Eckmann-Hilton and the Hopf Fibration

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The 2-sphere $\mathbb{S}^2$ is generated by a single 2-loop: $\mathsf{surf}_2 : \Omega^2(\mathbb{S}^2)$.

Yet $\mathbb{S}^2$ has non-trival loops in dimension ≥ 3. Where does this structure come from?

Answer:

the Hopf fibration: $\mathbb{S}^1 \to \mathbb{S}^3 \xrightarrow{\text{hpf}} \mathbb{S}^2$

But how does this arise from the intuitive notion of "generated by a 2-loop under operations on identifications"?

The Goal: Construct the Hopf fibration $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ using the Eckmann-Hilton argument.

And some reasons to care:

1. Simple description of the generator of $\pi_3(\mathbb{S}^2)$. From the fiber sequence of hpf.
2. Ditto the generator of $\pi_4(\mathbb{S}^3)$. From the Freudenthal suspension theorem.
3. $\pi_4(\mathbb{S}^3)$ has order *at most* 2. From Syllepsis.

The Plan

1. Use Eckmann-Hilton to construct $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$. This is equivalent to a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$.
2. Characterize the fiber by generalizing ideas from Kraus and Von Raumer's "Path Spaces of Higher Inductive Types".

The Eckmann-Hilton Identification

For $\alpha, \beta : \Omega^2(X)$, we have $\mathsf{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

Eckmann-Hilton in $\mathbb{S}^2$

$$\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$$

The type of this is identification is equivalent to $\Omega^3(\mathbb{S}^2)$.

The Eckmann-Hilton 3-loop

Define $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$ as the image of $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$ under said equivalence.

See agda-unimath for more.

The map hpf

The 3-loop eh is equivalent to a map, the Hopf fibration:

$\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$

Define a map $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ by $\mathbb{S}^3$-induction:

$$\mathrm{hpf}(\mathrm{base}_3) :\equiv \mathrm{base}_2$$

$$\mathrm{hpf}(\mathrm{surf}_3) := \mathrm{eh}$$

How to Compute the Fiber

"Path Spaces of Higher Inductive Types" outlines a method characterizing the family of based identity types of a HIT.

This can be seen as characterizing the family of fibers of a map unit $\to B$.

We will generalize this idea to work for general maps $A \to B$.

Kraus and Von Raumer's Computation of $\Omega(\mathbb{S}^1)$

1. Universal Property: $\mathsf{Id}_{\mathsf{b}_1} : \mathbb{S}^1 \to U$ is the inital type family equipped with a point over b_1.
2. Descent: $(\Omega(\mathbb{S}^1), \mathsf{tr}^{\mathsf{Id}}(\mathsf{loop}), \mathsf{refl})$ is the inital type equipped with an automorphism and a point.
3. Universal Property, pt. II: $(\mathbb{Z}, \mathsf{succ}, 0)$ is the initial type equipped with an automorphism and a point.

Our Computation of $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{base}_2)$

1. Universal Property: $\mathsf{fib}_{\mathsf{hpf}} : \mathbb{S}^2 \to U$ is the inital family equipped with a lift of hpf.
2. Characterize Lifts: lifts of hpf are equivalent to some data.
3. Descent: $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{base}_2)$ is the inital type equipped with some more data.
4. Universal Property, pt. II: $\mathbb{S}^1$ is the inital type equipped with the same data.

The Universal Property of the Family of Fibers

Fix a pointed map $h : A \to B$. Then:

Heuristic

$\mathrm{fib}_h(b_0)$ is like the loop space of B with extra identifications freely generated by the map h.

The Universal Property of the Family of Fibers

We have an induced type family $\mathrm{fib}_h \circ h : A \to U$.

This family always comes equipped with a section:

$$\lambda(a).(a, \mathrm{refl}) : (a : A) \to \mathrm{fib}_h \circ h(a)$$

called a lift of h to fib_h.

fib_h is the initial such type family

The Universal Property of the Family of Fibers

Wild Category of Families with Lifts

Objects: families $P : B \to U$ equipped with a lift $(a : A) \to P \circ h(a)$

Maps: families of maps $(b : B) \to P(b) \to Q(b)$ that preserve the lift

Universal Property of fib_h

The family fib_h with its canonical lift is intial in this wild category.

