ECKMANN-HILTON AND THE HOPF FIBRATION: AN OUTLINE

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Contents

Please note, this document currently serves as notes to myself and as an outline for the formalization project. Thus, keep in mind a few things: (i) this document is not fully completed, so should be looked at as a partial description of the current state of the proof, (ii) this document is not meant to be an introduction to the problem, but only an outline for those famliar with the topics, and (iii) this document is not at all polished for pubilcation.

1. Introduction

The goal is to show that the Eckmann-Hilton argument can be used to construct the Hopf fibration, and so a generator of $\Omega^3(\mathbb{S}^2)$. The main idea behind this proof can be stated simply: the connection between the Eckmann-Hilton argument and the Hopf fibration can be found in the naturality condition of certain homotopies. The first few sections are meant to provide a very rough introduction to the perspective we will use in the proof. Then, with this perspective in hand, we will delve into the formulation of the proof. Thus, we will prioritize conceptual clarity over rigour in the first few sections. As a consequnce, some statements may not be well typed, but are true in spirit. Then, in the formulation of the proof, we will provide some of this missing rigour. For a full dose of rigour, the reader is invited to read the accompanying agda-unimath formalization (which is currently in progress).

Part 1. Overview

2. Eckmann-Hilton

First we outline how Eckmann-Hilton relates to a naturality condition induced by 2-loops. Any 2-loop $\beta : \Omega^2(X, \bullet)$ induces a homotopy of type $\mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$ given by the formula:

$$\mathsf{whisker}_\beta :\equiv \lambda(p).1_p \star \beta$$

This can be depicted as follows:

$$\begin{tikzcd} \bullet \arrow[r, dash, "p"] & \bullet \arrow[r, dash, bend left, "1"] \arrow[r, dash, bend right, "1"'] \arrow[r, phantom, "\beta"] & \bullet \end{tikzcd}$$

This homotopy has a naturality condition induced by paths in $\Omega(X)$. In particular, for $\alpha : \Omega^2(X)$, the homotopy has a naturality condition which can be depicted as:

$$\begin{tikzcd} 1^2_\bullet \arrow[r, dash, "(\alpha\star 1^2_\bullet)"] \arrow[d, dash, "(1^2_\bullet\star\beta)"'] \arrow[dr, phantom, "\mathsf{nat\text{-}whisker}_\beta(\alpha)"] & 1^2_\bullet \arrow[d, dash, "(1^2_\bullet\star\beta)"] \\ 1^2_\bullet \arrow[r, dash, "(\alpha\star 1^2_\bullet)"'] & 1^2_\bullet \end{tikzcd}$$

In the formalization, we denote $\mathsf{nat\text{-}whisker}_\beta(\alpha)$ by $\mathsf{path\text{-}swap}(\beta)(\alpha)$. A standard part of the Eckmann-Hilton proof constructs paths $\beta = 1^2_\bullet \star \beta$ and $\alpha = \alpha \star 1^2_\bullet$. Together with the naturality condition, we obtain the Eckmann-Hilton path $\mathsf{EH}(\beta, \alpha) : \beta \cdot \alpha = \alpha \cdot \beta$. We can thus describe Eckmann-Hilton in the following way:

> Eckmann-Hilton is (more or less) the *naturality condition* of $\mathsf{whisker}$ when *applied to 2-loops*.

As a special case, we can consider $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$. The type of this term is equivalent to $\Omega^3(\mathbb{S}^2)$. By passing $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ through this equivalence, we thus obtain a 3-loop $\mathsf{eh} : \Omega(\mathbb{S}^2)$. This 3-loop allows us to define a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ such that $\mathsf{hpf}(\mathsf{surf}_3) = \mathsf{eh}$. This, as we will see, is the Hopf fibration.

3. Fibrations Over $\mathbb{S}^2$

This is all well and good, but what, then, does any of this have to do with the Hopf fibration? We will actually highlight a more general connection between fibrations over $\mathbb{S}^2$ and the naturality condition of $\mathsf{whisker}$. Then, in the case of the Hopf fibration, we will see that something extra nice happens. The following equivalence characterizes fibrations over $\mathbb{S}^2$:

$$\Big(\sum_{Z:U} Z \to \mathbb{S}^2\Big) \simeq (\mathbb{S}^2 \to U) \simeq \Big(\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X\Big)$$

Lets consider the righthand type in this equation. A term (X, H) of this type is called the descent data of its corresponding type family. Nearly all of what is special about families over $\mathbb{S}^2$ can be found in the homotopy $H : \mathsf{id}_X \sim \mathsf{id}_X$. Thus it is worth going on a brief digression about homotopies of type $\mathsf{id}_X \sim \mathsf{id}_X$

