u7. 1. Avger konvergens ella divergens.

(1) $\sum_{n=0}^{\infty} \frac{n}{4^{n+3}}$, kvotient testin gevur

$$\left| \frac{n+1}{4^{n+4}} \cdot \frac{4^{n+3}}{n} \right| = \frac{1}{4} \left| \frac{n+1}{n} \right| = \frac{1}{4}\left(1+\frac{1}{n}\right) \to \frac{1}{4} < 1 \quad \text{tá } n \to \infty.$$

Rekkjan er absolut konvergent per 4.30(i).

(2) $\sum_{n=0}^{\infty} (-1)^n n^2$, her gongur $n^2 \to \infty$ tá $n \to \infty$, so per 4.19 er rekkjan divergent.

(3) $\sum_{n=0}^{\infty} a_n$, har $a_n = \begin{cases} n^3 , & n = 1, 2, \ldots, 10000, \\ \frac{(-1)^n}{n^3} , & n \geq 10001. \end{cases}$

Vit hava $b_n = \frac{1}{n^3}$, og per 4.38 er rekkjan $\sum_{n=10001}^{\infty} (-1)^n b_n$ konvergent. Tískil er rekkjan $\sum_{n=0}^{\infty} a_n$ konvergent, tí hendan er ein sum av eini endaliga rekkju og eini konvergenta. Klárt absolut konvergent.

2. $\sum_{n=1}^{\infty} \frac{1}{a^n+1}$, har $a > 0$.

(1) Finn $a > 0$, har rekkjan er konvergent og divergent.

Divergent fyri $0<a<1$ og $a=1$, tí

$$\frac{1}{a^n+1} \to 1 \quad \text{og} \quad \frac{1}{a^n+1} \to \frac{1}{2} \quad \text{tá } n\to\infty \text{, og 4.19 gevur divergens.}$$

Lat $a>1$:

$$\left| \frac{1}{a^{n+1}+1} \cdot \frac{a^n+1}{1} \right| = \frac{1 + \frac{1}{a^n}}{a + \frac{1}{a^{n+1}}} \to \frac{1}{a} < 1 \quad \text{tá } n \to \infty.$$

Per 4.30 er rekkjan absolut konvergent fyri $a>1$.

(2) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$. Rokna við frávik á 0,1.

Lat $f(x) = \frac{1}{3^x+1}$. Funktiónin er kontinuert og fallandi, so vit brúka 4.35(ii).

$$f(2) = \frac{1}{3^2+1} = \frac{1}{10} = 0{,}1.$$

$$\int f(x)\,dx = x - \frac{\ln(3^x+1)}{\ln(3)} \quad , \quad \int_2^t f(x)\,dx = \left[x - \frac{\ln(3^x+1)}{\ln(3)} \right]_2^t = t - \frac{\ln(3^t+1)}{\ln(3)} - 2 + \frac{\ln(10)}{\ln(3)}$$

$$= \frac{\ln(3^t) - \ln(3^t+1)}{\ln(3)} - 2 + \frac{\ln(10)}{\ln(3)} \to \frac{\ln(10)}{\ln(3)} - 2 \quad \text{tá } t \to \infty.$$

$$\ln(3^t) - \ln(3^t+1)$$

$$= \ln\left(\frac{3^t}{3^t+1}\right) = \ln\left(\frac{1}{1+\frac{1}{3^t}}\right) \to \ln(1) = 0 \quad \text{tá } t \to \infty.$$

$$\sum_{n=1}^{1} \frac{1}{3^n+1} + \int_2^{\infty} f(x)\,dx = \frac{1}{4} + \frac{\ln(10)}{\ln(3)} - 2 = 0{,}3459.$$

3. Potensrekkjur, finn konvergens og sum. (Setn. 5.2)

(i) $\sum_{n=0}^{\infty} (-x)^n$, konv. um $|-x|<1$, sum er eins við $|x|<1$.

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x} \quad , \quad |x|<1.$$

Tá $|-x|>1 \iff |x|>1$ er rekkjan divergent.

