

Eckmann-Hilton and the Hopf Fibration

Raymond Baker

University of Colorado Boulder

<https://morphismz.github.io/talks/2024-04-03-hottuf>

HoTT/UF 2024

April 3rd, 2024

Conventions

\mathbb{S}^1

$b_1 : \mathbb{S}^1$

$\text{loop} : \Omega(\mathbb{S}^1)$

\mathbb{S}^2

$\text{base}_2 : \mathbb{S}^2$

$\text{surf}_2 : \Omega^2(\mathbb{S}^2)$

\mathbb{S}^3

$\text{base}_3 : \mathbb{S}^3$

$\text{surf}_3 : \Omega^3(\mathbb{S}^3)$

Conventions

refl

refl is the trivial path in $\Omega(X)$

refl^n is the trivial path in $\Omega^n(X)$

Transport

$\text{tr}^B(p) : B(x) \rightarrow B(y)$

$\text{tr}^2(\alpha) : \text{tr}(p) \sim \text{tr}(q)$

$\text{tr}^3(\gamma) : \text{tr}(\alpha) \sim \text{tr}(\beta)$

The Goal

And some reasons to care

The Goal: Construct the Hopf fibration $\text{hpf} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ using the Eckmann-Hilton argument.

Immediate Consequences:

- 1 Simple description of the generator of $\pi_3(\mathbb{S}^2)$. From the fiber sequence of hpf .
- 2 Ditto the generator of $\pi_4(\mathbb{S}^3)$. From the Freudenthal suspension theorem.
- 3 $\pi_4(\mathbb{S}^3)$ has order *at most* 2. From Syllepsis.

The Plan

- 1 Use Eckmann-Hilton to construct $eh : \Omega^3(\mathbb{S}^2)$. This is equivalent to a map $hpf : \mathbb{S}^3 \rightarrow \mathbb{S}^2$.
- 2 Generalize ideas from Von Raumer's "Path Spaces of Higher Inductive Types" to characterize the fiber.

The Eckmann-Hilton Identification

For $\alpha, \beta : \Omega^2(X)$, we have $\text{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

$$\text{EH}(\text{surf}_2, \text{surf}_2) : \text{surf}_2 \cdot \text{surf}_2 = \text{surf}_2 \cdot \text{surf}_2$$

The type of this identification is equivalent to $\Omega^3(\mathbb{S}^2)$.

The Eckmann-Hilton 3-loop

Define $\text{eh} : \Omega^3(\mathbb{S}^2)$ as the image of $\text{EH}(\text{surf}_2, \text{surf}_2)$ under said equivalence.

See [agda-unimath](#) for more.

The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

$$\text{hpf} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$$

Define a map $\text{hpf} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ by \mathbb{S}^3 -induction:

$$\text{hpf}(\text{base}_3) := \text{base}_2$$

$$\text{hpf}(\text{surf}_3) := \text{eh}$$

Generalize Kraus and Von Raumer's ideas

“Path Spaces of Higher Inductive Types” outlines a method characterizing the family of based identity types of a HIT.

This can be seen as characterizing the family of fibers of a map $\text{unit} \rightarrow B$.

We will generalize this idea to work for general maps $A \rightarrow B$.

Kraus and Von Raumer's Computation of $\Omega(\mathbb{S}^1)$

- 1 Universal Property: $\text{Id}_{b_1} : \mathbb{S}^1 \rightarrow U$ is the initial pointed type family
- 2 Descent: $\Omega(\mathbb{S}^1)$ is the initial pointed type equipped with an automorphism
- 3 Universal Property, pt. II: \mathbb{Z} is the initial pointed type equipped with an automorphism

Note, $\Omega(\mathbb{S}^1)$ is the fiber of $\text{unit} \rightarrow \mathbb{S}^1$

Our Computation of $\text{fib}_{\text{hpf}}(\text{base}_2)$

- 1 Universal Property: $\text{fib}_{\text{hpf}} : \mathbb{S}^2 \rightarrow U$ is the initial family equipped with a lift of hpf (a section over hpf).
- 2 Descent: $\text{fib}_{\text{hpf}}(\text{base}_2)$ is the initial type equipped with some data.
- 3 Universal Property, pt. II: \mathbb{S}^1 is the initial type equipped with some data.