3.1. **Homotopies of type $\mathsf{id}_X \sim \mathsf{id}_X$ and Eckmann-Hilton in the Universe.** Assume two homotopies $H, K : \mathsf{id}_X \sim \mathsf{id}_X$. By univalence, these are equivalent to 2-loops in the universe $\Omega^2(U, X)$. Thus we could apply the preceeding discussion on Eckmann-Hilton to the induced 2-loops. However, there is a much more direct route to making these homotopies commute, using only the homotopies themselves. Given a point $x : X$, K induces a 1-loop $K(x) : \Omega(X, x)$. Now, H is a homotopy, and so has a naturality condition induced by paths in X. We can thus apply the naturality condition of H to $K(x)$, lending the following 2-cell in X:

$$\begin{tikzcd} x \arrow[r, dash, "K(x)"] \arrow[d, dash, "H(x)"'] \arrow[dr, phantom, "\mathsf{nat}\text{-}H(K(x))"] & x \arrow[d, dash, "H(x)"] \\ x \arrow[r, dash, "K(x)"'] & x \end{tikzcd}$$

We can thus define a homotopy:

$$\mathsf{htpy\text{-}swap}(H)(K) :\equiv \lambda(x).\mathsf{nat}\text{-}H(K(x)) : H \ \cdot_h \ \mathsf{id} \ \cdot_l \ K \sim \mathsf{id} \ \cdot_l \ K \ \cdot_h \ H$$

Some coherence paths let us constuct $\mathsf{EH\text{-}htpy}(H)(K) : H \cdot_h K \sim K \cdot_h H$. Under univalence, this will be equal to the equalities we could have constructed using EH.

We can actually prove a more general claim. Fix a type A and a family $B : A \to U$. Any 2-loops $\alpha, \beta : \Omega^2 A$ induce homotopies

$$\mathsf{tr}^{(B)^2}(\alpha), \mathsf{tr}^{(B)^2}(\beta) : \mathsf{id}_{B(\mathsf{N}_2)} \sim \mathsf{id}_{B(\mathsf{N}_2)}$$

Then $\mathsf{tr}^{(B)^3}(\mathsf{EH}(\beta, \alpha))$ is given by (more or less) $\mathsf{htpy\text{-}swap}(\mathsf{tr}^{(B)^2}(\beta))(\mathsf{tr}^{(B)^2}(\alpha))$. I say "more or less" since I am ommiting a fair amount of coherence paths. Luckily, this lemma has already been (mostly) formalized in agda-unimath. The formalization freaks (I say this affectionately) can find (a partial formalization of) the main claim in section Coherences and algebraic identities for $\text{tr}^{\mathbf{3}}$ under the name $\text{tr}^{\mathbf{3}}$-htpy-swap-path-swap. The definition of Eckmann-Hilton for homotopies can be found in section Eckmann-Hilton for Homotopies, under the name eckmann-hilton-htpy. If we take $A \equiv U$ and $B \equiv \mathsf{id}$, we can deduce the original claim.

Now lets return our attention to a type family over $\mathbb{S}^2$ given by the descent data (X, H). Since H is of type $\mathsf{id}_X \sim \mathsf{id}_X$, we can apply the above reasoning to H on itself. That is, we can consider $\mathsf{EH\text{-}htpy}(H)(H) : H \ \bullet_h \ H \sim H \ \bullet_h \ H$, which (more or less) corresponds to $\mathsf{tr}^{(B)^3}(\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2))$. Since this type is equivalent to $\mathsf{refl\text{-}htpy}_{\mathsf{id}} \sim \mathsf{refl\text{-}htpy}_{\mathsf{id}}$, we can obtain a term $\mathsf{eh\text{-}htpy} : \mathsf{refl\text{-}htpy}_{\mathsf{id}} \sim \mathsf{refl\text{-}htpy}_{\mathsf{id}}$ from $\mathsf{EH\text{-}htpy}(H)(H)$. As it turns out, this is equal to $\mathsf{tr}^{(B)^3}(\mathsf{eh})$.

3.2. **Families of Fibers over $\mathbb{S}^2$.** The above remarks help highlight one of the important connections between fibrations over $\mathbb{S}^2$ and Eckmann-Hilton: Eckman-Hilton is present in every family over $\mathbb{S}^2$ in the form of the naturality condition of the 2-D descent data (i.e., the homotopy). To explicate this connection more, and pin point its importance in the case of the Hopf fibrations, lets consider a type family induced by a fibration. Consider again the equivalence

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

This is equivalence sends a fibration $h : Z \to \mathbb{S}^2$ to the pair consisting of the fiber $F :\equiv \mathsf{fib}_h(\mathsf{N}_2)$ and the homotopy $(\mathsf{fib})^2 :\equiv \mathsf{tr}^{(\mathsf{fib}_h)^2}(\mathsf{surf}_2)$. We can of course just apply the reasoning from the previous subsection to this family. But it may be helpful to spell things out in this special case. We can compute the action of homotopy on a point $(z, p) : \mathsf{fib}_h(\mathsf{N}_2)$ as

$$(\mathsf{fib})^2(z, p) = (1_z, 1_p \star \mathsf{surf}_2) \equiv (1_z, \mathsf{whisker}_{\mathsf{surf}_2}(p))$$

