### Eckmann-Hilton and the Hopf Fibration

#### Raymond Baker

University of Colorado Boulder https://morphismz.github.io/talks/2024-04-03-hottuf

> HoTT/UF 2024 April 3rd, 2024

### Conventions

#### $\mathbb{S}^1$

 $b_1:\mathbb{S}^1$ 

 $\mathsf{loop}:\Omega(\mathbb{S}^1)$ 

#### $\mathbb{S}^2$

 $base_2: \mathbb{S}^2$ 

 $surf_2: \Omega^2(\mathbb{S}^2)$ 

#### $\mathbb{S}^3$

 $base_3:\mathbb{S}^3$ 

 $\text{surf}_3:\Omega^3(\mathbb{S}^3)$ 

### Conventions

#### refl

refl is the trivial path in  $\Omega(X)$ 

refl<sup>n</sup> is the trivial path in  $\Omega^{n}(X)$ 

#### Transport

$$\operatorname{tr}^B(p): B(x) \to B(y)$$

$$\operatorname{tr}^2(\alpha) : \operatorname{tr}(p) \sim \operatorname{tr}(q)$$

$$\operatorname{tr}^3(\gamma) : \operatorname{tr}(\alpha) \sim \operatorname{tr}(\beta)$$

#### The Goal

#### And some reasons to care

<u>The Goal</u>: Construct the Hopf fibration hpf :  $\mathbb{S}^3 \to \mathbb{S}^2$  using the Eckmann-Hilton argument.

#### Immediate Consequences:

- 1 Simple description of the generator of  $\pi_3(\mathbb{S}^2)$ . From the fiber sequence of hpf.
- 2 Ditto the generator of  $\pi_4(\mathbb{S}^3)$ . From the Freudenthal suspension theorem.
- 3  $\pi_4(\mathbb{S}^3)$  has order at most 2. From Syllepsis.

#### The Plan

- 1 Use Eckmann-Hilton to construct a 3-loop eh :  $\Omega^3(\mathbb{S}^2)$ . This is equivalent to a map hpf :  $\mathbb{S}^3 \to \mathbb{S}^2$ .
- 2 Adapt ideas from Von Raumer's "Path Spaces of Higher Inductive Types" to characterize the fiber.

#### **Eckmann-Hilton**

#### The Eckmann-Hilton Identification

For  $\alpha, \beta : \Omega^2(X)$ , we have  $EH(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$ 

#### Eckmann-Hilton

$$EH(surf_2, surf_2) : surf_2 \cdot surf_2 = surf_2 \cdot surf_2$$

The type of this is identification is equivalent to  $\Omega^3(\mathbb{S}^2)$ .

#### The Eckmann-Hilton 3-loop

Define eh :  $\Omega^3(\mathbb{S}^2)$  as the image of  $\text{EH}(\text{surf}_2,\text{surf}_2)$  under said equivalence.

See agda-unimath for more.

### The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

 $\mathsf{hpf}:\mathbb{S}^3 \to \mathbb{S}^2$ 

Define a map hpf :  $\mathbb{S}^3 \to \mathbb{S}^2$  by  $\mathbb{S}^3$ -induction:

 $hpf(base_3) :\equiv base_2$ 

 $hpf(surf_3) := eh$ 

### Kraus and Von Raumer's Computation of $\Omega(\mathbb{S}^1)$

- **1** Universal Property:  $Id_{b_1} : \mathbb{S}^1 \to U$  is the inital pointed type family
- 2 Descent:  $\Omega(\mathbb{S}^1)$  is the inital pointed type equipped with an automorphism
- 3 Universal Property, pt. II:  $\mathbb{Z}$  is the initial pointed type equipped with an automorphism

Note,  $\Omega(\mathbb{S}^1)$  is the fiber of unit  $\to \mathbb{S}^1$ 

### Our Computation of fib<sub>hpf</sub>(base<sub>2</sub>)

- 1 Universal Property:  $fib_{hpf} : \mathbb{S}^2 \to U$  is the inital family equipped with a lift of hpf (a section over hpf).
- Descent: fib<sub>hpf</sub>(base<sub>2</sub>) is the inital type equipped with some data.
- 3 Universal Property, pt. II:  $S^1$  is the inital type equipped with some data.

