Differentiallíkningaskipanir

System

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + \cdots + a_{1n}x_n(t) + u_1(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + u_2(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + u_n(t) \end{cases}$$

Koefficientarnir a_{ij} við $i,j = 1,\dots,n$ eru reel tøl, og funktiónirnar $u_1,\dots,u_n$ eru defineraðar á einum intervalli $I \subseteq \mathbb{R}$. Vit skipa helst á vektor og matrix form, har vit leita eftir loysnunum $x_1,\dots,x_n$.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix},$$

ávirkan er vektor "Fleiri retningar sjálvandi"

har vit definera system matricuna A at vera

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Við at seta $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ og $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$,

so hava vit umset til lineera algebra. Vit skriva gjarna systemið

$$\dot{x} = Ax + u \quad \text{og} \quad \dot{x} = Ax$$

fyri inhomogena og homogena systemið, ávikavist.

Homogent system

Lat $\dot{x} = Ax$, so kenna vit framferðarháttin: finn røturnar í $P(\lambda)$, har

$$P(\lambda) = \det(A - \lambda I)$$

og loys eginvektorarnar, evt. við

$$(A - \lambda I)v = \underline{0}$$

Setn. 2.3

Um λ er ein rót í $P(\lambda)$ hjá A í $\dot{x} = Ax$ við eginvektorin v, so er

$$x(t) = e^{\lambda t} v$$

ein loysn. Loysnir við distinkt λ eru lineert óheftar.

Pf. Set fyri at λ er ein loysn við eginvektor v. Set nú $x(t) = e^{\lambda t} v$ í systemið, so hava vit
$$\dot{x} = \lambda e^{\lambda t} v = e^{\lambda t} \lambda v = e^{\lambda t} A v = A(e^{\lambda t} v) = Ax.$$

Um $v_1, \dots, v_k$ eru lineert óheftir vektorar, so vil
$$c_1 v_1 + \dots + c_k v_k = 0 \implies c_1 = 0, \dots, c_k = 0.$$
Minst til at eginvektorarnir eru lineert óheftir.
Um
$$d_1 e^{\lambda_1 t} v_1 + \dots + d_k e^{\lambda_k t} v_k = 0,$$
so er fyri eitthvørt fast $t_0 \in I$ galdandi at við $c_j = d_j e^{\lambda_j t_0}$, $j = 1, \dots, k$, so vil
$$c_1 v_1 + \dots + c_k v_k = 0 \implies c_j = d_j e^{\lambda_j t_0} = 0.$$
Men $e^{\lambda_j t_0} \neq 0$, so $d_j = 0$ fyri øll j. Altso eru loysnirnar óheftar. $\square$

Lemma 2.4 Lat $v = \operatorname{Re} v + i \operatorname{Im} v$ vera eginvektor hjá A við eginvirði λ. So er $\bar{v} = \operatorname{Re} v - i \operatorname{Im} v$ eginvektor hjá A við eginvirði $\bar{\lambda}$.

Pf. Lítil venjing í kompleks konjugering.

Dømi
$$\begin{cases} \dot{x}_1(t) = 3x_1(t) - x_2(t) \\ \dot{x}_2(t) = 4x_1(t) + 3x_2(t) \end{cases}, \qquad \dot{x} = \begin{bmatrix} 3 & -1 \\ 4 & 3 \end{bmatrix} x$$

$$(3-\lambda)^2 + 4 = \lambda^2 - 6\lambda + 13 = 0 \iff \lambda = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm \frac{\sqrt{-16}}{2} = 3 \pm 2i$$

$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ eginvektor hjá $3-2i$

$$x(t) = c_1 e^{(3+2i)t} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + c_2 e^{(3-2i)t} \begin{bmatrix} 1 \\ 2i \end{bmatrix} \qquad c_1, c_2 \in \mathbb{C}.$$

Setn. 2.6 Reellar loysnir til $\dot{x} = Ax$ fáast við

(i) fyri reel λ er ein loysn $x(t) = e^{\lambda t} v$.

(ii) fyri komplekst par $a \pm i\omega$ velja vit eina loysn. Við $a + i\omega$ og v eru loysnir
$$x(t) = \operatorname{Re}(e^{\lambda t} v) = e^{at}\left(\cos(\omega t) \operatorname{Re} v - \sin(\omega t) \operatorname{Im} v\right) \text{ og}$$
$$x(t) = \operatorname{Im}(e^{\lambda t} v) = e^{at}\left(\sin(\omega t) \operatorname{Re} v + \cos(\omega t) \operatorname{Im} v\right)$$

Pf. Skriva $v = \operatorname{Re} v + i \operatorname{Im} v$:
$$e^{\lambda t} v = e^{at}\left(\cos(\omega t) + i \sin(\omega t)\right)\left(\operatorname{Re} v + i \operatorname{Im} v\right) = e^{at}\left(\cos(\omega t) \operatorname{Re} v - \sin(\omega t) \operatorname{Im} v\right) + i e^{at}\left(\sin(\omega t) \operatorname{Re} v + \cos(\omega t) \operatorname{Im} v\right).$$
$\square$

