Stochastic Process

σ-field / σ-Algebra (F)

Family of subsets of Ω (sample space) which satisfies:

(i) $\phi \in F$

(ii) If $A \in F$, then $A^c \in F$.

(iii) If a countable sequence of sets $A_1, A_2 \ldots$ is in F, then $\bigcup_i A_i \in F$.

Smallest σ-field on Ω is $\{\phi, \Omega\}$; largest σ-field on Ω is the power set of Ω.

These subset of Ω which form the σ-field are called <u>F-events / F-Measurable sets</u>

Probability Measure (P)

let F be a σ-field over Ω. let P be a real valued function defined on F such that:

(i) $P(A) \geq 0, \forall A \in F,$

(ii) $P(\Omega) = 1,$

(iii) If $A_1, A_2, A_3 \ldots$ are mutually exclusive, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Then, P is called a Probability Measure. The triplet <u>(Ω, F, P)</u> is called a <u>Probability Space</u>.

Stochastic Process

let (Ω, F, P) be a probability space. A collection of random variables $\{x(t), t \in T\}$ defined on the probability space is called a stochastic process.

Given a probability space (Ω, F, P), the S.P. $\{X(t), t \in T\}$ can be identified as a real-valued function $X: T \times \Omega \to R$ of two independent variables $t \in T$, $\omega \in \Omega$ such that $X^{-1}(t)\{(-\infty, x]\}$ belongs to F for every $t \in T$. Here

$$X^{-1}(t)\{(-\infty, x]\} = \{\omega \in \Omega : X(t,\omega) \in (-\infty, x]\}$$

In this sense the S.P. $\{X(t), t \in T\}$ can be viewed as $\{X(t,\omega) : t \in T, \omega \in \Omega\}$.

$X(t,\omega)$

for fixed ω

$X(\cdot, \omega) : T \to R$

trajectory of S.P.

(not a random variable)

for fixed t

$X(t, \cdot) : \Omega \to R$

(a random variable)

Strict Sense Stationary Stochastic Process (Strong Stationary S.P.)

The S.P. is S.S.S. if for any $n \geq 1$, and for $0 < t_1 < t_2 \ldots < t_n$, the finite dimensional random vectors $(X(t_1), X(t_2), X(t_3), \ldots, X(t_n))$ and $(X(t_1+h), X(t_2+h), X(t_3+h), \ldots, X(t_n+h))$ have the same joint dist^n for $\forall\ h > 0$.

Wide Sense Stationary Stochastic Process (Weak Stationary S.P.)

The S.P. $\{X(t), t \geq 0\}$ is a wide sense S.P. if it follows the following:

(i) $E(X(t))$ is independent of t.

(ii) $cov(X(t), X(s))$ depends only on the time difference $|t-s|$ $\forall\ t, s \in T$.

(iii) $E((X(t))^2) < \infty$ (should be finite)

second order moment

For example,

consider the process $X(t) = A\cos(\theta t) + B\sin(\theta t)$, where A & B are uncorrelated random variables with mean 0 and variance 1, and θ is a positive constant. Is $\{X(t), t \geq 0\}$ a wide sense stationary process?

Sol:

(i) $E(X(t)) = E(A\cos\theta t) + E(B\sin\theta t)$
$= (\cos\theta t)\, E(A) + (\sin\theta t) \cdot E(B)$
$= 0 \quad (\because E(A) = E(B) = 0)$. $\Rightarrow$ independent of t.

$\therefore$ (i) is satisfied.

(ii) $\text{cov}(X(t), X(s))$

$\boxed{\text{NOTE}}$: $\text{cov}(X,Y) = E[XY] - E[X].E[Y]$

$\therefore \text{cov}(X(t), X(s)) = E(X(t).X(s)) - E(X(t)).E(X(s))$
$= E(X(t).X(s)) \quad (\because E(X(t)) = E(X(s)) = 0)$.

$E(X(t).X(s)) = (\cos\theta t)(\cos\theta s).E(A^2) + (\sin\theta t)(\sin\theta s).E(B^2) +$
$(\cos\theta t.\sin\theta s + \sin\theta t.\cos\theta s)\, E(AB)$
$= \cos\theta(t-s)$

$(\because E(A^2) = \text{var}(A) + (E(A))^2 \Rightarrow E(A^2) = 1$. Similarly, $E(B^2) = 1$.

