<u>Portfolio Optimization</u>

<u>*Definition:* Portfolio</u>

A portfolio is a collection of two or more assets say, $a_1, a_2, .. a_n$, represented by an ordered $n - tuple$ $\Theta = \Theta(x_1, x_2, .. x_n)$, where $x_i \in R, i = 1 \dots n$ is the number of units of the asset a_i $(i = 1, :.., n)$ owned by the investor.

We consider only a single period model, ie, in between the initial time taken as $t = 0$ and the final transaction time taken as $t = T$, no transaction ever takes place.

Let $V_i(0)$ and $V_i(T)$ be the values of the ith asset at $t = 0$ and $t = T$. respectively. Let $V_\Theta(0)$ and $V_\Theta(T)$ denote the values of the portfolio $\Theta = \Theta(x_1, x_2, .. x_n)$ at $t = 0$ and $t = T$, respectively. Then,

$$V_\Theta(0) = \sum_{i=1}^{n} x_1 V_i(0) \qquad \& \qquad V_\Theta(T) = \sum_{i=1}^{n} x_i V_i(T)$$

Then the quantity

$$r_\Theta(T) = \frac{V_\Theta(T) - V_\Theta(0)}{V_\Theta(0)}$$

is referred as the return of the portfolio $\Theta(x_1, x_2, .. x_n)$.

<u>*Definition:* Asset Weights</u>

The weight w_i of the asset a_i is the proportion of the value of the asset in the portfolio for $(i = 1, :.., n)$ at $t = 0$, i.e.

$$w_i = \frac{x_i V_i(0)}{V_\Theta(0)} = \frac{a_i V_i(0)}{\sum_{i=1}^{n} x_i V_i(0)} \quad , \quad i = 1, :.., n$$

It can be observed that $w_1 + w_2 + \cdots + w_n = 1$.

Therefore, a portfolio can now be represented by the weights as $(w_1, w_2, .. w_n)$.

§§ $w_i < 0$ for some 'i' is also possible in a portfolio, it indicates that the investor has taken a short position on the $i - th$ asset a_i.

Let r_i be the return on the $i - th$ asset. Then

$$r_i = \frac{V_i(T) - V_i(0)}{V_i(0)} \quad , \quad i = 1, .. n$$

Definition: Mean of the Portfolio Return

(*referred as return of the portfolio*)

Let $(w_1, w_2, .. w_n)$ be a portfolio of 'n' assets $a_1, a_2, .. a_n$. Let r_i, $(i = 1, .. n)$ be the return on the ith asset a_i and $E(r_i) = \mu_i$, $(i = 1, :.., n)$, be its expected value . Then the mean of the portfolio return is defined as

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i$$

[Since (total return of the portfolio) $R_i = \sum_{i=1}^{n} w_i r_i$, hence $\mu = E(R_i)$]

Definition: Variance of the Portfolio

(*referred as risk of the portfolio*)

Let $(w_1, w_2, .. w_n)$ be a portfolio of 'n' assets $a_1, a_2, .. a_n$. Then the variance of the portfolio defined as

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad , where \quad \sigma_{ij} = Cov(r_i, r_j)$$

Also, $\sigma^2{}_i = Var(r_i)$ & $\sigma^2{}_j = Var(r_j)$

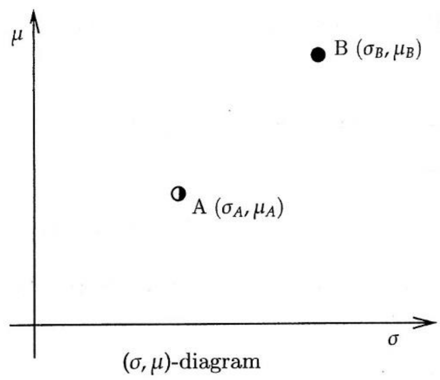
If ρ_{ij} is correlation coefficient between r_i & r_j , then

$$\rho_{ij} = \frac{Cov(r_i, r_j)}{\sigma_i \sigma_j} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \Rightarrow \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$$

Hence variance is also expressed as

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

Therefore given a portfolio $A: (w_1, w_2, .. w_n)$, we can compute its mean μ_A and standard deviation σ_A and therefore get the point A (σ_A, μ_A) in $(\sigma, \mu) - plane$. Thus irrespective of the number assets, a portfolio can always be identified as a point in the $(\sigma, \mu) - plane$.



(σ, μ)-diagram

The portfolio optimization problem refers to the problem of determining weights $w_i, (i = 1,..n)$ such that the return of the portfolio is maximum and the risk of the portfolio is minimum. Thus we aim to solve the following optimization problem,

i. Minimize the risk, ie, $min \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$
ii. Maximize the return, ie, $max \sum_{i=1}^{n} w_i \mu_i$

subject to, $w_1 + w_2 + \cdots + w_n = 1$

[*the search of combination of weights* $(w_1, w_2, .. w_n)$]

Two Assets Portfolio Optimization

Consider a portfolio with two assets , say, a_1 & a_2 with weights w_1 & w_2 returns r_1 & r_2 and standard deviations σ_1 & σ_2 respectively. Then the portfolio expected return μ and portfolio variance σ^2 are given by

$$\mu = E(w_1 r_1 + w_2 r_2) = w_1\mu_1 + w_2\mu_2 \qquad .....(1)$$

$$\sigma^2 = Var(w_1 r_1 + w_2 r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2\, \rho\, \sigma_1\sigma_2 \qquad .....(2)$$

Here ρ is the coefficient of correlation between r_1 & r_2 and lies in $[-1, 1]$.