Proof: follows from the standard equivalence $A \simeq \sum_{b:B} \mathrm{fib}_h(b)$.
Formalized in agda-unimath

Loop Spaces are a Special Case

If $A \equiv \text{unit}$ and $h : \text{unit} \to B$ defined by $h(\star) \equiv b_0$:

$$((a : \text{unit}) \to P \circ h(a)) \simeq P(b_0)$$

So fib_h is the inital type family equipped with a point over b_0

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$ and define h by $s : \Omega^3(B, b_0)$.

Lifts $((a : \mathbb{S}^3) \to P \circ h(a))$ are equivalent to dependent 3-loops:

a point $u : P(b_0)$

an identification $\mathrm{tr}^3(s)(u) = \mathrm{refl}_u^2$

So fib_h is the inital type family equipped with a point over b_0 and an identification as above.

Specializing the Universal Property

Let $A \equiv \mathbb{S}^3$, $B \equiv \mathbb{S}^2$ and $h \equiv \mathrm{hpf}$.

Then $\mathrm{fib}_{\mathrm{hpf}}$ is the inital:

family over $\mathbb{S}^2$

point $u : \mathrm{fib}_{\mathrm{hpf}}(\mathrm{base}_2)$

identification $\mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}_u^2$

Interlude, descent data of $\mathbb{S}^2$

A type family P over $\mathbb{S}^2$ is equivalent to:

Descent data of $\mathbb{S}^2$

a type X, the value of $P(\mathrm{base}_2)$

a 2-automorphism $\mathrm{id}_X \sim \mathrm{id}_X$, the transport $\mathrm{tr}^2(\mathrm{surf}_2)$

A Characterization of $\mathsf{fib}_{\mathsf{hpf}}$

Then $\mathsf{fib}_{\mathsf{hpf}}$ is the inital data:

type F

2-automorphism $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $t : \mathrm{tr}^3(\mathrm{eh})(u) = \mathrm{refl}^2_u$

The latter identification is equivalent to an identification

$$\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \cdot \mathrm{surf}_2)(u)}$$

The Eckmann-Hilton Argument

Eckmann-Hilton

For $\alpha, \beta : \Omega^2(X)$, we have $\mathsf{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$

But where does this identification come from?

Where does Path Concatenation come from?

Fix a pointed type $(X, \bullet)$ and consider $\mathsf{Id}_\bullet : X \to U$.

A loop $p : \Omega(X)$ induces:

$\mathrm{tr}^{\mathsf{Id}_\bullet}(p) : \Omega(X) \simeq \Omega(X)$

This is path concatenation:

for $q : \Omega(X)$ we have:

$\mathrm{tr}(p)(q) = q \cdot p.$

Where does Eckmann-Hilton come from?

Up one dimension:

a 2-loop $\alpha : \Omega^2(X, \bullet)$ induces:

$\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

This is Eckmann-Hilton:

for $\beta : \Omega^2(X)$, we have:

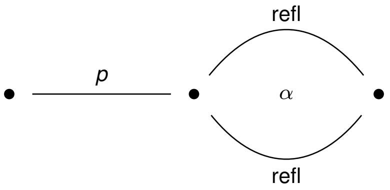
$\text{nat-}[\mathrm{tr}^2(\alpha)](\beta) = \mathrm{EH}(\alpha, \beta)$

(modulo coherence paths)

A formula for $\mathrm{tr}^2(\alpha)$

Computing $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

$$\mathrm{tr}^2(\alpha) = \mathrm{whisker}_\alpha = \lambda(p).\mathrm{refl}_p \star \alpha$$



$\mathrm{tr}^2(\alpha)(\mathrm{refl}_\bullet) = \alpha$

The naturality condition of $\mathrm{tr}^2(\alpha) : \mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$

