Eigenvalues and eigenvectors

Ex 1. Geogebra sheet 1.

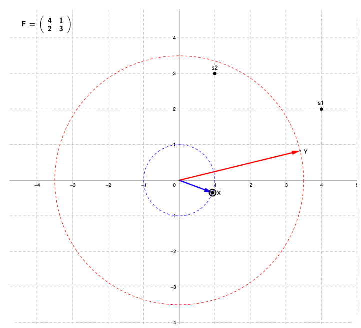
a) 1. How many times are $\underline{x}$ and $f(\underline{x})$ parallel during animation?

Twice, so $\underline{\underline{F}}$ consists of lin. indept. vectors. (Half circle, otherwise 4)

2. Move $\underline{x}$ and determine the ratio between $|\underline{x}|$ and $|\underline{y}|$.

First we have $1:5$, and second is $1:2$.

3. Explain that we can determine eigenvalues of f by letting $\underline{x}$ pass a semicircle of radius 1.

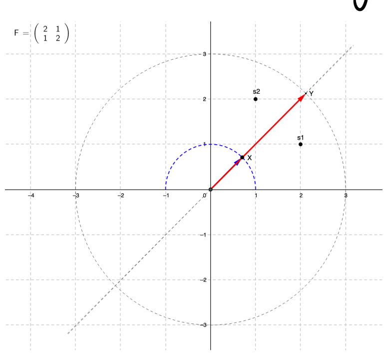


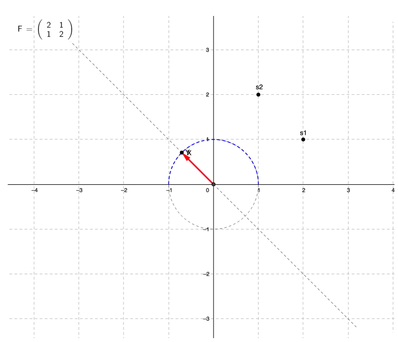
This is the case, as an eigenvalue is how much space is scaled along one dimension. So we only need a semicircle, as the opposite side of the circle corresponds to the same eigenspace, and using a radius of 1 means that the scaling factor is exactly the eigenvalue.

Note that not all maps have n distinct eigenvalues/vectors.

b) Geogebra sheet 2.

1. Rotate $\underline{x}$ to find eigenvalues. State the eigenvectors.





We have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ with eigenvectors $\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The vectors are not unique.

2. Check λ_1, λ_2 by hand with $P(\lambda)$.

$$P(\lambda) = \det(\underline{\underline{F}} - \lambda \underline{\underline{I}}) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 1$$

$$P(\lambda) = 0 \iff (2-\lambda)^2 - 1 = 0 \iff \lambda = 1 \vee \lambda = 3.$$

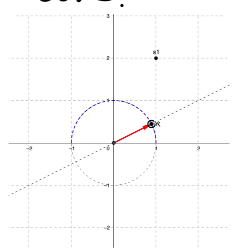
3. Check the eigenvectors.

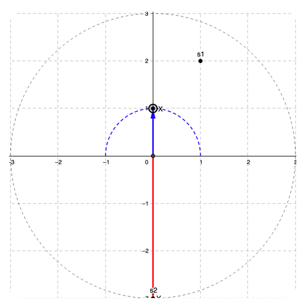
We solve $(\underline{\underline{F}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0}$ for $\lambda_1 = 3$ and $\lambda_2 = 1$.

$$\lambda_1 = 3: \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \underline{v} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$$\lambda_2 = 1: \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow \underline{w} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

4. Change $\underline{\underline{F}}$ and redo the above.

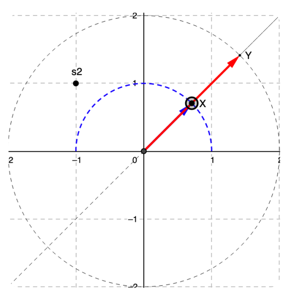




$\underline{\underline{F}} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ then we get

$\lambda_1 = 1$, $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$\lambda_2 = -3$, $\underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



$\underline{\underline{F}} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ then we get

$\lambda = 2$, $\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\lambda = 2$ is a double root in this case.

