Eckmann-Hilton and the Hopf Fibration

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Conventions

\mathbb{S}^1

 $b_1:\mathbb{S}^1$

 $\mathsf{loop}:\Omega(\mathbb{S}^1)$

\mathbb{S}^2

 $base_2: \mathbb{S}^2$

 $surf_2 : \Omega^2(\mathbb{S}^2)$

\mathbb{S}^3

 $base_3:\mathbb{S}^3$

 $\text{surf}_3:\Omega^3(\mathbb{S}^3)$

Conventions

refl

refl is the trivial path in $\Omega(X)$

reflⁿ is the trivial path in $\Omega^n(X)$

Transport

$$\operatorname{tr}^B(p): B(x) \to B(y)$$

$$\operatorname{tr}^2(\alpha) : \operatorname{tr}(p) \sim \operatorname{tr}(q)$$

$$\operatorname{tr}^3(\gamma) : \operatorname{tr}(\alpha) \sim \operatorname{tr}(\beta)$$

The Goal

And some reasons to care

<u>The Goal</u>: Construct the Hopf fibration hpf : $\mathbb{S}^3 \to \mathbb{S}^2$ using the Eckmann-Hilton argument.

Immediate Consequences:

- 1 Simple description of the generator of $\pi_3(\mathbb{S}^2)$. From the fiber sequence of hpf.
- 2 Ditto the generator of $\pi_4(\mathbb{S}^3)$. From the Freudenthal suspension theorem.
- 3 $\pi_4(\mathbb{S}^3)$ has order at most 2. From Syllepsis.

The Plan

- 1 Use Eckmann-Hilton to construct a 3-loop eh : $\Omega^3(\mathbb{S}^2)$. This is equivalent to a map hpf : $\mathbb{S}^3 \to \mathbb{S}^2$.
- 2 Adapt ideas from Von Raumer's "Path Spaces of Higher Inductive Types" to characterize the fiber.

Eckmann-Hilton

The Eckmann-Hilton Identification

For $\alpha, \beta : \Omega^2(X)$, we have $EH(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Eckmann-Hilton

$$EH(surf_2, surf_2) : surf_2 \cdot surf_2 = surf_2 \cdot surf_2$$

The type of this is identification is equivalent to $\Omega^3(\mathbb{S}^2)$.

The Eckmann-Hilton 3-loop

Define eh : $\Omega^3(\mathbb{S}^2)$ as the image of $\text{EH}(\text{surf}_2,\text{surf}_2)$ under said equivalence.

See agda-unimath for more.

The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

 $hpf: \mathbb{S}^3 \to \mathbb{S}^2$

Define a map hpf : $\mathbb{S}^3 \to \mathbb{S}^2$ by \mathbb{S}^3 -induction:

 $hpf(base_3) :\equiv base_2$

 $hpf(surf_3) := eh$

Kraus and Von Raumer's Computation of $\Omega(\mathbb{S}^1)$

- **1** Universal Property: $Id_{b_1} : \mathbb{S}^1 \to U$ is the inital pointed type family
- 2 Descent: $\Omega(\mathbb{S}^1)$ is the inital pointed type equipped with an automorphism
- 3 Universal Property, pt. II: \mathbb{Z} is the initial pointed type equipped with an automorphism

Note, $\Omega(\mathbb{S}^1)$ is the fiber of unit $\to \mathbb{S}^1$

Our Computation of fib_{hpf}(base₂)

- 1 Universal Property: $fib_{hpf} : \mathbb{S}^2 \to U$ is the inital family equipped with a lift of hpf (a section over hpf).
- Descent: fib_{hpf}(base₂) is the inital type equipped with some data.
- 3 Universal Property, pt. II: S^1 is the inital type equipped with some data.

The Universal Property of the Family of Fibers

Fix a pointed map $h: A \rightarrow B$. Then:

Heuristic

 $fib_h(b_0)$ is like the loop space of B with extra identifications freely generated by the map h

The Universal Property of the Family of Fibers

A map $h: A \rightarrow B$ is equivalent to a type family fib_h: $B \rightarrow U$.

This family always comes equipped with a lift:

$$\lambda(a).(a, refl): (a:A) \rightarrow fib_h \circ h(a)$$

 fib_h is the initial such type family

The Universal Property of the Family of Fibers

Wild Category of Families with Lifts

Objects: families $P: B \rightarrow U$ equipped with a lift $(a: A) \rightarrow P \circ h(a)$

Maps: families of maps $(b:B) \rightarrow P(b) \rightarrow Q(b)$ that preserve the lift

Universal Property of fib_h

The family fib_h with its canonical lift is intial in this wild category.

Proof: follows from the standard equivalence $A \simeq \sum_{b:B} fib_h(b)$. Formalized in agda-unimath

Specializing the Universal Property

If $A \equiv \text{unit and } h(\star) \equiv b_0$:

$$((a : \mathsf{unit}) \to P \circ h(a)) \simeq P(b_0)$$

So fib_h is the inital type family equipped with a point over b_0

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$ and define h by $s : \Omega^3(B, b_0)$.