The value of ρ provides a measure of the extent of diversification of the portfolio so as to reduce risk. Larger the value of ρ with negative sign, smaller will be the value of σ^2 .

Since , $w_1 + w_2 = 1$

Moreover, in case of short selling, the weights can be negative, hence

Let $w_2 = s$ $\quad then \quad w_1 = 1 - s \;,\; s \in R.$

$$\mu = (1 - s)\mu_1 + s\mu_2 \qquad .....(3)$$

$$\sigma^2 = (1 - s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1 - s)\, \rho\, \sigma_1\sigma_2 \qquad .....(4)$$

or, $$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\, \rho\, \sigma_1\sigma_2)s^2 - 2(\sigma_1 - \rho\sigma_2)\sigma_1 s + \sigma_1^2$$

Without loss of generality we assume that $0 < \sigma_1 \leq \sigma_2$.

We discuss equ. (3 & 4) under two cases

(1) $\rho = \pm 1$ (2) $-1 < \rho < 1$

<u>Case (1)</u>: for $\rho = \pm 1$, equ. (3 & 4) reduces to

$$\mu = (1 - s)\mu_1 + s\mu_2$$

$$\sigma = |(1 - s)\sigma_1 \pm s\sigma_2\, |$$

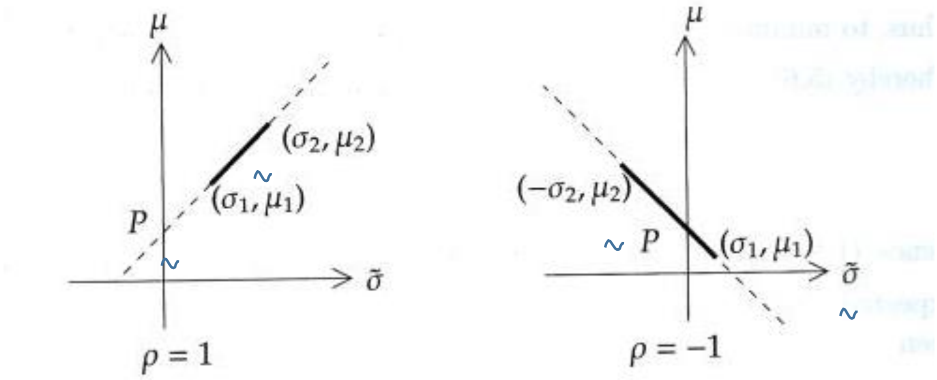
For $s \in [0,1]$ both weights are non-negative (no short selling). We have

$$\mu = (1-s)\mu_1 + s\mu_2$$

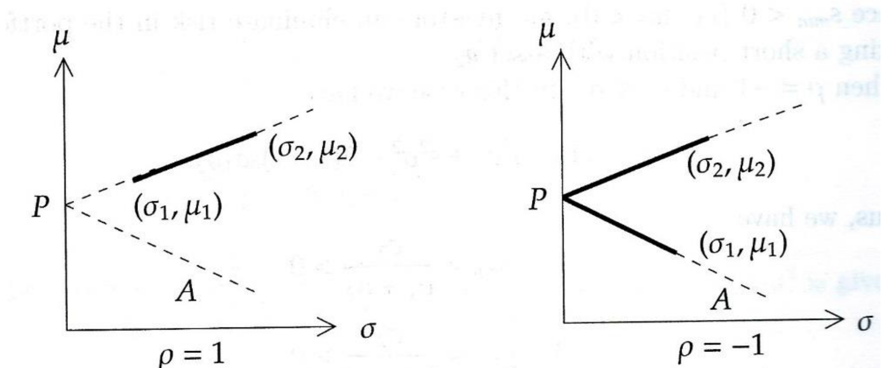
$$\tilde{\sigma} = (1-s)\sigma_1 + s\sigma_2 \quad (\rho = 1) \qquad or \qquad \tilde{\sigma} = (1-s)\sigma_1 - s\sigma_2 \quad (\rho = -1)$$

For, we plot the points $(\tilde{\sigma}, \mu)$ (obtained by considering the different values of s) in $(\tilde{\sigma}, \mu) - plane$.

We get the following fig. (graphs are essentially straight lines). The bold parts corresponding to $s \in [0,1]$.



Subsequently, we plot on $(\sigma, \mu) - plane$, and we the following graph



(σ, μ)-graph for$\rho = \pm 1$.

(i) $\rho = 1 \quad and \quad \sigma_1 < \sigma_2$, we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\sigma_1\sigma_2 = \{(1-s)\sigma_1 + s\sigma_2\}^2 \qquad .....(5)$$

For extremum, $\frac{d\sigma^2}{ds} = 2(\sigma_1 - \sigma_2)\{(1-s)\sigma_1 + s\sigma_2\} = 0$ or

$$s_{min} = -\frac{\sigma_1}{\sigma_2 - \sigma_1} < 0 \qquad .....(6)$$

and

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_2 - \sigma_1)^2 > 0$$

hence,

$$1 - s_{min} = 1 + \frac{\sigma_1}{\sigma_2 - \sigma_1} = \frac{\sigma_2}{\sigma_2 - \sigma_1} > 0$$

Let μ_{min} & ${\sigma^2}_{min}$ denote the expected return and variance of the portfolio, and

$$w_1 = 1 - s_{min} = \frac{\sigma_2}{\sigma_2 - \sigma_1} > 0 \ ,$$

$$\& \qquad w_2 = s_{min} = -\frac{\sigma_1}{\sigma_2 - \sigma_1} < 0 \qquad \ldots..(7)$$

$$\mu_{min} = \frac{\sigma_2\mu_1 - \sigma_1\mu_2}{\sigma_2 - \sigma_1} \qquad \& \qquad {\sigma^2}_{min} = 0 \qquad \ldots..(8)$$

Since $s_{min} < 0$ ie, $w_2 < 0$ an investor can eliminate risk in the portfolio by taking a short position with respect to asset a_2 .

