Binomial Model

The binomial model is defined, and work under ***two conditions****;*

1. *The one step return* $k(n)$ *are independent and identically distributed (i.i.d.) random variable, for each time step* 'n', where $-1 < d < u$ & $0 < p < 1$.
2. *The one step return on risk free investment* 'r' *is same at each time step and* $d < r < u$.

Dynamics of Stock Price

The current price of a stock $S(0)$ at $t = 0$ is fixed and decided by market forces. At $t = \tau$ the price of stock will be $S(\tau)$ which is purely random variable with condition $S(\tau) > 0$.

At expiry of each τ- period , the price of stock can take two values, either it will increase or decrease and u & d if are factors by which it goes up or down respectively with probability p and $(1-p)$ respectively.

i.e $$S(0) \begin{array}{l} \xrightarrow{\;p\;} (1+u)S(0) \\ \xrightarrow{\;1-p\;} (1+d)S(0) \end{array}$$ where $-1 < d < u$

$0 < p < 1$

This phenomena continues with every subsequent τ- period .

If r is the risk free return of one period, then $d < r < u$. Otherwise an arbitrage situation will be created.

i.e $$A(0) \begin{array}{l} \xrightarrow{\;p\;} (1+r)A(0) \\ \xrightarrow{\;1-p\;} (1+r)A(0) \end{array}$$

$(1+u)$, $(1+d)$, & $(1+r)$ are growth factor.

Now if we represent the growth factor by u & d directly then	$S(0) \begin{array}{l} \xrightarrow{\;p\;} uS(0) \\ \xrightarrow{\;1-p\;} dS(0) \end{array}$ also $1 + r = R$

Expected return $E\big(k(1)\big) = p.u + (1-p).d$

Now if for some probability $\boldsymbol{p}^*$ the average return on a risky asset is equal to risk free return, then such a probability is known as *risk neutral probability*.

$$E^*(k(1)) = p^*.u + (1-p^*).d = r \Longrightarrow p^* = \frac{r-d}{u-d}$$

{ In terms of growth factor $p^* = \frac{(1+r)-(1+d)}{(1+u)-(1+d)} = \frac{R-d}{u-d}$, u & d are growth factor. }

$$E(S(1)) = p.(1+u)S(0) + (1-p).(1+d)S(0)$$
$$= S(0)(p + p.u + (1-p) + (1-p).d)$$
$$= S(0)\{\,1 + E(k(1))\,\}$$

§§ Expected stock price for $n = 0,1,2\ldots$ is given by

$$E(S(n)) = S(0)\{1 + E(k(1))\}^n$$

Proof: Since the step returns $k(1), k(2), \ldots.$ are independent and so are the random variables $(1+k(1)), (1+k(2)), \ldots.$

$$E(S(n)) = E\{\,S(0)(1+k(1))(1+k(2))\ldots(1+k(n))\}$$
$$= S(0)E(1+k(1)).E\,(1+k(2))\ldots E(1+k(n))$$

$\because k(n),\ \forall n$ are $i.i.d.$ r.v. $\Longrightarrow$ $E(k(1)) = E(k(2)) = \cdots = E(k(n))$

Hence, $$E(S(n)) = S(0)\{1 + E(k(1))\}^n$$

In case of risk neutral probability p^* ,

$$E^*(S(n)) = S(0)\{1+r\}^n \qquad (\because\ E^*(k(1)) = r\ )$$

§§ Extending the result of 2-step model to 'n' time steps, stock price becomes $S(n)$, then the risk neutral expectation of the price $S(n+1)$ will be

$$E^*(S(n+1)/S(n)) = S(n)(1+r)$$

Proof: Suppose that $S(n) = x$ after n-time steps. Then

$$E^*(S(n+1)/S(n)) = p^*.(1+u).x + (1-p^*)(1+d).x$$
$$= x\{\,p^*.(1+u) + (1-p^*)(1+d)\}$$
$$= x(1+r) \qquad (\because\ 1+r\ = p^*.(1+u) + (1-p^*)(1+d))$$

Option Pricing

We will consider pricing of general European derivative security, in particular the option.