Lifts $((a: \mathbb{S}^3) \to P \circ h(a))$ are equivalent to dependent 3-loops:

a point $u: P(b_0)$

an identification $tr^3(s)(u) = refl_u^2$

So fib_h is the inital type family equipped with a point over b_0 and an identification as above.

Specializing the Universal Property

Let
$$A \equiv \mathbb{S}^3$$
, $B \equiv \mathbb{S}^2$ and $h \equiv hpf$.

Then fib_{hpf} is the inital:

family over \mathbb{S}^2

point u: fib_{hpf}(base₂)

identification $tr^3(eh)(u) = refl_u^2$

Interlude, descent data of S²

A type family P over \mathbb{S}^2 is equivalent to:

Descent data of S²

a type X, the value of $P(base_2)$

a homotopy $id_X \sim id_X$, the transport $tr^2(surf_2)$

Now we apply Kraus and Von Raumer's ideas using the descent data of \mathbb{S}^2 .

A Characterization of fib_{hpf}

Then fib_{hof} is the inital data:

type F

homotopy $H : id_F \sim id_F$

point u: F

identification $tr^3(eh)(u) = refl_u^2$

The latter identification is equivalent to an identification

$$\mathsf{tr}^3(\mathsf{EH}(\mathsf{surf}_2,\mathsf{surf}_2))(u) = \mathsf{refl}_{\mathsf{tr}^2(\mathsf{surf}_2\,\boldsymbol{\cdot}\,\mathsf{surf}_2)(u)}$$

The Eckmann-Hilton Argument

Eckmann-Hilton

For $\alpha, \beta : \Omega^2(X)$, we have $EH(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Where does Eckmann-Hilton come from?

Fix a pointed type (X, \bullet) and consider $Id_{\bullet}: X \to U$, the family of based identity types.

A loop $p: \Omega(X)$ induces:

$$\mathsf{tr}^{\mathsf{Id}_{\bullet}}(p) : \Omega(X) \simeq \Omega(X)$$

This is path concatination:

for $q: \Omega(X)$ we have:

$$tr(p)(q) = q \cdot p.$$

Where does Eckmann-Hilton come from?

Up one dimension:

a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$$\mathsf{tr}^{(\mathsf{Id}_{\bullet})^2}(\alpha) : \mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$$

This is Eckmann-Hilton:

for $\beta : \Omega^2(X)$, we have:

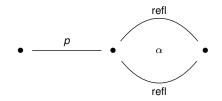
$$nat-[tr^2(\alpha)](\beta) = EH(\alpha,\beta)$$

(modulo coherence paths)

Eckmann-Hilton from tr²

Computing $\operatorname{tr}^2(\alpha) : \operatorname{id}_{\Omega(X)} \sim \operatorname{id}_{\Omega(X)}$

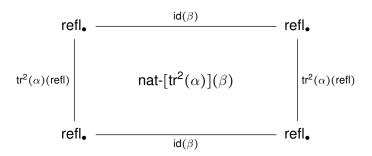
$$\mathsf{tr}^2(\alpha) = \mathsf{whisker}_{\alpha} = \lambda(p).\mathsf{refl}_p \star \alpha$$



$$\operatorname{tr}^2(\alpha)(\operatorname{refl}_{\bullet}) = \alpha$$

The naturality condition of $tr^2(\alpha) : id_{\Omega(X)} \sim id_{\Omega(X)}$

For $\beta : \Omega^2(X)$:



Plus coherence paths, this ends

$$\mathsf{EH}(\alpha,\beta):\alpha \cdot \beta = \beta \cdot \alpha$$

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Eckmann-Hilton in the Universe

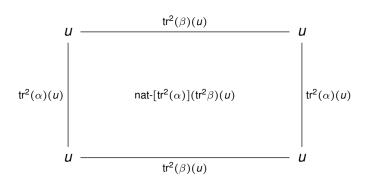
Let $P: X \to U$ with $u: P(\bullet)$ and $\alpha, \beta: \Omega^2(X)$. Then:

$$\operatorname{tr}^2(\alpha) : \operatorname{id}_{\Omega(P(ullet))} \sim \operatorname{id}_{\Omega(P(ullet))}$$

and

$$\operatorname{tr}^2(\beta)(u):\Omega(P(ullet))$$

Eckmann-Hilton in the Universe



$$\mathsf{nat}\text{-}[\mathsf{tr}^2(\alpha)](\mathsf{tr}^2(\beta)(u)) = \mathsf{tr}^3(\mathsf{EH}(\alpha,\beta))(u).$$
 (modulo coherence paths)

Proof: See agda-unimath



The Eckmann-Hilton 3-loop

We can EH to construct a 3-path in \mathbb{S}^2 :

$$surf_2 \cdot surf_2 \xrightarrow{EH(surf_2, surf_2)} surf_2 \cdot surf_2$$

This type is equivalent to $\Omega^3(\mathbb{S}^2)$

$\operatorname{eh}:\Omega^3(\mathbb{S}^2)$

Define eh as the image of EH(surf₂, surf₂) under this equivalence.