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda)+1 = 3-3\lambda-\lambda+\lambda^2+1$$

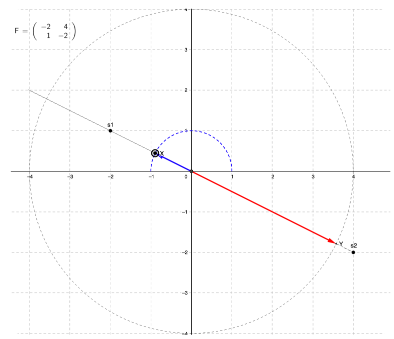
$$= \lambda^2-4\lambda+4 = (\lambda-2)^2.$$

Indeed $\lambda = 2$ is a double root!

$\underline{\underline{F}} = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$, the columns are scalar multiples.

$\lambda_1 = -4$, $\underline{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda_2 = 0$, $\underline{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



5. Set $\underline{\underline{F}} = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$. Rotate and read the eigenvalues.

Well there aren't any, let's see why.

$$P(\lambda) = (2-\lambda)\cdot(4-\lambda)+2 = \lambda^2-6\lambda+10$$

$$D = 36-40 = -4$$

$$\lambda = \frac{6 \pm 2i}{2} = 3 \pm i.$$

Ex 2. Given $\underline{\underline{A}} = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$.

a) Set up the characteristic matrix.

$$\begin{bmatrix} 2-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

b) Set up $P(\lambda)$ of $\underline{\underline{A}}$. As seen in Ex 1. b) 5. $P(\lambda) = \lambda^2-6\lambda+10$.

c) Characteristic equation and eigenvalues.

$$\lambda^2 - 6\lambda + 10 = 0 \iff \lambda = 3 \pm i \quad \text{(done in Ex1. G) 5.)}$$

d) Find an eigenvector, and then e) state the other without computation.

$$\begin{bmatrix} 2-(3+i) & 2 \\ -1 & 4-(3+i) \end{bmatrix} = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \underline{v} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

The other eigenspace for $\lambda = 3-i$ is given by $t\begin{bmatrix} 1+i \\ 1 \end{bmatrix}, t \in \mathbb{R}.$

f) Check with Maple.

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> Eigenvectors(<2,2;-1,4>,output=list);
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$$\left[\left[3+I, 1, \left\{\begin{bmatrix} 1-I \\ 1 \end{bmatrix}\right\}\right], \left[3-I, 1, \left\{\begin{bmatrix} 1+I \\ 1 \end{bmatrix}\right\}\right]\right]$$

Ex3. A linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

a) Determine $P(\lambda)$ and find eigenvalues. State alg. mult.

Determine real eigenspaces of real eigenvalues and state geom. mult.

$$P(\lambda) = (1-\lambda)(4-\lambda)(3-\lambda) + (3-\lambda)\cdot 2 = (4 - 5\lambda + \lambda^2)(3-\lambda) + 6 - 2\lambda$$

$$= 12 - 15\lambda + 3\lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 + 6 - 2\lambda$$

$$= -\lambda^3 + 8\lambda^2 - 21\lambda + 18 = (\lambda - 2)\cdot(\lambda-3)^2$$

$\lambda_1 = 3 \qquad am(3) = 2$

$\lambda_2 = 2 \qquad am(2) = 1$

$$\lambda_1: \quad \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \text{ or } \underline{v} = t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad gm(3) = 2.$$

$$\lambda_2: \quad \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad gm(2) = 1. \qquad t \in \mathbb{R}.$$

b) Choose a basis for $\mathbb{R}^3$ s.t. $\underline{\underline{A}}$ becomes a diagonal matrix.

We can use the eigenbasis $v = \left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$ so that

$${}_v\underline{\underline{A}}_v = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(order matters!)}$$

Now we're given $\underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$.

c) Redo a) for matrix $\underline{\underline{B}}$.

```
> B:=<1,1,0;2,-1,-1;0,2,1>:
  Eigenvectors(B,output=list);
```

$$\left[\left[1, 2, \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \right], \left[-1, 1, \left\{ \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\} \right] \right]$$

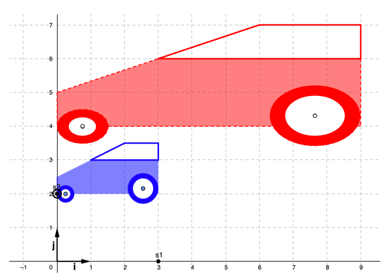
d) If possible create $\underline{\underline{V}}$ and Λ s.t. $\underline{\underline{V}}^{-1} \underline{\underline{B}}\, \underline{\underline{V}} = \underline{\underline{\Lambda}}$.

