ECKMANN-HILTON AND THE HOPF FIBRATION: AN OUTLINE

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Contents

Please note, this document currently serves as notes to myself and as an outline for the formalization project. Thus, keep in mind a few things: (i) this document is not fully completed, so should be looked at as a partial description of the current state of the proof, (ii) this document is not meant to be an introduction to the problem, but only an update for those famliar with the topics, and (iii) this document is not at all polished for pubilcation.

1. Introduction

The goal is to show that the Eckmann-Hilton argument can be used to construct the Hopf fibration. The main idea behind this proof can be stated simply: the connection between the Eckmann-Hilton argument and the Hopf fibration can be found in the naturality condition of certain 2-dimensional objects, 2-paths and homotopies, respectively. The first few sections will provide a very rough introduction to the perspective we will use in the proof. Thus, we will prioritize conceptual clarity over rigour in this first pass. So many statements may not be technically well typed, but are true in spirit. Then, in the formulation of the proof, we will provide some of this missing rigour. For a full dose of rigour, the reader is invited to read the accompanying agda-unimath formalization.

2. Eckmann-Hilton

First we outline how Eckmann-Hilton relates to a naturality condition on 2-paths. Any 2-loop $\beta : \Omega^2(X, \bullet)$ induces a homotopy of type $\mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$ given by the formula:

$$\mathsf{whisker}_\beta :\equiv \lambda(p).1_p \star \beta$$

This can be depicted as follows:

$$\begin{tikzcd} \bullet \arrow[r, dash, "p"] & \bullet \arrow[r, bend left, dash, "1"] \arrow[r, bend right, dash, "1"'] \arrow[r, phantom, "\beta"] & \bullet \end{tikzcd}$$

This homotopy has a naturality condition induced by paths in $\Omega(X)$. In particular, for $\alpha : \Omega^2(X)$, the homotopy has naturality condition which can be depicted as:

$$\begin{tikzcd} 1^2_\bullet \arrow[rr, dash, "(\alpha\star 1^2_\bullet)"] \arrow[d, dash, "(1^2_\bullet\star\beta)"'] & \arrow[d, phantom, "\mathsf{nat\text{-}whisker}_\beta(\alpha)"] & 1^2_\bullet \arrow[d, dash, "(1^2_\bullet\star\beta)"] \\ 1^2_\bullet \arrow[rr, dash, "(\alpha\star 1^2_\bullet)"'] & & 1^2_\bullet \end{tikzcd}$$

In the formalization, we denote $\mathsf{nat\text{-}whisker}_\beta(\alpha)$ by $\mathsf{path\text{-}swap}(\beta)(\alpha)$. A standard part of the Eckmann-Hilton proof constructs paths $\beta = 1^2_\bullet \star \beta$ and $\alpha = \alpha \star 1^2_\bullet$. Together with the naturality condition, we obtain the Eckmann-Hilton path $\mathsf{EH}(\beta, \alpha) : \beta \cdot \alpha = \alpha \cdot \beta$. We can thus describe Eckmann-Hilton in the following way:

> Eckmann-Hilton is (more or less) the *naturality condition* of $\mathsf{whisker}$ when *applied to 2-loops.*

As a special case, we can consider $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$. This type is equivalent to $\Omega^3(\mathbb{S}^2)$. By passing $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ through this equivalence, we thus obtain a 3-loop $\mathsf{eh} : \Omega(\mathbb{S}^2)$. This is 3-loop allows us to define a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ such that $\mathsf{hpf}(\mathsf{surf}_3) = \mathsf{eh}$. This, as we will see, is the Hopf fibration.

3. Fibrations Over $\mathbb{S}^2$

This is all well and good, but what, then, does the Hopf fibration have to do with the naturality condition of $\mathsf{whisker}$? We will actually highlight a more general connection between fibrations over $\mathbb{S}^2$ and this naturality condition. Then, in the case of the Hopf fibration, we will see that something extra nice happens. The following equivalence characterizes fibrations over $\mathbb{S}^2$.

