

Stochastic Process (SP)

Definition: σ – field

A σ – field F (or σ – algebra) is a family of subset of Ω (sample space) which satisfy the following properties,

- i. $\phi \in F$
- ii. If $A \in F$ the $A^c \in F$
- iii. If $A_1, A_2 \dots$ are in F and is a countable sequence, then $\bigcup_i A_i \in F$

eg; A fair coin is tossed 3-times generating a sample space

$$\Omega = \{ HHH, HHT, HTH, HTT, THH, TTH, THT, TTT \}$$

$$\text{Let } A_1 = \{ \text{first toss head} \} = \{ HHH, HHT, HTH, HTT \}$$

$$\& \quad A_2 = \{ \text{first toss tail} \} = \{ THH, TTH, THT, TTT \}$$

then , $F = \{ \phi, \Omega, A_1, A_2 \}$ is a σ – field .

Definition: Stochastic Process (SP)

Let (Ω, F, P) where P is probability measure defined on F be a given probability space. A collection of random variables (r.v.s) $\{ X_t, t \in T \}$, T is Index set defined on the probability space (Ω, F, P) is called a Stochastic Process (SP).

$$X_t = X_t(\omega), \text{ where } \omega \in R$$

$$\text{Hence , } \{ X_t, t \in T \} = \{ X_t(\omega), \omega \in \Omega, t \in T \} \quad \dots\dots\dots(1)$$

It is clear from representation that a Stochastic Process (SP) is function of two variables t , ω which are independent .

$$X: T \times \Omega \rightarrow R \quad \dots\dots\dots(2)$$

The mapping X gives rise to 2- mappings

- i. $X(\cdot, \omega) \rightarrow \text{fixed } \omega$ (trajectory is called sample path)
- ii. $X(t, \cdot) \rightarrow \text{fixed } t$ (is a random varriable)

Parameter Space and State Space

Let $\{X_t, t \in T\}$ be a given a Stochastic Process (SP). The set $\{t \in T\}$ is called the parameter space or index set. The collection of all possible values of X_t for $\forall t \in T$ is called state space denoted by S .

This gives rise to four situations

- i. discrete-time, discrete state
- ii. discrete time, continuous state
- iii. continuous time, discrete state
- iv. continuous time, continuous state

Whenever is state space or parameter space is finite are countably infinite then it is said to have discrete nature.

eg. If $t \in \{0, \pm 1, \pm 2, \dots\}$ is discrete.

And when t or ω takes values on real line (whole) are partially (in an interval) then it is a continuous situation.

- i. Continuous time discrete space Stochastic process (SP)

Total number of share $\{X_t, t \in [0, \infty)\}$ held by an investor at any time t .

or Number of cars passing through a signal in one cycle.

- ii. Continuous time continuous space Stochastic Process (SP)

The price of a stock (particular item) at any time t .

or Variation of humidity in an AC room between two cut off of AC.

- iii. Discrete time continuous Stochastic Process (SP)

The value of one US Dollar in Rupees at the end of day in a month.

or Temperature recorded of a city at 7.0 am every day in a month.

- iv. Discrete time discrete state Stochastic Process (SP)

Total number of share held by an investor at end of day in a month

Definition: Independent increment

If for all 'n' and $t_1 < t_2 \dots < t_n$ the random variables (r.v.s) $X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_n) - X(t_{n-1})$ are independent random variables (r.v.s) then the process is said to have independent increment.

Definition: Strict Sense Stationary Stochastic Process (SP)

(also called is strong stationary Stochastic Process (SP))

The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is called *Strict Sense Stationary Stochastic Process* if for arbitrary $0 < t_1 < \dots < t_n$ the finite dimensional random vectors $\{X(t_1), X(t_2), \dots, X(t_n)\}$ and $\{X(t_1 + h), X(t_2 + h), \dots, X(t_n + h)\}$ have the same joint distribution for all $h > 0$ and all $0 < t_1 < \dots < t_n$.

Definition: Wide Sense Stationary Stochastic Process (SP)

The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is *Wide Sense Stationary Stochastic Process* if it satisfies the following ,

- i. $E(X_t) = \mu(t)$ is independent of t .
- ii. $Cov(X_t, X_s)$ depends only on $|t - s|$ for all t, s .
- iii. $E(X_t^2) < \infty$ (finite second order moment)

A wide sense stationary Stochastic Process is also called covariance stationary or weak stationary or second order stationary Stochastic Process (SP).

Example: $X_t = A \cos \omega t + B \sin \omega t$, where A, B are un-correlated random variables with

expectation '0' and variance 1. ω is a positive constant.

Sol:

- i. $E(X_t) = E(A \cos \omega t + B \sin \omega t) = \cos \omega t E(A) + \sin \omega t E(B) = 0$
- ii. $Cov(X_t, X_s) = E(X_t \cdot X_s) - E(X_t)E(X_s) = E(X_t \cdot X_s)$ ($\because E(X_t) = 0$)
 $= \cos \omega t \cos \omega s E(A^2) + \sin \omega t \sin \omega s E(B^2) + (\cos \omega t \sin \omega s + \sin \omega t \cos \omega s)E(AB)$
 $= \cos \omega(t - s)$ ($\because E(A^2) = E(B^2) = 1$, & $E(AB) = 0$ A&B are uncorrelated)
- iii. $E(X_t^2) = Var(X_t) = 1$

Hence the given stochastic process is wide sense stationary.

Definition: Markov property

A given Stochastic Process (SP) $\{X_t, t \in T\}$ is said to have Markov property if for all 'n' and for all $0 < t_1 < \dots < t_n < t$ the CDF satisfies

$$P\{X_t \leq x / X(0) = x_0, X(t_1) = x_1 \dots X(t_n) = x_n\} = P\{X_t \leq x / X(t_n) = x_n\}$$

That is, future predictions depend only on the current state of the Stochastic Process (SP) and does not depend on the past information.

Definition: Random Walk

If each trial has more than two possible outcomes, $Y_i, i = 1, 2, \dots, n$ be the set of independent discrete random variables (r.v.s)

$$S_n = \sum_{i=1}^n Y_i$$

Then $\{S_n, n = 0, 1, 2, \dots\}$ where $S_0 = 0$ is called *Random Walk*.

Definition: Symmetric Random Walk

consider a random experiment of tossing a fair coin in finitely many times. Let the successive outcomes be denoted by $\omega = (\omega_1, \omega_2, \dots)$ (eg. (H,T,H,H,T,T...) or (T,T,T,H,H,T..)) we now define for $j = 1, 2, \dots$

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H \\ -1 & \text{if } \omega_j = T \end{cases},$$

and $P(X_j = 1) = P(\omega_j = H) = 0.5$, $P(X_j = -1) = P(\omega_j = T) = 0.5$

Set $S_0 = 0$. Let

$$S_k = \sum_{j=1}^k X_j, \quad k = 1, 2, \dots$$

Then $\{S_n, n = 0, 1, 2, \dots\}$ is known as a *symmetric random walk* .

Theorem: Let $\{S_k, k = 0, 1, 2, \dots\}$ be a asymmetric random walk. Then

- i. For each k , $E(S_k) = 0$ and $Var(S_k) = k$
- ii. it has independent increment
- iii. It has stationary increment
- iv. It is a Markov process

Proof:

- i. For $j = 1, 2, \dots$, $E(X_j) = 0$ and $Var(X_j) = 1$

Therefore

$$E(S_k) = E\left(\sum_{j=1}^k X_j\right) = \sum_{j=1}^k E(X_j) = 0$$

$$Var(S_k) = \sum_{j=1}^k Var(X_j) = k$$

- ii. We choose an arbitrary positive integer 'n' and then choose non negative $0 = k_0 < k_1 < \dots < k_n$ integers then

$$S_{k_{i+1}} - S_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

Since X_j are independent and identically distributed (i.i.d.) random variables (r.v.s) having Bernoulli distribution .

$S_{k_1} - S_{k_0}, S_{k_2} - S_{k_1}, \dots, S_{k_n} - S_{k_{n-1}}$ are mutually independent variables. Hence the Stochastic Process (SP) $\{S_n, n = 0, 1, 2, \dots\}$ has independent increment property.

