Functions of two variables

Ex 1. For $(x,y) \in \mathbb{R}^2$ we consider

$$f(x,y) = x^2 + y^3 \quad \text{and} \quad g(x,y) = y \cos(x).$$

a) Determine the partial derivatives of f and g. State the gradients as well.

$$\frac{\partial}{\partial x} f(x,y) = 2x \,, \quad \frac{\partial}{\partial y} f(x,y) = 3y^2 \,, \quad \nabla f(x,y) = \begin{bmatrix} 2x \\ 3y^2 \end{bmatrix}.$$

$$\frac{\partial}{\partial x} g(x,y) = -y \sin(x), \quad \frac{\partial}{\partial y} g(x,y) = \cos(x) \,, \quad \nabla g(x,y) = \begin{bmatrix} -y \cdot \sin(x) \\ \cos(x) \end{bmatrix}.$$

b) Determine the second order partial derivatives of f and g.

$$\frac{\partial}{\partial x^2} f = 2 \,, \quad \frac{\partial}{\partial y^2} f = 6y \,, \quad \frac{\partial}{\partial x \partial y} f = 0 = \frac{\partial}{\partial y \partial x} f.$$

$$\frac{\partial}{\partial x^2} g = -y \cos(x) \,, \quad \frac{\partial}{\partial y^2} g = 0 \,, \quad \frac{\partial}{\partial x \partial y} g = -\sin(x) = \frac{\partial}{\partial y \partial x} g.$$

c) Two derivatives of b) are equal, which? Is this circumstantial?

Both mixed derivatives are equal. By proposition 19.34 this is always true on a domain D, if the second order derivatives are continuous on D.

Ex 2. We consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^2 - 2y,$$

and its level curves

$$f(x,y) = c, \quad c \in \mathbb{R}.$$

a) Show that the level curve for a $c \in \mathbb{R}$ can be described by $y = g_c(x)$, where g_c is a real function of x. Draw the curves for $c \in \{-2, -1, 0, 1, 2\}$.

$$x^2 - 2y = c \iff y = \frac{x^2}{2} - \frac{c}{2}$$

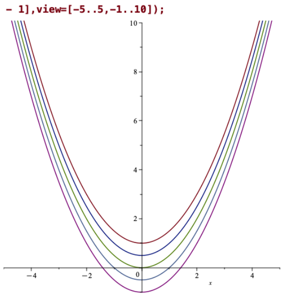
So the level curve is given by

$$y = g_c(x) = \frac{x^2}{2} - \frac{c}{2}$$

```
> seq(g(x),c=-2..2);
```

$$\frac{x^2}{2}+1, \frac{x^2}{2}+\frac{1}{2}, \frac{x^2}{2}, \frac{x^2}{2}-\frac{1}{2}, \frac{x^2}{2}-1$$

```
> plot([x^2/2 + 1, x^2/2 + 1/2, x^2/2, x^2/2 - 1/2, x^2/2 - 1],view=[-5..5,-1..10]);
```



b) Show that $P(2,1)$ is on $g_2(x)$, and provide a parametric representation of the curve.

$$g_2(2) = \frac{2^2}{2} - \frac{2}{2} = 2 - 1 = 1, \quad \text{so } P \text{ is on } g_2.$$

We have

$$y = g_2(x) = \frac{x^2}{2} - 1,$$

so let $t \in \mathbb{R}$ be a parameter, then a representation is

$$r(t) = \begin{bmatrix} t \\ \frac{t^2}{2} - 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

c) Determine the tangent vector at P to the parametric representation, and show that it is orthogonal to the gradient of f at P.

$$r'(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \quad \text{and} \quad r(2) = P,$$

so we have the tangent vector

$$r'(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

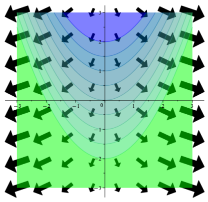
We find $\nabla f(x,y) = \begin{bmatrix} 2x \\ -2 \end{bmatrix} \Rightarrow \nabla f(2,1) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$

Since $r'(2) \cdot \nabla f(2,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 4 - 4 = 0,$