A Characterization of fib_{hpf}

Recall that fib_{hpf} is the inital data:

type F

homotopy $H: id_F \sim id_F$

point u: F

 $\mathsf{identification}\;\mathsf{tr}^3(\mathsf{EH}(\mathsf{surf}_2,\mathsf{surf}_2))(u) = \mathsf{refl}_{\mathsf{tr}^2(\mathsf{surf}_2 \, \cdot \, \mathsf{surf}_2)(u)}$

A Characterizaton of fib_{hpf}

Finally, fib_{hpf} is the initial data:

type F

point u: F

homotopy $H: id_F \sim id_F$

identification nat- $H(H(u)) = refl_{H(u)} \cdot H(u)$

We can package this as a HIT F generated by the subsquent data.

The Plan, Part III

Show that S^1 is generated by a point, a homotopy, and an identification as before.

Two approaches:

- 1 Using a HIT and directly constructing an equivalence
- 2 Show S^1 is initial amoung F-algebras

Using a HIT

In cubical agda, thanks to Tom Jack

In Book HoTT, good luck. In agda-unimath, not possible.

F-algebras

 \mathbb{S}^1 forms an *F*-algebra:

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type - \mathbb{S}^1
homotopy - L
point - b_1
identification - defn_L : nat-L(L(b_1)) = refl_{loop \cdot loop}
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F-algebra

There is a map from \mathbb{S}^1 to any other F-algebra

Let (X, K, x_0, p) be an F-algebra:

① Define $f: \mathbb{S}^1 \to X$ via $f(b_1) := x_0$ and $defn_f: f(loop) = K(x_0)$

Need:

2
$$f \cdot_I L \sim K \cdot_r f$$

3
$$f(b_1) = x_0$$

4 A witness that the trivialization is preserved

F-algebra

Any two F-algebra maps from \mathbb{S}^1 to X are equal.

Proof: Path algebra and universal property of \mathbb{S}^1 . Omitted due to time constraints.

Future Work

- 1 Adapting the James construction and Wärn's Zig Zag Construction
- 2 $\pi_4(\mathbb{S}^3)$
- 3 Higher Hopf Fibrations and Higher Coherences

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

Suspension Preserve Eckmann-Hilton

For the unit $\sigma: \mathbb{S}^2 \to \Omega(\mathbb{S}^3)$, we have $\sigma(eh) = eh_{surf_3}$

Proof: all functions preserve Eckmann-Hilton.

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

 eh_{surf_3} generates $\pi_4(surf_3)$ and its square is trival.

Proof: the map $\pi_3(\sigma):\pi_3(\mathbb{S}^2)\to\pi_4(\mathbb{S}^3)$ is surjective. So $\pi_3(\sigma)(\mathsf{eh})=\mathsf{eh}_{\mathsf{surf}_3}$ is a generator. Syllepsis implies that $\mathsf{eh}_{\mathsf{surf}_3}$ is its own inverse.

Non-Trivality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family $B: \Omega(\mathbb{S}^3) \to U$ such that

$$nat-[tr^2(surf_3)](tr^2(surf_3)(u))$$

is non-trivial, for some u : B(refl)

Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ should also arise from higher coherences.

The E_4 coherence, corresponding to $\mathbb{S}^7 \to \mathbb{S}^4$, was constructed by Sojakova.

Up, Up, and Away

There is a correspondence between the E_n coherence and the descent data of Id over the n-sphere.

E_1

Group structure, given by path concatination in $\Omega(\mathbb{S}^1)$

E_2

Braiding, given by Eckmann-Hilton in $\Omega^2(\mathbb{S}^2)$

E_3

Syllepsis, given by sypllepsis in $\Omega^3(\mathbb{S}^3)$

Syllepsis and Descent over S³

The family $Id : \mathbb{S}^3 \to U$ is equivalent to:

Descent over S3

$$\Omega(\mathbb{S}^3)$$

$$\operatorname{tr}^3(\operatorname{surf}_3):\operatorname{Refl} \sim \operatorname{Refl} \equiv (x:X) \to \Omega^2(X,x)$$

Claim: the 2-D naturality condition of this homotopy is syllepsis.

l.e. nat^2 - $[tr^3(surf_3)](surf_3)$ is equivalent to

$$Sy(surf_3, surf_3) : EH(surf_3, surf_3) = EH(surf_3, surf_3)^{-1}$$

Proof: Horrible path algebra

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 community

The End

Questions? Comments?