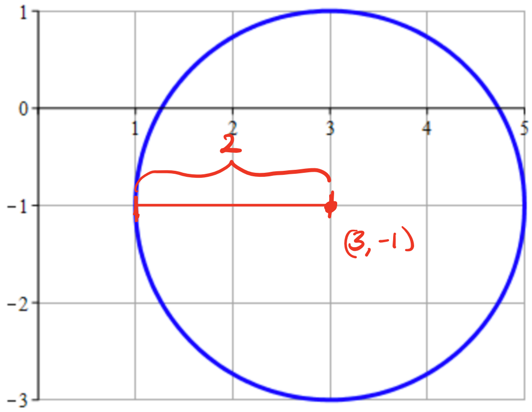
Conic Sections

Ex1.

a) State the standard equation of the circle.



The equation is given by

$$(x-3)^2+(y+1)^2=2^2$$

b) A circle has the equation

$$x^2+y^2+8x-6y=0.$$

Bring it to standard form and state center and radius.

We just complete the squares.

$$x^2+y^2+8x-6y=0 \iff (x+4)^2-16+(y-3)^2-9=0$$

$$\iff (x+4)^2+(y-3)^2=5^2$$

So $C=(-4,3)$ and $r=5$.

To be clear we just use that

$$(a+b)^2=a^2+b^2+2ab,$$

where $2ab$ in our case is

$$\left.\begin{aligned} 8x &= 2\cdot 4\cdot x \\ -6y &= 2\cdot(-3)\cdot y \end{aligned}\right\}$$ Hence 4 and -3 are obvious choices.

c) A sphere has the equation

$$x^2+y^2+z^2-2x+4y-6z+13=0.$$

Bring it to standard form and state center and radius.

Applying the same logic as above we have

$$x^2+y^2+z^2-2x+4y-6z+13=0$$

$$\iff (x-1)^2-1+(y+2)^2-4+(z-3)^2-9+13=0$$

$$\Rightarrow (x-1)^2+(y+2)^2+(z-3)^2=1^2.$$

Therefore $C=(1,-2,3)$ and $r=1$.

Ex 2.

a) An ellipse is given by the equation

$$4x^2+y^2+8x-6y+9=0.$$

Complete the square, bring to standard form, and state C and semi axes and axes of symmetry.

$$4x^2+y^2+8x-6y+9=0$$

$$\iff 4(x^2+2x)+(y-3)^2-9+9=0$$

$$\iff 4\left((x+1)^2-1\right)+(y-3)^2=0$$

$$\iff 4(x+1)^2+(y-3)^2=4$$

$$\iff \frac{(x+1)^2}{1^2}+\frac{(y-3)^2}{2^2}=1$$

This amounts to the ellipse with $C = (-1, 3)$, which is symmetric about $x = -1$ and $y = 3$. The semi axes are

$$a = 1 \quad \text{and} \quad b = 2$$

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> implicitplot(4*x^2+y^2+8*x-6*y+9=0,x=-3..1,y=0..6,scaling=constrained);
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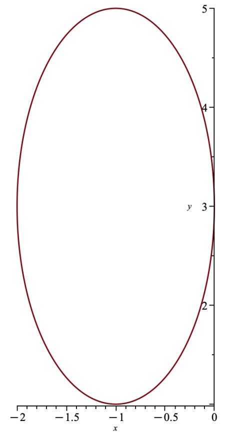

b) A hyperbola has the equation

$$x^2 - y^2 - 4x - 4y = 4.$$

Complete the square, bring to standard form, and state C and semi axes and axes of symmetry.

$$x^2 - y^2 - 4x - 4y = 4 \iff (x-2)^2 - 4 - (y+2)^2 + 4 = 4$$

$$\iff \frac{(x-2)^2}{2^2} - \frac{(y+2)^2}{2^2} = 1.$$

We have $C = (2, -2)$ and symmetry axes $x = 2$ and $y = -2$.

