Financial Engineering

No Arbitrage Principle :

No risk free profits are allowed without initial investment.

There is no admissible portfolio with initial value $V(0)=0$, such that $V(1) > 0$ with non-zero probability.

Other assumptions :

Randomness (R), Positivity of Price (PP), Divisibility (D) Liquidity (L)

Shortselling (S), solvency (S), Discrete unit price (D)

↓

(R P P D S D)

Risk and Return :

The price of a stock may acquire different values during its duration, with different non-zero probability. The return for final price $S(1)_i$ will be

$$K_{S_i} = \frac{S(1)_i - S(0)}{S(0)}$$

and expected return will be $E(K_s) = \sum p_i K_{S_i}$

Then, we calculate the standard deviation using variance as:

$$\sigma^2 = \sum [K_{S_i} - E(K_s)]^2 p_i$$

and find the σ (standard deviation).

> If the standard deviation is greater than the expected return, the investment is of high risk and we don't invest. otherwise, we can invest in the stock.

zero coupon Bonds :

Risk free asset in which the investor is guaranteed payment of money by the bond issuing body upon some initial investment. The fixed payment by issuing body is called Face value (F) and the time at which this payment is made (i.e. the bond matures) is called the maturity date (T).

In simple terms the investor is lending some money to the issuing body for a period T and interest rate (γ).

If compounded anually,

$$\underbrace{B(t,T)}_{\substack{\text{value of bond at time}\\ \text{'t' prior to maturity date}\\ \text{'T'.}}} = \underbrace{B(T)}_{\substack{\text{Face Value /}\\ B(T,T)/\\ \text{Maturity value}}} \cdot (1+\underbrace{\gamma}_{\text{interest rate.}})^{-(T-t)}$$

Incase of continuous compounding,

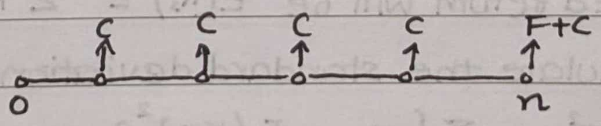
$$B(t,T) = B(T) \cdot e^{-(T-t)\gamma}$$

$B(0,T) = e^{-\gamma T}$ → discount factor (with face value = 1)

and $(B(0,T))^{-1} = e^{\gamma T}$ → growth factor (u)

Coupon Bonds :

Bonds promising a sequence of payments. This payment consits of face value (F) at maturity and coupons paid regularly (annually, half-yearly, quarterly, etc). Last coupon is paid at maturity.



$$V(0) = Ce^{-\gamma} + Ce^{-2\gamma} + Ce^{-3\gamma} + \cdots (F+C)e^{-n\gamma}$$

('γ' compounded continuously, with C = coupon value).

similarly,

$$V(1) = Ce^{-\gamma} + Ce^{-2\gamma} + \cdots (F+C)e^{-(n-1)\gamma}$$

$$\therefore \underbrace{V(1)+C}_{\text{Total wealth at } t=1.} = V(0)e^{\gamma}$$

One-step Binomial Model :

$A(0)$, $A(1)$, $S(0)$ have their usual meanings;

suppose the possible 'up' and 'down' prices of stock at $t=1$ are:

$$S(1) = \begin{cases} S^u, & \text{with prob. } p \\ S^d, & \text{with prob. } 1-p \end{cases}$$

where $S^d < S^u$ and $0 < p < 1$.

(Note: S^d maybe greater than $S(0)$; the term 'down' refers to the price being lower relative to the other ('up'/'higher') price of the stock.)

Proposition : If $S(0) = A(0)$, then $S^d < A(1) < S^u$, else an arbitrage opportunity can arise.

We prove that $A(1) > S^d$ by contradiction.

Suppose $A(1) \leq S^d$. Let $A(0) = S(0) = 100\$$.

At $t=0 \Rightarrow$ Borrow a bond (risk free) for $100\$$.

Invest those $100\$$ to buy a stock at $S(0) = 100\$$.

So, we have a portfolio with $x=1$ and $y=-1$ (-1 because we are the issuing body and we took a loan for $100\$$ by giving out a bond)

So, $V(0) = 0$ (No initial investment from our pocket).

At $t=1 \Rightarrow$

$$V(1) = \begin{cases} S^u - A(1), & \text{stock} \uparrow \\ S^d - A(1), & \text{stock} \downarrow \end{cases}$$

If $S^d \geq A(1)$, then both the possible values of $V(1)$ are positive (non-negative); ie. $V(1) > 0$, with $0 < p < 1$. This violates the No Arbitrage principle. Hence our assumption that $S^d \geq A(1)$ was wrong. Hence $A(1) > S^d$.

Similarly, we prove that $S^u > A(1)$ by contradiction.

Suppose $A(1) \geq S^u$. Let $A(0) = S(0) = 100\$$

At $t=0 \Rightarrow$ Sell one stock for $100\$$ $(x=-1)$

Use the $100\$$ to buy a bond (risk-free) $(y=1)$.

So, $V(0) = 0$

At $t=1 \Rightarrow$

$$V(1) = \begin{cases} A(1) - S^u, & \text{stock} \uparrow \\ A(1) - S^d, & \text{stock} \downarrow \end{cases}$$

Both the values of $v(1)$ will be non-negative (second one being strictly positive) if $A(1) \geq S^u$, giving rise to Arbitrage opportunity. Hence, this violates No Arbitrage Principle. So, our assumption was wrong. Hence $\underline{S^u > A(1)}$.

Forward Contracts :

Agreement to buy/sell a risky asset at a specified future time, known as delivery date (T), for a fixed price (F) at the present moment, called the forward price.

- Investor who buys the asset and pays the money → long forward posn
- Investor who sells the asset and delivers the good → short forward posn
- No money is paid initially, at the time of the exchange of contract.

The principal reason for entering into a forward contract is to become independent of the unknown future price of a risky asset.

