ecomes significant.

relevant texts for this chapter are Capinski and Zastawniak [25], Hull [65] ...d Luenberger [85].

1.7 Exercises

Exercise 1.1 *Let $B(0) =$ Rs 100, $B(1) =$ Rs 110 and $S(0) =$ Rs 80. Also let*

$$S(1) = \begin{cases} 100, & \text{with probability } p = 0.80 \\ 60, & \text{with probability } p = 0.20. \end{cases}$$

Design a portfolio with initial wealth of Rs 100,000, split fifty-fifty between stock and bond. Compute the expected return and the risk of the portfolio so constructed.

Exercise 1.2 *Let $B(0)$, $B(1)$ $S(0)$ and $S(1)$ be as specified in Exercise 1.1. Also let C and P respectively be a European call and European put with $K =$ Rs 100 and $T = 1$ year.*

(i) Determine $C(0)$ and $P(0)$.

(ii) Find the final wealth of an investment with initial capital of Rs 900 being

Hence by the forward price formula

$$\begin{aligned} F &= S(0)\, e^{rT} \\ &= (18000)\, e^{(0.08\times 3/4)} \\ &= (18000)\, e^{0.06} \\ &= \text{Rs } 19113.06. \end{aligned}$$

□

Example 2.3.2 *Find the forward price of a non-dividend paying stock traded today at Rs 100, with the continuously compounded interest rate of 8% per year, for a contract expiring seven months from today.*

Solution We have $S(0) = 100$, $r = 8\%$ per year and $T = 7/12$ year. This gives

$$F = 100\, e^{(0.08)(7/12)} = \text{Rs } 104.78.$$

□

Example 2.3.3 *The current price of sugar is Rs 60 per Kg and its carrying cost is 10 paisa per Kg per month to be paid at the beginning of each month. Let the constant interest rate r be 9% per annum. Find the forward price of the sugar (Rupees per Kg) to be delivered in 5 months.*

Solution The interest rate is $(.09)/12 = .0075$ per month. Therefore the reciprocal of one month discount rate for any month is (1.0075). Therefore we have

$$\begin{aligned} F(0,5) &= (1.0075)^5(60) + \Big(\sum_{i=1}^{5}(1.0075)^i\Big)(.1) \\ &= \text{Rs } 62.79. \end{aligned}$$

□

Example 2.3.4 *Let the price of a stock on 1st April 2010 be 10% lower than it was on 1st January 2010. Let the risk free rate be constant at $r = 6\%$. Find the percentage drop of the forward price on 1st April 2010 as compared to the one on 1st January 2010 for a forward contract with delivery on 1st October 2010.*

Solution It is convenient to take 1st January 2010 as $t = 0$. Then 1st October 2010 is 9 months, i.e. 3/4 year. Thus $t = 3/4$. Also $t = \tau$ is 1st April 2010, i.e. 3/12 = 1/4. Therefore using Theorem 2.3.1 we get

(iii) enter into a long forward contract with delivery time $t = T$ and forward price as $F(\tau, T)$.

[Th]en $V_Q(\tau) = 0$ and $V_Q(T)$ is given by

$$\begin{aligned} V_Q(T) &= \frac{f(\tau)}{d(\tau,T)} + (S(T) - F(\tau,T)) + (F(0,T) - S(T)) \\ &= \frac{f(\tau)}{d(\tau,T)} + F(0,T) - F(\tau,T)\,, \end{aligned}$$

which is strictly positive and risk free. This is not possible due to no arbitrage principle.