### The Universal Property of the Family of Fibers

Fix a pointed map  $h: A \rightarrow B$ . Then:

#### Heuristic

 $fib_h(b_0)$  is like the loop space of B with extra identifications freely generated by the map h

### The Universal Property of the Family of Fibers

A map  $h: A \rightarrow B$  is equivalent to a type family fib<sub>h</sub>:  $B \rightarrow U$ .

This family always comes equipped with a lift:

$$\lambda(a).(a, refl): (a:A) \rightarrow fib_h \circ h(a)$$

 $fib_h$  is the initial such type family

### The Universal Property of the Family of Fibers

#### Wild Category of Families with Lifts

Objects: families  $P: B \rightarrow U$  equipped with a lift  $(a: A) \rightarrow P \circ h(a)$ 

Maps: families of maps  $(b:B) \rightarrow P(b) \rightarrow Q(b)$  that preserve the lift

#### Universal Property of $fib_h$

The family  $fib_h$  with its canonical lift is intial in this wild category.

Proof: follows from the standard equivalence  $A \simeq \sum_{b:B} fib_h(b)$ . Formalized in agda-unimath

### Specializing the Universal Property

If  $A \equiv \text{unit and } h(\star) \equiv b_0$ :

$$((a : \mathsf{unit}) \to P \circ h(a)) \simeq P(b_0)$$

So fib<sub>h</sub> is the inital type family equipped with a point over  $b_0$ 

### Specializing the Universal Property

Let  $A \equiv \mathbb{S}^3$  and define h by  $s : \Omega^3(B, b_0)$ .

Lifts  $((a: \mathbb{S}^3) \to P \circ h(a))$  are equivalent to dependent 3-loops:

a point  $u: P(b_0)$ 

an identification  $tr^3(s)(u) = refl_u^2$ 

So fib<sub>h</sub> is the inital type family equipped with a point over  $b_0$  and an identification as above.

## Specializing the Universal Property

Let 
$$A \equiv \mathbb{S}^3$$
,  $B \equiv \mathbb{S}^2$  and  $h \equiv hpf$ .

Then fib<sub>hpf</sub> is the inital:

family over  $\mathbb{S}^2$ 

point u: fib<sub>hpf</sub>(base<sub>2</sub>)

identification  $tr^3(eh)(u) = refl_u^2$ 

### Interlude, descent data of S<sup>2</sup>

A type family P over  $\mathbb{S}^2$  is equivalent to:

#### Descent data of S<sup>2</sup>

a type X, the value of  $P(base_2)$ 

a homotopy  $id_X \sim id_X$ , the transport  $tr^2(surf_2)$ 

Now we apply Kraus and Von Raumer's ideas using the descent data of  $\mathbb{S}^2$ .

### A Characterization of fib<sub>hpf</sub>

Then fib<sub>hof</sub> is the inital data:

type F

homotopy  $H : id_F \sim id_F$ 

point u: F

identification  $tr^3(eh)(u) = refl_u^2$ 

The latter identification is equivalent to an identification

$$\mathsf{tr}^3(\mathsf{EH}(\mathsf{surf}_2,\mathsf{surf}_2))(u) = \mathsf{refl}_{\mathsf{tr}^2(\mathsf{surf}_2\,\boldsymbol{\cdot}\,\mathsf{surf}_2)(u)}$$

### The Eckmann-Hilton Argument

#### **Eckmann-Hilton**

For  $\alpha, \beta : \Omega^2(X)$ , we have  $EH(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$ 

#### Where does Eckmann-Hilton come from?

Fix a pointed type  $(X, \bullet)$  and consider  $Id_{\bullet}: X \to U$ , the family of based identity types.

