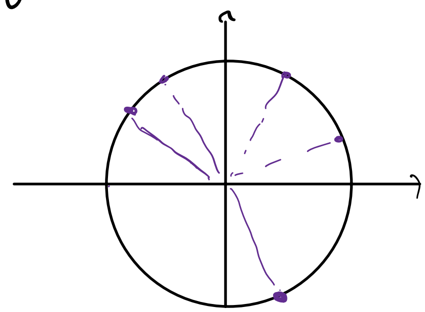
Complex Numbers in Polar form

Ex 1. $\sin\left(\frac{3\pi}{2}\right) = -1$

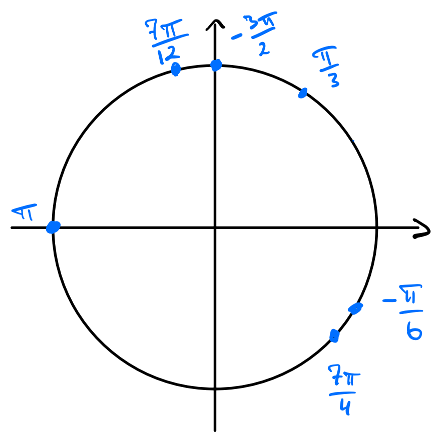
Ex 2. a) Radian values of angles in degrees:

Degrees	Radians	
30	$\frac{\pi}{6}$	
60	$\frac{\pi}{3}$	
120	$\frac{2\pi}{3}$	
135	$\frac{2\pi}{3} + \frac{\pi}{12} = \frac{3\pi}{4}$	$180 - 135 = 45 \ldots$
300	$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$	$360 - 300 = 60$



b) Point and degrees corresponding to

$\pi, \frac{\pi}{3}, -\frac{\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{12}, -\frac{3\pi}{2}, \frac{7\pi}{4}$

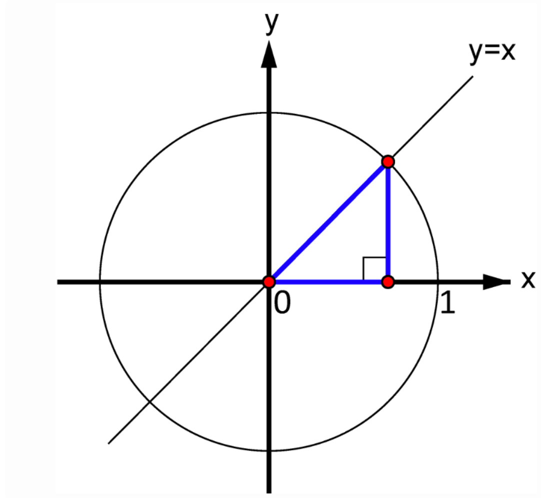


$\frac{\pi}{6}$ is 30°, so $\frac{\pi}{12}$ is 15°

Radians	Degrees
π	180
$\frac{\pi}{3}$	60
$-\frac{\pi}{6}$	-30 or 330
$\frac{7\pi}{12}$	105
$-\frac{3\pi}{2}$	-270 or 90
$\frac{7\pi}{4}$	315 or -45

Ex 3. Sine and Cosine

a) Use the triangle to compute $\sin(\frac{\pi}{4})$ and $\cos(\frac{\pi}{4})$.



Since $x = y$ it follows that

$$\cos(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4})$$

by definition. Additionally the Pythagorean theorem yields

$$\cos^2(\tfrac{\pi}{4}) + \sin^2(\tfrac{\pi}{4}) = 1$$, let's use x as unknown.

$$\Leftrightarrow \quad 2x^2 = 1$$

$$\Leftrightarrow \quad x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$ (Note positive solution!)

b) Compute solutions to pairs via

$$\cos(p\tfrac{\pi}{4}) \text{ and } \sin(p\tfrac{\pi}{4})$$

for $p = 3, 5, 7, -1, -3, -5, -7$ via symmetry.

