Eginvirðir og eginvektorar

Vit sóu, at tá ein lineer avmyndan $\underline{\underline{A}}$ hevur rætningar, sum eru varðveittir upp til ein konstant λ,

$$\underline{\underline{A}}\,\underline{r} = \lambda\,\underline{r},$$

so er λ eginvirði og $\underline{r}$ eginvektorur hjá $\underline{\underline{A}}$. Vit kunnu ikki altíð finna hesi, t.d. broytir ein rotatión ella spegling allar rætningin hjá øllum vektorunum.

Tá eginvirði eru at finna, so eru hesi loysn hjá karakterlikningini

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0,$$

ella ekvivalent, at λ er rót í karakteristiska polynomið

$$p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}).$$

Dømi

Lat $\underline{\underline{A}} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$, so er $p(\lambda) = \det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$

$= \lambda^2 - 3\lambda - 4$

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-4)}}{2} = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \begin{cases} 4 \\ -1 \end{cases}$$

Vit seta altso $\lambda_1 = 4$ og $\lambda_2 = -1$. Nú er $\underline{r}$ ein eginvektorur, um hesin er loysn hjá

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{r} = \underline{0}.$$

Set inn í tveimum umførum. $\lambda_1 = 4$:

$$\begin{bmatrix} 2-4 & 3 \\ 2 & 1-4 \end{bmatrix} \underline{r} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \underline{r} = c_1 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}, \quad c_1 \in \mathbb{R}.$$

$\lambda_2 = 1$:

$$\begin{bmatrix} 2-(-1) & 3 \\ 2 & 1-(-1) \end{bmatrix} \underline{r} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \underline{r} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_2 \in \mathbb{R}.$$

Dømi 7.2

Lat $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\qquad p(\lambda) = (1-\lambda)(2-\lambda) = 0 \iff \lambda_1 = 2$ ella $\lambda_2 = 1$.

$$\lambda_1 = 2: \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \underline{r}_1 = \underline{0} \iff \underline{r}_1 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad c_1 \in \mathbb{R}.$$

$$\lambda_2 = 1: \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \underline{r}_2 = \underline{0} \iff \underline{r}_2 = c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_2 \in \mathbb{R}.$$

Dømi

Lat $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so er $p(\lambda) = \lambda^2 + 1 \neq 0$, um $\lambda \in \mathbb{R}$.

Lat nú $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$, so er $p(\lambda) = (1-\lambda)^2 = 0 \iff \lambda = 1$.

Vit siga, at $\lambda_1 = \lambda_2 = 1$ er dupultrót í $p(\lambda)$, ella at rótin hevur ein algebraiskan multiplicitet á 2.

$$(A - I)\,\underline{r} = \underline{0} \iff \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \underline{r} = \underline{0} \iff \underline{r} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c \in \mathbb{R}.$$

Dømi 7.3 Lat $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, so er $p(\lambda) = (\frac{1}{2} - \lambda)^2 - \frac{1}{2} = \lambda^2 - \lambda = \lambda(\lambda - 1)$

$$p(\lambda) = 0 \iff \lambda_1 = 1 \ \vee \ \lambda_2 = 0.$$

$$\lambda_1 = 1: \quad \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \underline{r}_1 = \underline{0} \iff \underline{r}_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c_1 \in \mathbb{R}.$$

$$\lambda_2 = 0: \quad \underline{\underline{A}}\, \underline{r}_2 = \underline{0} \iff \underline{r}_2 = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_2 \in \mathbb{R}.$$

Symmetriskar matricur

Givið, at $A = A^T$, so finna vit tvey reel eginvirði og vit fáa gjørt eigendekompositión av $\underline{\underline{A}}$:

$$\underline{\underline{A}} = \underline{\underline{R}}\, \Lambda\, R^{-1}.$$

$$A = \begin{bmatrix} a & c \\ c & d \end{bmatrix}$$

Vit hava

$$(a - \lambda)(d - \lambda) - c^2 = 0$$
$$\iff (a - \lambda)(d - \lambda) - c^2 = 0$$
$$\iff \lambda - (a+d)\lambda + (ad - c^2) = 0$$
$$\iff \lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4 \cdot (ad - c^2)}}{2}$$

$$a^2 + 2ad + d^2 - 4ad + 4c^2$$
$$= a^2 + d^2 - 2ad + 4c^2$$
$$= (a-d)^2 + 4c^2 \quad > 0$$

So eginvirðini eru reel!

Givið

$$\underline{\underline{A}}\,\underline{r}_1 = \lambda_1 \underline{r}_1$$

$$\underline{\underline{A}}\,\underline{r}_2 = \lambda_2 \underline{r}_2$$

har $\underline{\underline{A}}$ er symmetrisk, so er

$$\begin{aligned}
& (\underline{\underline{A}}\,\underline{r}_1)^T = (\lambda_1 \underline{r}_1)^T \\
\Leftrightarrow\quad & \underline{r}_1^T \underline{\underline{A}}^T = \lambda_1 \underline{r}_1^T \\
\Leftrightarrow\quad & \underline{r}_1^T \underline{\underline{A}} = \lambda_1 \underline{r}_1^T \\
\Leftrightarrow\quad & \underline{r}_1^T \underline{\underline{A}}\,\underline{r}_2 = \lambda_1 \underline{r}_1^T \underline{r}_2
\end{aligned}$$

og sama við seinnu líkningina

$$\underline{r}_1^T \underline{\underline{A}}\,\underline{r}_2 = \lambda_2 \underline{r}_1^T \underline{r}_2 .$$

Nú er $\quad \lambda_1 \underline{r}_1^T \underline{r}_2 = \lambda_2 \underline{r}_1^T \underline{r}_2$

$$(\lambda_1 - \lambda_2)\,\underline{r}_1^T \underline{r}_2 = \underline{0} \quad \Rightarrow \quad \underline{r}_1^T \underline{r}_2 = 0$$

Lat $\quad R = \begin{bmatrix} r_{11} & r_{21} \\ r_{21} & r_{22} \end{bmatrix}$, so er

$$\underline{\underline{A}}\,\underline{\underline{R}} = [\lambda_1 \underline{r}_1 , \lambda_2 \underline{r}_2]$$

Við $\quad R = [\underline{r}_1 , \underline{r}_2]$ og $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, so er

$$\underline{\underline{A}}\,\underline{\underline{R}} = \underline{\underline{R}}\,\underline{\Lambda} \Leftrightarrow \underline{\underline{A}} = \underline{\underline{R}}\,\underline{\underline{\Lambda}}\,\underline{\underline{R}}^{-1}$$

Vit normalisera altíð $\underline{r}_1$ og $\underline{r}_2$, so

$$\underline{r}_1^T \underline{r}_1 = 1 \quad \text{og} \quad \underline{r}_2^T r_2 = 1,$$

$$\underline{r}_1^T \underline{r}_2 = 0 \quad \text{og} \quad \underline{r}_2^T \underline{r}_1 = 0$$

So vit fáa $\quad R^T R = I \quad$ og $\quad R^{-1} = R^T.$

Eisini er $\quad \Lambda = R^{-1} A R.$

$2D \to 3D$ Vit seta $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ og $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ at vera standardbasis hjá $\mathbb{R}^3$.

Longd á $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Krossprodukt $\wedge$

$$v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{og} \quad w = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$