#### A loop $p: \Omega(X)$ induces:

$$\mathsf{tr}^{\mathsf{Id}_{\bullet}}(p) : \Omega(X) \simeq \Omega(X)$$

This is path concatination:

#### for $q : \Omega(X)$ we have:

$$tr(p)(q) = q \cdot p.$$

#### Where does Eckmann-Hilton come from?

Up one dimension:

#### a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$$\mathsf{tr}^{(\mathsf{Id}_{\bullet})^2}(\alpha) : \mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$$

This is Eckmann-Hilton:

for  $\beta : \Omega^2(X)$ , we have:

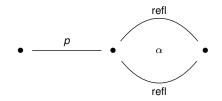
$$nat-[tr^2(\alpha)](\beta) = EH(\alpha,\beta)$$

(modulo coherence paths)

### Eckmann-Hilton from tr<sup>2</sup>

Computing  $\operatorname{tr}^2(\alpha) : \operatorname{id}_{\Omega(X)} \sim \operatorname{id}_{\Omega(X)}$ 

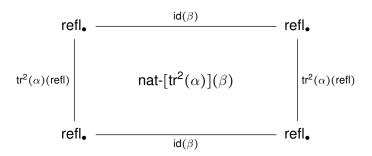
$$\mathsf{tr}^2(\alpha) = \mathsf{whisker}_{\alpha} = \lambda(p).\mathsf{refl}_p \star \alpha$$



$$\operatorname{tr}^2(\alpha)(\operatorname{refl}_{\bullet}) = \alpha$$

## The naturality condition of $tr^2(\alpha) : id_{\Omega(X)} \sim id_{\Omega(X)}$

For  $\beta : \Omega^2(X)$ :



Plus coherence paths, this ends

$$\mathsf{EH}(\alpha,\beta):\alpha \cdot \beta = \beta \cdot \alpha$$

(ロ) (部) (注) (注) 注 り(())

#### Eckmann-Hilton in the Universe

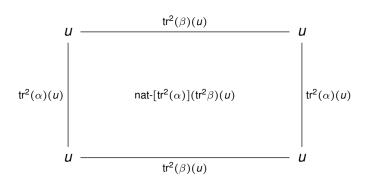
Let  $P: X \to U$  with  $u: P(\bullet)$  and  $\alpha, \beta: \Omega^2(X)$ . Then:

$$\operatorname{tr}^2(\alpha) : \operatorname{id}_{\Omega(P(ullet))} \sim \operatorname{id}_{\Omega(P(ullet))}$$

and

$$\operatorname{tr}^2(\beta)(u):\Omega(P(ullet))$$

#### Eckmann-Hilton in the Universe



$$\mathsf{nat}\text{-}[\mathsf{tr}^2(\alpha)](\mathsf{tr}^2(\beta)(u)) = \mathsf{tr}^3(\mathsf{EH}(\alpha,\beta))(u).$$
 (modulo coherence paths)

Proof: See agda-unimath



### The Eckmann-Hilton 3-loop

We can EH to construct a 3-path in  $\mathbb{S}^2$ :

$$surf_2 \cdot surf_2 \xrightarrow{EH(surf_2, surf_2)} surf_2 \cdot surf_2$$

This type is equivalent to  $\Omega^3(\mathbb{S}^2)$ 

### $\operatorname{eh}:\Omega^3(\mathbb{S}^2)$

Define eh as the image of EH(surf<sub>2</sub>, surf<sub>2</sub>) under this equivalence.

### A Characterization of fib<sub>hpf</sub>

Recall that fib<sub>hpf</sub> is the inital data:

type F

homotopy  $H: id_F \sim id_F$ 

point u: F

 $\mathsf{identification}\;\mathsf{tr}^3(\mathsf{EH}(\mathsf{surf}_2,\mathsf{surf}_2))(u) = \mathsf{refl}_{\mathsf{tr}^2(\mathsf{surf}_2 \, \cdot \, \mathsf{surf}_2)(u)}$ 

### A Characterizaton of fib<sub>hpf</sub>

Finally, fib<sub>hpf</sub> is the initial data:

type F

point u: F

homotopy  $H: id_F \sim id_F$ 

identification nat- $H(H(u)) = refl_{H(u)} \cdot H(u)$ 

We can package this as a HIT F generated by the subsquent data.

### The Plan, Part III

Show that  $S^1$  is generated by a point, a homotopy, and an identification as before.

Two approaches:

- 1 Using a HIT and directly constructing an equivalence
- 2 Show  $S^1$  is initial amoung F-algebras

### Using a HIT

In cubical agda, thanks to Tom Jack

In Book HoTT, good luck. In agda-unimath, not possible.