We can similarly characterize the naturality condition of this homotopy on a loop in the fiber. Though this can be done in general, it brings the most conceptual clarity to consider the case when we have a $z_0 : Z$ such that $h(z_0) \equiv \mathsf{N}_2$, like we do in the case of the Hopf fibration. Then a 2-loop $\alpha : \Omega^2(\mathbb{S}^2)$ determines a 1-loop $(1_{z_0}, \alpha)$ in $\mathsf{fib}_h(\mathsf{N}_2)$. The naturality condition of $(\mathsf{fib})^2$ on this loop can be described as (essentially)

$$\mathsf{nat}\text{-}(\mathsf{fib})^2(1_{z_0}, \alpha) = (1_{z_0}^2, \mathsf{nat}\text{-}\mathsf{whisker}_{\mathsf{surf}_2}(\alpha))$$

By throwing in the extra coherence paths, we can indeed return the Eckmann-Hilton path in the second component. It is hard to say much more for the case of arbitary Z. To bring things back down to Earth, lets consider a simple example. Consider the map $\bar{\mathsf{N}_2} : \mathsf{unit} \to \mathbb{S}^2$ that selects out the base point. Then the fiber over the base point is

$$\sum_{t:\mathsf{unit}} \bar{\mathsf{N}_2}(t) = \mathsf{N}_2$$

This is equivalent to the loop space $\Omega(\mathbb{S}^2)$. Under this equivalence, the homotopy mentioned above simplifies to $\mathsf{whisker}_{\mathsf{surf}_2}$, and the naturality condition, say on α, simplifies to $\mathsf{nat}\text{-}\mathsf{whisker}_{\mathsf{surf}_2}(\alpha)$. Thus, in the case of the trivial fibration $\mathsf{unit} \to \mathbb{S}^2$, Eckman-Hilton is literally, at least up to equivalence, the naturality condition of the 2-D descent data.

Of course the fibration $\mathsf{unit} \to \mathbb{S}^2$ is quite special. The induced type family is the initial pointed type family. Changing the domain from unit to a more exotic type, will have the effect of forcing non-trival equalites, or imposing relations, on the type family. Thus, the fiber over the base point of a more general fibration will act like a loop space with imposed relations. Further, we will still be able to describe the family of fibers as the initial type family generated by some data, where the extra data imposes the relations conditions. We will make this more precise in the next section.

But, to first get a sense of what this means, and to relate this back to the Hopf fibration, lets apply this to the map hpf. Since we have $\mathsf{hpf}(\mathsf{N}_3) \equiv \mathsf{N}_2$, we can apply the earlier characterization of the 2-D descent data and its naturality condition. Further, note that we have $(1_{\mathsf{N}_3}, \mathsf{surf}_2) : \Omega(\mathsf{fib}_{\mathsf{hpf}}(\mathsf{N}_2))$. Applying the naturality condition of the descent data to this loop lends

$$\mathsf{nat}\text{-}(\mathsf{fib})^2(1_{\mathsf{N}_3}, \mathsf{surf}_2) = (1^2_{\mathsf{N}_3}, \mathsf{nat}\text{-}\mathsf{whisker}_{\mathsf{surf}_2}(\mathsf{surf}_2))$$

Again, from this and some equivalence, we can coax out the 2-loop

$$(1^2_{\mathsf{N}_3}, \mathsf{eh})$$

Now, lets consider the trivializations of this 2-loop. That is, the type $(1^2_{\mathsf{N}_3}, \mathsf{eh}) = (1^2_{\mathsf{N}_3}, 1^3_{\mathsf{N}_2})$. Some standard computations reveal that this type is equivalent to $\mathsf{fib}_h(\mathsf{eh})$. Thus, this 2-loop is trivial if, and only if, eh is in the image of hpf. So, at least in the case of hpf, we can see that indeed $(1^2_{\mathsf{N}_3}, \mathsf{eh})$ is trivial. This equivalently implies the naturality condition of $(\mathsf{fib})^2$ (applied to the loop derived from surf_2) is trivial.

We can now state a few equivalent perspectives on the same phenomena. We have our characterization of fibrations and type families over $\mathbb{S}^2$:

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\mathbb{S}^2 \to U) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

For now, fix something in the middle type, i.e., a family $B : \mathbb{S}^2 \to U$, and suppose wa have a point $b : B(\mathsf{N}_2)$. Then, the the following are equivalent:

(i) trivializations of $\mathsf{tr}^{(B)^3}(\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2))(b)$

(ii) trivializations $\mathsf{tr}^{(B)^3}(\mathsf{eh})(b)$

(iii) trivializations of the naturality condition of the 2-D descent data. That is, if (X, H) is the descent data of B, the trivializations of $\mathsf{nat}\text{-}(H)(H(b))$

(iv) witnesses that eh is in the image of the induced fibration. That is, if $h : Z \to \mathbb{S}^2$ is the fibration induced by B, the type $\mathsf{fib}_h(\mathsf{eh})$.