(ii) $\rho = -1$ & $\sigma_1 \leq \sigma_2$ we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 - 2s(1-s)\sigma_1\sigma_2 = \{(1-s)\sigma_1 - s\sigma_2\}^2$$

$$\frac{d\sigma^2}{ds} = 2(\sigma_1 + \sigma_2)\{(1-s)\sigma_1 - s\sigma_2\} = 0 \qquad \ldots..(9)$$

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_2 + \sigma_1)^2 > 0$$

$$w_2 = s_{min} = \frac{\sigma_1}{\sigma_2 + \sigma_1} > 0 \ ,$$

$$\& \qquad w_1 = 1 - s_{min} = \frac{\sigma_2}{\sigma_2 + \sigma_1} > 0 \qquad \ldots..(10)$$

$$\mu_{min} = \frac{\sigma_2\mu_1 + \sigma_1\mu_2}{\sigma_2 + \sigma_1} \qquad \&$$

$${\sigma^2}_{min} = 0 \qquad \ldots..(11)$$

Since w_1 & w_2 both are positive hence the investor can eliminate the risk without selling.

<u>Case (2)</u>: $-1 < \rho < 1$ from equ. (3 & 4)

$$\mu = (1-s)\mu_1 + s\mu_2$$

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)s^2 - 2(\sigma_1 - \rho\sigma_2)\sigma_1 s + \sigma_1^2$$

It represents the parametric equation of a parabola in $(\sigma^2, \mu) - plane$.

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow s = \frac{(\sigma_1 - \rho\sigma_2)\sigma_1}{(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)}$$

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2) = 2\left[(\sigma_1 - \rho\sigma_2)^2 + (\sigma_2^2 - \rho^2\sigma_2^2)\right] > 0 \quad (\because \rho < 1)$$

With the result we get

$$w_2 = s_{min} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2 = (\sigma_1 - \rho\sigma_2)\sigma_1}{(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)} \qquad \&$$

$$w_1 = 1 - s_{min} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2 = (\sigma_2 - \rho\sigma_1)\sigma_2}{(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)} \qquad \ldots..(12)$$

$$\mu_{min} = (\mu_2 - \mu_1)s_{min} + \mu_1 \qquad \&$$

$$\sigma^2{}_{min} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)} \qquad \ldots..(13)$$

i. If $-1 \le \rho < \frac{\sigma_1}{\sigma_2}$ then using equ. (12) we have $0 < s_{min} < 1$ then minimum risk is attained without short selling. Also in this case

$$\sigma^2{}_{min} = 0 \quad for \quad \rho = -1$$

ii. If $\rho = \frac{\sigma_1}{\sigma_2} \Leftrightarrow s_{min} = 0 \Leftrightarrow \sigma^2{}_{min} = \sigma_1^2 \qquad (\because\ s_{min} = w_2 = 0)$

iii. If $\frac{\sigma_1}{\sigma_2} < \rho \le 1 \Longrightarrow s_{min} < 0$, ie, In this case the investor has taken a short position on asset a_2 in order to minimize the portfolio risk.

$$\sigma^2{}_{min} = 0 \quad for \quad \rho = 1$$

Some Results and Discussion

§§ The variance σ^2 of a portfolio cannot exceed the greater of the variances σ_1^2 & σ_2^2 of the components, ie,

$\sigma^2 \leq \max\{\sigma_1^2, \sigma_2^2\}$ if short selling is not allowed.

Proof:

Let us assume that $\sigma_1^2 \leq \sigma_2^2$. If short sales are not allowed, then w_1 & $w_2 > 0$ and

$$w_1\sigma_1 + w_2\sigma_2 \leq (w_1 + w_2)\sigma_2 = \sigma_2$$

Since the correlation coefficient satisfies $-1 \leq \rho \leq 1$, hence

$$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\,\rho\,\sigma_1\sigma_2$$
$$\leq (w_1\sigma_1 + w_2\sigma_2)^2 \leq \sigma_2^2$$

If $\sigma_1 > \sigma_2$, the proof is analogous.

§§ For $-1 < \rho < 1$ the portfolio with minimum variance is attained at

$$s_g = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{(\sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\sigma_2)}$$

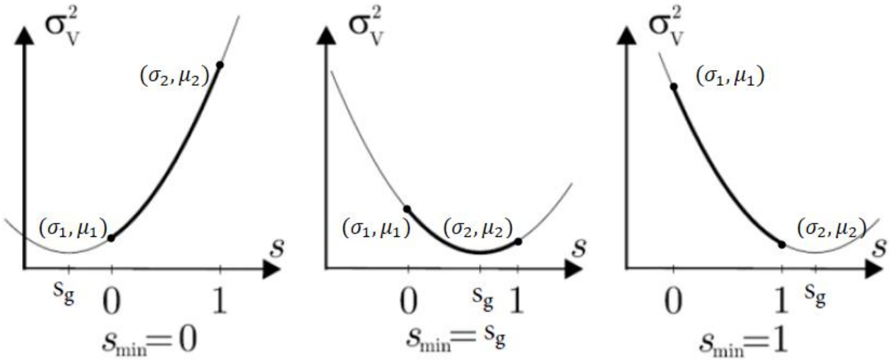
<u>*If short sales are not allowed*</u>, then the smallest variance is attained at

$$s_{min} = \begin{cases} 0 & if\ s_g < 0 \\ s_g & if\ 0 \le s_g \le 1 \\ 1 & if\ s_g > 1 \end{cases}$$

Proof: The relation for s_g has already been obtained for global minimum (*since* σ^2 *is a quadratic function of*) in equ. (12).