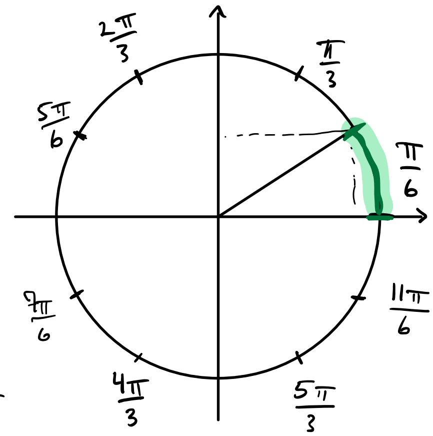
$p=3,-5$: $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$p=5,-3$: $\cos\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$p=7,-1$: $\cos\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$p=1,-7$: $\cos\left(\frac{-7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(\frac{-7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

c) Given $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, draw $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$ and find similar variations for $p = 2, 4, 5, 7, 8, 10, 11$



$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$

$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$, $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

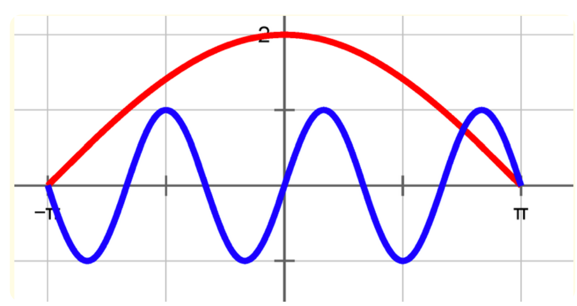
$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$, $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$

d) $x \mapsto A\cos(bx)$

$x \mapsto C\sin(dx)$

Determine the coefficients from the graphs.



cosine since $\neq 0$ for $x = 0$. Amplitude $A = 2$ and angular frequency $b = \frac{1}{2}$. (Half a period)

sine since $= 0$ for $x = 0$. Amplitude $C = 1$ and angular frequency $d = 3$. (Three periods)

Ex 4. Polar coordinates

a) Given $z_0 = 1+i\sqrt{3}$, $z_1 = -1+i\sqrt{3}$, $z_2 = -1-i\sqrt{3}$ and $z_3 = 1-i\sqrt{3}$

1. State the circle centered at 0, which the numbers lie on.

$$\sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$$

$$C = \{ z \in \mathbb{C} \mid |z| = 2 \}$$

2. Compute $\arg(z_0)$ and state the principal argument of z_1, z_2 and z_3. Additionally what are the polar coordinates?

Firstly $z_0 = 1+i\sqrt{3}$ with $|z_0| = 2$. Let's denote the angle by v, then

$$\cos(v) = \frac{1}{2} \quad \text{and} \quad \sin(v) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \arg(z_0) = \frac{\pi}{3} + p \cdot 2\pi, \ p \in \mathbb{Z}.$$

See arg vs. Arg in note (1-17) following definition 1.27.

$$\operatorname{Arg}(z_1) = \frac{2\pi}{3}, \quad \operatorname{Arg}(z_2) = -\frac{2\pi}{3}, \quad \operatorname{Arg}(z_3) = -\frac{\pi}{3}$$

since $\operatorname{Arg}(z)$ is the angle for which $\arg(z) \in]-\pi, \pi]$.

By definition 1.26 the polar coordinates are

$$\left(2, \frac{\pi}{3}\right), \ \left(2, \frac{2\pi}{3}\right), \ \left(2, -\frac{2\pi}{3}\right), \ \left(2, -\frac{\pi}{3}\right)$$

b) For $z = 2-2i$ a student gets the argument $\frac{\pi}{4}$. Clearly the student didn't remember to evaluate both for sine and cosine. The argument $\frac{\pi}{4}$ satisfies

$$\cos\left(\frac{\pi}{4}\right) = \frac{2}{2\sqrt{2}},$$

however this is not the case for

$$\sin\left(\frac{\pi}{4}\right) \neq \frac{-2}{2\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right)$$

c) $$|-2+2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

$$\operatorname{Arg}(-2+2i) = \frac{3\pi}{4}$$

$$\left|-\frac{1}{6} + \frac{1}{2\sqrt{3}}i\right| = \sqrt{\left(-\frac{1}{6}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2} = \sqrt{\frac{1}{36} + \frac{3}{36}} = \frac{2}{6} = \frac{1}{3}$$

$$\cos(v) = \frac{-\frac{1}{6}}{\frac{1}{3}} = -\frac{1}{2} \qquad \operatorname{Arg}\left(-\frac{1}{6} + \frac{1}{2\sqrt{3}}i\right) = \frac{2\pi}{3}$$

