Eckmann-Hilton and the Hopf Fibration

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Conventions

$\mathbb{S}^1$

$b_1 : \mathbb{S}^1$

$\mathrm{loop} : \Omega(\mathbb{S}^1)$

$\mathbb{S}^2$

$\mathrm{base}_2 : \mathbb{S}^2$

$\mathrm{surf}_2 : \Omega^2(\mathbb{S}^2)$

$\mathbb{S}^3$

$\mathrm{base}_3 : \mathbb{S}^3$

$\mathrm{surf}_3 : \Omega^3(\mathbb{S}^3)$

Conventions

refl

refl is the trivial path in $\Omega(X)$

refl^n is the trivial path in $\Omega^n(X)$

Transport

$\text{tr}^B(p) : B(x) \to B(y)$

$\text{tr}^2(\alpha) : \text{tr}(p) \sim \text{tr}(q)$

$\text{tr}^3(\gamma) : \text{tr}(\alpha) \sim \text{tr}(\beta)$

The Goal: Construct the Hopf fibration $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ using the Eckmann-Hilton argument.

Immediate Consequences:

1. Simple description of the generator of $\pi_3(\mathbb{S}^2)$. From the fiber sequence of hpf.
2. Ditto the generator of $\pi_4(\mathbb{S}^3)$. From the Freudenthal suspension theorem.
3. $\pi_4(\mathbb{S}^3)$ has order *at most* 2. From Syllepsis.

The Plan

1. Use Eckmann-Hilton to construct a 3-loop $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$. This is equivalent to a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$.

2. Adapt ideas from Von Raumer's "Path Spaces of Higher Inductive Types" to characterize the fiber.

Eckmann-Hilton

The Eckmann-Hilton Identification

For $\alpha, \beta : \Omega^2(X)$, we have $\mathrm{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Eckmann-Hilton

$$\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$$

The type of this is identification is equivalent to $\Omega^3(\mathbb{S}^2)$.

The Eckmann-Hilton 3-loop

Define $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$ as the image of $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ under said equivalence.

See agda-unimath for more.

The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

$\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$

Define a map $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ by $\mathbb{S}^3$-induction:

$\mathrm{hpf}(\mathrm{base}_3) :\equiv \mathrm{base}_2$

$\mathrm{hpf}(\mathrm{surf}_3) := \mathrm{eh}$

Kraus and Von Raumer's Computation of $\Omega(\mathbb{S}^1)$

1. Universal Property: $\mathrm{Id}_{b_1} : \mathbb{S}^1 \to U$ is the inital pointed type family
2. Descent: $\Omega(\mathbb{S}^1)$ is the inital pointed type equipped with an automorphism
3. Universal Property, pt. II: $\mathbb{Z}$ is the initial pointed type equipped with an automorphism

Note, $\Omega(\mathbb{S}^1)$ is the fiber of unit $\to \mathbb{S}^1$

Our Computation of $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{base}_2)$

1. Universal Property: $\mathsf{fib}_{\mathsf{hpf}} : \mathbb{S}^2 \to U$ is the inital family equipped with a lift of hpf (a section over hpf).
2. Descent: $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{base}_2)$ is the inital type equipped with some data.
3. Universal Property, pt. II: $\mathbb{S}^1$ is the inital type equipped with some data.

The Universal Property of the Family of Fibers

Fix a pointed map $h : A \to B$. Then:

Heuristic

$\mathrm{fib}_h(b_0)$ is like the loop space of B with extra identifications freely generated by the map h

The Universal Property of the Family of Fibers

A map $h : A \to B$ is equivalent to a type family $\mathrm{fib}_h : B \to U$.

This family always comes equipped with a lift:

$$\lambda(a).(a, \mathrm{refl}) : (a : A) \to \mathrm{fib}_h \circ h(a)$$

fib_h is the initial such type family

The Universal Property of the Family of Fibers

Wild Category of Families with Lifts

Objects: families $P : B \to U$ equipped with a lift $(a : A) \to P \circ h(a)$

Maps: families of maps $(b : B) \to P(b) \to Q(b)$ that preserve the lift

Universal Property of fib_h

The family fib_h with its canonical lift is intial in this wild category.

Proof: follows from the standard equivalence $A \simeq \sum_{b:B} \mathrm{fib}_h(b)$.
Formalized in agda-unimath

Specializing the Universal Property

If $A \equiv \text{unit}$ and $h(\star) \equiv b_0$:

$$((a : \text{unit}) \to P \circ h(a)) \simeq P(b_0)$$

So fib_h is the inital type family equipped with a point over b_0

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$ and define h by $s : \Omega^3(B, b_0)$.

Lifts $((a : \mathbb{S}^3) \to P \circ h(a))$ are equivalent to dependent 3-loops:

a point $u : P(b_0)$

an identification $\mathrm{tr}^3(s)(u) = \mathrm{refl}_u^2$

So fib_h is the inital type family equipped with a point over b_0 and an identification as above.

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$, $B \equiv \mathbb{S}^2$ and $h \equiv \mathrm{hpf}$.

