Uncertainties in Measurements: The Ping Pong and the Pendulum

Introduction:

This experiment is divided into two parts: In Part I the time between two bounces of a ping pong will be measured, in order to understand some fundamental definitions and properties of statistics, with the goal of obtaining an overall result and its uncertainty. In Part II these ideas are extended to a case in which more than one value has an uncertainty. Here a pendulum's length and period are measured and then combined to obtain the acceleration of gravity, and its uncertainty.

Part I: The Ping Pong

Purpose:

The objectives of this experiment are to determine the time between two events, and to make a quantitative (numerical) estimate of the uncertainty in this time.

Equipment:

Stopwatch and Ping-pong ball.

Preliminary measurements:

Your laboratory instructor will drop a ping-pong ball from a height of 1 meter above the floor. Using your stop watch, have one person in your group determine the time between the first and second bounce of the ball. Your instructor may ask you to write your result on the blackboard.

Each group now has a value for the time, but so far you have no good way of determining how close your result is to the "true" time between the bounces. Your group almost certainly does not have exactly the same result as does everybody else.

The rest of this laboratory session will be spent trying to make a good estimate of the uncertainty in your result.

Before you go on with the experiment, you need to understand the different terms that are used when discussing experimental errors and uncertainties.

Errors:

The error in a measurement is the difference between the experimentally determined value, t_{exp} and the "true" value, t_{true}. The error is positive if the experimental value is too large, and negative if the experimental value is too small.

Absolute Errors

Often, it does not matter if your result is too big or too small, but just the fact that you are unable to get exactly the true value. We then talk about the absolute error in the measurement,

$$|\Delta t| = |t_{exp} - t_{true}| \tag{2.1}$$

The relative error is the error divided by the true value, and may be expressed as a percentage.

$$\text{relative error}\,(\%) = \frac{\Delta t}{t_{true}} \cdot 100\% \tag{2.2}$$

Systematic Error

A systematic error arises from using defective measuring equipment, faulty experimental technique, or incorrect assumptions about the experiments, etc., resulting in measurements either too high or too low.

It is important to note that repeating a measurement a number of times and averaging the results does not eliminate systematic errors.

Examples of systematic errors:

- A thermometer that reads 99˚C when immersed in boiling water at standard conditions is improperly calibrated.
- On a hot day all measurements taken with a meter ruler are potentially in error due to thermal expansion of the ruler.
- If the effects of friction in a mechanical experiment are neglected, the results may be either too high or too low.

Random Errors

Random errors arise from unpredictable and unknown conditions affecting the experimental situation, such as: estimating to the tenths of the smallest scale division; unpredictable fluctuations in temperature; line voltage or vibrations. Generally, it is found that small random errors occur more frequently than large ones, and positive and negative random errors of about the same size are about equal in number.

You can get a good idea of the random errors involved in a particular measurement by taking several independent readings of the same quantity. Taking the average of your independent measurements will reduce the effect of the random errors. Observing the variations in your readings will give you an idea of what kind of random errors may be affecting your measurements.

Statistical functions:

In discussing measurements which are affected by random errors, there are several standard terms used to describe the results.

The Arithmetic Mean

For a series of N independent measurements, t_i, we will assume that the average, or arithmetic mean, is our best estimate of the true value.

$$\overline{t} = \frac{1}{N}\sum_{i=1}^{N} t_i = \frac{t_1 + t_2 + t_3 + \dots\dots\dots + t_N}{N} \tag{2.3}$$

The Standard Deviation

The standard deviation is a measure of the effect of random errors. If the true value is known, the standard deviation can be calculated by

$$\sigma_t = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(t_i - t_{true})^2} \tag{2.4}$$

The standard deviation is the square root of the average of the square error. This gives you an idea of how far from the true value you can expect an individual measurement to be. It is generally assumed that, ***if there are no systematic errors affecting the experiment***, 2/3 of all measurements will fall no more than one standard deviation from the true value. 95% of all measurements fall within $2\sigma_t$, and 99% fall within $3\sigma_t$ of the true value.

