Computing Gröbner Bases and Free Resolutions for OI-Modules

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Introduction

Fix a field K, and consider a sequence of related ideals I_n defined over a sequence of related polynomial rings P_n, e.g.

$$I_n = \langle x_i x_j \mid 1 \leq i < j \leq n \rangle \subset K[x_1, \ldots, x_n] = P_n.$$

Given such a sequence, we wish to simultaneously compute (in finite time) a free resolution of each I_n. Of course, in order to talk about free resolutions, one must introduce modules. Thus, our general process is as follows.

The Process:

1. Given a sequence of related modules M_n over a sequence of related polynomial rings P_n, express every M_n in terms of $M_0, \ldots, M_\ell$ for some fixed $\ell \geq 0$.
2. Using the generators of $M_0, \ldots, M_\ell$ and an analog of Buchberger's Algorithm, compute finite Gröbner bases $G_0, \ldots, G_w$ for $M_0, \ldots, M_w$ for some $w \geq \ell$. Every M_n will then have a finite Gröbner basis G_n expressible in terms of the G_i with $0 \leq i \leq w$.
3. Using $G_0, \ldots, G_w$ and an analog of Schreyer's Theorem, compute finite Gröbner bases $G'_0, \ldots, G'_{w'}$ for the modules of syzygies of $M_0, \ldots, M_{w'}$ for some $w' \geq 0$. Each $\text{Syz}(M_n)$ will then have a finite Gröbner basis G'_n expressible in terms of the $G'_0, \ldots, G'_{w'}$. Repeat this step to simultaneously compute free resolutions of each M_n out to desired homological degree.

Running Example

From now on, let $\mathbf{P} = (\mathbf{P}_n)_{n\geq 0}$ be the sequence of rings defined by

$$\mathbf{P}_n = K\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \end{bmatrix}$$

and let $\mathbf{F} = (\mathbf{F}_n)_{n\geq 0}$ be the sequence of free modules defined by

$$\mathbf{F}_n = \bigoplus_{i=1}^{n} \mathbf{P}_n e_{n,i} \cong \mathbf{P}_n^n.$$

Define a sequence of submodules $\mathbf{M} = (\mathbf{M}_n)_{n\geq 0}$ by

$$\mathbf{M}_n = \langle x_{1,j} x_{1,i} e_{n,j} + x_{2,j} x_{2,i} e_{n,i} \mid 1 \leq i < j \leq n \rangle \subset F_n.$$

For example, we have

$$\mathbf{M}_2 = \langle x_{1,2} x_{1,1} e_{2,2} + x_{2,2} x_{2,1} e_{2,1} \rangle \subset \mathbf{P}_2^2.$$

We will simultaneously compute free resolutions of each $\mathbf{M}_n$.

Background

- The sequence $\mathbf{F}$ is an example of a *free* OI-*module* over the OI-*algebra* $\mathbf{P}$, and the sequence $\mathbf{M}$ is an OI-*submodule* of $\mathbf{F}$ (see [1, 2]).
- OI denotes the category whose objects are intervals $[n]$ and whose morphisms are strictly increasing maps $[m] \to [n]$.
- Given any OI-algebra $\mathbf{A}$, there is an abelian category of OI-modules over $\mathbf{A}$. This provides the framework for our theory.

Step 1: Finding Generating Sets

Denote by $\text{Hom}(m, n)$ the set of all strictly increasing maps $[m] \to [n]$. For any $\varepsilon \in \text{Hom}(m, n)$ define the ring map

$$\varepsilon_* : \mathbf{P}_m \to \mathbf{P}_n \quad \text{via} \quad x_{i,j} \mapsto x_{i,\varepsilon(j)}$$

and define the K-linear map

$$\mathbf{F}(\varepsilon) : \mathbf{F}_m \to \mathbf{F}_n \quad \text{via} \quad f e_{m,i} \mapsto \varepsilon_*(f) e_{n,\varepsilon(i)}.$$

Now for $n \geq 2$, observe that

$$\mathbf{M}_n = \langle \mathbf{F}(\varepsilon)(f) \mid \varepsilon \in \text{Hom}(2, n) \rangle$$

where $f = x_{1,2} x_{1,1} e_{2,2} + x_{2,2} x_{2,1} e_{2,1} \in \mathbf{M}_2$. Thus, the element f *generates* the OI-module $\mathbf{M}$.

Step 2: Computing Gröbner Bases

Our goal is to simultaneously compute finite Gröbner bases for each $\mathbf{M}_n$. If we were to apply the classical Buchberger's Algorithm to each $\mathbf{M}_n$, this would require infinitely many calculations. However, now that we have a generator for $\mathbf{M}$ as an OI-module, we can express every S-polynomial for $\mathbf{M}_n$ in terms of finitely many S-polynomials.

Key Idea:

Consider an S-polynomial for $\mathbf{M}_n$, i.e. $S(\mathbf{F}(\sigma)(f), \mathbf{F}(\tau)(f))$ where $\sigma, \tau \in \text{Hom}(2, n)$. Then by the OI-Factorization Lemma [2] there are maps $\overline{\sigma}, \overline{\tau} \in \text{Hom}(2, k)$ and $\rho \in \text{Hom}(k, n)$ such that

$$\mathbf{F}(\rho)(S(\mathbf{F}(\overline{\sigma})(f), \mathbf{F}(\overline{\tau})(f))) = S(\mathbf{F}(\sigma)(f), \mathbf{F}(\tau)(f))$$

and $2 \leq k \leq 4$. Thus, all higher S-polynomials can be expressed in terms of finitely many S-polynomials, and an OI-analog of Buchberger's Criterion exists and can be checked in *finite time*. This gives an OI-analog of Buchberger's Algorithm [2] which terminates in finite time since $\mathbf{P}$ is a Noetherian OI-algebra [1].

