Gröbner Bases and Syzygies of Sequences of Related Ideals

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Introduction

Suppose we have a sequence of related ideals $I_0, I_1, I_2, \ldots$ living in a sequence of related polynomial rings $R_0, R_1, R_2, \ldots$ (i.e. $I_n \subseteq R_n$ $\forall n \geq 0$).

Simultaneously compute:

1. a finite Gröbner basis for each I_n, and
2. a finite Gröbner basis for the module of syzygies of each I_n.

P-Ideals

Notation: for any $n \in \mathbb{N}$ define $[n] = \{1, \ldots, n\}$ and set $[0] = \emptyset$.

Fix a field K and an integer $c > 0$. Let $\mathbf{P} : \mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \ldots$ be the sequence defined by

$$\mathbf{P}_n = K[x_{i,j} \;:\; i \in [c],\, j \in [n]] \quad \forall n \geq 0.$$

Example

If $c = 2$ then

$$\mathbf{P}_0 = K,\; \mathbf{P}_1 = K\begin{bmatrix} x_{1,1} \\ x_{2,1} \end{bmatrix},\; \mathbf{P}_2 = K\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix},\; \ldots$$

P-Ideals

There are maps $\mathbf{P}(\varepsilon) : \mathbf{P}_m \to \mathbf{P}_n$ for each strictly increasing function $\varepsilon : [m] \to [n]$. Specifically, $\mathbf{P}(\varepsilon)$ is induced by $x_{i,j} \mapsto x_{i,\varepsilon(j)}$. We often write ε_* instead of $\mathbf{P}(\varepsilon)$.

Definition

A sequence of ideals $\mathbf{I} : \mathbf{I}_0, \mathbf{I}_1, \mathbf{I}_2, \ldots$ with $\mathbf{I}_n \leq \mathbf{P}_n$ is called a **P**-*ideal* if $\mathbf{I}_0 = 0$ and $\varepsilon_*(\mathbf{I}_m) \subseteq \mathbf{I}_n$ for all $m \leq n$ and $\varepsilon : [m] \to [n]$.

Example

Let $c = 1$. Then the sequence $\mathbf{I}_n = \langle x_i x_j \ : \ 1 \leq i < j \leq n \rangle$ is a **P**-ideal.

Finite Generation of $\mathbf{P}$-Ideals

A *subset* of a $\mathbf{P}$-ideal $\mathbf{I}$ is a set $G \subseteq \bigsqcup_{n\geq 0} \mathbf{I}_n$. For all $n \geq 0$ we define the *n-orbit* of G to be the set

$$\operatorname{Orb}(G, n) = \{\varepsilon_*(g) \ : \ g \in G \cap \mathbf{I}_m, \ \varepsilon : [m] \to [n]\}.$$

Definition

A $\mathbf{P}$-ideal $\mathbf{I}$ is *finitely generated* if there is a finite subset $G \subset \mathbf{I}$ such that $\mathbf{I}_n = \langle \operatorname{Orb}(G, n) \rangle$ for all $n \geq 0$. In this case we write $\mathbf{I} = \langle G \rangle_{\mathbf{P}}$.

Example

The $\mathbf{P}$-ideal $\mathbf{I}_n = \langle x_i x_j \ : \ 1 \leq i < j \leq n \rangle$ is finitely generated by $\{x_1 x_2\}$.

Monomial Orders

A monomial in $\mathbf{P}$ is a monomial in some $\mathbf{P}_n$.

Definition

A total order $<$ on the monomials of $\mathbf{P}$ is called a *monomial order* on $\mathbf{P}$ if for all monomials $\mu, \nu \in \mathbf{P}_m$ with $\mu < \nu$ we have

1. $\mu < a\mu < a\nu$ for all monomials $1 \neq a \in \mathbf{P}_m$, and
2. $\mu < \varepsilon_*(\mu) < \varepsilon_*(\nu)$ for all $\varepsilon : [m] \to [n]$ with $m < n$.

Example

Give each $\mathbf{P}_n$ the lex order with $x_{i,j} < x_{i',j'}$ if $i < i'$ or $i = i'$ and $j < j'$. For any monomials $\mu \in \mathbf{P}_m$ and $\nu \in \mathbf{P}_n$ we declare $\mu < \nu$ if $m < n$ or $m = n$ and $\mu < \nu$ in $\mathbf{P}_n$.

Division with Remainder

Remark

Any monomial order $<$ on $\mathbf{P}$ restricts to a monomial order $<_n$ on $\mathbf{P}_n$ for all $n \geq 0$.