We introduce a function f called payoff function with stock S as the underlying asset. $D(T)$ is a random variable, then

$D(T) = f(S(T))$.

In particular for call option $f(S) = (S - X)^+$

For put option $f(S) = (X - S)^+$

Or if forward contract $f(S) = S - X$ (for long position)

Now a stock with current price $S(0)$ will be $S(1)$ after time $t = 1$ and may take two values

$$S(1) = \begin{cases} S(0)(1+u) = S^u & \text{with probability } p \\ S(0)(1+d) = S^d & \text{with probability } (1-p) \end{cases}$$

To replicate a general derivative security with payoff f we have to solve the following system of equations for $x(1)$ & $y(1)$

$$x(1)S^u + y(1)(1+r) = f(S^u)$$

$$x(1)S^d + y(1)(1+r) = f(S^d)$$

This gives

$$x(1) = \frac{f(S^u) - f(S^d)}{S^u - S^d}$$

Which is the replicating position in the stock is called the *delta* of the option

We also have money market position,

$$y(1) = -\frac{(1+d)f(S^u) - (1+u)f(S^d)}{(u-d)(1+r)}$$

The initial value of the replicating portfolio is

$$x(1)S(0) + y(1)$$

$$D(0) = \frac{f(S^u) - f(S^d)}{u-d} - \frac{(1+d)f(S^u) - (1+u)f(S^d)}{(u-d)(1+r)}$$

Theorem: The expectation of the discounted pay off computed with respect to the risk neutral probability is equal to the present value of European derivative security (or contingent claim).

$$D(0) = E^*\{(1+r)^{-1} f(S(1))\}$$

Proof: We have

$$D(0) = \frac{f(S^u) - f(S^d)}{u-d} + \frac{(1+u)f(S^d) - (1+d)f(S^u)}{(u-d)(1+r)}$$

$$= \frac{1}{1+r}\left[\frac{(r-d)f(S^u)}{u-d} + \frac{(u-r)f(S^d)}{u-d}\right]$$

$$= \frac{1}{1+r}[p^* f(S^u) + (1-p^*)f(S^d)]$$

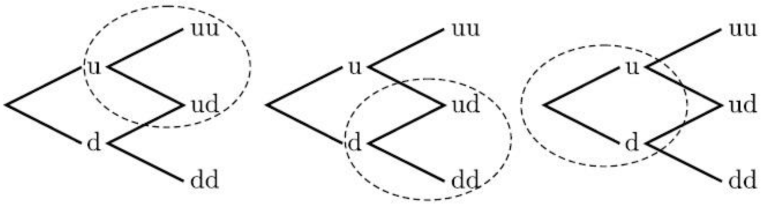
$$= \frac{1}{1+r} E^*\left(f(S(1))\right)$$

$$= E^*\{(1+r)^{-1} f(S(1))\}$$

Two step Binomial model

The stock price $S(2)$ has three possibilities,

$$S^{uu} = S(0)(1+u)^2, \;\; S^{ud} = S(0)(1+u)(1+d), \;\; S^{dd} = S(0)(1+d)^2$$



Branchings in the two-step binomial tree

For each subtree as above we can use one step replicating procedure. At time $t = 2$, the derivative security is represented by its payoff ,

$$D(2) = f(S(2))$$

which has three possible values.

