Eckmann-Hilton and the Hopf Fibration

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The Goal

And some reasons to care

The Goal: Construct the Hopf fibration $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ using the Eckmann-Hilton argument.

And some reasons to care:

1. Simple description of the generator of $\pi_3(\mathbb{S}^2)$. From the fiber sequence of hpf.
2. Ditto the generator of $\pi_4(\mathbb{S}^3)$. From the Freudenthal suspension theorem.
3. $\pi_4(\mathbb{S}^3)$ has order *at most* 2. From Syllepsis.

The Plan

1. Use Eckmann-Hilton to construct $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$. This is equivalent to a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$.
2. Characterize the fiber by generalizing ideas from Kraus and Von Raumer's "Path Spaces of Higher Inductive Types".

The Eckmann-Hilton Argument

Eckmann-Hilton

For $\alpha, \beta : \Omega^2(X)$, we have $\mathsf{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

But where does this identification come from?

Where does Path Concatenation come from?

Fix a pointed type $(X, \bullet)$ and consider $\mathsf{Id}_\bullet : X \to U$.

A loop $p : \Omega(X)$ induces:

$\mathrm{tr}^{\mathsf{Id}_\bullet}(p) : \Omega(X) \simeq \Omega(X)$

This is path concatenation:

for $q : \Omega(X)$ we have:

$\mathrm{tr}(p)(q) = q \cdot p.$

Where does Eckmann-Hilton come from?

Up one dimension:

a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

This is Eckmann-Hilton:

for $\beta : \Omega^2(X)$, we have:

$\text{nat-}[\mathrm{tr}^2(\alpha)](\beta) = \mathrm{EH}(\alpha, \beta)$

(modulo coherence paths)

A formula for $\mathrm{tr}^2(\alpha)$

Computing $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

$$\mathrm{tr}^2(\alpha) = \mathrm{whisker}_\alpha = \lambda(p).\mathrm{refl}_p \star \alpha$$

$$\begin{tikzcd}
\bullet \arrow[r, dash, "p"] & \bullet \arrow[r, dash, bend left, "\mathrm{refl}"] \arrow[r, dash, bend right, "\mathrm{refl}"'] \arrow[r, phantom, "\alpha"] & \bullet
\end{tikzcd}$$

$\mathrm{tr}^2(\alpha)(\mathrm{refl}_\bullet) = \alpha$

The naturality condition of $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

For $\beta : \Omega^2(X)$:

$$\begin{tikzcd}
\mathrm{refl}_\bullet \arrow[r, dash, "\mathrm{id}(\beta)"] \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"'] \arrow[dr, phantom, "\text{nat-}[\mathrm{tr}^2(\alpha)](\beta)"] & \mathrm{refl}_\bullet \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"] \\
\mathrm{refl}_\bullet \arrow[r, dash, "\mathrm{id}(\beta)"'] & \mathrm{refl}_\bullet
\end{tikzcd}$$

Plus coherence paths, this defines

$$\mathrm{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$$

.

Eckmann-Hilton in $\mathbb{S}^2$

$$\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$$

The type of this is identification is equivalent to $\Omega^3(\mathbb{S}^2)$.

The Eckmann-Hilton 3-loop

Define $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$ as the image of $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ under said equivalence.

See agda-unimath for more.

The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

hpf : $\mathbb{S}^3 \to \mathbb{S}^2$

Define a map $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ by $\mathbb{S}^3$-induction:

$$\mathrm{hpf}(\mathrm{base}_3) :\equiv \mathrm{base}_2$$

$$\mathrm{hpf}(\mathrm{surf}_3) := \mathrm{eh}$$

The Universal Property of the Family of Fibers

Fix a pointed map $h : A \to B$. Then:

Heuristic

$\mathrm{fib}_h(b_0)$ is like the loop space of B with extra identifications freely generated by the map h.

The Universal Property of the Family of Fibers

We have an induced type family $\mathrm{fib}_h \circ h : A \to U$.

This family always comes equipped with a section:

$$\lambda(a).(a, \mathrm{refl}_{h(a)}) : (a : A) \to \mathrm{fib}_h \circ h(a)$$

called a lift of h to fib_h.