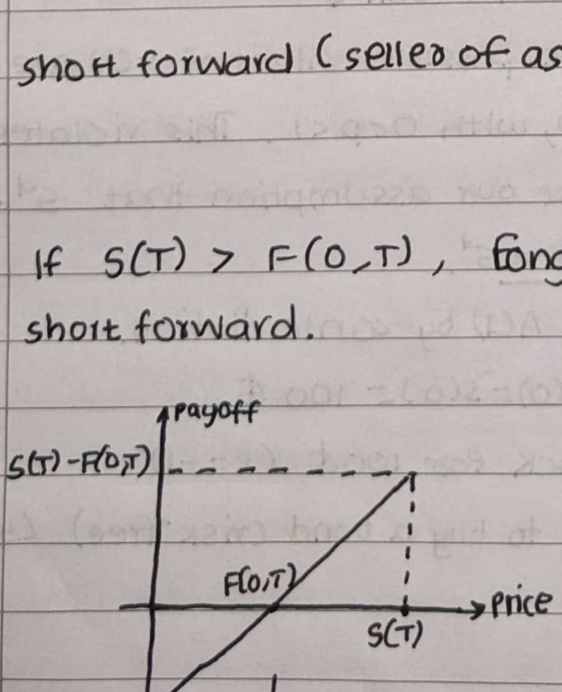
Suppose forward contract is exchanged at time '0' for a fixed price of $F(0,T)$ at a delivery date/time 'T.' The time 't' market price of the asset will be denoted by $S(t)$.

Now, at $t=T$, the payoffs for both parties will be :

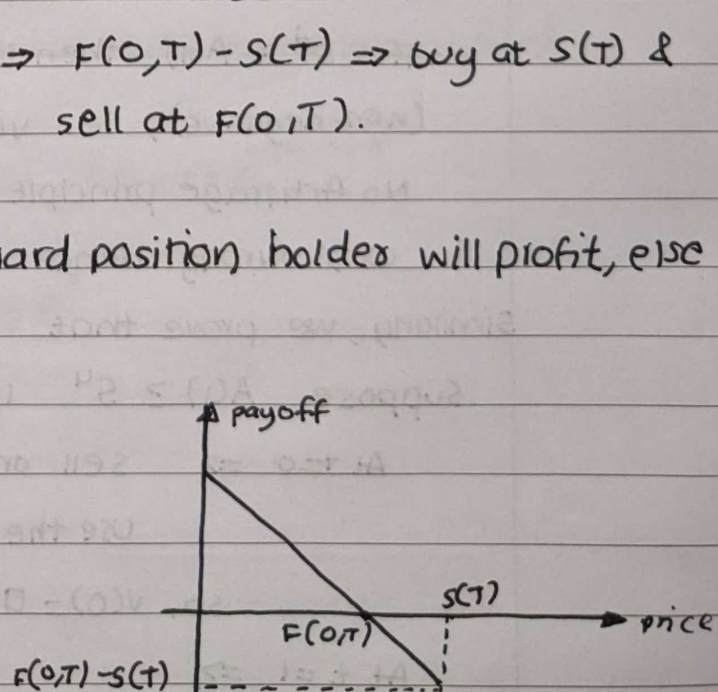
Long forward (buyer of good/asset) => $S(T) - F(0,T)$ => buy at $F(0,T)$ & sell at $S(t=T)$.

Short forward (seller of asset) => $F(0,T) - S(T)$ => buy at $S(T)$ & sell at $F(0,T)$.

If $S(T) > F(0,T)$, long forward position holder will profit, else short forward.



long forward



short forward

Forward Pricing (For stocks paying No Dividends):

For a stock paying no dividends, the forward price $F(0,T)$ is

$$\boxed{F(0,T) = S(0).e^{rT}}$$

where r = constant risk free interest rate under continuous compounding

If contract is initiated at $t=t \leq T$, then,

$$\boxed{F(t,T) = S(t).e^{r(T-t)}}$$

Proof: we shall prove that $F(0,T) = S(0).e^{rT}$ using contradiction.

Suppose $F(0,T) > S(0).e^{rT}$ — (1)

At $t=0$, we can design the portfolio as : $(1,-1,1)$.

- Borrow the amount $S(0)$ for time T.
- Buy one unit of stock at $S(0)$ price.
- enter a forward contract in short forward pos^n and sell one share for $F(0,T)$ at $t=T$.

Then at $t=T$,

- sell the stock for $F(0,T)$ using the contract.
- pay $S(0).e^{rT}$ to clear the loan with interest.

This shall bring a risk-free profit of $F(0,T) - S(0).e^{rT} > 0$ w/o any initial investment. Hence $F(0,T) \not> S(0).e^{rT}$. — (2)

Suppose $F(0,T) < S(0).e^{rT}$ — (3)

At $t=0$, we can design the portfolio as : $(-1,1,1)$.

- short sell one unit of asset for $S(0)$.
- invest $S(0)$ in risk-free investment.
- Enter a forward contract in long pos^n and get one asset for $F(0,T)$ at $t=T$.

Then, at $t=T$,

- encash the risk-free investment at a value of $S(0).e^{rT}$
- buy asset under forward contract and pay $F(0,T)$.

This shall bring a risk free profit of $S(0).e^{rT} - F(0,T) > 0$ w/o any initial investment. Hence $S(0).e^{rT} \not> F(0,T)$. — (4)

$\therefore$ From (2) & (4) $\Rightarrow$ $F(0,T) = S(0).e^{rT}$

Forward Pricing (with stocks paying Dividends) (not imp)

Let the asset with price $S(0)$ at $t=0$ pay a dividend of 'div' \$ at time 'τ', $0<\tau<T$. Then,

$$\boxed{F(0,T) = \{S(0) - div.\, e^{-rt}\}.\, e^{rt}}$$

If the asset pays dividend continuously at a rate γ_{div}, then

$$\boxed{F(0,T) = S(0).\, e^{(\gamma - \gamma_{div})T}}$$

Forward Pricing (with carrying cost) (not imp)

let an asset carry a holding cost of $C(i)$ per unit in period i ($i=0, 1, 2 \ldots (n-1)$).

$$F(0,T) = S(0).\, e^{rT} + \sum_{i=0}^{n-1} c(i).\, e^{\gamma(n-i)}$$

Value of a Forward contract

Every forward contract has 0 value when initiated. As time goes by, the price of the underlying asset may change. Along with it, the value of the forward contract will vary and will no longer be 0, in general.

let us have 2 forward contracts; one initiated at $t=0$ and another initiated at $t=t$; where $0<t<T$, for the same delivery date T. Let $f(t)$ be the value of the forward contract at $t=t$ & $F(t,T)$ and $F(0,T)$ be the forward prices of the respective contracts, then,

$$\boxed{f(t) = [F(t,T) - F(0,T)]\, e^{-\gamma(T-t)}}$$

(γ = riskfree interest rate compounded continuosisly)

Proof: We will prove this by contradiction.

suppose $f(t) < [F(t,T) - F(0,T)]e^{-\gamma(T-t)}$.

not $t=0$. ← At $t=t$, we can design a portfolio as:

- borrow $f(t)$ at risk free interest rate to enter fwd contract at t
- enter a forward contract in long pos^n with fwd price $F(0,T)$ by paying $f(t)$, value of fwd. contract at $t=t$.
- enter a forward contract in short pos^n with fwd. price

$F(t,T)$. Nothing has to be paid to enter such a fwd contract at $t=t$

$V_p(t)$ will be 0.

