Portfolio Optimization

$(k_1, k_2 \ldots k_n)$
$(a_1, a_2 \ldots a_n)$

$(w_1, w_2 \ldots w_n)$

$$V_p = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

$$w_i = \frac{a_j k_j}{V_p} = \frac{a_j k_j}{\sum a_i k_i}$$

$$w_1 + w_2 \ldots + w_n = 1$$

* A portfolio is collection of 2 or more assets $a_1, a_2, \ldots a_n$ represented by an ordered n-tuple vector. $\theta = (a_1, a_2 \ldots a_n)$ or it can be represented as $\theta = (k_1, k_2 \ldots k_n)$ [no. of assets] can also be represented by $(w_1, w_2, \ldots w_n)$

Return: $$r_V = \frac{V_p(T) - V_p(0)}{V_p(0)} = \frac{\sum k_i (a_{iT} - a_{i0})}{V_p(0)}$$

$$w_i = \frac{k_i V_i(0)}{V_p(0)}$$

mean of the portfolio return: let $(w_1, w_2 \ldots w_n)$ be a portfolio of n assets $(a_1, a_2 \ldots a_n)$ r_i $i = 1$ to n be the return on ith asset a_i & exp. val $E(r_i) = \mu_i$ $i = 1, 2 \ldots n$ be the exp. val

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then mean of the portfolio return is defined as $\mu = E\left(\sum_{i=1}^{n} w_i r_i\right)$

$$= \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i$$

Total return on the portfolio $R_i = \sum_i w_i r_i$

Variance of the portfolio return

$$Var(R_i) = Var\left(\sum_{i=1}^{N} w_i r_i\right)$$

$$= \sum_i \sum_j w_i w_j \sigma_{ij}$$

σ_{ij} represents cov of ~~R_i, R_j~~ r_i, r_j

σ_i^2, σ_j^2

$$\rho_{ij} = \frac{cov(r_i, r_j)}{\sqrt{var\, r_i}\sqrt{var\, r_j}} = \frac{cov(r_i, r_j)}{\sigma_i \sigma_j}$$

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$$

$$Var(P_i) = \sum\sum w_i w_j \sigma_i \sigma_j \rho_{ij}$$

The portfolio optimization refers (i) to minimising the risk: $\min \sum\sum w_i w_j \sigma_{ij}$

and (ii) maximise the return: $\max \sum w_i \mu_i$

subject to the constraint $\sum w_i = 1$

2-Asset Portfolio Optimization

A portfolio is consisting of 2 assets with w_1, w_2 return r_1, r_2, SD σ_1, σ_2

Then portfolio return μ & variance σ^2 is given as

$$\mu = E(w_1 r_1 + w_2 r_2)$$
$$= w_1 \mu_1 + w_2 \mu_2$$

$$\sigma^2 = Var(w_1 r_1 + w_2 r_2)$$
$$= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12}$$

where ρ is coeff. of correlation b/w $r_1 r_2$ lying in the closed intvl $[-1, 1]$

Since $w_1 + w_2 = 1$ & since short selling is allowed. let $w_2 = s \Rightarrow w_1 = 1 - s$

$$\mu = (1-s)\mu_1 + s\mu_2 \quad \text{---(1)}$$
$$\sigma^2 = (1-s)^2 \sigma_1^2 + s^2 \sigma_2^2 + 2s(1-s)\sigma_1 \sigma_2 \rho \quad \text{---(2)}$$
$$= s^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) + 2s(\sigma_1\sigma_2\rho - \sigma_1^2) + \sigma_1^2$$

(i) $\rho = 1$

$$\mu = (1-s)\mu_1 + s\mu_2$$
$$\sigma^2 = (1-s)^2 \sigma_1^2 + s^2 \sigma_2^2 + 2s(1-s)\sigma_1 \sigma_2$$
$$\sigma^2 = \{(1-s)\sigma_1 + s\sigma_2\}^2$$

$$\sigma = \{(1-s)\sigma_1 \pm s\sigma_2\}$$
$$s \in [0,1] \qquad 0 < \sigma_1 < \sigma_2$$

$$\tilde{\sigma} = (1-s)\sigma_1 + s\sigma_2 \qquad \rho = 1$$
$$\rho = 1$$
$$\tilde{\sigma} = (1-s)\sigma_1 - s\sigma_2 \qquad \rho = -1$$

