} = 4\%$ *respectively. Also the current rate of exchange is Rs 46 to a dollar, i.e.* $\frac{1}{46}$ *dollars to a rupee. Find the forward price of dollar-rupee exchange rate.*

Solution Here US dollars play the role of underlying. Therefore from (2.9) we get

$$F(0,1/2) = (1/46)\, e^{(0.5\times(0.06-0.04))}$$
$$= (1/46)\, e^{(0.5\times 0.02)}$$
$$= 0.022,$$

dollars to a rupee, i.e. $\frac{1}{(0\cdot 022)} = 45\cdot 54$ rupees to a dollar. □

2.4 The Value of a Forward Contract

Let a forward contract be initiated at $t = 0$ with delivery $t = T$. Also let $F(0,T)$ be the forward price of this contract. Consider an intermediate time $0 < \tau < T$ and let $F(\tau, T)$ be the forward price of the contract initiated at $t = \tau$ with delivery $t = T$.

Thus we have two forward contracts; one initiated at $t = 0$ and other initiated at $t = \tau$, both having the same delivery date, namely $t = T$. Then $F(0,T)$ and $F(\tau,T)$ are the respectively the forward prices of these two contracts. Now as time progress, *the value of the forward contract* initiated at $t = 0$ will be changing. Let at $t = \tau$, its value be $f(\tau)$. Is there a relationship connecting $f(\tau)$, $F(0,T)$ and $F(\tau,T)$? The below given theorem is precisely the answer to this question.

Theorem 2.4.1 (Value of a Forward Contract) *Let $f(\tau)$, $F(0,T)$ and $F(\tau,T)$ be as explained above. Then*

$$f(\tau) = (F(\tau,T) - F(0,T))\ d(\tau,T), \tag{2.10}$$

where $d(\tau,T)$ is the risk free discount factor over the period $t = \tau$ to $t = T$.

Proof. If possible let

$$f(\tau) < (F(\tau,T) - F(0,T))\ d(\tau,T).$$

Now at $t = \tau$ we construct a portfolio P as follows

(i) borrow the amount $f(\tau)$ to enter into a long forward contract with forward price $F(0,T)$ and delivery $t = T$,
(ii) enter into a short forward position with forward price $F(\tau,T)$ for which there is no cost as per the definition of forward contract.

Then the value of the portfolio P at $t = \tau$ is given by

$$V_P(\tau) = 0.$$

Next at time $t = T$, we

(i) close the forward contracts by collecting (or paying, depending upon the sign of pay-offs) the amounts $S(T) - F(0,T)$ for the long forward position and $-S(T) + F(\tau,T)$ for the short forward position,
(ii) pay back the loan amount with interest, i.e. amount $(f(\tau)/d(\tau,T))$.

Therefore the value of the portfolio at $t = T$ is

$$V_P(T) = F(\tau,T) - F(0,T) - (f(\tau)/d(\tau,T)),$$

which is strictly positive and risk free. This violates no arbitrage principle.

Let us now consider the second case, namely $f(\tau) > [F(\tau,T) - F(0,T)]\,d(\tau,T)$. In this case our strategy is to construct a portfolio Q at $t = \tau$ as follows

(i) sell the forward contract which was initiated at $t = 0$ for the amount $f(\tau)$,
(ii) invest this amount $f(\tau)$ risk free from $t = \tau$ to $t = T$,

(iii) enter into a long forward contract with delivery time $t = T$ and forward price as $F(\tau, T)$.

Then $V_Q(\tau) = 0$ and $V_Q(T)$ is given by

$$V_Q(T) = \frac{f(\tau)}{d(\tau, T)} + (S(T) - F(\tau, T)) + (F(0, T) - S(T))$$
$$= \frac{f(\tau)}{d(\tau, T)} + F(0, T) - F(\tau, T),$$

which is strictly positive and risk free. This is not possible due to no arbitrage principle.

□

Example 2.4.1 *Let at the beginning of the year, a stock be sold for Rs 45 and risk free interest rate be 6%. Consider a forward contract on this stock with delivery date as one year. Find its forward price. Also find its value after 9 months if it is given that the stock price at that time turns out to be Rs 49.*

Solution From (2.2), the initial forward price $F(0, 1)$ is given by

$$F(0, 1) = S(0)\ e^{rT} = 45\ e^{0.06} = Rs\ 47.78.$$

Also it is given that $S(9/12) = 49$ and hence

$$F(9/12, 1) = 49\ \exp((\cdot 06)(3/12)) = Rs\ 49 \cdot 74.$$

Therefore by Theorem 2.4.1, the value of the forward contract after 9 months is

$$f(9/12) = [F(9/12, 1) - F(0, 1)]\ \exp((-0 \cdot 06)(1 - (9/12)))$$
$$= 1 \cdot 93.$$

□

Example 2.4.2 *Consider the data of Example 2.4.1 and assume that a dividend of Rs 2 is being paid after 6 months. Find the forward price of the contract and also its value after 9 months.*

Solution By (2.7), the initial forward price $F(0, 1)$ is given by

$$F(0, 1) = [S(0) - (div)e^{-rt}]\ e^{rT}$$
$$= \left[45 - 2e^{-0.06(1/2)}\right] e^{\cdot 06}$$
$$= Rs\ 45 \cdot 72.$$

Also by (2.3)

$$F(9/12,1) = S(9/12)\ e^{(0\cdot06)(1-9/12)} = Rs\ 49\cdot74.$$

Hence by Theorem 2.4.1,

$$f(9/12) = Rs\ 3\cdot96.$$

□

2.5 Futures Contract

It is obvious that when two parties enter into a forward contract, one of them is certainly going to lose money. Therefore a forward contract is always exposed to a risk of default by the party suffering a loss. To eliminate this risk of default, futures contracts have been introduced which are managed by an organized exchange. Thus even though futures contracts are very much in the spirit of forward contracts, their working is entirely different. Here the role of an organized exchange becomes very important because individual contracts are made with the exchange, which itself becomes counter party to both long and short trades. Therefore individuals themselves do not need to search for an appropriate counterparty because this job is taken care off by the exchange itself. The exchange also takes appropriate measures so as to eliminate the risk of counterparty default. This is done by a process called *marking to market*. Before we explain the working of the process, we list the basic features of a futures contract.

(i) Similar to a forward contact, a futures contract also has an underlying asset (e.g. stock) and a delivery date, say time $t = T$.

(ii) In addition to the (underlying) asset prices, the market also dictates the futures price $f(t,T)$ at discrete time steps, say $f(n,T)$ for $n = 0,1,2,\ldots$ with $nt \leq T$. In practice these discrete time steps may refer to day1, day2 etc.

(iii) $f(0,T)$ is known but $f(n,T)$ for $n = 1,2\ldots$ with $nt \leq T$ are unknown and are treated as random variables. Also the (underlying) asset prices $S(n)$ for $n = 1,2\ldots$ with $nt \leq T$ are random variables.

(iv) As in the case of a forward contract, there is no cost involved in initiating a futures contract, but there is a major difference in the cash flow. A long forward contract involves just a single payment $S(T) - F(0,T)$ at the delivery time $t = T$, whereas a futures contract involves a random cash flow at each time step $n = 0,1,2\ldots$ with $nt \leq T$. Thus the holder of a long futures position will receive the amount $f(n,T) - f(n-1,T)$ if positive, or will have to pay if it is negative. For a short futures position, exactly opposite payments apply. These payments are managed by a clearing house for the futures market.

(v) The futures price satisfy the condition that $f(t,T) = S(T)$. This condition has to hold because the futures cost of immediate delivery of goods has to be the *market price* or *spot price*.

(vi) It costs nothing to close, open or alter a futures position at any time step between $t = 0$ and $t = T$. This condition can be met if at each time step $n = 0, 1, 2\ldots$ with $nt \leq T$, the value of futures position is zero. For $n \geq 1$, this value is computed after marking to market.

But what is the physical meaning of futures price $f(n,T)$? Suppose a forward contract initiated at time $t = 0$ has forward price $F(0,T)$, where time $t = T$ is the delivery date. Let the forward price for a new contract initiated at time $t = 1$ with delivery date time $t = T$ be $F(1,T)$. Now the clearing house comes into picture. At the second day it revises all earlier contracts to the new delivery price $F(1,T)$ and accordingly an investor which holds a long forward contract initiated at $t = 0$ receives or pays the difference of two prices depending upon if the change in price reflects a loss or gain. So if $F(1,T) > F(0,T)$, then an investor holding a long forward contract receives $F(1,T) - F(0,T)$ from the clearing house because at the delivery he/she has to pay $F(1,T)$ rather than $F(0,T)$. Continuing in this manner and assuming that the investor stays until maturity, the profit/loss pay-off of a long futures position will be $[F(1,T) - F(0,T)] + [F(2,T) - F(1,T)] + \ldots + [F(T,T) - F(T-1,T)]$, which equals $S(T) - F(0,T)$ because $F(T,T) = S(T)$.

The above discussion demonstrates that the pay-off of a futures contract is the pay-off of the corresponding forward contract, except that the pay-off is paid throughout the life of the contract rather than at maturity. However there is no requirement that the investor of a futures contract has to stay till maturity. He/she can come out of the contract any time by taking the opposite position in the contract with the same maturity. Thus $f(n,t)$, $n = 0, 1, 2, \ldots$ with $nt \leq T$ are essentially the forward prices as perceived by the market.

Here it may be noted that $f(n,T)$ are not known and are taken as random variables dictated by the market. This is because interest rate r is rarely constant and, in general, it is stochastic in nature.

This process of adjusting the contract is called *marking to market*. Here an individual is required to open a *margin account* with the clearing house. This account must have a specified amount of cash for each futures contract. In practice it is about 5 to 10% of the value of futures contract. The margin account is compulsory for all contract holders whether they have long or short position. If the price of futures contract increased that day, then the parties having long position receive an amount which equals (change in price)× (the contract quantity) which is deposited in their account. The short parties loose the same amount and therefore this amount is deducted from their account. This process is termed as *marking of accounts to the market* which is carried out at the end of each trading day. This guarantees that both parties of the contract cover their obligations.

Thus each margin account value fluctuates from day to day according to change in futures price. At the delivery date, delivery is made at the futures contract price at that time which may be different from the futures price when the contract was first initiated.

Margin accounts serve a dual role. They serve as accounts to collect or pay out daily profits and also guarantee that contract holders do not default on their obligations. If the value of a margin account drops below a pre-defined margin level (in practice about 75% of the initial margin requirement), a *margin call* is issued to the contract holder demanding additional margin. Otherwise futures position will be closed by taking an equal and opposite position. Also any excess amount above the initial margin can be withdrawn by the investor. This margin account is totally managed by the futures clearing house. There are other rules/procedures/practices for managing the market account but these details are not presented here.

Example 2.5.1 *Suppose that the initial margin is set at* 10% *and the maintenance margin at* 5% *of the futures price. Suppose for* $n = 0, 1, 2, 3, 4,$ *the futures prices are* 140, 138, 130, 140 *and* 150 *respectively. Show the working of marking to market and margin account in a tabular form.*

Solution In the below given table, the two columns termed as Margin 1 and Margin 2, respectively refer to the deposits at the beginning and end of the day.

n	f(n,T)	Cash Flow	Margin 1	Payment	Margin 2
0	140	opening	0	-14	14
1	138	-2	12	0	12
2	130	-8	4	-9	13
3	140	10	23	9	14
4	150	10	24	9	15
		closing	15	15	0
			Total	10	

Here on day 0, a futures position is opened and 10% deposit (i.e. *Rs* 14) paid to open the market account. On day 1, the futures price drops by *Rs* 2 which is subtracted from the deposit. On day 2, there is further drop of *Rs* 8 in the futures price which makes the deposit below 5%. Therefore there is a margin call and the investor has to pay *Rs* 9 to restore the deposit to 10% level. On day 3, the futures

3
Basic Theory of Option Pricing-I

3.1 Introduction

We have already seen some examples of European options in Chapter 1. Though these examples were specific to single step discrete time scenario, they were general enough to guide the basic principle of pricing methodology. There the strategy had been to replicate the given option in terms of stock and bond so that at the end of expiry, the pay-off of the option matches with the value of the replicating portfolio. Then, by no arbitrage principle, the initial value of the portfolio became the price of the given option.

