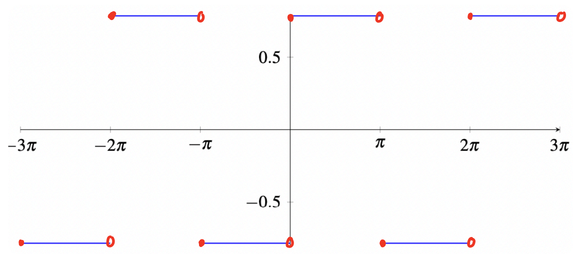
u.11. 1. Lat u vera tann 2π-periodiska funktiónin $u(t) = \begin{cases} \frac{\pi}{4}, & t \in [0,\pi[, \\ -\frac{\pi}{4}, & t \in [\pi, 2\pi[\end{cases}$



(i) Finn effektina hjá u.

Per definitión 7.10 er effektin givin við

$$P(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(t)|^2\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\pi}{4}\right)^2 dt = \frac{1}{2\pi}\left[\frac{\pi^2}{16}t\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3}{16} - \left(-\frac{\pi^3}{16}\right)\right) = \frac{\pi^2}{16}.$$

(ii) Finn Fourierrekkjuna hjá u.

Legg til merkis, at u er ólika, so $a_n = 0 \ \forall n \in \mathbb{N}_0$. Vit hava nú, at

$$b_n = \frac{2}{\pi}\int_0^{\pi} u(t)\sin(nt)\, dt = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\sin(nt)\, dt = \frac{1}{2}\left[-\frac{1}{n}\cos(nt)\right]_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{-(-1)^n + 1}{n}\right) = \frac{1-(-1)^n}{2n}.$$

$$\therefore\ u \sim \sum_{n=1}^{\infty} \frac{1-(-1)^n}{2n}\sin(nt).$$

(iii) Hvussu stórur partur av effektini hjá u er í $S_3(t)$?

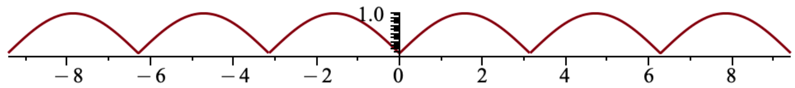
Vit brúka lemma 7.11 og Parsevals setning.

$$P(S_3) = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{3}\left(|a_n|^2 + |b_n|^2\right) = \frac{1}{2}\left(1^2 + \left(\frac{1}{3}\right)^2\right) = \frac{5}{9}.$$

Vit fáa nú lutfallið

$$\frac{P(S_3)}{P(u)} = \frac{5}{9}\cdot\frac{16}{\pi^2} = \frac{80}{9\pi^2} = 0{,}9006.$$

2. (i) Finn Fourierrekkjuna hjá $f(x) = |\sin(x)|$.



Funktiónin er lika, so $b_n = 0 \quad \forall n \in \mathbb{N}$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{2}{\pi}(1+1) = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{\cos((n-1)x)}{n-1} - \frac{\cos((n+1)x)}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} - \frac{1-(-1)^{n-1}}{n-1}\right) = \frac{2\left((-1)^{n+1}-1\right)}{(n^2-1)\pi} = \begin{cases} 0, & n = 2k-1 \\ \dfrac{4}{(n^2-1)\pi}, & n = 2k \end{cases}$$

$$\therefore \quad f \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2nx)$$

(ii) Vís, at f er eins við sína Fourierrekkju.

Funktiónin er 2π-periodisk, stykkivís differentiabul og kontinuert, so per korollar 6.13 konvergerar Fourierrekkjan ímóti $f(x)$ fyri øll $x \in \mathbb{R}$.

(iii) Rokna $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

Fyri $x = 0$ er $f(0) = 0$, og við (ii) fáa vit, at

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2n\cdot 0)$$

$$\Leftrightarrow \quad -\frac{2}{\pi}\cdot\left(-\frac{\pi}{4}\right) = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

(iv) Nýt Parsevals setning til at rokna $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin^2(x)\,dx = \frac{1}{2\pi}\Big[x - \sin(x)\cos(x)\Big]_0^{\pi} = \frac{1}{2}.$$

$$\Rightarrow \quad \frac{1}{4}\left(\frac{4}{\pi}\right)^2 + \frac{1}{2}\left(\frac{4}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{1}{2} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\frac{\pi^2}{8} = \frac{\pi^2}{16} - \frac{1}{2}.$$

(v) Skriva Fourierrekkjuna á kompleksan form.