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$$\frac{d\sigma^2}{ds} = 0 \qquad \text{for } \rho = 1$$

$$\Rightarrow 2(\sigma_2 - \sigma_1)\{(1-s)\sigma_1 + s\sigma_2\}$$

$$(1-s)\sigma_1 + s\sigma_2 = 0$$

$$\Rightarrow s = \frac{-\sigma_1}{\sigma_2 - \sigma_1} < 0$$

$$1 - s = \frac{\sigma_2}{\sigma_2 - \sigma_1}$$

$$\sigma^2_{min} = 0$$

$$\mu = \frac{\sigma_2 \mu_1 - \sigma_1 \mu_2}{\sigma_2 - \sigma_1}$$

different use for $\rho = -1$

$-1 < \rho < 1$

$$\mu = (1-s)\sigma_1 + s\sigma_2$$

$$\sigma^2 = s^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) - 2s(\sigma_1^2 - \sigma_1\sigma_2\rho) + \sigma_1^2$$

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow s = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

Case-I:

$$-1 \le \rho < \sigma_1/\sigma_2$$

$$s_{min} = \frac{\sigma_1(\sigma_1 - \sigma_2\rho)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho} > 0$$

$$1 - s = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

$$\mu_{min} = (\mu_2 - \mu_1)s_{min} + \mu_1$$

$$\sigma^2_{min} = \frac{(\sigma_1^2 - \sigma_1\sigma_2\rho)^2 - 2(\sigma_1^2 - \sigma_1\sigma_2\rho)^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

$$\sigma^2_{min} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

Case-II:

$$\rho = \sigma_1/\sigma_2$$

$$s_{min} = 0$$

$$1 - s = 1$$

$$\mu_{min} = \mu_1$$

$$\sigma^2_{min} = \sigma_1^2$$

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Case III

$$1 \geq \rho > \sigma_1 / \sigma_2$$

$$\varepsilon_{min} = \frac{\sigma_1(\sigma_1 - \rho\sigma_2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho} < 0$$

$$\mu_{min} = (\mu_2 - \mu_1)\,\varepsilon_{min} + \mu_1$$

$$\sigma^2_{min} = \frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

$$= 0 \quad \text{when } \rho = 1$$

Q: $S_1(0) = \$10$ $\quad S_2(0) = \$20$

$x_1 = 50$ $\quad x_2 = 25$

Calc. $w_1\, w_2$ $(50, 25)$

$$V_p(0) = x_1 S_1 + x_2 S_2$$
$$= 1000$$

Value of asset 1 $= x_i V_i(0)$

$= 50 \times 10 = 500$

2 $= x_2 \times V_2(0) = 500$

$$w_1 = 1/2, \quad w_2 = 1/2$$

Then $S_1(1) = 12$ rs

$S_2(1) = 22$ rs

$$W_1 = \frac{12 \times 50}{12 \times 50 + 22 \times 25} = \frac{600}{1150}$$

$$W_2 = \frac{22 \times 25}{12 \times 50 + 22 \times 25} = \frac{550}{1150}$$

Q. compute V_1 of portfolio worth \$100 initially
$w_1 = 25\%$: $w_2 = 75\%$
Given $S_1(0) = \$45$ $S_2(0) = 33\$$
$S_1(1) = \$48$ $S_2(1) = 32\$$

$$45x + 33y = 100$$

$$\left(\frac{45x}{45x+33y}\right)0.25 + \left(\frac{33y}{45x+33y}\right)0.75 = 1$$

$$\frac{45x}{4} + \frac{99y}{4} = 45x + 33y$$

$$\frac{-135x}{4} = \frac{33y}{4}$$

$$x = \frac{33}{-135}y$$

$$45\left(\frac{-33}{135}\right)y + 33y = 100 \Rightarrow 22y = 100$$

$$y = \frac{50}{11}$$

$$x = \frac{50}{11} \times \frac{33}{-135} = \frac{-10}{9}$$

Q. Compute wt. in a pf
Exp Ret. $= 0.2 = \mu$
Given that

	Prob.	K_1	K_2
w_1	0.1	-10%	10%
w_2	0.5	0	20%
w_3	0.4	20%	30%

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$$E(K_V) = W_1\mu_1 + W_2\mu_2$$
$$\mu_1 = E(K_1)$$
$$\mu_2 = E(K_2)$$

