Polynomials

Ex 1. $(z-3)(z^2+1) = 0 \iff z-3=0 \vee z^2+1=0$

$\iff z = 3 \vee z = \pm i$

Ex 2. $P(z) = (2-i)z + i$

a) Solve $P(z) = 0 \iff (2-i)z + i = 0$

$$\iff z = -\frac{i}{2-i} = -\frac{i(2+i)}{4+1}$$

$$= \frac{1}{5} - \frac{2}{5}i$$

b) Solve $P(z) = 2$ and $P(z) = -2+2i$.

$$(2-i)z + i = 2 \iff z = \frac{2-i}{2-i} = 1$$

$$(2-i)z + i = -2+2i \iff z = \frac{-2+i}{2-i} = -1$$

Ex 3.

a) Let $r>0$ and explain why $z^2 = -r$ has exactly two solutions given by $z_0 = -i\sqrt{r}$ and $z_1 = i\sqrt{r}$.

Recall from polar coordinates that z^2 is given by (r, π). Since multiplication of complex numbers equates to multiplying the moduli and adding arguments, then $z^2 = z \cdot z$ is represented by $(\sqrt{r}, \pm\frac{\pi}{2})$ (or simply apply proposition 1.52). Therefore z_0 and z_1 are solutions. Uniqueness follows by the fundamental theorem of algebra (2.11).

b) Solve $z^2 = 16$ and $z^2 = -16$

$z^2 = 16 \iff z = \pm 4.$

$z^2 = -16 \iff z = \pm 4i$ by a).

c) Solve $z^2 = 17$ and $z^2 = -17$

$z^2 = 17 \iff z = \pm\sqrt{17}$

$z^2 = -17 \iff z = \pm i\sqrt{17}$ by a).

d) Solve $z^2 = 625$ and $z^2 = -625$

$z^2 = 625 \iff z = \pm 25$

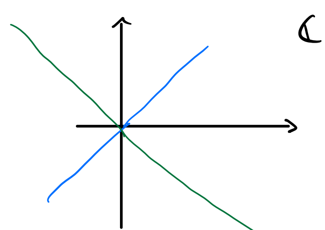
$z^2 = -625 \iff z = \pm 25i$ by a).

e) Let b be a real number. Show that the solutions to

$$z^2 = ib$$

lie on the line $y = x$ for $b > 0$ and $y = -x$ for $b < 0$.

Assume $b > 0$, then $z^2 = be^{\frac{\pi}{2}i}$ and the solutions amount to



$$z_0 = \sqrt{b}\, e^{\frac{\pi}{4}i} \quad \text{and} \quad z_1 = \sqrt{b}\, e^{\frac{5\pi}{4}i}$$

by 1.52. Observe that the solutions are on the $\frac{\pi}{4}$, or $45°$, line $y = x$.

Similarly we get the solutions $z_2 = \sqrt{b}\, e^{-\frac{\pi}{4}i}$ and $z_3 = \sqrt{b}\, e^{\frac{3\pi}{4}i}$ for $b < 0$.

Ex 4. Show that $-1+2i$ is a root of $P(z) = 3(z^2+2z+5)$.

a) State the second root and factorize $P(z)$.

$$P(-1+2i) = (-1+2i)^2 + 2\cdot(-1+2i)+5$$
$$= 1-4i-4-2+4i+5$$
$$= 0.$$

The second root is $-1-2i$, thus

$$P(z) = 3\cdot(z-(-1+2i))\,(z-(-1-2i)).$$

b) Given that i and $1+i$ are roots of

$$Q(z) = z^2 - z - 2iz - 1 + i$$

simplify $\dfrac{Q(z)}{z-1-i}$.

We factor $Q(z)$ such that

$$\frac{z^2 - z - 2iz - 1 + i}{z-1-i} = \frac{(z-i)(z-(1+i))}{(z-(1+i))} = z-i.$$

Note that two roots are given for a degree 2 polynomial, and so 2.11 ensures these are all the roots.