The subsequent result for no short sell can be visualized through the following graph drawn in the $(s, \sigma^2) - plane.$

Since the expression for σ^2 is quadratic in s, hence the graph is parabola, and bold parts is corresponding to the portfolio's with no short selling, ie, $0 \le s \le 1,$ and we only obtain a segment of the curve. As s increases from 0 to 1, the corresponding point (σ, μ) travels along the curve in the direction from (σ_1, μ_1) to (σ_2, μ_2) .



The minimum of σ_V^2 as a function of s

§§ $-1 \leq \rho \leq 1$ we have the following possibilities for $\sigma_1 \leq \sigma_2$

i. If $-1 \leq \rho < \frac{\sigma_1}{\sigma_2}$, then there is a portfolio without short selling such that $\sigma < \sigma_1$.

ii. If $\rho = \frac{\sigma_1}{\sigma_2}$, then $\sigma \geq \sigma_1$ for each portfolio.

iii. If $\frac{\sigma_1}{\sigma_2} < \rho \leq 1$, then there is a portfolio with short selling such that $\sigma < \sigma_1$.

Proof:

i. If $-1 \leq \rho < \frac{\sigma_1}{\sigma_2}$, then from expression of s_g

$$0 < s_g < \frac{\sigma_1}{\sigma_1 + \sigma_2} \ , \qquad \because \frac{\sigma_1}{\sigma_1 + \sigma_2} < 1$$

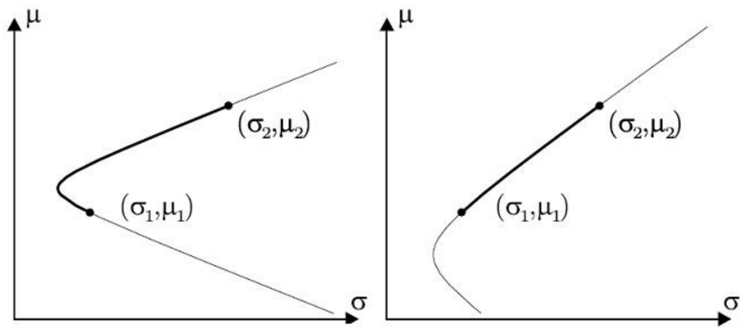
Hence, $0 < s_g < 1$

That means no short selling and $\sigma < \sigma_1$.

ii. If $\rho = \frac{\sigma_1}{\sigma_2}$, then $s_g = 0$ as a result we have $\sigma \geq \sigma_1$ for every portfolio because minimum variance is σ_1^2 .

iii. If $1 \geq \rho > \frac{\sigma_1}{\sigma_2}$ then $s_g < 0$, that means the portfolio with minimum variance that correspond to s_g involves short selling of security a_2 and satisfy $\sigma < \sigma_1$. And if short selling is not allowed then minimum variance will be σ_1, hence $\sigma > \sigma_1$ for all other portfolio.

shows two typical examples of such curve, with ρ close to but greater than -1 (left) and with ρ close to but smaller than $+1$ (right). Portfolios without short selling are indicated by the bold line segments.



Typical portfolio lines with $-1 < \rho_{12} < 1$

Multi Asset Portfolio Optimization

The weights of the various assets $a_1, a_2, .. a_n$ in the portfolio are written in the

vector form $w^T = [w_1, w_2, .. w_n]$.

Let

$e^T = (1, \dots, 1) \in R^n$, then $w_1 + w_2 + \dots w_n = 1$ can be expressed as $e^T w = 1$.

Let $m^T = (\mu_1, \mu_2, .. \mu_n)$ be the expected return vector of the portfolio , where, $\mu_i = E(r_i)$ and $C = [c_{ij}]$ denotes the $n\,x\,n$ variance-covariance matrix, ie,

$c_{ij} = Cov(r_i, r_j)\,,\ i, j = 1, .. n$.

Obviously C is a symmetric matrix. Now the expected return μ of the portfolio is given by

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i = m^T w$$

and the variance of the portfolio is

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_j \sigma_{ij} = w^T C\, w$$

Here C is certainly positive semidefinite. In practice, it is also assumed to be positive definite (and hence invertible) because the minimum risk of a general n-asset portfolio is rarely zero.

The Feasible Region of a Portfolio Problem

Let $W = \{w \in R^n : e^T w = 1\}$ be the collection of all portfolios. Each portfolio corresponds to a point in the $(\sigma, \mu) - plane$ say (σ^w, μ^w) . Then the set $\{(\sigma^w, \mu^w) : w \in W\}$ is called the feasible region.

Consider the n-dimensional hyperplane $e^T w = 1$, in which the weight vector w

resides. Let f be the mapping that takes each weight vector in the weight hyperplane to the corresponding portfolio point in the $(\sigma, \mu) - plane$. Our aim is to find the image of any straight line in the weight hyperplane $e^T w = 1$ under the mapping f.