The aim of this chapter is to formalize the above pricing methodology and introduce single and multi-period binomial lattice models for pricing of European and American call/put options. This discussion is continued in Chapter 4 as well, where first the Cox-Ross-Rubinstein (C.R.R.) model is presented, and then the celebrated Black-Scholes formula of option pricing is introduced. The Black-Scholes formula will be re-visited in Chapter 10 after readers have acquired the necessary background in stochastic calculus.

writer holder
call option seller Buyer
Seller
put option Buyer

3.2 Basic Definitions and Preliminaries

In this section we define European and American call/put options and discuss some of their simple properties.

Definition 3.2.1 (European Call Option) *A European call option is a contract giving the holder the right to buy an asset, called the underlying (e.g. stock), for a price K fixed in advance, called the strike price or exercise price, at a specified future time T, called the exercise time or the expiry time.*

Here the term 'underlying' has a general meaning. It could be stock, commodity, foreign currency, stock index or even interest rate. However, unless it is otherwise stated, we shall always mean stock option, i.e. the underlying asset is being taken as stock. Thus a European (stock) call option is a *derivative security* whose underlying security is stock. It gives the holder the right, to buy stock under specified terms as prescribed in Definition 3.2.1. There is absolutely no obligation for the holder to buy stock, but he/she has got the right to buy if he/she so wishes.

A *European put option* is exactly similar to a European call option, except that the word 'buy' changes to 'sell'. Therefore a European put option is a contract giving the holder the right to sell the underlying asset for the strike price K at the exercise time T.

Along with the European call/put options another term, namely *American call/put options*, is also very commonly used. The basic distinction between a European option and an American option is that an American option allows exercise at any time before and including the expiry; whereas a European option can be exercised only at the expiry. Here we may note that the word *European* and *American* refer to two different conventions of exercise rather than having any geographical significance. Thus the words *European* and *American* have become two standards in option market, referring to two different structures, no matter where they are issued.

The above discussion suggests that an option can be described by describing its four basic features. These are as listed below.

(i) The description of the underlying asset.
(ii) The nature of the option - whether a call or a put; a European or an American.
(iii) The strike or exercise price.
(iv) The exercise or expiry date.

Next we try to understand the meaning of the term *option pricing*. By definition, the holder of an option gets the right to buy or sell the asset (depending upon the nature of the option) but has no obligation, so some amount has to be paid at the time of contract to get this right. This amount is termed as the *premium* or *price* of the option, and the problem of option pricing revolves around the methodologies to find this price in a *fair* way.

An option has two sides or parties. The party that grants the option is said to *write* the option and the party that obtains the option is said to *purchase* it. The party that purchases the option becomes its holder, and it has no risk of loss other than the original premium paid because it has the right to exercise the option and has no obligation attached to it. Whereas the party that writes the option has major risk. This is because if the option, say a call, is exercised then the writer has to arrange for the asset. If the writer does not already own the asset then it might have to be acquired from the market at a price higher than the agreed strike price. Similarly in the case of a put option, the writer may have to accept the asset at a much lower price (namely the strike price) than what is prevailing in the market. The problem of option pricing is to be *fair* to both the parties and determine the *fair* price of the option under consideration. There is another terminology used for the writer and the purchaser of the option. The buyer (purchaser) of the option is said to have taken the *long position* and the seller (writer) is said to have taken the *short position*. Thus we have terms like *long call*, *short call*, *long put* and *short put* etc.

The next important concept to understand is the *pay-off* or the *value of the option at expiry*. To fix our ideas, let us consider the case of a European call option with strike price K and expiry as T. Let $S(T)$ denote the price of the underlying asset (stock) at the expiry. Then, the holder will not exercise the option at the expiry if $S(T) \leq K$ but will certainly exercise the option if $S(T) > K$. Therefore the pay-off to the holder is zero for $S(T) \leq K$ and $S(T) - K$ for $S(T) > K$. If we now introduce the notation x^+ as

$$x^+ = \begin{cases} x, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

then the pay-off of the given call option can be written as $(S(T) - K)^+$.

Definition 3.2.2 (Pay-off of a European Call Option) *Let C be a European call option with specifications as prescribed in Definition 3.2.1. Then*

$$C(T, K, S) = (S(T) - K)^+ = Max\,(S(T) - K,\ 0)\,,$$

is called the value or the pay-off of the call option C. Here $S(T)$ denotes the price of the underlying at the exercise time T.

It is simple to define the pay-off or the value of a European put option P. Obviously the holder of a European put option will exercise the option only when $K > S(T)$ and get the profit as $K - S(T)$. Therefore the pay-off or the value of the given European put option is defined as

$$P(T, K, S) = (K - S(T))^+ = \mathrm{Max}(K - S(T),\ 0)\,.$$

The left hand side diagram of Fig. 3.1 depicts the pay-off of a European call option whereas the pay-off of a European put option is depicted in the right hand side diagram of Fig. 3.1.

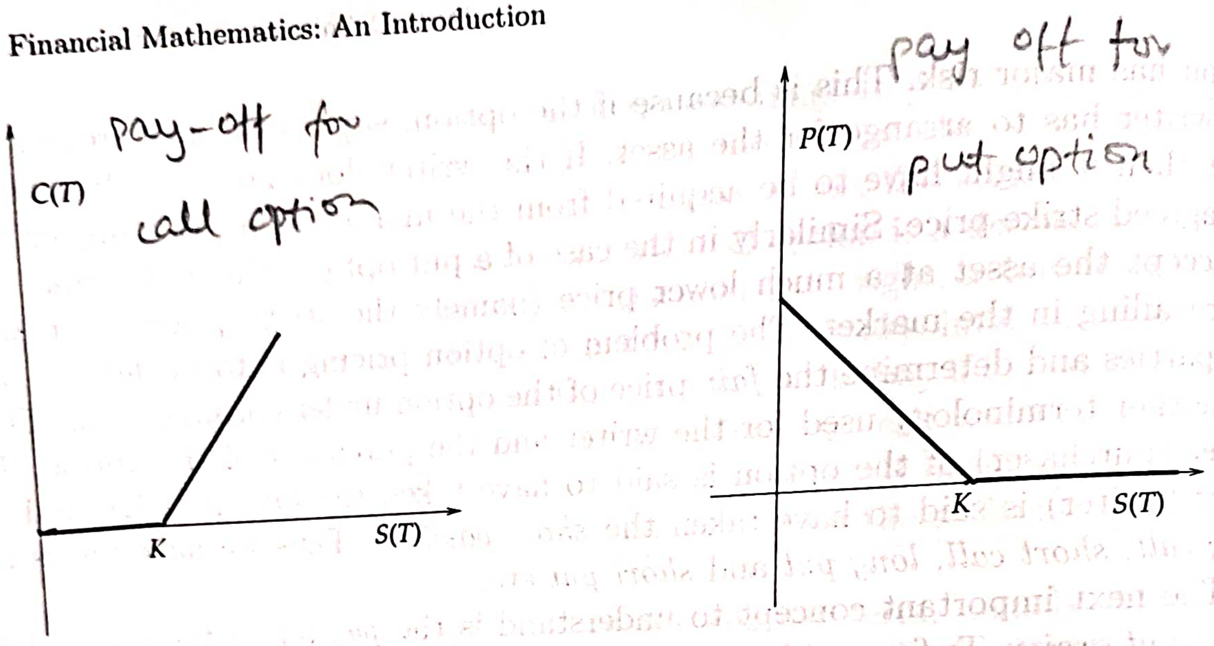

Fig. 3.1. Pay-off curves of call and put options

GAIN OF AN OPTION

We say that a European call option is *in the money, at the money* or *out* (*money* depending upon $S(T) > K$, $S(T) = K$ or $S(T) < K$. A European put option i *in the money, at the money* or *out of money* depending upon $S(T) < K$, $S(T) =$ l or $S(T) > K$.

We also define the *gain* of an option. The gain of an option buyer is the pay off modified by the premium $C(0)$ or $P(0)$ paid for the option. Thus for the ca option C, the gain is $(S(T) - K)^+ - C(0)e^{rT}$. Similarly the gain of the buyer of th put option P is $(K - S(T))^+ - P(0)e^{rT}$.

We shall discuss the pay-off corresponding to an American option at a late place. Also sometimes to make the context specific, we shall write C^E or C^A (re spectively P^E or P^A) to identify whether the option is European or America However, if only C or P is used, it will always mean a European option.

Though there are many results connecting C^E, P^E and C^A, P^A, the followin lemma, called *put-call parity* is interesting and useful.

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Lemma 3.2.1. *(Put-Call Parity) Let C^E and P^E be the prices of a Europea call and a European put defined over the same stock with price S. Let the give call and the given put have the same strike price K and the same expiry date T. Further, let the underlying stock pay no dividend. Then*

questions asked in exams

$$\boxed{C^E(0) - P^E(0) = S(0) - Ke^{-rT}}$$

where r is the constant risk-free interest rate under continuous compounding.

Though a formal proof could be given to the above lemma, we present here only an intuitive argument. Suppose we construct a portfolio by writing and selling one put and buying one call option, both with the same strike price K and expiry date T. Now if $S(T) \geq K$, then the call will pay $S(T) - K$ and the put will be worthless. If $S(T) < K$ then the call will be worthless and the writer of the put will need to pay $K - S(T)$. In either case, the value of the portfolio will be $S(T) - K$ at expiry. But $S(T) - K$ is also the pay-off of a long forward contract with forward price K and delivery time T. Therefore by no arbitrage principle, the current value of the constructed portfolio of options should be that of the forward contract, i.e. $C^E(0) - P^E(0)$ should be $S(0) - Ke^{-rT}$. Thus $C^E(0) - P^E(0) = S(0) - Ke^{-rT}$. Fig. 3.2. depicts this argument.

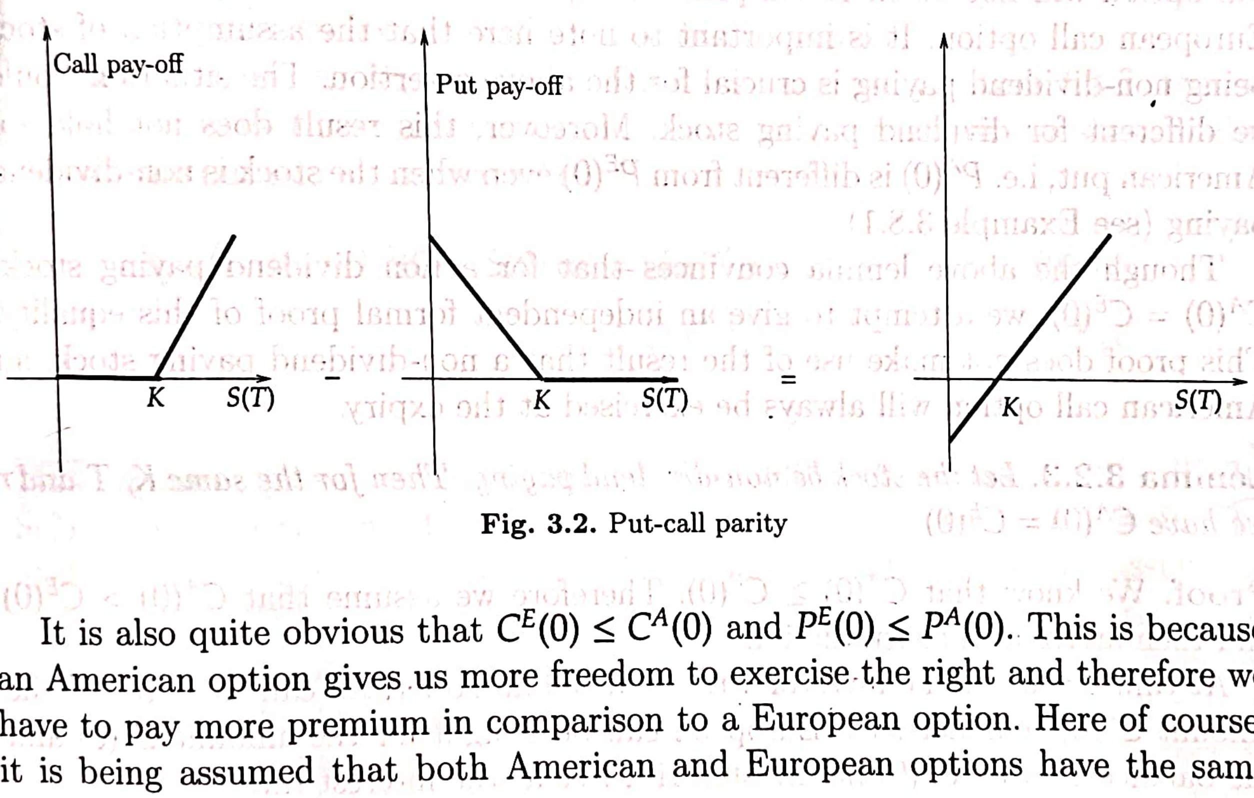

Fig. 3.2. Put-call parity

It is also quite obvious that $C^E(0) \leq C^A(0)$ and $P^E(0) \leq P^A(0)$. This is because an American option gives us more freedom to exercise the right and therefore we have to pay more premium in comparison to a European option. Here of course, it is being assumed that both American and European options have the same parameters, T, K and S.