- iii. Choose non-negative integers $k_1 < k_2$ then

$$S_{k_2} - S_{k_1} = \sum_{j=k_1+1}^{k_2} X_j$$

Since X_j are i.i.d. random variables (r.v.s) having Barnoulli distribution , $S_{k_2} - S_{k_1}$ has the same distribution of $S_{k_2-k_1} - S_0$ hence the Stochastic Process (SP) $\{S_n, n = 0, 1, 2, \dots\}$ has the stationary increment property.

iv. We have for $k = 1, 2, \dots$

$$S_k = S_{k-1} + X_k$$

$$\text{Now } P\{S_k \leq x_k / S_{k-1} = x_{k-1}, S_{k-2} = x_{k-2}, \dots, S_1 = x_1\}$$

$$= \frac{P\{S_k \leq x_k, S_{k-1} = x_{k-1}, S_{k-2} = x_{k-2}, \dots, S_1 = x_1\}}{P\{S_{k-1} = x_{k-1}\}P\{S_{k-2} = x_{k-2}\}, \dots, P\{S_1 = x_1\}}$$

$$= \frac{P\{S_k \leq x_k / S_{k-1} = x_{k-1}\} \dots \dots P\{S_2 = x_2 / S_1 = x_1\}P\{S_1 = x_1\}}{P\{S_{k-1} = x_{k-1}\}P\{S_{k-2} = x_{k-2}\}, \dots, P\{S_1 = x_1\}}$$

$$= P\{S_k \leq x_k / S_{k-1} = x_{k-1}\}$$

Hence $\{S_n, n = 0, 1, 2, \dots\}$ is a Markov process.

Definition: Poisson process

A Stochastic Process (SP) $\{S(t), t \geq 0\}$ is said to be a *Poisson Process* with intensity or rate (parameter) $\lambda > 0$ if it satisfies the following properties,

- i. It starts from zero i.e. $X(0) = 0$.
- ii. For all 'n' and for all $0 \leq t_0 < t_1 < \dots < t_n$ increments $S(t_i) - S(t_{i-1})$, $i = 1, 2, \dots, n$ are independent and stationary
- iii. For $0 < s < t$, $S(t) - S(s)$ is a poisson distributed random variable with parameter

$\lambda(t - s)$ i.e.,

$$P\{S(t) - S(s) = n\} = \frac{e^{-\lambda(t-s)}\{\lambda(t-s)\}^n}{n!}, \quad n = 0, 1, 2, \dots$$

§§ By virtue of property (ii) the Stochastic Process (SP) $\{S(t), t \geq 0\}$ satisfies the Markov property and hence it is a Markov process.

Definition: Brownian Motion (BM) or Wiener Process

A Stochastic Process (SP) $\{W(t), t \geq 0\}$ is said to be a Brownian Motion (BM) if it satisfies the following properties

- i. $W(0) = 0$ ie, it starts from zero.
- ii. for $t > 0$ the sample path of $W(t)$ is continuous.
- iii. $W(t), t \geq 0$ has independent and stationary increment.
- iv. for $0 \leq s < t < \infty$, $W(t) - W(s)$ is normally distributed random variable with mean '0' and variance $(t - s)$.

The path is always continuous but it is nowhere differentiable that is it is not possible to define unique tangent line at any point on a curve for this we can use the convergence of second order moment we shall show that

$\lim_{\Delta t \rightarrow 0} Var\left\{\frac{W(t_0 + \Delta t) - W(t_0)}{\Delta t}\right\}$ does not exist for any point t_0 .

Since we know that $W(t_0 + \Delta t) - W(t_0)$ it has normal distribution with mean '0' and variance $(t - s)$ that is $\sim N(0, \Delta t)$. Hence

$\lim_{\Delta t \rightarrow 0} Var\left\{\frac{W(t_0 + \Delta t) - W(t_0)}{\Delta t}\right\} = \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{(\Delta t)^2} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t}$. Which does not exist.

§§ The W.P. is not wide sense stationary because for $s < t$ the $Cov\{W(t), W(s)\}$ is not a function of $(t - s)$.

$$\begin{aligned} \text{For } Cov\{W(t), W(s)\} &= E[\{W(t) - E(W(t))\}\{W(s) - E(W(s))\}] \\ &= E\{W(s)W(t)\} = E[\{W(t) - W(s) + W(s)\}W(s)] \\ &= E[W(t) - W(s)]E\{W(s)\} + E\{W(s)W(s)\} \\ &= 0 + s \end{aligned}$$

Hence, $Cov\{W(t), W(s)\} = \min\{s, t\}$

§§ Given $W(t)$ the future $W(t + h)$ having $h > 0$ only depends on the increment $W(t + h) - W(t)$ which is independent of the past. Hence $\{W(t), t \geq 0\}$ is a Markov process.

Definition: Brownian Motion (BM) with drift μ and volatility σ

A Stochastic Process (SP) $\{X(t), t \geq 0\}$ is said to be a Brownian Motion (BM) with drift μ and volatility σ if $\{X(t) = \mu t + \sigma W(t)\}$ where

- i. $W(t)$ is a standard Brownian Motion (BM)
- ii. $-\infty < \mu < \infty$ is a constant
- iii. $\sigma > 0$ is a constant

This is a generalization of standard Brownian Motion (BM) in this process

$$E\{W(t)\} = \mu t, \quad \text{and}$$

$$\text{Cov}\{X(t), X(s)\} = \sigma^2 \text{Cov}\{W(t), W(s)\} = \sigma^2 \min\{s, t\}, \quad s, t > 0$$

Definition: Geometric Brownian Motion (GBM)

A Stochastic Process (SP) $\{X(t), t \geq 0\}$ is said to be a Geometric Brownian Motion (GBM) if

$$X(t) = X(0) e^{w(t)} \quad \text{where } W(t) \text{ is a standard Brownian Motion (BM).}$$

§§ for any $h > 0$, we have

$$\begin{aligned} X(t+h) &= X(0) e^{w(t+h)} = X(0) e^{w(t+h)-w(t)+w(t)} \\ &= X(0) e^{w(t)} e^{w(t+h)-w(t)} = X(t) e^{w(t+h)-w(t)} \end{aligned}$$

§§ We know that Brownian Motion (BM) has independent increments. Hence given $X(t)$ the future $X(t+h)$ only depends on the future increment of the Brownian Motion (BM). Thus future is independent of the past and therefore the Markov property is satisfied hence $\{X(t), t \geq 0\}$ is a Markov process.

GBM To model stock price

Let the stock price $S(t)$ at time 't' is given by $S(t) = S(0)e^{H(t)}$, $S(0)$ is initial price and $H(t) = \mu t + \sigma W(t)$ is Brownian Motion (BM) with drift.

In this case $H(t)$ represents a continuously compounded rate of interest of the stock price over the period $[0, t]$. Hence

$$\ln(S(t)) = \ln(S(0)) + H(t)$$

Therefore $\ln(S(t))$ has normal distribution with mean $\mu t + \ln(S(0))$ and variance $\sigma^2 t$.

(If a random variable X has property that $\ln(X)$ has normal distribution then the random variable X is said to have lognormal distribution. Accordingly $\left(\frac{S(t)}{S(0)}\right)$ is log normally distributed random variable.)

§§ If $S(t) = S(0)e^{H(t)}$ at any time 't' where $S(0)$ is initial price and $H(t) = \mu t + \sigma W(t)$ is a Brownian Motion (BM) with drift μ and volatility σ then

- i. $E\{S(t)\} = S(0)\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\}$
- ii. $Var\{S(t)\} = \left[S(0)\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\}\right]^2 \{\exp(\sigma^2 t) - 1\}$

Proof: Since for every 't' $W(t)$ is normally distributed with mean '0' and variance 't',

$H(t) = \mu t + \sigma W(t)$ normally distributed with mean μt and variance $\sigma^2 t$.