The derivative security price $D(1)$ has two values,

$$\frac{1}{1+r}[p^* f(S^{uu}) + (1-p^*)f(S^{ud})] \quad \& \quad \frac{1}{1+r}[p^* f(S^{ud}) + (1-p^*)f(S^{dd})]$$

This gives,

$$D(1) = \frac{1}{1+r}[\, p^* f(S(1)(1+u)) + (1-p^*) f(S(1)(1+d))\,]$$
$$= g\big(S(1)\big)$$

Where,

$$g(x) = \frac{1}{1+r}[p^* f(x(1+u)) + (1-p^*) f(x(1+d))]$$

As a result $D(1)$ can be regarded as a derivative Security expiring at time $t = 1$ with pay off $'g'$ this means that one step procedure can be applied once again to the single subtree at the root of the tree. We have, therefore

$$D(0) = \frac{1}{1+r} E^*\{g\big(S(1)\big)\}$$

$$= \frac{1}{1+r}\{p^* g\big(S(0)(1+u)\big) + (1-p^*) g\big(S(0)(1+d)\big)\}$$

It follows that

$$D(0) = \frac{1}{1+r}[p^* g(S^u) + (1-p^*) g(S^d)]$$
$$= \frac{1}{(1+r)^2}\left[p^{*2} f(S^{uu}) + 2p^*(1-p^*) f(S^{ud}) + (1-p^*)^2 f(S^{dd})\right]$$

§§ The expectation of the discounted free of computer with respect to the risk neutral probability is equal to the present value of derivative security

$$D(0) = E^*\left\{\frac{1}{(1+r)^2} f\big(S(2)\big)\right\}$$

Multi Period (N – Step) Binomial model

Following the similar process of beginning with payoff at the final step and moving backwards, solving the one step problem at each level repeatedly, we can generalize the result for N - step as following,

$$D(0) = \frac{1}{(1+r)^N} \sum_{k=0}^{N} \binom{n}{k} p^{*k} (1-p^*)^{N-k} f\{S(0)(1+u)^k (1+d)^{N-k}\}$$

§§ The value of a European derivative security with payoff $f(S(N))$ in N - step binomial model is the expectation of discounted payoff under the risk neutral probability,

$$D(0) = E^* \left\{ \frac{1}{(1+r)^N} f\big(S(N)\big) \right\}$$

<u>Cox-Ross-Rubinstein Formula (CRR model)</u>

[C. Cox, Stephen A. Ross, and Mark Rubinstein]

The assumptions on the financial market made for single period Binomial model are carried forward here as following;

1. The underlying stock on which option is written is perfectly divisible.
2. The underlying stock pays no dividend.
3. There is no transaction cost in buying or selling the option and no taxes.
4. Short selling is allowed.
5. The risk free interest rate is known and constant till time of expiration of option.
6. No arbitrage principle holds

We divide the time horizon interval $[0, T]$ into 'n' sub intervals, each of length

$$\Delta t = \frac{T}{N}$$

And assume that in each sub interval the stock price changes like one period binomial case. Thus in each case stock price either moves up by a constant factor u or moves down by constant factor d with positive probability. We define

$$E_k = \begin{cases} u & \text{with probabiity } p \\ d & \textit{with probability } (1-p) \end{cases} \quad , \quad k = 1,2, \dots \quad , \quad \& \quad 0 < p < 1 \qquad \dots(1)$$

Now E_k is a Bernoulli random variable .

At time of expiration T, the stock price $S(T)$ is given by

$$S(T) = S(0)E_1.E_2 \dots E_n$$
$$= S(0)e^H$$

where, $e^H = E_1.E_2 \dots E_n$.

$$\ln\big(S(T)\big) = \ln\big(S(0)\big) + H \; , \; \textit{here} \quad H = \textstyle\sum_{k=1}^{n} \ln(E_k) \qquad \dots(2)$$

H represents the logarithmic growth of the stock price.

Now

$$E(\ln(E_k)) = p.\ln(u) + (1-p).\ln(d) \qquad \dots(3A)$$

$$Var(\ln(E_k)) = p(1-p)[\ln(u) - \ln(d)]^2 \qquad \dots(3B)$$

Now we introduce new parameters μ & σ^2 as following

$$\mu.\Delta t = E(\ln(E_k)) \quad \& \quad \sigma^2.\Delta t = Var(\ln(E_k)) \qquad \dots(4)$$

The parameter μ is called <u>*drift*</u> and σ is called <u>*volatility*</u>.