The parametric equation of any line in the weight hyperplane is of the form

$$l(\xi) = (s_1\xi + b_1, s_2\xi + b_2, \ldots + s_n\xi + b_n)^T$$

$$= s\xi + b \ , \qquad -\infty < \xi < \infty$$

Where , $\quad s = (s_1, s_2, .. s_n)^T \quad \& \quad b = (b_1, b_2, .. b_n)^T$

Let w be any point this line, Then,

$$\mu = m^T w$$

$$= m^T(s\xi + b)$$

$$= \xi(m^T s) + (m^T b)$$

Let , $\quad (m^T s)^{-1} = \alpha \quad \& \quad -(m^T b)(m^T s)^{-1} = \beta \quad$, then

$$\xi = \alpha\mu + \beta$$

Now,

$$\sigma^2 = w^T C\, w$$

$$= (s\xi + b)^T C\,(s\xi + b)$$

$$= (s^T Cs)\xi^2 + (s^T Cb + b^T Cs)\xi + b^T Cb$$

$$= \gamma\xi^2 + \delta\xi + \eta$$

Substituting the value of ξ , we have

$$\sigma^2 = \gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta \qquad \ldots..(A)$$

For $\quad -\infty < \xi < \infty$, the ordered pair (σ^2, μ) traces a parabola given by equ. (A) with axis parallel to σ -axis in $(\sigma^2, \mu) - plane$.

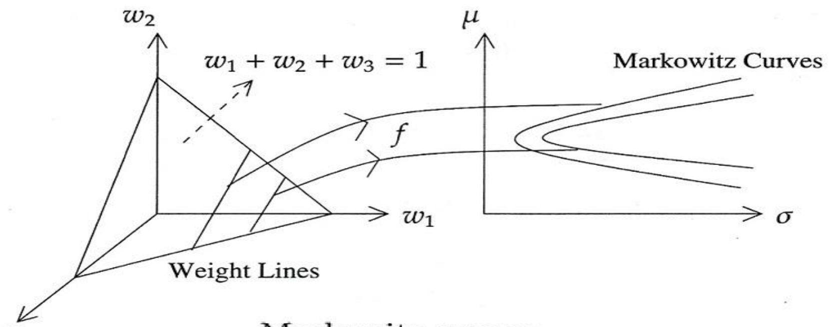
For $(\sigma, \mu) - plane$

$$\sigma = \sqrt{\gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta} \qquad \ldots.(B)$$

The curve represented by equ.(B) in $(\sigma, \mu) - plane$ is called *Markowitz Curve*.

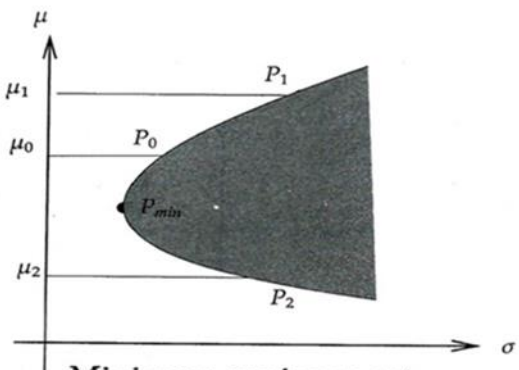
(The Markowitz curve given by equ.(B) is not a parabola in $(\sigma, \mu) - plane$. The M.C. behaves almost as a straight line as $\mu \to \infty$, whereas it is always possible to draw tangent on parabola for any value of μ .

As we cover all possible weight lines of the weight hyperplane, we trace a family of Markowitz curves in the $(\sigma, \mu) - plane$, and this generates a solid region in the $(\sigma, \mu) - plane$, and its shape will be like a bullet, which is appropriately called the *Markowitz bullet*.)



Markowitz curves.

The solid region generated by all possible weight vectors. Its left boundary is called the minimum variance set. The return corresponding to points $P_0, P_1, \& P_2$ are $\mu_0, \mu_1, \& \mu_2$ respectively with known level of risk . There is a point P_{min} which has the least variance. This point is called the minimum variance point.



Minimum variance set.

To find the minimum variance point we need to solve the following risk minimization problem

$$\left.\begin{array}{lc} min & \sigma^2 = w^T C\, w \\ subject\ to & e^T w = 1 \end{array}\right\} \ldots.(1)$$

Theorem: Portfolio with minimum risk has weight given by

$$w = \frac{C^{-1}e}{e^T C^{-1} e}$$

Proof: Using the method of Lagrange Multiplier to solve equ. (1), we have

$$L(w, \lambda) = w^T C\, w + \lambda(1 - e^T w)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_j \sigma_{ij} \ + \lambda(1 - e^T w)$$

differentiating w.r.t. $'w'$ and equating to zero, we get

$$2w^T C - \lambda e^T = 0 \Longrightarrow w = \tfrac{\lambda}{2} C^{-1} e \qquad \ldots..(2)$$

substituting in second equation of (1)

$$\frac{\lambda}{2} e^T C^{-1} e = 1$$

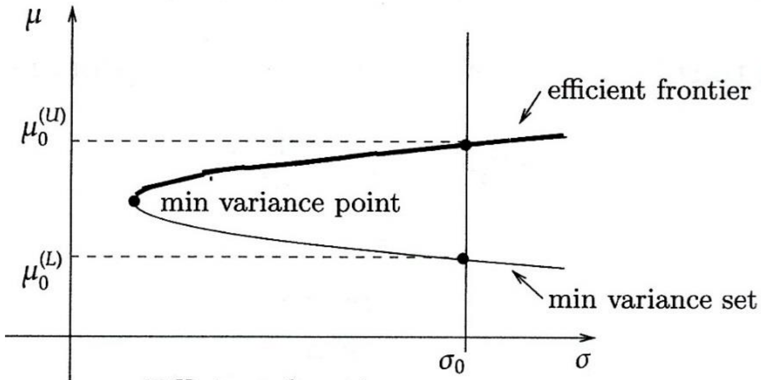
$$\frac{\lambda}{2} = \frac{1}{e^T C^{-1} e} \qquad \ldots..(3)$$

Putting in equ (2) from (3)

$$w = \frac{C^{-1}e}{e^T C^{-1} e}$$

Markovitz Efficient Frontier

Looking at the minimum variance set in the $(\sigma, \mu) - plane$,



Efficient frontier.

we observe that for a given level of risk, (say σ_1), there are two values of returns $\mu_0^L < \mu_0^U$. Since we want to maximize the return hence obvious choice is μ_0^U . Therefore in the minimum variance set, it is only the upper half which is of importance for investment.