(ii) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$, vit hava $\sum_{n=0}^{\infty} (-x^2)^n$ er konv. um $|-x^2|<1 \iff |x|<1$, annars divergent fyri $|x|>1$.

$$\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \quad , \quad |x|<1.$$

4. $\sum_{n=1}^{\infty} \frac{x^{2n}}{3^n(n^2+1)}$

(i) Finn ρ. Vit brúka kvotientkriteriið.

$$\left| \frac{x^{2n+2}}{3^{n+1}(n^2+2n+2)} \cdot \frac{3^n(n^2+1)}{x^{2n}} \right| = \frac{1}{3} \frac{n^2+1}{n^2+2n+2} x^2 \to \frac{1}{3} x^2 \quad \text{tá } n\to\infty.$$

Konvergent, um $\frac{x^2}{3} < 1 \iff |x|<\sqrt{3}$. Tvs. absolut konvergens fyri $\rho=\sqrt{3}$.

(ii) Konv. í $\rho = \pm\sqrt{3}$. Vit seta inn og fáa

$$\sum_{n=1}^{\infty} \frac{(\pm\sqrt{3})^{2n}}{3^n(n^2+1)} = \sum_{n=1}^{\infty} \frac{3^n}{3^n(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Rekkjan er konv. per 4.20(i) við rekkjuni $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(iii) Fyri $x\in(-\rho,\rho)$ er $s(x)$ summurin hjá rekkjuni. Vís, at $\frac{\pi}{4} \le s(\sqrt{3}) \le \frac{\pi}{4}+\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, lat $f(x) = \frac{1}{x^2+1}$ og við integralkriteriið:

$$\int_1^{\infty} f(x)dx \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \int_1^{\infty} f(x)\,dx + f(1) \quad , \quad \int_1^t f(x)dx = [\arctan(x)]_1^t \to \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

$$\iff \quad \frac{\pi}{4} \le s(\sqrt{3}) \le \frac{\pi}{4} + \frac{1}{2}$$

5. Brúka 5.2 og 5.17

(i) Vís, at $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$, $x\in(-1,1)$

Vit hava, at $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ og $\sum_{n=1}^{\infty} n x^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$.

Vit fáa tí $\sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = \frac{x}{(1-x)^2}$, $x \in (-1,1)$.

(ii) Vís, at $\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}$, $x \in (-1,1)$.

Úr (i) er $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$, $x \in (-1,1)$, so við S.17 aftur

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} = \frac{1+x}{(1-x)^3} , \quad x \in (-1,1).$$

So fáa vit $\sum_{n=1}^{\infty} n^2 x^n = x \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{x+x^2}{(1-x)^3}$, $x \in (-1,1)$.

(iii) Brúka (i) og (ii) til at rokna $\sum_{n=1}^{\infty} \frac{n}{2^n}$ og $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Set $x = \frac{1}{2}$, so er við (i)

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n x^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

Við (ii)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} n^2 x^n = \frac{\frac{1}{2} + (\frac{1}{2})^2}{(1-\frac{1}{2})^3} = \frac{\frac{3}{4}}{\frac{1}{8}} = 6.$$

6. (i) Finn ρ hjá $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

$$\left| \frac{2^{n+1}}{n+1} x^{n+1} \cdot \frac{n}{2^n \cdot x^n} \right| = 2 \cdot \frac{1}{1+\frac{1}{n}} |x| \to 2|x| \quad \text{tá} \quad n \to \infty.$$

$2|x| < 1 \iff |x| < \frac{1}{2}$ abs. konv. og $\rho = \frac{1}{2}$.

(ii) $f(x) = \sum_{n=1}^{\infty} \frac{2^n}{n} x^n$, $x \in (-\frac{1}{2}, \frac{1}{2})$.

Finn $f'(x)$. Per S.17 við $|x| < \rho$ fáa vit

$$f'(x) = \sum_{n=1}^{\infty} 2^n x^{n-1} = 2 \sum_{n=1}^{\infty} (2x)^{n-1} = 2 \sum_{n=0}^{\infty} (2x)^n$$

$$\overset{5.2}{=} \frac{2}{1-2x} , \quad |x| < \frac{1}{2}.$$

(iii) Finn $f(x)$.

Fyri $x=0$ er $f(0) = \sum_{n=1}^{\infty} \frac{2^n}{n} 0^n = 0$. So kunnu vit integrera

$$f(x) = f(0) + \int f'(x)\,dx = \int \frac{2}{1-2x}\,dx = -\ln(|1-2x|) = -\ln(1-2x) , \quad |x| < \frac{1}{2}.$$