Definition

Let $f \in \mathbf{P}_n$ and let $G \subseteq \mathbf{P}$. A *remainder of f modulo G* (with respect to $<$) is defined to be a remainder of f modulo $\mathrm{Orb}(G, n)$ (with respect to $<_n$).

Remainders of f modulo G can be computed in finite time [CLO].

Gröbner Bases

Given a monomial order $<$ on $\mathbf{P}$ one defines $\mathrm{lm}(f) \in \mathbf{P}_n$, $\mathrm{lt}(f) \in \mathbf{P}_n$ and $\mathrm{lc}(f) \in K$ for any $f \in \mathbf{P}_n$.

Definition

Fix a monomial order $<$ on $\mathbf{P}$ and let $\mathbf{I}$ be a $\mathbf{P}$-ideal. A subset $G \subseteq \mathbf{I}$ is called a *Gröbner basis* for $\mathbf{I}$ (with respect to $<$) if

$$\langle \mathrm{lm}(\mathbf{I}) \rangle_{\mathbf{P}} = \langle \mathrm{lm}(G) \rangle_{\mathbf{P}}$$

where $\mathrm{lm}(\mathbf{I}) = \{\mathrm{lm}(f) \ : \ f \in \mathbf{I}\}$ and $\mathrm{lm}(G) = \{\mathrm{lm}(g) \ : \ g \in G\}$.

Remark

A set $G \subseteq \mathbf{I}$ forms a Gröbner basis for $\mathbf{I}$ if and only if $\mathrm{Orb}(G, n)$ forms a Gröbner basis for $\mathbf{I}_n$ with respect to $<_n$ for all $n \geq 0$.

S-Polynomials and Critical Pairs

Definition

The *S-polynomial* of $f, g \in \mathbf{P}_n$ is the combination

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g))}{\operatorname{lt}(f)} f - \frac{\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g))}{\operatorname{lt}(g)} g.$$

Definition

Let $B \subseteq \mathbf{P}$ and let $m \geq 0$. A tuple

$$(\sigma_*(f), \tau_*(g)) \in \operatorname{Orb}(B,m) \times \operatorname{Orb}(B,m)$$

is called a *critical pair* if $m = |\operatorname{im}(\sigma) \cup \operatorname{im}(\tau)|$. The set of all critical pairs of B is denoted $\mathcal{C}(B)$.

Important: the set $\mathcal{C}(B)$ is finite if B is finite.

Buchberger's Criterion

Theorem (M, Nagel)

A generating set G of a $\mathbf{P}$-ideal $\mathbf{I}$ forms a Gröbner basis for $\mathbf{I}$ if and only if each $S(f, g)$ with $(f, g) \in \mathcal{C}(G)$ has a remainder of zero modulo G.

Key idea: any S-polynomial $S(\sigma_*(f), \tau_*(g))$ can be written as $\rho_*(S(\overline{\sigma}_*(f), \overline{\tau}_*(g))$ where $(\overline{\sigma}_*(f), \overline{\tau}_*(g))$ is a critical pair.

Buchberger's Algorithm

Let $<$ be a monomial order on $\mathbf{P}$ and let $G \subset \mathbf{P}$ be a finite set.

1. Are there $(f, g) \in \mathcal{C}(G)$ such that $S(f, g)$ has a nonzero remainder modulo G?
2. If so, append the remainder to G and repeat.
3. Terminates by a Noetherianity argument (see [NR19]).
4. Computes a finite Gröbner basis for $\langle G \rangle_{\mathbf{P}}$.

Gröbner Basis Example

Let $c = 2$ and let $B = \{x_{2,1}^2 + x_{1,1} \in \mathbf{P}_1, x_{2,2} + x_{1,2}x_{1,1} \in \mathbf{P}_2\}$. Using Macaulay2, we can compute a Gröbner basis for $\langle B \rangle_{\mathbf{P}}$. It consists of the elements

$$\begin{aligned} x_{2,1}^2 + x_{1,1} &\in \mathbf{P}_1 \\ x_{2,2} + x_{1,2}x_{1,1} &\in \mathbf{P}_2 \\ x_{1,2}^2x_{1,1}^2 + x_{1,2} &\in \mathbf{P}_2 \\ x_{1,3}^2 + x_{1,3}x_{1,1} &\in \mathbf{P}_3 \\ x_{1,3}x_{1,2} - x_{1,3}x_{1,1} &\in \mathbf{P}_3 \\ x_{1,3}x_{1,1}^3 + x_{1,3} &\in \mathbf{P}_3 \end{aligned}$$

https://github.com/morrowmh/OIGroebnerBases

What is a Syzygy?