This upper half portion of the minimum variance set is called the *Markowitz efficient frontier*.

Sometimes investor is more concern about the return. Hence for a given return μ ,we are to find the value of the weights for minimum risk.

Theorem: For a given expected return , the portfolio with minimum risk has weights given by

$$w = \frac{det\begin{pmatrix}\mu & m^T C^{-1} e \\ 1 & e^T C^{-1} e\end{pmatrix} C^{-1}m + det\begin{pmatrix}m^T C^{-1} m & \mu \\ e^T C^{-1} m & 1\end{pmatrix} C^{-1}m}{det\begin{pmatrix}m^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e\end{pmatrix}}$$

Proof. We have to solve the following quadratic programming problem

$$min \qquad \sigma^2 = \frac{1}{2} w^T C\, w \qquad .....(1)$$

Subject to,

$$m^T w = \mu \qquad .....(2)$$

$$e^T w = 1 \qquad ....(3)$$

We define the Lagrangian with 2-parameters

$$L(w, \alpha, \beta) = \frac{1}{2} w^T C\, w + \alpha(\mu - m^T w) + \beta(1 - e^T w), \qquad \alpha, \beta \in R$$

$$\frac{\partial L}{\partial w} = 0 = w^T C - \alpha m^T - \beta e^T$$

or

$$w = C^{-1}(\alpha m + \beta e) \qquad ...(4)$$

substituting in equ (2 & 3) from equ (4)

$$(m^T C^{-1} m)\alpha + (m^T C^{-1} e)\beta = \mu \qquad ...(5)$$

$$(e^T C^{-1} m)\alpha + (e^T C^{-1} e)\beta = 1 \qquad ...(6)$$

Solving for α & β we get

$$\alpha = \frac{det\begin{pmatrix}\mu & m^T C^{-1} e \\ 1 & e^T C^{-1} e\end{pmatrix}}{det\begin{pmatrix}m^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e\end{pmatrix}}, \qquad \& \qquad \beta = \frac{det\begin{pmatrix}m^T C^{-1} m & \mu \\ e^T C^{-1} m & 1\end{pmatrix}}{det\begin{pmatrix}m^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e\end{pmatrix}}$$

Putting the value in equ. (4) we get the result

Karush-Kuhn-Tucker optimality conditions. The result is known as the two fun d theorem.

Theorem 5.6.3 (Two Fund Theorem) Two efficient portfolios can be established so that any other efficient portfolio can be duplicated, in terms of mean and variance, as a linear combination of these two assets. In other words, it says that, an investor seeking an efficient portfolio need to invest only in the combination of these two assets. To be decided

Capital Asset Pricing Model (CAPM)

So far we considered the portfolio consisting of risky assets. Now we consider the portfolio containing risk free asset also.

Consider a portfolio with n risky assets $a_1, a_2, .. a_n$ with weight $w_1, w_2, .. w_n$ and

one risk-free asset a_{rf} with weight w_{rf} · Then

$$w_{risky} + w_{rf} = \sum_{i=1}^{n} w_i + w_{rf} = 1 \qquad \dots (1)$$

the expected return and the variance associated with this portfolio are given by

$$\mu = \sum_{i=1}^{n} w_i \mu_i + w_{rf}.\mu_{rf} \qquad \dots (2)$$

and

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i + w_{rf}\mu_{rf}\right) = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sigma^2{}_{risky} \qquad \dots (3)$$

respectively.

If we remove the risk-free asset from the portfolio and readjust the weight of the risky assets so that their sum remain 1, the resultant portfolio so obtained is referred to as the <u>derived risky portfolio</u>. We shall use μ_{der} & σ_{der}^2 to represent the expected return and risk of the derived risky portfolio. Then

$$\mu = w_{risky} \sum_{i=1}^{n} \frac{w_i}{w_{risky}} \mu_i + w_{rf}.\mu_{rf}$$

$$= w_{risky}.\mu_{der} + w_{rf}.\mu_{rf}$$

$$= w_{risky}.\mu_{der} + (1 - w_{risky}).\mu_{rf}$$

$$= w_{risky}(\mu_{der} - \mu_{rf}) + \mu_{rf} \qquad \dots (4)$$

Also,

$$\sigma^2 = Var\left(w_{risky} \sum_{i=1}^{n} \frac{w_i}{w_{risky}} r_i\right) = w_{risky}^2 Var\left(\sum_{i=1}^{n} \frac{w_i}{w_{risky}} r_i\right)$$

$$= w_{risky}^2 \sigma_{der}^2 \Longrightarrow w_{risky} = \frac{\sigma}{\sigma_{der}} \qquad \dots(5)$$

Putting in equ. (4) we get

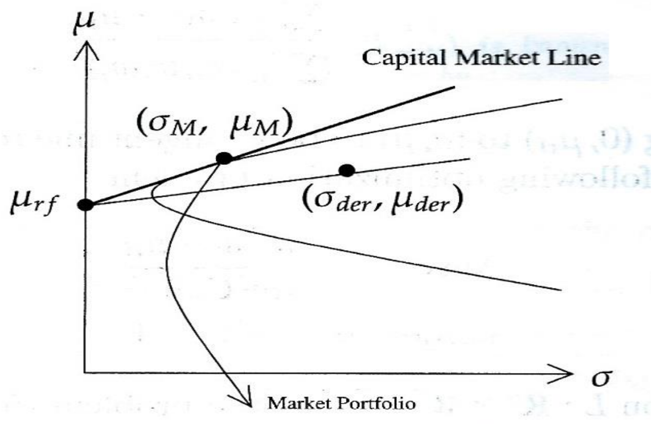
$$\mu = \mu_{rf} + \frac{(\mu_{der} - \mu_{rf})\sigma}{\sigma_{der}} \qquad \dots(7)$$

It is an equation of the line in $(\sigma, \mu) - plane$ joining the points $(0, \mu_{rf})$ & $(\sigma_{der}, \mu_{der})$.