For $\beta : \Omega^2(X)$:

$$
\begin{tikzcd}
\mathrm{refl}_\bullet \arrow[rr, dash, "\mathrm{id}(\beta)"] \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"'] \arrow[drr, phantom, "\text{nat-}[\mathrm{tr}^2(\alpha)](\beta)"] & & \mathrm{refl}_\bullet \arrow[d, dash, "\mathrm{tr}^2(\alpha)(\mathrm{refl})"] \\
\mathrm{refl}_\bullet \arrow[rr, dash, "\mathrm{id}(\beta)"'] & & \mathrm{refl}_\bullet
\end{tikzcd}
$$

Plus coherence paths, this defines

$$\mathrm{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$$

.

Eckmann-Hilton in the Universe

For $P : X \to U$ with $u : P(\bullet)$ and $\alpha, \beta : \Omega^2(X, \bullet)$:

$$
\begin{tikzcd}
\mathrm{tr}^2(\alpha \cdot \beta)(u) \arrow[r, dash, "\mathrm{tr}^2\text{-concat}"] \arrow[d, dash, "\mathrm{tr}^3(\mathrm{EH}(\alpha,\beta))(u)"'] \arrow[dr, phantom, "\mathrm{tr}^3\text{-EH}"] & \mathrm{tr}^2(\alpha)(u) \cdot \mathrm{tr}^2(\beta)(u) \arrow[d, dash, "\text{nat-}[\mathrm{tr}^2(\alpha)](\mathrm{tr}^2(\beta)(u))"] \\
\mathrm{tr}^2(\beta \cdot \alpha)(u) \arrow[r, dash, "\mathrm{tr}^2\text{-concat}"'] & \mathrm{tr}^2(\beta)(u) \cdot \mathrm{tr}^2(\alpha)(u)
\end{tikzcd}
$$

Proof: See agda-unimath

A Characterization of $\mathrm{fib}_{\mathrm{hpf}}$

Recall that $\mathrm{fib}_{\mathrm{hpf}}$ is the inital data:

type F

homotopy $H : \mathrm{id}_F \sim \mathrm{id}_F$

point $u : F$

identification $\mathrm{tr}^3(\mathrm{EH}(\mathrm{surf}_2, \mathrm{surf}_2))(u) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2 \cdot \mathrm{surf}_2)(u)}$

Last equality is equivalent to:

$$\text{nat-}[\mathrm{tr}^2(\mathrm{surf}_2)](\mathrm{tr}^2(\mathrm{surf}_2)(u)) = \mathrm{refl}_{\mathrm{tr}^2(\mathrm{surf}_2)(u) \cdot \mathrm{tr}^2\mathrm{surf}_2(u)}$$

A Characterizaton of $\mathrm{fib}_{\mathrm{hpf}}$

Finally, $\mathrm{fib}_{\mathrm{hpf}}$ is the initial data:

type F

point $u : F$

homotopy $H : \mathrm{id}_F \sim \mathrm{id}_F$

identification $\text{nat-}H(H(u)) = \mathrm{refl}_{H(u) \cdot H(u)}$

We can package this as a HIT F generated by the subsequent data.

The Fiber is $\mathbb{S}^1$

Want $F \simeq \mathbb{S}^1$

Two approaches:

1. Using a HIT and directly constructing an equivalence
2. Show $\mathbb{S}^1$ is initial in the wild category of F-algebras

Using a HIT

In cubical agda: thanks to Tom Jack

In Book HoTT: possible . . .

In agda-unimath (and other common HoTT repos): not possible

F-algebras

Want to show $\hom_{F\text{-alg}}(\mathbb{S}^1, X)$ is contractible for every F-algebra X.

Have a definition of F-algebras.

Need a definition of the hom type between F-algebras.

Morphisms of F-algebras

Consider F-algebras (X, K, x_0, p) and (Y, M, y_0, q).