□

Example 2.4.1 *Let at the beginning of the year, a stock be sold for Rs 45 and risk free interest rate be 6%. Consider a forward contract on this stock with delivery date as one year. Find its forward price. Also find its value after 9 months if it is given that the stock price at that time turns out to be Rs 49.*

Solution From (2.2), the initial forward price $F(0,1)$ is given by

$$F(0,1) = S(0)\, e^{rT} = 45\, e^{0.06} = Rs\ 47.78.$$

Also it is given that $S(9/12) = 49$ and hence

$$F(9/12, 1) = 49\ \exp\left((\cdot 06)(3/12)\right) = Rs\ 49 \cdot 74.$$

Therefore by Theorem 2.4.1, the value of the forward contract after 9 months is

$$\begin{aligned} f(9/12) &= [F(9/12,1) - F(0,1)]\ \exp((-0 \cdot 06)(1 - (9/12))) \\ &= 1 \cdot 93. \end{aligned}$$

□

Example 2.4.2 *Consider the data of Example 2.4.1 and assume that a dividend of Rs 2 is being paid after 6 months. Find the forward price of the contract and also its value after 9 months.*

Solution By (2.7), the initial forward price $F(0,1)$ is given by

$$\begin{aligned} F(0,1) &= [S(0) - (div)e^{-rt}]\ e^{rT} \\ &= \left[45 - 2e^{-0.06(1/2)}\right]\ e^{\cdot 06} \\ &= Rs\ 45 \cdot 72. \end{aligned}$$

Let us take n sufficiently large (equivalently, the time steps between successive trading instances approach zero) so that $(\Delta t)^2$ terms can be neglected. We get

$$U = ln(u) = \sigma\sqrt{\Delta t} \quad \text{and} \quad D = -U = -\sigma\sqrt{\Delta t}.$$

Consequently, we have

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}} \quad \text{and} \quad p = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}. \tag{4.6}$$

We take a small break here for some examples to illustrate the choice of the parameters.

Example 4.3.1 *A non-dividend paying stock is currently selling at Rs* 100 *with annual volatility* 20%. *Assume that the continuously compounded risk-free interest rate is* 5%. *Using a two-period CRR binomial option pricing model, find the price of one European call option on this stock with a strike price of Rs* 80 *and time to expiration 4 years.*

Solution We are given $S(0) = Rs\ 100,\ K = Rs\ 80,\ T = 4,\ \Delta t = 2,\ r = 0.05,\ \sigma = 0.2$. Applying the CRR model, the up-factor and the down-factor are respectively given by

$$u = e^{\sigma\sqrt{\Delta t}} = 1.3269, \quad d = \frac{1}{u} = 0.7536.$$

Now,

$$S_{uu} = u^2S(0) = 176.0664,\ S_{ud} = udS(0) = 100.00,\ S_{dd} = 56.7913.$$

Thus,

$$\begin{aligned} C_{uu} &= \text{Max}\{S_{uu} - K, 0\} = 96.0664, \\ C_{ud} &= \text{Max}\{S_{ud} - K, 0\} = 20.00, \\ C_{dd} &= \text{Max}\{S_{dd} - K, 0\} = 0. \end{aligned}$$

Since the annual rate of risk-free interest is continuously compounded (so instead of $1 + r$ or R we shall use $e^{r\Delta t}$), the risk neutral probability is

$$\hat{p} = \frac{e^{r\Delta t} - d}{u - d} = 0.6132.$$

Hence for a two period binomial model,

$$\begin{aligned} C(0) &= \cancel{e^{-r\Delta t}}((\hat{p})^2C_{uu} + 2\hat{p}(1 - \hat{p})C_{ud} + (1 - \hat{p})^2C_{dd}) \\ &= \cancel{e^{-0.1}}((0.3760)(96.0664) + (0.4744)(20.00) + 0) \\ &= \text{Rs}\ \cancel{41}.27. \quad 27.34 \end{aligned}$$

□

For computational purpose, we take the annualized risk-free interest rate and ... 365 days for counting a year but the exchanges across

Observe that the expected rate of return μ on the stock does not appear in the Black-Scholes formula. Moreover all parameters except for σ, the volatility of the underlying stock, can be observed in the market. Regarding σ, we assume to estimate it from historical market data (historical volatility) even though it may not present the true picture. We shall briefly be touching the concept of '*implied volatility* at the end of this chapter.

The next example underlines the strength of the Black-Scholes formula.

