Uppg. 1 Vís, at y er ein loysn hjá differentiallíkningini.

(i) $y' = \cos x + y$, $y = e^x + \frac{\sin x}{2} - \frac{\cos x}{2}$.

Vit hava $y' = e^x + \frac{\cos x}{2} + \frac{\sin x}{2}$, og við at seta í líkningina er

$$y' = \cos x + y$$

$$\iff e^x + \frac{\cos x}{2} + \frac{\sin x}{2} = \cos x + e^x + \frac{\sin x}{2} - \frac{\cos x}{2}$$

$$\iff e^x + \frac{\cos x}{2} + \frac{\sin x}{2} = e^x + \frac{\sin x}{2} + \frac{\cos x}{2}.$$

Altso er y ein loysn hjá differentiallíkningini.

(ii) $y' = y \cdot \sin x$, $y = \pi e^{-\cos x}$.

Vit fáa $y' = \pi e^{-\cos x} \cdot \sin x$, so við innsetan fáa vit, at

$$y' = y \cdot \sin x$$

$$\pi e^{-\cos x} \cdot \sin x = \pi e^{-\cos x} \cdot \sin x.$$

Altso er y ein loysn hjá differentiallíkningini.

Uppg. 2 Loys við panserformlinum.

(i) Rokna fullkomuligu loysnina og partikuleru loysnina við $x(1) = -1$ hjá

$$x' + \frac{1}{t} x = -2t^2 , \quad t > 0.$$

Lat $p(t) = \frac{1}{t}$ og $q(t) = -2t^2$. Vit hava nú $P(t) = \ln t$ og loysnin er

$$x(t) = e^{-\ln t} \int e^{\ln t} \cdot (-2t^2)\, dt + c e^{-\ln t} , \quad c \in \mathbb{R}$$

$$= \frac{1}{t} \cdot \left(-\frac{1}{2} t^4\right) + \frac{c}{t}$$

$$= -\frac{t^3}{2} + \frac{c}{t}.$$

Set $x(1) = -1$, so

$$-\frac{1}{2} + c = -1 \iff c = -\frac{1}{2}.$$

Vit hava nú partikulera loysnina

$$x(t) = -\frac{t^3}{2} - \frac{1}{2t}.$$

(ii) Rokna loysnina $z(t)$, har $c \in \mathbb{C}\setminus\{0\}$ í $z' - c \cdot z = 2$, $t \in \mathbb{R}$. Finn fyri $c = i-1$ loysnina, tá $z(0) = i$.

Vit hava, at $p(t) = -c$ og $q(t) = 2$. Við $P(t) = -ct$ fáa vit loysnina

$$z(t) = e^{ct} \int e^{-ct} \cdot 2 \, dt + k e^{-ct} \quad , \quad k \in \mathbb{C}$$

$$= -\frac{2}{c} + k e^{-ct}.$$

Set nú $c = i-1$ og $z(0) = i$, so er

$$-\frac{2}{i-1} + k = i \iff \frac{2+2i}{2} + k = i$$

$$\iff 1 + i + k = i \iff k = -1.$$

Partikulera loysnin er

$$z(t) = 1 + i - e^{(i-1)t}.$$

Uppg. 3 Ein separabul differentiallíkning er givin við $\frac{dy}{dx} = 6y^2x$.

(i) Loys vha. separatión av variablum.

$$\frac{dy}{dx} = 6y^2x \iff y^{-2}\frac{dy}{dx} = 6x \quad , \quad y \neq 0$$

$$\iff \int y^{-2} \, dy = \int 6x \, dx$$

$$\iff -\frac{1}{y} = 3x^2 + c \quad , \quad c \in \mathbb{R}$$

$$\iff y = -\frac{1}{3x^2 + c}.$$

(ii) Finn loysnina, har $y(1) = \frac{1}{25}$, og avger intervallið har loysnin gevur meining.

Vit kunnu seta í

$$-\frac{1}{y} = 3x^2 + c \Rightarrow c + 3 = -25 \Leftrightarrow c = -28.$$

Altso er partikulera loysnin

$$y(t) = \frac{1}{28 - 3x^2},$$

har $y(0) = \frac{1}{28}$. Vit fáa $x \in \left]-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right[$.

(iii) Kanna loysnirnar við $y(-4) = -\frac{1}{20}$ og $y(6) = -\frac{1}{80}$. Vís eisini á teirri definitiónsmongd.