The Universal Property of the Family of Fibers

Fix a pointed map $h: A \rightarrow B$. Then:

Heuristic

$\text{fib}_h(b_0)$ is like the loop space of B with extra identifications freely generated by the map h

The Universal Property of the Family of Fibers

A map $h : A \rightarrow B$ is equivalent to a type family $\text{fib}_h : B \rightarrow U$.

This family always comes equipped with a lift:

$$\lambda(a).(a, \text{refl}) : (a : A) \rightarrow \text{fib}_h \circ h(a)$$

fib_h is the initial such type family

The Universal Property of the Family of Fibers

Wild Category of Families with Lifts

Objects: families $P : B \rightarrow U$ equipped with a lift $(a : A) \rightarrow P \circ h(a)$

Maps: families of maps $(b : B) \rightarrow P(b) \rightarrow Q(b)$ that preserve the lift

Universal Property of fib_h

The family fib_h with its canonical lift is initial in this wild category.

Proof: follows from the standard equivalence $A \simeq \sum_{b:B} \text{fib}_h(b)$.

Formalized in agda-unimath

Loop Spaces are a Special Case

If $A \equiv \text{unit}$ and $h : \text{unit} \rightarrow B$ defined by $h(\star) \equiv b_0$:

$$((a : \text{unit}) \rightarrow P \circ h(a)) \simeq P(b_0)$$

So fib_h is the initial type family equipped with a point over b_0

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$ and define h by $s : \Omega^3(B, b_0)$.

Lifts $((a : \mathbb{S}^3) \rightarrow P \circ h(a))$ are equivalent to dependent 3-loops:

a point $u : P(b_0)$

an identification $\text{tr}^3(s)(u) = \text{refl}_u^2$

So fib_h is the initial type family equipped with a point over b_0 and an identification as above.

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$, $B \equiv \mathbb{S}^2$ and $h \equiv \text{hpf}$.

Then fib_{hpf} is the initial:

family over \mathbb{S}^2

point $u : \text{fib}_{\text{hpf}}(\text{base}_2)$

identification $\text{tr}^3(\text{eh})(u) = \text{refl}_u^2$

Interlude, descent data of \mathbb{S}^2

A type family P over \mathbb{S}^2 is equivalent to:

Descent data of \mathbb{S}^2

a type X , the value of $P(\text{base}_2)$

a 2-automorphism $\text{id}_X \sim \text{id}_X$, the transport $\text{tr}^2(\text{surf}_2)$

Now we apply Kraus and Von Raumer's ideas using the descent data of \mathbb{S}^2 .

A Characterization of fib_{hpf}

Then fib_{hpf} is the initial data:

type F

2-automorphism $H : \text{id}_F \sim \text{id}_F$

point $u : F$

identification $t : \text{tr}^3(\text{eh})(u) = \text{refl}_u^2$

The latter identification is equivalent to an identification

$$\text{tr}^3(\text{EH}(\text{surf}_2, \text{surf}_2))(u) = \text{refl}_{\text{tr}^2(\text{surf}_2 \cdot \text{surf}_2)}(u)$$

The Eckmann-Hilton Argument

Eckmann-Hilton

For $\alpha, \beta : \Omega^2(X)$, we have $\text{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Where does Eckmann-Hilton come from?

Fix a pointed type (X, \bullet) and consider $\text{Id}_\bullet : X \rightarrow U$, the family of based identity types.

A loop $p : \Omega(X)$ induces:

$$\text{tr}^{\text{Id}_\bullet}(p) : \Omega(X) \simeq \Omega(X)$$

This is path concatenation:

for $q : \Omega(X)$ we have:

$$\text{tr}(p)(q) = q \cdot p.$$

Where does Eckmann-Hilton come from?