Let us introduce another random variable,

$$X_k = \frac{\ln(E_k) - E(\ln(E_k))}{\sqrt{Var(\ln(E_k))}} \qquad \dots(5)$$

Using equ. (3) in (5) we get,

$$X_k = \begin{cases} \frac{(1-p)}{\sqrt{p(1-p)}} & \text{with probabiity } p \\ \frac{-p}{\sqrt{p(1-p)}} & \text{with probabiity } (1-p) \end{cases} \quad , \quad k = 1,2,\dots n$$

Clearly for each $k = 1,2,\dots n$

$$E(X_k) = 0 \qquad \& \qquad Var(X_k) = 1$$

Now

$$H = \sum_{k=1}^{n} \ln(E_k)$$

$$= \sum_{k=1}^{n} \left(\mu.\Delta t + \sigma\sqrt{\Delta t}\, X_k\right) \qquad using \;\; (5)\&(4)$$

$$= \mu.T + \sigma\sqrt{\Delta t}\, \Upsilon \qquad \dots(6)$$

Where, $\Upsilon = \sum_{k=1}^{n} X_k$ is a <u>*Simple Random Walk*</u>. $\dots(7)$

<u>***Definition*: Random Walk**</u>

A random walk is a stochastic process $\{S_n\,,\ n = 0,1,2,..\}$ with $S_0 = 0$ defined by

$$S_n = \sum_{k=1}^{n} X_k \qquad \dots(A)$$

where $\{X_k\}$ are independent and identically distributed (i.i.d.) random variables.
The random walk is simple if for each $k = 1,2,\dots n$, X_k takes value from $\{a, b\}$, $a, b \in R$ with $P(X_k = a) = p$ & $P(X_k = b) = (1-p)$.

Also from (A) we have, $S_{n+1} = S_n + X_{n+1}$.

§§ For CRR model with probability of uptick equals p and probability of down tick $(1-p)$, life time T and time increment $\Delta t = \frac{T}{N}$, the stock price is given by

$$S(T) = S(0)\exp\big(\mu.T + \sigma\sqrt{\Delta t}\,\Upsilon\big) \qquad \dots(B)$$

Where μ is drift and σ is volatility described by equ.(4) and Υ is a simple random walk given by equ. (7).

Now we calculate expectation and variance of random variable H .

$$E\{\,ln\frac{S(T)}{S(0)}\} = E(H)$$

$$= E(\mu.T + \sigma\sqrt{\Delta t}\,\Upsilon)$$

$$= \mu.T + \sigma\sqrt{\Delta t}\,E(\textstyle\sum_{k=1}^{n} X_k)$$

$$= \mu T \qquad \because E(X_k) = 0\,,\ \forall\, k \qquad \dots(8)$$

And, $$Var\left\{\,ln\frac{S(T)}{S(0)}\right\} = Var(H) = Var(\mu.T + \sigma\sqrt{\Delta t}\,\Upsilon)$$

$$= \sigma^2.\Delta t.Var(\Upsilon) = \sigma^2.\Delta t\textstyle\sum_{k=1}^{n} Var(X_k)$$

$$= \sigma^2 T\,, \quad \because \quad Var(X_k) = 1 \;\;\&\;\; n.\Delta t = T \qquad \dots(9)$$

From equation (8 & 9) we observe that drift μ is the expectation of logarithmic return and volatility σ is standard deviation of the logarithmic return .

