Diagonalization by Orthogonal Substitution

Ex 1.

a) In $\mathbb{R}$ provide a basis for the orthogonal complement of $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

This is just $\operatorname{span}\left\{\begin{bmatrix} -7 \\ 3 \end{bmatrix}\right\}$, since their dot product is zero for any scalar multiple, i.e. orthogonal.

b) Find a basis for the orthogonal complement of $\underline{v} = (1, 2, 3)$.

We solve $x_1 + 2x_2 + 3x_3 = 0$. Let $x_2 = t_2$ and $x_3 = t_3$ be free parameters, then

$$\underline{x} = t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad t_2, t_3 \in \mathbb{R}.$$

A basis for the orthogonal complement is thus $((-2,1,0), (-3,0,1))$ as the vectors a clearly not linearly dependent.

c) Find a basis for the orthogonal complement of $(1,1,0)$ and $(0,2,1)$ in $\mathbb{R}^3$.

The cross product is sufficient.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$ is a basis for the orthogonal complement in $\mathbb{R}^3$.

d) Find the orthogonal complement of $(1, -1, 2, 5)$ and $(0, 1, 0, -2)$ in $\mathbb{R}^4$.

We have the augmented system $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 5 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right]$, so we simply solve for a basis again.

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 0 & -2 \end{bmatrix} \xrightarrow{+R_2} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

Let $x_3 = t_3$ and $x_4 = t_4$ be free parameters, then

$$\underline{x} = t_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad t_3, t_4 \in \mathbb{R}^4 .$$

A basis is then given by $((-2,0,1,0), (-3,2,0,1))$.

Ex2.

a) Why is it easy to diagonalize a symmetric $n \times n$ matrix by orthogonal substitution, if it has n distinct eigenvalues?

The eigenvectors are in turn pairwise orthogonal, so we need only normalize the vectors.

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> A:=<<3,1,-1>|<1,3,-1>|<-1,-1,5>>:
  Eigenvectors(A,output=list);
```

$$\left[\left[6, 1, \left\{\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}\right], \left[3, 1, \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right], \left[2, 1, \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right]\right]$$

b) State $\underline{\underline{A}}$ from the Maple prompt, and explain that it is symmetric.

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix} = \underline{\underline{A}}^T .$$

Since $\underline{\underline{A}} = \underline{\underline{A}}^T$ it follows that $\underline{\underline{A}}$ is symmetric.

c) Let f be the map with mapping matrix $\underline{\underline{A}}$. Determine an ONB for $\mathbb{R}^3$ in terms of eigenvectors of f and state the mapping matrix in this new basis.

There are 3 eigenvalues, so we normalize the 3 eigenvectors for our ONB. We know the lengths from the last session: $\sqrt{3/2}$, $\sqrt{3}$ and $\sqrt{2}$, so we multiply by the factors $\sqrt{6}/3$, $\sqrt{3}/3$ and $\sqrt{2}/2$. Thus the ONB is

$$\left(\begin{bmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \right).$$

The mapping matrix for f wrt. this basis is $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

d) Determine $\underline{\underline{Q}}$ and $\underline{\underline{\Lambda}}$ such that $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}}$.

From the above we have

$$\underline{\underline{Q}} = \begin{bmatrix} -\sqrt{6}/6 & \sqrt{3}/3 & -\sqrt{2}/2 \\ -\sqrt{6}/6 & \sqrt{3}/3 & \sqrt{2}/2 \\ \sqrt{6}/3 & \sqrt{3}/3 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\underline{\Lambda}} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Ex3.

```
> B:=<<6,2,4>|<2,9,-2>|<4,-2,6>>:
  Eigenvectors(B,output=list);
```

$$\left[\left[10, 2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\} \right], \left[1, 1, \left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} \right] \right]$$

a) State $\underline{\underline{B}}$ and explain that it is symmetric.

$$\underline{\underline{B}} = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 9 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \underline{\underline{B}}^T.$$

Since $\underline{\underline{B}} = \underline{\underline{B}}^T$ it follows that $\underline{\underline{B}}$ is symmetric.

b) Determine $\underline{\underline{Q}}$ and $\underline{\underline{\Lambda}}$ such that $\underline{\underline{B}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$

```
> with(LinearAlgebra):
> B:=<6,2,4| 2,9,-2| 4,-2,6>;
```

$$B := \begin{bmatrix} 6 & 2 & 4 \\ 2 & 9 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