The semi axes are

$$a = 2 \quad \text{and} \quad b = 2.$$

```
> implicitplot(x^2-y^2-4*x-4*y=4,x=-3..7,y=-6..3,scaling=constrained);
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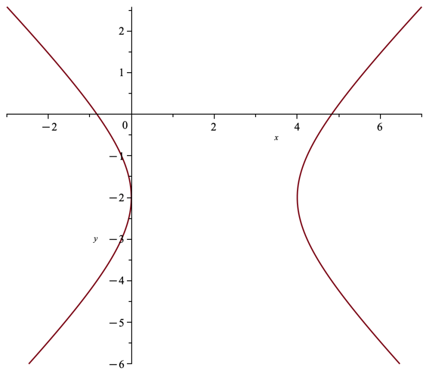


c) A parabola is given by

$$2x^2 + 12x - y + 17 = 0.$$

Complete the square, bring to standard form, and state T (top point) and axes of symmetry.

$$2x^2 + 12x - y + 17 = 0 \iff 2\left((x+3)^2 - 9\right) = y - 17$$

$$\iff y + 1 = 2(x+3)^2.$$

So $T = (-3,-1)$ and the parabola is symmetric about $x = -3$.

Ex 3. We're given the equation

$$9x^2 + 16y^2 - 24xy - 40x - 30y + 250 = 0.$$

a) State $k(x,y)$, i.e. the quadratic form, and determine its Hessian matrix.

$$k(x,y) = 9x^2 + 16y^2 - 24xy \Rightarrow \underline{\underline{H}} = \begin{bmatrix} 18 & -24 \\ -24 & 32 \end{bmatrix}.$$

b) Determine $\underline{\underline{A}}$ such that

$$k(x,y) = \begin{bmatrix} x & y \end{bmatrix} \underline{\underline{A}} \begin{bmatrix} x \\ y \end{bmatrix},$$

and find $\underline{\underline{Q}}$ and $\underline{\underline{\Lambda}}$ for which $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}}$.

Immedictely have

$$\underline{\underline{A}} = \frac{1}{2} \underline{\underline{H}} = \begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}.$$

Since $\underline{\underline{A}}$ is symmetric there exist a diagonalization by eigendecomposition.

```
> A:=<9,-12; -12,16>:
  ev:=Eigenvectors(A,output=list);
```

$$ev := \left[\left[0, 1, \left\{\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}\right\}\right], \left[25, 1, \left\{\begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}\right\}\right]\right]$$

```
> 1/Norm(ev[1,3,1],2)*ev[1,3,1],
  1/Norm(ev[2,3,1],2)*ev[2,3,1];
```

$$\begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

Let's choose

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}, \text{ so that } \underline{\underline{Q}} = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \quad (\underline{\underline{Q}} \text{ positive}).$$

c) Reduce k.

We just get $k(\tilde{x},\tilde{y}) = 25\tilde{x}^2$ with respect to this change of basis.

d) State the new ONB and determine a new equation for the conic section.

From $\underline{\underline{Q}}$ we have the basis given by $\left(\begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}\right)$.

We compute the linear terms and add the constant term to $k(\tilde{x},\tilde{y})$ in c).

$$[-40 \quad -30]\, \underline{\underline{Q}} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = [-40 \quad -30] \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

$$= [0 \quad -50] \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

$$= -50\, \tilde{y}.$$

We have

$$25\,\tilde{x}^2 - 50\,\tilde{y} + 250 = 0$$

e) Which type of conic section is this? Characterize and plot.

$$25\,\tilde{x}^2 - 50\,\tilde{y} + 250 = 0 \iff \tilde{y} = \frac{1}{2}\tilde{x}^2 + 5$$

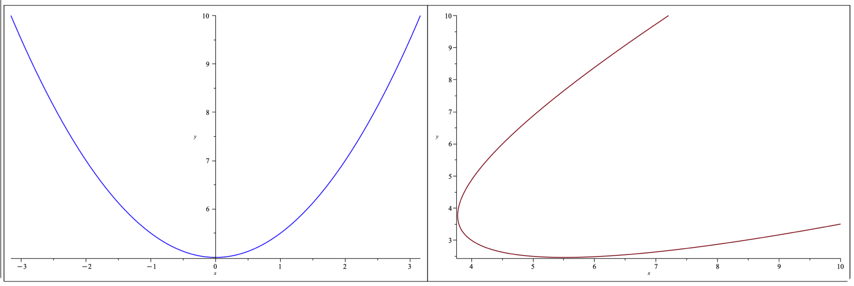
A parabola symmetric about $\tilde{x} = 0$ and $T = (0,5)$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{Q}} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \underline{\underline{Q}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}.$$

Thus in original coordinates $T = (4,3)$ and the axis of symmetry is given by the line

$$r(t) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}, \quad t \in \mathbb{R}.$$