The Sample Standard Deviation

You can always calculate the mean from a set of measured values, but you will usually not be able to calculate the standard deviation since you do not know the true value. The sample standard deviation, s_t, provides us with a "best estimate" of the standard deviation Ft. It can be calculated as follows:

$$s_t = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(t_i - \overline{t})^2} \tag{2.5}$$

As you see, the sample standard deviation uses the mean, in place of the true value. This would have a tendency to underestimate σ_t, so to compensate, we divide by $N - 1$ rather than by N.

Back to your experiment:

Each group has a measured value for the time between two bounces; the question now is, how good is your result?

If you could determine the error in your time measurement, you would of course correct for it. You would then no longer have any error in your result. But normally you are unable to determine the error, and the best you can do is to Estimate the Uncertainty in your measurement.

From now on, we will talk about UNCERTAINTIES rather than ERRORS.

Your instructor will drop the ping-pong ball 29 more times (for a total of 30), from the same height of 1 meter. Have the same person in your group measure the time between the first and second bounce each time.

Finding the Average and Sample Standard Deviation

Your instructor may ask you to do these calculations using the Statistical Functions on your scientific calculator or using Excel®.

Using a calculator with built-in statistical functions:

> Entering the 30 time values into your calculator, it will automatically calculate the average and the standard deviation (σ) as well as the sample standard deviation (often called σ_{n-1} or s_{n-1})

Using Excel®: Put all your times in a single column in the spreadsheet (A1 through A30). Excel® has all the statistical functions built-in:

> =SUM(A1..A30) will calculate the sum of the values,
> =AVERAGE(A1..A30) will calculate the average,
> =STDEV.S(A1..A30) gives you the ***sample*** standard deviation

You have now made an estimation of the true value (the average is your best estimate), and the uncertainty in a single time measurement (s_t is your best estimate of σ_t). For your initial time measurement, write your result (including units) as

$$t_1 \pm s_t \tag{2.6}$$

Does your initial time measurement fall within one standard deviation of your own best estimate of the true value (your average)? If it doesn't, remember that only about two thirds of all measurements fall within one standard deviation of the average. On the other hand, 95% fall within 2σ and 99% within 3σ.

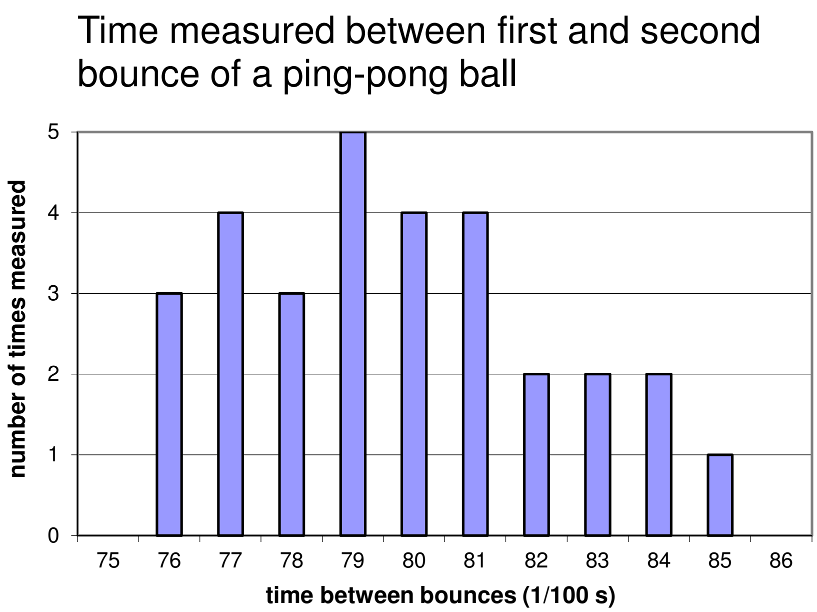

Figure 1: A 'bar graph' showing the distribution of times between the 1st and 2nd bounce of a ping-pong ball.

