Discounted Portfolio Process

Let the stock having a price $S(t)$ per unit follows a generalized GBM with constant mean return 'μ' and a constant volatility $\sigma > 0$. The price is governed by SDE

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t). \quad t \in [0,T] \qquad -(1)$$

Also, let $\beta(t)$ be the price of risk free asset which satisfy the ordinary d.e.

$$d\beta(t) = r\beta(t) \cdot dt. \qquad —(2)$$

where r is constant risk free interest rate.

Suppose at time 't' we take a portfolio consisting of $a(t)$ shares of stock and $b(t)$ shares of risk free asset. Let $V(t)$ be the value of this portfolio at 't', i.e.,

$$V(t) = a(t) \cdot S(t) + b(t) \cdot \beta(t), \quad t \in [0,T] \qquad —(3)$$

then

$$dV(t) = a(t) \cdot dS(t) + b(t) \cdot d\beta(t) \qquad —(4)$$

The discounted price of one share of stock is

$$\tilde{S}(t) = e^{-rt} S(t), \quad t \in [0,T] \qquad —(5)$$

Applying Itô-Doeblin formula of ~~first~~ second variant. $[f(t,x) = e^{-rt} x = \tilde{S}(t)]$

$$d\tilde{S}(t) = -re^{-rt} S(t) dt + e^{-rt} dS(t). \qquad —(6)$$

using (1)

$$= -re^{-rt} S(t) dt + e^{-rt} [\mu S(t) dt + \sigma S(t) dW(t)]$$

$$= \tilde{S}(t) [(\mu - r) dt + \sigma dW(t)]$$

$$= \sigma \tilde{S}(t) \cdot d\tilde{W}(t) \qquad -(7)$$

where $\tilde{W}(t) = \frac{\mu - r}{\sigma} t + W(t), \quad t \in [0,T].$

Now $(\mu - r)$ is expected return minus risk free return or we call it risk premium

therefore $\frac{\mu - r}{\sigma}$ is the risk premium per unit of risk and is called the market price of risk.

Feynman - Kac Theorem (R Feynman & M. Kac)

It establishes a link between parabolic pde and S.P.

Let the S.P. $\{X(t), 0 \le t \le T\}$ satisfy the following SDE

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)$$

where $\mu(t, X(t))$ & $\sigma(t, X(t))$ are functions on $[0,T) \times R$ where called drift and diffusion function respectively. Also $X(0) = x$, for some $x \in R$. Then the solution of the following pde.

$$g_t(t,x) + \mu(t,x)\, g_x(t,x) + \frac{1}{2}\sigma^2(t,x)\, g_{xx}(t,x) - r g(t,x) = 0 \quad \text{— (8)}$$

Subject to the boundary condition

$$\cancel{g(T, X(T) = x)}\; g(T, X(T) = x) = h(x), \quad x \in R$$ is

a function $g: [0,T] \times R \to R$ given by

$$g(t,x) = E\left[e^{-r(T-t)} \cdot h(X(T) \mid X(t) = x)\right]. \quad \text{— (9)}$$

We define an operator as following (called generator of the process)

$$A = \mu(t, x(t)) \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x(t)) \frac{\partial^2}{\partial x^2}$$

Remark Feynman-Kac th. implies both way i.e. If pde is given then sol is known and if a sol satisfying the boundary condition the pde whose sol is this is known.

Then equ (8) can be written as

$$\frac{\partial g}{\partial t} + Ag - rg = 0 \,.$$

Derivation of Black-Scholes formula for a derivative Security.

Let the stock price $S(t)$ be driven by the process

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t).$$

using equ (7) where $\tilde{W}(t) = \frac{\mu - r}{\sigma} + W(t)$ the risk neutral process is given by

$$dS(t) = r S(t) \cdot dt + \sigma S(t) \cdot d\tilde{W}_t \qquad (10)$$

suppose a derivative is written on this stock. Let $V(t, S(t))$ be the price of this security at any $t \in [0,T]$ and $V(T, S(T))$ be its payoff on maturity.

here $V : [0,T] \times R_+ \longrightarrow R_+$ where R_+ is non negative real no.

Using Ito lemma we have

$$dV(t) = dV(t, S(t)) = V_t \cdot dt + V_x \cdot dS(t) + \frac{1}{2} \cdot V_{xx}\, dS(t) \cdot dS(t)$$

$$= \left[\frac{\partial V}{\partial t} + r S(t) \cdot \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 S^2(t) \cdot \frac{\partial^2 V}{\partial x^2}\right] dt + \sigma S(t) \frac{\partial V}{\partial x} d\tilde{W}(t)$$

$$\left[dS(t) \cdot dS(t) = \sigma^2 S^2(t) \cdot \underbrace{dW\, dW}_{dt} \right] \qquad (11)$$

suppose the derivative security can be hedged. We replicate the portfolio taking $a(t)$ shares of stock and $b(t)$ shares of risk free asset whose price is governed by ode. $d\beta(t) = r\beta(t) \cdot dt$.

Then we have

$$\left[\text{value of the portfolio } V(t) = a(t) \cdot S(t) + b(t) \cdot \beta(t) \right]$$

$$dV(t) = a(t) \cdot dS(t) + b(t) \cdot r \cdot \beta(t) \cdot dt$$

$$= a(t) \cdot [r S(t)\, dt + \sigma S(t) \cdot d\tilde{W}(t)] + r\, b(t)\, \beta(t) \cdot dt \qquad (12)$$

Comparing (11) & (12)

$$\frac{\partial V}{\partial x} = a(t), \quad \& \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2(t) \cdot \frac{\partial^2 V}{\partial x^2} = r\, b(t) \cdot \beta(t) \qquad \text{— (13)}$$

Now using (12) $b(t)\,\beta(t) = V(t) - a(t)\,S(t) = V(t) - S(t) \cdot \frac{\partial V}{\partial x}$ $\left[\text{using } a(t) = \frac{\partial V}{\partial x} \right]$

putting the value of $b(t) \cdot \beta(t)$ in (13) we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2(t) \cdot \frac{\partial^2 V}{\partial x^2} = \left[V(t) - S(t) \cdot \frac{\partial V}{\partial x} \right] \times r$$

or
$$\boxed{\frac{\partial V}{\partial t} + r S \cdot \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r \cdot V = 0} \qquad \text{— (14)}$$

which is Black-Scholes p.d.e. for derivative price.

The generator of the process is given by

$$A = r S \frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}$$

By the Feynman-Kac th., the time t value of the derivative is the solution

$$V(t, S(t)) = e^{-r(T-t)} \cdot E_{\tilde{P}}\left(h(S(t)) / F_t\right).$$

where $\tilde{P}$ is risk neutral probability measure. (RNPM), and $h(S(t))$ is payoff of the derivative security on maturity.

Read Ch 12 & 13 of Hull. along with problem.