5.3.1 (a) $f(n) = 1001$

$f(2n) = 1001 = f(n)$

(b) $f(n) = 3n$

$f(2n) = 3 \cdot 2n = 6n = 2f(n)$

(c) $f(n) = 5n^2$

$f(2n) = 5 \cdot (2n)^2 = 20n^2 = 4f(n)$

(d) $f(n) = 2{,}5n^3$

$f(2n) = 2{,}5 \cdot (2n)^3 = 8f(n)$

5.3.3 Show $g(n) = n!$ is $O(n^n)$.

Since $n! = 1 \cdot 2 \cdot 3 \cdots n \leq n \cdot n \cdot n \cdots n = n^n$ for $n \in \mathbb{Z}_+$ it follows by definition that $g(n)$ is $O(n^n)$.

5.3.6 Show $g(n) = n^2(7n-2)$ is $O(n^3)$.

Since $n^2 \cdot (7n-2) = 7n^3 - 2n^2 \leq 7n^3$ for $n \in \mathbb{Z}_+$, then $g(n)$ is $O(n^3)$.

5.3.8 Show $f(n) = n^{100}$ is $O(g)$ for $g(n) = 2^n$, but also that g is not $O(f)$.

We have that $\Theta(n^k) < \Theta(a^n)$ for $k>0$ and $a>1$. Thus n^{100} is $O(2^n)$, but 2^n is not $O(n^{100})$.

5.3.9 Show f and g have the same order for

$$f(n) = 5n^2+4n+3 \quad \text{and} \quad g(n) = n^2+100n.$$

Recall:

$\Theta(n^a) < \Theta(n^b)$ if $0<a<b$.

$\Theta(r \cdot f) = \Theta(f)$ for function f and constant $r \neq 0$.

$\Theta(f+g) = \Theta(g)$ if $\Theta(f) < \Theta(g)$.

$\Rightarrow \quad \Theta(f) = \Theta(n^2) = \Theta(g)$

5.3.4 Show $h(n) = 1+2+3+\cdots+n$ is $O(n^2)$.

Since $h(n) \leq n+n+n+\cdots+n = n^2$ for $n \in \mathbb{Z}_+$, so h is $O(n^2)$.

5.3.11-12 Determine which functions are in the same Θ-class, and rank them.

$f_1(n) = 5n\lg(n)$, $f_2(n) = 6n^2-3n+7$, $f_3(n) = 1{,}5^n$, $f_4(n) = \lg(n^4)$,

$f_5(n) = 13\,463$, $f_6(n) = -15n$, $f_7(n) = \lg(\lg(n))$, $f_8(n) = 9n^{0{,}7}$,

$f_9(n) = n!$, $f_{10}(n) = n+\lg(n)$, $f_{11}(n) = \sqrt{n}+12n$, $f_{12}(n) = \lg(n!)$.

We make use of the "rules for Θ-classes".

We have $\Theta(f_6) = \Theta(f_{10}) = \Theta(f_{11})$ by 2, 3, 6 and 8.

As $\lg(n!) = \lg(1\cdot 2\cdot 3\cdot \cdots \cdot n) \leq \lg(n^n) = n\lg(n) \leq 5n\lg(n)$, for $n \geq 1$, then f_{12} is $O(f_1)$.

Conversely $\lg(n!) = \lg(1) + \lg(2) + \cdots + \lg(n) = \sum_{k=1}^{n} \lg(k) \geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \lg(k) \geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \lg(\frac{n}{2}) \geq \frac{n-1}{2} \lg(\frac{n}{2}) = \frac{1}{2} n \lg(n) - \frac{1}{2}\lg(n)$.
So $5n\lg(n)$ is $O(\lg(n!))$ and $\Theta(f_1) = \Theta(f_{12})$.

$\lim_{n\to\infty} \frac{13463}{\lg(\lg(n))} = 0$, so $\Theta(f_5) < \Theta(f_7)$.

For $n \geq 2$ $\lg(\lg(n)) \leq \lg(n^4)$, so f_7 is $O(f_4)$. However, the converse is not the case, since for 2^n

$$\lg(\lg(2^n)) = \lg(n) \quad \text{and} \quad \lg((2^n)^4) = 4n,$$

which is not $O(f_7)$. So $\Theta(f_7) < \Theta(f_4)$.

By 2 and 6 $\Theta(f_4) < \Theta(f_8)$. Now 3 and 6 yield $\Theta(f_8) < \Theta(f_6)$.
Since $\Theta(1) < \Theta(\lg(n))$ 6, 7 and 8 give $\Theta(f_{10}) < \Theta(f_1)$.

With $\Theta(\lg(n)) < \Theta(n)$ 3, 6, 7 and 8 give us $\Theta(f_1) < \Theta(f_2)$. By 3, 4, 6 and 8 $\Theta(f_2) < \Theta(f_3)$.
We have $\frac{1{,}5^n}{n!} \leq 1{,}5\left(\frac{1{,}5}{2}\right)^{n-1}$ for $n \geq 2$, so $\lim_{n\to\infty} \frac{1{,}5^n}{n!} = 0$, so $\Theta(f_3) < \Theta(f_9)$.

Ultimately $\Theta(f_5) < \Theta(f_7) < \Theta(f_4) < \Theta(f_8) < \Theta(f_6) = \Theta(f_{10}) = \Theta(f_4) < \Theta(f_1) = \Theta(f_{12}) < \Theta(f_2) < \Theta(f_3) < \Theta(f_9)$.

5.3.24 Prove $\Theta(a^n)$ is lower than $\Theta(b^n)$ if and only if $0 < a < b$.

Assume $0 < a < b$. Then $0 < \frac{a}{b} < 1$ and so

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0.$$

Therefore $\Theta(a^n) < \Theta(b^n)$.

The opposite direction follows by the contrapositive.
Suppose $a \geq b$, then $\Theta(a^n) \geq \Theta(b^n)$ by the above argument. Thus $\Theta(a^n) < \Theta(b^n) \iff 0 < a < b$. □