Differentiability and inverses

Ex 1. Differentiate $(x^2+7)^{13}$.

$$\left((x^2+7)^{13}\right)' = 13\,(x^2+7)^{12} \cdot 2x$$

Ex 2. Compute derivatives.

1. $f_1(x) = (x^2+1)\sin(x)$

$$f_1'(x) = 2x\sin(x) + (x^2+1)\cos(x)$$

2. $f_2(x) = \dfrac{e^x}{x^2}$

$$f_2'(x) = \frac{e^x x^2 - e^x \cdot 2x}{x^4} = \frac{x^2-2x}{x^4}\,e^x = \frac{x-2}{x^3}\,e^x$$

3. $f_3(x) = \cos(\ln(x)+1)$

$$f_3'(x) = -\sin(\ln(x)+1)\cdot\frac{1}{x}$$

4. $f_4(x) = \cos(\cos(\cos(x)))$

$$f_4'(x) = -\sin(\cos(\cos(x)))\cdot(-\sin(\cos(x)))\cdot(-\sin(x))$$
$$= -\sin(\cos(\cos(x)))\cdot\sin(\cos(x))\cdot\sin(x)$$

Ex 3. Compute derivatives.

1. $f_1(t) = t^2 + i\sin(t)$

$$f_1'(t) = 2t + i\cos(t)$$

2. $f_2(t) = 1 + i\,t^5$

$$f_2'(t) = 5i\,t^4$$

3. $f_3(t) = t^5 - i$

$f_3'(t) = 5t^4$

4. $f_4(t) = 3e^{it}$

$f_4'(t) = 3ie^{it}$

5. $f_5(t) = ie^{2t+3it}$

$f_5'(t) = i(2+3i)\,e^{2t+3it} = (-3+2i)\,e^{2t+3it}$

Ex4. Determine the derivative of arcsin using $(f^{-1})'(y) = \frac{1}{f'(x)}$.

$$(\arcsin^{-1})'(y) = \frac{1}{(\sin(x))'} = \frac{1}{\cos(x)}$$

$$= \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-y^2}}$$

Recall that sine is the function relating values y to input x by $y = \sin(x)$, hence the last substitution. So we have $(\arcsin^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$, $x \in]-1,1[$.

Ex5.

a) Show that $\tan'(t) = 1 + \tan^2(t)$ where defined.

$$\tan'(t) = \left(\frac{\sin(t)}{\cos(t)}\right)' = \frac{\cos(t)\cdot\cos(t) - \sin(t)\cdot(-\sin(t))}{\cos^2(t)}$$

$$= \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} = 1 + \tan^2(t) \quad \left(\text{also } \frac{1}{\cos^2(t)}\right)$$

$\cos(t) \neq 0$.

b) Determine $\arctan'(x)$.

$$\arctan'(y) = \frac{1}{\tan'(x)} = \frac{1}{1+\tan^2(x)}, \quad y = \tan(x)$$

$$= \frac{1}{1+y^2}, \quad y \in \mathbb{R}.$$

Thus we have $\arctan'(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Ex6. Show by definition that $f(x) = x^2$ is differentiable for every $x_0 \in \mathbb{R}$ with $f'(x_0) = 2x_0$.

Using $f(x) = f(x_0) + a(x-x_0) + \varepsilon(x-x_0)(x-x_0)$

we have for arbitrary $x_0 \in \mathbb{R}$

$$x^2 = x_0^2 + a(x-x_0) + \varepsilon(x-x_0)(x-x_0)$$

$$\Leftrightarrow \quad x^2 - x_0^2 = a(x-x_0) + \varepsilon(x-x_0)(x-x_0)$$

$$\Leftrightarrow \quad (x+x_0)(x-x_0) = a(x-x_0) + \varepsilon(x-x_0)(x-x_0)$$

$$\Leftrightarrow \quad x + x_0 = a + \varepsilon(x-x_0)$$

$$\rightarrow \quad 2x_0 - a = \varepsilon(0) \quad \text{for } x \to x_0.$$

With $a = 2x_0$ we have $\varepsilon(0) = 0$ and $|\varepsilon(x-x_0)| \to 0$ for $x \to x_0$. Thus differentiability is aserted by definition 1.59.

Ex 7.

a) Solve quadratics in $\mathbb{R}$ and $\mathbb{C}$.

1. $2x^2 + 9x - 5 = 0$ $\qquad D = 9^2 - 4 \cdot 2 \cdot (-5) = 121$

$$x = \frac{-9 \pm 11}{4} = \begin{cases} \frac{1}{2} \\ -5 \end{cases} \quad \text{in } \mathbb{R} \text{ and } \mathbb{C}$$

2. $x^2 - 4x = 0 \iff x(x-4) = 0 \iff x = 0 \lor x = 4$ in $\mathbb{R}$ and $\mathbb{C}$.

3. $x^2 - 4x + 13 = 0$ $\qquad D = (-4)^2 - 4 \cdot 13 = -36$

No real solutions. In $\mathbb{C}$ we have

$$x = \frac{4 \pm 6i}{2} = 2 \pm 3i .$$

b) Solve $2(x+1-i)(x+1+i) = 0$ and show it's actually real coefficients.

We have

$$2(x - (-1+i))(x - (-1-i)) = 0$$

$$\iff x = -1+i \lor x = -1-i .$$

By multiplying out we find the coefficients.

$$2(x+1-i)(x+1+i) = 2(x^2 + x + ix + x + 1 + i - ix - i + 1)$$

$$= 2x^2 + 4x + 4 .$$

Ex 8.

a) Solve $z^2 - (1+5i)z = 0 \iff z(z - (1+5i)) = 0$

$$\iff z = 0 \lor z = 1+5i$$

b) Solve $z^2 + (2+2i)z - 2i = 0$.

$$D = (2+2i)^2 - 4 \cdot (-2i) = 4 - 4 + 8i + 8i = 16i.$$

We have $w^2 = 16i$ and one solution w_0 is $4e^{\frac{\pi}{4}i} = \sqrt{8} + i\sqrt{8}$,

the other is $-\sqrt{8} - i\sqrt{8}$.

$$z_0 = \frac{-2-2i+\sqrt{8}+i\sqrt{8}}{2} = \frac{-2+\sqrt{8}}{2} + \frac{-2+\sqrt{8}}{2} i = -1+\sqrt{2} + (\sqrt{2}-1)i$$

$$z_1 = \frac{-2-2i-\sqrt{8}-i\sqrt{8}}{2} = \frac{-2-\sqrt{8}}{2} + \frac{-2-\sqrt{8}}{2} i = -1-\sqrt{2} - (\sqrt{2}+1)i$$

Ex 9.

a) Show that $f_1(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not an ε-function.

$|f_1(x)| \to \pm\infty \neq 0$ for $x \to 0^+$ and $x \to 0^-$ respectively.

b) Show that $f_2(x) = 1 - \cos(x)$ is an ε-function.

$f_2(0) = 1 - \cos(0) = 1 - 1 = 0$ and

$|f_2(x)| = |1 - \cos(x)| \to |1-1| = 0$ for $x \to 0$.

c) Show that $f_3(x) = ie^{ix} - i$ is an ε-function.

$f_3(0) = ie^{i \cdot 0} - i = i \cdot 1 - i = 0$ and

$$|f_3(x)| = |ie^{ix} - i| = |i(e^{ix}-1)| = |i|\,|e^{ix}-1|$$
$$= |e^{ix}-1| \to |e^{i\cdot 0}-1| = 0 \text{ for } x \to 0.$$