Hence

$$M_{H(t)}(\theta) = E(e^{\theta H(t)}) = \exp\left(\mu t \theta + \frac{1}{2} \sigma^2 t \theta^2\right) \quad (\text{mgf of Normal dist.}) \quad \dots\dots\dots(1)$$

$$\text{i. } E(S(t)) = E(S(0)e^{H(t)}) = S(0)E(e^{H(t)})$$

Putting $\theta = 1$ in equation (1) we get

$$E(e^{H(t)}) = \exp\left(\mu t + \frac{1}{2} \sigma^2 t\right)$$

$$\text{Hence, } E(S(t)) = \exp\left(\mu t + \frac{1}{2} \sigma^2 t\right)$$

$$\begin{aligned} \text{ii. } Var(S(t)) &= E\left((S(t))^2\right) - \left(E(S(t))\right)^2 \\ &= E\left(S^2(0)e^{2H(t)}\right) - \left(S(0)\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\}\right)^2 \\ &= S^2(0) \exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\} - S^2(0) \exp\{(2\mu t + \sigma^2 t)\} \\ &= S^2(0) \exp\{(2\mu t + \sigma^2 t)\} [\exp(\sigma^2 t) - 1] \\ &= \left[S(0)\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\}\right]^2 \{\exp(\sigma^2 t) - 1\} \end{aligned}$$

Filtration

Definition: Filtration in Discrete Time

Let Ω be the sample space and $F_0 = \{\phi, \Omega\}$. Then a filtration in discrete time is an increasing sequence of $F_0 \subset F_1 \subset F_2 \subset \dots$ of σ -field, one part time instant.

e.g.: $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$E_H = \text{head in first toss}$ & $E_T = \text{tail in first toss}$

$F_0 = \{\phi, \Omega\}$, $F_1 = \{\phi, \{E_H\}, \{E_T\}, \Omega\}$,

$F_2 = \{\phi, \{E_H\}, \{E_T\}, \{E_{HH}\}, \{E_{TT}\}, \{E_{HH}^c\}, \{E_{TT}^c\}, \{E_{HH} \cup E_{TT}\}\Omega\}$, then $F_0 \subset F_1 \subset F_2$

Definition: Filtration in continuous time

Let Ω be the sample space. Let T be a fixed positive number and assume that for $t \in [0, T]$, there is σ -field F_t . Assume further that, if $s \leq t$ then every set of F_s is in F_t . Then the collection of σ -field $\{F_t: t \geq 0\}$ is called filtration in continuous time

Thus a collection of σ -field $\{F_t: t \geq 0\}$ is called filtration in continuous time if $F_s \subset F_t$ for all $0 \leq s \leq t$.

§§ Filtration is used to model the flow of information over time.

e.g.: we think X_t as the price of an asset at time t and F_t as the information obtained by watching all prices in the market upto time t .

Definition: σ -field generated by a Stochastic Process (SP)

Let $\{Y_t: t \in T\}$ with the given stochastic process. Then σ -field generated by the Stochastic Process (SP) $\{Y_t: t \in T\}$ is the smallest σ -field containing all sets of the form

$\{w: \text{the sample path } (Y_t: t \in T) \text{ belongs to } C\}$ for all suitable sets C of on function on T .

Definition: σ – field generated by Brownian Motion (BM)

Let $W = \{W(s): 0 < s \leq t\}$ with the given Brownian Motion (BM) on $[0, t]$ then σ – field generated by all sets of the form

$$A_{t_2, t_3, \dots, t_n} = \{\omega \in \Omega: W(t_1, \omega), W(t_2, \omega) \dots W(t_n, \omega) \in C\}$$

For any n-dimensional Borel Set C , and for any choice of $t_i \in [0, t], i \geq 1$ is called σ – field generated by Brownian Motion (BM) ‘ W ’.

Definition: X_t is F_t - measurable

Let F_t be a σ – field of subsets of Ω . Then a random variable X_t is F_t – measurable if every set in $\sigma(X_t)$ is also in F_t that is a random variable is F_t – measurable iff the information in F_t is sufficient to determine the value of X_t .

Definition: Adapted process

A sequence of random variables (r.v.s) X_1, X_2, \dots are said to be adapted to a filtration F_1, F_2, \dots if X_n is F_n – measurable for each $n = 1, 2, \dots$

Or

A discrete time Stochastic Process (SP) $\{X_0, X_1, X_2, \dots\}$ is said to be adapted to a given filtration $\{F_n; n = 0, 1, \dots\}$ if the σ – field generated by X_n is a subset of F_n ie., $(X_n) \subset F_n$.

(same is for continuous SP)

Definition: Natural filtration

Natural filtration corresponding to a process is the smallest filtration to which it is adapted.

Martingales

Definition: Discrete-time Martingale

Let (Ω, \mathcal{F}, P) be a probability space. Let $\{X_n; n = 0, 1, \dots\}$ be a Stochastic Process (SP) and $\{\mathcal{F}_n; n = 0, 1, \dots\}$ be the filtration. The Stochastic Process (SP) $\{X_n; n = 0, 1, \dots\}$ is said to be a Martingale corresponding to the filtration $\{\mathcal{F}_n; n = 0, 1, \dots\}$ if it satisfies the following conditions,

1. For every n , $E(X_n)$ exists
2. Each X_n is \mathcal{F}_n – measurable
3. For every n , $E(X_{n+1}/\mathcal{F}_n) = X_n$

§§ Using the property of conditional expectation

$$E(E(X/Y)) = E(X)$$

In the definition of Martingale we observe that if $\{X_n\}$ is a Martingale then

$E(X_{n+1}) = X_n$, for every n . It implies that $E(X_n) = c$ (constant), therefore, if for some $n > 0$, $E(X_n) < \infty$ and the increments $X_{n+1} - X_n$ of the Martingale $\{X_n\}$ are bounded then $E(X_n) = E(X_0)$

Sub-Martingale and Super-Martingale

The 3rd condition of the def of Martingale is

if $E(X_{n+1}/\mathcal{F}_n) \geq X_n$ then $\{X_n; n = 0, 1, \dots\}$ is a sub-Martingale.

While if $E(X_{n+1}/\mathcal{F}_n) \leq X_n$ then $\{X_n; n = 0, 1, \dots\}$ is a super-Martingale

Ex.(8.6.2)

Let X_1, X_2, \dots be a sequence of i.i.d. random variables (r.v.s) each taking values $+1$ & -1 with equal probabilities. Let us define $S_0 = 0$ and

$$S_n = \sum_{i=1}^n X_i, \quad n = 1, 2, \dots$$

This discrete-time stochastic Process (SP) is a symmetric random walk. Prove that $\{S_n; n = 0, 1, \dots\}$ is a Martingale with respect to $\{\mathcal{F}_n; n = 1, 2, \dots\}$.

Solution: we have

$$E(|S_n|) \leq E(|X_1|) + E(|X_2|) \dots + E(|X_n|) < \infty$$

Also

$$\begin{aligned} E(S_{n+1} / X_1, X_2 \dots X_n) &= E(S_n + X_{n+1} / X_1, X_2 \dots X_n) \\ &= E(S_n / X_1, X_2 \dots X_n) + E(X_{n+1} / X_1, X_2 \dots X_n) \\ &= S_n + E(X_{n+1}) \quad (\because X_i \text{ are independent}) \\ &= S_n \quad (\because E(X_{n+1}) = 0) \end{aligned}$$

Hence $\{S_n; n = 0, 1, \dots\}$ is a Martingale w.r.t $\{X_n; n = 1, 2, \dots\}$.