The equivalence of the first two is induced by the equivalence of the types of EH and eh. The third is equivalent to the first two by the reasoning behind Eckmann-Hilton in the universe. And the equivalence of the fourth follows from the preceeding reasoning.

We can now start to see what makes hpf special; the induced type family, in particular its 2-dimensional descent data, has a trivial naturality condition. But this alone is not what makes hpf special. The fiber of the map hpf is the initial type family over $\mathbb{S}^2$ with a trivial naturality condition. This uniquely characerizes the fiber of hpf (along with hpf itself). Thus, if we are able to show that some other type family $\mathcal{H} : \mathbb{S}^2 \to U$ has this universal property, this will allow us to construct a fiberwise equivalence $(x : \mathbb{S}^2) \to_{\simeq} \mathsf{fib}_{\mathsf{hpf}}(x) \simeq H(x)$. This is the approach we will take in characterizing the fiber of hpf.

4. Type Families Equipped with Sections

We have alluded to a universal property enjoyed by the fiber of hpf, but we have not so far precisely explicated what this universal property is. This is the task to which we now turn. The universal property that we have alluded to is in fact an instance of the universal property enjoyed by the family of fibers of any map. For now, lets return our considerations to an arbitrary map $h : Z \to \mathbb{S}^2$. This map induces the family of fibers $\mathsf{fib}_h : \mathbb{S}^2 \to U$. First note that this family comes equipped with a section

$$\mathsf{triv}_h :\equiv \lambda(z).(z, 1_{hz}) : (z : Z) \to \mathsf{fib}_h(h(z))$$

We claim that fib_h is the intial type equipped with such a section. That is, it is initial amoung type families of the form $B : \mathbb{S}^2 \to U$ equipped with sections of type $(z : Z) \to B \circ h(z)$. This is fairly easy to show. Consider an arbitrary type family $B : \mathbb{S}^2 \to U$ and the type of fiberwise maps $(x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x)$. Using some type arithmetic, we have the following equivalence:

$$
\begin{aligned}
(x:\mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x) &\simeq (y:\sum_{x:\mathbb{S}^2} \mathsf{fib}_h(x)) \to B \circ \mathsf{pr}_1(y) \\
&\simeq (z:Z) \to B \circ h(z)
\end{aligned}
$$

where the second equivalence is induced by:

$$
\begin{aligned}
\sum_{x:\mathbb{S}^2} \mathsf{fib}_h(x) &\equiv \sum_{x:\mathbb{S}^2} \sum_{z:Z} hz = x \\
&\simeq \sum_{z:Z} \sum_{x:\mathbb{S}^2} hz = x \\
&\simeq Z
\end{aligned}
$$

It is easy to trace the first equivalence and see that it is given, definitionally, by the formula

$$G \mapsto \lambda(z).G(h(z))(z, 1_{h(z)}) \equiv G \circ \mathsf{triv}_h$$

Call this map $\mathsf{ev}_{\mathsf{triv}}$. Of course, for any type family $B : \mathbb{S}^2 \to U$ equipped with a section $\delta : (z : Z) \to B \circ h(z)$, we have a similar map $\mathsf{ev}_\delta :\equiv \lambda(G).G \circ \delta$. We can now state the universal proprety of fib_h precisely.

Lemma 4.1 (Universal Propery of fib_h)**.** *For every $B : \mathbb{S}^2 \to U$, the map*

$$\mathsf{ev}_{\mathsf{triv}} :\equiv \lambda(G).G \circ \mathsf{triv}_h : ((x:\mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x)) \to (z:Z) \to B \circ h(z)$$

is an equivalence.

If we then restrict our attention to families B with sections, indeed we can see that the type of section preserving fiberwise maps out of fib_h is contractible. As with any universal property, this suffices to uniquely characterize fib_h up to unique equivalence.

Lemma 4.2. *Suppose the family B over $\mathbb{S}^2$ with section δ is such that ev_δ is an equivalence (for any other type family). Then there is a unique fiberwise equivalence $(x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \simeq B(x)$ that commutes with the sections.*

4.0.1. *An example and a digression.* What exactly this universal property means depends a lot on the type Z. Since there is not much one can say about an arbitrary type Z, there isn't much more we can say in full generality. However, if Z itself enjoys a nice mapping out universal proprety, this allows us to characterize the sections and unpack the universal property further. Thus, it it worthwhile considering a special case where the total space has a very simple universal property. Consider again the map $\bar{\mathsf{N}_2} : \mathsf{unit} \to \mathbb{S}^2$ that selects the point N_2. We know that the induced family of fibers is just the (based) path space family $\mathsf{Id}(\mathsf{N}_2)$, and the fiber over N_2 is $\mathsf{Id}(\mathsf{N}_2)(\mathsf{N}_2) \equiv \Omega\mathbb{S}^2$. By our statement of the universal property, $\mathsf{Id}(\mathsf{N}_2)$ is the initial type family over $\mathbb{S}^2$ equipped with a section $(t : \mathsf{unit}) \to \mathsf{Id}(\mathsf{N}_2)(\bar{\mathsf{N}_2}(t))$. But this type is equivalent to $\mathsf{Id}(\mathsf{N}_2)(\mathsf{N}_2) \equiv \Omega\mathbb{S}^2$. Thus, $\mathsf{Id}(\mathsf{N}_2)$ is the initial type family with a point over N_2.

