Geometry in the plane and 3D

Ex 1.

a) Calculate the determinant by expansion.

Let's just take row 1.

$$\det\left(\begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & 2 \\ 3 & -5 & 1 \end{bmatrix}\right) = 2\cdot\det\left(\begin{bmatrix} 2 & 2 \\ -5 & 1 \end{bmatrix}\right) + 3\cdot\det\left(\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\right)$$

$$= 2\cdot(2+10) + 3(-1-6) + (5-6)$$

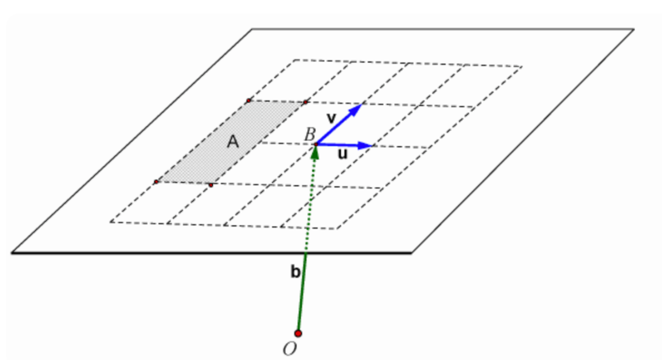
$$= 24 - 21 - 1 = 2$$

b) Compute the rest by any method in your head.

$$\det\left(\begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\right) = 8 \qquad \det\left(\begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 9 & -4 & 3 \end{bmatrix}\right) = 30$$

$$\det\left(\begin{bmatrix} 2-i & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 5 \qquad \det\left(\begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\right) = 0$$

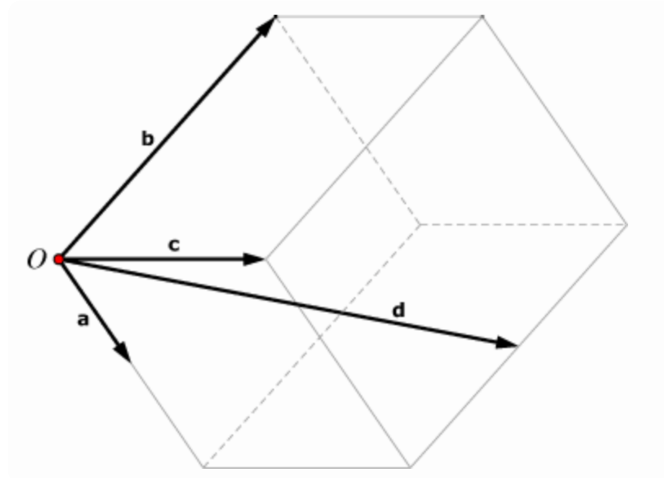
Ex 2. Provide a parametric representation for A.



$$A = \{P \mid \vec{OP} = \underline{b} + x\underline{u} + y\underline{v},\ x \in [-2,-1] \text{ and } y \in [-1,1]\}.$$

Ex 3.

a) Give the coordinates for $_{\underline{m}}\underline{d}$ in the basis $m = (\underline{a}, \underline{b}, \underline{c})$.

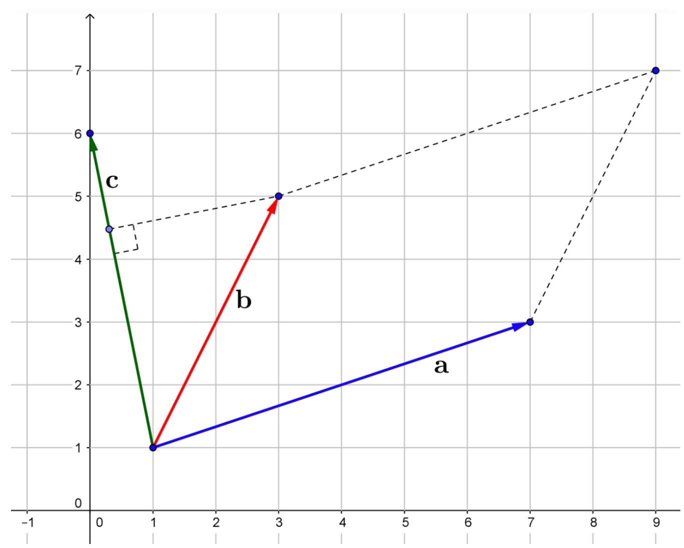


$$_{\underline{m}}\underline{d} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

b) Let $n = (\underline{a}, \underline{b}, \underline{d})$ be a new basis. Determine $_{\underline{n}}\underline{c}$.

$$_{\underline{n}}\underline{c} = \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Ex 4. Three geometric vectors are given.



a) Coordinates.

$$\underline{a} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

b) Length.

$$|\underline{a}| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

$$|\underline{b}| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

$$|\underline{c}| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}$$

c) Compute the angle between $\underline{a}$ and $\underline{b}$.

$$\cos(v) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{20}{4\sqrt{50}} = \frac{20}{2\sqrt{2} \cdot \sqrt{100}} = \frac{20}{20\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow v = \frac{\pi}{4} \text{ or } 45^\circ.$$

d) Compute the area of the parallelogram spanned by $\underline{a}$ and $\underline{b}$.

$$\det\left(\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\right) = 24 - 4 = 20.$$

e) Compute the length of $\underline{b}_{\underline{c}}$.