The below given lemma states that for a non-dividend paying stock, the American call option will never be exercised prior to expiry. This being so, it should be equivalent to the European call option, i.e. $C^A(0) = C^E(0)$.

Lemma 3.2.2. *Let C^E and C^A be respectively the prices of a European call and an American call defined over the same stock with price S. Let both these calls have the same strike price and the same expiry date. Further, let the stock be non-dividend paying. Then $C^A(0) = C^E(0)$.*

Again rather than giving a formal proof, let us give an intuitive argument f the above lemma. We have already seen that $C^A(0) \geq C^E(0)$ because in comparis to a European call option, an American option gives more right with regard the time of exercise of the given option. Also, by using the put-call parity, $C^E(0)$ $S(0) - Ke^{-rT}$, as $P^E(0) \geq 0$. Therefore $C^A(0) \geq C^E(0) \geq S(0) - Ke^{-rT}$. But $r > 0$ an hence this last inequality gives $C^A(0) > S(0) - K$.

But $C^A(0) > (S(0)-K)$ implies that the price of the given American call option greater than the pay-off. Therefore the option should be sold sooner than exercise at $t = 0$. Here the choice of $t = 0$ is for the purpose of reference time and is totall arbitrary. Hence taking any $t < T$ instead of $t = 0$, the above arguments show that the American call option will not be exercised at time t, i.e. the America call option will not be exercised prior to expiry, and then it is equivalent to th European call option. It is important to note here that the assumption of stock being non-dividend paying is crucial for the above assertion. The situation would be different for dividend paying stock. Moreover, this result does not hold for American put, i.e. $P^A(0)$ is different from $P^E(0)$ even when the stock is non-dividend paying (see Example 3.8.1).

Though the above lemma convinces that for a non dividend paying stock, $C^A(0) = C^E(0)$, we attempt to give an independent formal proof of this equality. This proof does not make use of the result that a non-dividend paying stock, an American call option will always be exercised at the expiry.

Lemma 3.2.3. *Let the stock be non-dividend paying. Then for the same K, T and r we have* $C^A(0) = C^E(0)$.

Proof. We know that $C^A(0) \geq C^E(0)$. Therefore we assume that $C^A(0) > C^E(0)$ and then arrive at a contradiction.

At time $t = 0$, let the investor write and sell an American call so as to get the amount $C^A(0)$. Further let a European call be bought for the amount $C^E(0)$ and the balance $C^A(0) - C^E(0)$ be invested risk free at the interest rate r.

If the American call is exercised at time $t < T$, then the investor borrows a share of stock and sells the same for K to the buyer so as to settle his/her obligation as writer of the American call option. Further the investor invests this amount K risk free up to time T. Now at time $t = T$, the investor exercises his/her European call option to buy a share of stock for K and closes the short position on stock. This will result in an arbitrage profit of $\left(C^A(0) - C^E(0)\right)e^{rT} + Ke^{r(T-t)} - K > 0$. If the American option is not exercised at all, then the investor will end up with the European call and even an arbitrage profit $\left(C^A(0) - C^E(0)\right)e^{rT} > 0$. Therefore $C^A(0) = C^E(0)$. □

An obvious question at this stage is as follows : Do we have put call parity for American option as well? Unfortunately there is no exact put call parity for American option. But we have certain estimates for $C^A(0) - P^A(0)$ as detailed in the below given lemma.

Lemma 3.2.4. ***(Put-Call parity Estimate for American Option)*** *Let the stock be non dividend paying. Then for the same K, T and S, we have*

$$S(0) - Ke^{-rT} \geq C^A(0) - P^A(0) \geq S(0) - K.$$

Proof. We shall first prove that $S(0) - Ke^{-rT} \geq C^A(0) - P^A(0)$. For this, we assume that $C^A(0) - P^A(0) - S(0) + Ke^{-rT} > 0$, and arrive at some contradiction. Now at $t = 0$, we write and sell an American call, buy an American put, buy a share of stock, and finance the transactions in the money market.

If the holder of the American call chooses to exercise it at time $t \leq T$, then we shall receive K for the share of stock and settle the money market position, ending up with the put and a positive amount

$$\begin{aligned} K + \left(C^A(0) - P^A(0) - S(0)\right)e^{rt} &= \left(Ke^{-rt} + C^A(0) - P^A(0) - S(0)\right)e^{rt} \\ &\geq \left(Ke^{-rT} + C^A(0) - P^A(0) - S(0)\right)e^{rt} \\ &> 0. \end{aligned}$$

This argument presumes that $C^A(0) - P^A(0) - S(0) > 0$. In case $C^A(0) - P^A(0) - S(0) < 0$, then this much money is borrowed from the bank at $t = 0$. Then at t, we get K because of the sold American call. But under our assumption $K > \left(C^A(0) - P^A(0) - S(0)\right)e^{rt}$ and therefore we close the position in the money market and still having a positive amount. This violates no arbitrage principle.

Next suppose that $C^A(0) - P^A(0) - S(0) + K < 0$. In this case at $t = 0$, we write and sell a put, buy a call, sell short one share of stock and finance the transactions in the money market. If the American put is exercised at $t \leq T$, then we can withdraw K from the money market to buy a share of stock and close the short sale. We shall be left with the call option and a positive amount $\left(-C^A(0) + P^A(0) + S(0)\right)e^{rt} - K > Ke^{rt} - K \geq 0$. If the put is not exercised at all, then we can buy a share of stock for K by exercising the call at time T and close the short position on stock. On closing the money market position, we shall also end up with a positive amount. This again contradicts the no arbitrage principle.

□

3.3 Behavior of Option Prices With Respect to Variabl

The price of an option depends on a number of variables. These are

(i) the strike price K
(ii) the expiry time T
(iii) the current stock price $S(0)$
(iv) the dividend rate r_{div}, and
(v) the risk free rate r.

Both from theoretical as well as from applications point of view, it should be usefu to analyze the behavior of option price as function of one of the variables, keepin the remaining variables constant. We now present a sample of some results an refer to the text by Capinski and Zastawniak [25] for further details. The notatio $C^E(K)$ denotes the price of European call options over the same underlying an with the same exercise time, but with different values of the strike price K. Simila interpretation is given for other notations, namely $C^E(S)$ and $C^E(T)$ etc.

Lemma 3.3.1. *Let $K_1 < K_2$. Then*

(i) $C^E(K_1) \geq C^E(K_2)$,
(ii) $P^E(K_1) \leq P^E(K_2)$.

Thus $C^E(K)$ is a non-increasing and $P^E(K)$ is a non-decreasing function of K. These assertions are obvious because to have the right to buy at a lower price, we have to pay more premium than to have the right to buy at a higher price.

Lemma 3.3.2. *Let $K_1 < K_2$. Then*

(i) $C^E(K_1) - C^E(K_2) \leq e^{-rT}(K_2 - K_1)$,
(ii) $P^E(K_2) - P^E(K_1) \leq e^{-rT}(K_2 - K_1)$.

Proof. From put-call parity, we have

$$C^E(K_1) - P^E(K_1) = S(0) - K_1 e^{-rt},$$

and

$$C^E(K_2) - P^E(K_2) = S(0) - K_2 e^{-rt}.$$

If we now subtract the above two equations, we get

$$C^E(K_1) - C^E(K_2) + P^E(K_2) - P^E(K_1) = (K_2 - K_1)e^{-rt}.$$

But then an application of Lemma 3.3.1 gives the result. □

X**Lemma 3.3.3.** *Let $K_1 < K_2$ and $0 \le \alpha \le 1$. Then $C^E(K)$ and $P^E(K)$ satisfy*

(i) $C^E(\alpha K_1 + (1-\alpha)K_2) \le \alpha C^E(K_1) + (1-\alpha)C^E(K_2)$

(ii) $P^E(\alpha K_1 + (1-\alpha)K_2) \le \alpha P^E(K_1) + (1-\alpha)P^E(K_2)$.

This means that $C^E(K)$ and $P^E(K)$ are convex functions of K.

Proof. Let us denote $\widehat{K} = \alpha\, K_1 + (1-\alpha)\, K_2$. Suppose that $C^E(\widehat{K}) > \alpha\, C^E(K_1) + (1-\alpha)\, C^E(K_2)$.

Now at $t = 0$, let the investor write and sell the option with strike $\widehat{K}$ and get the amount (premium) $C^E(\widehat{K})$. Then let the investor purchase α option with strike price K_1 and $(1-\alpha)$ options with strike price K_2. This will involve a payment (premium) of $\alpha\, C^E(K_1) + (1-\alpha)\, C^E(K_2)$. But then under the assumed inequality the investor is still left with the amount $C^E\,(\widehat{K}) - (\alpha\, C^E(K_1) + (1-\alpha)\, C^E(K_2)) > 0$ which he/she invests risk free.

At the expiry, if the option with the strike price $\widehat{K}$ is exercised, then the investor shall have to pay the amount $\left(S(T) - \widehat{K}\right)^+ = \mathrm{Max}\left(S(T) - \widehat{K}, 0\right)$. But he/she can raise the amount $\alpha\;(S(T) - K_1)^+ + (1-\alpha)\;(S(T) - K_2)^+$ by exercising α calls with strike price K_1 and $(1-\alpha)$ calls with strike price K_2. But it is known that

$$\left(S(T) - \widehat{K}\right)^+ \le \alpha\;(S(T) - K_1)^+ + (1-\alpha)\;(S(T) - K_2)^+,$$

because the function $(S(T) - K)^+$ is a convex function of K. Therefore the investor will realize an arbitrage profit of $\left[C^E(\widehat{K}) - (\alpha\, C^E(K_1) + (1-\alpha)\, C^E(K_2))\right] e^{rT}$ at expiry. Therefore our supposition is wrong and we get the result.

□

The below given Lemmas can also be proved on the similar lines.

X**Lemma 3.3.4.** *Let $T_1 < T_2$. Then*

(i) $C^E(T_1) \le C^E(T_2)$

(ii) $P^E(T_1) \le P^E(T_2)$.

Proof. If possible let $C^E(T_1) > C^E(T_2)$. We write and sell one option expiring at time T_1 and buy one with the same strike price but expiry as T_2, investing the balance without risk. If the written option is exercised at T_1, we can exercised the option immediately to cover our liability. The balance $(C^E(T_1) - C^E(T_2)) > 0$ invested without risk will be our arbitrage profit.

The inequality (*ii*) can be proved on similar lines.

□

Lemma 3.3.5. *Let $0 < x_1 < x_2$. Further let at time $t = 0$, $\overline{S} = x_1 S(0)$ and $\hat{S} = x_2 S(0)$. Then*

(i) $C^E(\overline{S}) \le C^E(\hat{S})$

(ii) $P^E(\overline{S}) \ge P^E(\hat{S})$.

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Proof. We shall prove the first inequality. If possible let $C^E(\overline{S}) > C^E(\hat{S})$. We can write and sell a call on a portfolio with x_1 shares and buy a call on a portfolio with x_2 shares having the same strike price K and exercise time T. Also we invest the amount $C^E(\overline{S}) - C^E(\hat{S})$ risk-free. As $x_1 < x_2$, we have $(x_1 S(T) - K)^+ \le (x_2 S(T) - K)^+$. If the sold option is exercised at time T, we can exercised other option to cover our liability. The balance $C^E(\overline{S}) - C^E(\hat{S}) > 0$ invested risk-free will be our arbitrage profit.

□

Lemma 3.3.6. *Let $\overline{S}$ and $\hat{S}$ be as defined in Lemma 3.3.5. Then*

(i) $C^E(\hat{S}) - C^E(\overline{S}) \le (\hat{S} - \overline{S})$

(ii) $P^E(\overline{S}) - P^E(\hat{S}) \le (\hat{S} - \overline{S})$.