Matching of CRR model with a multi period Binomial model

We are to determine three parameters $u, d, \& p$ using with question (3), assuming that σ is known from past data of the stock. Since we need to solve three variables with two equations uniquely, hence let us denote

$$U = \ln(u) \quad \& \quad D = \ln(d) \quad \text{and assume that}$$

$$D = -U \Rightarrow d = \frac{1}{u}$$

That means we are matching the CRR model to a particular multi period Binomial model where $d = \frac{1}{u}$. Now equation (4) becomes

$$(2p-1)U = \mu.\Delta t \qquad \dots(10\text{A})$$

$$4p(1-p)U^2 = \sigma^2 \Delta t \qquad \ldots(10B)$$

Squaring (10A) and adding to (10B)

$$(2p-1)^2 U^2 + 4p(1-p)U^2 = \mu^2(\Delta t)^2 + \sigma^2 \Delta t$$

Or $$U = \sqrt{\mu^2(\Delta t)^2 + \sigma^2 \Delta t}\ , \qquad U = -D \qquad \ldots(11)$$

From (10A) & (11) we have

$$p = \frac{1}{2} + \frac{\mu.\Delta t}{2\sqrt{\mu^2(\Delta t)^2 + \sigma^2 \Delta t}} \qquad \ldots.(12)$$

Let us take 'n' sufficiently large , i.e., $\Delta t = \frac{T}{N}$ as n increases Δt decreases, and so $(\Delta t)^2$ can be neglected. From equ. (11) we get

$$U = \ln(u) = \sigma\sqrt{\Delta t} \Longrightarrow u = e^{\sigma.\sqrt{\Delta t}} \qquad \ldots(13A)$$

$$d = \frac{1}{u} \Longrightarrow d = e^{-\sigma.\sqrt{\Delta t}} \qquad \ldots(13B)$$

$$p = \frac{1}{2} + \frac{\mu.\sqrt{\Delta t}}{2\sigma} \qquad \ldots(13C)$$

Black-Scholes formula from CRR model

Now we define a counter on up and down tick movements on stock price at time $k.\Delta t\, , k = 1,2, \dots n$ as a Bernoulli random variable.

$$Y_k = \begin{cases} 1 & if \text{ stock goes up with probabiity } p \\ 0 & \text{if stock goes down with probabiity } (1-p) \end{cases}$$

Then,

$$S(T) = S(n.\Delta t) = S(0)u^{\sum_{k=1}^{n} Y_k}.\, d^{(n-\sum_{k=1}^{n} Y_k)}$$

$$= S(0)d^n \left(\frac{u}{d}\right)^{\sum_{k=1}^{n} Y_k}$$

$$\frac{S(T)}{S(0)} = d^{T/\Delta t} \left(\frac{u}{d}\right)^{\sum_{k=1}^{T/\Delta t} Y_k} \qquad \dots(14)$$

here, $\Upsilon = \sum_{k=1}^{n} Y_k$ is a simple random walk with

$$E(\Upsilon) = p.\frac{T}{\Delta t} \quad \& \quad Var(\Upsilon) = p(1-p)\frac{T}{\Delta t}$$

From equ. (14) we have

$$ln\frac{S(T)}{S(0)} = -\frac{\sigma T}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{k=1}^{\frac{T}{\Delta t}} Y_k \qquad (using\ (13)) \qquad \dots(15)$$

Hence from equ. (15)

$$E\left\{ln\frac{S(T)}{S(0)}\right\} = -\frac{\sigma T}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{k=1}^{\frac{T}{\Delta t}} E(Y_k)$$

$$= -\frac{\sigma T}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}.\, p.\frac{T}{\Delta t}$$

$$= (2p-1)\frac{\sigma T}{\sqrt{\Delta t}}$$

$$= \mu T \qquad (using\ \ (13))$$

$$Var\left\{ln\frac{S(T)}{S(0)}\right\} = 4\sigma^2\Delta t\sum_{k=1}^{n} Var(Y_k)$$

$$= 4\sigma^2\Delta t.\, p(1-p)\frac{T}{\Delta t}$$

$$= 4\sigma^2 T . p(1-p)$$

$$\to \sigma^2 T \quad \text{as } p \to \frac{1}{2} \quad \text{when } \; n \to \infty$$