Up one dimension:

a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$$\mathrm{tr}^{(\mathrm{Id}_\bullet)^2}(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$$

This is Eckmann-Hilton:

for $\beta : \Omega^2(X)$, we have:

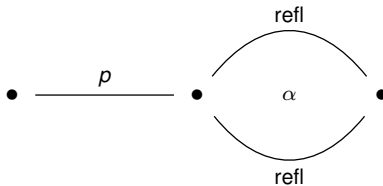
$$\mathrm{nat}[\mathrm{tr}^2(\alpha)](\beta) = \mathrm{EH}(\alpha, \beta)$$

(modulo coherence paths)

Eckmann-Hilton from tr^2

Computing $\text{tr}^2(\alpha) : \text{id}_{\Omega(X)} \sim \text{id}_{\Omega(X)}$

$$\text{tr}^2(\alpha) = \text{whisker}_\alpha = \lambda(p).\text{refl}_p \star \alpha$$



$$\text{tr}^2(\alpha)(\text{refl}_\bullet) = \alpha$$

The naturality condition of $\text{tr}^2(\alpha) : \text{id}_{\Omega(X)} \sim \text{id}_{\Omega(X)}$

For $\beta : \Omega^2(X)$:

$$\begin{array}{ccc} \text{refl}_\bullet & \xrightarrow{\text{id}(\beta)} & \text{refl}_\bullet \\ \text{tr}^2(\alpha)(\text{refl}) \downarrow & \text{nat-}[\text{tr}^2(\alpha)](\beta) & \downarrow \text{tr}^2(\alpha)(\text{refl}) \\ \text{refl}_\bullet & \xrightarrow{\text{id}(\beta)} & \text{refl}_\bullet \end{array}$$

Plus coherence paths, this ends

$$\text{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$$

.

Eckmann-Hilton in the Universe

Let $P : X \rightarrow U$ with $u : P(\bullet)$ and $\alpha, \beta : \Omega^2(X)$. Then:

$$\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(P(\bullet))} \sim \mathrm{id}_{\Omega(P(\bullet))}$$

and

$$\mathrm{tr}^2(\beta)(u) : \Omega(P(\bullet))$$

Eckmann-Hilton in the Universe

$$\begin{array}{ccc} u & \xrightarrow{\text{tr}^2(\beta)(u)} & u \\ \text{tr}^2(\alpha)(u) \downarrow & \text{nat-}[\text{tr}^2(\alpha)](\text{tr}^2\beta)(u) & \downarrow \text{tr}^2(\alpha)(u) \\ u & \xrightarrow{\text{tr}^2(\beta)(u)} & u \end{array}$$

$$\text{tr}^3(\text{EH}(\alpha, \beta))(u) = \text{nat-}[\text{tr}^2(\alpha)](\text{tr}^2(\beta)(u)). \text{ (modulo coherence paths)}$$

Proof: See [agda-unimath](#)

A Characterization of fib_{hpf}

Recall that fib_{hpf} is the initial data:

type F

homotopy $H : \text{id}_F \sim \text{id}_F$

point $u : F$

identification $\text{tr}^3(\text{EH}(\text{surf}_2, \text{surf}_2))(u) = \text{refl}_{\text{tr}^2(\text{surf}_2 \cdot \text{surf}_2)}(u)$

Now $\text{tr}^3(\text{EH}(\text{surf}_2, \text{surf}_2))(u) = \text{nat}[\text{tr}^2(\text{surf}_2)](\text{tr}^2(\text{surf}_2)u)$

A Characterization of fib_{hpf}

Finally, fib_{hpf} is the initial data:

type F

point $u : F$

homotopy $H : \text{id}_F \sim \text{id}_F$

identification $\text{nat-}H(H(u)) = \text{refl}_{H(u)} \cdot H(u)$

We can package this as a HIT F generated by the subsequent data.

The Fiber is \mathbb{S}^1

Want $\mathbb{S}^1 \simeq F$

Two approaches:

- 1 Using a HIT and directly constructing an equivalence
- 2 Show \mathbb{S}^1 is initial in the wild category of F -algebras

Using a HIT

In cubical agda, thanks to Tom Jack

In Book HoTT, good luck. In agda-unimath, not possible.

Want to show $\text{hom}_{F\text{-alg}}(\mathbb{S}^1, X)$ is contractible for every F -algebra X .

Have a definition of F -algebras.

Need a definition of the hom type between F -algebras.

Morphisms of F -algebras

Consider F -algebras (X, K, x_0, p) and (Y, M, y_0, q) .