Also, $E(AB) = \underbrace{\text{cov}(A,B)}_{0} + \underbrace{E(A).E(B)}_{0} = 0)$.
0 (as they are uncorrelated)

$\therefore$ (ii) is satisfied.

(iii) $E((X(t))^2) = \text{var}(X(t)) + (\underbrace{E(X(t))}_{0})^2$

(x, y, are constants) $\leftarrow$ $\boxed{\text{NOTE}}$: $\text{var}(x_1X + y_1Y) = x_1^2\,\text{var}(X) + y_1^2.\text{var}(Y) + 2x_1y_1\text{cov}(X,Y)$

$\therefore E((X(t))^2) = \text{var}(A\cos\theta t + B\sin\theta t) = \cos^2\theta t.\,\text{var}(A) + \sin^2\theta t.\,\text{var}(B)$
$+ 2\cos\theta t\sin\theta t\, \underbrace{\text{cov}(A,B)}_{0}$
$= \cos^2\theta t + \sin^2\theta t \quad (\because \text{var}(A) = \text{var}(B) = 1)$
$= 1$.

$\therefore$ (iii) is satisfied $\quad \therefore$ S.P. is a wide sense stationary S.P.

Independent Increments

If for all n and for $t_1 < t_2 < \dots t_n$, $x(t_2) - x(t_1)$, $x(t_3) - x(t_2) \dots$ $x(t_n) - x(t_{n-1})$ are independent random variables, then the process is said to have Independent Increments.

Markov Property

A given S.P. is said to have Markov Property if for all n and for all $0 < t_1 < t_2 \dots < t_n < t$, the conditional CDF (Cumulative Dist^n Func.) satisfies:

$$P(x(t) \le x \mid x(0) = x_0, x(t_1) = x_1, \dots x(t_n) = x_n)$$
$$= P(x(t) \le x \mid x(t_n) = x_n).$$

That is,

if the future prediction depends only on the current state of the stochastic process and doesn't depend on the past information, then it has the Markov property.

A Markov Process is a S.P. with property that, given the value of $X(s)$, the values of $x(t)$, $t > s$, do not depend on the values of $X(u)$, $u < s$, i.e.

- "the probability of any particular future behaviour of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behaviour."

Brownian Motion (Wiener Process)

A S.P. $\{W(t), t \ge 0\}$ is said to be a B.M. if it satisfies the following properties:

(i) $W(0) = 0$; starts from 0.

(ii) for $t > 0$, sample path of $W(t)$ is continuous.

(iii) $W(t), t \ge 0$, has independent and stationary increments.

(iv) for $0 \le s < t < \infty$, $W(t) - W(s)$ is a normally distributed random variable with mean 0 and variance $t-s$

NOTE: • The path followed by a Brownian Process is always continuous but nowhere differentiable.

- weiner process is not wide sense stationary S.P. as $Cov[W(t), W(S)] = \min\{s, t\}$
- Wiener process is a Markov Process because given $W(t)$, the future $W(t+h)$ for any $h>0$ only depends on the increment $W(t+h) - W(t)$, and is independent of the past.
- $E(W(t)) = 0$ because for $t = t$, $s = 0$, $E[W(t-s)] = 0$ and $var(W(t-s)) = t-s$ $(0 \le s < t < \infty)$.

Brownian Motion with drift μ and volatility σ (Generalized Brownian Mtn.)

A S.P. $\{X(t), t \ge 0\}$ is said to be a B.M. with drift 'μ' and volatility 'σ' if

$$X(t) = \mu t + \sigma W(t) \quad \text{where}$$

(i) $W(t)$ is a standard B.M.

(ii) $-\infty < \mu < \infty$ is a constant.

(iii) $\sigma > 0$ is a constant.