Proof. Using put-call parity we have

$$C^E(\hat{S}) - P^E(\hat{S}) = \hat{S} - Ke^{-rT},$$

and

$$C^E(\overline{S}) - P^E(\overline{S}) = \overline{S} - Ke^{-rT}.$$

On subtraction, we get

$$\left(C^E(\hat{S}) - C^E(\overline{S})\right) + \left(P^E(\overline{S}) - P^E(\hat{S})\right) = (\hat{S} - \overline{S}),$$

which gives inequalities (i) and (ii) because both terms on the right hand side are non-negative.

□

Results similar to above Lemmas also hold for American options. For these details we may refer to Capinski and Zastawniak [25].

3.4 Pay-off Curves of Options Combinations

The graph between the pay-off and the stock price at the expiry is called the *pay-off curve of the option*. Often we need to invest in combinations of options to take care of specific hedging or speculative strategies. Knowing the pay-off curves of individual options we can obtain the pay-off curve of the combinations of options. This curve will be a combination of connected straight line segments depending upon the number and nature of securities in the specific combination. This pay-off curve of the combination is called its *spread*. We now give examples of some of the most common spreads.

Bull Spread

Consider a scenario in which the investor expects the stock price to rise and wants to speculate on that. Obviously the investor should buy a call option say C_{K_1}, with strike price K_1 which is close to the current stock price. Here we have used the symbol C_{K_1} rather than $C(K_1)$ for the sake of simplicity. The premium may be reduced by selling a call option, say C_{K_2}, with strike price $K_2 > K_1$. This strategy should bring good returns provided the stock price increases are moderate. The spread of the combination $C_{K_1} - C_{K_2}$ is called the *bull spread*, which is depicted in Fig 3.3.

Fig. 3.3. Bull spread

Bear Spread

The pay-off curve of the combination $C_{K_1} - C_{K_2}$ with $(K_1 > K_2)$ gives rise to a *bear spread*. This is usually employed by an investor who is expecting a moderate decline in the stock price. The bear spread is depicted in Fig. 3.4.

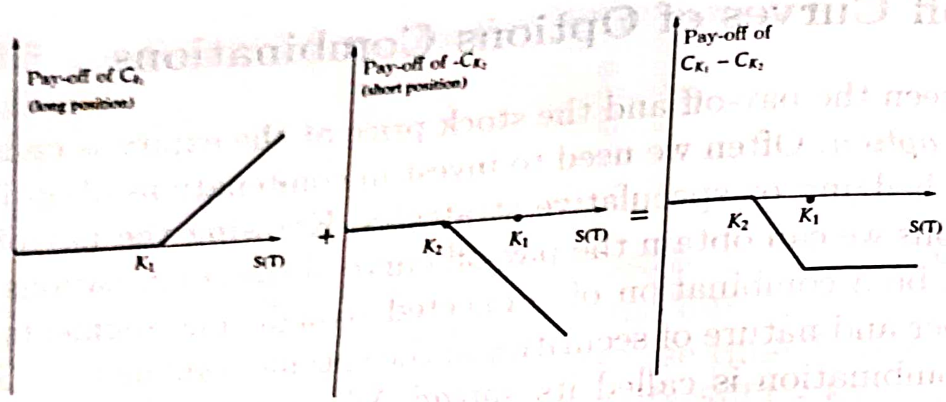


Fig. 3.4. Bear spread

Butterfly Spread

Let $K_1 < K_2 < K_3$. Let C_{K_1}, C_{K_2} and C_{K_3} be call options with strike prices K

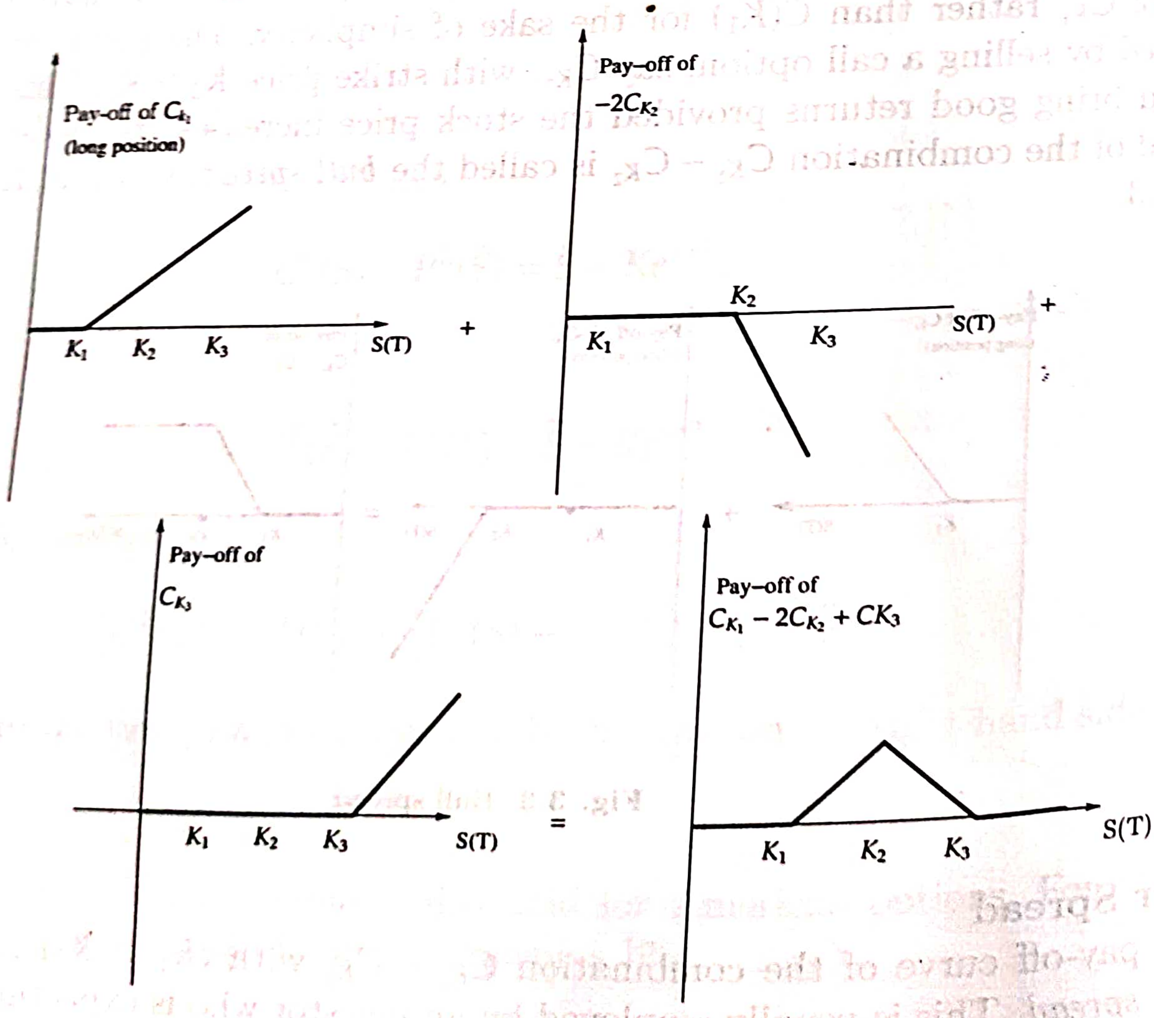


Fig. 3.5. Butterfly spread

K_2 and K_3 respectively. Let these calls be defined over the same stock and have common expiry T. Then the pay-off curve of the combination $C_{K_1} - 2C_{K_2} + C_{K_3}$ gives rise to a *butterfly spread*. This option combination is used by an investor who feels that the stock price will generally remain unaltered, i.e. it will not change significantly. The butterfly spread is depicted in Fig 3.5.

The above discussion suggests that by forming the combinations of options, any pay-off function can be approximated by a sequence of straight line segments. In other words, any continuous pay-off function can be made close to pay-off of an appropriate option combination.

Now similar to pay-off curve, we can also sketch *gain curve* of a given option combination. The readers can sketch the gain curves for the above examples in an obvious manner by utilizing the definition of gain of an option.

Example 3.4.1 *Let the sale and purchase of options over the same stock and having the same expiry be given by the expression*

$$-P_{100} + P_{120} + 2C_{150} - C_{180}.$$

Sketch the spread of the given option combination.

Solution To draw the spread, we draw the individual pay-off curves and then combine them. The details in Fig. 3.6 are self explanatory. □

3.5 Single Period Binomial Lattice Model for Option Pricing

The simple example of pricing a European option, presented in Chapter 1, is in fact an example of a single period binomial lattice model. Here we formally develop the theory for the single period case and extend the same for multi-period case in the next section. For this model we assume the following

(i) The initial value of the stock is $S(0)$, i.e. the stock price at $t = 0$ is $S(0)$.
(ii) At the end of the period, the price is either $u\,S(0)$ with probability p, or $d\,S(0)$ with probability $(1 - p)$, $0 < p < 1$.
(iii) $u > d > 0$.
(iv) At every period, it is possible to borrow or lend at a common risk free interest rate r. Let $R = (1 + r)$, and $u > R > d$. This assumption is needed to avoid arbitrage.

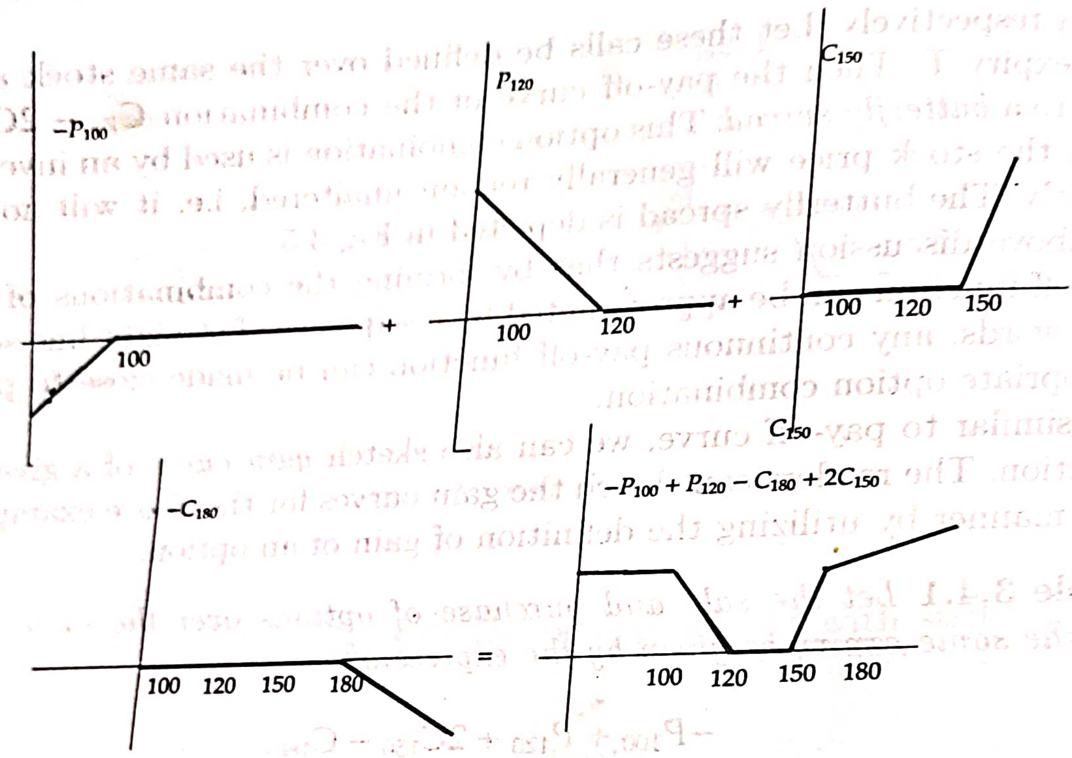


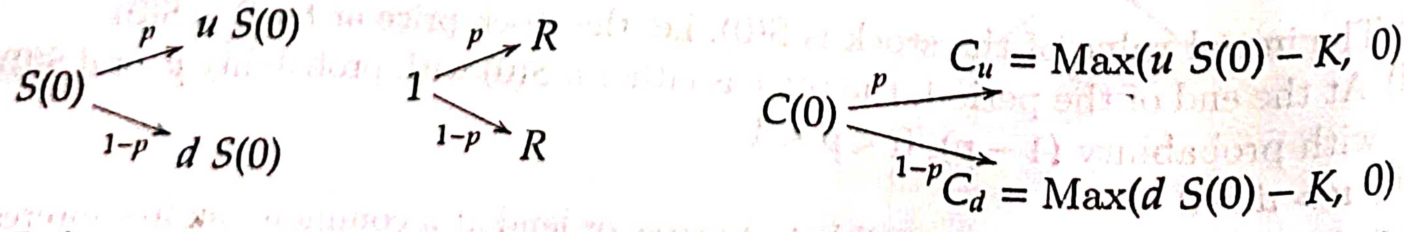
Fig. 3.6. Spread of Example 3.4.1

Let C be the given European call option on the stock with strike price K and expiry at the end of period (i.e. $t = 1$). Our aim is to find $C(0)$. Since at $t = 1$, the stock price could be either $u\,S(0)$ (with probability p) or $d\,S(0)$ (with probability $(1-p)$). We have

$$C(1) = \begin{cases} C_u = \mathrm{Max}(u\,S(0) - K,\ 0), & \text{with probability } p \\ C_d = \mathrm{Max}(\cancel{u\,S(1)} - K,\ 0), & \text{with probability } (1-p). \end{cases}$$

d S(0)

Though the value of the risk free asset (say bond) is deterministic, we can take it as a (degenerate) derivative of the stock-degenerate in the sense that the same value R is assigned at the end of each arc. Thus we have the following figures



Before we actually derive the pricing formula for the given scenario, let us try to justify the assumption $u > R > d$. For this we have below given lemma.