Vit brúka aftur $-\frac{1}{y} = 3x^2 + c$ og fáa

$$1: \quad c + 48 = 20 \qquad\qquad 2: \quad c + 108 = 80$$
$$\Leftrightarrow \quad c = -28 \qquad\qquad \Leftrightarrow \quad c = -28$$

Loysnirnar eru tískil sama sum í (ii), men vit fáa intervallini

$$1: \quad y(t) = \frac{1}{28 - 3x^2}, \quad x < -\sqrt{\frac{28}{3}}.$$

$$2: \quad y(t) = \frac{1}{28 - 3x^2}, \quad \sqrt{\frac{28}{3}} < x.$$

Uppg. 4 Ein separabul differentiallíkning er givin við

$$y' = \frac{3x^2 + 4x - 4}{2y - 4}, \quad y \neq 2.$$

Finn loysnina, har $y(1) = 3$, og definitiónsmongdina.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \Leftrightarrow \int 2y - 4 \, dy = \int 3x^2 + 4x - 4 \, dy$$
$$\Leftrightarrow y^2 - 4y = x^3 + 2x^2 - 4x + c, \; c \in \mathbb{R}$$

Fyri $y(1) = 3$ fáa vit

$$9 - 12 = 1 + 2 - 4 + c$$
$$\Leftrightarrow c = -2.$$

Vit hava nú

$$y^2 - 4y - (x^3 + 2x^2 - 4x - 2) = 0$$

$$\Rightarrow \quad y(t) = \frac{4 \pm \sqrt{16 + 4\cdot(x^3 + 2x^2 - 4x - 2)}}{2}$$

$$= \frac{4 \pm 2\sqrt{4 + x^3 + 2x^2 - 4x - 2}}{2}$$

$$= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}\,.$$

Vit kanna fyri $x = 1$, hvat fyri loysn svarar til $y(1) = 3$.

$$2 \pm \sqrt{1 + 2 - 4 + 2} = \begin{cases} 3 \\ 1 \end{cases}$$

Tann partikulera loysnin er tískil

$$y(t) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}\,,$$

har $x > -3.36523$ (5 des.)

Uppg. 5 Ein differentiallíkningaskipan er givin við

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 3x_1 + 2x_2\,. \end{cases}$$

(i) Avger fullkomuligu loysnina hjá skipanini.

Vit hava $\underline{x}' = \underline{\underline{A}}\,\underline{x}$, har $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Vit loysa eginvirðini og eginvektorarnar.

$$p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (1-\lambda)(2-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4\,.$$

Nú er $p(\lambda) = 0 \iff \lambda = \dfrac{3 \pm \sqrt{9 - 4\cdot(-4)}}{2} = \dfrac{3 \pm 5}{2} = \begin{cases} 4 \\ -1 \end{cases}.$

Fyri eginvektorarnar loysa vit $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{v} = \underline{0}$.

$$\lambda_1 = 4 : \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \underline{v}_1 = \underline{0} \iff \underline{v}_1 = k \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad k \in \mathbb{R}.$$

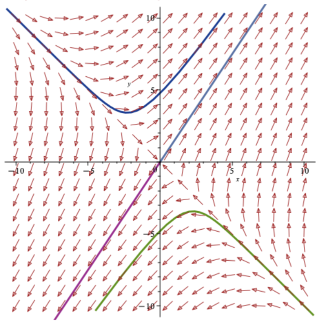
$$\lambda_2 = -1 : \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \underline{v}_2 = \underline{0} \iff \underline{v}_2 = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad k \in \mathbb{R}.$$

Fullkomuliga loysnin er tískil

$$x(t) = c_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

(ii) Ger eitt fasuportret/DEplot í Maple og vís nakrar loysnir.

```
> DEplot([D(x)(t)=x(t)+2*y(t),D(y)(t)=3*x(t)+2*y(t)],[x(t),y(t)],t=-2..2,[[x(0)=-5,y(0)=5.1],[x(0)=5,y(0)=-5.1],[x(0)=1,y(0)=3/2],[x(0)=-1,y(0)=-3/2]],x=-10..10,y=-10..10,arrows=medium,animatecurves=true)
```



(iii) Finn loysnina við $x_1(0) = 0$ og $x_2(0) = -4$.

$$x(t) = c_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} x_1(t) &= c_1 e^{4t} \cdot 2 + c_2 e^{-t} \cdot 1 \\ x_2(t) &= c_1 e^{4t} \cdot 3 + c_2 e^{-t} \cdot (-1) \end{aligned}$$

Vit hava nú : $\begin{cases} 2c_1 + c_2 = 0 \\ 3c_1 - c_2 = -4 \end{cases} \Rightarrow 5c_1 = -4 \iff c_1 = -\frac{4}{5}$,

og so $2\cdot(-\frac{4}{5}) + c_2 = 0$
$\iff c_2 = \frac{8}{5}$.

Altso hava vit

$$x(t) = -\frac{4}{5} e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{8}{5} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Uppg. 6 Skriva sum lineerar differentiallíkningar av fyrstu ordan. Loys síðani við Maple.