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\mathbb{S}^2 \to U) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

Lets consider the righthand type in this equation. A term (X, H) of this type is called the descent data of its corresponding type family. Nearly all of what is special about families over $\mathbb{S}^2$ can be found in the homotopy $H : \mathsf{id}_X \sim \mathsf{id}_X$. Thus it is worth going on a breif digression about homotopies of type $\mathsf{id}_X \sim \mathsf{id}_X$

3.1. **Eckmann-Hilton in the Universe.** Assume two homotopies $H, K : \mathsf{id}_X \sim \mathsf{id}_X$. By univalence, these are equivalent to 2-loops in the universe $\Omega^2(U, X)$. Thus we could apply the preceeding discussion on Eckmann-Hilton to the induced 2-loops. However, there is a much more direct route to making these homotopies commute, using only the homotopies themselves. Given a point $x : X$, both H and K induce 1-loops $H(x), K(x) : \Omega(X, x)$. But, H is a homotopy, and thus has a naturality condition induced by paths, and so loops, in X. We can thus apply the naturality condition of H to $K(x)$, lending the following 2-cell in X:

$$\begin{array}{ccc} x & \xrightarrow{K(x)} & x \\ {\scriptstyle H(x)}\Big| & \mathsf{nat}\text{-}H(K(x)) & \Big|{\scriptstyle H(x)} \\ x & \xrightarrow[K(x)]{} & x \end{array}$$

We can thus define a homotopy:

$$\mathsf{htpy\text{-}swap}(H)(K) :\equiv \lambda(x).\mathsf{nat}\text{-}H(K(x)) : H \ \bullet_h \ \mathsf{id} \circ K \sim \mathsf{id} \circ K \ \bullet_h \ H$$

Some coherence paths let us constuct $\mathsf{EH\text{-}htpy} : H \ \bullet_h \ K \sim K \ \bullet_h \ H$. Under univalence, this will be equal to the equalities constructed using EH. We will actually prove a more general claim. Fix a type A and a family $B : A \to U$. Any 2-loops $\alpha, \beta : \Omega^2 A$ induce homotopies

$$\mathsf{tr}^{(B)^2}(\alpha), \mathsf{tr}^{(B)^2}(\beta) : \mathsf{id}_{B(\mathsf{N}_2)} \sim \mathsf{id}_{B(\mathsf{N}_2)}$$

Then $\mathsf{tr}^{(B)^3}(\mathsf{EH}(\beta, \alpha))$ is given by (more or less) $\mathsf{htpy\text{-}swap}(\mathsf{tr}^{(B)^2}(\beta))(\mathsf{tr}^{(B)^2}(\alpha))$. I say "more or less" since I am ommiting a fair amount of coherence paths. Luckily, this lemma has already been (mostly) formalized in agda-unimath. The formalization freaks (I say this affectionately) can find (a partial formalization of) the main claim it in section Coherences and algebraic identities for $\mathrm{tr}^{\mathbf{3}}$ under the name $\mathrm{tr}^{\mathbf{3}}$-htpy-swap-path-swap. The definition of Eckmann-Hilton for homotopies can be found in section Eckmann-Hilton for Homotopies, under the name eckmann-hilton-htpy.

If we take $A \equiv U$ and $B \equiv \mathsf{id}$, we can deduce the original claim.

3.2. **Families of Fibers over $\mathbb{S}^2$.** The above remarks hint at one of the important connections between fibrations over $\mathbb{S}^2$ and Eckmann-Hilton: Eckman-Hilton is present in every family over $\mathbb{S}^2$ in the form of the naturality condition of the 2-D descent data (i.e., the homotopy). To explicate this connection more, and pin point its importance in the case of the Hopf fibrations, lets return our attention to fibrations. Consider again the equivalence

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

This is equivalence sends a fibration $h : Z \to \mathbb{S}^2$ to the pair consisting of the fiber $F :\equiv \mathsf{fib}_h(\mathsf{N}_2)$ and the homotopy $(\mathsf{fib})^2 :\equiv \mathsf{tr}^{(\mathsf{fib}_h)^2}(\mathsf{surf}_2)$. We can of course just apply the reasoning from the previous subsection to this family. But it may be helpful to spell things out in this special case. We can compute the action of homotopy on a point $(z, p) : \mathsf{fib}_h(\mathsf{N}_2)$ as

$$(\mathsf{fib})^2(z, p) = (1_z, 1_p \star \mathsf{surf}_2) \equiv (1_z, \mathsf{whisker}_{\mathsf{surf}_2}(p))$$