```
> v1:=1/sqrt(2)*<1,0,1>;
  v3:=1/(3/2)*<-1,1/2,1>;
  v2:=1*CrossProduct(v1,v3);
```

$$v1 := \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$v3 := \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$v2 := \begin{bmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \\ \frac{\sqrt{2}}{6} \end{bmatrix}$$

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> Q:=<v1|v2|v3>;
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$$Q := \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & -\frac{2}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & \frac{2}{3} \end{bmatrix}$$

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> 'Lambda'=Transpose(Q).B.Q;
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$$\Lambda = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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> 'B'=Q.rhs(%).Transpose(Q);
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$$B = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 9 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

Essentially we have orthogonal eigenvectors, but only two. The cross product provides the third, then we normalize to construct $\underline{\underline{Q}}$.

Ex4. A linear map f has the mapping matrix $\begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$.

a) Find all 8 ONB from eigenvectors of f. Draw them.

$$P(\lambda) = (5-\lambda)(7-\lambda) - 3 = \lambda^2 - 12\lambda + 32$$

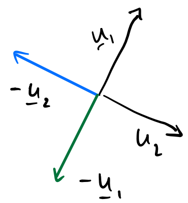
$$P(\lambda) = 0 \iff \lambda = \frac{12 \pm 4}{2} = \begin{cases} 8 \\ 4 \end{cases}$$

$$\lambda = 8: \begin{bmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \Rightarrow \underline{v}_1 = t \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}. \qquad \lambda = 4: \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}.$$

We normalize and get $\underline{u}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ and $\underline{u}_2 = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$.

The 8 ONB are $(\underline{u}_1, \underline{u}_2)$, $(\underline{u}_1, -\underline{u}_2)$, $(-\underline{u}_1, \underline{u}_2)$, $(-\underline{u}_1, -\underline{u}_2)$,

$(\underline{u}_2, \underline{u}_1)$, $(\underline{u}_2, -\underline{u}_1)$, $(-\underline{u}_2, \underline{u}_1)$, $(-\underline{u}_2, -\underline{u}_1)$.



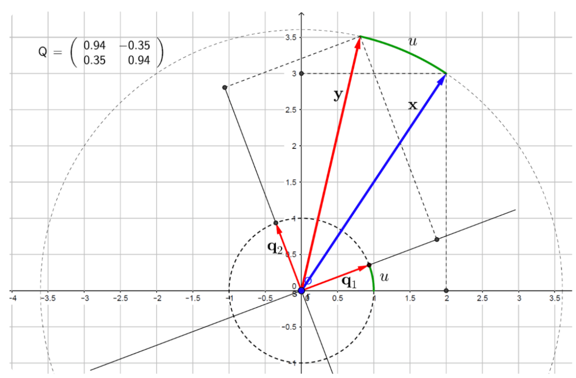
b) Four bases have standard orientation and four don't. (show $\det(\underline{\underline{Q}}) = \pm 1$)

$$\det([\underline{u}_1 \; -\underline{u}_2]) = \det([-\underline{u}_1 \; \underline{u}_2]) = \det([\underline{u}_2 \; \underline{u}_1]) = \det([-\underline{u}_2 \; -\underline{u}_1]) = 1.$$

$$\det([\underline{u}_1 \; \underline{u}_2]) = \det([-\underline{u}_1 \; -\underline{u}_2]) = \det([\underline{u}_2 \; -\underline{u}_1]) = \det([-\underline{u}_2 \; \underline{u}_1]) = -1.$$

Ex5.

a) Explain that $\underline{y} = \underline{\underline{Q}}\,\underline{x}$ appears by rotating by u.



We can use the given base $q = (\underline{q}_1, \underline{q}_2)$ and the standard basis. Then $\underline{\underline{Q}} = {}_e\underline{\underline{M}}_q$, so also

$${}_e\underline{y} = \underline{\underline{Q}}\,{}_e\underline{x} = {}_e\underline{\underline{M}}_q\,{}_e\underline{x}.$$

In q we have

$${}_q\underline{y} = {}_q\underline{\underline{M}}_e\,{}_e\underline{y} = {}_q\underline{\underline{M}}_e\left({}_e\underline{\underline{M}}_q\,{}_e\underline{x}\right) = {}_e\underline{x}.$$

Thus we obtain y by rotating precisely u radians in

$$\underline{\underline{Q}} = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}.$$

b) Geogebra sheet.

$\underline{\underline{Q}}$ rotates by u and $\underline{\underline{Q}}^T$ rotates by $-u$. This follows from transposing and noting $-\sin u = \sin(-u)$, so we swapped the action across the diagonal.

The maps preserve lengths, but alter the angle as stated. The angle between $\underline{q}_1$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is u, and all the basis vectors are tied to $\underline{q}_1$ so that this dictates the angle of rotation.

Ex6. Assume $\underline{\underline{A}}$ is 2x2 and $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}}$.

a) Show that $\underline{\underline{A}} = \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T$.

Recall that $\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}$ for orthogonal matrices. As such

$$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}} \Leftrightarrow \underline{\underline{Q}}\, \underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}}\, \underline{\underline{Q}}^T = \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T$$

$$\Leftrightarrow \quad \underline{\underline{A}} = \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T.$$

b) Explain that a symmetric map is composed by

1. A rotation of $-u$
2. Scaling by λ_1 and λ_2 along $\underline{i}$ and $\underline{j}$.
3. A rotation of u.

This follows from all orthogonal matrices being of the type

$$\underline{\underline{Q}} = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}$$

So $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$ implies $\underline{\underline{A}}\,\underline{x} = \underline{\underline{Q}}(\underline{\underline{\Lambda}}(\underline{\underline{Q}}^T\underline{x}))$. Hence firstly a rotation $-u$, then $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ scales along the first and second axis. Lastly $\underline{\underline{Q}}$ rotates back by u.

c) Consider $\underline{\underline{B}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find the angle u and scaling factors.

$$P(\lambda) = (2-\lambda)^2 - 1 = 0 \iff (2-\lambda)^2 = 1 \iff \lambda_1 = 3 \vee \lambda_2 = 1.$$

The eigenvalues are the scaling factors, so $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

$\lambda_1 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underline{q}_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $\underline{q}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$.

The angle u is $\frac{\pi}{4}$ radians by the coordinates of $\underline{q}_1$.

d) Maple sheet. Follow steps and try $u = -\frac{\pi}{3}$, $a = 5$, $b = -2$.

This is just to see the steps in action. Try the Maple sheet.