401.
$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u(t) \\ y = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases}$$

(a) Finn $H(s)$. $\quad (A - sI) = \begin{pmatrix} 2-s & 3 \\ 0 & 4-s \end{pmatrix}, \quad (A-sI)^{-1} = \frac{1}{(2-s)(4-s)} \begin{pmatrix} 4-s & -3 \\ 0 & 2-s \end{pmatrix}.$

$$H(s) = -\begin{pmatrix} 3 & 4 \end{pmatrix} \frac{1}{(2-s)(4-s)} \begin{pmatrix} 4-s & -3 \\ 0 & 2-s \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{(2-s)(4-s)} \cdot \left(-\begin{pmatrix} 3 & 4 \end{pmatrix}\right) \begin{pmatrix} 8-2s \\ 0 \end{pmatrix} = \frac{1}{(2-s)(4-s)} \cdot (6s - 24)$$

$$= \frac{-6(4-s)}{(2-s)(4-s)} = \frac{6}{s-2}, \quad s \notin \{2, 4\}.$$

(b) Finn y til $u(t) = e^{st}$.

$$H(S) = \frac{6}{s-2} = 2 \implies y(t) = H(S)\, e^{st} = 2e^{st}.$$

402.
$$\begin{cases} \frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = x_2(t) \end{cases}$$

(i) Finn $H(s)$. $\quad A - sI = \begin{pmatrix} -s & 1 \\ 0 & 1-s \end{pmatrix}, \quad (A-sI)^{-1} = \frac{1}{s(s-1)} \begin{pmatrix} 1-s & -1 \\ 0 & -s \end{pmatrix}.$

$$H(s) = -\begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{s(s-1)} \begin{pmatrix} 1-s & -1 \\ 0 & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{-1}{s(s-1)} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -s \end{pmatrix} = \frac{-1}{s(s-1)} \cdot (-s)$$

$$= \frac{1}{s-1}, \quad s \notin \{0, 1\}.$$

(ii) Finn y til $u(t) = e^{-t}$.

$$H(-1) = \frac{1}{-1-1} = -\frac{1}{2} \implies y(t) = H(-1)\, e^{-t} = -\frac{1}{2} e^{-t}.$$

411. Let $\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = 2x_1 - x_3 \\ \dot{x}_3 = -x_1 \end{cases}$

(iii) Er systemið stabilt?

$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ hevur eginvirðini $\lambda = 0 \vee \lambda = \pm i$ fylgi (i), sum øll hava multiplicitetin $p = 1 = q$. Per setning 2.34 er systemið stabilt, tí $Re(\lambda) = 0$ fyri øll λ og algebraiski multiplicteturin er eins við tann geometriska.

(iv) Er systemið asymptodiskt stabilt?

Setningur 2.36 sigur, at systemið bert er asymptodiskt stabilt, um $Re(\lambda) < 0$ fyri øll eginvirðir λ. Vit hava fyri øll okkeru λ, at $Re(\lambda) = 0$, so systemið er ikki asymptodiskt stabilt.

417. $\begin{cases} \dfrac{dx}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t) \\ y(t) = 2x_1(t) - x_2(t) \end{cases}$

(i) Finn $H(s)$. $(A - sI) = \begin{pmatrix} 2-s & -1 \\ 3 & 1-s \end{pmatrix}$, $(A - sI)^{-1} = \dfrac{1}{(2-s)(1-s)+3} \begin{pmatrix} 1-s & 1 \\ -3 & 2-s \end{pmatrix}$.

$$H(s) = \frac{1}{s^2 - 3s + 5} \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} 1-s & 1 \\ -3 & 2-s \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$ $\det(A - sI) = 0 \iff s = \frac{3 \pm \sqrt{9-20}}{2} = \frac{3}{2} \pm \frac{\sqrt{11}}{2} i$

$$= \frac{1}{s^2 - 3s + 5} \left(2 + (s - 2) \right) = \frac{s}{s^2 - 3s + 5}, \quad s \notin \left\{ \frac{3}{2} \pm \frac{\sqrt{11}}{2} i \right\}.$$

(ii) Finn y til $u(t) = \cos(t)$. Fylgja setn. 2.24.

$$H(i) = \frac{i}{i^2 - 3i + 5} = \frac{i}{-3i + 4} = \frac{i\,(3i + 4)}{9 + 16} = -\frac{3}{25} + \frac{4}{25} i$$

Vit fáa nú $|H(i)| = \sqrt{\frac{9}{25^2} + \frac{16}{25^2}} = \sqrt{\frac{25}{25^2}} = \frac{1}{5}$, $\arg(H(i)) = \arctan\left(-\frac{4}{3} + \pi\right)$.

$$\therefore \; y(t) = \frac{1}{5} \cos\left(t + \arctan\left(-\frac{4}{3} + \pi\right)\right).$$

Annars við $Re\left(H(i) e^{it} \right) = Re\left(\left(-\frac{3}{25} + \frac{4}{25} i\right) \left(\cos(t) + i \sin t \right) \right)$

$$= -\frac{3}{25} \cos(t) - \frac{4}{25} \sin(t).$$

4.22. Kanna um systemið er stabilt ella asymptodiskt stabilt.
Vit minna á, at eitt asymptodiskt stabilt system er stabilt.

