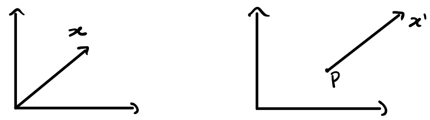
Affin avmyndan Vit fara nú at gera anboð, so at avmyndanir hjá okkum ikki nýtast at vera knýttar at origo.

Teknikkurin við affina avmyndan er einfald. Vit fremja translatión av origo til eitt punkt p og fremja lineera avmyndan.



Her er $x' = p + \underline{\underline{I}}x$

Ein affin avmyndan er tiskil ein sum kann skrivast

$$x' = p + \underline{\underline{A}}x \qquad (\text{punktið } x \text{ kann lesast sum vektorur } \underline{x} - \underline{o}).$$

Target box Senda vit $[e_1, e_2]$ í target box við (min_1, min_2) og (max_1, max_2), so er affin avmyndan, sum sendir í hesi target box

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p + Ax = \begin{bmatrix} min_1 \\ min_2 \end{bmatrix} + \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Hetta er rektangulert! Um A er skew ella eitthvørt annað, so kunnu vit lættliga skriva slíka avmyndan.

Dømi Lat $p = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ og $A = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}$.

Fyri $\underline{u} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix}$ fáa vit

$$x = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}.$$

Hvat um $\underline{u} = k \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix}$? So fáa vit $\begin{bmatrix} 1+k \\ 8+k \end{bmatrix}$.

Lineer avmyndan A er ein lineer avmyndan, um

$$\underline{\underline{A}}(\underline{u} + \underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} \quad \text{og} \quad \underline{\underline{A}}(c\underline{u}) = c\,\underline{\underline{A}}\,\underline{u}$$

Ein affin avmyndan manglar linearitet

$$\underline{\underline{A}}(\underline{u} + \underline{v}) + p = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} + p \neq \underline{\underline{A}}\,\underline{u} + p + \underline{\underline{A}}\,\underline{v} + p$$

$$\underline{\underline{A}}(c\underline{u}) + p = c\underline{\underline{A}}\,\underline{u} + p \neq c(\underline{\underline{A}}\,\underline{u} + p)$$

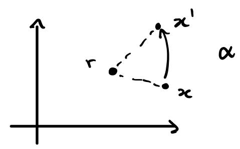
Ein affin avmyndan varðveitir tó lutføll. Taka vit trý punkt, sum liggja á somu linju, so at $p_2 = (1-t)p_1 + tp_3$ í barycentriskum koordinatum, so er við avmyndan $x' = \underline{A}x + p$ givið, at

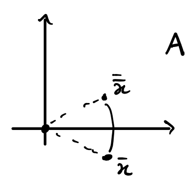
$$\begin{aligned} p_2' &= \underline{A}\left((1-t)p_1 + tp_3\right) + p \\ &= (1-t)\underline{A}p_1 + t\underline{A}p_3 + (1-t+t)p \\ &= (1-t)(\underline{A}p_1 + p) + t(\underline{A}p_3 + p) \\ &= (1-t)p_1' + t p_3' \end{aligned}$$

Rotation

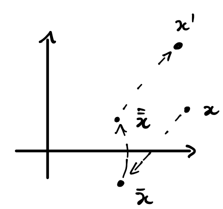
Vit klára at rotera um origo, og um vit vilja rotera úti í $\mathbb{R}^2$ um eitt annað punkt, so flyta vit affint hartil.
Lat okkum rotera x um eitt punkt r, so er $x-r$ ein translatión til origo. Rotera sum vanligt við A og flyt uppá pláss aftur

$$x' = A(x-r) + r$$



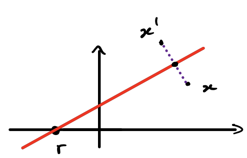


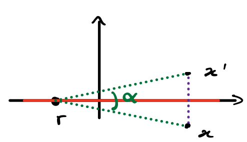
$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$



Reflection

Tað loysir seg ikki altíð at arbeiða affint. Reflektera vit x um eina linju l, so ber til at gera fylgjandi:





Rotera síðani uppá pláss.

Enn betur er at brúka røttu amboðini til uppgávuna. Lat p vera ortogonal projektiónin av x á l. So er $p = \frac{1}{2}x + \frac{1}{2}x'$ í barycentriskum koordinatum, so

$$x' = 2p - x.$$

Vit finna p á linjuni l við: $t = \frac{v \cdot w}{\|v\|^2}$, sí 3.7.

Triangle

Vit kunnu avmynda skap í flatanum. Fyri sovitt nýtast vit best tríkantar, tí restin byggja vit úr tríkantar.

Trý punkt definera ein tríkant, men so definer tveir vektorar eisini ein tríkant.

$$(a_1, a_2, a_3) \longrightarrow \quad \underline{v}_2 = a_2 - a_1 \quad \text{og} \quad \underline{v}_3 = a_3 - a_1 .$$

Um vit vilja avmynda yvir í ein annan tríkant við $\underline{v}_2'$ og $\underline{v}_3'$, so seta vit upp

$$A[\underline{v}_2, \underline{v}_3] = [\underline{v}_2', \underline{v}_3']$$
$$\Leftrightarrow A\,V = V'$$
$$A = V'V^{-1}$$

Altso rokna vit A uttan víðari.