Then, at $t=T$,

→ close the long fwd contract by collecting/paying the amount $S(T)-F(0,T)$ and acquire asset to close the short fwd contract by getting $F(t,T)-S(T)$.

→ pay back the loan with interest $f(t).e^{r(T-t)}$.

The final balance $\cancel{S(T)} - F(0,T) + F(t,T) - \cancel{S(T)} - f(t).e^{r(T-t)}$ is the arbitrage profit and strictly greater than 0 (as assumed).

ie. $$F(t,T) - F(0,T) - f(t)\,e^{r(T-t)} > 0$$

$$\Rightarrow \quad f(t) < [F(t,T) - F(0,T)]\,e^{-r(T-t)}$$

should not be allowed. — (1)

Also, suppose $f(t) > [F(t,T) - F(0,T)]\,e^{-r(T-t)}$.

At $t=t$; we can construct the following portfolio:

→ short sell the fwd contract which was initiated at $t=0$ (with a face value/fwd price of $F(0,T)$ and use the cash obtained in form of fwd contract value. $f(t)$ further, encashing $F(0,T)-S(T)$ too.

→ invest $f(t)$ so obtained in risk-free asset.

→ Enter a fwd contract in long posn. with fwd. price $F(t,T)$ and delivery time $t=T$.

The worth of the portfolio is 0.

Then at $t=T$,

→ cash out the risk free investment with interest $f(t).e^{r(T-t)}$

→ close the forward contract with fwd price $F(t,T)$ initiated at $t=t$ and collect/pay $-(F(t,T))+S(T)$.

The final balance will be

$$\cancel{S(T)} - F(t,T) + F(0,T) - \cancel{S(T)} + f(t)e^{r(T-t)}$$

which is >0.

Hence, $$f(t) \ngtr [F(t,T) - F(0,T)]\,e^{-r(T-t)}. \quad -(2)$$

∴ using (1) & (2),

$$f(t) = [F(t,T) - F(0,T)]\,e^{-r(T-t)}$$

Options (call and put)

An option is a contract that gives the holder a right to trade, without any obligation to buy (call option) or sell (put option) an asset at an agreed price on / before a fixed time.

Let $A(0) = 100$, $A(1) = 110$, $S(0) = 100$ and

$$S(1) = \begin{cases} 120, & p \\ 80, & 1-p \end{cases} \qquad 0 < p < 1$$

CALL

A call option with strike price of 100\$ and exercise time 1 is a contract giving the holder a right (but no obligation) to purchase a share of a stock for 100\$ at time 1.

Payoff of a call option for its holder: let $S(0)$, $S(1)$ have usual meanings and let 'E' be the exercise price of call option. Then at time 1, we have 3 possibilities,

(i) $S(1) = 120 > E = 100$. In this case call option should be exercised to get a profit of $S(1) - E = 20$ \$, which is the payoff of the call option for the buyer.

(ii) $S(1) = 80 < E = 100$. In this case call option should not be exercised as it will attract loss. Hence, payoff is 0.

(iii) $S(1) = E$, payoff will be 0.

Hence, value of call option at expiration will be

$$C(1) = \begin{cases} S(T) - E, & \text{stock } \uparrow \\ 0, & \text{stock } \downarrow \end{cases}$$

Payoff of a call option for seller: $\min[E - S(1), 0]$

NOTE: A call option may resemble a long forward position. But it has the following key differences:

- Buyer is not obliged to exercise the call option, but in a forward contract payment is mandatory upon maturation.
- Initial investment of a premium is required to buy a call option, which is not required in a fwd. contract

Our task will be to find the time 0 price, $c(0)$ of the call option consistent with the assumptions of market & no arbitrage principle. Because the holder of the call option has a certain right, but never an obligation, it is reasonable to expect that $c(0)$ will be positive : one needs to pay a premium to acquire this right.

The option price $C(0)$ can be found in 2 steps :

[I] <u>Replicating the option</u> : construct an investment in 'x' stocks and 'y' bonds such that the value of the investment at time 1 is the same as of the option,

$$x(S(1)) + y(A(1)) = c(1).$$

For our case, time 1 value of stocks & bonds will be

$$xS(1) + y(A(1)) = \begin{cases} 120x + 110y, & \text{stock} \uparrow \\ 80x + 110y, & \text{stock} \downarrow \end{cases}$$

Thus, the equality $xS(1) + yA(1) = C(1)$ b/w 2 random variables can be written as :

$$\begin{cases} 120x + 110y = 20 & , \text{stock} \uparrow \\ 80x + 110y = 0 & , \text{stock} \downarrow \end{cases}$$

These need to be simultaneously true. So, solving for x and y,

$$x = \frac{1}{2}, \quad y = -\frac{4}{11}$$

So, to replicate the option, we need to buy $\frac{1}{2}$ a share of stock and take a short pos^n of $-\frac{4}{11}$ in bonds (or borrow $\frac{4}{11} \times 100 \simeq \frac{400}{11}$ \$ in cash)

[II] <u>Pricing the option</u> : compute the time 0 value of the investment in stock and bonds. <u>It must be equal to the option price</u>,

$$x\,S(0) + y\,A(0) = c(0)$$

<u>an arbitrage opportunity would be available otherwise.</u>

For our case, we calculated $x = \frac{1}{2}$ & $y = -\frac{4}{11}$ in 1st step. So.

$$xS(0) + y\,A(0) = \frac{1}{2} \times 100 - \frac{4}{11} \times 100 \cong \underline{13.6364}$$

PUT A put option with strike price 100 \$ and exercise time 1 gives the right to sell one share of stock for 100 \$ at time 1. This kind of option is worthless if the stock goes up, but it brings profit otherwise, the payoff being

$$P(1) = \begin{cases} 0, & \text{stock} \uparrow \\ 20, & \text{stock} \downarrow \end{cases}$$

given $A(0), A(1), S(0), S(1)$ are same as in the previous case.
The replicating and pricing procedures for put follows the same pattern as for call options.

payoff of put option for buyer : $P(1) = \max[E - S(1), 0]$

NOTE: category of options :

(i) European option : when an option is allowed to be exercised only on the maturity date.