By application of Central limit Theorem, we can assume that Y_k follows a normal distribution when time step approaches to Zero. Thus,

$$ln \frac{S(T)}{S(0)} \sim N(\mu T, \sigma^2 T)$$

Now,

$$u = e^{\sigma\sqrt{T/n}} \cong 1 + \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2 T}{2n} \qquad \text{(neglecting } \Delta t^{3/2} \text{ and higher)} \quad \ldots(16)$$

$$d = e^{-\sigma\sqrt{\frac{T}{n}}} \cong 1 - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2 T}{2n} \qquad \ldots(17)$$

So the risk neutral probability measure (RNPM) is given by

$$p^* = \frac{R-d}{u-d}$$

$$= \frac{1 + \frac{rT}{n} - d}{u-d} \qquad (Since\ R = e^{r\Delta t} = 1 + r\Delta t)$$

$$\cong \frac{\frac{rT}{n} + \sigma\sqrt{\frac{T}{n}} - \frac{\sigma^2 T}{2n}}{2\sigma\sqrt{\frac{T}{n}}}$$

Thus,

$$p^* = \frac{1}{2} + \frac{(2r - \sigma^2)}{4\sigma}\sqrt{T/n} \qquad \ldots(18)$$

Now European call option price for n-period binomial model is described as,

If K is the strike price,

$$C(0) = \frac{1}{(1 + \frac{rT}{n})^n} E^*\{(S(T) - K)^+\}$$

$$= (1 + rT/n)^{-n} E^*\{(S(0)u^{\Upsilon} d^{n-\Upsilon} - K)^+\}$$

$$= (1 + rT/n)^{-n} E^*\left\{\left(S(0)\left(\frac{u}{d}\right)^{\Upsilon} d^n - K\right)^+\right\}$$

Using the equ. (16 & 17) for u & d , we get,

$$C(0) = (1 + rT/n)^{-n} E^*\{(S(0)e^{\omega} - K)^+\} \qquad \ldots(19)$$

where $$\omega = 2\sigma\sqrt{\frac{T}{n}}\Upsilon - \sigma\sqrt{nT}$$

Now $$E(\omega) = 2\sigma\sqrt{\frac{T}{n}}E(\Upsilon) - \sigma\sqrt{nT}$$

$$= 2\sigma\sqrt{\frac{T}{n}}.n.p - \sigma\sqrt{nT}$$

$$= (2p-1)\sigma\sqrt{nT}$$

$$= \mu\sqrt{\Delta t}.\sqrt{nT} \qquad using\ (13C)$$

$$= \mu T$$

Also, $$Var(\omega) = 4\sigma^2\frac{T}{n}Var(\Upsilon)$$

$$= 4\sigma^2\frac{T}{n}.p.(1-p)\frac{T}{\Delta t}$$

$$= 4\sigma^2 p(1-p)T$$

$$\rightarrow \sigma^2 T \qquad \text{as } p \rightarrow \frac{1}{2}, \quad \Delta t \rightarrow 0 \quad \text{when} \quad n \rightarrow \infty,$$

Consider now,

$$E^*(\omega) = 2\sigma\sqrt{\frac{T}{n}}E^*(\Upsilon) - \sigma\sqrt{nT}$$

$$= 2\sigma\sqrt{\frac{T}{n}}.n.p^* - \sigma\sqrt{nT}$$

$$= 2\sigma\sqrt{nT}\left(\frac{1}{2} + \frac{(2r-\sigma^2)}{4\sigma}\sqrt{T/n}\right) - \sigma\sqrt{nT} \qquad \text{using (18)}$$

$$= \left(r - \frac{\sigma^2}{2}\right)T$$

And,

$$Var^*(\omega) = 4\sigma^2\frac{T}{n}Var^*(\Upsilon)$$

$$= 4\sigma^2\frac{T}{n}p^*(1-p^*)\frac{T}{\Delta t}$$

$$= 4\sigma^2 p^*(1-p^*)T$$

$$\rightarrow \sigma^2 T \qquad \text{as } p^* \rightarrow \frac{1}{2}, \quad \Delta t \rightarrow 0 \quad \text{when} \quad n \rightarrow \infty$$

