Domains and Tangent planes

Ex 1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = x^2 + y^2 .$$

a) Describe the level curves $f(x,y) = c$ for $c \in \{1,2,3,4,5\}$.

Recall that $x^2 + y^2 = r^2$ describes a circle of radius r with center $C(0,0)$. Hence $f(x,y) = c$ is a circle of radius

$$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}.$$

b) Determine $\nabla f(1,1)$, and the directional derivative at $(1,1)$ determined by $\underline{e} = (1,0)$.

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \quad \text{so} \quad \nabla f(1,1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Then we have

$$f'((1,1), \underline{e}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2.$$

Consider now $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^2 - 4x + y^2.$$

c) Describe $f(x,y) = c$ for $c \in \{-3,-2,-1,0,1\}$.

Firstly we have

$$x^2 - 4x + y^2 = c \iff x^2 - 4x - c + y^2 = 0$$

If we factor, then

$$x^2 - 4x - c + y^2 = 0 \iff (x-2)^2 + y^2 = 4 + c.$$

We get circles centered at $(2,0)$ with radii

$$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5},$$

since $r^2 = 4-3,\ 4-2,\ 4-1,\ 4+0,\ 4+1.$

d) Determine $\nabla f(1,2)$ and $f'((1,2), \underline{e})$, where $\underline{e}$ points to the origin.

$$\nabla f(x,y) = \begin{bmatrix} 2x-4 \\ 2y \end{bmatrix}, \quad \text{so} \quad \nabla f(1,2) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

The vector $\underline{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ points to the origin from $(1,2)$, so

let $\underline{e} = \dfrac{\underline{v}}{|\underline{v}|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Then we have

$$f'((1,2), \underline{e}) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -\frac{6}{\sqrt{5}} = -\frac{6\sqrt{5}}{5}.$$

Ex 2. For $(x,y) \in \mathbb{R}^2$ we consider

$$f(x,y) = \exp(-x + \sin(y)).$$

a) Determine the approximating polynomial of 1-deg of f with expansion point $(x,y) = (0,0)$.

$P_{1,(x_0,y_0)}(x,y)$ is defined by (19-39).

$$P_{1,(0,0)}(x,y) = f(0,0) + f'_x(0,0)(x-0) + f'_y(0,0)(y-0)$$

$$= 1 - x + y.$$

b) Determine an equation for the plane tangent to the graph of f in $(0,0,f(0,0))$. Also state a normal vector.

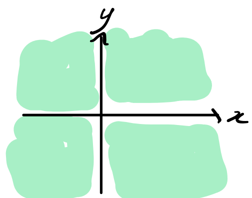
By definition 19.39 the equation is

$$z = 1 - x + y \iff x - y + z = 1,$$

and we find a normal vector to be $\underline{n} = (1,-1,1)$.

Ex 3. Given four sets in the (x,y)-plane do the following:

- Sketch
- Determine A°, ∂A, $\bar{A}$
- State whether A is open, closed or neither
- State whether A is bounded or not

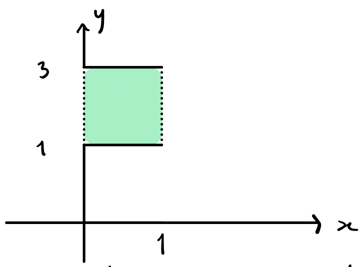


1. $A = \{(x,y) \mid xy \neq 0\}$.

These are all point except for the coordinate axes.

We have $A = A^\circ$ and ∂A are the coordinate axes, while $\bar{A} = \mathbb{R}^2$.

The set is open and unbounded.



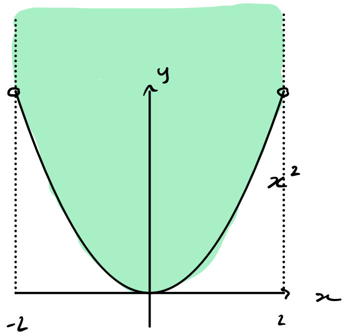
2. $A = \{(x,y) \mid 0 < x < 1 \wedge 1 \leq y \leq 3\}$.

The interior is just A excluding the lines.

The boundary is all 4 lines bounding A. The closure is the entire rectangle.

The set A is not open or closed, and it is bounded.

3. $A = \{(x,y) \mid y \geq x^2 \wedge |x| < 2\}$.