(ii) American option : when an option can be exercised at anytime before its maturity date.

Exercise of option results into 3 possibilities :

(i) In-the-money → advantageous for investor to exercise

(ii) out-of-the-money → not advantageous for investor to exercise

(iii) At-the-money → exercising does not lead to any gain / loss.

Options General Properties

The payoff of a European Call option is

$$\begin{cases} S(T) - X, & S(T) > X \\ 0, & \text{otherwise} \end{cases}$$

This payoff is a random variable. It is convenient to use the notation

$$x^+ = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the payoff of a European call option can be written as $(S(T)-X)^+$ and that of a put option be written as $(X-S(T))^+$.

Since the payoff are non-negative, a <u>premium</u> must be paid to buy an option, else an investor purchasing an option could never lose money and infact make a profit whenever the payoff is positive. This would contradict the no Arbitrage Principle. Hence, the premium is the market price of the option. The prices (premiums) of call and put options will be denoted by C^E, P^E resp. (E stands for European options).

Taking in consideration the time value of premium at an interest rate 'r' compounded continuously, the gain of an option (call) buyer at time T will be $(S(T)-X)^+ - C^E e^{rT}$. For buyer of a put option, the gain is $(X-S(T))^+ - P^E e^{rT}$. For writers of a call option and put option, the gains are $\underline{C^E e^{rT} - (S(T)-X)^+}$ and $\underline{P^E e^{rT} - (X-S(T))^+}$ respectively.

<u>NOTE</u>: It is to be noted that the potential loss for a buyer of a call/put is always limited to the premium paid. For a writer of an option, the loss can be much higher, even unbounded in case of a call option.

<u>Put-Call Parity</u>

For the stocks paying no dividends, the following relation holds b/w C^E and P^E with exercise price X and exercise time T :

$$\boxed{C^E - P^E = S(0) - Xe^{-rT}}$$

<u>Proof</u> : we will prove this by contradicting the no arbitrage principle

let us suppose $C^E - P^E > S(0) - Xe^{-rT}$.

At $t=0$, we can construct the following arbitrage opportunity:

→ write and sell one call option for C^E \$

→ Buy one share for $S(0)$

- Buy a put option for p^E
- invest (or borrow if negative) the balance $c^E - (p^E + s(0))$ in a risk free asset at rate 'r' and for time T.

The worth of portfolio $V_p(0) = 0$.

Then at time T,

- cash out the risk free investment (payout if borrowed) of $(c^E - p^E - s(0)) e^{Tr}$
- sell the stock and obtain x \$ either by using the put option (if $x > s(T)$) or by closing the short position on call option (if $s(T) > x$; ie. the buyer will pay us x \$ to buy the stock instead of it's market value of $s(T)$).

Hence, the balance or portfolio worth at time T is

$$(c^E - p^E - s(0)) e^{rT} + x > 0$$ according to our assumption. But since this presents an arbitrage opportunity,

$$c^E - p^E \not> s(0) - x e^{-rT} \quad - (1)$$

Now suppose $c^E - p^E < s(0) - x e^{-rT}$.

At $t=0$, we can construct the following scheme:

- sell short one share for $s(0)$.
- sell a put option for p^E.
- buy a call option for c^E
- invest (or borrow if negative) the balance $s(0) + p^E - c^E$ in a risk-free asset at rate 'r' for time T.

The worth of portfolio $V_p(0) = 0$.

Then at time T,

- cash out (or pay out if borrowed) the risk free investment $(s(0) + p^E - c^E) e^{rT}$.
- buy one stock for x \$ buy either using the call option if $(s(T) > x)$ or settling the put option by paying x \$ to its holder when $(x > s(T))$. This closes the short position on the stock that we took initially.

The balance $(S(0) + P^E - C^E) e^{rT} - X > 0$ by our assumption.
This violates the no Arbitrage law. Hence,

$$C^E - P^E \nless S(0) - Xe^{-rT} \quad -(2)$$

From (1) & (2) $\Rightarrow$

$$C^E - P^E = S(0) - Xe^{-rT}.$$

Bounds on option Prices (European options)

(I) $C^E \geq 0$, $P^E \geq 0$ (premiums are to be non-negative)

(II) $C^E < S(0)$, otherwise an arbitrage opportunity would pop out as:

- write and sell a call option for C^E \$.
- Buy a stock for $S(0)$.
- invest the balance $C^E - S(0)$ risk-free at rate 'r' for time T

At time T,

- cash out the risk free investment at $(C^E - S(0)) e^{rT}$.
- close the call option by selling the stock for $\min\{S(T), X\}$.

Net non-negative arbitrage profit $= (C^E - S(0)) e^{rT} + \min\{S(T), X\} > 0$

$\therefore$ To avoid this $C^E < S(0)$ holds. (upper bound for C^E)

(III) From $P^E \geq 0$, we get $C^E \geq S(0) - Xe^{-rT}$ (lower bound for C^E)

(IV) From (II), we get $P^E < Xe^{-rT}$ (upper bound on P^E).

(V) From (I) we get $C^E \geq 0 \Rightarrow P^E \geq Xe^{-rT} - S(0)$ (lower bound on P^E).

combining the results we get:

$$\left[\begin{array}{l} \max\{0, S(0) - Xe^{-rT}) \leq C^E < S(0) \\ \max\{0, Xe^{-rT} - S(0)\} \leq P^E < Xe^{-rT} \end{array}\right]$$

Variable determining option Prices

The option price depends upon a number of variables like strike price X, expiry time T, the current price of underlying asset $s(0)$, risk-free rate r, etc.

(i) Dependence on Strike Price (X)

consider options on the same underlying asset with same exercise time T, but with different values of strike prices. The call and put prices (premiums) will be denoted by $C^E(X)$ & $P^E(X)$ to show dependence on X. Other symbols & variables hold their usual meanings.

Result 1: If $X_1 < X_2$, then

$$C^E(X_1) > C^E(X_2)$$
$$P^E(X_1) < P^E(X_2)$$

which is obvious because the right to buy at a lower price (X_1 here) is more valuable than the right to buy at a higher price (X_2). Similarly, it is better to sell the asset at a higher price (X_2) than at a lower one (X_1).