Dømi

kompleks → reel

$$e^{(3+2i)t}\begin{bmatrix}1\\-2i\end{bmatrix} = e^{2t}\left(\cos(2t)+i\sin(2t)\right)\begin{pmatrix}1\\-2i\end{pmatrix}$$

$$= e^{2t}\left(\cos(2t)+i\sin(2t)\right)\left(\begin{pmatrix}1\\0\end{pmatrix}+i\begin{pmatrix}0\\-2\end{pmatrix}\right)$$

$$= e^{3t}\begin{pmatrix}\cos 2t\\2\sin 2t\end{pmatrix} + i\,e^{3t}\begin{pmatrix}\sin(2t)\\-2\cos(2t)\end{pmatrix}$$

$$x(t) = c_1 e^{3t}\begin{pmatrix}\cos 2t\\2\sin 2t\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}\sin 2t\\-2\cos 2t\end{pmatrix}, \quad c_1,c_2\in\mathbb{R}.$$

Dømi

$$\dot{x} = Ax = \begin{pmatrix}-1 & -2\\2 & -1\end{pmatrix}x$$

$$\det(A-\lambda I) = (-1-\lambda)^2+4 = \lambda^2+2\lambda+5 = 0$$

$$\Leftrightarrow \lambda = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i$$

$$-1+2i: \quad \begin{pmatrix}-2i & -2\\2 & -2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \underline{0} \Rightarrow \begin{matrix}v_1 = i\\v_2 = 1\end{matrix}$$

$$\therefore\ x(t) = c_1 e^{(-1+2i)t}\begin{pmatrix}i\\1\end{pmatrix} + c_2 e^{(-1-2i)t}\begin{pmatrix}-i\\1\end{pmatrix}, \quad c_1,c_2\in\mathbb{C}.$$

Reelt:

$$e^{(-1+2i)t}\begin{pmatrix}i\\1\end{pmatrix} = e^{-t}\left(\cos(2t)+i\sin(2t)\right)\left(\begin{pmatrix}0\\1\end{pmatrix}+i\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= e^{-t}\begin{pmatrix}-\sin(2t)\\\cos(2t)\end{pmatrix} + i\,e^{-t}\begin{pmatrix}\cos(2t)\\\sin(2t)\end{pmatrix}$$

$$\therefore\ x(t) = c_1 e^{-t}\begin{pmatrix}-\sin(2t)\\\cos(2t)\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}\cos(2t)\\\sin(2t)\end{pmatrix}, \quad c_1,c_2\in\mathbb{R}.$$

So mugu vit aftur klára multiplicitet.

Setn. 2.10 (a) Um ein reel rót λ hevur $p\geq 2$ og $q<p$, so finnast vektorar $b_{ij}\in\mathbb{R}^n$, so at

$$x_1(t) = b_{11}e^{\lambda t}$$
$$x_2(t) = b_{21}e^{\lambda t} + b_{22}te^{\lambda t}$$
$$\vdots$$
$$x_p(t) = b_{p1}e^{\lambda t} + b_{p2}te^{\lambda t} + \dots + b_{pp}t^{p-1}e^{\lambda t}.$$

(b) Kompleks rót λ

$$x_1(t) = Re(b_1 e^{\lambda t}) \qquad x_{p+1}(t) = Im(b_1 e^{\lambda t})$$
$$x_2(t) = Re(b_{21}e^{\lambda t} + b_{22}te^{\lambda t}) \qquad x_{p+2}(t) = Im(b_{21}e^{\lambda t} + b_{22}te^{\lambda t})$$
$$\vdots$$
$$x_p(t) = Re(b_{p1}e^{\lambda t} + b_{p2}te^{\lambda t} + \dots + b_{pp}t^{p-1}e^{\lambda t}) \qquad x_{2p}(t) = Im(b_{p1}e^{\lambda t} + b_{p2}te^{\lambda t} + \dots + b_{pp}t^{p-1}e^{\lambda t})$$

Setn. 2.11 Fullkomuliga loysnin hjá $\dot{x} = Ax$ er linearkombinatión av loysnunum í 2.10.

Dømi $\dot{x} = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} x$, $\det(A-\lambda I) = (1-\lambda)(2-\lambda)(-\lambda) + (1-\lambda) = 0$

$\iff (1-\lambda)\left((2-\lambda)(-\lambda)+1\right) = 0$, $\lambda = 1$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0 \iff (\lambda-1)^2 = 0$, $\lambda = 1$ $p = 3$

Eginvektor: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} v = \underline{0} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Tvær loysnir $e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ og $e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Vit gita nú $x(t) = b_1 e^t + b_2 t e^t$ og fáa

$\dot{x} = b_1 e^t + b_2 e^t + b_2 t e^t$ meðan $Ax = A(b_1 + b_2 t) e^t$.