Lemma 3.5.1. *If $u > R > d$ does not hold then "no arbitrage principle" is violated.*

Proof. We first consider the case $R \geq u > d$. Now construct the portfolio P : $\left(x = -\dfrac{1}{S(0)},\ y = \dfrac{1}{B(0)}\right)$, where $B(0)$ denotes the price of the bond at $t = 0$. Let $V_P(0)$ and $V_P(1)$ respectively denote the value of the portfolio P at $t = 0$ and $t = 1$. Then

$$V_P(0) = x\ S(0) + y\ B(0) = \left(-\frac{1}{S(0)}\right)S(0) + \left(\frac{1}{B(0)}\right)B(0) = 0,$$

and

$$\begin{aligned} V_P(1) &= x\ S(1) + y\ B(1) \\ &= \begin{cases} -\dfrac{1}{S(0)}u\ S(0) + \dfrac{1}{B(0)}R\ B(0), & \text{with probability } p \\ -\dfrac{1}{S(0)}d\ S(0) + \dfrac{1}{B(0)}R\ B(0), & \text{with probability } (1-p) \end{cases} \\ &= \begin{cases} R - u, & \text{with probability } p \\ R - d, & \text{with probability } (1-p). \end{cases} \end{aligned}$$

But under the assumption $R \geq u > d$, $(R-u) \geq 0$ and $(R-d) > 0$. As $0 < p < 1$, $V_P(1) \geq 0$ with probability 1 and it can take positive value with positive probability even though $V_P(0) = 0$. This clearly violates the "no arbitrage principle".

We next consider the case $u > d \geq R$, and construct the portfolio Q : $\left(x = \dfrac{1}{S(0)}, y = -\dfrac{1}{B(0)}\right)$ and note that $V_Q(0) = 0$. Further $V_Q(1) \geq 0$ with probability 1 and it can take positive value $(u - R)$ with positive probability, which again violates the "no arbitrage principle". Therefore to avoid the arbitrage opportunity, we need to assume that $u > R > d$. □

Finding C(0)

Now to find the price $C(0)$ of the call, we need to replicate it in terms of stock and bond so that at $t = 1$, the value of this replicating portfolio equals the pay-off of the call at $t = 1$. Let the replicating portfolio be $RP : (x = a, y = b)$, where a is the number of shares of stock and b is the units of bond. Then we need to find a and b such that $V_{RP}(1) = C(1)$. But

$$\begin{aligned} V_{RP}(1) &= a\ S(1) + b\ B(1) \\ &= \begin{cases} a\ u\ S(0) + b\ R\ B(0), & \text{with probability } p \\ a\ d\ S(0) + b\ R\ B(0), & \text{with probability } (1-p), \end{cases} \end{aligned}$$

and

$$C(1) = \begin{cases} C_u, & \text{with probability } p \\ C_d, & \text{with probability } (1-p). \end{cases}$$

Hence $V_{RP}(1) = C(1)$ gives

$$a\,u\,S(0) + b\,R\,B(0) = C_u \quad (3.1)$$
$$a\,d\,S(0) + b\,R\,B(0) = C_d. \quad (3.2)$$

Solving the above system we obtain

$$a = \frac{C_u - C_d}{S(0)(u-d)}, \quad (3.3)$$

and

$$b = \frac{u\,C_d - d\,C_u}{R\,(u-d)B(0)}. \quad (3.4)$$

Also

$$V_{RP}(0) = a\,S(0) + b\,B(0). \quad (3.5)$$

Therefore using (3.3), (3.4) and (3.5) we have

$$\begin{aligned} V_{RP}(0) &= a\,S(0) + b\,B(0) \\ &= \left(\frac{C_u - C_d}{S(0)(u-d)}\right)S(0) + \left(\frac{u\,C_d - d\,C_u}{R\,(u-d)\cancel{A(0)}_{B(0)}}\right)B(0) \\ &= \left(\frac{C_u - C_d}{(u-d)}\right) + \left(\frac{u\,C_d - d\,C_u}{R\,(u-d)}\right) \\ &= \frac{1}{R}\left(\left(\frac{R-d}{u-d}\right)C_u + \left(\frac{u-R}{u-d}\right)C_d\right) \\ &= \frac{1}{R}(\hat{p}\,C_u + (1-\hat{p})\,C_d), \end{aligned} \quad (3.6)$$

where $\hat{p} = \left(\frac{R-d}{u-d}\right)$ and $(1-\hat{p}) = \left(\frac{u-R}{u-d}\right)$.

Since at $t = 1$, the value $V_{RP}(1)$ of the replicating portfolio and the pay-off $C(1)$ of the call are equal, by no arbitrage principle, we should have $C(0) = V_{RP}(0)$. Hence from (3.6)

europian call at t = 0

$$\boxed{C(0) = \frac{1}{R}(\hat{p}\,C_u + (1-\hat{p})\,C_d).} \quad (3.7)$$

In (3.7) we should note that $0 < \hat{p} < 1$ because $u > R > d$. Hence $\hat{p}$ can be onsidered a probability. This probability $\hat{p}$ is called the *risk neutral probability*. Before we discuss its interpretation and nomenclature, we observe from (3.7) that $C(0) = \frac{1}{R}E_{\hat{p}}(C(1))$, where $E_{\hat{p}}$ denotes the expectation under risk neutral probability $\hat{p}$. The formula

$$\boxed{C(0) = \frac{1}{R}E_{\hat{p}}(C(1))} \tag{3.8}$$

s very general and is valid for any derivative security with "appropriate" modifications.

In the above derivation, there is no role of $B(0)$. In fact we may take $B(0) = 1$, giving $B(1) = R$. Therefore we can think of bond as cash, as Rs 1 gives $Rs\ (1+r) = Rs\ R$ after one time period. In this light, formula (3.8) tells that to find the price $C(0)$ of the call, we should first take the expectation of the pay-off at expiry with respect to the risk neutral probability, and then discount the same according to the risk free rate.

Here we should note that risk neutral probability $\hat{p}$ is different from p. The actual probability p describes the (stochastic) price movement but the risk neutral probability $\hat{p}$ has a totally different interpretation. In fact the actual probability p enters nowhere in the derivation of $C(0)$. However it gives a motivation to introduce risk neutral probability. This is because one would invest in stock only if the expected growth rate of stock is higher than that of the rate at money market (bank/bond/cash), i.e.

$$p\,(u\ S(0)) + (1-p)\,(d\ S(0)) > R\ S(0),$$

i.e.

$$S(0) < \frac{1}{R}[p\,(u\ S(0)) + (1-p)\,(d\ S(0))]. \tag{3.9}$$

But

$$\hat{p}\,(u\ S(0)) + (1-\hat{p})\,(d\ S(0)) = \left(\frac{R-d}{u-d}\right)(u\ S(0)) + \left(\frac{u-R}{u-d}\right)(d\ S(0)),$$

which on simplification gives

p_u $\quad$ p_d

$$\boxed{\frac{1}{R}[\hat{p}\,(u\ S(0)) + (1-\hat{p})\,(d\ S(0))] = S(0).} \tag{3.10}$$

Equation (3.10) tells that under risk neutral probability $\hat{p}$, the mean rate of growth of stock equals the rate of growth in the money market account which is risk free. Now if this is the case then the investor must be neutral about the risk and hence the name *risk neutral probability*. The formula (3.7) is therefore referred as the *risk neutral pricing formula* for European call option.

The role of risk neutral probability is very important because it provides for opportunity to the writer of the option to *hedge* his/her risk and thereby provides a *fair* price of the call. The hedging problem in the present context is to find an equivalent replicating portfolio, i.e. a portfolio of stock and bond at $t = 0$ such that the value of this portfolio at $t = 1$ matches with the pay-off of the call. This process of replicating the option is called the *hedging problem*, and the replicating portfolio is called the *hedge of the given option.*

Remark 3.5.1 *Though the actual probabilities p and $(1-p)$ have not entered in the pricing formula (3.7), they have played an important role indirectly. Thus the investor will like to buy a call option provided he/she feels that there is high probability of the stock price going up at $t = 1$. Also the agreed strike price K depends to some extent on these probabilities. This is because if there is a feeling that at $t = 1$, the stock price will go up with high value of p then he/she may agree for a higher value of K.*

The above procedure of finding the call price is valid provided we guarantee that the risk neutral probability measure (RNPM) always exists and is unique. The below given result tells that under no arbitrage principle RNPM always exists. Further if in addition, the market is *complete*, then the RNPM is unique.

Lemma 3.5.2. *A risk neutral probability measure exists if and only if no arbitrage principle holds. Further if the market is complete then, the RNPM is unique.*

We shall partly prove this Lemma in Section 3.7 by making use of the concept of duality in linear programming. Specifically we shall prove that RNPM exists if and only if, no arbitrage principle holds. The discussion about the uniqueness of RNPM will be postponed till the concept of completeness of the market is introduced.

Example 3.5.1 *Find the price of a European call option with the given data as* $B(0) = 100$, $B(1) = 110$, $S(0) = 100$, $K = 100$, $T = 1$ *and*

$$S(1) = \begin{cases} 120, & \text{with probability } 0 \cdot 6 \\ 80, & \text{with probability } 0 \cdot 4. \end{cases}$$

Will the price change if the probabilities p and $(1-p)$ are taken as 0.3 and 0.7 respectively?

Solution As $B(1) = 110$, $B(0) = 100$, we get $r = 10\%$, i.e. $R = 1 + r = 1.1$. Further $S(0) = 100$, $u\ S(0) = 120$ and $d\ S(0) = 80$ yield $u = 1.2$ and $d = 0.8$. Here the risk neutral probability $\hat{p}$ is $\frac{3}{4}$ as shown below.

$$\hat{p} = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.2-0.8} = \frac{3}{4}$$

i.e. $1 - \hat{p} = 1 - \frac{3}{4} = \frac{1}{4}$.

Also

$$C_u = \text{Max}(u\ S(0) - K,\ 0) = \text{Max}(120 - 100, 0) = 20$$
$$C_d = \text{Max}(d\ S(0) - K,\ 0) = \text{Max}(80 - 100, 0) = 0.$$

Therefore

$$\begin{aligned} C(0) &= \frac{1}{R}[\hat{p}\ C_u + (1-\hat{p})\ C_d] \\ &= \frac{1}{1.1}\left[\frac{3}{4}(20) + \frac{1}{4}(0)\right] \\ &= 13.63. \end{aligned}$$

We exhibit the above details in the form of following tables

t	0	1
$S(t)$		$\xrightarrow{p}$ 120
	100	
		$\xrightarrow{(1-p)}$ 80

t	0	1
$C(t)$		$\xrightarrow{\hat{p}=\frac{3}{4}}$ $C_u(= 20)$
	$C(0)$ (?)	
		$\xrightarrow{(1-\hat{p})=\frac{1}{4}}$ $C_d(= 0)$

This gives $C(0) = Rs\ 13.64$. As call price does not depend on p, it will not change if ~~0.6 and 0.4 are changed~~ to 0.7 and 0.3 respectively.

□

To find the price of a European put option we can follow exactly the same derivation as for the European call. We need to use pay-off of the put at expiry instead of the pay-off of the call to get the following pricing formula

$$P(0) = \frac{1}{R}[\hat{p}\ P_u + (1-\hat{p})\ P_d], \tag{3.11}$$

where $P_u = \text{Max}(K - u\ S(0),\ 0)$ and $P_d = \text{Max}(K - d\ S(0),\ 0)$.