A morphism of F -algebras comprises:

- ① $g : X \rightarrow Y$
- ② $G : g \cdot_I K \sim M \cdot_I g$
- ③ $g_0 : g(x_0) = y_0$
- ④ A witness that “ p is sent to q ”

\mathbb{S}^1 forms an F -algebra

type - \mathbb{S}^1

homotopy - L

point - b_1

identification - $\text{defn}_L : \text{nat} \rightarrow L(L(b_1)) = \text{refl}_{\text{loop}} \cdot \text{loop}$

$$\text{hom}_{F\text{-alg}}(\mathbb{S}^1, X) \simeq \text{unit}$$

a map: $(g : \mathbb{S}^1 \rightarrow X, G : g \cdot_l L \sim K \cdot_r g, g_0 : g(b_1) = x_0, t)$

g is equivalent to $g(b_1) : X$ and $g(\text{loop}) : \Omega(X, x)$.

$(g(b_1), g_0)$ is a contractible pair.

G is equivalent to $G(b) : g(\text{loop}) = K(g(b_1))$ and $\text{nat-}G(\text{loop})$.

$(g(\text{loop}), G)$ is a contractible pair.

Claim: $\text{nat-}G(\text{loop})$ and t form a contractible pair.

- 1 Adapting the James construction and Wörn's Zig Zag Construction
- 2 $\pi_4(\mathbb{S}^3)$
- 3 Higher Hopf Fibrations and Higher Coherences

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

Suspension Preserve Eckmann-Hilton

For the unit $\sigma : \mathbb{S}^2 \rightarrow \Omega(\mathbb{S}^3)$, we have $\sigma(eh) = eh_{\text{surf}_3}$

Proof: all functions preserve Eckmann-Hilton.

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

eh_{surf_3} generates $\pi_4(\text{surf}_3)$ and its square is trivial.

Proof: the map $\pi_3(\sigma) : \pi_3(\mathbb{S}^2) \rightarrow \pi_4(\mathbb{S}^3)$ is surjective. So $\pi_3(\sigma)(eh) = eh_{\text{surf}_3}$ is a generator. Syllepsis implies that eh_{surf_3} is its own inverse.

Non-Triviality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family $B : \Omega(\mathbb{S}^3) \rightarrow U$ such that

$$\text{nat}[\text{tr}^2(\text{surf}_3)](\text{tr}^2(\text{surf}_3)(u))$$

is non-trivial, for some $u : B(\text{refl})$

Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ and $\mathbb{S}^{15} \rightarrow \mathbb{S}^8$ should also arise from higher coherences.

The E_4 coherence, corresponding to $\mathbb{S}^7 \rightarrow \mathbb{S}^4$, was constructed by Sojakova.

Up, Up, and Away

There is a correspondence between the E_n coherence and the descent data of Id over the n -sphere.

E_1

Group structure, given by path concatenation in $\Omega(\mathbb{S}^1)$

E_2

Braiding, given by Eckmann-Hilton in $\Omega^2(\mathbb{S}^2)$

E_3

Syllepsis, given by syllepsis in $\Omega^3(\mathbb{S}^3)$

Syllepsis and Descent over \mathbb{S}^3

The family $\text{Id} : \mathbb{S}^3 \rightarrow U$ is equivalent to:

Descent over \mathbb{S}^3

$\Omega(\mathbb{S}^3)$

$\text{tr}^3(\text{surf}_3) : \text{Refl} \sim \text{Refl} \equiv (x : X) \rightarrow \Omega^2(X, x)$

Claim: the 2-D naturality condition of this homotopy is syllepsis.

I.e. $\text{nat}^2\text{-}[\text{tr}^3(\text{surf}_3)](\text{surf}_3)$ is equivalent to

$$\text{Sy}(\text{surf}_3, \text{surf}_3) : \text{EH}(\text{surf}_3, \text{surf}_3) = \text{EH}(\text{surf}_3, \text{surf}_3)^{-1}$$

Proof: Horrible path algebra

Acknowledgements

- My Advisor: Professor Jonathan Wise, CU Boulder
- Tom Jack, for many helpful discussions
- Egbert Rijke, Fredrik Bakke, Vojtěch Štěpančík, and the agda-unimath community

The End

Questions? Comments?