Quite unnecessary to check sine, but let's show it's $\frac{\sqrt{3}}{2}$:

$$\sin(v) = \frac{\frac{1}{2\sqrt{3}}}{\frac{1}{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

d) Rectangular form

$$\left(4, -\pi\right) = -4, \qquad \left(2, \frac{4\pi}{3}\right) = -1 - i\sqrt{3}$$

$$\left(6, \frac{21\pi}{4}\right) = \left(6, \frac{5\pi}{4}\right) = -3\sqrt{2} - 3\sqrt{2}\,i$$

Ex 5. a) $e^{\frac{\pi}{2}i} = i$ and $3e^{1+\pi i} = -3e$

b) Convert $z_0 = 1 + i\sqrt{3}$, $z_1 = -\sqrt{3} + i$, $z_2 = -1 - i\sqrt{3}$, $z_3 = \sqrt{3} - i$ to exponential form.

$$z_0 = 2e^{\frac{\pi}{3}i}, \quad z_1 = 2e^{\frac{5\pi}{6}i}, \quad z_2 = 2e^{-\frac{2\pi}{3}i}, \quad z_3 = 2e^{-\frac{\pi}{6}i}$$

Show that the binomial equation

$$z^4 = w,$$

where w is a complex scalar, exists to which all four numbers are a solution.

$$z_0^4 = z_1^4 = z_2^4 = z_3^4 = 16e^{-\frac{2\pi}{3}i}$$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \text{and} \quad \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

For $w = a + ib$ we have

$$a = -\frac{1}{2} \cdot 16 = -8 \quad \text{and} \quad b = -\frac{\sqrt{3}}{2} \cdot 16 = -8\sqrt{3}.$$

Thus the rectangular form is

$$w = -8 - 8\sqrt{3}\, i.$$

Ex 6. a) Let $w = 1 - i$ and compute

$$|w| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \mathrm{Arg}(w) = -\frac{\pi}{4}$$

$$|e^w| = |e \cdot e^{-i}| = |e| \cdot |e^{-i}| = e$$

$$\mathrm{Arg}(e^w) = -1$$

b) Given $w_1 = 1$, $w_2 = e$, $w_3 = i$ and $w_4 = 2i$ determine all solutions to

$$e^z = w_n \quad \text{for } n = 1, \dots, 4.$$

$$e^z = 1 \iff z = 2p\pi i, \quad p \in \mathbb{Z}$$

$$e^z = e \iff z = 1 + 2p\pi i, \quad p \in \mathbb{Z}$$

$$e^z = i \iff z = \left(\frac{\pi}{2} + 2p\pi\right) i, \quad p \in \mathbb{Z}$$

$$e^z = 2i \iff z = \ln(2) + \left(\frac{\pi}{2} + 2p\pi\right) i, \quad p \in \mathbb{Z}$$

Determine all solutions to

$$(e^z - 1) \cdot (e^z - i) = 0$$

$$\iff \quad z = 2p\pi i \ \vee \ z = \left(\frac{\pi}{2} + 2p\pi\right) i, \quad p \in \mathbb{Z}.$$

c) Show that $e^z \neq 0$ for all $z \in \mathbb{C}$.

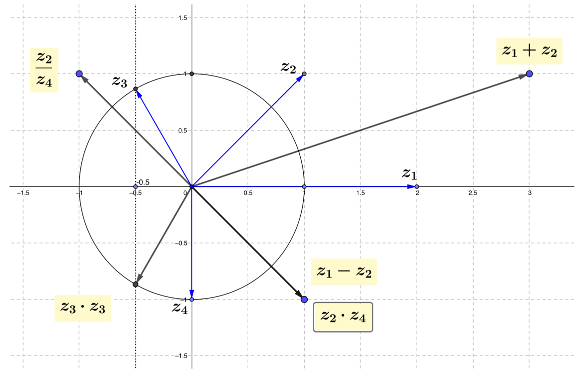
Let $z = a + ib$ for $a, b \in \mathbb{R}$, then

$$e^{a+ib} = e^a \left(\cos(b) + i \sin(b) \right)$$

$$= e^a \cos(b) + i \, e^a \sin(b).$$

Note that the result is zero only if both real and imaginary parts are zero. However, for given $b \in \mathbb{R}$ there is no value for which both $\cos(b) = 0 = \sin(b)$. Thus at least one of the terms must remain non-zero, i.e. $e^z \neq 0$ for all $z \in \mathbb{C}$ as $e^a \neq 0$ for all $a \in \mathbb{R}$.