§§ Suppose F_k is the σ -field of information corresponding to the first k random variables (r.v.s) X_k , we have for non negative integers $k < n$, $E(S_n / F_k) = S_k$

Ex.(8.6.3)

Consider a symmetric random walk $\{S_n; n = 0, 1, \dots\}$ which is a Martingale with respect to filtration $\{F_n; n = 0, 1, \dots\}$ where $F_0 = \{\phi, \Omega\}$ and $F_n = \sigma(X_1, X_2 \dots X_n)$, $n \geq 1$ is the σ -field of information corresponding to the n random variable $X_1, X_2 \dots X_n$. Verify if $\{S_n^2; n = 0, 1, \dots\}$ is a Martingale with respect to filtration $\{F_n; n = 0, 1, \dots\}$

Solution: For each $n = 1, 2, \dots$, S_n^2 is F_n -measurable. Also

$$E(S_n^2) = \sum_{i=1}^n E(X_i^2) < \infty \quad (\text{since } E(X_i, X_j) = 0 \text{ for } i \neq j)$$

Now, $E(S_{n+1}^2 / F_n) = E[(S_{n+1} - S_n + S_n)^2 / F_n]$

$$\begin{aligned} &= E[(S_{n+1} - S_n)^2 / F_n] + 2 E[S_n(S_{n+1} - S_n) / F_n] + E[S_n^2 / F_n] \\ &= E[(X_{n+1})^2 / F_n] + 2 E[S_n X_{n+1} / F_n] + E[S_n^2 / F_n] \end{aligned}$$

X_{n+1} is independent of F_n and S_n^2 is F_n -measurable, we have

$$\begin{aligned} E(S_{n+1}^2 / F_n) &= E[(X_{n+1})^2 / F_n] + 2 S_n E[X_{n+1} / F_n] + S_n^2 \\ &= 1 + S_n^2 \end{aligned}$$

Since ,

$$E(S_{n+1}^2 / F_n) \neq S_n^2 \quad \text{hence } \{S_n^2; n = 0, 1, \dots\} \text{ is not a Martingale.}$$

However

$$E(S_{n+1}^2 / F_n) > S_n^2 \quad \text{hence } \{S_n^2; n = 0, 1, \dots\} \text{ is a sub-Martingale.}$$

Ex.(8.6.5)

Let a person start with Rs.1. A fair coin is tossed infinitely many times. A person gets Rs.2 if it turns up head and gets nothing if it is a tail in n^{th} toss. Let Y_n be his / her fortune at the end of the n^{th} . Prove that Y_n is a Martingale.

Solution:

Let X_1, X_2, \dots be a sequence of i.i.d. random variables (r.v.s) each defined as

$$X_n = \begin{cases} 2 & \text{with probability } 0.5 \\ 0 & \text{with probability } 0.5 \end{cases}$$

according to problem

$$Y_n = X_1 X_2 \dots X_n, \quad n = 1, 2, \dots$$

let F_n be σ -field generated by X_1, X_2, \dots, X_n . We have

$$0 \leq Y_n \leq 2^n \quad \text{and} \quad E(X_i) = 1$$

$$\text{Now, } E(Y_{n+1} / F_n) = E(Y_n X_{n+1} / F_n) = Y_n E(X_{n+1} / F_n)$$

$$= Y_n E(X_{n+1}) \quad \text{since } X_{n+1} \text{ is independent of } F_n .$$

$$= Y_n \quad \text{since } E(X_{n+1}) = 1$$

hence $\{Y_n; n = 1, 2, \dots\}$ is a Martingale.

Ex.(8.6.6)

Consider a binomial lattice model. let S_n be the stock prices at period 'n' and

$$S_{n+1} = \begin{cases} uS_n & \text{with probability } p \\ dS_n & \text{with probability } 1 - p \end{cases}$$

define a related process R_n as $R_n = \ln(S_n) - n[p \ln u + (1 - p) \ln d]$

prove that $\{\ln S_n; n = 1, 2, \dots\}$ is not a Martingale whereas $\{R_n; n = 1, 2, \dots\}$ is a Martingale

where as $\{R_n; n = 1, 2, \dots\}$ is a Martingale with respect to $\{S_n; n = 1, 2, \dots\}$. Also prove that the discounted stock price $S_0, S_1 e^{-r}, S_2 e^{-2r}, \dots$ is a Martingale only if

$$p = \frac{e^r - d}{u - d} \quad \text{Where 'r' is the nominal interest rate.}$$

Solution:

In this binomial lattice model $\{S_0, S_1, S_2, \dots\}$ with the natural filtration $\{F_0, F_1, F_2, \dots\}$ we have

$$P(S_{n+1} = uS_n / F_n) = 1 - P(S_{n+1} = dS_n / F_n) = p \quad \dots\dots\dots(1)$$

hence

$$E(S_{n+1} / F_n) = p \cdot uS_n + (1 - p) \cdot dS_n = S_n \{ p \cdot u + (1 - p) \cdot d \} \quad \dots\dots\dots(2)$$

we consider the variable $\ln(S_n)$ in and observe that

$$E\left\{\ln\left(\frac{S_n}{S_{n-1}}\right) / S_{n-1}, S_{n-2}, \dots, S_0\right\} = p \cdot u + (1 - p) \cdot d$$

$$\text{therefore, } E\{\ln(S_{n+1}) / S_{n-1}, S_{n-2}, \dots, S_0\} = \ln S_n + p \cdot u + (1 - p) \cdot d \quad \dots\dots\dots(3)$$

hence $\{\ln S_n; n = 1, 2, \dots\}$ is not a Martingale. Depending upon the value of u & d it can be Sub Martingale or super Martingale.

Consider

$$E\{R_n / R_{n-1}, R_{n-2}, \dots, R_0\} = E[\{\ln(S_n) - n[p \cdot \ln u + (1 - p) \cdot \ln d]\} / R_{n-1}, R_{n-2}, \dots, R_0]$$

since the information of $R_{n-1}, R_{n-2}, \dots, R_0$ is killed by history of $S_{n-1}, S_{n-2}, \dots, S_0$ and vice versa, so we get

$$\begin{aligned} E\{R_n / R_{n-1}, R_{n-2}, \dots, R_0\} &= \ln(S_{n-1}) - (n - 1)[p \cdot \ln u + (1 - p) \cdot \ln d] \quad \dots\dots\dots(4) \\ &= R_{n-1} \end{aligned}$$

therefore $\{R_n; n = 1, 2, \dots\}$ is a Martingale.

Consider the discounted process $(S_0, S_1 e^{-r}, S_2 e^{-2r}, \dots)$ where 'r' is the interest rate. We have

$$E(e^{-(n+1)r} S_{n+1} / F_n) = p \cdot u \cdot e^{-(n+1)r} S_n + (1 - p) \cdot d \cdot e^{-(n+1)r} S_n$$

$$= e^{-nr} S_n e^{-r} \{ p \cdot u + (1 - p) \cdot d \} \dots\dots\dots(5)$$

Therefore the discounted process will be Martingale if the RHS of equation (5) is $e^{-nr} S_n$ ie.

$$p \cdot u + (1 - p) \cdot d = e^r$$

Or
$$p = \frac{e^r - d}{u - d}$$

Ex.(8.6.7)

Prove that under RNPM Q the discounted stock price process $\{(1 + r)^{-k} S_k; k = 1, 2, \dots\}$ is a Martingale.

Solution: Let Ω be the sequence of heads - H and tail - T such that the stock price goes up by factor u and goes down by factor d with the occurrence of H or T respectively.

Let S_0 be a fix number. Define

$$S_k(\omega) = u^j d^{k-j} S_0$$

where the first k element $\omega \in \Omega$ has 'j' occurrence of H and ' $k - j$ ' occurrence of T .

Let F_n be the σ - field generated by S_0, S_1, \dots, S_k and

$$\bar{p} = \frac{(1+r)-d}{u-d}, \quad \& \quad \bar{q} = \frac{u-(1+r)}{u-d}, \quad \text{also} \quad Q(\omega) = \bar{p}^j \bar{q}^{n-j}$$

ie, for any sequence of 'n' movements $H \rightarrow j \text{ times}$ and $T \rightarrow n - j \text{ times}$ appears

now the random variable

$$Y_k = S_{k+1}/S_k \dots\dots\dots(A)$$

(defined like this) are i.i.d. with probability mass function as

$$P(Y = u) = \bar{p}, \quad P(Y = d) = \bar{q}$$

Since Y_k are i.i.d. random variable therefore they satisfy

$$P(Y_1 = Y_1, \dots, Y_n = Y_n) = P(Y_1 = Y_1) \cdot P(Y_2 = Y_2) \dots P(Y_n = Y_n)$$

$$= \bar{p}^j \bar{q}^{n-j}$$

also from equation (A) we get that \mathbb{Y}_k is the factor by which the stock price is moving.