Using above relation (19) gives

$$C(0) = e^{-rT} \int_{-\infty}^{\infty} (S(0)e^{\omega} - K)^{+} \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} \; e^{-\frac{1}{2}\left\{\frac{\omega-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\}^2} \; d\omega$$

$(S(0)e^{\omega} - K)^{+}$ will be non- zero iff,

$$S(0)e^{\omega} > K \Rightarrow \omega > \ln\left(\frac{K}{S(0)}\right) = \omega_1(say)$$

$$C(0) = e^{-rT} \int_{\omega>\omega_1}^{\infty} (S(0)e^{\omega} - K) \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} \; e^{-\frac{1}{2}\left\{\frac{\omega-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\}^2} \; d\omega$$

Substitute $\quad y = \frac{\omega-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, then

$$\omega = y\sigma\sqrt{T} + (r - \frac{\sigma^2}{2})T \Rightarrow d\omega = \sigma\sqrt{T}.dy$$

Where $\omega > \omega_1$ gives $y > y_1$, while

$$y_1 = \frac{1}{\sigma\sqrt{T}}\{\ln\left(\frac{K}{S(0)}\right) - (r - \frac{\sigma^2}{2})T\}$$

$$C(0) = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} (S(0)e^{y\sigma\sqrt{T}+\left(r-\frac{\sigma^2}{2}\right)T} - K) \; e^{-\frac{y^2}{2}} \, dy$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} S(0)e^{y\sigma\sqrt{T}+\left(r-\frac{\sigma^2}{2}\right)T} \; e^{-\frac{y^2}{2}} \, dy - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} e^{-\frac{y^2}{2}} \; dy$$

$$= I - Ke^{-rT}\phi(-y_1) \qquad \ldots(20)$$

Where $\phi(x)$ is distribution function (CDF) of a standard normal r.v. given as,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \; dz$$

$$I = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} S(0)e^{y\sigma\sqrt{T}+\left(r-\frac{\sigma^2}{2}\right)T} \; .e^{-\frac{y^2}{2}} \, dy$$

$$= \frac{S(0)}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} e^{-\frac{(y-\sigma\sqrt{T})^2}{2}} dy$$

Substitute $\quad y - \sigma\sqrt{T} = s \Longrightarrow dy = ds$

For $\quad y > y_1$, we have $\quad s > s_1 = y_1 - \sigma\sqrt{T}$

$$I = \frac{S(0)}{\sqrt{2\pi}} \int_{s>s_1}^{\infty} e^{-\frac{s^2}{2}} ds$$

$$= S(0)\phi(-s_1) = S(0)\phi\left(\sigma\sqrt{T} - y_1\right)$$

$$\left(\sigma\sqrt{T} - y_1 = \sigma\sqrt{T} - \frac{1}{\sigma\sqrt{T}}\{\ln\left(\frac{K}{S(0)}\right) - (r - \frac{\sigma^2}{2})T\}\right)$$

$$= S(0)\phi(d_1)$$

where, $$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad \dots(21)$$

hence finally equ. (19) becomes

$$\left.\begin{aligned} C(0) &= S(0)\phi(d_1) - Ke^{-rT}\phi(d_2) \\ d_2 &= -y_1 = d_1 - \sigma\sqrt{T} \end{aligned}\right\} \qquad \dots(22)$$

Where d_1 is given by (21).

The expressions **(21 & 22)** are known as ***Black-Scholes formula*** for pricing the European call option.