We can similarly characterize the naturality condition of this homotopy on a loop in the fiber. Though this can be done in general, it brings the most conceptual clarity to consider the case when we have a $z_0 : Z$ such that $h(z_0) \equiv \mathsf{N}_2$, like we do in the case of the Hopf fibration. Then a 2-loop $\alpha : \Omega^2(\mathbb{S}^2)$ determines a 1-loop $(1_{z_0}, \alpha)$ in $\mathsf{fib}_h(\mathsf{N}_2)$. The naturality condition of $(\mathsf{fib})^2$ on this loop can be described as (essentially)

$$\mathsf{nat}\text{-}(\mathsf{fib})^2(1_{z_0}, \alpha) = (1^2_{z_0}, \mathsf{nat}\text{-}\mathsf{whisker}_{\mathsf{surf}_2}(\alpha))$$

By throwing in the extra coherence paths, we can indeed return the Eckmann-Hilton path in the second component. This again shows that Eckmann-Hilton is present in every fibration (and type family) over $\mathbb{S}^2$. Now, we can apply this homotopy to the loop $(1_{z_0}, \mathsf{surf}_2)$ and ultimately obtain a 2-loop $(1^2_{z_0}, \mathsf{eh})$ in the fiber. Now, to relate this back to the Hopf fibration, lets consider the trivializations of this 2-loop. That is, the type $(1^2_{z_0}, \mathsf{eh}) = (1^2_{z_0}, 1^3_{\mathsf{N}_2})$. Some computations reveal that this type is equivalent to $\mathsf{fib}_h(\mathsf{eh})$. Thus, this 2-loop is trivial if, and only if, eh is in the image of h. So, in the case of hpf, we can see that indeed $(1^2_{z_0}, \mathsf{eh})$ is trivial. This means that the naturality condition of $(\mathsf{fib})^2$ is trivial.

We now have a few equivalent perspectives on the same phenomena. It will be helpful to summarize them. We have our characterization of fibrations and type families over $\mathbb{S}^2$:

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\mathbb{S}^2 \to U) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

For now, consider something in the middle type, i.e., a family $B : \mathbb{S}^2 \to U$. The 3-path $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ induces a homotopy with (more or less) type:

$$\mathsf{tr}^{(B)^3}(\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)) : \mathsf{tr}^{(B)^2}(\mathsf{surf}_2) \ \cdot_h \ \mathsf{tr}^{(B)^2}(\mathsf{surf}_2) \sim \mathsf{tr}^{(B)^2}(\mathsf{surf}_2) \ \cdot_h \ \mathsf{tr}^{(B)^2}(\mathsf{surf}_2)$$

Then, the the following are equivalent

(i) $\mathsf{tr}^{(B)^3}(\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2))$ is trivial

(ii) $\mathsf{tr}^{(B)^3)}(\mathsf{eh})$ is trivial

(iii) the naturality condition of the 2-D descent data is trivial. That is, $\mathsf{nat\text{-}tr}^{(B)^2}\mathsf{surf}_2$ is trivial.

(iv) eh is in the image of the induced fibration. That is, if $h : Z \to \mathbb{S}^2$ is the fibration corresponding to B, the type $\mathsf{fib}_h(\mathsf{eh})$ is inhabited.

We can now start to see what makes hpf special; the induced type family, in particular its 2-dimensional descent data, has a trivial naturality condition. But this alone is not what makes hpf special. The fiber of the map hpf is the initial type family over $\mathbb{S}^2$ with a trivial naturality condition. This uniquely characerizes the fiber of hpf (along with hpf itself). Thus, if we are able to show that some other type family $\mathcal{H} : \mathbb{S}^2 \to U$ has this universal property, this will allow us to construct a fiberwise equivalence $(x : \mathbb{S}^2) \to_\simeq \mathsf{fib}_{\mathsf{hpf}}(x) \simeq H(x)$. This is the approach we will take in characterizing the fiber of hpf.

4. Type Families Equipped with Sections

We have alluded to a universal property enjoyed by the fiber of hpf, but we have not so far precisely explitated what this universal property is. This is the task to which we now turn. The universal property that we have alluded to is in fact an instance of the universal property enjoyed by the family of fibers of any map. For now, lets return our considerations to an arbitrary map $h : Z \to \mathbb{S}^2$, though the following holds true under a change of codomain. This map induces the family of fibers $\mathsf{fib}_h : \mathbb{S}^2 \to U$. First note that this family comes equipped with a section

$$\mathsf{triv}_h :\equiv \lambda(z).(z, 1_{hz}) : (z : Z) \to \mathsf{fib}_h(h(z))$$