To get a clearer picture of your data, you should make a graph of it, similar to Figure 1.

Again, your instructor may ask you to produce this graph by hand or using Excel®, or both. First you need to sort your data:

By Hand: Simply count how many times you got the exact same time reading, and make a table like the one below.

Using Excel®: In the same spreadsheet that contains your original time data, make a column with the possible times recorded like the left column in the table below. Let's say your original time data is in Column A, and your possible time values are in Column B. Your Column B might look like the first column below.

To count the number of times a given measurement was recorded, use the FREQUENCY function. This is an example of an array function, and is a little difficult to use for the first time. So you must follow these steps, exactly. Put the cursor in Column C, in the cell to the right of the "75". Use the mouse to highlight the cells from there down to the cell to the right of the one containing the "86". Type the following formula: =FREQUENCY and a left parenthesis. Next use the mouse to highlight the numbers in Column A, and then type a comma. Use the mouse to highlight the numbers in Column B, then type a right parenthesis. Finally, press the CONTROL+SHIFT+ENTER keys simultaneously. Excel® will then count the number of 75's, 76's, 77's, etc., displaying the values in your Column C. You will have to provide the table name, column headings, etc.

Time between 1st and 2nd bounce [1/100s]	Number of times a given measurement was recorded
75	0
76	3
77	4
78	3
79	5
80	4
81	4
82	2
83	2
84	2
85	1
86	0

To make a graph like Figure 1 using Excel®, select Insert Chart or use the Chart Wizard icon. Select the Column type of chart, then click Next. Input the data, with your Column B on the horizontal axis and Column C on the vertical axis. Label the axes, give the graph a title, etc.

Look at your graph and compare it with your calculated values for the mean and the sample standard deviation. The data shown in Figure 1 has an average of 0.7973 seconds and a standard deviation of 0.025 seconds. (The difference between σ_t and s_t are negligible. They are 0.0249 and 0.0253 seconds, respectively.) Does your graph look like 2/3 of your data is within one standard deviation of your mean?

The Uncertainty in the Average

Ok, so you now have an estimate of the uncertainty in a single time measurement, but you wouldn't just use a single measurement when you have made 30 of them.

Your best estimate of the true time is the mean. But what is the uncertainty in the mean? As discussed above, the effect of random errors decreases as you average more and more measurements. The sample standard deviation of the average of *N* independent measurements is

$$s_{\bar{t}} = \frac{s_t}{\sqrt{N}} \tag{2.7}$$

This value can be used to define the UNCERTAINTY in $\bar{t}$, or $\Delta\bar{t}$. Note however that if this value is used, then the true value t_{true} will fall within this uncertainty only about two thirds of the time. If instead the uncertainty is given by $\Delta\bar{t} = 2s_{\bar{t}}$, then t_{true} will fall within this range about 95% of the time (see p. 151, *An Introduction to Error Analysis*, J. Taylor).

If we choose the uncertainty to be $s_{\bar{t}}$, then for the data above $\Delta\bar{t} = s_{\bar{t}} = s_t / \sqrt{30} = 0.0046$s, and the best estimate of the time would be

$$\bar{t} \pm \Delta\bar{t} = 0.7973\text{s} \pm 0.0046\text{s} \tag{2.8}$$

State your result in the same way. Put your average and sample standard deviation on the chalkboard. Compare your result with that of the other groups' averages. Does your result agree with theirs? Do they agree when you include the estimated uncertainties? If they do not, this is probably due to systematic errors.

Estimating the Systematic Errors

So far, your analysis has only included looking at random errors. As you average more and more measurements the effects of random errors become less and less. Systematic errors on the other hand remain just as large for the average as in a single measurement.

Estimate the uncertainty due to systematic errors by the largest difference between any two laboratory groups' averages.