$$E(K_1) = -0.1 \times 0.1 + 0 \times 0.5 + 0.4 \times \frac{20}{100}$$

$$= 0.07$$

$$E(K_2) = 0.1 \times 0.1 + 0.2 \times 0.5 + 0.4 \times 0.3$$
$$= 0.23$$

$$0.2 = W_1(0.07) + W_2(0.23)$$
$$W_1 + W_2 = 1$$

Th. The variance σ_V^2 of a portfolio cannot exceed the greater of the variance σ_1^2 & σ_2^2 of the components if short selling is not allowed

$$\sigma^2 \leq \max(\sigma_1^2, \sigma_2^2)$$

Proof: let us assume that $\sigma_1^2 \leq \sigma_2^2$

since SS is not allowed $W_1, W_2 > 0$

$$W_1\sigma_1 + W_2\sigma_2 \leq (W_1 + W_2)\sigma_2$$
$$\leq \sigma_2$$

$$-1 \leq \rho \leq 1$$

$$\sigma^2 = W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + 2\sigma_1\sigma_2 W_1 W_2 \rho$$
$$\leq (W_1\sigma_1 + W_2\sigma_2)^2$$
$$\leq \sigma_2^2$$

thus $\sigma^2 \leq \max(\sigma_1^2, \sigma_2^2)$

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Q:

	Prob.	K_1	K_2
W_1	0.4	−10%	20%
W_2	0.2	0	20%
W_3	0.4	20%	16%

(1) $w_1 = 40\%$ $\quad w_2 = 60\%$

$\sigma_1^2 = 0.0184 \quad \sigma_2^2 = 0.0024 \quad \rho = -0.96306$

$\sigma_V^2 = 0.000736$

(2) $w_1 = 80\%$ $\quad w_2 = 20\%$

$\sigma_1^2 = 0.009824$

(3) $w_1 = -50\%$ $\quad w_2 = 150\%$

$\sigma_V^2 = 0.0196$

(4) $w_1 = 150\%$ $\quad w_2 = -50\%$

$\sigma_V^2 = 0.0516$

Multi Asset Portfolio Optimization

$(a_1, a_2 \dots a_n) \rightarrow$ assets

weights of these assets are in an n-tuple vector

$w^T = (w_1, w_2 \dots w_n) \in \mathbb{R}^n$

let $\quad e^T = (1, 1 \dots 1)$

$$w_1 + w_2 + \dots + w_n = 1$$

$$e^T w = 1$$

$m^T = (\mu_1, \mu_2 \dots \mu_n)$ be the exp. return vector of the portfolio where

$\mu_i = E(r_i)$, $i = 1, 2, \dots n$

$C = [c_{ij}] \quad i, j = 1 \dots n$

c_{ij} represents the covariance of r_i & r_j
$\rightsquigarrow n \times n$ matrix, symmetric $c_{ij} = c_{ji}$

$$\mu = w_1\mu_1 + w_2\mu_2 + \dots + w_n\mu_n$$
$$= m^T w$$

$$\sigma^2 = \sum_i^n \sum_j^n w_i w_j \sigma_{ij}$$
$$= w^T C w$$

min σ^2 or $w^T C w$ subject to $e^T w = 1$

feasible region of the portfolio problem

$w_1 = 0.4$, $w_2 = -0.2$, $w_3 = 0.8$
$\mu_1 = 0.08$ $\mu_2 = 0.1$ $\mu_3 = 0.06$
$\sigma_1 = 1.5$ $\sigma_2 = 0.5$ $\sigma_3 = 1.2$
$\rho_{12} = 0.3$ $\rho_{23} = 0$ $\rho_{31} = -0.2$

$$\sigma^2 = w^T C w$$
$$w^T C = \begin{bmatrix} 0.4 & -0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 2.25 & 0.225 & -0.36 \\ 0.225 & 0.5 & 0 \\ -0.35 & 0 & 1.44 \end{bmatrix}$$

let $W = \{w \in \mathbb{R}^n : e^T w = 1\}$
be the collection of all portfolio. Each portfolio correspond to a point in (σ, μ) plane. Then the set $\{(\sigma^w, \mu^w), w \in W\}$ is called the feasible region.