Ex 5.

a) Show $x_0 = 1$ is a root of $P(x) = x^3 - x^2 + x - 1$, and determine Q such that $P(x) = (x-1)\,Q(x)$.

We just follow 2.6 directly to determine Q. First we have

$$P(1) = 1^3 - 1^2 + 1 - 1 = 0.$$

We have $n=3$, so $b_2 = 1$ $\quad (b_{n-1} = a_n)$

$b_1 = -1 + 1 \cdot 1 = 0$ $\quad (b_k = a_{k+1} + z_0 \cdot b_{k+1},\ k = n-2, \dots, 0)$

$b_0 = 1 + 1 \cdot 0 = 1$

$\Rightarrow$ $Q(x) = x^2 + 1$ i.e. $P(x) = (x-1)(x^2+1)$.

b) Compute all the roots of

$$P(z) = (z^6 - z^5 + z^4 - z^3)(z-1).$$

Write on factorized form and state root multiplicity.

$$P(z) = z^3 \cdot (z-1)\, Q(z) \cdot (z-1) \quad \text{using a).}$$

$z=0$ is a root of multiplicity 3.

$z=1$ is a root of multiplicity 2.

$z=\pm i$ are roots of multiplicity 1, see $Q(z)$.

$$P(z) = z^3 (z-i)(z+i)(z-1)^2.$$

c) Show that 2 is a double root of $P(z) = 2z^4 - 4z^3 - 16z + 32$.

Let's just do descent. Clearly 2 is a root.

$$\left.\begin{aligned} b_3 &= 2 \\ b_2 &= -4 + 2 \cdot 2 = 0 \\ b_1 &= 0 + 2 \cdot 0 = 0 \\ b_0 &= -16 + 2 \cdot 0 = -16 \end{aligned}\right\} \quad Q(z) = 2z^3 - 16$$

$P(z) = (z-2)\, Q(z)$. Now descend Q as well.

Note that 2 is indeed a root of Q.

$$\left.\begin{aligned} b_2 &= 2 \\ b_1 &= 0 + 2\cdot 2 = 4 \\ b_0 &= 0 + 2 \cdot 4 = 8 \end{aligned}\right\} \quad R(z) = 2z^2 + 4z + 8.$$

It follows by 2.6 that 2 is a double root of $P(z)$, since

$$P(z) = (z-2)^2 \cdot (2z^2 + 4z + 8).$$

d) Find all solutions to $P(z) = 2z^4 - 4z^3 - 16z + 32 = 0$.

We've established that 2 is a double root in c).

We need only solve $R(z) = 0$. Let's apply proposition 2.24.

$$D = 4^2 - 4 \cdot 2 \cdot 8 = 16 - 64 = -48 \Rightarrow w_0 = \pm i\sqrt{48} \quad [3.a)]$$

$$z = \frac{-4 \pm i\sqrt{48}}{2 \cdot 2} = -1 \pm \sqrt{\frac{48}{16}}\, i = -1 \pm \sqrt{3}\, i.$$

$$P(z) = (z-2)^2 \cdot \left(z - (-1 + \sqrt{3} i)\right) \cdot \left(z - (-1 - \sqrt{3} i)\right).$$

Ex 6.
- A polynomial is a function, and it represents a formula for all values given a specific input. In an extended sense this represents all the possible equations. An equations puts forward a question regarding which input corresponds to a given value.
- Anywhere from 0 to n.

- Anywhere from 0 to n if the degree is even.
  Anywhere from 1 to n if the degree is odd.
- n roots counting multiplicity.
- Not necessarily, since you could just scale one, and then they're different, but the roots are intact:

$$P(z) = (z-z_1)^{k_1} \cdot (z-z_2)^{k_2} \cdots (z-z_m)^{k_m}$$
$$\neq c \cdot (z-z_1)^{k_1} \cdot (z-z_2)^{k_2} \cdots (z-z_m)^{k_m}, \quad c \in \mathbb{C}.$$

Ex 7.