Of course this is already well known, since it is just a rephrasing of path induction. But it is worth taking a moment to consider what kind of structure is imposed by "being initial". Egbert Rijke has wonderful blog post elucidating exactly what this means. I invite you to read it here. Since we will essentially be using the same idea, but extendend to fibers of more general maps, I will reiterate some of the main points here. We know that $\mathsf{Id}(\mathsf{N}_2)$ is the initial family equipped with a point, meaning it is freely generated by said point.

But we have an equivalent description of families over $\mathbb{S}^2$ in terms of the descent data $\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X$. So the descent data of $\mathsf{Id}(\mathsf{N}_2)$ should be freely generated by a point. Thus the descent data of $\mathsf{Id}(\mathsf{N}_2)$ should just be the initial type X equipped with a homotopy

$\mathsf{id}_X \sim \mathsf{id}_X$ and a point $x_0 : X$. This means that X should be a higher inductive type generated by such a homotopy and point. Since X corresponds to the fiber over N_2, which is $\Omega\mathbb{S}^2$ in this case, this gives us a HIT description of $\Omega\mathbb{S}^2$ as a type X with constructors $x_0 : X$ and $H : \mathsf{id}_X \sim \mathsf{id}_X$.

If this idea is unfamiliar, it is very much worth reading Rijke's blog post and considering the more familiar case of $\Omega\mathbb{S}^1$ (which is well covered in Rijke's post). We will essentially be using Rijke's idea and extending it to more general maps, not requiring the domain to be unit. Of course, unless Z has a nice universal property, there is not a nice way to give a HIT presentation of $\mathsf{fib}_h(\mathsf{N}_2)$. At least, not one that I am aware of.

4.0.2. *Returning to Hopf.* Thus, it is worth returning our attention to the case when $Z \equiv \mathbb{S}^3$ and considering the map hpf. We know $\mathsf{fib}_{\mathsf{hpf}}$ is the initial family with a section $(z : \mathbb{S}^3) \to \mathsf{fib}_{\mathsf{hpf}}(\mathsf{hpf}(z))$. Since $\mathbb{S}^3$ itself enjoys a universal property, we can unpack the section $(z : \mathbb{S}^3) \to \mathsf{fib}_{\mathsf{hpf}}(\mathsf{hpf}(z))$. A section of the afformentioned type is equivalent to a free dependent 3-loop. This consists of a point $a : \mathsf{fib}_{\mathsf{hpf}}(\mathsf{hpf}\mathsf{N}_3) \equiv \mathsf{fib}_{\mathsf{hpf}}(\mathsf{N}_2)$ and dependent 3-loop:

$$\mathsf{tr}^{(\mathsf{fib})^3}(\mathsf{hpf}(\mathsf{surf}_3))(a) \equiv \mathsf{tr}^{(\mathsf{fib})^3}(\mathsf{eh})(a) = 1_a^2$$

Our digression in subsection 3.1 "Eckmann-Hilton in the Universe" gave us three equivalent descriptions of this type. In particular, this is equivalent to having $\mathsf{nat}\text{-}\mathsf{tr}^{(\mathsf{fib})^2}\mathsf{surf}_2(\mathsf{tr}^{(\mathsf{fib})^2}\mathsf{surf}_2(a))$ be trivial. Thus, the fiber of hpf over N_2 should be equivalent to a HIT X with constructors:

a point $x_0 : X$

a homotopy $H : \mathsf{id}_X \sim \mathsf{id}_X$

a trivialization of type $\mathsf{nat}\text{-}H(H(x_0))$

Note how this HIT has the same constructors as the HIT preseting $\Omega(\mathbb{S}^2)$, plus an additonall constructor imposing a relation.

In theory, we could prove prove that hpf is the Hopf fibration by showing that the above HIT induces a type family with the desired universal proprety, and then showing that this HIT is equivalent to $\mathbb{S}^1$. This approach may very well be worth exploring, since Tom Jack has already provided a short cubical proof that the above HIT is equvalent to $\mathbb{S}^1$. Thus, all that is left to show is that the induced type family has universal property of $\mathsf{fib}_{\mathsf{hpf}}$. However, since I aim to formalize this in the "book HoTT" agda-unimath library, in which such recursive HITs are not easy to work with, we will not use this method in our formulation of the proof. Instead, we will construct a type family $\mathcal{H}$ "by hand", along with a section, and then show that this $\mathcal{H}$ has the desired universal property. But, the prespective and intuition (hopefully) communicated above will be instrumental in guiding this process. Thus, the above passages were (again, hopefully) not a waste of the readers time.

