Machine Learning 2 — Homework 4

Maurice Frank
11650656
maurice.frank@posteo.de

September 29, 2019

Problem 1.

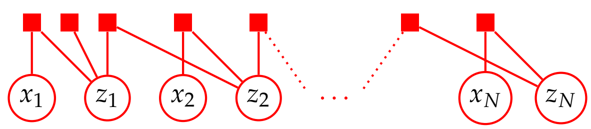
We have $\boldsymbol{X} = \{x_1, \ldots, x_N\}$ and $\boldsymbol{Z} = \{z_1, \ldots, z_N\}$.

1.

$$p(\boldsymbol{Z}, \boldsymbol{X}) = p(z_1) \cdot \left(\prod_{i=2}^{N} p(z_i | z_{i-1}) \right) \cdot \left(\prod_{i=1}^{N} p(x_i | z_i) \right)$$

2.

We present the factor graph for the Markov chain:



3.

$$p(\boldsymbol{X}) = f_1(z_1) \cdot \left(\prod_{i=2}^{N} f_i(z_i, z_{i-1}) \right) \cdot \left(\prod_{i=1}^{N} f_{N+1}(z_i, x_i) \right)$$

4.

$$
\begin{aligned}
p(z_n|\boldsymbol{X}) &= \frac{p(\boldsymbol{X}|z_n)p(z_n)}{p(\boldsymbol{X})} \\
&= \frac{p(x_1,\ldots,x_N|z_n)p(z_n)}{p(\boldsymbol{X})} \\
&= \frac{p(x_1,\ldots,x_n|z_n)p(x_{n+1},\ldots,x_N|z_n)p(z_n)}{p(\boldsymbol{X})} \qquad (1) \\
&= \frac{\alpha(z_n)\beta(z_n)}{p(\boldsymbol{X})} \\
&\implies \\
\alpha(z_n) &:= p(x_1,\ldots,x_n|z_n) \\
\beta(z_n) &:= p(x_{n+1},\ldots,x_N|z_n)p(z_n) \\
&= p(x_{n+1},\ldots,x_N,z_n)
\end{aligned}
$$

Step (1) is possible as z_n d-seperates the Markov chain at x_n, x_{n+1}.

$$
\begin{aligned}
\alpha(z_n) &= p(x_1,\ldots,x_n|z_n) \\
&= \sum_{z_{n-1}} p(x_1,\ldots,x_n|z_n,z_{n-1})p(z_{n-1}) \\
&= \sum_{z_{n-1}} \frac{p(x_1,\ldots,x_n,z_n|z_{n-1})}{p(z_n)}p(z_{n-1}) \\
&= \sum_{z_{n-1}} \frac{p(x_1,\ldots,x_{n-1}|z_{n-1})p(x_n,z_n|z_{n-1})}{p(z_n)}p(z_{n-1}) \qquad (2) \\
&= \sum_{z_{n-1}} p(x_1,\ldots,x_{n-1}|z_{n-1})p(x_n,z_{n-1}|z_n) \\
&= \sum_{z_{n-1}} \alpha(z_{n-1})p(x_n,z_{n-1}|z_n)
\end{aligned}
$$

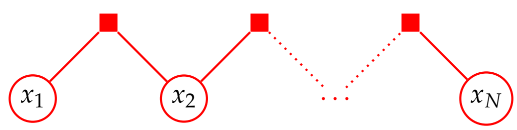
Again Step (2) is possible as z_{n-1} d-seperates $\{x_1,\ldots,x_n,z_n\}$ into $\{x_1,\ldots,x_{n-1}\}$ and $\{x_n,z_n\}$.

$$
\begin{aligned}
\beta(z_n) &= p(x_{n+1}, \ldots, x_N, z_n) \\
&= \sum_{z_n+1} p(x_{n+1}, \ldots, x_N, z_n | z_{n+1}) p(z_{n+1}) \\
&= \sum_{z_n+1} p(x_{n+2}, \ldots, x_N | z_{n+1}) p(x_{n+1}, z_n | z_{n+1}) p(z_{n+1}) \qquad (3) \\
&= \sum_{z_n+1} p(x_{n+2}, \ldots, x_N, z_{n+1}) p(x_{n+1}, z_n | z_{n+1}) \\
&= \sum_{z_n+1} \beta(z_{n+1}) p(x_{n+1}, z_n | z_{n+1})
\end{aligned}
$$

Again Step (3) is possible as z_{n+11} d-seperates $\{x_{n+1}, \ldots, x_N, z_n\}$ into $\{x_{n+2}, \ldots, x_N\}$ and $\{x_{n+1}, z_n\}$.

Problem 2.

First we build the implicit factor graph for the chain with the implicit factors $\{f_{1,2}, \ldots, f_{N-1,N}\}$:



1.