a) $$P(x) = 25x^4 - 33x^3 + 49x^2 - 97x + 96, \quad x \in \mathbb{R}.$$
$$P'(x) = 100x^3 - 99x^2 + 98x - 97.$$

b) Determine $f'(t)$ and $f'(0)$ given

$$f(t) = t - 3t^3 + 1 + i(t^2 - 5t - 1), \quad t \in \mathbb{R}.$$
$$f'(t) = 1 - 9t + i(2t - 5).$$
$$f'(0) = 1 - 5i.$$

c) Given $g(t) = (i \cdot t^2 + t - i) \cdot ((1+i)t - i)$, $t \in \mathbb{R}$, there is exactly one solution to $g'(t) = -6 - i$ exist. Find it.

$$g(t) = (i-1)t^3 + t^2 + (1+i)t^2 - it + (1-i)t - 1$$
$$= -t^3 + 2t^2 + t - 1 + i(t^3 + t^2 - 2t)$$
$$g'(t) = -3t^2 + 4t + 1 + i(3t^2 + 2t - 2)$$

$$-3t^2+4t+1 + i(3t^2+2t-2) = -6 - i$$

$$\Leftrightarrow \begin{cases} -3t^2+4t+1 = -6 \Leftrightarrow -3t^2+4t-7=0 \Leftrightarrow t = \frac{-4\pm 10}{-6} = \begin{cases} -1 \\ \frac{7}{3} \end{cases} \\ 3t^2+2t-2 = -1 \Leftrightarrow 3t^2+2t-1=0 \Leftrightarrow t = \frac{-2\pm 4}{6} = \begin{cases} \frac{2}{3} \\ -1 \end{cases} \end{cases}$$

The solution is $t=-1$.

d) Let $Q(x) = (1+2i)x^2 + (2-5i)x - (1-i)$, $x \in \mathbb{R}$.

Show that we can differentiate by treating i as a constant.

$$Q(x) = x^2 + 2x - 1 + i(2x^2 - 5x + 1)$$

Let's see that Q' behaves nicely in both instances.

$Q'(x) = (2+4i)x + (2-5i)$ treated as constant.

$Q'(x) = 2x + x + i(4x-5) = 2x + x + 4ix - 5i$.

Yes these methods agree.

Ex 8.

a) Solve $z^2 = 3-4i$. (Assuming a solution should be based on ch.2)

$$z^2 = (x+iy)^2 = x^2-y^2+2xyi = 3-4i$$

$$\Rightarrow x^2-y^2 = 3 \wedge 2xy = -4$$

Let $y = -\frac{2}{x}$ and substitute:

$$x^2 - \left(-\frac{2}{x}\right)^2 = 3 \Leftrightarrow x^2 - \frac{4}{x^2} = 3 \Leftrightarrow x^4 - 4 = 3x^2, \quad u=x^2$$

$$\Leftrightarrow u^2 - 3u^2 - 4 = 0 \Leftrightarrow u = \frac{3\pm 5}{2} = \begin{cases} 4 \\ -1 \end{cases}$$

Since $u = x^2$ then -1 is not a solution, as x and y are real numbers. For $u = 4$ we have $x = \pm 2$, and so $y = \mp 1$.

The solutions are

$$z_1 = 2 - i \quad \text{and} \quad z_2 = -2 + i.$$

b) Compute the roots of $P(z) = z^2 - (1+2i)z - \frac{3}{2} + 2i$.

We use 2.24 directly.

$$D = \left(-(1+2i)\right)^2 - 4 \cdot 1 \cdot \left(-\frac{3}{2} + 2i\right)$$

$$= 1 - 4 + 4i + 6 - 8i = 3 - 4i$$

Since $w_0^2 = D$ we can apply a) in the solution.

If we use the formula, then only one solution w_0 is necessary!