The random variable \mathbb{Y}_k are independent of F_k .

$$\begin{aligned} E_Q\{(1+r)^{-k+1}S_{k+1}/F_k\} &= (1+r)^{-k+1}E_Q\{S_{k+1}/F_k\} \\ &= (1+r)^{-k+1}S_k E_Q\{\frac{S_{k+1}}{S_k}/F_k\} \\ &= (1+r)^{-k}S_k (1+r)^{-1}E_Q\{\frac{S_{k+1}}{S_k}\} \quad \text{since } \mathbb{Y}_k \text{ are independent of } F_k \\ &= (1+r)^{-k}S_k (1+r)^{-1}\{\bar{p}.u + \bar{q}.d\} \\ &= (1+r)^{-k}S_k \quad \text{since } 1+r = \{\bar{p}.u + \bar{q}.d\} \end{aligned}$$

the process $\{(1+r)^{-k}S_k; k = 1, 2, \dots\}$ is a Martingale.

Wealth process

Let Δ_k be the number of shares of a stock held between time k and $k+1$. We assume that Δ_k is F_k -measurable and X_0 is the amount of money we have started with time $t = 0$ now $\Delta_k \cdot S_{k+1}$ will be the worth of the stock at time $k+1$ where S_{k+1} is the price of the stock.

Now the amount of cash we hold between k and $k+1$ is X_k minus the amount held in stock ie, $X_k - \Delta_k \cdot S_k$.

hence the worth of this amount at time $k+1$ is

$$(1+r)(X_k - \Delta_k \cdot S_k)$$

Therefore, the amount of money we have at time $k+1$ is

$$X_{k+1} = \Delta_k \cdot S_{k+1} + (1+r)(X_k - \Delta_k \cdot S_k) \quad \dots\dots\dots (A)$$

When $r = 0$, it reduces to $X_{k+1} = \Delta_k (S_{k+1} - S_k) + X_k$.

$$\text{Thus, } X_{k+1} = X_0 + \sum_{i=0}^k \Delta_i (S_{i+1} - S_i)$$

The Stochastic Process (SP) $\{X_k; k = 0, 1, 2, \dots\}$ is called the wealth process.

§§ Discounted wealth process is a Martingale under RNPM Q .

$$\begin{aligned}
 E_Q[X_{k+1} - X_k/F_k] &= E_Q[\Delta_k (S_{k+1} - S_k)/F_k] \\
 &= \Delta_k E_Q[(S_{k+1} - S_k)/F_k] && \text{since } \Delta_k \text{ is } F_k - \text{measurable} \\
 &= \Delta_k [E_Q(S_{k+1}/F_k) - E_Q(S_k/F_k)] \\
 &= \Delta_k (S_k - S_k) && \text{since } \{S_k; k = 0, 1, 2, \dots\} \text{ is a Martingale.} \\
 &= 0 \text{ (earlier proved that discounted stock price process is a} \\
 &\text{Martingale)}
 \end{aligned}$$

Now noting that $r > 0$, from equation (A) we have

$$\begin{aligned}
 E_Q[\{(1+r)^{-(k+1)}X_{k+1} - (1+r)^{-k}X_k\}/F_k] \\
 &= E_Q[\Delta_k \{(1+r)^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k\}/F_k] \\
 &= \Delta_k E_Q[\{(1+r)^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k\}/F_k] \\
 &= \Delta_k [E_Q\{(1+r)^{-(k+1)}S_{k+1}/F_k\} - E_Q\{(1+r)^{-k}S_k/F_k\}] \\
 &= \Delta_k [(1+r)^{-k}S_k - (1+r)^{-k}S_k] \\
 &= 0
 \end{aligned}$$

(since $\{(1+r)^{-k}S_k; k = 1, 2, \dots\}$ is a Martingale.)

Or we have

$$E_Q[(1+r)^{-(k+1)}X_{k+1}/F_k] = (1+r)^{-k}X_k$$

Hence discounted wealth process $\{(1+r)^{-k}X_k; k = 1, 2, \dots\}$ is a Martingale.

Definition: Continuous time Martingale

Let (Ω, \mathcal{F}, P) be a probability space. Let $\{X_t, t \geq 0\}$ be a Stochastic Process (SP) and $\{F_t, t \geq 0\}$ be a filtration. The Stochastic Process (SP) $\{X_t, t \geq 0\}$ is said to be a Martingale corresponding to the filtration $\{F_t, t \geq 0\}$ if it satisfies the following conditions

1. For every t , $E(X(t))$ exists.
2. Each $X(t)$ is F_t - measurable .
3. For every $0 < s < t$ $E(X(t)/F_s) = X(s)$.

e.g. Prove that $\{W(t), t \geq 0\}$ is a Martingale

Solution: for $0 < s < t$

$$\begin{aligned} E(W(t)/F_s) &= E[\{W(t) - W(s) + W(s)\}/F_s] \\ &= E[\{W(t) - W(s)\}/F_s] + E[W(s)/F_s] = W(s) \\ &\quad [\text{since } \{W(t) - W(s)\} \text{ is normally distributed with mean 0}] \end{aligned}$$

Hence $\{W(t), t \geq 0\}$ is a Martingale .

Prob. Show that $\exp(W(t) - \frac{t}{2})$ is a Martingale .

Sol: Let $0 < s < t$. Since $\{W(t) - W(s)\}$ is independent of F_s and $W(s)$ is F_s - measurable . We have

$$\begin{aligned} E(e^{W(t)}/F_s) &= E(e^{W(t)-W(s)+W(s)}/F_s) \\ &= e^{W(s)} E(e^{W(t)-W(s)}/F_s) \\ &= e^{W(s)} E(e^{W(t)-W(s)}) \quad \text{since } \{W(t) - W(s)\} \text{ has independent increment} \end{aligned}$$

Since $\{W(t) - W(s)\}$ has normal distribution with mean 0 and variance $(t - s)$, we have $E(e^{W(t)-W(s)}) = e^{(t-s)/2}$

[using mgf for normal distribution $E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$]

hence , $E(e^{W(t)}/F_s) = e^{W(s)} e^{(t-s)/2}$

Therefore for $0 < s < t$,

$$E(e^{W(t)-t/2}/F_s) = e^{-t/2} E(e^{W(t)}/F_s) = e^{-t/2} e^{W(s)} e^{(t-s)/2} = e^{W(s)-s/2}$$

Hence it is a Martingale.

Stochastic calculus

Let the time interval be $[0, T]$ and its one of the partition is

$$\pi = \{ 0 = t_0 < t_1 < \dots < t_n = T \} \quad \dots\dots\dots(1)$$

Now Π is collection of all such partition ie, $\pi \in \Pi$. Now the norm of π is given as

$$\|\pi\| = \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \quad \text{where } \pi \text{ is an arbitrary partition.} \quad \dots\dots\dots(2)$$

The quadratic variation for Brownian Motion (BM) $\{W(t), t \geq 0\}$ over the interval $[0, T]$ is denoted by $[W, W](T)$ and is given by

$$[W, W](T) = \lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = \lim_{\|\pi\| \rightarrow 0} Q_\pi \quad \dots\dots\dots(3)$$

Where

$$Q_\pi = \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 \quad \dots\dots\dots(4)$$

(as $n \rightarrow \infty, \|\pi\| \rightarrow 0$)

Definition: Let $\{X_n, n \geq 1\}$ and X be random variables (r.v.s) defined on a common space (Ω, \mathcal{F}, P) we say that X_n converges to X in mean square sense if

$$\lim_{n \rightarrow \infty} E(|X_n - X|)^2 = 0 \quad \dots\dots\dots(5)$$

Theorem: Let Q_π is defined as in equation (4) then

- 1) $E(Q_\pi) = T$
- 2) $Var(Q_\pi) \leq 2 \|\pi\| T$

Proof:

- 1) We have from equation (4)

$$E(Q_\pi) = \sum_{i=0}^{n-1} E\{W(t_{i+1}) - W(t_i)\}^2$$

Since $W(t_{i+1}) - W(t_i)$ is normally distributed with mean 0 and variance $(t_{i+1} - t_i)$ for fixed i .