Then $\mathrm{fib}_{\mathrm{hpf}}$ is the inital:

family over $\mathbb{S}^2$

point $u : \mathrm{fib}_{\mathrm{hpf}}(\mathrm{base}_2)$

identification $\mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}_u^2$

Interlude, descent data of $\mathbb{S}^2$

A type family P over $\mathbb{S}^2$ is equivalent to:

Descent data of $\mathbb{S}^2$

a type X, the value of $P(\mathsf{base}_2)$

a homotopy $\mathrm{id}_X \sim \mathrm{id}_X$, the transport $\mathrm{tr}^2(\mathsf{surf}_2)$

Now we apply Kraus and Von Raumer's ideas using the descent data of $\mathbb{S}^2$.

A Characterization of $\mathrm{fib}_{\mathrm{hpf}}$

Then $\mathrm{fib}_{\mathrm{hpf}}$ is the inital data:

type F

homotopy $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $\mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}_u^2$

The latter identification is equivalent to an identification

$$\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \cdot \mathrm{surf}_2)(u)}$$

The Eckmann-Hilton Argument

Eckmann-Hilton

For $\alpha, \beta : \Omega^2(X)$, we have $\mathsf{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Where does Eckmann-Hilton come from?

Fix a pointed type $(X, \bullet)$ and consider $\mathsf{Id}_\bullet : X \to U$, the family of based identity types.

A loop $p : \Omega(X)$ induces:

$\mathsf{tr}^{\mathsf{Id}_\bullet}(p) : \Omega(X) \simeq \Omega(X)$

This is path concatination:

for $q : \Omega(X)$ we have:

$\mathsf{tr}(p)(q) = q \cdot p.$

Where does Eckmann-Hilton come from?

Up one dimension:

a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$\mathrm{tr}^{(\mathrm{Id}_\bullet)^2}(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

This is Eckmann-Hilton:

for $\beta : \Omega^2(X)$, we have:

$\mathrm{nat}\text{-}[\mathrm{tr}^2(\alpha)](\beta) = \mathrm{EH}(\alpha, \beta)$

(modulo coherence paths)

Eckmann-Hilton from tr^2

Computing $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

$$\mathrm{tr}^2(\alpha) = \mathrm{whisker}_\alpha = \lambda(p).\mathrm{refl}_p \star \alpha$$

$$\begin{tikzcd}
\bullet \arrow[r, "p", no head] & \bullet \arrow[r, bend left=50, "\mathrm{refl}", no head] \arrow[r, bend right=50, "\mathrm{refl}"', no head] \arrow[r, phantom, "\alpha"] & \bullet
\end{tikzcd}$$

$$\mathrm{tr}^2(\alpha)(\mathrm{refl}_\bullet) = \alpha$$

The naturality condition of $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

For $\beta : \Omega^2(X)$:

$$\begin{tikzcd}
\mathrm{refl}_\bullet \arrow[rr, dash, "\mathrm{id}(\beta)"] \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"'] & \text{nat-}[\mathrm{tr}^2(\alpha)](\beta) & \mathrm{refl}_\bullet \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"] \\
\mathrm{refl}_\bullet \arrow[rr, dash, "\mathrm{id}(\beta)"'] & & \mathrm{refl}_\bullet
\end{tikzcd}$$

Plus coherence paths, this ends

$$\mathrm{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$$

.

Eckmann-Hilton in the Universe

Let $P : X \to U$ with $u : P(\bullet)$ and $\alpha, \beta : \Omega^2(X)$. Then:

$$\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(P(\bullet))} \sim \mathrm{id}_{\Omega(P(\bullet))}$$

and

$$\mathrm{tr}^2(\beta)(u) : \Omega(P(\bullet))$$

Eckmann-Hilton in the Universe

$$
\begin{tikzcd}
u \arrow[rr, dash, "\mathrm{tr}^2(\beta)(u)"] \arrow[d, dash, "\mathrm{tr}^2(\alpha)(u)"'] & & u \arrow[d, dash, "\mathrm{tr}^2(\alpha)(u)"] \\
u \arrow[rr, dash, "\mathrm{tr}^2(\beta)(u)"'] \arrow[urr, phantom, "\text{nat-}[\mathrm{tr}^2(\alpha)](\mathrm{tr}^2\beta)(u)"] & & u
\end{tikzcd}
$$

$\text{nat-}[\mathrm{tr}^2(\alpha)](\mathrm{tr}^2(\beta)(u)) = \mathrm{tr}^3(\mathrm{EH}(\alpha, \beta))(u).$ (modulo coherence paths)

Proof: See agda-unimath

The Eckmann-Hilton 3-loop

We can EH to construct a 3-path in $\mathbb{S}^2$:

$$\mathsf{surf}_2 \cdot \mathsf{surf}_2 \xrightarrow{\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)} \mathsf{surf}_2 \cdot \mathsf{surf}_2$$

This type is equivalent to $\Omega^3(\mathbb{S}^2)$

$\mathsf{eh} : \Omega^3(\mathbb{S}^2)$

Define eh as the image of $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ under this equivalence.