Result 2: If $X_1 < X_2$, then

$$C^E(X_1) - C^E(X_2) < e^{-rT}(X_2 - X_1)$$
$$P^E(X_2) - P^E(X_1) < e^{-rT}(X_2 - X_1)$$

Proof:

using put-call parity, $C^E - P^E = S(0) - Xe^{-rT}$,

$$C^E(X_1) - P^E(X_1) = S(0) - X_1 e^{-rT}$$
$$C^E(X_2) - P^E(X_2) = S(0) - X_2 e^{-rT}$$

Subtracting,

$$\{C^E(X_1) - C^E(X_2)\} + \{P^E(X_2) - P^E(X_1)\} = (X_2 - X_1)e^{-rT}$$

both the terms on the LHS are >0 following Result 1.

Hence, each term is smaller than RHS indivisually.

$\therefore$ Result 2 holds.

Result 3: let $X_1 < X_2$ and $\alpha \in (0,1)$, then

$$c^E(\alpha X_1 + (1-\alpha)X_2) \le \alpha c^E(X_1) + (1-\alpha)c^E(X_2)$$

$$p^E(\alpha X_1 + (1-\alpha)X_2) \le \alpha p^E(X_1) + (1-\alpha)p^E(X_2).$$

ie. $c^E(X)$ & $p^E(X)$ are <u>convex functions of X.</u>

Proof: for brevity, put $X = \alpha X_1 + (1-\alpha)X_2$.

suppose $c^E(X) > \alpha c^E(X_1) + (1-\alpha)c^E(X_2)$

we can construct the following portfolio :

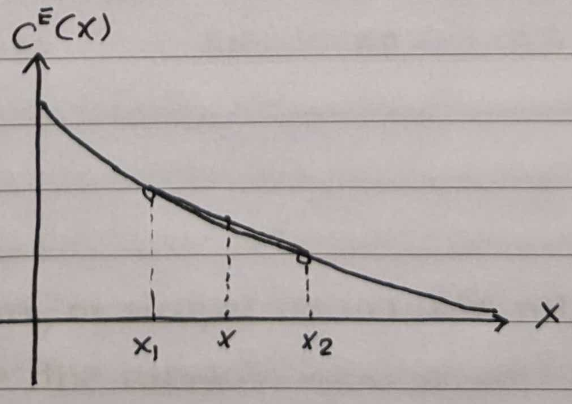
- write and sell call option with strike price X.
- purchase α call options with strike price X_1 and $1-\alpha$ call options with strike price X_2.
- invest the balance $c^E(X) - (\alpha c^E(X_1) + (1-\alpha)c^E(X_2)) > 0$ (due to our assumption) risk free.

upon expiry (at T), if option with strike price X is exercised, then

- If $S(T) > X$, we pay $(S(T)-X)^+$
- we can encash the amount $\alpha(S(T)-X_1)^+ + (1-\alpha)(S(T)-X_2)^+$ by closing α and $(1-\alpha)$ call options.

$\therefore$ our net balance portfolio worth will be :

$$\alpha(S(T)-X_1)^+ + (1-\alpha)(S(T)-X_2)^+ - (S(T)-X)^+$$



Geometrically, convexity means that if 2 points on the graph of the func. are made and joined with a straigh line, then graph of function lies below that line b/w those two points.

Dynamics of a Stock Price

price of a stock at $t \rightarrow S(t)$, $S(t) > 0$, $\forall t$.

$S(0)$ is the current stock price , $t=0$ is the present time.

Mathematically, $S(t)$ can be represented as a +ve random variable on a probability space Ω, ie.,

$$S(t) : \Omega \rightarrow (0, \infty)$$

The probability space Ω consists of all the feasible price movement 'scenarios' (like boom, recession, stagnation etc) $\omega \in \Omega$. we shall write $S(t, \omega)$ to denote the price at a time 't' under the market scenario 'ω' $\in \Omega$.

<u>NOTE</u>: We assume that time runs in a distinct manner, $t = n\tau$, where $n = 0, 1, 2, \ldots$ and τ is a fixed time-stamp , namely a year, a month, a week, a day or even a second. Because we take <u>one year as the unit measure of time</u>, a month corresponds to $\tau/12$, a week $\tau/52$ and so on. To simplify our notation, we shall write $S(0), S(1), S(2) \ldots$ instead of $S(0), S(\tau), S(2\tau) \ldots$, identifying $n\tau$ with n.

As an example, suppose there are 3 possible market scenarios, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, the stock prices taking the following values over two time steps :

	Scenario	$S(0)$	$S(1)$	$S(2)$
(boom)	ω_1	55	58	60
(stagnation)	ω_2	55	58	52
(recession)	ω_3	55	52	53

Return

The rate of return, or simply return $K(n, m)$ over a time interval $[n, m]$ is defined to be the random variable as :

$$K(n, m) = \frac{S(m) - S(n)}{S(n)}$$

The retuin over a single time step $[n-1, n]$ will be denoted by $K(n)$

$$K(n) = K(n, n-1) = \frac{S(n) - S(n-1)}{S(n-1)}$$

$$\Rightarrow \quad S(n) = S(n-1)(1 + K(n))$$

Expected Return

Suppose that the prob. dist. of K (return) over a certain period is known, then we can find the expectation / average of the return $E(K)$.

As an example, suppose we estimate the probabilities of recession, stagnation & boom to be $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. If predicted annual returns on some stocks in these scenarios are -6%, 4%, 30% respect., then $E(K) = -6\% \times \frac{1}{4} + 4\% \times \frac{1}{2} + 30\% \times \frac{1}{4}$
$= 8\%$

Binomial Tree Model

The model is defined and works under two conditions :

$\boxed{\text{I}}$ The one step returns $K(n)$ on stock are identically distributed independent random variables such that

$$K(n) = \begin{cases} u, & \text{with prob. } p \\ d, & \text{with prob. } (1-p) \end{cases}$$

at each time step n, where $-1 < d < u$ and $0 < p < 1$

This means that if we write $S(n) = S(n-1) \cdot (1 + K(n))$, the return may go up and attain value 'u' or may go down to attain value 'd' (these are returns, both u and d).

Corresponding to the 2 values (u and d) of $K(n)$, the stock price $S(n)$ may go up or down by a factor of $1+u$ or $1+d$ resp.

$$S(n) = \begin{cases} S(n-1)(1+u), & p \\ S(n-1)(1+d), & 1-p \end{cases}$$

($1+u$, $1+d$ are called growth factors)

(III) The one-step return 'r' on a risk-free investment is the same at each time step and $d < r < u$.