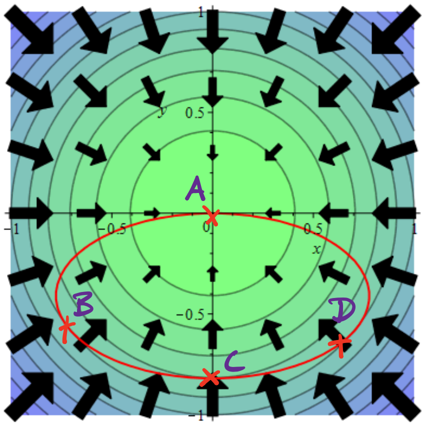
it follows that $r'(2) \perp \nabla f(2,1)$.

d) In Maple make a plot of the level curves and gradient vector field.

```
> C:=contourplot(f(x,y),x=-3..3,y=-3..3,coloring=[blue,green],scaling=constrained,filled=true,contours=[-4,-3,-2,-1,0,1,2,3,4],transparency=0.5):
  G:=gradplot(f(x,y),x=-3..3,y=-3..3,grid=[8,8],arrows=thick):
  display(C,G);
```



Ex 3. The image shows an elevation map.



a) Where on the red trail is the rate of increase equal to 0?

When walking parallel to level curves, i.e. tangent to level curves.

The points are marked x.

If we go counterclockwise from A → B → C → D → A, then:

A → B is downhill,

B → C is uphill,

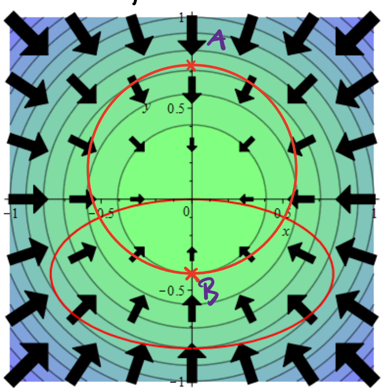
C → D is downhill,

D → A is uphill.

b) Why are the gradient vectors perpendicular to the level curves?

The path along a level curve is at a constant level by definition, and so travelling along such a path is entirely horizontal. The gradient at any point yields the direction of greatest change/increase, which is naturally the steepest/direct ascent to the top of the mountain.

c) Draw a new path. Where does it ascend/descend?



A → B uphill,

B → A downhill.

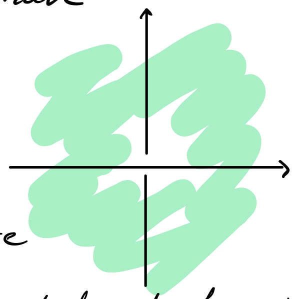
Ex4. Two real functions f and g of two real variables are given by

$$f(x,y) = \arctan\left(\frac{x}{y}\right) \quad \text{and} \quad g(x,y) = \ln\left(\sqrt{x^2+y^2}\right).$$

a) Determine the domains of f and g. Sketch the domains.

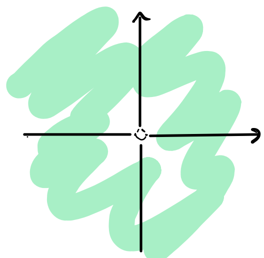
Since arctan is defined on $\mathbb{R}$, we need only worry about $y \neq 0$ to avoid division by 0. So we have

$$D(f) = \{(x,y) \in \mathbb{R}^2 \,/\, y \neq 0\}.$$



As for g we see that all input values are positive by squaring, so the square root is defined for all $(x,y) \in \mathbb{R}^2$. However the logaritm is not defined for $(x,y) = (0,0)$, so

$$D(g) = \mathbb{R}^2 \setminus \{(0,0)\}$$

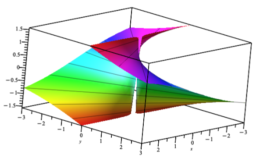


b) Determine the gradients of f and g.

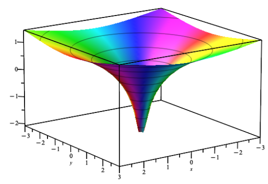
$$\nabla f = \begin{bmatrix} \dfrac{y}{x^2+y^2} \\[2ex] -\dfrac{x}{x^2+y^2} \end{bmatrix}, \quad \nabla g = \begin{bmatrix} \dfrac{x}{x^2+y^2} \\[2ex] \dfrac{y}{x^2+y^2} \end{bmatrix}$$

c) Plot the functions. Also use gradient vector fields with contours.