Consider the n-dimensional plane $e^T w = 1$ in which the weight vector lies. Let f be the mapping that takes each weight vector to a point in (σ, μ) plane

our aim is to find img. of any straight line of the weight hyperplane under

The parametric equation of any line in the weight hyperplane is of the form

$$l(\xi_1) = (s_1\xi_1 + b_1,\ s_2\xi_1 + b_2,\ \dots + s_n\xi_1 + b_n)^T$$
$$= \xi_1 s + b$$
$$s = (s_1, s_2 \dots s_n)^T,\quad b = (b_1, b_2 \dots b_n)$$

let w be any point on this line

then $\mu = m^T w$
$$\mu = m^T(\xi_1 s + b)$$
$$\mu = \xi_1 (m^T s) + m^T b$$

let $(m^T s)^{-1} = \alpha$, $-(m^T b)(m^T s)^{-1} = \beta$

$$\xi_1 = \mu\alpha + \beta$$

$$\sigma^2 = w^T C w$$
$$= (\xi_1 s + b)^T C (\xi_1 s + b)$$
$$= \xi_1^2 (s^T C s) + \xi_1 (s^T C b + b^T C s) + b^T C b$$
$$\sigma^2 = \gamma \xi_1^2 + \delta \xi_1 + \eta$$
$$\sigma^2 = \gamma(\mu\alpha + \beta)^2 + \delta(\mu\alpha + \beta) + \eta \qquad -\infty < \xi_1 < \infty$$

$\sigma = \sqrt{(\quad)} \rightarrow$ generates Markovitz curve

all w will generate a solid known as Markovitz bullet

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Q. Consider a portfolio of 2 assets a_1 a_2 with the following : $\mu_1 = 5\%$. $\mu_2 = 10\%$.

$\sigma_1 = 10\%$ $\sigma_2 = 40\%$ $\rho = -0.5$

draw the risk return curve

A. $\mu = s\mu_1 + (1-s)\mu_2$

$\mu = \frac{5s}{100} + (1-s)\frac{10}{100} = 0.1 - 0.05s = \mu$

$s = 2 - 20\mu \Leftarrow s = \frac{0.1-\mu}{0.05}$

$\sigma^2 = s^2(0.01) + (1-s)^2(0.16) + \cancel{2}s(1-s)\left(\frac{-1}{\cancel{2}}\right)(0.1)(0.4)$

$\sigma^2 = s^2(0.01) + 0.16 + s^2(0.16) - 2s(0.16)$
$+ s^2(0.04) - s(0.04)$

$\sigma^2 = s^2(0.21) - s(0.36) + 0.16$

$\sigma^2 = (2-20\mu)^2(0.21) - (2-20\mu)(0.36) + 0.16$

$\sigma = \sqrt{(0.084)^2 + 84(\mu - 0.057)^2}$

$\mu = 0.057 \rightarrow \sigma = 0.084$

minimum value of $\sigma = 0.084$

$\sigma^2 = W^T C W$

$e^T W = 1$

$W = \frac{C^{-1}e}{e^T C^{-1} e}$

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Proof: $L(\lambda, \omega) = \omega^T C \omega + \lambda(1 - e^T\omega)$

$$2\omega^T C - \lambda e^T = 0$$

$$\omega^T C = \frac{\lambda e^T}{2}$$

$$\omega = \frac{\lambda C^{-1} e}{2}$$

$$e^{-1} C^{-1} e = 1$$

$$\frac{\lambda}{2} = \frac{1}{e^T C^{-1} e}$$

Q. $\mu_1 = 0.06$ $\mu_2 = 0.1$ $\mu_3 = 0.15$

$\sigma_1 = 1$ $\sigma_2 = 1.5$ $\sigma_3 = 2$

$\rho_{12} = 0 = \rho_{23} = \rho_{13}$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.25 & 0 \\ 0 & 0 & \cancel{0.25}\ 4 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.444 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$e^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\omega = \cancel{1.69}\ C^{-1}e / 1.69$$