NOTE :

$$\begin{tikzcd} & S(0)(1+u) \arrow[d, dashed] \\ S(0) \arrow[ur, "p"] \arrow[dr, "1-p"'] & \vdots \\ & S(0)(1+d) \arrow[d, dashed] \\ & S(1) \end{tikzcd}$$

one-step binomial tree.

In an n-step tree of stock prices, each scenario (or path of tree) with exactly 'i' upward and '$n-i$' downward price movements produces the same stock price

$$S(n) = S(0)\,(1+u)^{i}\,(1+d)^{n-i} \quad \text{with prob. } {}^{n}C_{i}\; p^{i}(1-p)^{n-i}.$$

for $i = 0, 1, 2, \ldots, n$.

$S(n)$ at time n is a discrete random variable. The number 'i' of upward movements and that '$n-i$' of downward movements, both follow binomial distribution being seperate random variables.

Result 1 : Expected value of stock price for $n = 0, 1, 2 \ldots$ is

$$\boxed{E(S(n)) = S(0)\,\{1 + E(K(1))\}^{n}}$$

proof :

since the one-step returns $K(1), K(2) \ldots$ are i.i.d r.v., so are $1+K(1)$, $1+K(2)$, ...

$$E(S(n)) = E\big(S(0)(1+K(1))(1+K(2)) \cdots (1+K(n))\big)$$
$$= S(0).\,E\big((1+K(1)).(1+K(2)) \cdots (1+K(n))\big)$$
$$= S(0).\,E(1+K(1)).\,E(1+K(2)).\,E(1+K(3)) \cdots E(1+K(n))$$

$\because K(n), \forall n$ are i.i.d r.v $\Rightarrow E(K(1)) = E(K(2)) = E(K(3)) = \cdots E(K(n))$

$$\therefore \quad E(S(n)) = S(0).\,[\,E(1+K(1))\,]^{n}$$

Risk-Neutral Probability.

we shall note that

$$E(S(1)) = S(0).\,[1 + E(K(1))]$$

and

$$E(K(1)) = p.u + (1-p).d$$

If the amount $S(0)$ were to be invested risk-free at time 0, it would grow to $S(0)(1+r)^n$ after n time steps. We can compare $E(S(n))$, our expected value of risky investment with the risk-free value $S(0).(1+r)^n$ by only comparing the return rates, $E(K(1))$ and r.

The border case of a market, where $E(K(1)) = r$, is referred to as risk-neutral.

We use a special symbol p^* for the probability as well as E^* for the corresponding expectation satisfying:

$$E^*(K(1)) = p^* u + (1-p^*) d = r$$

for risk-neutrality, which implies that,

$$p^* = \frac{r-d}{u-d}$$

p^* is called the risk-neutral probability.

↓

(in terms of growth factor $p^* = \frac{(1+r)-(1+d)}{(1+u)-(1+d)}$).

Option Pricing

$D(T) = f(S(T))$, where f is a given function called the payoff.

$D(T)$ is the time T value of a DERIVATIVE SECURITY / CONTINGENT CLAIM.

A derivative security is a financial instrument whose value depends upon the value of an other asset.

Examples of derivative securities are Forward contracts, Future Contracts, options (call/put) etc.

Also $f(S(T))$ is the payoff for the security under consideration.

for call option, $f(S(T)) = (S(T) - X)^+$

for put option, $f(S(T)) = (X - S(T))^+$

NOTE: **Result:** Suppose for any contingent claim $D(T)$, there exists a replication strategy, that is, an admissible portfolio $x(t), y(t)$ with final value $V(T) = D(T)$. Then, the price $D(0)$ of the contingent claim at time 0 must be equal to that of the replication strategy, i.e.

$$\boxed{D(0) = V(0)}$$

European option Pricing using Binomial Tree Model.

- **One step Model:**

We assume that $S(1)$ may take 2 values:

$$S(1)\begin{cases} \xrightarrow{\ p\ } S^u = S(0)(1+u) \\ \xrightarrow{\ 1-p\ } S^d = S(0)(1+d) \end{cases}$$

To replicate a general derivative security with payoff 'f', we need to solve the system of eq^{n}s:

$$\begin{cases} x(1).S^u + y(1).A(0).(1+r) = f(S^u) \\ x(1).S^d + y(1).A(0).(1+r) = f(S^d) \end{cases}$$

for $x(1) \Rightarrow$ no of stocks at $t=1$ and $y(1) \Rightarrow$ no. of bonds at $t=1$,

we get,

$$\boxed{x(1) = \frac{f(S^u) - f(S^d)}{S^u - S^d}}$$

which is the replicating pos^n in stocks, called Delta of the option.

and,

$$\boxed{y(1) = -\frac{(1+d)f(S^u) - (1+u)f(S^d)}{(u-d)A(0)(1+r)}}$$

which is the money market pos^n.

Hence, the initial value of the replicating portfolio, $V(0) = x(1).S(0) + y(1).A(0)$.

Also, $D(0) = V(0) \Rightarrow$

$$D(0) = S(0).\left(\frac{f(S^u) - f(S^d)}{S^u - S^d}\right) + A(0).\left[-\frac{(1+d)f(S^u)-(1+u)f(S^d)}{(u-d).A(0).(1+r)}\right]$$

$$= S(0).\left(\frac{f(S^u) - f(S^d)}{S(0)(1+u-(1+d))}\right) - \left(\frac{(1+d)f(S^u) - (1+u)f(S^d)}{(u-d)(1+r)}\right)$$

$$= \frac{f(S^u) - f(S^d)}{u-d} - \frac{(1+d)f(S^u) - (1+u).f(S^d)}{(u-d)(1+r)}$$

$$= \frac{1}{r+1}\left(\frac{(r-d).f(S^u)}{(u-d)} + \frac{(u-r).f(S^d)}{(u-d)}\right)$$

$$= \frac{1}{r+1}\left(p^*.f(S^u) + (1-p^*).f(S^d)\right)$$

$$= \frac{1}{r+1}\left(E^*(f(S(1)))\right) = E^*\left((1+r)^{-1}.f(S(1))\right)$$

$$\therefore \boxed{D(0) = E^*\left((1+r)^{-1}f(S(1))\right)}$$

ie., the present value of the derivative security is equal to the expectation of discounted payoff computed wrt. risk-neutral probability.