Við lemma 6.22:

$$c_0 = \frac{1}{2} a_0 = \frac{2}{\pi}$$

$$c_{\pm n} = 0 \quad \text{fyri} \quad n = 2k-1 \quad \text{við} \quad k \in \mathbb{Z}.$$

$$c_{\pm n} = \frac{1}{2}(a_n \mp i b_n) = \frac{1}{2} \frac{2\left((-1)^{n+1} - 1\right)}{(n^2-1)\pi} = \frac{(-1)^{n+1} - 1}{(n^2-1)\pi} = \frac{-2}{(n^2-1)\pi}, \quad n = 2k.$$

So
$$f(x) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{4n^2-1} e^{2inx}.$$

(vi) Finn $N \in \mathbb{N}$, so at $|f(x) - S_N(x)| \leq 0{,}1$ fyri øll $x \in \mathbb{R}$.

Vit fáa við 6.17, at

$$|f(x) - S_{2N-2}(x)| \leq \sum_{n=2N-1}^{\infty} (|a_n| + |b_n|) \leq \frac{4}{\pi} \sum_{n=N}^{\infty} \frac{1}{4n^2-1}, \quad \forall x \in \mathbb{R}.$$

Við (iii) hava vit, at

$$\sum_{n=4}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} - \frac{1}{3} - \frac{1}{15} - \frac{1}{35}$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{15} - \frac{1}{35} =$$

$$= \frac{15}{210} = \frac{1}{14} < 0{,}1.$$

Tá $N \geq 2 \cdot 4 - 2 = 6$, so er frávikið undir $0{,}1$ $\forall x \in \mathbb{R}$

(vii) Rokna, hvussu stóran part av effektini hjá f er í S_4.

Við lemma 7.11 fáa vit, at

$$\sum_{n=-4}^{4} |c_n|^2 = \left(\frac{2}{\pi}\right)^2 + 2\left(\frac{2}{3\pi}\right)^2 + 2\left(\frac{2}{15\pi}\right)^2 = \frac{1108}{225\pi^2} = 0{,}49895$$

Lutfallið verður tí
$$\frac{0{,}49895}{0{,}5} = 0{,}9979 \quad \text{ella} \quad 99{,}79\,\%.$$

3. Ein funktión f hevur perioduna $T = \frac{2\pi}{\omega}$ og er givin við

$$f(t) = \begin{cases} t, & 0 \le t < \frac{T}{2}, \\ 0, & \frac{T}{2} \le t < T. \end{cases}$$

(i) Finn effektina hjá f.

$$\frac{1}{T}\int_0^T |f(t)|^2\,dt = \frac{1}{T}\int_0^{\frac{T}{2}} t^2\,dt = \frac{1}{3T}\left[t^3\right]_0^{T/2} = \frac{T^2}{24} = \frac{\pi^2}{6\omega^2}.$$

(ii) Finn reellu Fourierkoefficientarnar.

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{2T}\left[t^2\right]_0^{T/2} = \frac{T}{4} = \frac{\pi}{2\omega}.$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cdot\cos(n\omega t)dt = \frac{2}{T}\int_0^{\frac{T}{2}} t\cos(n\omega t)\,dt$$

$$= \frac{2}{Tn^2\omega^2}\left[tn\omega\sin(n\omega t) + \cos(n\omega t)\right]_0^{T/2} = \frac{(-1)^n - 1}{n^2\pi\omega}.$$

$$b_n = \frac{2}{T}\int_0^T f(t)\cdot\sin(n\omega t)dt = \frac{2}{T}\int_0^{\frac{T}{2}} t\sin(n\omega t)\,dt$$

$$= \frac{2}{Tn^2\omega}\left[\sin(n\omega t) - tn\omega\cos(n\omega t)\right]_0^{T/2} = \frac{(-1)^{n+1}}{n\omega}.$$

(iii) Skriva 4 avsnit í røkkjuni.

Hetta svarar til $a_3\cos(3\omega t) + b_3\sin(3\omega t) = \frac{-2}{9\pi\omega}\cos(3\omega t) + \frac{1}{3\omega}\sin(3\omega t)$.

(iv) Finn $S_N(x)$, so at hendan hevur 85% av effektini hjá f.