Example 4.3.3 *Suppose a non-dividend paying stock is currently selling at Rs 100 and the stock's volatility is 24%. Assume that the continuously compounded risk-free interest rate is 5%. A European call option is offered on this stock with time to maturity 3 months and strike price Rs 125. Calculate the price of the block of 100 options in the Black-Scholes framework.*

Solution Here, $S(0) = Rs\ 100,\ K = Rs\ 125,\ T = 3/12 = 0.25,\ r = 0.05,\ \sigma = 0.24.$

$$d_1 = \frac{ln(100/125) + (0.05 + \frac{1}{2}(0.24)^2)\, 0.25}{0.24\sqrt{0.25}} = -1.695363$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -1.815363.$$

Because d_1 and d_2 are negative, we use, $\Phi(d_1) = 1 - \Phi(-d_1)$ and $\Phi(d_2) = 1 - \Phi(-d_2)$. We round off d_1 to 1.70 and d_2 to 1.82 before looking up the standard normal distribution table. Thus,

$$\Phi(d_1) = 1 - 0.9554 = 0.0446, \quad \Phi(d_2) = 1 - 0.9656 = 0.0344.$$

Invoking (4.13), we get

$$C(0) = 100(0.0446) - 125e^{(-0.05)(0.25)}(0.0344) = 0.2134.$$

Therefore, the cost of the block of 100 options is Rs 21.34.

□

In case we wish to compute the European call option price at any time t, $0 \leq t < T$, then we simply need to make a small change in the Black-Scholes formula as follows.

$$C(t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

where $S(t)$ is the spot price of the stock at time t and $T-t$ is the remaining time to maturity.

The price of the European put option with the strike price K and time to maturity $T-t$, $0 \leq t < T$, can either be calculated using the put-call parity or directly by the following Black-Scholes formula.

$$P(t) = Ke^{-r(T-t)}\Phi(-d_2) - S(t)\Phi(-d_1)$$

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Obviously if we enter into the European put option on the day it is offered then the put price $P(0)$ can be obtained by taking $t = 0$ in the above formula. Here, we encouraged the readers to derive the Black-Scholes formula for pricing European put option using the analogous arguments as done above in the derivation of $C(0)$.

So far, we have discussed a European call or a European put options cases. It is well known that, for a non-dividend paying stock, the value of an American call option is same as that of a European call option with the same strike price and time to expiration. Hence (4.13) can be used to compute price of an American call option on a non-dividend paying underlying stock. Notably, the Black-Scholes formula for an American put option is not the same as for European put option as it may pay to exercise them early. An American put price has to be approximated using the binomial method explained in previous chapter wherein we can simulate sufficiently large binomial lattice by taking sufficiently small time steps for more accurate approximation.

4.4 Black-Scholes Formula for Dividend Paying Stock

One of the assumptions in the Black-Scholes basic model is that the underlying stock pays no dividend. We ...

$$0.3\sqrt{5}$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.4922.$$

$$\Phi(d_1) = \Phi(0.18) = 0.5714, \quad \Phi(d_2) = 1 - \Phi(-d_2) = 1 - 0.6879 = 0.3121.$$

$$C(0) = S_a\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

Thus,

$$C(0) = (56.082)(0.5714) - (62.304)(0.3121) = \text{Rs } 12.60.$$

□

Remark 4.4.1 *Suppose in the above example the underlying stock pays no dividend in the first two years but thereafter pays a dividend of Rs 30 in 3 years. Then,*

$$S_a = 74.1787,\ d_1 = 0.5955,\ d_2 = -0.0753.$$

$$\Phi(d_1) = 0.7123,\ \Phi(d_2) = 1 - \Phi(-d_2) = 0.4721 \text{ and } C(0) = \text{Rs } 23.42.$$

Now assume the stock pays no dividend in its entire lifespan of 5 years, that is, div = 0. Then

$$d_1 = \frac{\ln(100/80) + (0.05 + 0.5(0.3)^2)5}{0.3\sqrt{5}} = 1.0407, \quad d_2 = 0.3699.$$

$$\Phi(d_1) = 0.8508,\ \Phi(d_2) = 0.6443,$$

and hence $C(0) = \text{Rs } 44.94$.

Thus, the dividend paid by the underlying stock gets reflected in price reduction of the European call option.