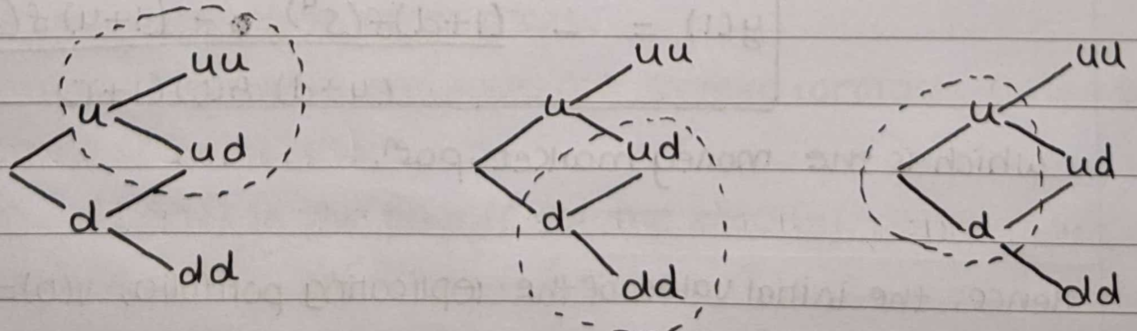
- **Two Steps Model :**

we start with two time steps, $t=1$ & $t=2$. The stock price $S(2)$ has 3 possible values:

$$S^{uu} = S(0).(1+u)^2, \quad S^{ud} = S(0)(1+u)(1+d), \quad S^{dd} = S(0).(1+d)^2$$

and $S(1)$ has 2 possible values:

$$S^u = S(0)(1+u) \text{ and } S^d = S(0)(1+d).$$



used for finding $D(1)$ by 1 step.

used For finding $D(0)$ by 1 step

$$D(2) = f(S(2))$$

D(1) has two possible values :

$$\frac{1}{1+r}\left[p^* f(S^{uu}) + (1-p^*) f(S^{ud})\right], \frac{1}{1+r}\left[p^* f(S^{du}) + (1-p^*) f(S^{dd})\right]$$

$$\Rightarrow D(1) = \frac{1}{1+r}\left[p^* f(S^{uu}) + (1-p^*) f(S^{ud}) + p^* f(S^{du}) + (1-p^*) f(S^{dd})\right]$$

$$= \frac{1}{1+r}\left[p^*\left(f(S^{uu}) + f(S^{du})\right) + (1-p^*)\left(f(S^{ud}) + f(S^{dd})\right)\right]$$

$$= \frac{1}{1+r}\left[p^* f(S(1).(1+u)) + (1-p^*) f(S(1).(1+d))\right]$$

$$= g(S(1)).$$

D(1) can be regarded as a derivative security expiring at time 1 with payoff g (though it can't be exercised at time 1, because it is a European option, the derivative security can be sold for $D(1) = g(S(1))$).

where $g(x) = \frac{1}{1+r}\left[p^* f(x(1+u)) + (1-p^*) f(x(1+d))\right]$

Applying one-step procedure again,

$$D(0) = \frac{1}{1+r}\left[p^* g(S(0).(1+u)) + (1-p^*) g(S(0)(1+d))\right]$$

$$= \frac{1}{1+r}\left[p^* g(S^u) + (1-p^*) g(S^d)\right]$$

$$= \frac{1}{(1+r)^2}\underbrace{\left[(p^*)^2 f(S^{uu}) + 2p^*(1-p^*) f(S^{ud}) + (1-p^*)^2 f(S^{dd})\right]}_{E(f(S(2)))}$$

NOTE: Note that $E(f(S(N))) = \sum_{i=0}^{N} {}^N C_i (p^*)^i (1-p^*)^{N-i} . f\left(S(0).(1+u)^i(1+d)^{N-i}\right)$

where $S(N) = S(0).(1+u)^i(1+d)^{N-i}$, with prob. ${}^N C_i (p^*)^i (1-p^*)^{N-i}$

for $i = 0, 1, \ldots N$.

and $f(S(N)) = f\left(S(0).(1+u)^i(1+d)^{N-i}\right)$

f = payoff function

$$\therefore \boxed{D(0) = E^*\left[(1+r)^{-2} \cdot f(S(2))\right]}$$

- General N-step Model :

$$\boxed{D(0) = E^*\left[(1+r)^{-N} \cdot f(S(N))\right]}$$

Cox-Ross-Rubinstein Model

The following assumptions of financial market hold :

(i) underlying stock is perfectly divisible

(ii) stock pays no dividend

(iii) no transaction costs or taxes

(iv) short selling is allowed

(v) risk free interest rate 'r' is known & constant till expiration of option.

(vi) No-Arbitrage Principle holds at all times.

Divide time interval $[0, T]$ into 'n' sub-intervals : $\Delta t = \frac{T}{n}$

In each time interval (Δt), stock price either goes up by a constant factor 'u' or goes down by a constant factor 'd', with positive probability.

We define $E_K = \begin{cases} u & , p \\ d & , 1-p \end{cases}$ which is a Bernoulli r.v. — (1)

At $t = T$, $S(T) = S(0) . E_1 . E_2 . E_3 \cdots E_n = S(0) . e^H$

where $e^H = E_1 E_2 E_3 \cdots E_n$

Now, $\left.\begin{aligned} E(\ln(E_K)) &= p . \ln(u) + (1-p) . \ln(d) \\ var(\ln(E_K)) &= p . (1-p) . [\ln(u) - \ln(d)]^2 \end{aligned}\right]$ —(2)

Now we introduce new parameters μ and σ^2 as :

$$\mu \Delta t = E(\ln(E_K)) \quad \& \quad \sigma^2 \Delta t = Var(\ln(E_K)) \quad -(3)$$

$\underline{\mu \to \text{drift}}$, $\underline{\sigma \to \text{volatility}}$

Let us introduce another random variable

$$X_K = \frac{\ln(E_K) - E(\ln(E_K))}{\sqrt{var(\ln(E_K))}} \quad -(4)$$

using (2) in (4) $\Rightarrow$

$$X_K = \begin{cases} \dfrac{1-p}{\sqrt{p(1-p)}} , & \text{with prob. } p \\[2ex] \dfrac{-p}{\sqrt{p(1-p)}} , & \text{with prob. } (1-p) \end{cases}$$

$K = 1, 2, ---n.$

$\underline{E(X_K) = 0}$ and $\underline{var(X_K) = 1}$

Now, $H = \sum_{K=1}^{n} \ln(E_K) \quad (\because e^H = E_1.E_2.E_3 ... E_n)$

$$= \sum_{K=1}^{n} (\mu \Delta t + \sigma\sqrt{\Delta t}\, X_K) \quad \text{(using (3) \& (4))}$$

$$= \mu T + \sigma\sqrt{\Delta t} \sum_{K=1}^{n} X_K = \underline{\mu T + \sigma\sqrt{\Delta t}\, Y}$$

where $Y = \sum_{K=1}^{n} X_K$ is a simple random walk.