Dømi 6.4 Tríkanturin T givin við $a_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ og $a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ skal avmyndast í T', sum er givin við $a_1' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $a_2' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ og $a_3' = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Vit fáa $\underline{v}_2 = \begin{bmatrix} -1-0 \\ -1-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ og $\underline{v}_3 = \begin{bmatrix} 1-0 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so $V = \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix}$.

$\underline{v}_2' = \begin{bmatrix} 1-0 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ og $\underline{v}_3' = \begin{bmatrix} -1-0 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so $V' = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$.

Vit finna V^{-1}:

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ -2 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \cdot(-\frac{1}{2}) \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -\frac{1}{2} \end{array}\right] \begin{array}{l} \\ -R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} \frac{1}{2}R_2 \\ \cdot\frac{1}{2} \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

Tískil verður $A = V'V^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Ein $180°$ rotatión, so um vit royna $x = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix}$ í T, tvs. a_1 er origo, so avmyndast til

$$x' = A(x - a_1) + a_1' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0-0 \\ -\frac{1}{3}-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{7}{3} \end{bmatrix}.$$

Eigenvalues and eigenvectors

Bæði sum avmyndan og líkningaskipan hevur ein matrica A information, um sínar eginleikar. Innbygd information, sum er fjald handan virðir hjá A.
Til tíðir finna vit ein rætning, har vektorarnir næstan ikki ávirkast av A, bert upp til ein konstant λ.
Tvs. vektorar, har $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ liggja í eini dimensión, sum er útspent av $\underline{v}$.
Slíkir vektorar fortelja okkum geometriina hjá $\underline{\underline{A}}$.
Typiskt fyri 2D matricu $\underline{\underline{A}}$ er, at hendan hevur tveir rætningar, har vektorar bert ávirkast við skalar multiplikatión. Hetta er ymiskt frá, at vit taka $[\underline{e}_1, \underline{e}_2]$ til $[\underline{a}_1, \underline{a}_2]$, tí um

$$\underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v}_1 \quad \text{og} \quad \underline{\underline{A}}\,\underline{v}_2 = \lambda_2 \underline{v}_2,$$

so eru $[\underline{v}_1, \underline{v}_2]$ eitt vektorpar, sum er í báðum skipanum, og bera beinleiðis geometrisku boðini hjá A í sær.

Vit definera við $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ skalarin λ at vera eginvirði hjá $\underline{\underline{A}}$, meðan $\underline{v}$ er eginvektor hjá $\underline{\underline{A}}$. Eitt eginvirði hevur ein tilknýttan eginvektor.

Eginvirðir

Her skalararnir λ eru ikki so torførir at leita fram. Vit skriva eindarmatricuna inn í líkningina

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$
$$\Leftrightarrow \quad \underline{\underline{A}}\,\underline{v} = \lambda \underline{\underline{I}}\,\underline{v}$$
$$\Leftrightarrow \quad \underline{\underline{A}}\,\underline{v} - \lambda \underline{\underline{I}}\,\underline{v} = \underline{0}$$
$$\Leftrightarrow \quad (\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0}$$

Tann lineera avmyndanin $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ skal senda $\underline{v}$ í $\underline{0}$, har $\underline{v} \neq \underline{0}$, so avmyndanin hevur ikki fullan rank. Ergo er determinanturin null, um ein loysn finst.
Vit definera nú karakteristiska polynomið

$$p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}),$$

og karakter líkningina

$$p(\lambda) = 0.$$

Tær loysnir λ hjá karakter líkningini eru tískil eginvirðini.

Dømi 7.1

Lat $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, so er $p(\lambda) = |\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$

Vit loysa λ í karakter líkningini

$$p(\lambda) = 0$$
$$\Leftrightarrow \lambda^2 - 4\lambda + 3 = 0$$
$$\Leftrightarrow \lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 3}}{2\cdot 1} = \frac{4 \pm \sqrt{4}}{2} = \begin{cases} 3 \\ 1 \end{cases}$$
$$\lambda_1 = 3 \quad \text{og} \quad \lambda_2 = 1$$

$$(2-\lambda)^2 - 1 = 0$$
$$\Leftrightarrow (2-\lambda)^2 = 1$$
$$\Leftrightarrow \lambda_2 = 1 \lor \lambda_1 = 3$$

Eginvectorar Tá eginvirðini eru funnin, so er skjótt at loysa eginvektorar út frá líkningini

$$(\underline{\underline{A}}-\lambda \underline{\underline{I}})\,\underline{v} = \underline{0}.$$

Set λ_1 inn og ger eina avmyndan $B = \underline{\underline{A}}-\lambda\underline{\underline{I}}$. Loys síðani homogena líkningaskipanina. Sama gera vit fyri hvørt eginvirði.

Dømi Lat $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, so er $\lambda_1 = 3$ og $\lambda_2 = 1$.

λ_1: $\underline{\underline{A}} - 3\underline{\underline{I}} = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \underline{v}_1 = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ +R_1 \end{array} \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{les} \quad -1\cdot v_1 + 1\cdot v_2 = 0$$

λ_1: $\underline{\underline{A}} - \underline{\underline{I}} = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \underline{v}_2 = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \end{array} \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{les} \quad 1\cdot v_1 + 1\cdot v_2 = 0$$

Vit normalisera gjarna eginvektorar, so

$$\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{og} \quad \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$