$$\omega = C^{-1}e / 1.69 = \begin{bmatrix} 0.59 \\ 0.26 \\ 0.15 \end{bmatrix}$$

$$\sigma^2 = \omega^T C \omega = 0.59$$

$$\mu = m^T \omega = \sum \mu_i w_i$$

$$\rightarrow = \begin{bmatrix} 0.06 & 0.1 & 0.15 \end{bmatrix} \begin{bmatrix} \ \end{bmatrix}$$

$$= 0.0839$$

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Th: for a given expected return μ, the minimum risk has weights given by

$$W = \det\begin{pmatrix} \mu & m^{-1}C^{-1}e \\ 1 & e^TC^{-1}e \end{pmatrix} C^{-1}m + \det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1 \end{pmatrix} C^{-1}e$$

$$\det\begin{pmatrix} m^TC^{-1}e & m^TC^{-1}m \\ e^TC^{-1}e & e^TC^{-1}m \end{pmatrix}$$

$\sigma^2 = \frac{1}{2} w^TCw$

$m^Tw = \mu$

$e^Tw = 1$

Proof: $L(w, \alpha, \beta) = w^TCw + \alpha(\mu - m^Tw) + \beta(1 - e^Tw)$

$$= w^TC\cancel{w} - \alpha m^T - \beta e^T = 0$$

$$w^TC = \alpha m^T + \beta e^T$$

$$w = \alpha C^{-1}m + \beta C^{-1}e$$

$$m^Tw = \alpha(m^TC^{-1}m) + \beta(m^TC^{-1}e) = \mu$$

$$e^Tw = \alpha(e^TC^{-1}m) + \beta(e^TC^{-1}e) = 1$$

Using Cramer's rule

$$\alpha = \frac{\det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}e \end{pmatrix}}{\det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}} \longrightarrow (\quad)$$

$$\beta = \frac{\det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1 \end{pmatrix}}{}$$

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Capital Asset Pricing Model

Consider a portfolio with n-risky assets $a_1, a_2 \ldots a_n, a_{rf}$ with weights $w_1 w_2 \ldots w_n\ w_{rf}$

$$W_{risky} + W_{rf} = 1 = \sum_{i=1}^{n} w_i + w_{rf} \quad -(1)$$

Exp return & variance

$$\mu = \sum w_i \mu_i + w_{rf} \mu_{rf}$$

return of risky asset, return of risk free asset

$$= \mu_{risky} + w_{rf} \mu_{rf} \quad -(2)$$

$$\sigma^2 = Var(\sum w_i r_i + w_{rf} \mu_{rf})$$

$$= Var(\sum w_i r_i) = \sigma^2_{risky} \quad -(3)$$

If we remove the risk free asset & adjust the risky weights so that their sum is 1 the portfolio so created is known as derived risky portfolio. $\mu_{der}, \sigma^2_{der}$

$$\mu = w_{risky} \sum_{i=1}^{n} \frac{w_i \mu_i}{W_{risky}} + w_{rf} \mu_{rf}$$

$$\mu = W_{risky} \mu_{der} + w_{rf} \mu_{rf}$$

$$\mu = W_{risky} (\mu_{der} - \mu_{rf}) + \mu_{rf} \quad -(4)$$

$$\sigma^2 = Var\left(w_{risky} \sum_{i=1}^{n} \frac{w_i r_i}{W_{risky}} + w_{rf} \mu_{rf} \right)$$

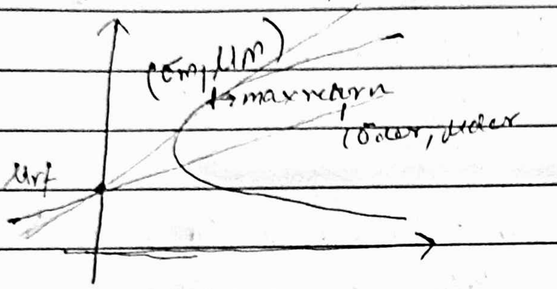
$$= W^2_{risky} Var\left(\sum_{i=1}^{n} \frac{w_i r_i}{W_{risky}} \right)$$

$$= w^2_{risky} \sigma^2_{der}$$

$$w_{risky}^2 = \frac{\sigma^2}{\sigma_{der}^2} \quad \text{— (5)}$$

$$\mu = \mu_{rf} + (\mu_{der} - \mu_{rf}) \cdot \frac{\sigma}{\sigma_{der}} \quad \text{— (6)}$$

