ECKMANN-HILTON AND THE HOPF FIBRATION: AN OUTLINE

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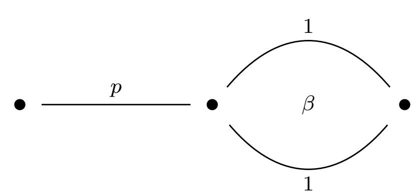
Please note, this document current serves as notes to myself and as an outline for the formalization project. Thus, keep in mind two things: (i) this document is not fully completed, so should be looked at as a partial description of the current state of the proof, and (ii) this document is not meant to be an introduction to the problem, but only an update for those famliar with the topics.

The goal is to show that the Eckmann-Hilton argument can be used to construct the Hopf fibration. The main idea behind this proof can be stated simply: the connection between the Eckmann-Hilton argument and the Hopf fibration can be found in the naturality condition of certain 2-dimensional objects, 2-paths and homotopies, respectively. First we outline how Eckmann-Hilton relates to a naturality condition on 2-paths. Any 2-loop $\beta : \Omega^2(X, \bullet)$ induces a homotopy of type $\mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$ given by the formula:

$$\mathsf{whisker}_\beta :\equiv \lambda(p).1_p \star \beta$$

This can be depicted as follows:



We can describe Eckmann-Hilton in the following way:

> Eckmann-Hilton is (more or less) the *naturality condition* of this homotopy when *applied to 2-loops*.

Lets see why. The above homotopy has a naturality condition induced by paths in $\Omega(X)$. In particular, for $\alpha : \Omega^2(X)$, the homotopy has naturality condition which can be depicted as:

$$
\begin{tikzcd}
1^2_\bullet \arrow[r, dash, "(\alpha\star 1^2_\bullet)"] \arrow[d, dash, "(1^2_\bullet\star\beta)"'] \arrow[dr, phantom, "\mathsf{nat\text{-}whisker}_\beta(\alpha)"] & 1^2_\bullet \arrow[d, dash, "(1^2_\bullet\star\beta)"] \\
1^2_\bullet \arrow[r, dash, "(\alpha\star 1^2_\bullet)"'] & 1^2_\bullet
\end{tikzcd}
$$

A standard part of the Eckmann-Hilton proof constructs paths $\beta = 1^2_\bullet \star \beta$ and $\alpha = \alpha \star 1^2_\bullet$. Together with the naturality condition, we obtain the Eckmann-Hilton path $\mathsf{EH}(\beta, \alpha) : beta \cdot \alpha = \alpha \cdot \beta$. Thus, Eckmann-Hilton ultimately comes from the naturality of $\mathsf{whisker}_\beta$.

As a special case, we can consider $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$. This type is equivalent to $\Omega^3(\mathbb{S}^2)$. By applying this equivalence to $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$, we thus obtain a 3-loop $\mathsf{eh} : \Omega(\mathbb{S}^2)$. This is 3-loop allows us to define a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ such that $\mathsf{hpf}(\mathsf{surf}_3) = \mathsf{eh}$.

This is all well and good, but what, then, does the Hopf fibration have to do with the naturality condition of $\mathsf{whisker}$? We will actually highlight a more general connection between fibrations over $\mathbb{S}^2$ and this naturality condition. Then, in the case of the Hopf fibration, we will see that something extra nice happens. The following equivalence characterizes fibrations over $\mathbb{S}^2$.

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\mathbb{S}^2 \to U) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

This is equivalence sends a fibration $h : Z \to \mathbb{S}^2$ to the pair consisting of the fiber $F :\equiv \mathsf{fib}_h(\mathsf{N}_2)$ and the homotopy $(\mathsf{fib})^2 :\equiv \mathsf{tr}^{(\mathsf{fib}_h)^2}(\mathsf{surf}_2)$. We can compute the action of this homotopy on a point $(z, p) : \mathsf{fib}_h(\mathsf{N}_2)$ as

$$(\mathsf{fib})^2(z, p) = (1_z, 1_p \star \mathsf{surf}_2) \equiv (1_z, \mathsf{whisker}_{\mathsf{surf}_2}(p))$$

We can similarly characterize the naturality condition of this homotopy on loop in the fiber. Though this can be done in general, though it brings the most conceptual clarity to consider the case when we have a $z_0 : Z$ such that $h(z_0) \equiv \mathsf{N}_2$. Then a 2-loop $\alpha : \Omega^2(\mathbb{S}^2)$ determines a 1-loop $(1_{z_0}, \alpha)$ in $\mathsf{fib}_h(\mathsf{N}_2)$. The naturality condition of $(\mathsf{fib})^2$ on this loop can be described as (essentially)

$$\mathsf{nat}\text{-}(\mathsf{fib})^2(1_{z_0}, \alpha) = (1^2_{z_0}, \mathsf{EH}(\mathsf{surf}_2, \alpha))$$

This equation is not quite well typed; we need to throw in the standard coherence paths to make the boundaries agree, but the spirit of the equation is correct. Thus, Eckmann-Hilton is present in every fibration (and type family) over $\mathbb{S}^2$ in the form of the naturality condition of the two dimensional descent data (i.e., the homotopy), or, at least, something equivalent to said naturality condition. Of course we can apply this homotopy to the loop $(1_{z_0}, \mathsf{surf}_2)$ and ultimately obtain a 2-loop $(1^2_{z_0}, \mathsf{eh})$ in the fiber.

Now let us consider the trivializations of this 2-loop. That is, the type $(1^2_{z_0}, \mathsf{eh}) = (1^2_{z_0}, 1^3_{\mathsf{N}_2})$. Some computations reveal that this type is equivalent to $\mathsf{fib}_h(\mathsf{eh})$. Thus, this 2-loop is trivial if, and only if, eh is in the image of h. Thus, in the case of hpf, we can see that indeed $(1^2_{z_0}, \mathsf{eh})$ is trivial. This further implies that the naturality condition of $(\mathsf{fib})^2$ is trivial.

We can now start to see what makes hpf special; the induced type family, in particular its 2-dimensional descent data, has a trivial naturality condition. But this alone is not what makes hpf special. The fiber of the map hpf is the initial type family over $\mathbb{S}^2$ with a trivial naturality condition. This uniquely characerizes the fiber of hpf (along with hpf itself). Thus, if we are able to show that some other type family $\mathcal{H} : \mathbb{S}^2 \to U$ has this universal property, this will allow us to construct a fiberwise equivalence $(x : \mathbb{S}^2) \to H(x) \simeq \mathsf{fib}_{\mathsf{hpf}}(x)$. This is the approach we will take in characterizing the fiber of hpf.

First we need to explicate much of the preceeding explination in more detailed terms. First lets review more closely the universal property of the fiber of hpf.

- the idea of initial type family equipped with a section, and connect it with previous description

- give eh in universe

then move on to motivating family $\mathcal{H}$