The geometric multiplicity sums to $2 < 3$, so we don't have 3 lin. indept. vectors for a basis.

Ex 4. Geogebra sheet 3.

a) 1. Map the blue car onto the dashed red field.

Use $\underline{\underline{F}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$



2. Try $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Flipped image — width same as original.

3. In general $\underline{\underline{F}} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ and $P(\lambda) = (a-\lambda)(b-\lambda)$.

Thus $P(\lambda) = 0 \iff \lambda = a$ or $\lambda = b$. Eigenvectors follow

as $\begin{bmatrix} 0 & 0 \\ 0 & b-a \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$.

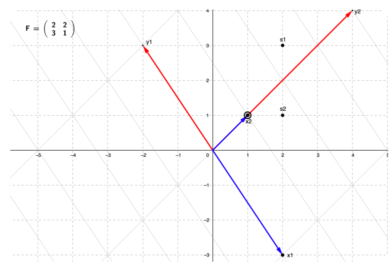
$\begin{bmatrix} a-b & c \\ 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} c \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

In the case of diagonal matrices $\lambda = a$ is the scaling along $\underline{x}_1$

and $\lambda = b$ is the scaling along $\underline{x}_2$.

Sheet 4

b) 1. Move $\underline{x}_1$ and $\underline{x}_2$ to get eigenvectors.



Here we have

$\lambda_1 = -1$ with $\underline{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and

$\lambda_2 = 4$ with $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. Which coordinates does $(6,1)$ have in the new system?

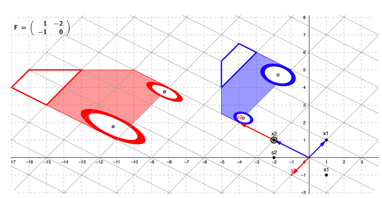
This would be $(1,4)$ since

$$\underline{v}_1 + 4\,\underline{v}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$

3. What about the image of $(6,1)$ in the new system?

$$\underset{v=v}{F} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 16 \end{bmatrix}.$$

c) Sheet 5



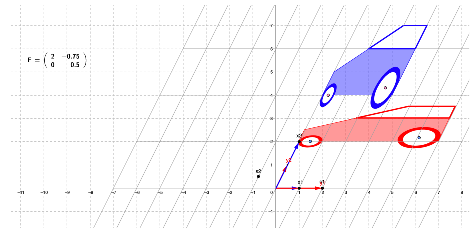
1. Set to $\underline{\underline{F}} = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$.

2. Move $\underline{x}_1$ and $\underline{x}_2$ to get eigenvectors and values.

$\lambda_1 = -1$, $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 2$, $\underline{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$$ {}_{v}\underline{\underline{F}}_{v} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} $$, the first axis flips and the second scales by 2.

3. Repeat for sheet 6.



$\lambda_1 = 2$, $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = \frac{1}{2}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$ {}_{v}\underline{\underline{F}}_{v} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} . $$

4. Eigenvectors constitute a basis and are unaffected by their linear map aside from scaling by the eigenvalue. Thus they can tell us exactly how the map perturbates the space.

Ex 5. Consider $f(x(t)) = x'(t) - x(t)$.

a) Explain that e^{kt} is an eigenfunction of f for all $k \in \mathbb{R}$. State the eigenvalue.

$$ (e^{kt})' - e^{kt} = k\,e^{kt} - e^{kt} = (k-1)\,e^{kt} \Rightarrow \lambda = k-1 . $$

b) Explain that e^{kt} for $k \in \{-1, 0, 1, 2\}$ are lin. indept.

These all have distinct eigenvalues, so they are lin. indept. by proposition 13.11.1.

c) Let $U \subseteq C^{\infty}(\mathbb{R})$ have the basis $v = (e^{-t}, 1, e^{t}, e^{2t})$. Show $f(U) \subseteq U$. Determine ${}_{v}\underline{\underline{F}}_{v}$.

These are eigenvectors of f, so clearly $f(U) \subseteq U$.

Eigenvalues of e^{kt} are $k-1$, so we have

$$ {}_v\underline{\underline{F}}_v = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

d) Coordinate vector for $q(t) = -6e^{-t} + e^{2t} + 2$ in v.