Gröbner Basis Computation

Using the `OIGroebnerBases.m2` script for Macaulay2 [3], we compute finite Gröbner bases as follows:

```
i1: P = makePolynomialOIAlgebra(2, x, QQ);
i2: F = makeFreeOIModule(e, {1}, P);
i3: installBasisElements(F, 2);
i4: f = x_(1,2)*x_(1,1)*e_(2,{2},1)+x_(2,2)*x_(2,1)*e_(2,{1},1);
i5: oiGB {f}
```

Step 2: Continued

The Macaulay2 code above produces Gröbner bases $G_2 = \{f\} \subset \mathbf{M}_2$ and

$$\begin{aligned} G_3 = &\{(x_{2,3} x_{2,2}^2 x_{2,1} + x_{2,3} x_{2,1} x_{1,2}^2) e_{3,1},\ x_{2,3} x_{2,2} x_{1,1} e_{3,2} - x_{2,3} x_{2,1} x_{1,2} e_{3,1}\} \\ &\cup \{\mathbf{F}(\varepsilon)(f) \mid \varepsilon \in \text{Hom}(2, 3)\} \subset \mathbf{M}_3 \end{aligned}$$

so that each $\mathbf{M}_n$ has finite Gröbner basis

$$G_n = \{\mathbf{F}(\varepsilon)(g) \mid g \in G_2, \varepsilon \in \text{Hom}(2, n)\} \cup \{\mathbf{F}(\varepsilon)(g) \mid g \in G_3, \varepsilon \in \text{Hom}(3, n)\}.$$

Step 3: Computing Free Resolutions

- Let $\mathbf{G} = (\mathbf{G}_n)_{n\geq 0}$ be a free OI-module equipped with surjective maps $\varphi_n : \mathbf{G}_n \to \mathbf{M}_n$ sending the generators of $\mathbf{G}_n$ to the elements of the Gröbner bases G_n so that $\ker(\varphi_n) = \text{Syz}(\mathbf{M}_n)$.
- The OI-analog of Schreyer's Theorem [2] constructs finite Gröbner bases $G'_0, \ldots, G'_{w'}$ in terms of G_2 and G_3 for $\text{Syz}(\mathbf{M}_0), \ldots, \text{Syz}(\mathbf{M}_{w'})$ and some $w' \geq 0$. Moreover, each $\text{Syz}(\mathbf{M}_n)$ has a finite Gröbner basis expressible in terms of the $G'_0, \ldots, G'_{w'}$.
- Iterating this process, one is able to simultaneously compute free resolutions of each $\mathbf{M}_n$.

Free Resolution Computation

Using the same Macaulay2 session from earlier, we compute free resolutions as follows:

```
i6: oiRes({f}, 6)
```

Step 3: Continued

The Macaulay2 code above simultaneously computes resolutions of each $\mathbf{M}_n$ out to homological degree 6. As an example, we display the resolutions of $\mathbf{M}_i$ for $4 \leq i \leq 8$ below.

$$0 \to \mathbf{P}_4^2 \to \mathbf{P}_4^6 \to \mathbf{M}_4 \to 0$$

$$0 \to \mathbf{P}_5^5 \to \mathbf{P}_5^{10} \to \mathbf{P}_5^{10} \to \mathbf{M}_5 \to 0$$

$$0 \to \mathbf{P}_6^9 \to \mathbf{P}_6^{30} \to \mathbf{P}_6^{30} \to \mathbf{P}_6^{15} \to \mathbf{M}_6 \to 0$$

$$0 \to \mathbf{P}_7^{14} \to \mathbf{P}_7^{63} \to \mathbf{P}_7^{105} \to \mathbf{P}_7^{70} \to \mathbf{P}_7^{21} \to \mathbf{M}_7 \to 0$$

$$0 \to \mathbf{P}_8^{20} \to \mathbf{P}_8^{112} \to \mathbf{P}_8^{252} \to \mathbf{P}_8^{280} \to \mathbf{P}_8^{140} \to \mathbf{P}_8^{28} \to \mathbf{M}_8 \to 0$$

Background

The code `oiRes({f}, 6)` computes a free resolution of $\mathbf{M}$ as an OI-module over $\mathbf{P}$ out to homological degree 6 whose ranks are minimal out to homological degree 5. By restricting this resolution "width-wise", we obtain resolutions for each $\mathbf{M}_n$. The restricted resolutions displayed above are minimal, but this need not be the case in general.

References

[1] U. Nagel, T. Römer, *FI- and OI-modules with varying coefficients*, J. Algebra **535** (2019), 286-322.

[2] M. Morrow, U. Nagel, *Computing Gröbner Bases and Free Resolutions of OI-Modules*, Preprint, arXiv:2303.06725, 2023.

[3] M. Morrow, *OIGroebnerBases.m2*, Macaulay2 package; available at https://github.com/morrowmh/OIGroebnerBases.