Hence,

$$E(Q_\pi) = \sum_{i=0}^{n-1} (t_{k+1} - t_k) = T \quad \dots\dots\dots(6)$$

2)

$$Var(Q_\pi) = \sum_{i=0}^{n-1} Var \{W(t_{i+1}) - W(t_i)\}^2 \quad \dots \dots \dots (7)$$

But

$$\begin{aligned} & Var\{W(t_{i+1}) - W(t_i)\}^2 \\ &= E[\{W(t_{i+1}) - W(t_i)\}^4] - 2E[\{W(t_{i+1}) - W(t_i)\}^2 W(t_{i+1}) - W(t_i)] + (t_{k+1} - t_k)^2 \\ & \dots (8) \end{aligned}$$

(using the formula $VarX = E(X - E(X))^2$)

Since the fourth order moment of Normal Distribution with mean 0 and variance $(t_{k+1} - t_k)$ is

$$3(t_{k+1} - t_k)^2$$

Substituting in equation (8) we get

$$\begin{aligned} Var\{W(t_{i+1}) - W(t_i)\}^2 &= 3(t_{k+1} - t_k)^2 - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2 \\ &= 2(t_{k+1} - t_k)^2 \quad \dots \dots \dots (9) \end{aligned}$$

Putting in (7) we have

$$Var(Q_\pi) = \sum_{i=0}^{n-1} 2(t_{k+1} - t_k)^2 \leq 2 \|\pi\| \sum_{i=0}^{n-1} (t_{k+1} - t_k) = 2 \|\pi\| T \quad \dots \dots \dots (10)$$

§§ With above discussion we have for a Brownian Motion (BM) $\{W(t), t \geq 0\}$, since

$$Var(Q_\pi) = E[(Q_\pi - E(Q_\pi))^2] = E[(Q_\pi - T)^2]$$

Using (10)

$$\lim_{\|\pi\| \rightarrow 0} E[(Q_\pi - T)^2] = \lim_{\|\pi\| \rightarrow 0} Var(Q_\pi) = \lim_{\|\pi\| \rightarrow 0} 2 \|\pi\| T$$

$$\text{Therefore, } [W, W](T) = \lim_{\|\pi\| \rightarrow 0} (Q_\pi) = T$$

Hence we write $[W, W](T) = T$ almost surely.

§§ Now we know that for any given Brownian Motion (BM) $\{W(t), t \geq 0\}$

$$[W, W](T) = T \text{ ie., } \lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = T \quad \dots\dots\dots(11)$$

Also for $0 < T_1 < T_2$

$$[W, W](T_2) - [W, W](T_1) = T_2 - T_1$$

We say Brownian Motion (BM) accumulates $(T_2 - T_1)$ quadratic variation over the interval $[T_1, T_2]$.

Since it is true for every interval we inform that Brownian Motion (BM) accumulates quadratic variation at rate one per unit time this can be informally written as

$$dW(t) \cdot dW(t) = dt \quad \dots\dots\dots(12)$$

Theorem: Let $\{W(t), t \geq 0\}$ be the given Brownian Motion (BM) and $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$ be a partition of $[0, T]$, then

$$\text{i. } \lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\} = 0 \quad \dots\dots\dots(13)$$

$$\text{ii. } \lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 = 0 \quad \dots\dots\dots(14)$$

Proof: We observe that

$$\text{i. } |(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)| \leq \max_{0 \leq i \leq n-1} |W(t_{i+1}) - W(t_i)| (t_{i+1} - t_i)$$

Therefore

$$\left| \sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\} \right| \leq \max_{0 \leq i \leq n-1} |W(t_{i+1}) - W(t_i)| T$$

Since $w(t)$ is continuous hens RHS tends to zero for $\|\pi\| \rightarrow 0$

$$\lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)\} = 0$$

ii.

$$\sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 \leq \max_{0 \leq i \leq n-1} (t_{i+1} - t_i) \sum_{i=0}^{n-1} (t_{i+1} - t_i) \leq \|\pi\| T$$

as $\|\pi\| \rightarrow 0$,

$$\lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{t_{i+1} - t_i\}^2 = 0$$

§§ Using analogy of equation (12) we can write from (13) & (14)

$$dW(t) \cdot dt = 0 ,$$

$$dt \cdot dt = 0 \quad \dots\dots\dots(12A)$$

Definition: Stochastic Integral

Let $\{X(t), t \geq 0\}$ be a Stochastic Process (SP) which is adapted to the natural filtration $\{F_t, t \geq 0\}$ of Wiener process (BM) $\{W(t), t \geq 0\}$,

That is X_t is F_t - measurable .

Next we consider the partition π of $[0, T]$ where $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$ and form the sum under limit $\|\pi\| \rightarrow 0$ of

$$\lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{X(t_i) \cdot (W(t_{i+1}) - W(t_i))\} = \int_0^T X(s) dW(s) = I(T) \quad \dots\dots\dots(15)$$

In defining Stochastic Integral the convergence used is mean square convergence hence the equation (15) is stochastic integral or Ito integral of the Stochastic Process (SP) $\{X(t), t \geq 0\}$ with respect to Brownian Motion (BM) $\{W(t), t \geq 0\}$.

Properties of ITO integrals

The stochastic integral $I(t)$, $0 < t \leq T$ satisfies the following properties

- i. $E(I(t)) = 0$
- ii. $E \left\{ \int_0^t X(s) dW(s) \right\}^2 = E \left\{ \int_0^t X^2(s) dW(s) \right\}$, (Ito isometry)
- iii. Let $\{X^1(t), t \geq 0\}$ & $\{X^2(t), t \geq 0\}$ be two stochastic process having stochastic integral w.r.t. BM $\{W(t), t \geq 0\}$. Let α & β be the constant, then

$$\int_0^t [\alpha X^1(s) + \beta X^2(s)] dW(s) = \alpha \int_0^t X^1(s) dW(s) + \beta \int_0^t X^2(s) dW(s)$$
 (linearity of Ito Int)
- iv. $\int_0^t X(s) dW(s) = \int_0^{t_1} X(s) dW(s) + \int_{t_1}^t X(s) dW(s)$ for $0 < t_1 < t$
- v. The process $I(t)$ has a continuous sample path.
- vi. For each t , $I(t)$ is F_t - measurable .
- vii. $[I, I](t) = \int_0^t X^2(s) ds$
- viii. The process $\int_0^t X(s) dW(s)$, $t \in [0, T]$ is a Martingale with respect to natural Brownian filtration F_t , $0 \leq t \leq T$.

Prob: Find the value of integral $\int_0^T W(s) dW(s)$

Sol: Let $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$ be an arbitrary partition of $[0, T]$.

we have, $\int_0^T W(s) dW(s) = \lim_{\|\pi\| \rightarrow 0} \sum_{i=0}^{n-1} \{W(t_i) \cdot (W(t_{i+1}) - W(t_i))\} \dots \dots \dots (1)$

but for each 'i', $W(t_i)$ & $W(t_{i+1}) - W(t_i)$ are independent random variables (r.v.s) and are having normal distribution. Now

$$\begin{aligned} Q_\pi &= \sum_{i=0}^{n-1} \{W(t_{i+1}) - W(t_i)\}^2 = \sum_{i=0}^{n-1} [W^2(t_{i+1}) - W^2(t_i) - 2W(t_i)(W(t_{i+1}) - W(t_i))] \\ &= W^2(T) - W^2(0) - 2 \sum_{i=0}^{n-1} [W(t_i) \cdot (W(t_{i+1}) - W(t_i))] \end{aligned}$$

$$\text{ie. } \sum_{i=0}^{n-1} [W(t_i) \cdot (W(t_{i+1}) - W(t_i))] = \frac{1}{2} [W^2(T) - W^2(0) - Q_\pi]$$

taking limit as $\|\pi\| \rightarrow 0$ using equation (1)

$$\int_0^t W(s) dW(s) = \frac{1}{2} [W^2(T) - T]$$

Prob: Evaluate $\int_0^t W(1) dW(s)$, $0 \leq t \leq 1$.