Example 3.5.2 *For the data given in Example 3.5.1, find the price of the co-sponding European put option.*

Solution We have already obtained $\hat{p} = \frac{3}{4}$, $(1-\hat{p}) = \frac{1}{4}$ and $R = 1.1$. Furt $P_u = \text{Max}\,(K - u\ S(0),\ 0) = 0$ and $P_d = \text{Max}\,(K - d\ S(0),\ 0) = 20$. Therefore

$$\begin{aligned} P(0) &= \frac{1}{R}\,[\hat{p}\ P_u + (1-\hat{p})\ P_d] \\ &= \frac{1}{1.1}\left[\left(\frac{3}{4}\times 0\right) + \left(\frac{1}{4}\times 20\right)\right] \\ &= 4.54, \end{aligned}$$

and the corresponding table is

$$\begin{tikzcd} t & 0 & 1 \\ P(t) & & P_u(=0) \\ & P(0) \arrow[ur, "\hat{p}"] \arrow[dr, "(1-\hat{p})"'] & \\ & & P_d(=20) \end{tikzcd}$$

At this stage we can also verify the put-call parity. For the given data w have already obtained $C(0)$ =Rs 13.63 and $P(0)$ =Rs 4.54. Hence the value of th expression $C(0) - P(0) + d(0,1)K$ comes out to be $13.64 - 4.54 + (1.1)^{-1} \times 100 = 10($ which is same as the value $S(0)$. Here $d(0,1) = \frac{1}{R}$ is the discount factor betwee $t = 0$ and $t = 1$.

3.6 Multi Period Binomial Lattice Model for Option Pricing

We shall now extend the pricing formula (3.7) to multi period binomial lattice model. For this let us first take the two period case. Our notations are same as those in Section 3.5. Thus $S(0)$ is the price of the stock at $t = 0$ (initial time), u and d are up and down factors of the stock price movement, K is the strike price, $t = 2$ is expiry and $R = (1 + r)$. The following table illustrates the price movement of the stock along the binomial lattice

t	0	1	2
$S(t)$			$u^2S(0)$
		$u\,S(0)$	
	$S(0)$		$u\,d\,S(0)$
		$d\,S(0)$	
			$d^2S(0)$

Along the line of the single period case, the table for the call price should be

t	0	1	2
$C(t)$			C_{uu}
		C_u	
	$C(0)$ =?		C_{ud}
		C_d	
			C_{dd}

Here C_{uu}, C_{ud} and C_{dd} are pay-offs of the call at the expiry (i.e. at $t = 2$). As per our definition, it is obvious that

$$C_{uu} = \text{Max}(u^2S(0) - K,\ 0)$$
$$C_{ud} = \text{Max}(u\ d\ S(0) - K,\ 0)$$
$$C_{dd} = \text{Max}(d^2S(0) - K,\ 0).$$

But what are the values of C_u and C_d? For this let us re-look the binomial lattice for the price of the stock. We can visualize this bigger lattice as a combination of three binomial lattices of the following type

$$\begin{tikzcd}[row sep=small] & u\,S(0) \\ S(0) \arrow[ur] \arrow[dr] & \\ & d\,S(0) \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & u^2S(0) \\ u\,S(0) \arrow[ur] \arrow[dr] & \\ & u\,d\,S(0) \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & u\,d\,S(0) \\ d\,S(0) \arrow[ur] \arrow[dr] & \\ & d^2S(0) \end{tikzcd}$$

Let above lattices be referred as $L(1)$, $L(2)$ and $L(3)$ respectively. Treating lattice $L(3)$ as the single period binomial lattice and using formula (3.7), we obtain

$$C_d = \frac{1}{R}\,[\hat{p}\ C_{ud} + (1 - \hat{p})\ C_{dd}], \tag{3.12}$$

where as before $\hat{p}$ is the risk neutral probability given by $\hat{p} = \frac{R-d}{u-d}$. Similar using lattice $L(2)$ we obtain

$$C_u = \frac{1}{R}\left[\hat{p}\, C_{uu} + (1-\hat{p})\, C_{ud}\right]. \tag{3.13}$$

Finally using lattice $L(1)$ we obtain

$$C_0 = \frac{1}{R}\left[\hat{p}\, C_u + (1-\hat{p})\, C_d\right]. \tag{3.14}$$

Here the same risk neutral probability $\hat{p}$ has been used for all three lattices $L(1)$, $L(2)$ and $L(3)$. The reason being that for these single period lattices the up-tick probability p, the down-tick probability $1-p$ and the factors u and d do not change - they remain same for $t=0$ as well as for $t=1$. Now substituting for C_d and C_u from (3.12) and (3.13), equation (3.14) gives

$$C_0 = \frac{1}{R^2}\left[\hat{p}^2 C_{uu} + 2\hat{p}\,(1-\hat{p})\, C_{ud} + (1-\hat{p})^2 C_{dd}\right]. \tag{3.15}$$

Formula (3.15) can also be written as

$$C_0 = \frac{1}{R^2}\left[\sum_{j=0}^{2} \frac{2!}{j!(2-j)!}(\hat{p})^j (1-\hat{p})^{2-j}\left(u^j d^{2-j} S(0) - K\right)^+\right], \tag{3.16}$$

where $\left(u^j d^{2-j} S(0) - K\right)^+ = \text{Max}\left(u^j d^{2-j} S(0) - K,\ 0\right)$.

Remark 3.6.1 *Here, unlike the single period case, C_u and C_d are NOT the pay-offs - $Max(u\ S(0)-K,\ 0)$ and $Max(d\ S(0)-K,\ 0)$ at $t=1$, because the call is being exercised at $t=2$ only.*

In deriving formula (3.15), we have not made use of the concept of replicating portfolio directly. This has been possible because we have visualized the given 2-period binomial lattice as a combination of three 1-period binomial lattices which have already been studied. But it will be interesting to derive the same formula by employing the replicating portfolio strategy as well.

Though we are not giving the complete details here, we are giving enough hint to get the complete solution. In order to have replication at maturity $t=2$ we start replicating backwards, from the end of lattice. For this at $t=1$, we consider the upper node of the lattice L_2 and determine the scalars a_1 and b_1 such that

$$a_1\left(u^2S(0)\right) + b_1R = C_{uu},$$

nd

$$a_1\left(udS(0)\right) + b_1R = C_{ud}.$$

'his system can be solved to get

$$a_1 = \frac{C_{uu} - C_{ud}}{uS(0)(u-d)},$$

and

$$b_1 = \frac{C_{uu} - a_1(u^2S(0))}{R}.$$

Then $C_u = a_1\left(uS(0)\right) + b_1$ becomes the value of the replicating portfolio in the upper node at $t = 1$. We can similarly find the value of C_d. Then we need to replicate these two values in the first period. Thus we need to find a_0 and b_0 such that

$$a_0\left(uS(0)\right) + b_0 = C_u,$$

and

$$a_0\left(dS(0)\right) + b_0 = C_d.$$

The above system can be solved as before to get the values of a_0 and b_0. Now a little manipulation and the expression $C(0) = a_0S(0) + b_0R$ will give the formula (3.15).

Remark 3.6.2 *It is to be noted that in this model, the number of shares in the replicating portfolio is always equal to the ratio of ΔC and ΔS, where ΔC is the change in the future values of the call and ΔS is the change in the future values of the stock. This ratio is called the* delta *of the call, and it is different at different nodes of the lattice. Delta is one of the* Greeks *to be studied later for a derivative security defined over a given undertaking. Greeks have been used extensively in derivative pricing for hedging purposes.*

Example 3.6.1 *Find the price of a European call option with the given data as* $S(0) = 100$, $K = 100$, $u = 1.2$, $d = 0.8$, $r = 10\%$ *per year and time to expiry* $T = 2$ *years.*

Solution We have

$\frac{1}{(1.1)^2}$ $[(0.75)^2 (20) + 2(0.75)(0.25)(0) + (0.25)^2 (~~0.64~~)(64)(0)]$

$\frac{0.3}{0.4}$ $1~~20~~ -$ 0.96 $64 -$

96

$$
\begin{array}{c|l}
t & 0 \qquad\qquad 1 \qquad\qquad 2 \\ \hline
S(t) &
\begin{tikzcd}
 & & 144 \\
 & 120 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur, "1.2"] \arrow[dr, "0.8"'] & & 96 \\
 & 80 \arrow[ur] \arrow[dr] & \\
 & & 64
\end{tikzcd}
\end{array}
\qquad
\begin{array}{c|l}
t & 0 \qquad\qquad 1 \qquad\qquad 2 \\ \hline
C(t) &
\begin{tikzcd}
 & & C_{uu}\ 44 \\
 & C_u \arrow[ur, "3/4"] \arrow[dr, "1/4"] & \\
C(0) = ? \arrow[ur] \arrow[dr] & & C_{ud}\ 0 \\
 & C_d \arrow[ur, "3/4"] \arrow[dr, "1/4"] & \\
 & & C_{dd}\ 0
\end{tikzcd}
\end{array}
$$

From the given data $R = 1 + r = 1.1$ and $\hat{p} = \dfrac{R-d}{u-d} = \dfrac{1.1-0.8}{1.2-0.8} = \dfrac{3}{4}$. Also

$$
\begin{aligned}
C_{uu} &= \mathrm{Max}(u^2S(0) - K,\ 0) = \mathrm{Max}(144 - 100, 0) = 44 \\
C_{ud} &= \mathrm{Max}(u\, d\, S(0) - K,\ 0) = \mathrm{Max}(96 - 100, 0) = 0 \\
C_{dd} &= \mathrm{Max}(d^2S(0) - K,\ 0) = \mathrm{Max}(64 - 100, 0) = 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
C_u &= \frac{1}{R}[\hat{p}\ C_{uu} + (1-\hat{p})\ C_{ud}] = \frac{1}{1.1}\left[\left(\frac{3}{4}\times 44\right) + \left(\frac{1}{4}\times 0\right)\right] = 30 \\
C_d &= \frac{1}{1.1}\left[\left(\frac{3}{4}\times 0\right) + \left(\frac{1}{4}\times 0\right)\right] = 0,
\end{aligned}
$$

which gives

$$
\begin{aligned}
C(0) &= \frac{1}{R}[\hat{p}\ C_u + (1-\hat{p})\ C_d] \\
&= \frac{1}{1.1}\left[\left(\frac{3}{4}\times 30\right) + \left(\frac{1}{4}\times 0\right)\right] \\
&= 20.45.
\end{aligned}
$$

Alternatively, we can use the formula (3.15) to get

$$
\begin{aligned}
C(0) &= \frac{1}{R^2}\left[\hat{p}^2C_{uu} + 2\hat{p}\ (1-\hat{p})\ C_{ud} + (1-\hat{p})^2C_{dd}\right] \\
&= \frac{1}{1.1^2}\left[\left(\frac{9}{64}\times 44\right)\right] \\
&= 20.45.
\end{aligned}
$$

Example 3.6.2 *For the data given in Example 3.6.1, find the price of the corresponding European put option.*

Solution We have already obtained $\hat{p} = \frac{3}{4}$, $(1 - \hat{p}) = \frac{1}{4}$ and $R = 1.1$. Further

$$P_{uu} = \text{Max}(K - u^2 S(0),\ 0) = \text{Max}(-44, 0) = 0$$
$$P_{ud} = \text{Max}(K - u\ d\ S(0),\ 0) = \text{Max}(4, 0) = 4$$
$$P_{dd} = \text{Max}(K - d^2 S(0),\ 0) = \text{Max}(36, 0) = 36.$$

Therefore

$$P_u = \frac{1}{1.1}\left[\left(\frac{3}{4} \times 0\right) + \left(\frac{1}{4} \times 4\right)\right] = \frac{1}{1.1}$$
$$P_d = \frac{1}{1.1}\left[\left(\frac{3}{4} \times 4\right) + \left(\frac{1}{4} \times 36\right)\right] = \frac{12}{1.1}.$$

This gives

$$\begin{aligned} P(0) &= \frac{1}{R}\left[\hat{p}\ P_u + (1 - \hat{p})\ P_d\right] \\ &= \frac{1}{1.1}\left[\left(\frac{3}{4} \times \frac{1}{1.1}\right) + \left(\frac{1}{4} \times \frac{12}{1.1}\right)\right] \\ &= 3.10. \end{aligned}$$

Alternatively, we can get

$$\begin{aligned} P(0) &= \frac{1}{R^2}\left[\hat{p}^2 P_{uu} + 2\hat{p}\ (1 - \hat{p})\ P_{ud} + (1 - \hat{p})^2 P_{dd}\right] \\ &= \frac{1}{1.1^2}\left[\left(\left(\frac{3}{4}\right)^2 \times 0\right) + \left(2 \times \frac{3}{4} \times \frac{1}{4} \times 4\right) + \left(\left(\frac{1}{4}\right)^2 \times 36\right)\right] \\ &= 3.10. \end{aligned}$$

We can also use put-call parity to find $P(0)$, once $C(0)$ is known. Specifically

$$C(0) - P(0) + d(0,2)K = S(0),$$

gives

$$P(0) = C(0) + d(0,2)K - S(0) = 20.45 + d(0,2)100 - 100.$$

We can take $d(0,2) = (1 + 2r)^{-1}$ or e^{-2r} depending upon the nature of the compounding of the interest rate r; but it has to be same for the entire calculations. □

The development of the pricing methodology for the case of multi period b mial lattice model is similar to the one discussed for the two period lattice m The single period risk free discounting is carried out at every node of the la as has been done for the two period case. Obviously we need to start from final period ($t = N$) and then proceed backward till we reach the initial period ($t = 0$). For the case of European call option, this process will result ir following formula

$$C(0) = \frac{1}{R^N}\left[\sum_{j=0}^{N} \frac{N!}{j!(N-j)!}(\hat{p})^j(1-\hat{p})^{N-j}\left(u^j d^{N-j} S(0) - K\right)^+\right], \qquad (3$$

which reduces to (3.16) for $N = 2$.