Compute using ${}_v\underline{\underline{F}}_v$ solution in U for which $f(x(t)) = q(t)$.

$$ {}_v\underline{q} = (-6, 2, 0, 1) $$

Set ${}_v x = (3, -2, t, 1)$, for $t \in \mathbb{R}$, then $x(t)$ is a solution.

e) Compare with dsolve in maple. Why aren't there more solutions outside of U?

```
> eq:=diff(x(t),t)-x(t)=-6*exp(-t)+exp(2*t)+2;
  dsolve(eq,x(t));
```

$$ eq := \frac{d}{dt}x(t) - x(t) = -6e^{-t} + e^{2t} + 2 $$

$$ x(t) = e^{2t} + 3e^{-t} - 2 + e^{t}c_1 $$

There aren't other solutions, as no other functions map to the exponentials. We also know there is one particular solution x_0 to the inhomogeneous equation, and the rest, ce^t, is just the homogeneous solution.

Ex6. We have $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

a) Get an eigenbasis v. Determine $e^{\underline{\underline{M}}}v$.

$$ P(\lambda) = (1-\lambda)^2(2-\lambda) $$

$$ P(\lambda) = 0 \iff \lambda = 2 \vee \lambda = 1 $$

$$ \lambda = 2: \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}. $$

$$\lambda = 1: \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \vee \underline{v} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

These are lin. indept. so $\; {}_e\underline{\underline{M}}_v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

for basis in this order.

b) Find $\underline{\underline{V}}$ and $\underline{\underline{\Lambda}}$, so that $\underline{\underline{\Lambda}} = \underline{\underline{V}}^{-1} \underline{\underline{A}}\, \underline{\underline{V}}$.

$${}_v\underline{\underline{F}}_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad {}_e\underline{\underline{M}}_v \, {}_v\underline{\underline{F}}_v \, {}_v\underline{\underline{M}}_e = \underline{\underline{A}}$$

$$\iff {}_v\underline{\underline{F}}_v = {}_v\underline{\underline{M}}_e \, \underline{\underline{A}} \, {}_e\underline{\underline{M}}_v$$

So let ${}_v\underline{\underline{F}}_v = \Lambda$ and ${}_e\underline{\underline{M}}_v = \underline{\underline{V}}$ and we're done.

Ex 7. a) Use Maple to get eigenvalues and vectors.

```
> B:=< -1,-1,-6,3; 1,-2,-3,0; -1,1,0,1; -1,-1,-5,3 >;
  Eigenvectors(B,output=list);
```

$$B := \begin{bmatrix} -1 & -1 & -6 & 3 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -5 & 3 \end{bmatrix}$$

$$\left[\left[-1, 1, \left\{ \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} \right], \left[1, 1, \left\{ \begin{bmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{4} \\ 1 \end{bmatrix} \right\} \right], \left[0, 2, \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right] \right]$$

As a reminder this shows [value, alg. mult., vector] and the number of vectors corresponds to the geom. mult.

b) Can we diagonalize $\underline{\underline{B}}$?

No, there are 3 vectors and $\dim(\mathbb{R}^4) = 4$.

Ex 8. Assume $\lambda_1 \neq \lambda_2$ for $\underline{\underline{A}}$. Then $\underline{v}_1 \neq \underline{0}$ and $\underline{v}_2 \neq \underline{0}$, such that

$$\underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v} \quad \text{and} \quad \underline{\underline{A}}\,\underline{v}_2 = \lambda_2 \underline{v}_2 .$$

a) Show that $\underline{v}_1$ and $\underline{v}_2$ are lin. indept.

Assume for contradiction that $\underline{v}_2 = k \cdot \underline{v}_1$ for some $k \neq 0$.

We have that

$$\underline{\underline{A}}\,\underline{v}_2 = \underline{\underline{A}}(k\underline{v}_1) = k\lambda_1 \underline{v}_1 ,$$

$$\underline{\underline{A}}\,\underline{v}_2 = \lambda_2 \underline{v}_2 = k\lambda_2 \underline{v}_1 ,$$

but then $\underline{\underline{A}}\,\underline{v}_2 - \underline{\underline{A}}\,\underline{v}_2 = \underline{0} = k(\lambda_1 - \lambda_2)\underline{v}_1$.

This is a contradiction as $k \neq 0$ and $\lambda_1 - \lambda_2 \neq 0$ and $\underline{v}_1 \neq \underline{0}$.

Thus $\underline{v}_1$ and $\underline{v}_2$ are not linearly dependent, i.e. they are linearly independent.