# Eckmann-Hilton and the Hopf Fibration in Homotopy Type Theory

Raymond Baker

## 0.1 Introduction

This thesis explores the connection between the Hopf fibration and the Eckmann-Hilton argument through the lens of Homotopy Type Theory (HoTT). Both the Hopf fibration and the Eckmann-Hilton argument are familiar constructions from classical homotopy theory. The Hopf fibration is a non-trivial map $\mathbb{S}^3 \to \mathbb{S}^2$ whose fiber is $\mathbb{S}^1$. The fiber sequence (and induced long exact sequence of homotopy groups) of the Hopf fibration gives an equivalence $\Omega^n(\mathbb{S}^2) \cong \Omega^n(\mathbb{S}^3)$, for $n \geq 3$. This result implies, in particular, that $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. This is a surprising result. Given that $\mathbb{S}^2$ is generated by a single two dimensional loop, why should $\mathbb{S}^2$ have any non-trivial loops at dimension three, much less have the same loop space as $\mathbb{S}^3$ from there on out?

The Eckmann-Hilton argument provides an answer to this question. The Eckmann-Hilton argument can be phrased in different ways. A simple way to phrase it is simply that a monoid object in the category of monoids is necessarily a commutative monoid. However, a more apt phrasing is the following: for any space $X$ and $n \geq 2$, the concatenation of loops in $\Omega^n(X)$ is commutative (up to homotopy of paths). Taking $X$ to be $\mathbb{S}^2$ and $n$ equal to 2, Eckmann-Hilton tells us that the concatenation of loops in $\Omega^2(\mathbb{S}^2)$ is commutative. But this commutativity only holds up to homotopy. The homotopy itself is a three dimensional path and this three dimensional path lends a generator of $\Omega^3(\mathbb{S}^3)$.

This is a result known to hold classically. Proofs of this claim, however, are hard to come by. And discussion of the claim usually involves talk of higher groups, braided groups, and the free braided group on one generator. The goal of this thesis is to write out an elementary proof of the above claim in the language of HoTT. The proof will be elementary in the sense that it makes no reference to braided groups or the like. In theory, this proof will be accesible to any undergraduate familiar with the basics of HoTT: Martin-Löf Type Theory (MLTT), Univalence, Higher Inductive Types (HITs), and some basic background theory. To make this thesis self contained, I will introduce each of these concepts and all prerequisite background theory before turning to the Hopf fibration and Eckmann-Hilton.

The thesis is broken up into three parts. Part I serves as exposition and as an introduction to the basics of HoTT. Part II develops the background theory required for the proof that the Eckmann-Hilton path lends a generator of $\pi_3(\mathbb{S}^2)$. The thesis culminates in Part III which contains the desired proof.