(i) $2y'' - 5y' + y = 0 \quad , \quad y(3) = 6, \quad y'(3) = -1.$

Lat $\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -\tfrac{1}{2}y + \tfrac{5}{2}y' = -\tfrac{1}{2}x_1 + \tfrac{5}{2}x_2 \end{aligned}$

So er skipanin givin við $\underline{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \underline{x}, \quad \begin{aligned} x_1(3) &= 6 \\ x_2(3) &= -1. \end{aligned}$

```
> dsolve([diff(x__1(t),t)=x__2(t),diff(x__2(t),t)=-1/2*x__1(t)+5/2*x__2(t),x__1(3)=6,x__2(3)=-1]);
  sol1:=rhs(%[1]);
```

$$\left\{ x_1(t) = \frac{\left(-17+3\sqrt{17}\right)\sqrt{17}\,e^{\frac{\left(5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}+\frac{3\sqrt{17}}{4}}} + \frac{\left(17+3\sqrt{17}\right)\sqrt{17}\,e^{-\frac{\left(-5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}-\frac{3\sqrt{17}}{4}}}, x_2(t) = \frac{\left(-17+3\sqrt{17}\right)\sqrt{17}\left(\frac{5}{4}+\frac{\sqrt{17}}{4}\right)e^{\frac{\left(5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}+\frac{3\sqrt{17}}{4}}} + \frac{\left(17+3\sqrt{17}\right)\sqrt{17}\left(\frac{5}{4}-\frac{\sqrt{17}}{4}\right)e^{-\frac{\left(-5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}-\frac{3\sqrt{17}}{4}}} \right\}$$

$$sol1 := \frac{\left(-17+3\sqrt{17}\right)\sqrt{17}\,e^{\frac{\left(5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}+\frac{3\sqrt{17}}{4}}} + \frac{\left(17+3\sqrt{17}\right)\sqrt{17}\,e^{-\frac{\left(-5+\sqrt{17}\right)t}{4}}}{17\,e^{\frac{15}{4}-\frac{3\sqrt{17}}{4}}}$$

```
> dsolve([2*diff(y(t),t,t)-5*diff(y(t),t)+y(t)=0,y(3)=6,D(y)(3)=-1]);
  sol2:=rhs(%);
```

$$y(t) = \left(\sqrt{17}+3\right)e^{-\frac{(t-3)\left(-5+\sqrt{17}\right)}{4}} - e^{\frac{(t-3)\left(5+\sqrt{17}\right)}{4}}\left(\sqrt{17}-3\right)$$

$$sol2 := \left(\sqrt{17}+3\right)e^{-\frac{(t-3)\left(-5+\sqrt{17}\right)}{4}} - e^{\frac{(t-3)\left(5+\sqrt{17}\right)}{4}}\left(\sqrt{17}-3\right)$$

```
> is(sol1=sol2)
```

$$true$$

(ii) $y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3, \quad y'''(0) = 4.$

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y''' \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = x_4 \\ x_4' &= y^{(4)} = -8y + \sin t \cdot y' - 3y'' + t^2 \\ &= -8x_1 + \sin t \cdot x_2 - 3x_3 + t^2 \end{aligned}$$

Maple fær einki úrslit.

Uppg. 7 Ein homogen 2. ordans differentiallíkning er givin við

$$x''(t) + 6x'(t) + 34x(t) = 0, \quad t \in \mathbb{R}.$$

(a) Karakterlíkningin hevur diskriminantin D. Loys líkningina $\omega_0^2 = D$ og finn real og imaginer partin hjá einari rót í karakterlíkningini.

Líkningin: $\lambda^2 + 6\lambda + 34 = 0 \implies D = 6^2 - 4 \cdot 1 \cdot 34 = -100$

$$\omega_0^2 = D = -100 \iff \omega_0 = 10i \vee -10i$$

Ein loysn er $z = \dfrac{-6+10i}{2} = -3+5i$ og $\mathrm{Re}(z) = -3$ og $\mathrm{Im}(z) = 5$.

(b) Avger fullkomuliga loysn.

$$x(t) = c_1 e^{-3t} \sin(5t) + c_2 e^{-3t} \cos(5t), \quad c_1, c_2 \in \mathbb{R}.$$

(c) Finn partikuleru loysnina, har $x(0) = 0$ og $x'(0) = 1$.

$$x(0) = c_2 = 0 \implies x(t) = c_1 e^{-3t} \sin(5t)$$

$$x'(t) = c_1 \left(-3e^{-3t} \cdot \sin(5t) + e^{-3t} \cdot 5\cos(5t)\right)$$

$$x'(0) = c_1 (0 + 5) = 1 \iff c_1 = \frac{1}{5}$$

$$\therefore \quad x(t) = \frac{1}{5} e^{-3t} \sin(5t).$$

Uppg. 8 Endurtak uppg. 7 við $x''(t) + 4x'(t) + 29x(t) = 0$, $t \in \mathbb{R}$, $x(0) = 0$ og $x'(0) = -1$.

```
> solve(x^2+4*x+29=0,x);
  w__0=Im(%[1]),Im(%[2]);
```

$$-2+5\,\mathrm{I},\ -2-5\,\mathrm{I}$$

$$w_0 = 5,\ -5$$

```
> dsolve((D@@2)(x)(t)+4*D(x)(t)+29*x(t)=0):
  subs(_C1=c__1,_C2=c__2,%);
```

$$x(t) = c_1 e^{-2t} \sin(5t) + c_2 e^{-2t} \cos(5t)$$

```
> dsolve({diff(x(t),t,t)+4*diff(x(t),t)+29*x(t)=0,x(0)=0,D(x)(0)=-1},{x(t)});
```

$$x(t) = -\frac{e^{-2t}\sin(5t)}{5}$$