This should give at least a rough overview of the prespective and approach we will use to characterize the fiber of hpf. Before diving into the actual proof, we will give a motivation for the construction of the family $\mathcal{H}$.

5. The Family $\mathcal{H}$

TODO – use explanation in terms of HIT

Since we are building the type family for the total space of the Hopf fibration, we already know what this type family should look like. Just copy the type family used in 8.5 of the HoTT book. However, we will pretend that the results of that section are unavailable to us and try to motivate the construction of $\mathcal{H}$ "from scratch". The hope is that this will elucidate the proof that $\mathcal{H}$ has the desired universal property, and may hint at a method for characterizing the fibers of other maps.

Consider again the HIT presentation of the fiber:

a point $x_0 : X$

a homotopy $H : \mathsf{id}_X \sim \mathsf{id}_X$

a 3-cell of type $\mathsf{nat}\text{-}H(H(x_0)) = 1_{H(x_0)} \cdot H(x_a)$

By the Eckmann-Hilton in the universe idea, the 2-cell $\mathsf{nat}\text{-}H(H(x_0))$ corresponds to the path in the fiber induced by $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$. Thus, this type X, per the prespective of fibers as loop spaces with imposed relations, should behave like $\Omega\mathbb{S}^2$ with the imposed relation $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) = 1_{\mathsf{surf}_2 \cdot \mathsf{surf}_2}$. We expect $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$, or really its companion $\mathsf{eh} : \Omega^3(\mathbb{S}^3)$, to generate all the higher structure in $\mathbb{S}^2$. Since we are trivializing it, we should expect that all higher structure is trivialized as well. This should leave only the generating 2-loop surf_2, and all its powers, as non-trival. Then, in $\Omega\mathbb{S}^2$, this will correspond to a generating 1-loop. By we know already know that the type freely generated by a 1-loop is $\mathbb{S}^1$. Thus we should expect the fiber of hpf to be $\mathbb{S}^1$.

We know have half of the descent data of $\mathcal{H}$. For the 2-D descent data, we need a homotopy of type $\mathsf{id}_{\mathbb{S}^1} \sim \mathsf{id}_{\mathbb{S}^1}$. Since this correpsoponds to the homotopy which generates the fiber, expect the value of this homotopy to be loop at the base point b_1. What should the naturality condition be? We could cheat and say that, since $\mathbb{S}^1$ is known to be a 1-type, there is no real choice to be made and we should just take the center of contraction. But this would obfuscate an important conceptual point. Suppose we have some such filler for this naturality condition, i.e, a 2-cell of type $\mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$. This allows us to construct a homotopy $K : \mathsf{id}_{\mathbb{S}^1} \sim \mathsf{id}_{\mathbb{S}^1}$, which in tur induces a map $\mathbb{S}^1 \to \mathbb{S}^1 \to \mathbb{S}^1$ that sends b_1 to $\mathsf{id}_{\mathbb{S}^1}$ and sends loop to K. The map in turn is equivalent to a map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

Since $\mathbb{S}^1 \times \mathbb{S}^1$ is equivalent to the torus, which admits a HIT presentation, we can characterize this map in terms of the universal proprety of the torus. On the genarating 1-loop $(\mathsf{loop}, \mathsf{b}_1)$, the map will be equal to $K(\mathsf{b}_1)$, which we know should be equal to loop. On $(\mathsf{b}_1, \mathsf{loop})$, the map will be equal to $\mathsf{id}_{\mathbb{S}^1}(\mathsf{loop})$, which is just loop. Together, these two equalities imply that the map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ induced by K fits into the following triangle of pointed maps, with pointed homotopy filler:

$$
\begin{tikzcd}
\mathbb{S}^1 \vee \mathbb{S}^1 \arrow[r, "(\mathsf{id}_{\mathbb{S}^1}, \mathsf{id}_{\mathbb{S}^1})"] \arrow[d] & \mathbb{S}^1 \\
\mathbb{S}^1 \times \mathbb{S}^1 \arrow[ur] &
\end{tikzcd}
$$

The data of the diagonal map and homotopy is called an $(\mathsf{id}_{\mathbb{S}^1}, \mathsf{id}_{\mathbb{S}^1})$ extension. These are studied in the context of HoTT in the paper [1]. Building off the reasoning of the previous paragraph, we can prove that the type of extensions is equvialent to fillers of the naturality condition, which in turn is equivalent to the possible $K : \mathsf{id}_{\mathbb{S}^1} \sim \mathsf{id}_{\mathbb{S}^1}$ we can use as the 2-D descent data of $\mathcal{H}$. Using the universal property of the torus, a (pointed) map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ is equivalent to a pair of loops $p, q : \Omega\mathbb{S}^1$ and a 2-cell $p \cdot q = q \cdot p$. The filling homotopy is equivalent to the choice of vertical maps and fillers of the following to squares:

$$
\begin{tikzcd}
\mathsf{b}_1 \arrow[r, dash, "\mathsf{loop}"] \arrow[d, dash, "r"'] & \mathsf{b}_1 \arrow[d, dash, "r"] \\
\mathsf{b}_1 \arrow[r, dash, "p"'] & \mathsf{b}_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{b}_1 \arrow[r, dash, "\mathsf{loop}"] \arrow[d, dash, "r"'] & \mathsf{b}_1 \arrow[d, dash, "r"'] \\
\mathsf{b}_1 \arrow[r, dash, "q"] & \mathsf{b}_1
\end{tikzcd}
$$