$(0, \mu_{rf})$, $(\sigma_{der}, \mu_{der})$



Eq^n (6) represents a line in the σ-μ plane passing through 2 points $(0, \mu_{rf})$ & $(\sigma_{der}, \mu_{der})$. for a given risk σ if we choose various weights combination of risk free & risky assets for which (1) holds This generates different lines represented by equation (6) in σ-μ plane. The line that produces the highest exp return for a given risk is tangent to the upper portion of the Markowitz Bullet & is called the capital market line. Point of contact is called market portfolio.

Q. A portfolio comp. rf asset. $\mu_{rf} = 0.5$
3 mutt indp risk $\mu_1 = 1$ $\mu_2 = 2$ $\mu_3 = 3$
Var $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$
determine CML

A. $C = I$ $C^{-1} = I$ $e^T = (1,1,1)$ $m^T = (1,2,3)$

$$W_m = \frac{C^{-1}(m - \mu_{rf} e)}{e^T C^{-1}(m - \mu_{rf})} = \frac{\begin{bmatrix} 0.5 \\ 1.5 \\ 2.5 \end{bmatrix}}{4.5}$$

$\sigma_m = (w^T C w)^{1/2} = \sqrt{35}/9$

$W_m = \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right)$ $m^T w_m = 22/9$

$$\mu = \frac{1}{2} + \left(\frac{\mu_m - 1/2}{\sigma_m}\right)\sigma$$

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Th. for any expected risk free return μ_{rf} the weight vector w_m of the market portfolio is given by

$$w_m = \frac{C^{-1}(m - \mu_{rf}e)}{e^T C^{-1}(m - \mu_{rf}e)}$$

Proof: for any $\sigma - \mu$ in the Markovitz Bullet, the slope of the line joining $(0, \mu_{rf})$ & (σ, μ) will be given as

$$\frac{\mu - \mu_{rf}}{\sigma} = \frac{\sum_{i=1}^{n} m_i\mu_i - \mu_{rf}}{(\sum\sum w_i w_j c_{ij})^{1/2}}$$

for the line joining $(0, \mu_{rf})$ & (σ, μ) to be a tangent line to the markovitz bullet, we have to solve

$$\max \frac{m^T w - \mu_{rf}}{(w^T C w)^{1/2}}$$

subject to the condition $e^T w = 1$

$$L(\lambda, \mu) = \frac{m^T w - \mu_{rf}}{(w^T C w)^{1/2}} + \lambda(1 - e^T w)$$

$$\frac{\partial L}{\partial w} = \frac{1}{w^T C w}\left[(w^T C w)^{1/2} m - (m^T w - \mu_{rf})(w^T C w)^{-1/2} Cw\right] - \lambda e = 0$$

$$\sigma m - (\overset{m^T w}{\mu} - \mu_{rf})\frac{Cw}{\sigma} = \lambda\sigma^2 e \quad \text{—(2)}$$

~~On~~ Premultiply by w^T ; $w^T m = \mu$, $w^T e = 1$

$w^T C w = \sigma^2$

$$\cancel{\sigma}\mu - (\mu - \mu_{rf})\frac{\cancel{\sigma^2}}{\cancel{\sigma}} = \lambda\sigma^2$$

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$$\mu - (\mu - \mu_{rf}) = \lambda\sigma$$

$$\lambda = \frac{\mu_{rf}}{\sigma} \quad -(3)$$

Substituting this value of λ in eqⁿ 2 & 3 we can get the requisite expression

Note: If the market portfolio σ_m, μ_m is known, then the equation of Capital Market Line

$$\mu = \mu_{rf} + \left(\frac{\mu_{der} - \mu_{rf}}{\sigma_{der}}\right)\sigma$$

becomes $\mu = \mu_{rf} + \left(\frac{\mu_m - \mu_{rf}}{\sigma_m}\right)\sigma$

if the investor is willing to take a positive risk there will be an additional return

suppose an investor is ready to take risk σ_p then for this risk, the expected return μ_p is maximum if the point (σ_p, μ_p) lies on the CML

$$\mu_p = \mu_{rf} + \left(\frac{\mu_m - \mu_{rf}}{\sigma_m}\right)\sigma_p$$

$$w_p = \sigma_p / \sigma_m$$

if we let $w_p = \sigma_p/\sigma_m$, $\mu_p = (1 - w_p)\mu_{rf} + w_p \mu_m$

this means investor should invest $w_p = \sigma_p/\sigma_m$ proportion of investment in index or stock or risky asset & $(1 - w_p)$ proportion in risk free.