$$\therefore \boxed{S(T) = S(0).e^H = S(0).e^{(\mu T + \sigma\sqrt{\Delta t}\, Y)} , Y = \sum_{K=1}^{n} X_K}$$

Also, $\underline{E(H) = \mu T}$, $\underline{Var(H) = \sigma^2 T}$

Matching of CRR Model with a Multiperiod Binomial Model

let $U = \ln(u)$ and $D = \ln(d)$

<u>we assume</u> that $\underline{D = -U}$, ie. $d = \frac{1}{u}$

so, $\mu \Delta t = E(\ln(E_K)) = p.\ln(u) + (1-p)\ln(d) = pU - (1-p)U$
(from (3))

$\Rightarrow \quad (2p-1)U = \mu\Delta t \qquad -(5)$

and $\quad 4p(1-p)U^2 = \sigma^2\Delta t \qquad -(6)$

Squaring (5) and adding to (6) $\Rightarrow$

$$(2p-1)^2U^2 + 4p(1-p)U^2 = \mu^2(\Delta t)^2 + \sigma^2\Delta t$$

or

$$U = \sqrt{\sigma^2\Delta t + \mu^2(\Delta t)^2}$$

If we take $n \to$ large, $\Delta t = \frac{T}{n} \to 0$, $\therefore (\Delta t)^2$ can be ≈ 0

$$\therefore U = \sigma\sqrt{\Delta t} \Rightarrow \ln(u) = \sigma\sqrt{\Delta t}$$

$$\Rightarrow \boxed{u = e^{\sigma\sqrt{\Delta t}}}$$

and $\boxed{d = u^{-1} = e^{-\sigma\sqrt{\Delta t}}}$

where σ is the volatility.

and $\boxed{p = \frac{1}{2} + \frac{\mu\sqrt{\Delta t}}{2\sigma}}$ $-$ (from (5))

Risk Neutral Probability Measure

let Ω represent the state space of economic scenarios $\Omega = \{\omega_1, \omega_2 \ldots \omega$

As an example, a binomial model contains 2 scenarios, one up-tick (one period) movement and another down-tick movement. A 2-period binomial model may have 4 (uu, ud, du, dd).

Apart from bonds and underlying stocks, there can be multiple securities so, we represent price of a security (derivative) by

$$S_i^k(\omega_j)$$

$k=0$ represents that the security is bond, $k=1$ for stocks and such...

'i' represents the time.

'j' represents the economic scenario.

For ex., $S_1^0(\omega_1) \rightarrow$ price of bond at $t=1$ in ω_1 scenario.

A Risk Neutral Probability Measure (RNPM) is a vector p^*,

$$p^* = (p_1^*, p_2^*, p_3^*, \ldots p_m^*)$$

such that,

(i) $p_j^* \geq 0, \quad j = 1, 2 \ldots m.$

(ii) $\sum_{j=1}^{m} p_j^* = 1$

and for every security represented by $k = 0, 1, 2 \ldots,$

$$S_0^k = \frac{1}{R} \underbrace{\left[\sum_{j=1}^{m} p_j^* \, S_1^k(\omega_j) \right]}_{\text{Expected value of security 'k' at time 1.}}$$

Risk-free growth rate

This is the same as finding p^* from $E(S(1)) = S(0).[1 + E(K)]$

$$= S(0)[1 + r]$$

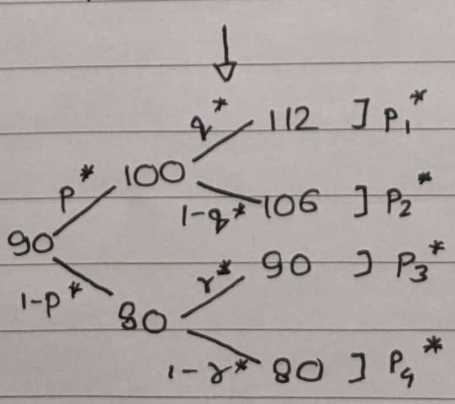
$$\Rightarrow \quad p^*(1+u) + (1-p^*)(1+d) = \underbrace{E(K)}_{\text{Expected rate}} = 1 + r = \text{risk-free rate}$$

we can find the risk neutral probabilities for all one-step transitions and then finally form an RNPM.

For example, let $A(0) = 100$, $A(1) = 110$, $A(2) = 121$

and for security $K = 1$, ie. Stocks :

scenario	S(0)	S(1)	S(2)
ω_1	90	100	112
ω_2	90	100	106
ω_3	90	80	90
ω_4	90	80	80



we have to compute

$$(p_1^*, p_2^*, p_3^*, p_4^*) = (p^* q^*, \; p^*(1-q^*), \; (1-p^*) r^*, \; (1-p^*)(1-r^*))$$

we just have to find the risk-neutral probability from one-step binomial transitions $S(0) \to S(1)$ and $S(1) \to S(2)$.

For $S_1(\omega_1) \to S_2(\omega_1)$ and $S_1(\omega_1) \to S_2(\omega_2) \Rightarrow$

$$q^* = \frac{r-d}{u-d}$$, where r = constant risk free rate

$$= \frac{121-110}{110} = \frac{110-100}{100} = \frac{1}{10}$$

$$d = \frac{106-100}{100} = 0.06 \quad , \quad u = \frac{112-100}{100} = 0.12$$

$$\therefore q^* = \frac{0.1-0.6}{0.12-0.6} = 2/3$$

$$\therefore 1-q^* = 1/3$$

Similarly, we calculate $p^* = \frac{19}{20}$ and $r^* = \frac{4}{5}$

$\therefore$ we get $P_1^* = p^* q^* = \frac{19}{20} \times \frac{2}{3} = 38/60$

$$P_2^* = p^*(1-q^*) = \frac{19}{20} \times \frac{1}{3} = 19/60$$

$$P_3^* = (1-p^*)(r^*) = (1/20) \times (4/5) = 4/100$$

$$P_4^* = (1-p^*)(1-r^*) = (1/20)(1/5) = 1/100$$

also, $P_1^* + P_2^* + P_3^* + P_4^* = 1$.

$\therefore p^* = (P_1^*, P_2^*, P_3^*, P_4^*)$ is an RNPM.