Now, for a given risk , if we choose various weight combinations of risk-free

asset and risky assets for which equ. (1) holds, we generate different lines represented by equ. (7) $(\sigma, \mu) - plane$.

The line that produces the point with highest expected return for a given risk is tangent to the upper portion of the Markowitz bullet.



Capital market line.

Definition: Capital Market Line (CML)

Among all the lines satisfying equ. (7) for various weight combinations of risk-free and risky assets, the line giving the highest return for a given risk is called the capital market line, and point of contact on the Markowitz bullet is said to represent Market Portfolio.

Theorem: One Fund Theorem

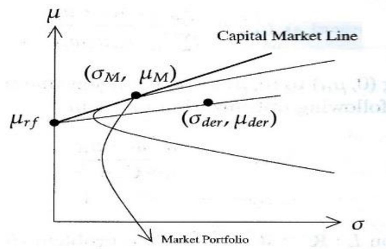
There exists a single portfolio, namely the market portfolio M, of risky assets such that any efficient portfolio can be constructed as a linear combination of the market portfolio M and the risk-free asset.

Theorem: For any expected risk-free return μ_{rf}, the weight vector w_M of the market portfolio is given by

$$w_M = \frac{C^{-1}(m - \mu_{rf}\, e)}{e^T C^{-1}(m - \mu_{rf}\, e)}$$

Proof:

For any (σ, μ) in the Morkovitz bullet, the slope of the line joining $(0, \mu_{rf})$ and (σ, μ) is



Capital market line.

$$\mu = \frac{\mu - \mu_{rf}}{\sigma} = \frac{\sum_{i=1}^{n} \mu_i w_i - \mu_{rf}}{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i\, w_j c_{ij}\right)^{1/2}}$$

For the line joining $(0, \mu_{rf})$ and (σ, μ) to be a tangent line to the Markovitz bullet, we to solve,

$$\left.\begin{array}{ll} \max & \dfrac{m^T w - \mu_{rf}}{(w^T C\, w)^{1/2}} \\ subject & e^T w = 1 \end{array}\right\} \quad \dots (1)$$

The Lagrange's function $L: R^n \times R \to R$ for equ. (1) will be

$$L(w, \lambda) = \frac{m^T w - \mu_{rf}}{(w^T C\, w)^{1/2}} + \lambda(1 - e^T w)$$

considering $\frac{\partial L}{\partial w} = 0$

$$\frac{1}{w^T C\, w}\left[(w^T C\, w)^{1/2} m - (m^T w - \mu_{rf})(w^T C\, w)^{-1/2} C w\right] = \lambda e$$

or $$\sigma\, m - (\mu - \mu_{rf})\frac{Cw}{\sigma} = \lambda \sigma^2 e$$

or $$\sigma^2 m - (\mu - \mu_{rf}) C w = \lambda \sigma^3 e$$

...(2)

or $$\sigma^2 w^T m - (\mu - \mu_{rf}) w^T C\, w = \lambda \sigma^3 w^T e$$

since $w^T m = \mu$, $e^T w = 1$, & $\sigma^2 = w^T C\, w$

$$\sigma^2 \mu - (\mu - \mu_{rf})\sigma^2 = \lambda \sigma^3$$

$$\lambda = \frac{\mu_{rf}}{\sigma} \quad \dots (3)$$

Hence the requisite value of weight vector w_M can be obtained using equ. (2 & 3)

§§ If the market portfolio (σ_m, μ_m) is k:nown. Then the equation of the capital market line is given by

$$\mu = \mu_{rf} + \frac{(\mu_M - \mu_{rf})}{\sigma_M}\sigma$$

If the investor is willing to take a positive risk σ, it will in an additional return $\frac{(\mu_M - \mu_{rf})}{\sigma_M}\sigma$ over and above the risk-free return.

§§ Now suppose an investor is willing to take risk σ_P, then for this risk the expected return μ_P is maximum if the point (σ_P, μ_p) lies on CML (Capital Market Line) , thus,

$$\mu_P = \mu_{rf} + \frac{(\mu_M - \mu_{rf})}{\sigma_M}\sigma_P$$

If we let $w_P = \frac{\sigma_P}{\sigma_M}$, then

$$\mu_P = \mu_{rf}(1 - w_P) + w_P\, \mu_M$$

That means investor should invest $w_P = \frac{\sigma_P}{\sigma_M}$ proportion of investment in index and $(1 - w_P)$ proportion in risk free.

