- Two important results in martingale theory namely Doob's decomposition and Doob's optional sampling theorem are presented. Optional sampling concept reveals that the expectation of a martingale remains constant. It has immediate implications concerning the pathwise behaviour of martingales, submartingales and supermartingales. We may refer to Roman [113] for some more details on martingale theory.

8.8 Exercises

Exercise 8.1 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables each having uniform distribution with values -2, -1, 0, 1, 2, 3. Let* $\{N(t), t \geq 0\}$ *be a Poisson process with parameter 2. Find the mean and variance of* $\sum_{i=1}^{N(t)} X_i$.

Exercise 8.2 *Consider two i.i.d random variables X and Y each having uniform distribution between the intervals 0 and 1. Define* $Z = X + Y$. *Prove that* $E(X/Z) = \frac{Z}{2}$.

Exercise 8.3 *Consider the successive rolling of an unbiased die. Let X and Y denote the number of rolls necessary to obtain a two and a three respectively. Obtain (i)* $E(X/Y = 2)$, *(ii)* $E(X/Y = 4)$.

Exercise 8.4 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let X be an integrable random variable and let* $\mathcal{G} \subseteq \mathcal{F}$ *be a σ-field. Prove that, the conditional expectation* $E(X/\mathcal{G})$ *exists. (Hint: Use Radon-Nikodym theorem)*

Exercise 8.5 *Consider* $\Omega = \{a, b, c, d\}$. *Construct 4 distinct σ-fields* $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_3, \mathcal{F}_4$ *such that* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}_4$.

Exercise 8.6 *Construct an example of σ-fields* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *such that* $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$ *and* $\mathcal{F}_2 \subset \mathcal{F}_1$.

Exercise 8.7 *Consider a binomial model with* $t = 1, 2, 3$. *Let* S_t *be the stock price at time t. Let* $\Omega = \{(u,u,u), (u,u,d), (u,d,u), (u,d,d), (d,u,d), (d,u,u), (d,d,u), (d,d,d)\}$. *Let σ-field* $\mathcal{F}$ *be the power set of* Ω *and* $P(w) = \frac{1}{8}$ *for all* $w \in \Omega$. *Let* $\mathcal{G} = \{\emptyset, \{(u,u,u), (u,u,d), (u,d,u), (u,d,d)\}, \{(d,u,d), (d,u,u), (d,d,u), (d,d,d)\}\Omega\}$. *Define a discrete random variable X as*

$$X(w_i) = i, \quad i = 1, 2, \ldots, 8 .$$

Find $E(X/\mathcal{G})$.

Exercise 8.8 *Let $\{X_n, n = 0, 1, \ldots\}$ and $\{Y_n, n = 0, 1, \ldots\}$ be stochastic processes. We say $\{X_n\}$ is a martingale with respect to $\{Y_n\}$ if*

(i) $E(|X_n|) < \infty$.
(ii) $E(X_{n+1}/Y_0, Y_1, \ldots, Y_n) = X_n$.

Prove that $\{X_n, n = 0, 1, \ldots\}$ is a martingale with respect to $\{Y_n, n = 0, 1, \ldots\}$ where $X_n = Y_1 + Y_2 + \ldots + Y_n$, $n \geq 1$, $Y_0 = 0$, $\{Y_i, i = 1, 2, \ldots\}$ are independent random variables with $E(Y_n) = 0$.

Exercise 8.9 *Let $\{S_n, n = 0, 1, \ldots\}$ be a symmetric random walk and $\mathcal{F}_n$ be a filtration. Show that $Y_n = (-1)^n \cos(\pi S_n)$ is a martingale with respect to $\mathcal{F}_n$.*

Exercise 8.10 *Consider the tossing of an unbiased coin three times. Suppose that the toss of a head wins Rs 1 while an outcome of a tail loses Rs 1. Let X_n denote the sum of the winnings at time n. Prove that $\{X_n, n = 1, 2, 3\}$ is an adapted process with respect to $\{\mathcal{F}_i, i = 1, 2, 3\}$ defined in Section 7.4. Further prove that $\{X_n, n = 1, 2, 3\}$ is a martingale with respect to $\{\mathcal{F}_i, i = 1, 2, 3\}$.*

Exercise 8.11 *$\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Prove that, $\{N(t) - \lambda t, t \geq 0\}$ is a martingale.*