- $E[W(t)] = E[\mu t + \sigma W(t)] = \mu t + \sigma \underbrace{E(W(t))}_{0 \text{ (W(t) is a standard B.M.)}} = \boxed{\mu t}$

- $cov\{X(t), X(S)\} = \sigma^2 cov\{W(S), W(t)\} = \boxed{\sigma^2 \min\{s, t\}}$, $s, t \ge 0$

[$cov[X(t), X(S)] = E[X(t). X(S)] - E[X(t)]. E[X(S)]$

$= E[(\mu t + \sigma W(t))(\mu S + \sigma W(S))] - E[\mu t + \sigma W(t)] E[\mu S + \sigma W(S)]$

$= E[\mu^2 st + \mu t \sigma W(S) + \sigma^2 W(t) W(S) + \sigma \mu S W(S)] - \cancel{E[\mu^2 t S +} \mu^2 st$

$= \mu^2 st + E[\sigma^2 W(t) W(S)] - \mu^2 st = \sigma^2 E[W(t). W(S)]$

$= \sigma^2 \min\{s, t\}$

]

Geometric Brownian Motion

A s.p $\{x(t), t \geq 0\}$ is said to be a GBM if

$$\boxed{x(t) = x(0).\, e^{W(t)}}, \text{ where}$$

$W(t)$ is a standard B.M. ~~for~~

- For any $h > 0$, we have

$$x(t+h) = x(0) e^{W(t+h)}$$
$$= x(0) e^{W(t) + W(t+h) - W(t)}$$
$$= x(t).\, e^{W(t+h) - W(t)}$$

Hence, this B.M. has independent increments. Also, given $x(t)$, the future $x(t+h)$ only depends upon the future increment $(W(t+h) - W(t))$ and not on the past. Hence GBM is a Markov Process.

GBM TO Model Stock Price

let the stock price $S(t)$ at time 't' is given by $\underline{S(t) = S(0)\, e^{H(t)}}$, where $S(0)$ = initial price , $H(t) = \mu t + \sigma W(t)$ is brownian motion with drift 'μ' and 'volalitity 'σ'.
($H(t)$ represents the continuously compounded rate of interest over $[0,t]$).

$$S(t) = S(0) e^{H(t)}$$

$$\Rightarrow \quad \ln(S(t)) = \ln(S(0)) + H(t).$$

- $\ln(S(t))$ has a normal dist^n with mean $\underline{\mu t + \ln(S(0))}$ and variance $\underline{\sigma^2 t}$.

- $\underbrace{\ln\left(\frac{S(t)}{S(0)}\right)}_{\text{normally distributed}} = \underbrace{H(t)}$. $\left(\frac{S(t)}{S(0)}\right)$ is lognormally distributed.

• $E(S(t)) \Rightarrow$

NOTE: The Moment Generating Function of a Normal Dist^n $H(t)$ can be expressed as :

$$M_{H(t)}(\theta) = E\left(e^{\theta H(t)}\right).$$

$$\therefore M_{H(t)}(\theta) = E\left(e^{\theta H(t)}\right) = \exp\left(\mu t\theta + \frac{1}{2}\sigma^2 t\theta^2\right)$$

where, ($H(t) = \mu t + \sigma W(t)$)

($W(t)$ is a normal dist^n with mean 0 and variance 't')

$$\Big(E\left(e^{\theta H(t)}\right) = E\left(e^{\theta.(\mu t + \sigma W(t))}\right) = E\left(e^{\theta\mu t} . e^{\theta\sigma W(t)}\right)$$

$$= e^{\theta\mu t} . E\left(e^{\theta\sigma W(t)}\right) = e^{\theta\mu t} . e^{\theta^2\sigma^2 t/2}$$

($\because W(t)$ is a Normal dist $\simeq N(0,t)$).

$$= \boxed{\exp.\left(\theta\mu t + \frac{\theta^2\sigma^2 t}{2}\right)} \quad — \text{Ⓐ} \Big)$$

$\mu = 0, \sigma^2 = t$

NOTE: $\boxed{\left(M_X(t) = e^{\mu t + \sigma^2 t^2/2} \right)}$

$X \sim N(\mu, \sigma^2)$ — μ: Mean, σ^2: Variance

$$S(t) = S(0)e^{H(t)} \Rightarrow E(S(t)) = E\left(S(0).e^{H(t)}\right) = S(0).E\left(e^{H(t)}\right)$$

$$\Rightarrow \boxed{E(S(t)) = S(0).\exp.\left(\mu t + \frac{\sigma^2 t}{2}\right)} \quad (\text{put } \theta = 1 \text{ in Ⓐ})$$