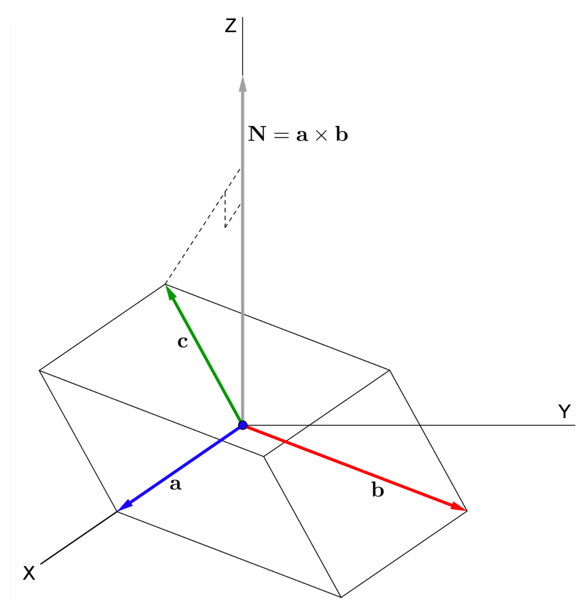
$$|\underline{b}_{\underline{c}}| = \frac{\underline{b} \cdot \underline{c}}{|\underline{c}|} = \frac{18}{\sqrt{26}} = \frac{18\sqrt{26}}{26} = \frac{9\sqrt{26}}{13}.$$

f) Compute $\underline{b}_{\underline{c}}$.

$$\underline{b}_{\underline{c}} = \frac{\underline{b} \cdot \underline{c}}{|\underline{c}|^2} \underline{c} = \frac{18}{26}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \frac{9}{13}\begin{bmatrix} -1 \\ 5 \end{bmatrix} \left(= \begin{bmatrix} -\frac{9}{13} \\ \frac{45}{13} \end{bmatrix} \right).$$

Ex5. We have

$$\underline{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \underline{c} = \begin{bmatrix} 3/2 \\ 1/2 \\ 3/2 \end{bmatrix}.$$



a) Find the volume of the parallelipiped by base area times height.

Base area is given by $|\underline{a} \times \underline{b}|$.

$$\underline{a} \times \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

So $|\underline{a} \times \underline{b}| = 2$. Since $\underline{a}$ and $\underline{b}$ are in the xy-plane, we can determine the height to be the length of $\underline{c}$ projected onto the z-axis.

$$\underline{c}_{\underline{k}} = \frac{\underline{c} \cdot \underline{k}}{|\underline{k}|^2} \underline{k} = \frac{3/2}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/2 \end{bmatrix}.$$

Now the volume can be determined as

$$|\underline{a} \times \underline{b}|\,|\underline{c}_{\underline{k}}| = 2\,\frac{3}{2} = 3.$$

b) Repeat but use determinants.

$$\det\left(\begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2 & 1/2 \\ 0 & 0 & 3/2 \end{bmatrix}\right) = 1 \cdot 2 \cdot \frac{3}{2} = 3.$$

Ex 6. Let

$$\underline{a} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} \text{ and } \underline{c} = \begin{bmatrix} -5 \\ 3 \\ 3 \end{bmatrix}.$$

a) Find the determinant of $[\underline{a}\ \underline{b}\ \underline{c}]$. Are the vectors linearly independent?

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> a:=<3,1,5>: b:=<2,3,9>: c:=<-5,3,3>:
> Determinant(<a|b|c>);
                                        0
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The determinant is 0, so the matrix is singular.

As such there is linear dependence among the vectors.

(They do not span a parallelipiped/volume)

b) Write one of the vectors as a linear combinations of the other two.

$$-3\,\underline{a} + 2\,\underline{b} = \underline{c}$$

c) Volume spanned by $\underline{a}$, $\underline{b}$ and $\underline{c}$ is 0 as determined in a).

d) Using $\underline{a}$ and $\underline{b}$ together with a vector perpendicular to these two to span a volume of 187.

$$\underline{n} = \underline{a} \times \underline{b} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -17 \\ 7 \end{bmatrix} \text{ and } \underline{n} \perp \underline{a} \text{ and } \underline{n} \perp \underline{b}.$$

$$\det\left(\begin{bmatrix} 3 & 2 & -6 \\ 1 & 3 & -17 \\ 5 & 9 & 7 \end{bmatrix}\right) = 63 - 170 - 54 + 90 + 459 - 14 = 374 \text{ and } \frac{187}{374} = \frac{1}{2}, \text{ so}$$

$$\underline{m} = \pm\frac{1}{2}\underline{n} \text{ is a solution.}$$