We claim that fib_h is the intial type equipped with such a section. That is, it is initial amoung type families of the form $B : \mathbb{S}^2 \to U$ equipped with sections of type $(z : Z) \to B \circ h(z)$. This is fairly easy to show. Consider an arbitrary type family $B : \mathbb{S}^2 \to U$ and the type of fiberwise maps $(x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x)$. Using some type arithmetic, we have the following equivalence:

$$\begin{aligned}(x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x) &\simeq (y : \sum_{x:\mathbb{S}^2} \mathsf{fib}_h(x)) \to B \circ \mathsf{pr}_1(y) \\ &\simeq (z : Z) \to B \circ h(z)\end{aligned}$$

where the second equivalence is induced by:

$$\begin{aligned}\sum_{x:\mathbb{S}^2} \mathsf{fib}_h(x) &\equiv \sum_{x:\mathbb{S}^2} \sum_{z:Z} hz = x \\ &\simeq \sum_{z:Z} \sum_{x:\mathbb{S}^2} hz = x \\ &\simeq Z\end{aligned}$$

It is easy to trace the first equivalence and see that it is given, definitionally, by the formula

$$G \mapsto \lambda(z).G(h(z))(z, 1_{h(z)}) \equiv G \circ \mathsf{triv}_h$$

Call this map $\mathsf{ev}_{\mathsf{triv}}$. Of course, for any type family $B : \mathbb{S}^2 \to U$ equipped with a section $\delta : (z : Z) \to B \circ h(z)$, we have a similar map $\mathsf{ev}_\delta :\equiv \lambda(G).G \circ \delta$. We can now state the universal proprety of fib_h precisely.

Lemma 4.1 (Universal Propery of fib_h)**.** *For every $B : \mathbb{S}^2 \to U$, the map*

$$\mathsf{ev}_{\mathsf{triv}} :\equiv \lambda(G).G \circ \mathsf{triv}_h : ((x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \to B(x)) \to (z : Z) \to B \circ h(z)$$

is an equivalence.

If we then restrict our attention to families B with sections, indeed we can see that the type of section preserving fiberwise maps out of fib_h is contractible. As with any universal proprety, this suffices to uniquely characterize fib_h up to unique equivalence.

Lemma 4.2. *Suppose the family B over $\mathbb{S}^2$ with section δ is such that $\mathsf{ev}_{\mathsf{triv}_\delta}$ is an equivalence (for any other type family). Then there is a unique fiberwise equivalence $(x : \mathbb{S}^2) \to \mathsf{fib}_h(x) \simeq B(x)$ that commutes with the sections.*

4.0.1. *An example and a digression.* What exactly this universal property means dependent a lot on the type Z. Since there is not much one can say about an arbitrary type Z, there isn't much more we can say in full generality. However, if Z itself enjoys a nice mapping out universal proprety, this allows us to characterize the sections. Thus, it it worthwhile considering a special case where the total space has a very simple universal property. Consider the map $\bar{\mathsf{N}}_2 : \mathsf{unit} \to \mathbb{S}^2$ that selects the point N_2. We know that the induced family of fibers is just the (based) path space family $\mathsf{Id}(\mathsf{N}_2)$, and the fiber over N_2 is $\mathsf{Id}(\mathsf{N}_2)(\mathsf{N}_2) \equiv \Omega\mathbb{S}^2$. By our statement of the universal property, $\mathsf{Id}(\mathsf{N}_2)$ is the initial type family over $\mathbb{S}^2$ equipped with a section $(t : \mathsf{unit}) \to \mathsf{Id}(\mathsf{N}_2)(\bar{\mathsf{N}}_2(t))$. But this type is just equivalent to $\mathsf{Id}(\mathsf{N}_2)(\mathsf{N}_2)$. Thus, $\mathsf{Id}(\mathsf{N}_2)$ is the initial type family with a point over N_2.

Of course this is already well known, since it is just a rephrasing of path induction. But it is worth taking a moment to consider what kind of structure is imposed by "being initial". Egbert Rijke has wonderful (now old) blog post elucidating exactly what this means. I invite you to read it here. Since we will essentially be using the same idea, but extendend to fibers of more general maps, I will reiterate some of the main points here. We know that $\mathsf{Id}(\mathsf{N}_2)$ is the initial family equipped with a point, meaning it is freely generated by said point.