From the factor graph we start the message passing of the sum-product algorithm from the left end to the root x_n: Left side::

$$
\begin{aligned}
\mu_{x_1 \to \psi_{1,2}} &= 1 \\
\mu_{\psi_{1,2} \to x_2} &= \sum_{x_1} \psi_{1,2}(x_1, x_2) \\
\mu_{x_i \to \psi_{i,i+1}} &= \mu_{\psi_{i-1,i} \to x_i} \quad \forall 1 < i < n-1 \\
\mu_{\psi_{i,i+1} \to x_{i+1}} &= \sum_{x_i} \psi_{i,i+1}(x_i, x_{i+1}) \mu_{x_i \to \psi_{i,i+1}} \quad \forall 1 < i < n
\end{aligned}
$$

With that we define μ_α:

$$\begin{aligned}
\mu_\alpha(x_n) &:= \mu_{\psi_{n-1,n}\to x_n} \\
&= \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\mu_{x_{n-1}\to\psi_{n-1,n}} \\
&= \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\mu_{\psi_{n-2,n-1}\to x_{n-1}} \\
&= \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\mu_\alpha(x_{n-1})
\end{aligned}$$

Same we do from the right side:

$$\begin{aligned}
\mu_{x_N\to\psi_{N-1,N}} &= 1 \\
\mu_{\psi_{N-1,N}\to x_{N-1}} &= \sum_{x_N} \psi_{N-1,N}(x_{N-1}, x_N) \\
\mu_{x_i\to\psi_{i-1,i}} &= \mu_{\psi_{i+1,i}\to x_i} \quad \forall n+1 < i < N \\
\mu_{\psi_{i-1,i}\to x_{i-1}} &= \sum_{x_i} \psi_{i-1,i}(x_{i-1}, x_i)\mu_{x_i\to\psi_{i-1,i}} \quad \forall n < i < N
\end{aligned}$$

With that we define μ_β:

$$\begin{aligned}
\mu_\beta(x_n) &:= \mu_{\psi_{n,n+1}\to x_n} \\
&= \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\mu_{x_{n+1}\to\psi_{n,n+1}} \\
&= \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\mu_{\psi_{n+2,n+1}\to x_{n+1}} \\
&= \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\mu_\beta(x_{n+1})
\end{aligned}$$

Given the two neighbours of x_n we can write (Bishop 8.63):

$$\begin{aligned}
p(x_n) &= \frac{1}{Z} \cdot \mu_{\psi_{n-1,n}\to x_n} \cdot \mu_{\psi_{n,n+1}\to x_n} \\
&= \frac{1}{Z} \cdot \mu_\alpha(x_n) \cdot \mu_\beta(x_n)
\end{aligned}$$

The normalizing constant Z can be computed from the complete joint distribution.

2.

Problem 3.

When x_N is observed this will only affect the last potential $\psi_{N-1,N}(x_{N-1}, x_N)$. Thus we can extend this to only pass the observed value. Assuming we have a observed value of $x_N = \chi$ we can introduce a indicator function $\mathbb{1}[x_N = \chi]$ which one-hot encode the observed value:

$$\mathbb{1}[x_N = \chi] = \begin{cases} 1 & \text{if } x_N = \chi \\ 0 & \text{else} \end{cases}$$

Given that the last potential becomes:

$$\psi_{N-1,N}(x_{N-1}, x_N) \leftarrow \psi_{N-1,N}(x_{N-1}, x_N)\mathbb{1}[x_N = \chi]$$

and we can keep all message passes after that.

Problem 4.

The variables $\boldsymbol{x}_s = \{x_1, \dots, x_N\}$ are connected to factor f_s.

$$\begin{aligned} p(\boldsymbol{x}_s) &= f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \mu_{x_i \to f_s(x_i)} \\ &= f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \prod_{j \in \mathrm{ne}(x_i) \setminus f_s} F_j(x_i, \boldsymbol{X}_j) \end{aligned}$$

We get the marginal distribution by summing over $\boldsymbol{x}$ without $\boldsymbol{x}_s$ and get:

$$\begin{aligned} p(\boldsymbol{x}_s) &= \sum_{\boldsymbol{x} \setminus \boldsymbol{x}_s} p(\boldsymbol{x}) \\ &= \sum_{\boldsymbol{x} \setminus \boldsymbol{x}_s} f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \prod_{j \in \mathrm{ne}(x_i) \setminus f_s} F_j(x_i, \boldsymbol{X}_j) \\ &= f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \prod_{j \in \mathrm{ne}(x_i) \setminus f_s} \left(\sum_{\boldsymbol{X}_j} F_j(x_i, \boldsymbol{X}_j) \right) \\ &= f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \prod_{j \in \mathrm{ne}(x_i) \setminus f_s} \mu_{f_j \to x_i}(x_i) \\ &= f_s(\boldsymbol{x}_s) \prod_{i \in \mathrm{ne}(f_s)} \mu_{x_i \to f_s}(x_i) \end{aligned}$$