$b_1 + b_2 = A b_1$ og $b_2 = A b_2$. Her er b_2 eginvektorur.

Set $b_2 = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, so finna vit at $b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ tí

$(A-I)b_1 = b_2 \iff \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} b_1 = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, har fyrri vektorurin ikki kann produccrast.

So $k_1 = 0$ og lat $k_2 = 1$. Vel nú $b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, so er hetta ein loysn.

Fullkomuliga loysnin er nú

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) e^t, \quad c_1, c_2, c_3 \in \mathbb{R}$$

Fundamental Matrican. Lat $x_1(t), \dots, x_n(t)$ vera n lineert óheftar loysnir hjá $\dot{x} = Ax$, so er

$$\Phi(t) = (x_1(t), \dots, x_n(t)), \quad \text{har } t \in I,$$

Fundamentalmatrican hjá systeminum.

Setn. 2.15 Φ er reguler fyri øll $t \in I$.

Pf. Søjlurnar eru lineert óheftar og Φ er $n \times n$, so Φ er invertibel.

Setn. 2.16 Fyri Φ galda

(i) $\dot{\Phi} = A\,\Phi$

(ii) Reella loysnin hjá $\dot{x} = Ax$ kann skrivast

$$x(t) = \Phi(t)\,c\,, \quad \text{har } c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n.$$

Ein partikuler loysn við $x(t_0) = x_0$ kann skrivast

$$x(t) = \Phi(t)\,[\Phi(t_0)]^{-1}\,x_0.$$

Inhomogen system $\quad \dot{x} = Ax + u \quad$ og vit kunnu enn gita loysnir sum vanligt.

Setn. 2.17 Fyri $t_0 \in I$ og $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ er ein loysn so at $x(t_0) = v$.

Setn. 2.19 Lat Φ vera fundamentalmatrica hjá $\dot{x} = Ax + u$ við reella ávirkan og $t_0 \in I$.

(i) Allar reellar loysnir eru givnar við

$$x(t) = \Phi(t)\,c + \Phi(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau\,, \quad t \in I, \quad c = (c_1, \ldots, c_n) \in \mathbb{R}^n.$$

(ii) Partikulera loysnin við $x(t_0) = x_0$ er givin við.

$$x(t) = \Phi(t)\,[\Phi(t_0)]^{-1}\,x_0 + \Phi(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau\,, \quad t \in I.$$

Pf. Lat $x(t) = \Phi(t)\,[\Phi(t_0)]^{-1}\,x_0 + \Phi(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau$, so hava vit

$$\dot{x}(t) = \dot{\Phi}(t)\,[\Phi(t_0)]^{-1}\,x_0 + \dot{\Phi}(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau + \Phi(t)\,[\Phi(t)]^{-1}\,u(t)$$

$$= A\,\Phi(t)\,[\Phi(t_0)]^{-1}\,x_0 + A\,\Phi(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau + u(t)$$

$$= A\left(\Phi(t)\,[\Phi(t_0)]^{-1}\,x_0 + \Phi(t)\int_{t_0}^{t} [\Phi(\tau)]^{-1}\,u(\tau)\,d\tau\right) + u(t). \qquad \square$$

Dømi 2.20 $\quad \ddot{y}(t) + a_1\,\dot{y}(t) + a_2\,y(t) = q(t)$

Fyrst hyggja vit at

$$\ddot{y}(t) + a_1\,\dot{y}(t) + a_2\,y(t) = 0.$$

Vit definera $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$. Nú er $\dot{x}_1 = x_2$.

Vit fáa soleiðis

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ q \end{pmatrix}$$

Givið loysnirnar $\begin{pmatrix} y_1 \\ \dot{y}_1 \end{pmatrix}$ og $\begin{pmatrix} y_2 \\ \dot{y}_2 \end{pmatrix}$ til systemið omanfyri hava vit

$$\Phi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} \implies \Phi(t)^{-1} = \frac{1}{W(t)} \begin{pmatrix} \dot{y}_2(t) & -y_2(t) \\ -\dot{y}_1(t) & y_1(t) \end{pmatrix} \quad \text{við } W(t) = \det(\Phi(t)).$$

$$\Phi(t)^{-1} \begin{pmatrix} 0 \\ q \end{pmatrix} = \frac{q(t)}{W(t)} \begin{pmatrix} -y_2(t) \\ y_1(t) \end{pmatrix}, \quad \text{so loysnin er}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} \left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \int_{t_0}^{t} \frac{q(\tau)}{W(\tau)} \begin{pmatrix} -y_2(\tau) \\ y_1(\tau) \end{pmatrix} d\tau \right)$$

Loysnin til upprunaliga systemið er nú

$$y = x_1 = c_1 y_1 + c_2 y_2 + y_1 \int_{t_0}^{t} \frac{q\tau}{W(\tau)} \left(-y_2(\tau)\right) d\tau + y_2 \int_{t_0}^{t} \frac{q(\tau)}{W(\tau)} y_1(\tau) d\tau .$$