Theorem: Suppose the market portfolio is (σ_M, μ_M) . The expected return of an asset a_i is given by

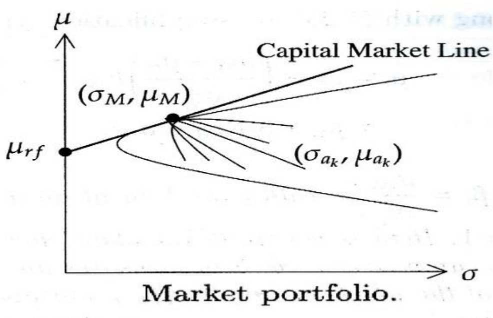
$$\mu_i = \mu_{rf} + \beta_i(\mu_M - \mu_{rf}), \quad where \quad \beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2}$$

Proof: Suppose an investor portfolio comprises of asset a_i with weight w and the market portfolio M with weight $1 - w$. Then the expected return and risk of the portfolio will be

$$\left.\begin{aligned} \mu &= w\mu_i + (1-w)\mu_M \\ \sigma^2 &= w^2\sigma_i^2 + (1-w)\sigma_M^2 + 2\rho w(1-w)\sigma_i\,\sigma_j \end{aligned}\right\} \quad \ldots(1)$$

Where ρ is coefficient of correlation between the returns of asset a_i and the market portfolio M.

As w varies, these values trace out a curve in the $(\sigma, \mu) - plane$. When w=0 the capital market line (CML) becomes tangent to the curve at M. So we have the condition that slope of the curve is slope of the CML at M, differentiating (1)



$$\left.\frac{d\mu}{d\sigma}\right|_{w=0} = \left.\frac{d\mu}{dw}\frac{dw}{d\sigma}\right|_{w=0} = (\mu_i - \mu_M)\left.\frac{dw}{d\sigma}\right|_{w=0}$$

$$also \qquad \left.\frac{d\sigma}{dw}\right|_{w=0} = \left.\frac{w\sigma_i^2 - (1-w)\sigma_M^2 + \rho\sigma_i\sigma_M(1-2w)}{\sigma}\right|_{w=0}$$

$$= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}, \qquad (\, since \;\; \sigma_{iM} = \rho\sigma_i\sigma_M \,)$$

$$therefore, \qquad \left.\frac{d\mu}{d\sigma}\right|_{w=0} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} \qquad \ldots (2)$$

Equating with slope of line, we get,

$$\frac{\mu_i - \mu_{rf}}{\sigma_M} = \left.\frac{d\mu}{d\sigma}\right|_{w=0} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

$$or \qquad \mu_i = \mu_{rf} + \frac{(\mu_M - \mu_{rf})\sigma_M}{\sigma_M^2}$$

$$or \qquad \mu_i = \mu_{rf} + \beta_i(\mu_M - \mu_{rf}) \qquad \ldots (3)$$

Here $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$ is called the ***beta of an asset***.

If $\boldsymbol{\beta_i = 0}$ ie, asset is completely uncorrelated with the market. Thus CAPM gives $\mu_i = \mu_{rf}$ $using$ (3)
Showing that however large is the risk σ_i , return will always be limited to risk free, ie, no risk premium.

If $\boldsymbol{\beta_i < 0}$, then $\mu_i < \mu_{rf}$, correlated negatively with the market. Hence can be used to reduce the overall risk of the portfolio when other assets are not doing well. For this reason it is called *insurance.*

(The above discussion suggests that though for a portfolio an appropriate mea- sure of risk is σ *but for an individual asset the proper measure of risk is its beta.)*

Example 5.7.2

Let the risk-free ·rate Pr! be 8% and the market has µM = 12% and aM = 15%. Let an asset a be given which has covariance of 0.045 with the market. Determine the expected rate of return of the given asset.

Therefore the expected rate of return of the given asset is 16%.

***Definition*: Beta of the Portfolio**

The overall β of the portfolio is defined as

$$\beta = \sum_{i=1}^{n} w_i \, \beta_i$$

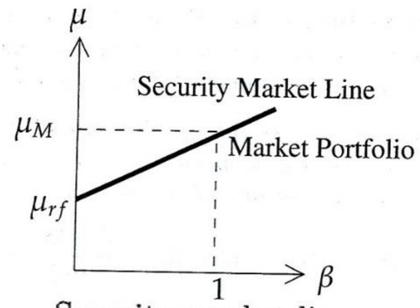
Ie, the weighted average of the betas of the individual assets in the portfolio with the weights being those that define the portfolio.

***Definition*: Security Market Line**

A linear equation

$$\mu = \mu_{rf} + \beta\left(\mu_M - \mu_{rf}\right), \qquad where \quad \beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2}$$

that describes the expected return for all assets in the market is called the *security market line*.



Security market line.

CAPM as a Pricing Formula

Let an asset be purchased at price P and later sold at price Q , then rate of return

$$r = \frac{Q - P}{P}$$

Here P is known but Q is random. If we write $E(Q) = \tilde{Q}$, then the CAPM formula gives

$$\frac{\tilde{Q} - P}{P} = \mu_{rf} + \beta(\mu_M - \mu_{rf})$$

or

$$P = \frac{\tilde{Q}}{1 + \mu_{rf} + \beta(\mu_M - \mu_{rf})}$$

β is the beta of given asset.

If $\tilde{Q}$ is known then price P can be determined , since $P = \frac{Q}{1+\mu_{rf}}$, hence

$\mu_{rf} + \beta(\mu_M - \mu_{rf})$ can be interpreted as *risk adjusted interest rate*.

Limitation of Markovitz Model

Markovitz model is known to be valid if returns r_i are normally distributed and the investor is 'risk averse' preferring lower standard deviation (S.D.). The idea of standard deviation being a measure of risk may not be very appealing to investor. This means the perception about risk of an investor is symmetric about mean. There are several empirical studies, which reveal that most r_i are not normally are even symmetrically distributed.

If it is not symmetric about me then consideration of skewness and kurtosis may also be considered. Thus the efficient Frontier in Markovitz model may be generated in (mean, variance, skewness, kurtosis)-space.