MC306 financial engineering

Assignment - 3

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Ans. 1) Consider $\Omega = \{a, b, c, d\}$

Condition $F_1 \subset F_2 \subset F_3 \subset F_4$

let $F_1 = \{\phi, \Omega\}$

$F_2 = \{\phi, \Omega, \{a\}, \{bcd\}\}$

$F_3 = \{\phi, \Omega, \{a\}, \{bcd\}, \{b\}, \{acd\}\}$

$F_4 = \{\phi, \Omega, \{a\}, \{b\}, \{c\}, \{bcd\}, \{acd\}, \{abd\}, \{ac\}, \{bc\}\}$

F_1, F_2, F_3, F_4 are all σ-fields satisfying the given condition.

Ans.2) $W(s) + W(t)$ can be written as:

$= [W(t) - W(s)] + W(s) + [W(s) - W(0)]$

$W(s)$ is independent increment

$W(s) + W(t) = 2(W(s) - W(0)) + [W(t) - W(s)]$

which is normally distributed

$\sim N(0, 2^2 s + (t-s))$

$\sim N(0, 3s + t)$

Ans - 3) X & Y are uniformly distributed with mean $E[X] = E[Y] = (\pi - \pi)/2 = 0$

$Var[X] = Var[Y] = [\pi - (-\pi)]^2/12 = 4\pi^2/12$
$= \pi^2/3$

$E[X^2] = Var X + E^2[X] = E[Y^2] = \pi^2/12$

$Z(t) = \cos(tX + Y)$

consider $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots$

$E[Z(t)] = E[\cos(tX+Y)] = E\left[1 - \frac{(tX+Y)^2}{2!} + \frac{(tX+Y)^4}{4!} \dots\right]$

$= 1 - \frac{1}{2} E[X^2t^2 + Y^2 + 2XYt] + \dots$

$= 1 - \frac{1}{2}\left[\frac{\pi^2}{3}t^2 + \frac{\pi^2}{3} + 0\right] + \dots$

$= 1 - \frac{\pi^2}{6}[t^2 + 1] + \dots$

$E[Z(t)]$ is a function of t^2 & not independent

Thus, this is not wide sense stationary process.

Ans.4) Given $Z(t)$ is normally distributed with mean $= 0$, $Var = 1$

$$E[Z^2] = Var Z + E^2[Z]$$
$$= 1$$

Now, $X = \sqrt{t} Z$

$$E[X] = E[\sqrt{t} Z] = \sqrt{t} E[Z] = 0$$

$Var X = t$

$X(0) = 0$

Var is of the form $\sigma^2 t$ & mean is $\mu\sqrt{t}$

thus, X is a Brownian Motion.

Ans. 5) let $0 < s < t$, $W(t) - W(s)$ is independent of F_s. $W(s)$ is F_s measurable & σ be any positive constant, we have

$$E(e^{\sigma W(t)} | F_s) = E(e^{\sigma(W(t) - W(s))} | F_s)$$
$$= e^{\sigma W(s)} E(e^{\sigma(W(t) - W(s))})$$

$$E(e^{\sigma(W(t) - W(s))}) = (-s/2)$$

$$W(t) - W(s) \sim N(0, t-s)$$

Hence, $E(e^{\sigma W(t)} | F_s) = e^{\sigma W(s)} e^{\sigma(t-s)/2}$

This gives,
$$E(e^{\sigma W(s) - \frac{\sigma^2}{2}} | F_s)$$
$$= e^{\sigma^2(W(s) - s/2)}$$

thus, $\exp(\sigma W(t) - \frac{t^2\sigma}{2})$ is a martingale

6) Using the Ito - Doeblin formula

$$dw^2(t) = dt + 2x\, w(t)\, d(t)$$

using equiv. integral equation with under $w(0) = v$, we get

$$w^2(t) = t + 2\int_0^t w(t)\, dw(t)$$

$$\int_0^t w(t)\, dw(t) = \frac{1}{2} w^2(t) - \frac{1}{2}(t)$$

$= w^2(t)$ is Ito process

a) $dS = rSdt + \sigma S dW$

which is the given SDE

Current price $S(0)$, $r = 0.04$, $\sigma = 0.1$

$K = 1.25 \times S(0)$ strike price

$T = 0.5$

Using 2nd version of Ito-Doeblin formula, solving (1) with it & substituting

$$S(t) = S(0) \exp\{(r - \frac{\sigma^2}{2})t + \sigma W(t)\}$$

Probability that call opn expires in the money,

$$P(S(t) > K) \text{ — } (2)$$

$$S(t) = S(0) \exp\{(r - \frac{\sigma^2}{2})t + \sigma W(t)\} > K$$

solving the inequality we get

$$W(t) > \frac{\left(\ln\left(\frac{S(0)}{K}\right) + (r - \frac{\sigma^2}{2})t\right)}{\sigma}$$