Sol: Since $W(1)$ is not adapted to filtration $\sigma\{W(s), 0 \leq s \leq t\}$, $0 \leq t \leq 1$ because it depends on future events hence this Ito integral does not exist.

Ito - Doeblin Formula for Brownian Motion (BM) : First Version

Let f be at least twice continuously differentiable function of 't' and $\{W(t), t \geq 0\}$ be a Wiener process. Then

$$df(W(t)) = f'(W(t)) \cdot dW(t) + \frac{1}{2} f''(W(t)) \cdot dt \dots \dots \dots (1)$$

Or equivalently

$$f(W(t)) = f(W(0)) + \int_0^t f'(W(t)) dW(t) + \frac{1}{2} \int_0^t f''(W(t)) dt \dots \dots \dots (2)$$

The first integral is Ito integral and second integral is Riemann integral.

Pob: Evaluate $\int_0^T W(t) dW(t)$ using Ito-Doeblin formula version one .

Sol: According to Ito Doeblin formula in equation (2)

If we take $f(x) = \frac{x^2}{2}$, we get $f'(x) = x$, $f''(x) = 1$ & $\int_0^t x dx = \frac{t^2}{2}$ where $x = W(t)$

We get ,

$$\frac{W^2(T)}{2} - 0 = \int_0^T W(t) dW(t) + \frac{1}{2} \int_0^T 1 dt \quad (\text{since } W(0) = 0)$$

$$\int_0^T W(t) dW(t) = \frac{1}{2} [W^2(T) - T]$$

Ito - Doeblin formula for Brownian Motion (BM) : Second Version

Let $f(t, x)$ have continuous partial derivatives of at least second order and $\{W(t), t \geq 0\}$ is a given Wiener process (W.P.). Then

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt \quad \dots\dots\dots(3)$$

where $x = W(t)$, or equivalently

$$f(t, W(t)) - f(0, W(0)) = \int_0^t f_t(u, W(u))du + \frac{1}{2} \int_0^t f_{xx}(u, W(u))du + \int_0^t f_x(u, W(u))dW(u) \quad \dots(4)$$

Version one and 2nd can be justified by considering Taylor's expansion of function of one variable or function of two variable respectively.

Pob: Evaluate $\int_0^T W(t) dW(t)$ using Ito-Doeblin formula version two .

Sol: Considering $f(t, x) = \frac{x^2}{2}$, we get $f_t(t, x) = 0$, $f_x(t, x) = x$, $f_{xx}(t, x) = 1$

where $x = W(t)$ and substituting in equation (4)

$$\frac{W^2(T)}{2} - f(0, W(0)) = \int_0^T 0 + \frac{1}{2}dt + \int_0^T W(u) dW(u)$$

$$\int_0^T W(u) dW(u) = \frac{W^2(T) - T}{2}$$

Stochastic Differential Equation

Consider an IVP $\frac{dx(t)}{dt} = f(t, x(t))$, $t \in [0, T]$ & $x(0) = x_0$

Where $f: [0, T] \times R \rightarrow R$ is continuous function. This ODE possess a solution

$x(t) = x_0 + \int_0^t f(s, x(s))ds$ Provided Lipschitz condition is met by function ' f '. ie.,

\exists constant $k > 0$ s.t. $|f(t, x) - f(t, y)| \leq k|x - y|$ for $\forall t \in [0, T]$ & $x, y \in R$

Or the solution can be obtained using standard Picard's method .

e.g., A circuit containing L-R is $L \frac{dI}{dt} + RI = aI_0 \sin \omega t$

or $\frac{L}{R} \frac{dV}{dt} + V = aI_0 \sin \omega t \Rightarrow \frac{dV}{dt} = \frac{R}{L} [aI_0 \sin \omega t - V]$

Now suppose x_0 or f is random then solution is not unique rather it will depend up on the value $\omega \in \Omega$ (sample space) .

ie, $\{X(t, \omega(t)), \omega \in \Omega \text{ \& } t \in [0, T]\}$, which becomes an Stochastic Process (SP) and such d.e. is called Random Differential Equation .

Adding an uncertainty by way of differential of Brownian Motion (BM) we get

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t)) \cdot \frac{dW(t)}{dt} \quad , \quad 0 \leq t \leq T \quad \dots\dots\dots(1)$$

Where $b: [0, T] \times R \rightarrow R$ and $\sigma: [0, T] \times R \rightarrow R$ are two given function. Equation (1) can be symbolically written as

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t)) \cdot dW(t) \quad \dots\dots\dots(2)$$

Equation (2) is Stochastic Differential Equation. It can equivalent be written as

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) \quad , \quad 0 \leq t \leq T \quad \dots\dots\dots(3)$$

Equation (3) is called Stochastic Integral Equation .

Strong Solution

The strong solution of SDE given by equation (2) is a Stochastic Process (SP) $\{X(t), t \in [0, T]\}$ which satisfies the following ,

- i. $\{X(t), t \in [0, T]\}$ is adapted to the Brownian Motion (BM) ie, at time 't' it is a function of $W(s), s \leq t$.
- ii. The integral given in equation (3) is well defined and satisfied by $\{X(t), t \in [0, T]\}$.
- iii. $\{X(t), t \in [0, T]\}$ is a function of underlying BM sample path and of the coefficient $b(t, x)$ & $\sigma(t, x)$.

Thus strong solution is an explicit function 'f' such that

$$X(t) = f(t, W(s)), s \leq t .$$

Since strong solution is based on the path of underlying BM therefore solution $\{X(t), t \in [0, T]\}$ is called unique strong solution if for any given other solution

$$\{Y(t), t \in [0, T]\} , \quad P\{X(t) = Y(t)\} = 1 \text{ for all } t \in [0, T]$$

Weak Solution

For a weak solution, the path behaviour is not essential. That means we are only interested in distribution of $X(t)$, which can determine expectation, variance and covariance of the process.

Diffusion

A solution of SDE (strong or weak) is called diffusion .

§§ Putting $b(t, x) = 0$ & $\sigma(t, x) = 1$ in equation (2) we see that BM is also a diffusion process.

Existence theorem

Let $E(X^2(0)) < \infty$ and $X(0)$ be independent of $\{W(t), t \geq 0\}$.

Let for all $t \in [0, T]$ and $x, y \in R$, $b(t, x)$ & $\sigma(t, x)$ be continuous and satisfy Lipschitz condition with respect to second variable ie,

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq k|x - y| \quad \forall t \in [0, T] , k \text{ is a constant .}$$

Then the SDE has unique strong solution $\{X(t), t \in [0, T]\}$.

Definition: Ito process

Let $\{W(t), t \geq 0\}$ be a BM and let $\{F_t, t \geq 0\}$ be the associated natural filtration. And Ito process is a Stochastic Process (SP) $\{X(t), t \geq 0\}$ of the form

$$X(t) = X(0) + \int_0^t \Delta(u) dW(u) + \int_0^t \Theta(u) ds$$

where $\{X(0)\}$ is non-random, $\Delta(u)$ & $\Theta(u)$ are adopted process.

The SDE form of the Ito process is $\{X(t), t \geq 0\}$ is

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt$$

SDE of GBM

Let $S(t)$ be the stock price at time 't'. Let $-\infty < \mu < \infty$ be the constant growth rate of the stock and $\sigma > 0$ be the volatility. Considered the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), S(0) \text{ is known.} \quad \dots\dots\dots(1)$$

We wish to find the strong solution of $S(t)$ if it exist.

Now the condition of existence theorem is verified since μ & σ are constant .