In case the European call option price is to be computed at any time $t; 0 < t < T$ then

$$C(t) = S(t)\phi(d_1) - Ke^{-r(T-t)}\phi(d_2) \qquad \dots(23)$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad ,$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$S(t)$ is price of the stock at time t .

The price of **put option** with strike price K and the time to maturity $(T - t)$ using ***Black-Scholes formula***

$$P(t) = Ke^{-r(T-t)}\phi(-d_2) - S(t)\phi(-d_1) \qquad \dots(24)$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad ,$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

For $t = 0$

$$P(0) = Ke^{-rT}\phi(-d_2) - S(0)\phi(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right)+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Black-Scholes formula **for dividend paying stock**

The derivation of Black-Scholes formula is based on the assumption that is stock pays no dividend. But that can be adjusted by reducing the stock price for the discounted amount of dividend for the difference period of $t = 0$ to ex-dividend date, that is, instead of $S(0)$ we have to consider

$$S(0) - div.e^{-rt_{div}}$$

div → amount of dividend & t_{div} → difference of time

If many dividends are paid in between expiry of option then, stock price will be adjusted for sum of all discounted values, i.e.;

$$S(0) - \sum_{i=1}^{n} div_i \cdot e^{-rt_{div_i}}$$

Cox-Ross-Rubinstein formula (Binomial approximation)

The payoff for a call option with strike price X satisfies $f(x) = 0$ for $x \le X$,

The summation starts with the least 'm' such that

$$S(0)(1+u)^m(1+d)^{N-m} > X$$

hence

$$C^E(0) = (1+r)^{-N} \sum_{k=m}^{N} \binom{n}{k} p^{*k}(1-p^*)^{N-k}\{S(0)(1+u)^k(1+d)^{N-k} - X\} \qquad \text{... (1)}$$

this can be written as

$$C^E(0) = x(1)S(0) + y(1) \qquad \text{...(2)}$$

relating the option price to the initial replicating portfolio $x(1)$ & $y(1)$ where

$$x(1) = (1+r)^{-N} \sum_{k=0}^{N} \binom{n}{k} p^{*k}(1-p^*)^{N-k}(1+u)^k(1+d)^{N-k} \qquad \text{... (3)}$$

$$y(1) = -(1+r)^{-N} \sum_{k=0}^{N} \binom{n}{k} p^{*k}(1-p^*)^{N-k}X \qquad \text{... (4)}$$

The expression for $x(1)$ can be written as

$$x(1) = \sum_{k=0}^{N} \binom{n}{k} (p^* \frac{1+u}{1+r})^k\{(1-p^*)\frac{1+d}{1+r}\}^{N-k}$$

$$= \sum_{k=0}^{N} \binom{n}{k} q^k(1-q)^{N-k} \qquad \text{... (5)}$$

Where $q = p^* \frac{1+u}{1+r}$, then $1 - q = (1-p^*)\frac{1+d}{1+r}$ ...(6)

Now $\phi(m, N, p)$ represents cumulative Binomial Distribution with N trials and probability 'p' of success

$$\phi(m, N, p) = \sum_{k=0}^{m} \binom{n}{k} p^{*k}(1-p^*)^{N-k}$$

Theorem : In the binomial model the price of European call option and put option with strike price X to be exercised after N and time steps is given by

$$C^E(0) = S(0)[1 - \phi(m-1, N, q)] - (1+r)^{-N} X[1 - \phi(m-1, N, p^*)]$$

$$P^E(0) = -S(0)\phi(m-1, N, q) + (1+r)^{-N} X\phi(m-1, N, p^*)$$

The initial replicating portfolio $x(1)$ & $y(1)$ is given by

	$\boldsymbol{x(1)}$	$\boldsymbol{y(1)}$
For call	$1 - \phi(m-1, N, q)$	$-(1+r)^{-N} X[1 - \phi(m-1, N, p^*)]$
For put	$-\phi(m-1, N, q)$	$(1+r)^{-N} X.\phi(m-1, N, p^*)$