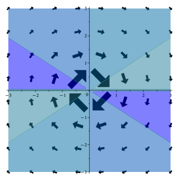
```
> f1:=plot3d(f(x,y),x=-3..3,y=-3..-0.01,scaling=constrained,style=patchcontour,color=f(x,y),projection=0.8):
  f2:=plot3d(f(x,y),x=-3..3,y=0.01..3,scaling=constrained,style=patchcontour,color=f(x,y),projection=0.8):
  display(f1,f2,orientation=[60,75,20]);
```



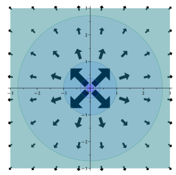
```
> plot3d(g(x,y),x=-3..3,y=-3..3,scaling=constrained,style=patchcontour,color=f(x,y),projection=0.8)
```



```
> Cf:=contourplot(f(x,y),x=-3..3,y=-3..3,coloring=[blue,green],scaling=constrained,filled=true,contours=[-4,-3,-2,-1,0,1,2,3,4],transparency=0.5):
  Gf:=gradplot(f(x,y),x=-3..3,y=-3..3,grid=[8,8],arrows=thick):
  display(Cf,Gf);
```



```
> Cg:=contourplot(g(x,y),x=-3..3,y=-3..3,coloring=[blue,green],scaling=constrained,filled=true,contours=[-4,-3,-2,-1,0,1,2,3,4],transparency=0.5):
  Gg:=gradplot(g(x,y),x=-3..3,y=-3..3,grid=[8,8],arrows=thick):
  display(Cg,Gg);
```



d) Can you determine an increase/decrease in the gradient plots at $P(0,2)$ in the direction $\underline{v} = (-1,-1)$?

Yes, in both cases there is a decrease in the direction of $\underline{v}$. (Move against the arrows essentially).

e) Determine the directional derivative at P given by $\underline{v}$ for both functions.

By definition 19.50: $\nabla f(x_0, y_0) \cdot \underline{e}$ where $\underline{e} = \dfrac{\underline{v}}{|\underline{v}|}$

$$\nabla f(0,2) \cdot \underline{e} = \begin{bmatrix} \dfrac{2}{0^2+2^2} \\ -\dfrac{0}{0^2+2^2} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\frac{1}{2\sqrt{2}}$$

$$\nabla g(0,2)\cdot \underline{e} = \begin{bmatrix} \frac{0}{0^2+2^2} \\ \frac{2}{0^2+2^2} \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\frac{1}{2\sqrt{2}}$$

f) "We can characterize the gradient as the vector that points in the direction in which the function f increases the most."

We know that the partial derivatives describe the rate of change in each direction relative to the standard basis. We need only check that this change is maximal in the direction of ∇f. Let $\underline{e}$ be an arbitrary unit vector, then at some point p we have

$$\begin{aligned} \nabla f(p)\cdot \underline{e} &= |\nabla f(p)|\cdot|\underline{e}|\cdot\cos(\theta) \\ &= |\nabla f(p)|\cdot\cos(\theta), \end{aligned}$$

since $|\underline{e}|=1$. Naturally this is maximal when $\nabla f(p) \parallel \underline{e}$, so that $\cos(\theta)=1$. So f increases the most in the direction of the gradient vector.

Ex5. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = x^2 - 4x + y^2$$

a) Show that f is differentiable and determine ∇f.

We use definition 19.27.

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= (x_0+\Delta x)^2 - 4(x_0+\Delta x) + (y_0+\Delta y)^2 - (x_0^2 - 4x_0 + y_0^2)$$

$$= x_0^2 + 2x_0\Delta x + \Delta x^2 - 4x_0 - 4\Delta x + y_0^2 + 2y_0\Delta y + \Delta y^2 - x_0^2 + 4x_0 - y_0^2$$

$$= 2x_0 \cdot \Delta x + 2y_0 \cdot \Delta y - 4 \cdot \Delta x + \Delta x^2 + \Delta y^2$$

$$= \begin{bmatrix} 2x_0 - 4 \\ 2y_0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + (\Delta x^2 + \Delta y^2)$$

$$= \begin{bmatrix} 2x_0 - 4 \\ 2y_0 \end{bmatrix} \cdot \underline{h} + |\underline{h}|^2 , \qquad \varepsilon(\underline{h}) = |\underline{h}|$$

is an ε-function, as $|\underline{h}| \to 0$ for $\underline{h} \to \underline{0}$. So we have

$$\Delta f = \begin{bmatrix} 2x_0 - 4 \\ 2y_0 \end{bmatrix} \cdot \underline{h} + \varepsilon(\underline{h}) \cdot |\underline{h}| ,$$

and by 19.27 is differentiable, and the gradient is

$$\nabla f(x,y) = (2x-4, 2y).$$

b) Show differentiability with Theorem 19.36.