But we have an equivalent description of families over $\mathbb{S}^2$ in terms of the descent data $\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X$. Thus, $\mathsf{Id}(\mathsf{N}_2)$ should be freely generated by this descent data and said point. So the descent data of $\mathsf{Id}(\mathsf{N}_2)$ should just be the initial type X equipped with a homotopy $\mathsf{id}_X \sim \mathsf{id}_X$ and a point $x_0 : X$. This means that X should be a higher inductive type generated by such a homotopy and point. Since X corresponds to the fiber over N_2, which is $\Omega\mathbb{S}^2$ in this case, this gives us a HIT description of $\Omega\mathbb{S}^2$ as a type X with constructors $x_0 : X$ and $(x : X) \to x = x$.

If this idea is unfamiliar, it is very much worth reading Rijke's blog post and considering the more familiar case of $\Omega\mathbb{S}^1$ (which is well covered in Rijke's post). We will essentially be using Rijke's idea and extending it to more general maps, not requiring the domain to be unit. Consider again a map $h : Z \to \mathbb{S}^3$. Now, unless Z has a nice universal property, there is not a nice way to give HIT presentation of $\mathsf{fib}_h(\mathsf{N}_2)$, since the section requires referencing the entire family.

4.0.2. *Returning to Hopf.* Thus, it is worth returning our attention to the case when $Z \equiv \mathbb{S}^3$ and consider the map hpf. Since $\mathbb{S}^3$ itself enjoys a universal property, we can unpack the section $(z : \mathbb{S}^3) \to \mathsf{fib}_{\mathsf{hpf}}(()\mathsf{hpf}(z))$. A section of the afformentioned type is equivalent to a free dependent 3-loop. This consists of a point $a : \mathsf{fib}_{\mathsf{hpf}}(\mathsf{hpf}\mathsf{N}_3) \equiv \mathsf{fib}_{\mathsf{hpf}}(\mathsf{N}_2)$ and dependent 3-loop:

$$\mathsf{tr}^{(\mathsf{fib})^3}(\mathsf{hpf}(\mathsf{surf}_3))(a) \equiv \mathsf{tr}^{(\mathsf{fib})^3}(\mathsf{eh})(a) = 1_a^2$$

Our digression in subsection 3.1 "Eckmann-Hilton in the Universe" gave us three equivalent descriptions of this type. In particular, this is equivalent to having $\mathsf{nat\text{-}tr}^{(B)^2}\mathsf{surf}_2(\mathsf{tr}^{(B)^2}\mathsf{surf}_2(a))$ be trivial. Thus, the fiber of hpf over N_2 should, more or less, be the initial type X equipped with:

> a point $x_0 : X$
>
> a homotopy $H : \mathsf{id}_X \sim \mathsf{id}_X$
>
> a trivialization of type $\mathsf{nat}\text{-}H(H(x_0))$

In theory, we could present the fiber of hpf as such a HIT, then show that this HIT is equivalent to $\mathbb{S}^1$. However, such "recursive" HITs are not easy to work with, so we will avoid this method. Instead, we construct a type family $\mathcal{H}$ "by hand", along with a section, and then show that this $\mathcal{H}$ has the desired universal property. But, the prespective and intuition (hopefully) communicated above will be instrumental in guiding this process. Thus, the above passages were (again, hopefully) not a waste of the readers time.

This should give at least a rough overview of the prespective and approach we will use to characterize the fiber of hpf. Before diving into the actual proof, we will give a motivation for the construction of the family $\mathcal{H}$.

5. The Family $\mathcal{H}$

TODO Since we are building the type family for the total space of the Hopf fibration, we already know what this type family should look like. Just copy the type family used in 8.5 of the HoTT book. However, we will pretend that the results of that section are unavailable to us and try to motivate the construction of $\mathcal{H}$ "from scratch". The hope is that this will hint at a method for characterizing the fibers of more general maps.

By our earlier reasoning, we need to construct an $\mathcal{H}$ along with a section $(z : \mathbb{S}^3) \to \mathcal{H} \circ \mathsf{hpf}(z)$ such that $\mathcal{H}$ is the initial type family equipped with such a section. This means that, for any other family $B : \mathbb{S}^2 \to U$, we have an equivalence:

$$((x:\mathbb{S}^2) \to \mathcal{H}(x) \to B(x)) \simeq ((z:\mathbb{S}^3) \to B \circ \mathsf{hpf}(z))$$

given by precomposing with the section.

Our first step in motivating the construction of $\mathcal{H}$ is to give a characterization of the left hand type using the universal property of $\mathbb{S}^2$.