• $var(S(t)) \Rightarrow \quad var(S(t)) = E\left((S(t))^2\right) - \left(E(S(t))\right)^2$

$$E\left((S(t))^2\right) = E\left((S(0))^2 . e^{2H(t)}\right) = (S(0))^2 . E\left(e^{2H(t)}\right)$$

put $\theta = 2$ in Ⓐ $\Rightarrow$

$$E\left((S(t))^2\right) = (S(0))^2 . \exp.\left(2\mu t + 2\sigma^2 t\right)$$

and,

$$\left(E(S(t))\right)^2 = (S(0))^2 . \exp\left(2\mu t + \sigma^2 t\right)$$

$$\therefore \quad var(S(t)) = (S(0))^2 \left[e^{(2\mu t + 2\sigma^2 t)} - e^{(2\mu t + \sigma^2 t)} \right]$$

$$= \boxed{\left(S(0) . \exp.\left(\left(\mu+\frac{\sigma^2}{2}\right)t\right)\right)^2 . \left(\exp.\left(\sigma^2 t\right) - 1\right)}$$

Filtration and Martingales

Filtration

consider the given probability space $(\Omega, F, \mathbb{P})$ and let $T > 0$.
Assume that for $0 \le t \le T$, $\exists$ a σ-algebra (σ-field), $F(t)$ (for continuous time) or F_t (for discrete time), such that $F(t) \subset F$ and for $\forall s$, $0 \le s \le t$, $F(s) \subseteq F(t)$, then, such a collection of σ-fields, $\{F(t)\}_{0 \le t \le T}$ is called a Filtration associated with $(\Omega, F, \mathbb{P})$.

$\downarrow$ less formally

As the process / experiment goes on with time, more and more information, (which can be considered as the subsets of the complete σ-algebra F) is revealed to us. For example, consider a toss of 3 coins.
These are the possible outcomes: $\{HHT, HTH, HHH, HTT, THT, TTH, TTT, THH\}$
This is the sample space Ω.
Now, suppose we say that the outcome of the toss of the 1^{st} coin is revealed to us, that is,

$$A_H \text{ (1^{st} coin comes out to be 'H')} = \{HHT, HTH, HHH, HTT\}$$

$$\text{or} \quad A_T \text{ (1^{st} coin comes out to be 'T')} = \{TTT, THT, THH, TTH\}$$

we can guess what will be the consequent outcomes; we can make an estimation. Therefore, the first σ-algebra that can be formed, using the information disclosed at this step is:

$$F_1 = \{\phi, \Omega, A_H, A_T\}.$$

Evidently, $F_1 \subset F$ which is to say that a part (some subsets) of the σ-algebra F is revealed when 'some' information is revealed.

If we further reveal the outcome of the toss of the 2^{nd} coin, then the following events can occur next:

$A_{HH} = \{HHH, HHT\}$

$A_{HT} = \{HTH, HTT\}$ → we further divided / refined / filtered the outcomes based on the information revealed.

$A_{TT} = \{TTH, TTT\}$

$A_{TH} = \{THH, THT\}$

corresponding to this, the σ-algebra $\mathcal{F}_2$ can be formed as:

$$\mathcal{F}_2 = \{\phi, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH}^c, A_{HT}^c, A_{TH}^c, A_{TT}^c, (A_{HH} \cup A_{TH}), (A_{HH} \cup A_{TT}), (A_{HT} \cup A_{TH}), (A_{HT} \cup A_{TT})\}$$

what was revealed, will continue to be known to us and will be carried along in every subsequent encoding.

Thus, $\mathcal{F}_2$ contains (encodes) all the information we could attain after two steps of revelations / conduction of experiment.

If we say that we reveal $\mathcal{F}_3$, we will basically be saying that we know the outcome of all the 3 tosses, ie. $\mathcal{F}_3 = \mathcal{F}$, having 256 elements.

We can also note that $\mathcal{F}_0$ implies when nothing is known. This compels us to estimate that nothing will happen OR everything happens, ie. $\mathcal{F}_0 = \{\phi, \Omega\}$.

As we can observe, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$. And the set $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ is called a FILTRATION.

NOTE: The σ-field $\mathcal{F}_n$ maybe thought of as the events of which the occurrence is determined at or before time 'n', ie., the "known events" at time n.