Since we have a pointed homotopy, there is an aditonal coherence that r is the composite of the base point preserving maps. The base point prerving path for both the map $(\mathsf{id}_{\mathbb{S}^1}, \mathsf{id}_{\mathbb{S}^1})$ and the map $\mathbb{S}^1 \times \mathbb{S}^1$ are 1_{b_1}, we have path $r = 1_{\mathsf{b}_1}$. This allows us to contract away the choice of r, meaning fillers for the two squares are equivalent to 2-cells $\mathsf{loop} = p$ and $\mathsf{loop} = q$. This allows us to contract away the choice of p and q. Thus we are only left with the 2-cell of type $\mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$.

Thus, the type of $(\mathsf{id}_{\mathbb{S}^1}, \mathsf{id}_{\mathbb{S}^1})$-extensions, and so the type of the possible 2-D descent data of $\mathcal{H}$, is equivalent to fillers of type $\mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$. The fillers of this type correspond to the naturality condition of the 2-D descent data of $\mathcal{H}$. In the paper [1], the authors prove that the type of $(\mathsf{id}_{\mathbb{S}^1}, \mathsf{id}_{\mathbb{S}^1})$-extensions is equivalent to the type of H-space structures (on $\mathbb{S}^1$). As the authors also prove in that paper, $\mathbb{S}^1$ is a central H-space, meaning there is a unique (contractible space of) H-space structures. We can choose as the center of contraction $1 : \mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$. We will denote the induced homotopy, and so the 2-D descent data of $\mathcal{H}$, by L. Thus, the family $\mathcal{H}$ is defined by the descent data $(\mathbb{S}^1, \mathsf{L})$.

This observation, the connection between the possible naturality conditions of the 2-D descent data of $\mathcal{H}$ and the possible H-space structures on $\mathbb{S}^1$, provides a bridge between our construction of the Hopf fibration and the construction in the HoTT book.

Of course this doesn't yet establish that $\mathcal{H}$ is the initial family with section over $\mathbb{S}^2$. It is easy to construct a section $(z : \mathbb{S}^3) \to \mathcal{H} \circ \mathsf{hpf}(z)$. By the universal property of $\mathbb{S}^3$, this is equivalent to a point in $\mathcal{H}(\mathsf{N}_2) \equiv \mathbb{S}^1$ and a dependent 3-loop $\mathsf{tr}^{\mathcal{H}}(\mathsf{eh})(\mathsf{b}_1) = 1$. As we've alread discussed, this later type is equivalent to $\mathsf{nat}\text{-}-\mathsf{L}(\mathsf{L}(\mathsf{b}_1)) = 1$. Whether or not this holds definitionaly depends on if you're working in cubical or book HoTT. But we have constructed L so that that this type is inhabited. This together defines a section $\mathsf{diag} : (z : \mathbb{S}^3) \to \mathcal{H} \circ \mathsf{hpf}(z)$.

The only thing left to show is that $\mathsf{ev}_{\mathsf{diag}}$ is an equivalence. It turns out that it is rather easy to constuct an equivalence

$$((x : \mathbb{S}^2) \to \mathcal{H}(x) \to B(x)) \simeq ((z : \mathbb{S}^3) \to B \circ \mathsf{hpf}(z))$$

simply by unpacking the universal property of $\mathbb{S}^2$ and $\mathbb{S}^1$. Showing that the underlying map of this equivalence is $\mathsf{ev}_{\mathsf{diag}}$ relies on a techinal lemma. Thus, we will review construct the above equivalence here, but defer proving it is $\mathsf{ev}_{\mathsf{diag}}$. The construction of the equivalence is essent

Suppose B is given by the descent data (X, K). By the universal property of $\mathbb{S}^2$, a term of type $(x : \mathbb{S}^2) \to \mathcal{H}(x) \to B(x)$ is equivalent to a point over N_2, i.e., a map $g : \mathbb{S}^1 \to X$ and a dependent 2-loop. The key is of course to characterize dependent 2-loops in this family. Some easy, but tedious (and thus deferred), computations reveal that a dependent 2-loop at g is equivalent to a homotopy

$$g \cdot_l \mathsf{L} \sim K \cdot_r g$$

Since both the map and the homotopy have domain $\mathbb{S}^1$, we can characterize these using $\mathbb{S}^1$'s universal property. The hardest to characterize is the homotopy. But, a few comptutations reveal that such a homotopy is equivalent to a choice of $p : g \cdot_l \mathsf{L}(\mathsf{b}_1) \equiv g(\mathsf{loop}) = K(g(\mathsf{b}_1))$ and a coherence cell:

$$\begin{tikzcd} g(\mathsf{loop}) \cdot g(\mathsf{loop}) \arrow[r, dash, "1_{g(\mathsf{loop}) \cdot g(\mathsf{loop})}"] \arrow[d, dash, "p \star 1_{g(\mathsf{loop})}"'] & g(\mathsf{loop}) \cdot g(\mathsf{loop}) \arrow[d, dash, "1_{g\mathsf{b}_1} \star p"] \\ K(g(\mathsf{b}_1)) \cdot g(\mathsf{loop}) \arrow[r, dash, "\mathsf{nat}\text{-}K(g(\mathsf{loop}))"'] & g(\mathsf{loop}) \cdot K(g(\mathsf{b}_1)) \end{tikzcd}$$

The path p is the component of the homotpy at b_1 and the coherence cell is the naturality condition of the homotopy on loop. The map g is equivalent to a choice of $b : X$, corresponding to $g(\mathsf{b}_1)$, and a $l : \Omega(X, b)$, corresponding to $g(\mathsf{loop})$. Now note that the choice of l and $p : g(\mathsf{loop}) \equiv l = K(g(\mathsf{b}_1))$ forms a contractible pair. After, contracting away, the above 2-cell becomes:

$$\begin{tikzcd} K(b) \cdot K(b) \arrow[r, dash, "1_{K(b) \cdot K(b)}"] \arrow[d, dash, "1_{K(b)} \star 1_{K(b)}"'] & K(b) \cdot K(b) \arrow[d, dash, "1_{K(b)} \star 1_{K(b)}"] \\ K(b) \cdot K(b) \arrow[r, dash, "\mathsf{nat}\text{-}K(K(b))"'] & K(b) \cdot K(b) \end{tikzcd}$$

Thus, the map together with the homotopy, is equivalent to the following data:

a point $b : X$

a 2-cell $\mathsf{nat}\text{-} - K(K(b)) = 1$

By tracing a $G : ((x : \mathbb{S}^2) \to \mathcal{H}(x) \to B(x))$ through this equivalence, we can see that it returns $G(\mathsf{N}_2)(\mathsf{b}_1)$ and, more or less $\mathsf{nat}\text{-}(G(\mathsf{surf}_2))(\mathsf{loop})$. We don't exactly get $\mathsf{nat}\text{-}(G(\mathsf{surf}_2))(\mathsf{loop})$ out the other side since we had to transport along the path p, as well as apply many coherence paths.

The 2-cell is equivalent to $\mathsf{tr}^{(B)^3}(\mathsf{eh})(b) = 1$. So, all together, the abovedata is equivalento ta section $(z : \mathbb{S}^3) \to B \circ \mathsf{hpf}(z)$. Thus, we have

$$((x : \mathbb{S}^2) \to \mathcal{H}(x) \to B(x)) \simeq ((z : \mathbb{S}^3) \to B \circ \mathsf{hpf}(z))$$

We will later need to show that the underlying map of this equivalence in $\mathsf{ev}_{\mathsf{diag}}$. Given $G : ((x : \mathbb{S}^2) \to \mathcal{H}(x) \to B(x))$, we have $\mathsf{ev}_{\mathsf{diag}}(G) \equiv G \circ \mathsf{diag} : (z : \mathbb{S}^3) \to B \circ \mathsf{hpf}(z)$. We will ultimately show that $G \circ \mathsf{diag}$ is equivalent to the section resulting from the above equivalence by showing that the corresponding free dependent 3-loops are equivalent. The map $G \circ \mathsf{diag}$ induces the free dependent 3-loop given by

$$G \circ \mathsf{diag}(\mathsf{N}_3) \equiv G(\mathsf{N}_2)(\mathsf{diag}(\mathsf{N}_3)) \equiv G(\mathsf{N}_2)(\mathsf{b}_1)$$

and

$$G \circ \mathsf{diag}(\mathsf{surf}_3) = G(\mathsf{hpf}(\mathsf{surf}_3))(\mathsf{diag}(\mathsf{surf}_3)) = G(\mathsf{eh})(\mathsf{diag}(\mathsf{surf}_3))$$

The two section agree definitionally on the base point. Thus, we essentially just need to show that they agree on surf_3. This amounts to compute $G(\mathsf{eh})$ and showing that it is givern, more or less, by $\mathsf{nat}\text{-}(G(\mathsf{surf}_2))(\mathsf{loop})$. We will give such a computation in the next section. If the reader has been sufficently convinced by our so far informal spelling out of the proof, they could skip directly to this computation, thus completing the proof.

Part 2. The Proof

In the previous part of this write up, we gave an informal overview of the construction of the Hopf fibration and the characterization of its fiber. In this section, we fully develop the proof, providing any lemmas that were deffered.

TO DO

References

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