Differentiability follows as the partial derivatives are continuous. Existence of partial derivatives is insufficient as demonstrated in example 19.37.

Ex6. A mountain is given by the graph of

$$f(x,y) = x^2 - y^2 + 4,$$

which is bounded by

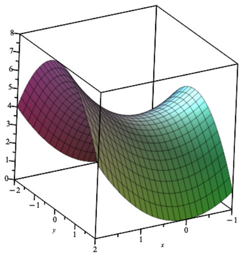
$$-1 \le x \le 2 \quad \text{and} \quad -2 \le y \le 2.$$

a) Draw in Maple.

```
> f:=(x,y)-> x^2 - y^2 +4:
  f(x,y);
```

$$x^2 - y^2 + 4$$

```
> plot3d(f(x,y),x=-1..2,y=-2..2);
```



b) What are the coordinates of the highest point B?

The maximum is either along the boundary, or an interior stationary point.

$$\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, \quad \nabla f = \underline{0} \iff (x,y) = (0,0).$$

Just for clarification $(0,0)$ is clearly not the maximum, since $(x,0)$ produces higher values. Now we check boundary points.

Corners: $f(-1,-2) = 1$, $f(-1,2) = 1$,

$f(2,-2) = 4$, $f(2,2) = 4$.

Curves: $f(-1,y) = -y^2 + 5$, max 5.

$f(2,y) = -y^2 + 8$, max 8.

$f(x,-2) = x^2$, max 4.

$f(x,2) = x^2$, max 4.

Thus the point B is $(2,0,8)$.

c) Show that

$$r(t) = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} , \quad t \in [0,2]$$

is entirely in the mountain surface, and it connects $A(0,-2,0)$ to B.

We plug (x,y) of $r(t)$ into f and see that f agrees with $r(t)$.

$$f(t, -2+t) = t^2 - (-2+t)^2 + 4$$
$$= 4t ,$$

so all coordinates agree, i.e. $r(t)$ is on the surface.

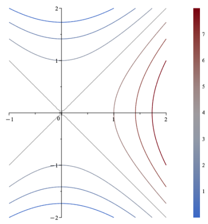
Note that $r(0) = A$ and $r(2) = B$.

d) Why is $r(t)$ the shortest path from A to B?

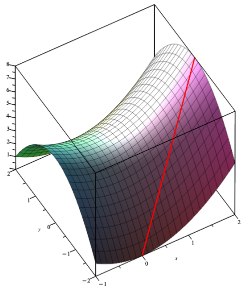
Because it is a straight line, which is always the shortest path between two points.

e) Use contourplot on f.

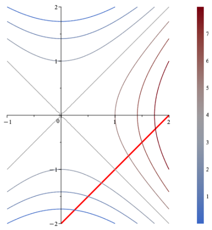
```
> levels:= contourplot(f(x,y),x=-1..2,y=-2...2,contours=7, scaling=constrained):
> display(levels);
```



```
> r:=t->t,-2+t,4*t:
  path := spacecurve(<r(t)>, t=0...2, color=red, thickness=3):
  display(path,mountain,orientation=[-120,40]);
```



```
> path2d := plot([t,t-2,t=0...2], color=red, scaling=constrained, thickness=3):
> display(path2d,levels);
```



f) Compute the gradient of f at three different points along $r(t)$.

$$\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix} \qquad r(0) = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad r(1) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad r(2) = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

$$\nabla f(0,-2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad \nabla f(1,-1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \nabla f(2,0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

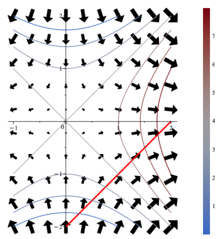
g) Show that there is one and only one point on the curve where ∇f is parallel to the (x,y) projection of $r(t)$.

This is precisely $(1,-1)$.

In 2D the line follows the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and ∇f is only parallel when $2x = -2y$

$\Rightarrow (x,y) = (1,-1)$, when on this particular line.

```
> gradfield:=gradplot(f(x,y),x=-1..2,y=-2...2, arrows=thick, grid=[10,10], scaling=constrained):
> display(path2d,levels,gradfield);
```



h) Doesn't this contradict the characterization of the gradient in Ex. 4 f) ?

No, the shortest path between A and B just happens to follow the steepest path at one particular point. This path is not dependent on level curves.