Let us assume that $S(t) = f(t, W(t))$

Using second version of Ito-Doebelin formula we get set

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt, \text{ where } x = W(t) \quad \dots\dots\dots(2)$$

Comparing with given SDE (1) we get

$$\frac{1}{2}f_{xx} + f_t = \mu S(t) = \mu f \quad \dots\dots\dots(3)$$

$$f_x = \sigma S(t) = \sigma f \quad \dots\dots\dots(4)$$

Solving equation (4) we get

$$f(t, x) = k(t)e^{\sigma x} \text{ for some function } k(t) \quad \dots\dots\dots(5)$$

From equation (5) we have

$$f_t = k'(t)e^{\sigma x} \quad \& \quad f_{xx} = \sigma^2 k(t)e^{\sigma x} \quad \dots\dots\dots(6)$$

substituting in equation (3) we get

$$\left[\frac{1}{2} \sigma^2 k(t) + k'(t) \right] e^{\sigma x} = \mu k(t) e^{\sigma x}$$

$$\text{or} \quad k'(t)e^{\sigma x} = \left(\mu - \frac{\sigma^2}{2} \right) k(t)e^{\sigma x} \quad \dots\dots\dots(7)$$

Solving equations (7) gives

$$k(t) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t} \quad \dots\dots\dots(8)$$

$$\begin{aligned} [\text{since } k(t) = Ae^{\left(\mu - \frac{\sigma^2}{2}\right)t}, \text{ at } t = 0, A = k(0), \text{ using (5) } S(t) = f(t, W(t)) \\ = k(t)e^{\sigma W(t)} \rightarrow k(0) = S(0)] \end{aligned}$$

Hence the required solution is, form equation (5)

$$S(t) = f(t, W(t)) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

We observe that for fixed 't', S(t) follows lognormal distribution.

$$\S\S \quad E(S(t)) = S(0)e^{\mu t}$$

$$\S\S \quad E(S^2(t)) = S^2(0)e^{(2\mu + \sigma^2)t}$$

$$\S\S \quad \text{Var}(S(t)) = E(S^2(t)) - [E(S(t))]^2 = S^2(0)e^{2\mu t} (e^{\sigma^2 t} - 1)$$

Discounted Portfolio Process

let the stock having price S(t) per unit follows a generalized GBM with constant mean return μ and a constant volatility $\sigma > 0$. The price is governed by SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad t \in [0, T] \quad \dots\dots\dots(1)$$

Also let $\beta(t)$ be the price of risk free asset which satisfy the ordinary d.e.

$$d\beta(t) = r \cdot \beta(t) \cdot dt \quad \dots\dots\dots(2)$$

where ' r ' is constant risk free interest rate .

Suppose at time ' t ' we take a Portfolio consisting of $a(t)$ shares of stock and $b(t)$ shares of risk free asset. Let $V(t)$ be the value of this portfolio at ' t ', that is

$$V(t) = a(t) S(t) + b(t) \beta(t), \quad t \in [0, T] \quad \dots\dots\dots(3)$$

Then,

$$dV(t) = a(t) dS(t) + b(t) d\beta(t) \quad \dots\dots\dots(4)$$

The discounted price of one share of stock is

$$\tilde{S}(t) = e^{-rt} S(t), \quad t \in [0, T] \quad \dots\dots\dots(5)$$

Applying Ito-Doebelin formula of second variant ,

$$d\tilde{S}(t) = -re^{-rt} S(t)dt + e^{-rt} dS(t) \quad \dots\dots\dots(6)$$

$$= -re^{-rt} S(t)dt + e^{-rt} [\mu S(t)dt + \sigma S(t)dW(t)] \quad \text{using equation (1)}$$

$$= \tilde{S}(t) [(\mu - r) dt + \sigma dW(t)]$$

$$= \sigma \tilde{S}(t) d\tilde{W}(t), \quad \text{where } \tilde{W}(t) = \frac{(\mu - r)}{\sigma} t + W(t), \quad t \in [0, T] \quad \dots\dots\dots(7)$$

Now $(\mu - r)$ is called risk premium .

Therefore $\frac{(\mu - r)}{\sigma}$ is the risk premium per unit of risk and is called the market price of risk .

Feynman-Kac Theorem (R. Feynman & M. Kac)

It establishes a link between parabolic p.d.e. and stochastic process .

Let the stochastic process $\{X(t), 0 \leq t \leq T\}$ satisfy the following SDE

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t)) \cdot dW(t)$$

Where $\mu(t, X(t))$ & $\sigma(t, X(t))$ are functions on $[0, T] \times R \rightarrow R$ called drift and diffusion function respectively . Also $X(0) = x$ for some $x \in R$. Then the solution of the following p.d.e.

$$g_t(t, x) + \mu(t, x)g_x(t, x) + \frac{1}{2}\sigma^2(t, x)g_{xx}(t, x) - r g(t, x) = 0 \quad \dots\dots(8)$$

Subject to the boundary condition

$$g(T, X(T)) = x = h(x) , \quad x \in R$$

Is a function $g: [0, T] \times R \rightarrow R$ given by

$$g(t, x) = E[e^{-r(T-t)} \cdot h(X(T)/(X(t) = x))] \quad \dots\dots(9)$$

We define an operator as following (called generator of the process)

$$\mathcal{A} = \mu(t, x(t)) \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x(t)) \frac{\partial^2}{\partial x^2}$$

§§ Feynman-Kac theorem implies both way, ie, if pde is given then solution is known and if a solution satisfying the boundary condition, then the pde whose solution is this is known.

Then equation (8) can be written as

$$\frac{\partial g}{\partial t} + \mathcal{A}g - rg = 0$$

Derivation of Black- Scholes formula for a derivative security

Let the stock price $S(t)$ be driven by the process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Using equations (7) where $\tilde{W}(t) = \frac{(\mu-r)}{\sigma} t + W(t)$ the risk neutral process is given by

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t) \quad \dots\dots\dots(10)$$

Suppose a derivative is written on this stock. Let $V(t, S(t))$ be the price of this security at any $t \in [0, T]$ and $V(T, S(T))$ be its pay off on maturity.

Here $V: [0, T] \times R_+ \rightarrow R_+$, R_+ is non-negative real number .

Using Ito-Lemma we have

$$dV(t) = dV(t, S(t)) = V_t dt + V_x dS(t) + \frac{1}{2}V_{xx}dS(t)dS(t)$$

Here $x = S(t)$

$$= \left[\frac{\partial V}{\partial t} + r \cdot S(t) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \cdot S^2(t) \frac{\partial^2 V}{\partial x^2} \right] dt + \sigma \cdot S(t) \frac{\partial V}{\partial x} d\tilde{W}(t) \quad \dots(11)$$

$$[dS(t)dS(t) = \sigma^2 S^2(t)d\tilde{W}(t)d\tilde{W}(t) = \sigma^2 S^2(t)dt \quad \text{squaring (10)}]$$

Suppose the derivative security can be hedged .We replicate the portfolio taking $a(t)$ shares of stock and $b(t)$ shares of risk free asset whose price is governed by *ode*

$$d\beta(t) = r \cdot \beta(t) \cdot dt$$

Then we have

$$dV(t) = a(t) dS(t) + b(t) \cdot r \cdot \beta(t) dt \quad (\text{using equation (2 \& 3)}) \quad \dots\dots\dots(12)$$

$$= a(t) [rS(t)dt + \sigma S(t)d\tilde{W}(t)] + r \cdot \beta(t) \cdot b(t)dt$$

comparing equation (11) and equation (12)

$$\frac{\partial V}{\partial x} = a(t), \quad \& \quad \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V}{\partial x^2} = r \cdot \beta(t) \cdot b(t) \quad \dots\dots\dots(13)$$

now using equation (13) in equation (3)

$$b(t) \beta(t) = V(t) - a(t) S(t) = V(t) - S(t) \frac{\partial V}{\partial x}$$

putting the value of $b(t)\beta(t)$ in equation (13) we get

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V}{\partial x^2} &= \left[V(t) - S(t) \frac{\partial V}{\partial x} \right] \cdot r \\ \frac{\partial V}{\partial t} + r \cdot S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V}{\partial S^2} - r \cdot V &= 0 \end{aligned}$$

which is **Black-Scholes** pde for derivative price.

The generator of the process is given by

$$\mathcal{A} = r \cdot S \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2}$$

By Feynman-Kac theorem the time 't' value of the derivative is the solution

$$V(t, S(t)) = e^{-r(T-t)} E_{\tilde{P}}[h(S(T))/F_t]$$

Where \tilde{P} is risk neutral probability measure, and $h(S(T))$ is pay-off of the derivative security at maturity.