(i) $\dot{x} = \begin{pmatrix} 1 & 7 \\ 3 & -2 \end{pmatrix} x$

$$\det(A-\lambda I) = (1-\lambda)(-2-\lambda) - 21 = \lambda^2 + \lambda - 23 = 0$$

$$\Leftrightarrow \quad \lambda = \frac{-1 \pm \sqrt{1-4(-23)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{93}}{2}.$$

Av tí at $Re(\lambda) > 0$, so er systemið óstabilt per setning 2.34.

(ii) $\dot{x} = \begin{pmatrix} -1 & 1 & 0 \\ -5 & -1 & 1 \\ -7 & 0 & 1 \end{pmatrix} x$

$$\begin{aligned}
\det(A-\lambda I) &= (-1-\lambda)^2(1-\lambda) - 7 + 5(1-\lambda) = 0 \\
\Leftrightarrow \quad & (\lambda^2+2\lambda+1)(1-\lambda) - 5\lambda - 2 = 0 \\
\Leftrightarrow \quad & -\lambda^3 - \lambda^2 + \lambda + 1 - 5\lambda - 2 = 0 \\
\Leftrightarrow \quad & -\lambda^3 - \lambda^2 - 4\lambda - 1 = 0 \\
\Leftrightarrow \quad & \lambda^3 + \lambda^2 + 4\lambda + 1 = 0
\end{aligned}$$

Koefficientarnir eru allir meiri enn 0 og $\det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 3 > 0$,
so per korollar 2.41 hava allar røturnar $Re(\lambda) < 0$.
Av tí at $Re(\lambda) < 0$ fyri øll eginvirði λ, so gevur setningur 2.36, at systemið er asymptodiskt stabilt.

(iii) $\dot{x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x$, vit útvikla í fyrsta rað.

$$A - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & -1 & -1 \\ -1 & 0 & -1-\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned}
\det(A-\lambda I) &= -\lambda \cdot \left((-1-\lambda)^2 \cdot (-\lambda) + (-1-\lambda) \right) \\
&\quad + (-1)^{1+4}\left(-1-\lambda \right) \\
&= \lambda^2(\lambda^2+2\lambda+1) - \lambda(-1-\lambda) + 1 + \lambda \\
&= \lambda^4 + 2\lambda^3 + \lambda^2 + \lambda + \lambda^2 + 1 + \lambda \\
&= \lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1
\end{aligned}$$

Positivir koefficientar og

$\det\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} = 1 > 0$, $\det\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} = 8 - 4 - 4 = 0$. Lat okkum fáa λ við $Re(\lambda) = 0$,

tí korollar 2.42 sigur, at $Re(\lambda) < 0$ kann ei staðfestast.

$$\det(A-\lambda I) = (\lambda^2+1)^2(1+\lambda)^2 \quad \Rightarrow \quad \lambda = \pm i \ \vee \ \lambda = -1$$

Systemið er stabilt per setning 2.34.

507. Lat $a \in \mathbb{R}$ og $\dot{x} = \begin{pmatrix} a & 2 \\ a & 1 \end{pmatrix} x$.

(i) Finn a, so at systemið er asymptodiskt stabilt.

$$\det(A-\lambda I) = (a-\lambda)(1-\lambda) - 2a = \lambda^2 + a - \lambda - a\lambda - 2a$$
$$= \lambda^2 - (1+a)\lambda - a = 0$$

Per korollar 2.40, so hava røturnar bert $Re(\lambda)<0$, um koefficientarnir

$$a_1 = -(1+a) > 0 \quad \text{og} \quad a_2 = -a > 0.$$

Her skal $a<-1$, so at $Re(\lambda)<0$, og tá er systemið asymptodiskt stabilt per setning 2.36.

(ii) Kanna stabilitet fyri $a=-1$.

$$A = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \Rightarrow \det(A-\lambda I) = \lambda^2+1 = 0 \iff \lambda = \pm i.$$

Her er $Re(\lambda)=0$, men $p=q=1$, so systemið er stabilt per setning 2.34.