The Wild Category of Families with Lifts

And the Universal Property of the Family of Fibers

Wild Category of Families with Lifts

Objects: families $P : B \to U$ equipped with a lift $(a : A) \to P \circ h(a)$

Maps: families of maps $(b : B) \to P(b) \to Q(b)$ that preserve the lift

Universal Property of fib_h

The family fib_h with its canonical lift is intial in this wild category.

Proof: follows from the standard equivalence $A \simeq \sum_{b:B} \mathrm{fib}_h(b)$.
Formalized in agda-unimath

Loop Spaces are a Special Case

If $A \equiv \text{unit}$ and $h : \text{unit} \to B$ defined by $h(\star) \equiv b_0$:

$$((a : \text{unit}) \to P \circ h(a)) \simeq P(b_0)$$

So fib_h is the inital type family equipped with a point over b_0

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$ and define h by $s : \Omega^3(B, b_0)$.

Lifts $((a : \mathbb{S}^3) \to P \circ h(a))$ are equivalent to dependent 3-loops:

a point $u : P(b_0)$

an identification $t : \mathrm{tr}^3(s)(u) = \mathrm{refl}_u^2$

So fib_h is the inital type family equipped with a point over b_0 and an identification as above.

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$, $B \equiv \mathbb{S}^2$ and $h \equiv \mathrm{hpf}$.

Then $\mathrm{fib}_{\mathrm{hpf}}$ is the inital:

family over $\mathbb{S}^2$

point $u : \mathrm{fib}_{\mathrm{hpf}}(\mathrm{base}_2)$

identification $t : \mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}_u^2$

Interlude, descent data of $\mathbb{S}^2$

A type family P over $\mathbb{S}^2$ is equivalent to:

Descent data of $\mathbb{S}^2$

a type X, the value of $P(\mathrm{base}_2)$

a 2-automorphism $\mathrm{id}_X \sim \mathrm{id}_X$, the transport $\mathrm{tr}^2(\mathrm{surf}_2)$

A Characterization of $\mathrm{fib}_{\mathrm{hpf}}$

Then $\mathrm{fib}_{\mathrm{hpf}}$ is the inital data:

type F

2-automorphism $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $t : \mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}_u^2$

The latter identification is equivalent to an identification

$$\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \cdot \mathrm{surf}_2)(u)}$$

Eckmann-Hilton in the Universe

For $P : X \to U$ with $u : P(\bullet)$ and $\alpha, \beta : \Omega^2(X, \bullet)$:

$$\begin{tikzcd}
\mathrm{tr}^2(\alpha \cdot \beta)(u) \arrow[r, dash, "\mathrm{tr}^2\text{-concat}_{\alpha,\beta}"] \arrow[d, dash, "\mathrm{tr}^3(\mathrm{EH}(\alpha,\beta))(u)"'] \arrow[dr, phantom, "\mathrm{tr}^3\text{-EH}"] & \mathrm{tr}^2(\alpha)(u) \cdot \mathrm{tr}^2(\beta)(u) \arrow[d, dash, "\text{nat-}[\mathrm{tr}^2(\alpha)](\mathrm{tr}^2(\beta)(u))"] \\
\mathrm{tr}^2(\beta \cdot \alpha)(u) \arrow[r, dash, "\mathrm{tr}^2\text{-concat}_{\beta,\alpha}"'] & \mathrm{tr}^2(\beta)(u) \cdot \mathrm{tr}^2(\alpha)(u)
\end{tikzcd}$$

Proof: See agda-unimath

A Characterization of $\mathsf{fib}_{\mathsf{hpf}}$

Recall that $\mathsf{fib}_{\mathsf{hpf}}$ is the inital data:

type F

2-automorphism $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \,\cdot\, \mathrm{surf}_2)(u)}$

Last equality is equivalent to:

$$\text{nat-}[\mathrm{tr}^2(\mathrm{surf}_2)](\mathrm{tr}^2(\mathrm{surf}_2)(u)) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2)(u)\,\cdot\,\mathrm{tr}^2\mathrm{surf}_2(u)}$$

A Characterizaton of $\mathrm{fib}_{\mathrm{hpf}}$

Finally, $\mathrm{fib}_{\mathrm{hpf}}$ is the initial data:

type F

point $u : F$

2-automorphism $H : \mathrm{id}_F \sim \mathrm{id}_F$

identification $\text{nat-}H(H(u)) = \mathrm{refl}_{H(u) \cdot H(u)}$

We can package this as a HIT F generated by the subsequent data.

