Próvtøka 29. mai 2020

Opg. 1. (i) $y'' - 6y' + 9y = 0$ við $P(\lambda) = (\lambda - 3)^2$

Dupultrót $\lambda = 3$, so við 1.15 er $y(t) = c_1 e^{3t} + c_2 t e^{3t}$, $c_1, c_2 \in \mathbb{R}$.

(ii) $\dot{x} = Ax$, $A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = (-1-\lambda)(-2-\lambda) - 2 = \lambda^2 + 3\lambda = 0 \iff \lambda = 0 \vee \lambda = -3.$$

$$(A - 0I)v = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - (-3)I)u = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} u = 0 \Rightarrow u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

(iii) $P(\lambda) = \lambda^3 + \lambda^2 + (4-c)\lambda + c$.

$$4 - c > 0 \iff c < 4 \quad \text{og} \quad c > 0 \Rightarrow 0 < c < 4$$

$$\det \begin{pmatrix} 1 & c \\ 1 & 4-c \end{pmatrix} = 4 - c - c = 4 - 2c > 0 \iff c < 2.$$

$0 < c < 2$.

(iv) $\sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2+1} = 2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}$, $\frac{1}{n^2+1} \leq \frac{1}{n^2}$ $\forall n \in \mathbb{N}$ og $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konv.

$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2+1}$ er abs. konv.

(v) $\sum_{n=0}^{\infty} \frac{3^n}{2^n} x^n = \sum_{n=0}^{\infty} \left(\frac{3x}{2}\right)^n$, sum fyri $\left|\frac{3x}{2}\right| < 1 \iff |x| < \frac{2}{3}$

er altso $\frac{1}{1 - \frac{3x}{2}} = \frac{1}{\frac{2-3x}{2}} = \frac{2}{2-3x}$.

(vi) $\sum_{n=1}^{\infty} \frac{1}{n^2} (2 \cos nx + 3 \sin nx)$

$\left|\frac{1}{n^2}(2\cos nx + 3 \sin nx)\right| \leq \frac{1}{n^2}(2+3) = \frac{5}{n^2}$. $\sum_{n=1}^{\infty} \frac{5}{n^2}$ er konv. majorant.

Opg. 2 $$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 30y = u(t)$$

(i) Reel loysn til homogenu líkningina.

$$P(\lambda) = \lambda^3 + \lambda^2 + 4\lambda + 30 = 0 \iff \lambda = -3, \ \lambda = 1 \pm 3i.$$

Per 1.15 hava vit

$$y(t) = c_1 e^{-3t} + c_2 e^t \cos(3t) + c_3 e^t \sin(3t), \quad c_1, c_2, c_3 \in \mathbb{R}.$$

(ii) Skriva $H(s)$.

Við (1.20) hava vit

$$H(s) = \frac{1}{s^3 + s^2 + 4s + 30}, \quad s \notin \{-3, 1 \pm 3i\}.$$

(iii) Stationert svar til $u(t) = \sin(2t)$.

Við 1.27(ii) er $$y(t) = \mathrm{Im}\left(H(2i)\, e^{2it}\right) = \mathrm{Im}\left(\frac{1}{26} e^{2it}\right)$$
$$= \frac{1}{26}\sin(2t).$$

(iv) Fullkomulig kompleks loysn hjá $D_3(y) = e^t$.

Við 1.24 er stationera svarið til $u(t) = e^t$ akkurát

$$y(t) = H(1) \cdot e^t = \frac{1}{36} e^t$$

Úr struktursetningínum og 1.15 er loysnin

$$y(t) = c_1 e^{-3t} + c_2 e^{(1+3i)t} + c_3 e^{(1-3i)t} + \frac{1}{36} e^t, \quad c_1, c_2, c_3 \in \mathbb{C}.$$

Opg. 3 Ein funktión f er 2π-per., stk.vís diff. og kont. Fourierrekkjan er

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \sin(nx), \quad x \in \mathbb{R}.$$

(i) Finn a_n og b_n. Hetta er ein ólíka funktión per korollar 6.4.
Altso $a_n = 0 \quad \forall n \in \mathbb{N}_0$. $b_n = \frac{1}{n^2+1} \quad \forall n \in \mathbb{N}$.

(ii) Grundgev fyri, at $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+1} \sin(nx) \quad , \quad x \in \mathbb{R}.$$

f er 2π-per., stk.vís diff. og kont., so talan er um uniforman konvergens. Korollar 6.13(i) gevur niðurstøðuna.

(iii) Avger um f er lika, ólíka ella hvørgin.

Aftur korollar 6.4 $\Rightarrow$ ólíka.

(iv) Finn $N \in \mathbb{N}$ so, at $|f(x) - S_N(x)| \leq 0{,}1 \quad \forall x \in \mathbb{R}$.

Við setning 6.17 er
$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2+1} \leq \int_{N+1}^{\infty} \frac{1}{1+x^2}\,dx + \frac{1}{(N+1)^2+1}$$
$$= \lim_{t\to\infty} \left[\arctan(x)\right]_{N+1}^{t} + \frac{1}{(N+1)^2+1}$$
$$= \frac{\pi}{2} - \arctan(N+1) + \frac{1}{(N+1)^2+1} \quad , \quad N \in \mathbb{N},\ x \in \mathbb{R}.$$

Her er $\frac{\pi}{2} - \arctan(N+1) + \frac{1}{(N+1)^2+1} \leq 0{,}1$ um $N \geq 10$.
Vel tískil $N = 10$.

(v) Vís, at $P(f) > 0{,}15$.

Við Parseval: $$P(f) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} > \frac{1}{2} \sum_{n=1}^{3} \frac{1}{(n^2+1)^2}$$
$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{25} + \frac{1}{100}\right) = 0{,}15.$$

Opg. 4 (i) Um $y(t) = \sum_{n=0}^{\infty} c_n t^n$ er loysn hjá $ty'' - 2y = 0$, vís $c_0 = 0$ og
$$c_{n+1} = \frac{2c_n}{(n+1)n}.$$

$$t\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} - 2\sum_{n=0}^{\infty} c_n t^n = \sum_{n=1}^{\infty} c_{n+1}(n+1)n\,t^n - 2c_0 - 2\sum_{n=1}^{\infty} c_n t^n$$
$$= -2c_0 + \sum_{n=1}^{\infty} \left(c_{n+1}(n+1)n - 2c_n\right)t^n = 0$$
$$\overset{5.21}{\Rightarrow} \quad c_0 = 0 \quad \text{og} \quad c_{n+1}(n+1)n - 2c_n = 0 \Leftrightarrow c_{n+1} = \frac{2c_n}{(n+1)n} \quad , \quad n \geq 1.$$

(ii) Finn konvergensradius hjá $y(t) = \sum_{n=0}^{\infty} c_n t^n = \sum_{n=1}^{\infty} c_n t^n$.

Kvotientkriteriið:

$$\left| \frac{c_{n+1}\, t^{n+1}}{c_n\, t^n} \right| = \left| \frac{\frac{2c_n}{(n+1)n}\, t^{n+1}}{c_n\, t^n} \right| = \frac{2}{(n+1)n} |t| \to 0 \quad \text{tá } n \to \infty \quad \forall t \in \mathbb{R}.$$

Tískil er $\rho = \infty$.