3.7 Existence of Risk Neutral Probability Measure

We have already seen in Section 3.5 that for the single period binomial pri model, risk neutral probability measure (RNPM) plays the most fundame role. We have also exhibited the RNPM $\hat{p}$ and used the same for deriving option pricing formula (3.7). Here it may be noted that in this scenario the s space Ω consists of only two points say $\Omega = \{\omega_1, \omega_2\}$ where ω_1 is the up-movement of the stock price and ω_2 is the down-tick movement of the stock pı But realistically Ω may consist of m points $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ or Ω may an interval $[a, b]$. Can we still guarantee the existence of RNPM? To answer question is important if we wish to discuss derivative pricing for more gen scenarios.

Let us consider a more general model where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, ω_j be the j^{th} state of economy. Thus Ω represents the finite set of possible values; these could be m possible values of the stock price $S(1)$ (at $t = 1$), namely $S_1(\omega$ $S_1(\omega_2)$, ..., $S_1(\omega_m)$. For the single period binomial lattice, $\Omega = \{\omega_1, \omega_2\}$ w $S_1(\omega_1) = uS(0)$ and $S_1(\omega_2) = dS(0)$.

Also we could have more than two securities, say $S^{(k)}$ $(k = 0, 1, 2, \ldots, n)$. L $S_1^{(k)}(\omega_j)$, for $(k = 0, 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$ denote the price of the security at $t = 1$ when the state of the economy is ω_j. We may think of k securiti as bond, stock, option, forward contracts, etc having different defining variabl but over the same underlying.

Let $S_0^{(k)}$ $(k = 0, 1, 2, \ldots, n)$ denote the current $(t = 0)$ price of the k^{th} securit Here we may note that $S_0^{(k)}$ is deterministic, but $S_1^{(k)}$ is a random variable tak ing values $\left\{S_1^{(k)}(\omega_j),\ (j = 1, 2, \ldots, m)\right\}$. Since all derivative securities are defined o the same underlying (stock), they all will be random variables taking m possible values.

As a convention, we take $k = 0$ for bond, $k = 1$ for the underlying (stock) and ($k = 2, \ldots, n$) for other (derivative) securities. Let r be the risk free interest rate for the period $t = 0$ to $t = 1$. It is convenient to assume that $S_0^{(0)} = 1$ and $S_1^{(0)}(\omega_j) = R = (1 + r)$, $(j = 1, 2, \ldots, m)$.

Definition 3.7.1 (Risk Neutral Probability Measure) *A risk neutral probability measure (RNPM) is a vector $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)^T$ such that*

(i) $\hat{p}_j > 0$ $(j = 1, 2, \ldots, m)$,

(ii) $\sum_{j=1}^{m} \hat{p}_j = 1$,

and for every security k $(k = 0, 1, 2, \ldots, n)$, we have

$$S_0^{(k)} = \frac{1}{R}\left(\sum_{j=1}^{m} \hat{p}_j S_1^{(k)}(\omega_j)\right). \tag{3.18}$$

If we denote by $E_{\hat{p}}\left(S_1^{(k)}\right)$ the expected value of $S_1^{(k)}$ with respect to RNPM $\hat{p}$, then (3.18) can be written as

$$S_0^{(k)} = \frac{1}{R} E_{\hat{p}}\left(S_1^{(k)}\right). \tag{3.19}$$

Here $S_1^{(k)}$ denotes the value of the k^{th} security at $t = 1$ and $S_0^{(k)}$ is the price of the same security at $t = 0$.

Regarding the existence and uniqueness of RNPM we have the following two main theorems

Theorem 3.7.1 (First Fundamental Theorem of Asset Pricing) *A risk neutral probability measure $\hat{p}$ exists if and only if no arbitrage principle holds.*

Theorem 3.7.2 (Second Fundamental Theorem of Asset Pricing) *The RNPM is unique if and only if the market is complete.*

Thus for an arbitrage free market, there is unique RNPM $\hat{p}$ if and only if the market is *complete*. We have not yet discussed the meaning of *market completeness* but that we postpone for the time being.

We shall now give a linear programming based proof of Theorem 3.7.1. Let us recall the following primal-dual pair of linear programming problem

$$\text{(LP)} \quad \text{Min} \quad c^T x \quad \text{subject to} \quad Ax \geq b, \quad x \geq 0,$$

and

$$\text{(LD)} \quad \text{Max} \quad b^T y \quad \text{subject to} \quad A^T y \leq c, \quad y \geq 0.$$

It is well known in duality theory that if (LP) and (LD) both are feasible then b[oth] have optimal solutions. The below given theorem, called *strict complementa[rity] theorem*, gives some additional information as well.

Theorem 3.7.3 (Goldman-Tucker Theorem) *Let (LP) and (LD) both be [fea]sible. Then they both have optimal solutions x^* (for(LP)) and y^* (for(LD)) sa[tis]fying*

$$x^* + (c - A^T y^*) > 0. \quad (3.[20])$$

The condition (3.20) is called the strict complementarity condition. Here it m[ay] be noted that Theorem 3.7.3 does not tell that (3.20) holds for every pair ($\bar{x}$ [, $\bar{y}$)] of optimal solution of (LP) and (LD). But rather it guarantees the existence o[f a] pair (x^*, y^*) of optimal solution of (LP) and (LD) for which (3.20) holds. We [may] refer to Goldman and Tucker [52] for the proof of Theorem 3.7.3.

We now proceed to prove Theorem 3.7.1. For this let us consider a portfo[lio] $P : (x_0, x_1, \ldots, x_n)$. Then the value of this portfolio at $t = 0$ is

$$V_P(0) = \sum_{k=0}^{n} x_k \, S_0^{(k)}.$$

Further its value at $t = 1$ will be one of the m values, namely $V_{P,1}(\omega_j)$ [=] $\sum_{k=0}^{n} x_k \, S_1^{(k)}(\omega_j)$, $(j = 1, 2, \ldots, m)$, depending upon the state of economy ω_j whe[re] $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$. We now consider the following linear programming proble[m] (LPP)

$$\text{Min} \quad V_P(0)$$

subject to

$$V_{P,1}(\omega_j) \geq 0 \quad (j = 1, 2, \ldots, m),$$

i.e.

$$\text{Min} \quad \sum_{k=0}^{n} x_k \, S_0^{(k)}$$

subject to

$$\sum_{k=0}^{n} x_k \, S_1^{(k)}(\omega_j) \geq 0 \qquad (j = 1,2,\ldots,m). \tag{3.21}$$

The dual of problem (3.21) is

$$\text{Max} \quad \sum_{j=1}^{m} 0.p_j$$

subject to

$$\sum_{j=1}^{m} S_1^{(k)}(\omega_j)\, p_j = S_0^{(k)} \qquad (k = 0,1,2,\ldots,n)$$
$$p_j \geq 0 \qquad (j = 1,2,\ldots,m). \tag{3.22}$$

Next we note that LPP (3.21) is feasible because $\{x_k = 0,\ (k = 0,1,2,\ldots,n)\}$ satisfies all its constraints. Also this is optimal because by definition to meet the assumption of solvency, only admissible portfolios are to be considered, i.e. $V_P(0) \geq 0$, i.e. $\sum_{k=0}^{n} x_k \, S_0^{(k)} \geq 0$. But (3.21) is a minimization problem and therefore $\{x_k = 0,\ (k = 0,1,2,\ldots,n)\}$ is optimal to (3.21) and the optimal value is zero. Therefore its dual LPP (3.22) also has an optimal solution with optimal value as zero.

We now write the primal-dual pair (3.21) - (3.22) in the form of the pair (LP)-(LD) and then apply the Goldman-Tucker theorem (Theorem 3.7.3) to this pair. This implies that there exists $\overline{x}$ optimal to (3.21) and $\overline{p}$ optimal to (3.22) such that

$$-\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\bar{x}_k \leq 0 \qquad (j = 1,2,\ldots,m),$$

$$\sum_{j=1}^{m} S_1^{(k)}(\omega_j)\,\bar{p}_j = S_0^{(k)} \qquad (k = 0,1,2,\ldots,n),$$

$$\bar{p}_j \geq 0 \qquad (j = 1,2,\ldots,m),$$

$$\sum_{j=1}^{m} 0.\bar{p}_j = \sum_{k=0}^{n} S_0^{(k)}\,\bar{x}_k = 0,$$

and

$$\bar{p}_j + \left(0 - \left(-\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\,\bar{x}_k\right)\right) > 0 \qquad (j = 1,2,\ldots,m). \qquad (3.$$

But from (3.26), $V_{\bar{p}}(0) = 0$ for the portfolio $(\bar{x}_k,\ (k = 0,1,2,\ldots,n))$. Hence by *arbitrage principle*, $V_{\bar{p}}(1)$ should be zero with probability 1, i.e. $\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\,\bar{x}$ $0\ (j = 1,2,\ldots,m)$. Therefore (3.27) gives $\bar{p}_j > 0\ (j = 1,2,\ldots,m)$.

Now we recall that $k = 0$ refers to the bond. Taking $S_0^{(0)} = 1$ we get $S_1^{(0)}(\omega_j$ $1 + r = R\ (j = 1,2,\ldots,m)$. Then (3.24) gives

$$\sum_{j=1}^{m} R\,\bar{p}_j = 1,$$

i.e.

$$\sum_{j=1}^{m} \bar{p}_j = \frac{1}{R}. \qquad (3.2$$

If we now define $\hat{p}$ such that $\hat{p}_j = R\,\bar{p}_j\ (j = 1,2,\ldots,m)$, then (3.24), (3.25) an (3.28) give $\hat{p}_j > 0\ (j = 1,2,\ldots,m)$, $\sum_{j=1}^{m} \hat{p}_j = 1$ and

$$\frac{1}{R}\left(\sum_{j=1}^{m} S_1^{(k)}(\omega_j)\,\hat{p}_j\right) = S_0^{(k)} \qquad (k = 0,1,2,\ldots,n). \qquad (3.29)$$

But then this shows that $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_j, \ldots, \hat{p}_m)$ is a risk neutral probability measure.

We can also write (3.29) in a more familiar notation as

$$S_0^{(k)} = \frac{1}{R} E_{\hat{p}}\left(S_1^{(k)}\right) \qquad (k = 0, 1, 2, \ldots, n), \tag{3.30}$$

where $E_{\hat{p}}$ denotes the expectation under RNPM $\hat{p}$. This proves that if no arbitrage principle holds then RNPM exists. The converse can be proved on similar lines.