Example

Let $c = 1$ and consider the elements $x_1 \in \mathbf{P}_1$ and $x_2x_3 \in \mathbf{P}_3$. We have a relation

$$x_2\varepsilon_*(x_1) - x_2x_3 = x_2x_3 - x_2x_3 = 0 \in \mathbf{P}_3$$

where $\varepsilon : [1] \to [3]$ is the map given by $1 \mapsto 3$. Such a relation is an example of a *syzygy* on the elements x_1 and x_2x_3.

Syzygies of **P**-ideals have been studied in the context of free resolutions and complexes in [FN21] and [FN22].

How do we compute them?

$\mathbf{P}$-Modules

Definition

A sequence $\mathbf{M} : \mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \ldots$ of modules is a $\mathbf{P}$-*module* if

1. each $\mathbf{M}_n$ is a $\mathbf{P}_n$-module,
2. there is a K-linear map $\mathbf{M}(\varepsilon) : \mathbf{M}_m \to \mathbf{M}_n$ for all strictly increasing $\varepsilon : [m] \to [n]$, and
3. for all $a \in \mathbf{P}_m$ the diagram

$$\begin{tikzcd} \mathbf{M}_m \arrow[r, "\mathbf{M}(\varepsilon)"] \arrow[d, "a\cdot"'] & \mathbf{M}_n \arrow[d, "\mathbf{P}(\varepsilon)(a)\cdot"] \\ \mathbf{M}_m \arrow[r, "\mathbf{M}(\varepsilon)"] & \mathbf{M}_n \end{tikzcd}$$

commutes, where the vertical maps are multiplication by the indicated element.

$\mathbf{P}$-Modules

Definition

Let $\mathbf{M}$ be a $\mathbf{P}$-module. A $\mathbf{P}$-*submodule* of $\mathbf{M}$ is a sequence of modules $\mathbf{N} : \mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, \ldots$ such that each $\mathbf{N}_n$ is a $\mathbf{P}_n$-submodule of $\mathbf{M}_n$ and $\mathbf{M}(\varepsilon)(\mathbf{N}_m) \subseteq \mathbf{N}_n$ for all $\varepsilon : [m] \to [n]$.

Example

The $\mathbf{P}$-ideals are precisely the $\mathbf{P}$-submodules of $\mathbf{P}$.

P-Linear Maps

Definition

Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{P}$-modules. A $\mathbf{P}$-*linear map* $\varphi : \mathbf{M} \to \mathbf{N}$ is a collection of $\mathbf{P}_n$-linear maps $\varphi_n : \mathbf{M}_n \to \mathbf{N}_n$ such that the diagram

$$\begin{tikzcd} \mathbf{M}_m \arrow[r, "\varphi_m"] \arrow[d, "\mathbf{M}(\varepsilon)"'] & \mathbf{N}_m \arrow[d, "\mathbf{N}(\varepsilon)"] \\ \mathbf{M}_n \arrow[r, "\varphi_n"] & \mathbf{N}_n \end{tikzcd}$$

commutes for all $\varepsilon : [m] \to [n]$.

Definition

The *kernel* of a $\mathbf{P}$-linear map $\varphi : \mathbf{M} \to \mathbf{N}$ is the $\mathbf{P}$-submodule of $\mathbf{M}$ given by $\ker(\varphi)_n = \ker(\varphi_n)$ for all $n \geq 0$.

Free $\mathbf{P}$-Modules

Definition

Fix integers $d_1, \ldots, d_s \geq 0$ and define the sequence $\mathbf{F} : \mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2$ via

$$\mathbf{F}_n = \bigoplus_{\substack{1 \leq i \leq s \\ \pi : [d_i] \to [n]}} \mathbf{P}_n e_{\pi,i}$$

with maps $\mathbf{F}(\varepsilon) : \mathbf{F}_m \to \mathbf{F}_n$ induced by $e_{\pi,i} \mapsto e_{\varepsilon \circ \pi, i}$. We call $\mathbf{F}$ a *free* $\mathbf{P}$*-module with basis* $\{e_{\mathrm{id}_{[d_i]},i}\}$.

Example

Let $\mathbf{F}$ have basis $\{e_{\mathrm{id}_{[2]}}\}$. Then $\mathbf{F}_n$ is a free $\mathbf{P}_n$-module of rank $\binom{n}{2}$ for all $n \geq 0$.