Exercise 8.12 *Let $X(t) = \mu t + \sigma W(t)$, $\quad -\infty < \mu < \infty$, $\quad 0 < \sigma < \infty$. Prove that $\{X(t), t \geq 0\}$ is a martingale for $\mu = 0$. Also prove that $\{X(t), t \geq 0\}$ is a submartingale for $\mu > 0$, and a supermartingale for $\mu < 0$.*

Exercise 8.13 *Show that the maximum of two submartingales (with respect to the same filtration) is a submartingale.*

Exercise 8.14 *Let $Y_1, Y_2, \ldots$ be a sequence of positive independent random variables with $E(Y_i) = 1$ for all i. Define $X_0 = 1$, and $X_n = \prod_{i=1}^{n} Y_i$ for $n \geq 1$. Show that $\{X_n, n = 0, 1, \ldots\}$ is a martingale with respect to its natural filtration.*

Exercise 8.15 *Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d random variables with $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$. Define $X_n = \prod_{i=1}^{n} \frac{Y_i}{p}$. Prove that, $E(X_{n+1}/Y_1, Y_2, \ldots, Y_n) = X_n$.*

Exercise 8.16 *Let X be a random variable and let ϕ be a convex function on* **R**. *Suppose both X and $\phi(X)$ are integrable, and $\mathcal{G} \subset \mathcal{F}$ is a σ-field. Prove that, $\phi(E(X/\mathcal{G})) \leq E(\phi(X)/\mathcal{G})$.*

Exercise 8.17 *Let $X_1, X_2, \ldots$ be a sequence of square integrable random variables. Show that if $\{X_n, n = 1, 2, \ldots\}$ is a martingale with respect to a filtration $\mathcal{F}_n$, then X_n^2 is a sub martingale with respect to the same filtration. (Hint: Use Jensen's inequality with convex function $\phi(x) = x^2$)*

Exercise 8.18 *Prove that, if $\{X_n, n = 1, 2, \ldots\}$ is a martingale, then $| X_n |$, X_n^2, e^{X_n} and e^{-X_n} are all submartingales.*

Exercise 8.19 *Prove that, if $X_n > 0$ and is a martingale, then $\sqrt{X_n}$ and $ln(X_n)$ are supermartingales.*

Exercise 8.20 *Prove that, if X_n is a submartingale and K a constant, then Max $\{X_n, K\}$ is a submartingale, while if X_n is a supermartingale, so is Min $\{X_n, K\}$.*

Exercise 8.21 *Let $X_1, X_2, \cdots$ be i.i.d random variables each having normal distribution with zero mean and unit variance. Show that the sequence $Y_n = \exp\left(\left(\sum_{i=1}^{n} X_i\right) - \frac{1}{2}n\right)$ forms a martingale.*

Exercise 8.22 *Prove that $\{W^2(t) - t, t \geq 0\}$ is a martingale, where $\{W(t), t \geq 0\}$ is a Brownian motion.*

Exercise 8.23 *Let $\{W(t), t \geq 0\}$ be a Wiener process. Is $\exp(\sigma W(t) - \frac{\sigma^2}{2}t)$ a martingale where σ is a positive constant?*

Exercise 8.24 *Let $\{N(t),\ t \geq 0\}$ be a Poisson process with parameter 1. Which of the following are martingales.*

(i) $\{N(t) - t,\ t \geq 0\}$.
(ii) $\{N(t)^2 - t,\ t \geq 0\}$.
(iii) $\{(N(t) - t)^2 - t,\ t \geq 0\}$.

Exercise 8.25 *Let X be a integrable random variable and $\mathcal{G}_1$ and $\mathcal{G}_2$ be two sub-σ-fields of $\mathcal{F}$. Prove that, if $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $E(E(X/\mathcal{G}_1)/\mathcal{G}_2) = E(E(X/\mathcal{G}_2)/\mathcal{G}_1) = E(X/\mathcal{G}_1)$.*

Exercise 8.26 *Let $\{\mathcal{F}_n, n = 1, 2, \ldots\}$ be a filtration. Let X be any random variable with $E(| X |) < \infty$. Define $X_n = E(X/\mathcal{F}_n), n = 1, 2, \ldots$ Prove that, $\{X_n, (n = 1, 2, \ldots)\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n,\ n = 1, 2, \ldots\}$.*

Exercise 8.27 *Let $\{Y_n, (n = 0, 1, \ldots)\}$ be an arbitrary sequence of random variables. Suppose X is a random variable with $E(| X |) < \infty$. Define*

$$X_n = E(X/Y_0, Y_1, \ldots, Y_n),\ (n = 0, 1, \ldots)\ .$$