substituting values we get

$$W(t) > -\frac{\left[\ln\left(\frac{4}{5}\right) + \left(\frac{4}{100} - \frac{1}{200}\right)\frac{1}{2}\right]}{1/10} = \frac{\ln\left(\frac{5}{4}\right) - \frac{7}{400}}{1/10}$$

$$W(t) > 2.056435 \simeq 2.06$$

using (2) we get

$$P(S(t) > K) = P(W(t) > 2.06) = P(W(t) \leq -2.06)$$

$$= \phi(-2.06) = 0.0197$$

which is the required probability

consider 1st version of Ito Doeblin formula

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t))\,dt \quad -(1)$$

Let us consider $f(x) = \frac{x^3}{3}$ & we get

$f'(x) = x^2$

$f''(x) = 2x$

where x is $W(t)$

substituting values of f, f', f'' in (1), we get

$$\int_0^T \frac{dW^3(t)}{3} = \int_0^T W^2(t)\,dW(t) + \int_0^T W(t)\,d(t)$$

$$\int_0^T W^2(t)\,dW(t) = \frac{W^3(t)}{3} - \int_0^T W(t)\,dt$$

Now, we have to find the value of $\int_0^T W(t)\,dt$

$$E\left[\int_0^T W(t)\,dt\right] = \int_0^T E(W(t))\,dt = 0$$

$$Var\left[\int_0^T W(t)\,dt\right] = \int_0^T (T-t)^2\,dt = \frac{T^3}{3}$$

i.e, from here we get

$$\int_0^T W(t)\,d(t) \sim N(0, T^3/3)$$

i.e, $\int_0^T W(t)$ is a random variable that is distributed normally with mean 0 & variance $T^3/3$

we have to find stochastic differential of $\sin(w(t))$

consider 1st version of the Ito Doeblin formula

$$d f(w(t)) = f'(w(t))\, d(w(t)) + \frac{1}{2} f''(w(t))\, dt \quad -(1)$$

let us consider $f(x) = \sin x$, we get

$f'(x) = \cos x$

$f''(x) = -\sin x$

where x is $w(t)$

substituting values of F, f', f'' in (1), we get

$$d(\sin(w(t))) = \cos(w(t))\, d(w(t)) \pm \frac{1}{2}(-\sin(w(t)))\, dt$$

which is the required stochastic differential

10) Given that stock price $S(0) = \$50$

Time T (in years) $= 2$

Exp. return $\mu = 0.18$

Valatility $= \sigma = 30\% = 0.3$

We use probability distribution of the stock price in 2 years using log-normal distribution

$$\ln S(T) = \phi\left(\ln 50 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

$$= \phi\left(\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2, 0.3^2 \times 2\right)$$

$$\ln S_T = \phi(4.18, 0.18)$$

The mean of stock price $E(S_T)$ is given by:

$$E(S_T) = S(0)\, e^{rT}$$

$$= 50 \exp(0.36) = \$71.67$$

SD is given by:

$$\sigma_{S_T} = S_0\, e^{rT}\sqrt{e^{\sigma^2 T} - 1} = 50\, e^{0.18 \times 2}\sqrt{e^{0.09 \times 2} - 1}$$

$$\sigma_{S_T} = 31.83$$

95% confidence level for $\ln S_T$

By Interval Table for critical value at $\alpha/2 = 0.05/2$

$= 0.025$ is 1.96

$$4.18 \pm 1.96 \times 0.42$$

$$3.35, 5.01$$

Corresponding 95% interval for S_T are

$$e^{3.35} \;\&\; e^{5.01}$$

$$= 28.52 \;\&\; 150.44$$