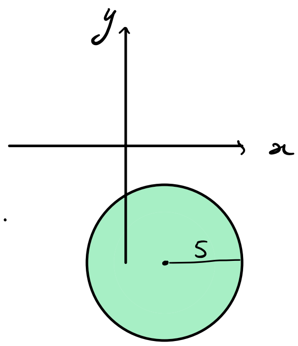
The interior does not include the parabola.

The boundary is the vertical lines and parabola.

The closure is $A^\circ \cup \partial A$. The set is not open or closed, and it is unbounded.

4. $A = \{(x,y) \mid x^2+y^2-2x+6y \leq 15\}$

$= \{(x,y) \mid (x-1)^2+(y+3)^2 \leq 5^2\}$



The interior is the disc with no perimeter.

The boundary is the perimeter.

The closure is the disc.

The set A is closed and bounded.

Ex 4. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \ln(9-x^2-y^2).$$

a) Determine $D(f)$ and characterize it.

Since $D(\ln) = \mathbb{R}_+$ we check

$$9-x^2-y^2 > 0 \iff x^2+y^2 < 9 = 3^2.$$

Thus $D(f) = \{(x,y) \mid x^2+y^2 < 3^2\}$, i.e. the open disk centered at the origin with a radius of 3, which is bounded.

Let $\underline{r}$ be the parametrized curve

$$\underline{r}(u) = (u, u^3), \quad u \in [-1.2, 1.2].$$

b) Which curve is this?

This is just the cubic $y = x^3$, $x \in [-1.2, 1.2]$.

Now consider $h(u) = f(\underline{r}(u))$.

c) Why is it fair to call h an elevation function?

If we consider the graph of f, which is given in coordinates (x, y, z), then h corresponds the z-value, i.e. height of f over the (x,y)-plane.

d) Determine $h'(1)$ by
1) Usual differentiation
2) Theorem 19.49

1) $$h(u) = \ln(9 - u^2 - u^6), \quad h'(u) = \frac{1}{9 - u^2 - u^6}(-6u^5 - 2u)$$

$$\Rightarrow \quad h'(1) = -\frac{8}{7}$$

2) $$\nabla f(x,y) = \begin{bmatrix} -\dfrac{2x}{9 - x^2 - y^2} \\ -\dfrac{2y}{9 - x^2 - y^2} \end{bmatrix}, \quad \underline{r}'(u) = \begin{bmatrix} 1 \\ 3u^2 \end{bmatrix}$$

$$\nabla f(\underline{r}(1)) \cdot r'(1) = \begin{bmatrix} -2/7 \\ -2/7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{8}{7}$$

Ex 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \frac{e^x}{y}.$$

a) Determine $D(f)$.

$D(f) = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$.

b) Compute the value at $A(1,1)$, $B(0,1)$ and $C(-1, \frac{1}{e})$. Two of the points are on the same level curve of f, describe the curve.

$$f(1,1) = \frac{e^1}{1} = e$$

$$f(0,1) = \frac{e^0}{1} = 1$$

$$f(-1, \tfrac{1}{e}) = \frac{e^{-1}}{\frac{1}{e}} = 1$$

Both B and C are on $f(x,y) = 1$, where

$$\frac{e^x}{y} = 1 \iff y = e^x,$$

i.e. the exponential function.

c) Determine $\nabla f(1,1)$ and directional derivative at $(1,1)$ in the direction of $\underline{s} = (1,-1)$.

$$\nabla f(x,y) = \begin{bmatrix} e^x/y \\ -e^x/y^2 \end{bmatrix}, \quad \text{so} \quad \nabla f(1,1) = \begin{bmatrix} e \\ -e \end{bmatrix}$$

$$f'\left((1,1), \frac{\underline{s}}{|\underline{s}|}\right) = \begin{bmatrix} e \\ -e \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2}\, e.$$

A curve is given by

$$\underline{r}(u) = (u, u), \quad u > 0.$$

Also we're given

$$h(u) = f(\underline{r}(u)).$$

d) Determine the point $\underline{r}(u_0)$ in the (x,y)-plane, for which $h'(u_0) = 0$.

Well this is just $u_0 = 1$, so $r(u_0) = (1,1)$.

This follows from

$$h(u) = \frac{e^u}{u} \Rightarrow h'(u) = \frac{e^u \cdot u - e^u}{u^2} = \frac{e^u(u-1)}{u^2},$$

and so

$$h'(1) = \frac{e^1(1-1)}{1^2} = 0.$$