Filtration is used to model the flow of information over time. As an example, we think of X_t as the price of some asset at time 't' and $\mathcal{F}_t$ as the information obtained by watching all the prices in the market upto time 't'.

- **X_t is $\mathcal{F}_t$-measurable** : we say that a discrete time S.P. $\{X_0, X_1, \ldots\}$ is $\mathcal{F}_t$-measurable, if the σ-field generated by X_t is a subset of $\mathcal{F}_t$, ie. $\sigma(X_t) \subset \mathcal{F}_t$, for $\forall t$. In a similar manner, a continuous time S.P. $\{X(t), t \geq 0\}$ is said to be $\mathcal{F}_t$-measurable if $\sigma(X(t)) \subset \mathcal{F}_t \ \forall t \geq 0$.

In simple terms, a r.v. X_t is $\mathcal{F}_t$-measurable iff the information in $\mathcal{F}_t$ is sufficient to determine the value of X_t.

Martingales

Discrete Time Martingale

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{X_n, n=0,1\ldots\}$ be a S.P. and $\{\mathcal{F}_n, n=0,1,\ldots\}$ be the filtration. The S.P. $\{X_n\}$ is said to be a martingale corresponding to the filtration $\{\mathcal{F}_n\}$, if it satisfies the following:

(i) For every n, $E(X_n)$ exists.

(ii) Each X_n is $\mathcal{F}_n$-measurable

(iii) $\forall n$, $E[X_{n+1} \mid \mathcal{F}_n] = X_n$.

NOTE: conditional expectations and calculation rules:

- $E(X) = \int_\Omega X(\omega)\, dP(\omega)$ on $(\Omega, \mathcal{F}, P)$.
- $E(aX_1 + bX_2 \mid Y) = aE(X_1 \mid Y) + bE(X_2 \mid Y)$
- $E[E[X \mid Y]] = E[X]$ (average property)
- $E[E[X \mid Y, Z] \mid Z] = E[X \mid Z]$ (tower property)
- $E[a \mid Y] = a$ (a is some constant)

this is basically $E[E[X_{n+1}|F_n]]$

- $E[X_n] = \int_\Omega X_n \, dP = \int_\Omega E[X_{n+1}|\mathcal{F}_n] \, dP = \int_\Omega X_{n+1} \, dP = E[X_{n+1}]$

(using average property).

$\therefore$ If $\{X_n\}$ is a Martingale, then

$$\boxed{E[X_0] = E[X_1] = E[X_2] = \ldots E[X_n]}$$

- $E[X_m | \mathcal{F}_n] = X_n \, , \; \forall \; m \geq n.$

which essentially translates to : we cannot hope to get something more than that we got at the n^{th} step, provided we have only the information upto n steps.
We cannot do better than what we know. This is the idea behind Martingales.

- If $E[X_{n+1} | \mathcal{F}_n] \geq X_n$ → Sub-martingale
  If $E[X_{n+1} | \mathcal{F}_n] \leq X_n$ → Super- Martingale

- **Examples :**

1) 8.6.2 (Pg 304) let X_1, X_2 -- be a seq^n of i.i.d. r.v.---.
Sol^n:

$$E(S_n) = E\left(\sum_i X_i\right) = \sum_i E(X_i) = 0 \quad \left(\because E(X_i) = \left(\tfrac{1}{2}\right)(+1) + \left(\tfrac{1}{2}\right)(-1) = 0\right)$$

Let us prove (iii) ⇒

$$E[S_{n+1} | \mathcal{F}_n] = E[S_n + X_{n+1} | \mathcal{F}_n] = E[S_n | \mathcal{F}_n] + E[X_{n+1} | \mathcal{F}_n]$$

$$(\mathcal{F}_n = \{(X_1, X_2 \ldots X_n)\})$$

$$E[S_{n+1} | \mathcal{F}_n] = \underbrace{E[S_n | \mathcal{F}_n]}_{= S_n} + \underbrace{E[X_{n+1} | (X_1, X_2, \ldots X_n)]}_{= E[X_{n+1}]}$$

$= S_n$ ($\because S_n$ will be definitely known given $(X_1, X_2, \ldots X_n) = \mathcal{F}_n$)

$= E[X_{n+1}]$ as all X_i s are i.i.d r.v.