With Δt as the sum of the systematic error and the sample standard deviation of the average, your best estimate of the time is then

$$\bar{t} \pm \Delta t = \bar{t} \pm (systematic\ error + \Delta\bar{t}) \tag{2.9}$$

Clearly state your result in this form.

Final Comments about Errors

In this laboratory experiment, the uncertainty due to random and systematic errors are both important for individual measurements. For the average on the other hand, the systematic errors probably dominated.

Often you can get a good idea of the random errors by taking repeated measurements, and you can reduce their effect on your results by taking an average of many independent measurements.

For systematic errors the story is much different. Usually, you will not know what they are. If you suspect that you are using faulty measuring equipment or technique, you can try to use another instrument or try to determine the same quantity in a different way.

Many times, the best you can do is to make an estimate (guess) of what the errors may be. Just the same, you should always record a numerical estimate of the uncertainty in anything that you have measured. You usually have a better idea of possible sources of errors at the time that you are doing the measurement than you will ever have later. Another person looking at your results will have to rely on your estimate of the accuracy of your measurements.

Part II: The Propagation of Uncertainties - The Pendulum

Purpose:

The purpose of this experiment is to investigate how uncertainties in measured quantities influence the uncertainty in a calculated quantity.

Equipment:

String, meter sticks, a pendulum bob, and timer.

Introduction:

In an earlier experiment you looked at how random and systematic errors affected the uncertainty in a single measured quantity. Often the thing that you are trying to determine in the laboratory cannot be measured directly but has to be calculated from other measured quantities. You then need to know how to determine the uncertainty in your final result from the uncertainties in your measured quantities. In this experiment you will use a simple pendulum to determine the value of the acceleration due to gravity.

The period T and the length L of a simple pendulum are related by

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{2.10}$$

This relationship is actually only true for a simple pendulum, which is an idealization consisting of a point mass m suspended by means of a massless, non-stretchable string of length L from a frictionless pivot. In addition, the amplitude θ of the oscillation (maximum angle that the pendulum is displaced from vertical) must be very small. If one includes the effect of a finite angular amplitude, Equation (2.11) becomes

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \tfrac{1}{4}\sin^2(\tfrac{\theta}{2}) + \tfrac{9}{64}\sin^4(\tfrac{\theta}{2}) + ...\right] \tag{2.11}$$

Equation (2.12) is presented here for future reference and to indicate why it is important that you restrict your pendulum to very small angular displacements, such as 1° or 2° from the vertical. In that case, Equation (2.11) and (2.12) are essentially identical.

In this experiment, you will use measurements of the period and length of a pendulum to determine g, the acceleration due to gravity. If Equation (2.11) is solved for g, the result is

$$g = 4\pi^2 \frac{L}{T^2} \tag{2.12}$$

Note that in this equation, the unknown quantity (g) is expressed in terms of the quantities to be measured in the laboratory (L and T). In this experiment you will measure both L and T as carefully as possible, yet each of these measured quantities will have some uncertainty. That means ultimately, that the quantity that you calculate from them, in this case g, will also have some uncertainty. You will be able to determine approximately how uncertain g is from estimations of the uncertainties in each of the measured quantities, L and T. In this case you will also know the accepted value of the unknown, or can look it up in some reference, and see if this accepted value falls within your experimental range, which you will express as $g_{\exp} \pm \Delta g$. If it does, your result agrees with the accepted value.

In the workplace, for example, you might design a certain piece of machinery to have a certain weight. That becomes the "accepted" value. Now, someone builds that device using the materials that are specified. But each dimension of the device must have some tolerance, and the density of the material is only approximately known. Thus, the final weight will also have some range of possible values. If the final weight is within the tolerance range you estimated, then there is agreement and the product meets the specifications.