Við Parsevals setning er effektin hjá f á $\frac{\pi^2}{6\omega^2}$ javn við

$$\frac{\pi^2}{16\omega^2} + \frac{1}{2\omega^2\pi^2}\sum_{n=1}^{N}\frac{(1-(-1)^n)^2}{n^4} + \frac{1}{2\omega^2}\sum_{n=1}^{N}\frac{1}{n^2}.$$

Vit fáa úrslitið

$$\frac{6}{16} + \frac{3}{\pi^4}\sum_{n=1}^{N}\frac{(1-(-1)^n)^2}{n^4} + \frac{3}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2}$$

Fyri $N=2$ fáa vit nú, at

$$\frac{6}{16} + \frac{12}{\pi^4} + \frac{3}{\pi^2}\left(1 + \frac{1}{4}\right) = 0{,}8781 > 0{,}85.$$

4. (i) Finn $H(s)$ fyri
$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u$$
$$y = (1 \ \ 0)\, x$$

Systemmatricen $A = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix}$ hevur $\det(A - sI) = \begin{vmatrix} -1-s & 0 \\ 2 & -2-s \end{vmatrix} = (-1-s)(-2-s)$.

Her eru eginvirdini $s=-1$ og $s=-2$, so $A-sI$ er invertibel fyri $s \notin \{-2,-1\}$.

$$H(s) = -d^T (A-sI)^{-1} b = -(1 \ \ 0) \begin{pmatrix} -1-s & 0 \\ 2 & -2-s \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= -(1 \ \ 0) \frac{1}{(-1-s)(-2-s)} \begin{pmatrix} -2-s & 0 \\ -2 & -1-s \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= -(1 \ \ 0) \frac{1}{(-1-s)(-2-s)} \begin{pmatrix} -4-2s \\ -5-s \end{pmatrix}$$
$$= \frac{-1}{(-1-s)(-2-s)} (-4-2s)$$
$$= \frac{-2\,(-2-s)}{(-1-s)(-2-s)} = \frac{-2}{-1-s} = \frac{2}{s+1}, \quad s \notin \{-2,-1\}.$$

(ii) Vís, at systemið er asymptodiskt stabilt.

Vit sóu, at $P(\lambda) = (\lambda+1)(\lambda+2)$, hvar røturnar eru -1 og -2.
Per setning 2.36 er systemið asymptodiskt stabilt.

(iii) Finn við eini óendeliga rekkju eina loysn til ávirkanini u, sum er 2π-periodisk og givin við $u(t) = \frac{1}{4}t^2 - \frac{\pi}{2}t$ tá $t \in [0, 2\pi[$.

Vit hava
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4}t^2 - \frac{\pi}{2}t \, dt = \frac{1}{24\pi}\left[t^3 - 3\pi t^2\right]_0^{2\pi}$$
$$= \frac{1}{24\pi}(8\pi^3 - 12\pi^3) = -\frac{\pi^2}{6}.$$
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\, e^{-int}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{4}t^2 - \frac{\pi}{2}t\right) e^{-int}\,dt$$
$$= \frac{1}{2\pi}\left[\frac{2nt - 2n\pi + (n^2t^2 - 2n^2\pi t - 2)i}{4n^3}\, e^{-int}\right]_0^{2\pi} = \frac{4n\pi}{8n^3\pi} = \frac{1}{2n^2}.$$

Ávirkanin u er kontinuert við $u(0) = u(2\pi) = 0$. Harafturat er u 2π-periodisk og stk.vís. diff., so tá skipanin er asymptodisk stabil gevur setningur 7.8 loysnina.
$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in)\, e^{int} = c_0 H(0) + \sum_{n=1}^{\infty} \left(c_n H(in) e^{int} + c_{-n} H(-in) e^{-int}\right)$$
$$= -\frac{\pi^2}{6}\cdot 2 + \sum_{n=1}^{\infty} \frac{1}{2n^2}\left(\frac{2}{1+in} e^{int} + \frac{2}{1-in} e^{-int}\right) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1-in}{1+n^2}(\cos(nt) + i\sin(nt)) + \frac{1+in}{1+n^2}(\cos(nt) - i\sin(nt))\right)$$
$$= -\frac{\pi^2}{3} \sum_{n=1}^{\infty} \frac{1}{n^2+n^4}\left(2\cos(nt) + 2n\sin(nt)\right)$$
$$= -\frac{\pi^2}{3} \sum_{n=1}^{\infty} \frac{2}{n^2+n^4}\left(\cos(nt) + n\cdot\sin(nt)\right).$$