□

Remark 3.7.1 *The formula (3.30) is very general. Apart from the fact that it holds for the stock and bond, it should hold for other securities ($k = 2, 3, \ldots, n$) as well, be it European call, put, forward contract etc. What we simply need to do is to find the expectation of the pay-off of the given security at expiry under RNPM and discount the same for $t = 0$.*

Remark 3.7.2 *For the case of single period binomial lattice model the linear programming problem to find RNPM is*

$$\begin{aligned} \textit{Max} \quad & 0\,\hat{p}_1 + 0\,\hat{p}_2 \\ \textit{subject to} \quad & \\ & (u\,S(0))\,\hat{p}_1 + (d\,S(0))\,\hat{p}_2 = RS(0) \\ & \hat{p}_1 + \hat{p}_2 = 1 \\ & \hat{p}_1 \geq 0, \hat{p}_2 \geq 0, \end{aligned}$$

such that at optimality $\hat{p}_1 > 0$, $\hat{p}_2 > 0$. It is not difficult to see that the unique optimal solution is $\left(\hat{p}_1 = \dfrac{R-d}{u-d}, \ \hat{p}_2 = \dfrac{R-u}{u-d}\right)$, *which is the same as obtained earlier by replicating portfolio arguments.*

Also if $k = 2$ and $k = 3$ respectively refer to European call and European put options, then formula (3.30) gives

$$C(0) = \frac{1}{R}\,[\hat{p}_1\,C_u + \hat{p}_2\,C_d],$$

and

$$P(0) = \frac{1}{R}\,[\hat{p}_1\,P_u + \hat{p}_2\,P_d].$$

Example 3.7.1 *Consider a forward contract with the given data as $S(0) = 100$, $u = 1.2$, $d = 0.8$, $T = 1$ year and $r = 10\%$ per year. Determine the forward price F.*

Solution We know that $F = RS(0) = (1.1) \times 100 = Rs\ 110$. But here we ha[ve to]
determine F by utilizing the formula (3.30). For this we note that the pay-[off of]
the forward contract is $(S - F)$ if $S > F$ and $-(F - S)$ if $S \leq F$. Also from the g[iven]
data ($\hat{p}_1 = \frac{3}{4}$, $\hat{p}_2 = \frac{1}{4}$). Therefore formula (3.30) gives

$$F(0) = \frac{1}{1.1}\left[\frac{3}{4}(120 - F) + \frac{1}{4}(80 - F)\right]. \qquad (3$$

But in the case of forward contract $F(0) = 0$, and then (3.31) gives $F = 110$.

Let us again look at formula (3.29) and concentrate for the case $k = 1$. We re[call]
that $k = 1$ refers to the stock and therefore if we write $S_0^1 = S(0)$ and $S_1^1 = S$
then

$$\begin{aligned} S(0) &= \frac{1}{R}E_{\hat{p}}(S(1)) \\ &= E_{\hat{p}}\left(\frac{S(1)}{R}\right) \\ &= E_{\hat{p}}\left(\frac{S(1)}{B(1)}\right) \\ &= E_{\hat{p}}(\widetilde{S}(1)). \end{aligned} \qquad (3.3$$

Here we have taken $B(0) = 1$, $\widetilde{S}(1) = (S(1)/B(1))$ and written $E_{\hat{p}}$ to emphasize th[e]
expectation has been taken with respect to RNPM $\hat{p}$. Therefore (3.32) gives

$$\widetilde{S}(0) = E_{\hat{p}}(\widetilde{S}(1)) \qquad (3.3$$

The expectation in (3.32) is in fact conditional because it is computed once th[e]
stock price $S(0)$ becomes known at $t = 0$. Therefore (3.33) is actually

$$\widetilde{S}(0) = E_{\hat{p}}\left(\widetilde{S}(1))/\widetilde{S}(0)\right),$$

which, in general, is expressed as

$$\widetilde{S}(l) = E_{\hat{p}}\left((\widetilde{S}(l+1))/\widetilde{S}(l)\right). \qquad (3.34)$$

The relationship (3.34) is expressed as: *the discounted stock prices $\widetilde{S}(0), \widetilde{S}(1), \widetilde{S}(2), \ldots$ form a martingale with respect to RNPM $\hat{p}$.* This holds for other securities as well.
i.e.

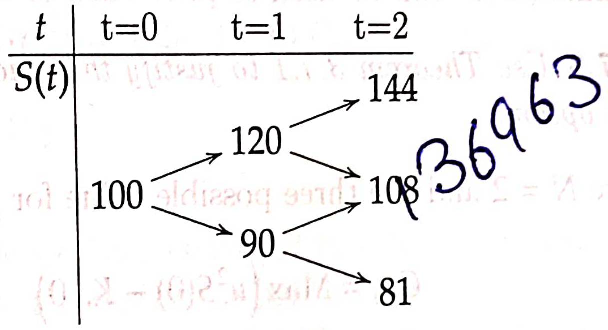
$$\widetilde{S}^{(k)}(l) = E_{\hat{p}}\left(\widetilde{S}^{(k)}(l+1)/\widetilde{S}^{(k)}(l)\right) \quad (k = 0,1,2,\ldots,n).$$

Therefore the problem of asset pricing gets translated into the problem of finding a unique RNPM $\hat{p}$ or to be precise $\hat{p}$-*martingale* on the set of scenarios Ω. Moreover $E_{\hat{p}}$ is called the *risk neutral* or *martingle expectation* with respect to $\hat{p}$. These concepts will probably get much clearer once we are familiar with basics of stochastic process and stochastic calculus.

Since there are no goods in the model other than money, it is convenient to pick one of the security as reference and normalize others with respect to it. The security so chosen for normalization purpose is called *numeraire*. In our context, the usual choice of numeraire is the bond price and that is what exactly we have done in defining $\widetilde{S}(0) = S(0)/B(0)$, $\widetilde{S}(1) = S(1)/B(1)$ etc.

Example 3.7.2 *Consider the data:* $S(0) = 100$, $u = 1.2$, $d = 0.9$, $r = 10\%$ *per year and* $T = 2$ *years. Determine RNPM* $\hat{p}$ *and show that discounted stock prices form a* $\hat{p}$*-martingale.*

Solution We have $\hat{p} = \frac{2}{3}$ and $(1-\hat{p}) = \frac{1}{3}$. We have the dynamics of the stock price as



We can check the following

$$\frac{1}{1.1}\left(\frac{2}{3}(144) + \frac{1}{3}(108)\right) = 120$$

$$\frac{1}{1.1}\left(\frac{2}{3}(108) + \frac{1}{3}(81)\right) = 90$$

$$\frac{1}{1.1}\left(\frac{2}{3}(120) + \frac{1}{3}(90)\right) = 100.$$

These give

$$E_{\hat{p}}\left[\left(\frac{S(2)}{(1.1)^2}\right)\Big/\left(\frac{S(1)}{1.1}\right) = \frac{120}{1.1}\right] = \frac{120}{1.1}$$

$$E_{\hat{p}}\left[\left(\frac{S(2)}{(1.1)^2}\right)\Big/\left(\frac{S(1)}{1.1}\right) = \frac{90}{1.1}\right] = \frac{90}{1.1}$$

etc. Thus

$$E_{\hat{p}}\left(\tilde{S}(2)/\tilde{S}(1)\right) = \tilde{S}(1).$$

The below given theorem is very general and is valid for any European deri
security.

Theorem 3.7.4 *Let D be a European derivative security whose pay-off i N-period binomial model is $f(S(N))$. Then*

$$D(0) = \frac{1}{R^N} E_{\hat{p}}\left(f\left(S(N)\right)\right).$$

The above theorem essentially tells that the price of a European deri security D with pay-off $f(S(N))$ in the N-period binomial model is the expec of the discounted pay-off under the risk neutral probably measure.

We know that for a European call, $f(S(N)) = (S(N)-K)^+$; for a Europea $f(S(N)) = (K-S(N))^+$ and for a forward contract $f(S(N)) = (S(N)-F)$. The the same formula (3.35) can be used to price each of these derivative securi

Example 3.7.3 *Use Theorem 3.7.1 to justify the pricing formula (3.15) f European call option.*

Solution Here $N = 2$ and the three possible value for $f(S(2))$ are

$$C_{uu} = \text{Max}\left(u^2 S(0) - K,\ 0\right)$$
$$C_{ud} = \text{Max}\left(u\,d\,S(0) - K,\ 0\right)$$
$$C_{dd} = \text{Max}\left(d^2 S(0) - K,\ 0\right).$$

Also $\hat{p} = \dfrac{(R-d)}{(u-d)}$. Therefore if we define $\hat{p}_1 = (\hat{p})^2$, $\hat{p}_2 = 2\hat{p}(1-\hat{p})$ and $\hat{p}_3 = (1$
then

(i) $\hat{p}_i > 0 \ (i = 1,2,3)$

(ii) $\sum_{i=1}^{3} \hat{p}_i = 1$ and

(iii) $E_{\hat{p}}(S(2)) = R^2 S(0)$.

The third assertion can be verified as follows

$$
\begin{aligned}
E_{\hat{p}}(S(2)) &= \hat{p}_1\left(u^2 S(0)\right) + \hat{p}_2(u\, d\, S(0)) + \hat{p}_3\left(d^2 S(0)\right) \\
&= S(0)\left[u^2(\hat{p})^2 + 2\hat{p}\,(1-\hat{p})\, u\, d\, S(0) + (1-\hat{p})^2 d^2 S(0)\right] \\
&= S(0)\,[u\,\hat{p} + (1-\hat{p})\, d]^2 \\
&= S(0)\left[\frac{u(R-d) + d(u-R)}{(u-d)}\right]^2 \\
&= S(0)\left[\frac{R(u-d)}{u-d}\right]^2 = R^2 S(0),
\end{aligned}
$$

i.e.

$$
S(0) = \frac{1}{R^2}\left(E_{\hat{p}}(S(2))\right).
$$

Therefore $\hat{p}_i$ $(i = 1,2,3)$ as defined above is in fact RNPM. Hence

$$
\begin{aligned}
C(0) &= \frac{1}{R^2}\left(E_{\hat{p}}(f(S(2)))\right) \\
&= \frac{1}{R^2}\left[(\hat{p})^2 C_{uu} + 2\hat{p}\,(1-\hat{p})\, C_{ud} + (1-\hat{p})^2 C_{dd}\right],
\end{aligned}
$$

as obtained by formula (3.15).

□

Remark 3.7.3 *Apparently we have employed two distinct approaches to price a given derivative. These are the replicating portfolio approach and the RNPM approach. Various illustrative examples presented above and also the derivation of single period binomial lattice model suggest that these two approaches are related. In fact for a complete market, the two approaches are equivalent. This is because the replicating portfolio approach uses the law of one price which is essentially a consequence of no arbitrage principle. This is because then the unique RNPM exists which can be used to price any contingent claim. For the existence of RNPM, the market has to be arbitrage free. Therefore if no arbitrage principle does not hold then we cannot price the given derivative even if the corresponding unique replicating portfolio exists. The below given example illustrates the point. We shall further discuss this aspect in Section 3.10.*

Example 3.7.4 *Let* $B(0) = 100, B(1) = 120,$ $S(0) = 100$ *and* $S(1)$ *take values* 120 *and* 80 *with probabilities* 0.8 *and* 0.2 *respectively. Let* C *be a European call with* $K = 100$ *and* $T = 1$ *year. Find the replicating portfolio* (x, y) *for the call* C. *Are we justifying in taking* $C(0) = x\, S(0) + y\, B(0)$? *Determine RNPM if it exists.*

Solution The pay-off of the call C is

$$C(1) = \begin{cases} 20, & \text{with probability } 0.8 \\ 0, & \text{with probability } 0.2. \end{cases}$$

Therefore if (x, y) is the replicating portfolio, then we have $x\,S(1) + y\,B(1) = C($
This gives the following two equations

$$120x + 120y = 20$$

$$80x + 120y = 0,$$

having solution as $(x = 1/2, y = -1/3)$. Here we cannot take $C(0) = 1/2S(0)$ $1/3B(0)$ because this utilizes the law of one price which is valid only under arbitrage principle. But as $u = 1.2$ and $R = 1 + r = 1.2$ we do not have t required condition $u > R > d$ for no arbitrage principle to hold.

To determine the RNPM we need to solve the system

$$\hat{p}_1 + \hat{p}_2 = 1$$

$$\frac{1}{1.2}(120\hat{p}_1 + 80\hat{p}_2) = 100$$

$$\hat{p}_1 > 0, \; \hat{p}_2 > 0.$$

□

The above system does not have a solution because the first two equations giv $\hat{p}_1 = 1$, $\hat{p}_2 = 0$. Therefore the RNPM does not exist. This is again because u $1.2 = R$ which violates the condition $u > R > d$ and therefore no arbitrage princip does not hold.