(This tells us that S_n is $\mathcal{F}_n$-measurable)

$$= S_n + E[X_{n+1}] = S_n + 0 = \underline{S_n}.$$

$\therefore \{S_n\}$ is a martingale.

2) 8.6.3 (Pg 304) consider a symmetric random walk $\{S_n, n = 0, 1 \ldots\}$ -----
sol^n:

$$S_n = \sum_{i=1}^{n} X_i \Rightarrow S_n^2 = (\sum_{i=1}^{n} X_i)^2$$

$$\therefore E(S_n^2) = E((\sum_{i=1}^{n} X_i)^2) = E(\sum_{i=1}^{n} X_i^2) = \sum_{i=1}^{n} E(X_i^2) = 1 \text{ (finite)}$$

$(\frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2)$

(because say for $n=2$, $S_n^2 = X_1^2 + X_2^2 + 2X_1X_2 \Rightarrow E(S_n^2) = E(X_1^2) + E(X_2^2) + 2\underbrace{E(X_1)}_{0}\underbrace{E(X_2)}_{0}$

$E(X_1^2) + E(X_2^2) =$

same can be said for $n = 3, 4 \ldots$)

Also, S_n^2 is $\mathcal{F}_n$-measurable because S_n is $\mathcal{F}_n$-measurable.

Now,

$$E[S_{n+1}^2 | \mathcal{F}_n] = E[(S_{n+1} - S_n + S_n)^2 | \mathcal{F}_n] = E[(S_{n+1} - S_n)^2 + S_n^2 + 2S_n(S_{n+1} - S_n) | \mathcal{F}_n]$$

$$= E[(S_{n+1} - S_n)^2 | \mathcal{F}_n] + E[S_n^2 | \mathcal{F}_n] + 2E[S_n(S_{n+1} - S_n) | \mathcal{F}_n]$$

$$= E[X_{n+1}^2 | \mathcal{F}_n] + E[S_n^2 | \mathcal{F}_n] + 2E[S_n X_{n+1} | \mathcal{F}_n]$$

$$= E[X_{n+1}^2] + S_n^2 + 2S_n \underbrace{E[X_{n+1}]}_{0}$$

(because X_{n+1} is independent of $(X_1, X_2 \ldots X_n)$)

(because given $\mathcal{F}_n$, we can calculate and hence know that value of S_n^2, making it a constant)

$$= 1 + S_n^2 \neq S_n^2$$

$\therefore \{S_n^2\}$ is not a Martingale

3) 8.6.5 (Pg 306) Let a person start with Rs. 1 ......

Solⁿ: let $X_1, X_2 \ldots$ be a seqⁿ of r.v. such that

$$X_i = \begin{cases} 2, & p = 0.5 \\ 0, & p = 0.5 \end{cases}$$

Since the game is double or nothing, his fortune at the end of n^{th} toss is given by

$$Y_n = X_1 X_2 \ldots X_n \quad (n = 1, 2 \ldots), (0 \le Y_n \le 2^n)$$

Let $\mathcal{F}_n$ be a σ-field generated by $(X_1, X_2 \ldots X_n)$

$$E[Y_n] = E[X_1 X_2 \ldots X_n] = E[X_1] E[X_2] \ldots E[X_n] = 1 \quad (\because E[X_i] = 1)$$

Also,

$$E[Y_{n+1} | \mathcal{F}_n] = E[Y_n X_{n+1} | \mathcal{F}_n] = Y_n E[X_{n+1} | \mathcal{F}_n] = Y_n E[X_{n+1}]$$

$$= Y_n$$

$\therefore \{Y_n\}$ is a Martingale.

Continuous Time Martingale

A s.p. $\{X(t), t \geq 0\}$ is said to be a Martingale, corresponding to the filtration $\{\mathcal{F}(t), t \geq 0\}$ on the probability space $(\Omega, \mathcal{F}, P)$ if it satisfies the following:

(i) $E[X(t)]$ exists $\forall t \geq 0$

(ii) $X(t)$ is $\mathcal{F}(t)$-measurable, $\forall t \geq 0$

(iii) $E[X(t) \mid \mathcal{F}(s)] = X(t) \quad \forall \quad 0 < s < t.$