Remark

Any $\varphi : \mathbf{F} \to \mathbf{M}$ is determined by where the $e_{\mathrm{id}_{[d_i]},i}$ are sent.

P-Module Concepts

The following concepts for **P**-ideals generalize to **P**-modules:

- Finite generation
- Monomial orders
- Division with remainder
- S-polynomials
- Gröbner bases

Computing Syzygies

Let $B = \{b_1, \ldots, b_s\} \subset \mathbf{P}$ be a Gröbner basis with $b_i \in \mathbf{P}_{d_i}$ and let $\mathbf{F}$ be a free $\mathbf{P}$-module with basis $\{e_{\mathrm{id}_{[d_i]},i}\}$.

Let $\varphi : \mathbf{F} \to \mathbf{P}$ be the map induced by $e_{\mathrm{id}_{[d_i]},i} \mapsto b_i$. We wish to compute a finite Gröbner basis for the syzygy module of $\langle B \rangle_{\mathbf{P}}$, i.e. $\mathrm{Syz}(B) := \ker(\varphi)$.

Computing Syzygies

Definition

For any $i, j \in [s]$, $\sigma : [d_i] \to [m]$ and $\tau : [d_j] \to [m]$ with $m \geq \max(d_i, d_j)$ we can use the division algorithm to write

$$S(\sigma_*(b_i), \tau_*(b_j)) = \sum_{\ell} a_{i,j,\ell}^{\sigma,\tau} \mathbf{P}(\pi_{i,j,\ell}^{\sigma,\tau})(b_{k_{i,j,\ell}^{\sigma,\tau}})$$

for some $a_{i,j,\ell}^{\sigma,\tau} \in \mathbf{P}_m$ and $\mathbf{P}(\pi_{i,j,\ell}^{\sigma,\tau})(b_{k_{i,j,\ell}^{\sigma,\tau}}) \in \mathrm{Orb}(B, m)$. Define

$$s_{i,j}^{\sigma,\tau} = m_{i,j}^{\sigma,\tau} e_{\sigma,i} - m_{i,j}^{\tau,\sigma} e_{\tau,j} - \sum_{\ell} a_{i,j,\ell}^{\sigma,\tau} e_{\pi_{i,j,\ell}^{\sigma,\tau}, k_{i,j,\ell}^{\sigma,\tau}} \in \mathbf{F}_m$$

where

$$m_{i,j}^{\sigma,\tau} = \frac{\mathrm{lcm}(\sigma_*(\mathrm{lm}(b_i)), \tau_*(\mathrm{lm}(b_j)))}{\sigma_*(\mathrm{lm}(b_i))} \in \mathbf{P}_m.$$

Computing Syzygies: Schreyer's Theorem

Theorem (M, Nagel)

There exists a monomial order on $\mathbf{F}$ *with respect to which the* $s_{i,j}^{\sigma,\tau}$ *with* $(\sigma_*(b_i), \tau_*(b_j)) \in \mathcal{C}(B)$ *form a finite Gröbner basis for* $\mathrm{Syz}(B)$.

For the classical version of this theorem, see [E95, Chapter 15].

Remark

If G forms a Gröbner basis for $\mathrm{Syz}(B)$ then $\mathrm{Orb}(G, n)$ forms a Gröbner basis for $\mathrm{Syz}(B)_n$ for all $n \geq 0$.

Syzygy Example

Let $\mathbf{F}$ be the free $\mathbf{P}$-module with basis $\{e_{\mathrm{id}_{[2]}}\}$ and let $B = \{x_{1,1}x_{2,2} - x_{1,2}x_{2,1} \in \mathbf{P}_2\}$ so that $(\langle B\rangle_{\mathbf{P}})_n$ is the ideal of $\mathbf{P}_n$ generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \end{bmatrix}.$$

Using Macaulay2 we compute a Gröbner basis for $\mathrm{Syz}(B)$:

$$x_{1,2}e_{13} - x_{1,1}e_{23} - x_{1,3}e_{12} \in \mathbf{F}_3$$
$$-x_{2,3}e_{12} + x_{2,2}e_{13} - x_{2,1}e_{23} \in \mathbf{F}_3.$$

Note: $e_{ij} \leftrightsquigarrow e_\pi$ where $\pi : [2] \to [3]$ is given by $1 \mapsto i$ and $2 \mapsto j$.

https://github.com/morrowmh/OIGroebnerBases

References

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