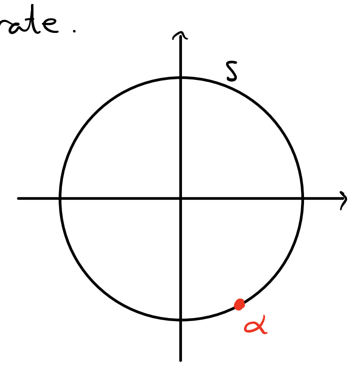
$$z_1 = \frac{1 + 2i - (2-i)}{2} = -\frac{1}{2} + \frac{3}{2}i$$

$$z_2 = \frac{1 + 2i + (2-i)}{2} = \frac{3}{2} + \frac{1}{2}i$$

Ex 9. We're given $\alpha = 3 - 4i$ and $S = \{z \in \mathbb{C} \mid |z| = 5\}$.

a) Show that $\alpha \in S$ and illustrate.

$$|\alpha| = \sqrt{3^2 + (-4)^2} = 5$$



b) The polynomial $P(z) = z^2 + az + b$ has real coefficients and $P(\alpha) = 0$.

State all the roots of $P(z)$ and compute a and b.

Firstly the second root is $\bar{\alpha} = 3 + 4i$. Using the roots we factorize P and compute a and b from there.

$$P(z) = (z - (3-4i))\,(z - (3+4i))$$

$$= z^2 - (3+4i)z - (3-4i)z + 3^2 + 4^2$$

$$= z^2 - 6z + 25.$$

So $a = -6$ and $b = 25$.

c) Let $c \in \mathbb{R}$ and $|c| \leq 10$. Show that the roots of

$$Q(z) = z^2 + cz + 25$$

belong to S.

Note that $D = c^2 - 100$, so we get if $|c| = 10$

$$z = \frac{-c \pm i\sqrt{100 - c^2}}{2} = -\frac{c}{2} \pm \sqrt{25 - \frac{c^2}{4}}\; i.$$

We used 3.a) to determine w_0 from $w^2 = c^2 - 100$, since $|c| \leq 10$, i.e. $D \leq 0$. For $|c| = 10$ we have $c = \pm 10$, so

$$z = -\frac{c}{2} \quad \text{and} \quad |z| = \left|-\frac{c}{2}\right| = 5 \quad \Rightarrow \quad z \in S.$$

For $|c| < 10$ we have

$$|z| = \sqrt{\left(-\frac{c}{2}\right)^2 + 25 - \frac{c^2}{4}} = 5 \quad \Rightarrow \quad z \in S \text{ as desired.}$$

d) Given the real quadratic equation

$$z^2 + mz + n = 0,$$

show that if $-2\sqrt{n} \leq m \leq 2\sqrt{n}$, then the solutions z can be found as the intersection of the vertical line through $-\frac{m}{2}$ and the circle centered at 0 with radius $\sqrt{n}$.

[This is a reformulation of c). The real part of the solution together with the modulus yield the desired result.]

We deal with this in two parts. First the extremes. Assume $|m| = 2\sqrt{n}$, then

$$D = m^2 - 4 \cdot n = 4n - 4n = 0$$

$$\Rightarrow \quad z = -\frac{m}{2} \quad \text{and} \quad |z| = \sqrt{\left(-\frac{m}{2}\right)^2} = \sqrt{n}.$$

There's one unique solution of multiplicity 2. The solution is at $-\frac{m}{2}$ and sits on the desired circle.

Now assume that $|m| < 2\sqrt{n}$, then

$$D = m^2 - 4 \cdot n < 0 \quad \text{and} \quad D \in \mathbb{R},$$

so we apply 3.a, i.e. $w^2 = D$ and $w = \pm i\sqrt{4n - m^2}$.

$$z = -\frac{m \pm \sqrt{4n - m^2}}{2} = -\frac{m}{2} \pm \sqrt{n - \frac{m^2}{4}}\, i, \quad \text{vertical line } -\frac{m}{2}.$$

$$|z| = \sqrt{\left(-\frac{m}{2}\right)^2 + \left(\sqrt{n - \frac{m^2}{4}}\right)^2} = \sqrt{n}, \quad \text{radius } \sqrt{n}.$$