Ex 7. Draw complex numbers in Geogebra



$$z_1 + z_4 = 2 + 1 + i = 3 + i$$

$$z_1 - z_2 = 2 - 1 - i = 1 - i$$

$$z_3 \cdot z_3 = \left(e^{\frac{2\pi}{3} i}\right)^2 = e^{\frac{4\pi}{3} i}$$

$$z_2 \cdot z_4 = \sqrt{2} e^{\frac{\pi}{4} i} \cdot e^{\frac{6\pi}{4} i} = \sqrt{2} e^{\frac{7\pi}{4} i} = \sqrt{2} e^{-\frac{\pi}{4} i}$$

$$\frac{z_2}{z_4} = \sqrt{2} e^{-\frac{5\pi}{4} i} = \sqrt{2} e^{\frac{3\pi}{4} i}$$

Ex 8. Double angles

a)

$$\sin(2v) = 2\sin(v)\cos(v), \quad \cos(2v) = \cos^2(v) - \sin^2(v)$$

Use this to determine $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{12}\right)$.

Note that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ can be written as $\cos\left(2\frac{\pi}{8}\right)$.

Thus in turn we have

$$\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$$

$$\Leftrightarrow \quad \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2} + \sin^2\left(\frac{\pi}{8}\right)$$

$$\Leftrightarrow \quad \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2} + 1 - \cos^2\left(\frac{\pi}{8}\right)$$

$$\Leftrightarrow \quad \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{4} + \frac{2}{4}$$

$$\Leftrightarrow \quad \cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$$

Similarly for $\sin\left(\frac{\pi}{12}\right)$ we use $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \cos\left(2\frac{\pi}{12}\right)$.

$$\cos^2\left(\frac{\pi}{12}\right) - \sin^2\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \quad \sin^2\left(\frac{\pi}{12}\right) = \cos^2\left(\frac{\pi}{12}\right) - \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \quad \sin^2\left(\frac{\pi}{12}\right) = 1 - \sin^2\left(\frac{\pi}{12}\right) - \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \quad \sin\left(\frac{\pi}{12}\right) = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

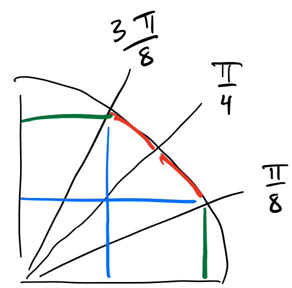
(Alternate form: $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$)

b) Use a) to find $\sin\left(\frac{\pi}{8}\right)$, $\cos\left(\frac{3\pi}{8}\right)$ and $\sin\left(\frac{3\pi}{8}\right)$.

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{1-\cos^2\left(\frac{\pi}{8}\right)} = \sqrt{1-\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right)^2}$$
$$= \sqrt{1-\frac{1}{2}-\frac{\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

By symmetry

$$\cos\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$$
$$\sin\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$$



c) Similarly find interesting angles of the form $\frac{p\pi}{12}$.

$$\cos\left(\frac{\pi}{12}\right) = \sqrt{1-\left(\frac{1}{2}\sqrt{2-\sqrt{3}}\right)^2} = \sqrt{1-\frac{1}{2}+\frac{\sqrt{3}}{4}}$$
$$= \frac{1}{2}\sqrt{2+\sqrt{3}}$$

$$\Rightarrow \quad \cos\left(\frac{5\pi}{12}\right) = \frac{1}{2}\sqrt{2-\sqrt{3}} \quad \text{and} \quad \sin\left(\frac{5\pi}{12}\right) = \frac{1}{2}\sqrt{2+\sqrt{3}}.$$

We note that only $p=1$ and 5 are interesting, since $p = 2, 3, 4$ and 6 are known ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$).