A morphism of F-algebras comprises:

1. $g : X \to Y$
2. $G : g \bullet_I K \sim M \bullet_I g$
3. $g_0 : g(x_0) = y_0$
4. A witness that "p is sent to q"

$\mathbb{S}^1$ forms an F-algebra

type - $\mathbb{S}^1$

homotopy - L

point - b_1

identification - $\text{defn}_L : \text{nat-L}(L(b_1)) = \text{refl}_{\text{loop}} \cdot \text{loop}$

$\hom_{F\text{-alg}}(\mathbb{S}^1, X) \simeq \mathrm{unit}$

a map: $(g : \mathbb{S}^1 \to X\ ,\ G : g \bullet_l L \sim K \bullet_r g\ ,\ g_0 : g(\mathrm{b}_1) = x_0\ ,\ t)$

g is equivalent to $g(\mathrm{b}_1) : X$ and $g(\mathrm{loop}) : \Omega(X, x)$.

$(g(\mathrm{b}_1), g_0)$ is a contractible pair.

G is equivalent to $G(b) : g(\mathrm{loop}) = K(g(\mathrm{b}_1))$ and nat-$G(\mathrm{loop})$.

$(g(\mathrm{loop}), G)$ is a contractible pair.

Claim: nat-G(loop) and t form a contractible pair.

Future Work

1. Adapting the James construction and Wärn's Zig Zag Construction
2. $\pi_4(\mathbb{S}^3)$
3. Higher Hopf Fibrations and Higher Coherences

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

Suspension Preserve Eckmann-Hilton

For the unit $\sigma : \mathbb{S}^2 \to \Omega(\mathbb{S}^3)$, we have $\sigma(\mathrm{eh}) = \mathrm{eh}_{\mathrm{surf}_3}$

Proof: all functions preserve Eckmann-Hilton.

$\pi_4(\mathbb{S}^3)$ has order ≤ 2

$\mathrm{eh}_{\mathrm{surf}_3}$ generates $\pi_4(\mathrm{surf}_3)$ and its square is trival.

Proof: the map $\pi_3(\sigma) : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective. So $\pi_3(\sigma)(\mathrm{eh}) = \mathrm{eh}_{\mathrm{surf}_3}$ is a generator. Syllepsis implies that $\mathrm{eh}_{\mathrm{surf}_3}$ is its own inverse.

Non-Trivality of $\pi_4(\mathbb{S}^3)$

Suffices to find a family $B : \Omega(\mathbb{S}^3) \to U$ such that

$$\text{nat-}[\mathrm{tr}^2(\mathrm{surf}_3)](\mathrm{tr}^2(\mathrm{surf}_3)(u))$$

is non-trivial, for some $u : B(\mathrm{refl})$

Higher Hopf Fibrations and their Coherences

The higher Hopf fibrations $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ should also arise from higher coherences.

The E_4 coherence, corresponding to $\mathbb{S}^7 \to \mathbb{S}^4$, was constructed by Sojakova.

Up, Up, and Away

There is a correspondence between the E_n coherence and the descent data of Id over the n-sphere.

E_1

Group structure, given by path concatination in $\Omega(\mathbb{S}^1)$

E_2

Braiding, given by Eckmann-Hilton in $\Omega^2(\mathbb{S}^2)$

E_3

Syllepsis, given by sypllepsis in $\Omega^3(\mathbb{S}^3)$

The family $\mathrm{Id} : \mathbb{S}^3 \to U$ is equivalent to:

Descent over $\mathbb{S}^3$

$\Omega(\mathbb{S}^3)$

$\mathrm{tr}^3(\mathrm{surf}_3) : \mathrm{Refl} \sim \mathrm{Refl} \equiv (x : X) \to \Omega^2(X, x)$

Claim: the 2-D naturality condition of this homotopy is syllepsis.

I.e. $\mathrm{nat}^2\text{-}[\mathrm{tr}^3(\mathrm{surf}_3)](\mathrm{surf}_3)$ is equivalent to

$$\mathrm{Sy}(\mathrm{surf}_3, \mathrm{surf}_3) : \mathrm{EH}(\mathrm{surf}_3, \mathrm{surf}_3) = \mathrm{EH}(\mathrm{surf}_3, \mathrm{surf}_3)^{-1}$$

Proof: Horrible path algebra

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The End

Questions? Comments?