Now, back to our experiment, in which you will measure the length and period of a pendulum, then determine the acceleration due to gravity and its uncertainty from these measurements. If you refer to Appendix B "Propagation of Uncertainties", you can follow the examples given there to show that in this case the relative uncertainty in g is given by:

$$\frac{\Delta g}{g} = \frac{\Delta L}{L} + 2\frac{\Delta T}{T} \tag{2.13}$$

where Δg, ΔL, and ΔT are the uncertainties in g, T, and L, respectively. The goal is to calculate Δg from experimental determinations of g, L, T, and estimations of ΔL and ΔT.

The experiment:

Obtain a piece of string about 1.5 m in length and attach it to the pendulum bob. Suspend the pendulum from one of the heavy wall mounts, using a pendulum clamp if available.

Finding the Length and its Uncertainty

Use a two-meter stick to measure L, the distance from the support to the center of the pendulum bob. Your laboratory partner will have to read the position of the center of the bob while you hold the other end of the stick even with the support. Actually, you should never trust the end of the stick to be correct, as it may be dented. Hold the 1.0 cm mark even with the support. Your lab partner must try to estimate the position of the center of the bob as precisely as possible, just as you must try to position the upper end of the stick as carefully as possible. (It will not be exactly 1.00 cm.) With some effort, this can be done to a fraction of a millimeter, say the nearest 0.1 or 0.2 mm. In any case, perform this measurement as carefully as you can. **Record the position of the center of the pendulum bob, and the position of the support**. Determine the length of the string by subtracting the position of the bob from position of the support.

Trade two-meter sticks with another lab group and repeat the measurement, again being as careful as possible. This time trade places with your lab partner. Again, read the positions of the support and the center of the bob then determine the length of the pendulum.

Keep trading two-meter sticks with other groups, until you have at least five independent sets of measurements from which to determine the length of the pendulum. Continue to trade places with your lab partner each time.

Find the average value length of the pendulum and estimate the uncertainty in the average length by using the sample standard deviation. Compare this uncertainty in the length to what you would expect from the fact that you can't read each position better than 0.1 or 0.2 mm. Using different meter sticks also decreases one possible source of systematic errors.

State your result as $L = L_{avg} \pm \Delta L$.

Do you think that there are any other significant sources of systematic errors in the length? If so, try to estimate them.

Finding the Period and its Uncertainty

Calculate how far (in centimeters) you will need to pull the pendulum bob to the side to produce an angle θ of one or two degrees. Mark this position by attaching a small piece of tape to the wall beside your pendulum. This way, you will be able to maintain a consistent value of θ for several trials.

Obtain a timer (you may use your watch or phone for a stopwatch if it has that capability).

Have your lab partner pull the pendulum aside the proper amount and release it. The pendulum may not swing smoothly for the first couple of cycles. This is all right – just wait until it does and then start the timer at the bottom of a swing and count oscillations from there. Note – the pendulum will have completed its first cycle the next time it passes through the bottom of the swing, ***in the same direction***. Determine the time required for at least 20 complete oscillations.

Repeat, trading places with your lab partner and using different timers, until you have at least five independent measurements. Make your measurements as carefully as possible. Try not to be influenced by the previous measurement.

Calculate the period (time for one complete oscillation) from each of your time measurements, and calculate the average (arithmetic mean) for the period. Estimate the uncertainty in the period by calculating the sample standard deviation of your period data.

State your result as $T = T_{avg} \pm \Delta T$.

Do you think that your estimate of the uncertainty in the period is reasonable? Do you think that there are any significant sources of systematic errors?

Finding the acceleration due to gravity and its uncertainty

Calculate your value of g_{exp} by substituting the average values of L and T into Eq. (2.12).

Calculate the uncertainty in g, by substituting for g, L, ΔL, T, and ΔT in Eq. (2.13).

Express your determination of g in the form $g = g_{\text{exp}} \pm \Delta g$

Compare your result with the "accepted value" for g. Be sure to give full reference information when you quote this "accepted value". Use correct and complete footnote procedure. If you have done all the measurements carefully and made all the calculations correctly, the "accepted value" should fall within your range.