```
> A:=implicitplot(y=1/2*x^2+5,x=-5..5,y=0..10,color=blue):
  B:=implicitplot(9*x^2+16*y^2-24*x*y-40*x-30*y+250=0,x=0..10,y=0..10):
  display(<A|B>);
```



Ex4. A curve is given by the equation
a)
$$52x^2 + 73y^2 - 72xy - 200x - 150y + 525 = 0.$$

Describe type and position. Provide parametric representation of axes of symmetry.

$$k(x,y) = 52x^2 + 73y^2 - 72xy$$

$$\Rightarrow \underline{\underline{H}} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix} \quad \text{and} \quad \underline{\underline{A}} = \begin{bmatrix} 52 & -36 \\ -36 & 73 \end{bmatrix}.$$

We get
$$\underline{\underline{\Lambda}} = \begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix} \quad \text{and} \quad \underline{\underline{Q}} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

Now
$$k(\tilde{x}, \tilde{y}) = 100\tilde{x}^2 + 25\tilde{y}^2.$$

Linear terms:
$$[-200 \quad -150] \, \underline{\underline{Q}} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = -250\tilde{y}.$$

This leaves us with a conic section in the ONB given by $\underline{\underline{Q}}$ as follows

$$100\tilde{x}^2 + 25\tilde{y}^2 - 250\tilde{y} + 525 = 0$$

$$\Leftrightarrow 100\tilde{x}^2 + 25\left((\tilde{y}-5)^2 - 25\right) = -525$$

$$\Leftrightarrow 100\tilde{x}^2 + 25(\tilde{y}-5)^2 = 100$$

$$\Leftrightarrow \frac{\tilde{x}^2}{1^2} + \frac{(\tilde{y}-5)^2}{2^2} = 1.$$

This is an ellipse with $C = (0,5)$ and symmetry axes are $\tilde{x} = 0$ and $\tilde{y} = 5$. The semi axes are $a = 1$ and $b = 2$.

Let's translate to the original ONB.

$$\underline{\underline{Q}}\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}, \quad \underline{\underline{Q}}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-3/5\\4/5\end{bmatrix} \text{ and } \underline{\underline{Q}}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4/5\\3/5\end{bmatrix}.$$

This gives us an ellipse with the same semi axes and $C=(4,3)$. The axes of symmetry in direction a and b are

$$r_a(t) = \begin{bmatrix}4\\3\end{bmatrix} + t\begin{bmatrix}-3/5\\4/5\end{bmatrix}, \quad t \in \mathbb{R} \quad \text{and}$$

$$r_b(s) = \begin{bmatrix}4\\3\end{bmatrix} + s\begin{bmatrix}4/5\\3/5\end{bmatrix}, \quad s \in \mathbb{R}.$$

b) Plot the conic sections. Compare with level sets of

$$f(x,y) = 52x^2 + 73y^2 - 72xy - 200x - 150y + 525.$$

```
> f:=(x,y)-> 52*x^2+73*y^2-72*x*y-200*x-150*y+525:f(x,y);
```

$$52x^2 - 72xy + 73y^2 - 200x - 150y + 525$$

```
> A:=implicitplot(100*x^2+25*y^2-250*y+525=0,x=-3..3,y=0..10,scaling=constrained,color=blue):
  B:=implicitplot(f(x,y)=0,x=0..10,y=0..10,scaling=constrained):
  ra:=plot([4-3/5*t,3+4/5*t,t=-5..5],color=red):
  rb:=plot([4+4/5*s,3+3/5*s,s=-5..5],color=blue):
  H:=display(A,B,ra,rb):
> K:=contourplot(f(x,y),x=-3..10,y=-1..10,contours=[0,100,200,300,400,500,600,700,900,1000,2000],view=[-1..8,-1..7],coloring=[blue,green],scaling=constrained,filled=true,
  transparency=0.5):
> display(<H|K>);
```