### *F*-algebras

 $\mathbb{S}^1$  forms an *F*-algebra:

```
type - \mathbb{S}^1
homotopy - L
point - b_1
identification - defn_L : nat-L(L(b_1)) = refl_{loop \cdot loop}
```

### F-algebra

There is a map from  $\mathbb{S}^1$  to any other F-algebra

Let  $(X, K, x_0, p)$  be an F-algebra:

① Define  $f: \mathbb{S}^1 \to X$  via  $f(b_1) := x_0$  and  $defn_f: f(loop) = K(x_0)$ 

Need:

2 
$$f \cdot_I L \sim K \cdot_r f$$

3 
$$f(b_1) = x_0$$

4 A witness that the trivialization is preserved

### *F*-algebra

Any two F-algebra maps from  $\mathbb{S}^1$  to X are equal.

Proof: Path algebra and universal property of  $\mathbb{S}^1$ . Omitted due to time constraints.

#### **Future Work**

- 1 Adapting the James construction and Wärn's Zig Zag Construction
- **2**  $\pi_4(\mathbb{S}^3)$
- 3 Higher Hopf Fibrations and Higher Coherences

### $\pi_4(\mathbb{S}^3)$ has order $\leq 2$

#### Suspension Preserve Eckmann-Hilton

For the unit  $\sigma: \mathbb{S}^2 \to \Omega(\mathbb{S}^3)$ , we have  $\sigma(eh) = eh_{surf_3}$ 

Proof: all functions preserve Eckmann-Hilton.

### $\pi_4(\mathbb{S}^3)$ has order $\leq 2$

 $eh_{surf_3}$  generates  $\pi_4(surf_3)$  and its square is trival.

Proof: the map  $\pi_3(\sigma):\pi_3(\mathbb{S}^2)\to\pi_4(\mathbb{S}^3)$  is surjective. So  $\pi_3(\sigma)(\mathsf{eh})=\mathsf{eh}_{\mathsf{surf}_3}$  is a generator. Syllepsis implies that  $\mathsf{eh}_{\mathsf{surf}_3}$  is its own inverse.

## Non-Trivality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family  $B: \Omega(\mathbb{S}^3) \to U$  such that

$$nat-[tr^2(surf_3)](tr^2(surf_3)(u))$$

is non-trivial, for some u : B(refl)

### Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations  $\mathbb{S}^7 \to \mathbb{S}^4$  and  $\mathbb{S}^{15} \to \mathbb{S}^8$  should also arise from higher coherences.

The  $E_4$  coherence, corresponding to  $\mathbb{S}^7 \to \mathbb{S}^4$ , was constructed by Sojakova.

### Up, Up, and Away

There is a correspondence between the  $E_n$  coherence and the descent data of Id over the n-sphere.

#### $E_1$

Group structure, given by path concatination in  $\Omega(\mathbb{S}^1)$ 

#### $E_2$

Braiding, given by Eckmann-Hilton in  $\Omega^2(\mathbb{S}^2)$ 

#### $E_3$

Syllepsis, given by sypllepsis in  $\Omega^3(\mathbb{S}^3)$ 

### Syllepsis and Descent over S<sup>3</sup>

The family  $Id : \mathbb{S}^3 \to U$  is equivalent to:

#### Descent over S3

$$\Omega(\mathbb{S}^3)$$

$$\operatorname{tr}^3(\operatorname{surf}_3):\operatorname{Refl} \sim \operatorname{Refl} \equiv (x:X) \to \Omega^2(X,x)$$

Claim: the 2-D naturality condition of this homotopy is syllepsis.

l.e.  $nat^2$ - $[tr^3(surf_3)](surf_3)$  is equivalent to

$$Sy(surf_3, surf_3) : EH(surf_3, surf_3) = EH(surf_3, surf_3)^{-1}$$

Proof: Horrible path algebra

### Acknowledgements

- My Advisor: Professor Jonathan Wise, CU Boulder
- Tom Jack, for many helpful discussions
- Egbert Rijke, Fredrik
   Bakke, Vojtěch Štěpančík,
   and the agda-unimath
   community

# The End

Questions? Comments?