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Th: Suppose the market portfolio (σ_M, μ_M). The expected return of an asset a_i is given by:

$$\mu_i = \mu_{rf} + \beta_i(\mu_M - \mu_{rf}) \quad \text{where } \beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2}$$

Proof: Suppose an investor portfolio comprises of asset a_i with weight w & market portfolio M with weight $1-w$.

Return : $\mu = W \cdot \mu_i + (1-W)\mu_M$

Risk : $\sigma^2 = W^2\sigma_1^2 + (1-W)^2\sigma_M^2 + 2W(1-W)\sigma_i\sigma_M \rho_{iM}$

where $\rho = \rho_{im}$ is the coefficient of correlation between return on asset a_i & market portfolio M. as W varies these values trace out a curve in the σ-μ plane.

If $W = 0$, CML will become tangential at the point M. So, we have the condition that slope of the curve is the slope of the CML at m.

$$\left.\frac{d\mu}{d\sigma}\right|_{W=0} = \left.\frac{d\mu}{dw}\cdot\frac{dw}{d\sigma}\right|_{W=0} \qquad \frac{d\sigma}{dw} = \frac{W\sigma_1^2 - (1-W)\sigma_M^2 + (1-2W)\sigma_{ij}}{\sigma}$$

$$\left.\frac{d\mu}{d\sigma}\right|_{W=0} = \frac{(\mu_i - \mu_m)\cdot\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\mu_M - \mu_{rf}}{\sigma_M}$$

$$(\mu_i) = \mu_M + \frac{(\mu_M - \mu_{rf})(\sigma_{iM} - \sigma_M^2)}{\sigma^2_M}$$

$$\mu_i = \cancel{\mu_M} + (\mu_M - \mu_{rf})\frac{\sigma_{iM}}{\sigma_M^2} - \cancel{(\mu_M)} + \mu_{rf}$$

$$\mu_i = \mu_{rf} + (\mu_M - \mu_{rf})\frac{\sigma_{iM}}{\sigma^2_M}$$

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$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$ is called the Beta of the asset

Case - I:

$\beta_i = 0$

asset is completely uncorrelated with the market

thus CAPM gives $\mu_i = \mu_{rf}$ showing that however large is the risk σ_i, return will always be limited to the risk free.

Case - II:

$\beta_i < 0$

$\mu_i < \mu_{rf}$

correlated negatively with the market

hence can be used to reduce the overall risk of the portfolio when other assets are not doing well. for this reason, it is called insurance

for a portfolio, appropriate measure of risk is σ but for individual assets, the proper measure of its risk is β.

The overall β of the portfolio is defined as

$$\beta = \sum_{i=1}^{n} w_i \beta_i$$

Security Market line:

A linear eqn: $\mu = \mu_{rf} + \beta(\mu_m - \mu_{rf})$ where $\beta = \frac{cov(r, r_M)}{\sigma_M^2}$ that describes the expected return of all assets is SML.

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Q. $\mu_{rf} = 8\% = 0.08$

$\mu_m = 12\% = 0.12$

$\sigma_m = 15\% = 0.15$

$0.045 \rightarrow$ cov with market

exp ror?

A. $\mu_i = \mu_{rf} + \beta_i(\mu_m - \mu_{rf})$

$$\mu_i = 0.08 + 0.04 \times \frac{0.045}{0.15 \times 0.15}$$

$$= 0.08 + \frac{\cancel{4}^{2} \times \cancel{45}\ \cancel{3}}{\cancel{15}\ \cancel{150}\ \cancel{50}\ 25}$$

$$= \frac{8}{100} + \frac{2 \times 4}{25 \times 4}$$

$$\mu_i = \frac{16}{100} = 0.16 \Rightarrow 16\%$$