ECKMANN-HILTON AND THE HOPF FIBRATION: AN OUTLINE

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Please note, this document current serves as notes to myself and as an outline for the formalization project. Thus, keep in mind two things: (i) this document is not fully completed, so should be looked at as a partial description of the current state of the proof, and (ii) this document is not meant to be an introduction to the problem, but only an update for those famliar with the topics.

The goal is to show that the Eckmann-Hilton argument can be used to construct the Hopf fibration. The main idea behind this proof can be stated simply: the connection between the Eckmann-Hilton argument and the Hopf fibration can be found in the naturality condition of certain 2-dimensional objects, 2-paths and homotopies, respectively. First we outline how Eckmann-Hilton relates to a naturality condition on 2-paths. Any 2-loop $\beta : \Omega^2(X, \bullet)$ induces a homotopy of type $\mathsf{id}_{\Omega(X)} \sim \mathsf{id}_{\Omega(X)}$ given by the formula:

$$\mathsf{whisker}_\beta :\equiv \lambda(p).1_p \star \beta$$

This can be depicted as follows:

$$\begin{tikzcd} \bullet \arrow[r, no head, "p"] & \bullet \arrow[r, bend left=50, no head, "1"] \arrow[r, bend right=50, no head, "1"'] \arrow[r, phantom, "\beta"] & \bullet \end{tikzcd}$$

We can describe Eckmann-Hilton in the following way:

> Eckmann-Hilton is (more or less) the *naturality condition* of this homotopy when *applied to 2-loops*.

Lets see why. The above homotopy has a naturality condition induced by paths in $\Omega(X)$. In particular, for $\alpha : \Omega^2(X)$, the homotopy has naturality condition which can be depicted as:

$$\begin{tikzcd}
1^2_\bullet \arrow[rr, dash, "(\alpha\star 1^2_\bullet)"] \arrow[d, dash, "(1^2_\bullet\star\beta)"'] & & 1^2_\bullet \arrow[d, dash, "(1^2_\bullet\star\beta)"] \\
1^2_\bullet \arrow[rr, dash, "(\alpha\star 1^2_\bullet)"'] \arrow[urr, phantom, "\mathsf{nat\text{-}whisker}_\beta(\alpha)"] & & 1^2_\bullet
\end{tikzcd}$$

A standard part of the Eckmann-Hilton proof constructs paths $\beta = 1^2_\bullet \star \beta$ and $\alpha = \alpha \star 1^2_\bullet$. Together with the naturality condition, we obtain the Eckmann-Hilton path $\beta \cdot \alpha = \alpha \cdot \beta$. Thus, Eckmann-Hilton ultimately comes from the naturality of $\mathsf{whisker}_\beta$.

What, then, does the Hopf fibration have to do with the naturality condition of $\mathsf{whisker}$? We will actually highlight a more general connection between fibrations over $\mathbb{S}^2$ and this naturality condition. Then, in the case of the Hopf fibration, we will see that something extra nice happens. The following equivalence characterizes fibrations over $\mathbb{S}^2$.

$$(\sum_{Z:U} Z \to \mathbb{S}^2) \simeq (\mathbb{S}^2 \to U) \simeq (\sum_{X:U} \mathsf{id}_X \sim \mathsf{id}_X)$$

This is equivalence sends a map h with total space type Z to the pair consisting of the fiber $F :\equiv \mathsf{fib}_h(\mathsf{N}_2)$ and the homotopy $(\mathsf{fib})^2 :\equiv \mathsf{tr}^{(\mathsf{fib}_h)^2}(\mathsf{surf}_2)$. We can compute the action of this homotopy on a point $(z,p) : \mathsf{fib}_{\mathsf{hpf}}(\mathsf{N}_2)$ as

$$(\mathsf{fib})^2(z,p) = (1_z, 1_p \star \mathsf{surf}_2) \equiv (1_z, \mathsf{whisker}_{\mathsf{surf}_2}(p))$$

Thus, Eckmann-Hilton is present in every fibration (and type family) over $\mathbb{S}^2$ in the form of the naturality condition of the two dimensional descent data (i.e., the homotopy).

This suffices to outline the basic connection between Eckmann-Hilton and general fibrations over $\mathbb{S}^2$, though in no way fully explicates the connection.

TO BE WRITTEN