(iii) Kanna stabilitet fyri $a \geq 0$.

$$\det(A-\lambda I) = \lambda^2 - (1+a)\lambda - a \quad \text{við } a \geq 0 \text{ gevur koefficientar}$$
$$a_1 = -(1+a) < 0 \quad \text{og} \quad a_2 = -a \leq 0.$$

Per korollar 2.40, so hevur systemið ikki bert eginvirðir λ við $Re(\lambda)<0$. Vit kunnu altso vátta at systemið ikki er asymptodiskt stabilt við setning 2.36.
Vit vísa, at eitt eginvirði λ hevur $Re(\lambda)>0$, so systemið er óstabilt.

$$\lambda = \frac{1+a \pm \sqrt{(1+a)^2+4a}}{2} = \frac{1}{2} + \frac{a}{2} \pm \frac{\sqrt{a^2+6a+1}}{2}$$

Av tí at $a \geq 0$, so er $\lambda = \frac{1}{2} + \frac{a}{2} + \frac{\sqrt{a^2+6a+1}}{2} \geq \frac{1}{2}$ fyri øll $a \geq 0$.
Systemið er altso óstabilt per setning 2.34.

430.

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u \\ y = x_1 \end{cases}$$

(a) Finn $H(s)$. $\quad (A - sI)^{-1} = \dfrac{1}{(-4-s)(-1-s)+2} \begin{pmatrix} -1-s & -2 \\ 1 & -4-s \end{pmatrix}$

$$H(s) = \frac{-1}{(s+4)(s+1)+2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1-s & -2 \\ 1 & -4-s \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{-1}{s^2+5s+6} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2s \\ 6+s \end{pmatrix}, \quad \frac{-5 \pm 1}{2} = \begin{cases} -2 \\ -3 \end{cases}$$

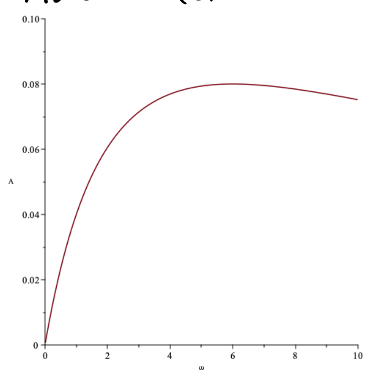
$$= \frac{-1}{(s+2)(s+3)} (-2s) = \frac{2s}{(s+2)(s+3)}, \quad s \notin \{-3, -2\}.$$

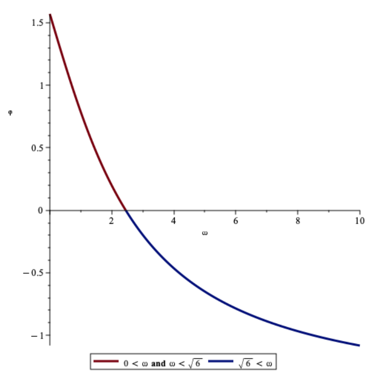
(b) Finn $A(\omega)$. $\quad A(\omega) = |H(i\omega)|$

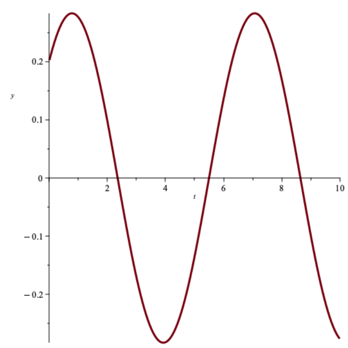
$$H(i\omega) = \frac{2i\omega}{(2+i\omega)(3+i\omega)} = \frac{2i\omega}{(6-\omega^2)+5i\omega}$$

$$|H(i\omega)| = \frac{\sqrt{(2\omega)^2}}{\sqrt{(6-\omega^2)^2+(5\omega)^2}} = \frac{2\omega}{\omega^2+13\omega+36}, \quad \omega > 0.$$

(c) Plots: $A(\omega)$ $\quad \varphi(\omega)$ $\quad y(t)$ vid $u(t) = \sin(t)$.







(d) Finn og plotta $\varphi(\omega)$. $\quad \varphi(\omega) = \arg(H(i\omega))$

$$H(i\omega) = \frac{2i\omega}{(6-\omega^2)+5i\omega},$$

$$\arg(H(i\omega)) = \arg(2i\omega) - \arg((6-\omega^2)+5i\omega) = \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{5\omega}{6-\omega^2}\right), & 0 < \omega < \sqrt{6} \\ -\frac{\pi}{2} - \arctan\left(\frac{5\omega}{6-\omega^2}\right), & \omega > \sqrt{6} \end{cases}$$

(e) Finn og plotta y til $u(t) = \sin(t)$.

$$y(t) = \mathrm{Im}\left(H(i)e^{it}\right) = \mathrm{Im}\left(\frac{2i}{5+5i}(\cos(t)+i\sin(t))\right) = \mathrm{Im}\left(\left(\frac{1}{5}+\frac{1}{5}i\right)(\cos(t)+i\sin(t))\right)$$

$$= \frac{1}{5}\sin(t) + \frac{1}{5}\cos(t).$$

ella $\quad y(t) = A(1)\sin(t+\varphi(1)) = \frac{1}{5}\sin\left(t+\frac{\pi}{2}-\arctan(1)\right) = \frac{1}{5}\sin\left(t+\frac{\pi}{2}-\frac{\pi}{4}\right)$

$$= \frac{1}{5}\sin\left(t+\frac{\pi}{4}\right).$$