A Characterization of $\mathrm{fib}_{\mathrm{hpf}}$

Recall that $\mathrm{fib}_{\mathrm{hpf}}$ is the inital data:

type F

homotopy $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \cdot \mathrm{surf}_2)(u)}$

A Characterizaton of $\mathsf{fib}_{\mathsf{hpf}}$

Finally, $\mathsf{fib}_{\mathsf{hpf}}$ is the initial data:

type F

point $u : F$

homotopy $H : \mathsf{id}_F \sim \mathsf{id}_F$

identification $\mathsf{nat}\text{-}H(H(u)) = \mathsf{refl}_{H(u) \cdot H(u)}$

We can package this as a HIT F generated by the subsquent data.

The Plan, Part III

Show that $\mathbb{S}^1$ is generated by a point, a homotopy, and an identification as before.

Two approaches:

1. Using a HIT and directly constructing an equivalence
2. Show $\mathbb{S}^1$ is initial amoung F-algebras

Using a HIT

In cubical agda, thanks to Tom Jack

In Book HoTT, good luck. In agda-unimath, not possible.

F-algebras

$\mathbb{S}^1$ forms an F-algebra:

type - $\mathbb{S}^1$

homotopy - L

point - b_1

identification - $\text{defn}_L : \text{nat-L}(L(b_1)) = \text{refl}_{\text{loop}} \cdot \text{loop}$

F-algebra

There is a map from $\mathbb{S}^1$ to any other F-algebra

Let (X, K, x_0, p) be an F-algebra:

1. Define $f : \mathbb{S}^1 \to X$ via $f(\mathrm{b}_1) :\equiv x_0$ and $\mathrm{defn}_f : f(\mathrm{loop}) = K(x_0)$

Need:

2. $f \bullet_l \mathrm{L} \sim K \bullet_r f$
3. $f(\mathrm{b}_1) = x_0$
4. A witness that the trivialization is preserved

F-algebra

Any two F-algebra maps from $\mathbb{S}^1$ to X are equal.

Proof: Path algebra and universal property of $\mathbb{S}^1$. Omitted due to time constraints.

Future Work

1. Adapting the James construction and Wärn's Zig Zag Construction
2. $\pi_4(\mathbb{S}^3)$
3. Higher Hopf Fibrations and Higher Coherences

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

Suspension Preserve Eckmann-Hilton

For the unit $\sigma : \mathbb{S}^2 \to \Omega(\mathbb{S}^3)$, we have $\sigma(\mathsf{eh}) = \mathsf{eh}_{\mathsf{surf}_3}$

Proof: all functions preserve Eckmann-Hilton.

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

$\mathsf{eh}_{\mathsf{surf}_3}$ generates $\pi_4(\mathsf{surf}_3)$ and its square is trival.

Proof: the map $\pi_3(\sigma) : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective. So $\pi_3(\sigma)(\mathsf{eh}) = \mathsf{eh}_{\mathsf{surf}_3}$ is a generator. Syllepsis implies that $\mathsf{eh}_{\mathsf{surf}_3}$ is its own inverse.

Non-Trivality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family $B : \Omega(\mathbb{S}^3) \to U$ such that

$$\text{nat-}[\text{tr}^2(\text{surf}_3)](\text{tr}^2(\text{surf}_3)(u))$$

is non-trivial, for some $u : B(\text{refl})$

Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ should also arise from higher coherences.

The E_4 coherence, corresponding to $\mathbb{S}^7 \to \mathbb{S}^4$, was constructed by Sojakova.

Up, Up, and Away

There is a correspondence between the E_n coherence and the descent data of Id over the n-sphere.

E_1

Group structure, given by path concatination in $\Omega(\mathbb{S}^1)$

E_2

Braiding, given by Eckmann-Hilton in $\Omega^2(\mathbb{S}^2)$

E_3

Syllepsis, given by sypllepsis in $\Omega^3(\mathbb{S}^3)$

Syllepsis and Descent over $\mathbb{S}^3$

The family $\mathrm{Id} : \mathbb{S}^3 \to U$ is equivalent to:

Descent over $\mathbb{S}^3$

$\Omega(\mathbb{S}^3)$

$\mathrm{tr}^3(\mathrm{surf}_3) : \mathrm{Refl} \sim \mathrm{Refl} \equiv (x : X) \to \Omega^2(X, x)$

Claim: the 2-D naturality condition of this homotopy is syllepsis.

I.e. $\mathrm{nat}^2\text{-}[\mathrm{tr}^3(\mathrm{surf}_3)](\mathrm{surf}_3)$ is equivalent to

$$\mathrm{Sy}(\mathrm{surf}_3, \mathrm{surf}_3) : \mathrm{EH}(\mathrm{surf}_3, \mathrm{surf}_3) = \mathrm{EH}(\mathrm{surf}_3, \mathrm{surf}_3)^{-1}$$

Proof: Horrible path algebra

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The End

Questions? Comments?