The Fiber is $\mathbb{S}^1$

Want $F \simeq \mathbb{S}^1$

Two approaches:

1. Using a HIT and directly constructing an equivalence
2. Show $\mathbb{S}^1$ is initial in the wild category of F-algebras

Using a HIT

In cubical agda: thanks to Tom Jack

In Book HoTT: possible . . .

In agda-unimath (and other common HoTT repos): not possible

F-algebras

Want to show $\mathsf{hom}_{F\text{-alg}}(\mathbb{S}^1, X)$ is contractible for every F-algebra X.

Have a definition of F-algebras.

Need a definition of the hom type between F-algebras.

Morphisms of F-algebras

Consider F-algebras (X, K, x_0, p) and (Y, M, y_0, q).

A morphism of F-algebras comprises:

1. $g : X \to Y$
2. $G : g \cdot_I K \sim M \cdot_I g$
3. $g_0 : g(x_0) = y_0$
4. t, a witness that "p is sent to q"

$\mathbb{S}^1$ forms an F-algebra

type - $\mathbb{S}^1$

2-automorphism - L

point - b_1

identification - $\mathrm{defn}_L : \text{nat-L}(L(b_1)) = \mathrm{refl}_{\mathrm{loop}} \cdot \mathrm{loop}$

$\hom_{F\text{-alg}}(\mathbb{S}^1, X) \simeq \text{unit}$

a map: $(g : \mathbb{S}^1 \to X\ ,\ G : g \cdot_l L \sim K \cdot_r g\ ,\ g_0 : g(\mathrm{b}_1) = x_0\ ,\ t)$

g is equivalent to $g(\mathrm{b}_1) : X$ and $g(\mathrm{loop}) : \Omega(X, x)$.

$(g(\mathrm{b}_1), g_0)$ is a contractible pair.

G is equivalent to $G(b) : g(\mathrm{loop}) = K(g(\mathrm{b}_1))$ and nat-$G(\mathrm{loop})$.

$(g(\mathrm{loop}), G)$ is a contractible pair.

Claim: nat-$G(\mathrm{loop})$ and t form a contractible pair.

Fiber Sequnce and the Calculation $\pi_3(\mathbb{S}^2)$

We now have a fiber sequence $\mathbb{S}^1 \to \mathbb{S}^3 \xrightarrow{\text{hpf}} \mathbb{S}^2$

Consequences:

It follows that $\Omega^3(\text{hpf}) : \Omega^3(\mathbb{S}^3) \simeq \Omega^3(\mathbb{S}^2)$

So eh : $\Omega^3(\mathbb{S}^2)$ generates $\pi_3(\mathbb{S}^2)$

Future Work

1. Adapting the James construction and Wärn's Zig Zag Construction
2. $\pi_4(\mathbb{S}^3)$
3. Higher Hopf Fibrations and Higher Coherences

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

The Generator of $\pi_4(\mathbb{S}^3)$

$\mathrm{eh}_{\mathrm{surf}_3}$ generates $\pi_4(\mathbb{S}^3)$

Proof: Freudenthal + functions preserve eh.

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

The square of $\mathrm{eh}_{\mathrm{surf}_3}$ is trival.

Proof: Syllepsis (see Sojakova)

Non-Trivality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family $B : \Omega(\mathbb{S}^3) \to U$ such that

$$\text{nat-}[\text{tr}^2(\text{surf}_3)](\text{tr}^2(\text{surf}_3)(u))$$

is non-trivial, for some $u : B(\text{refl})$

Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ should also arise from higher coherences.

The E_4 coherence, corresponding to $\mathbb{S}^7 \to \mathbb{S}^4$, was constructed by Sojakova.

E_n and Descent over $\mathbb{S}^n$

$\mathrm{surf}_n : \Omega^n(\mathbb{S}^n)$ induces an n-automorphism of $\Omega(\mathbb{S}^n)$

the E_n coherence is the $(n-1)$-dimensional naturality condition this.

easy to calculate for $n = 1, 2$. I've calculated this for $n = 3$ with much trouble. The case for $n \geq 4$ needs a motivated approach

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The End

Questions? Comments?