Particular Surfaces in Space

Ex1. Compute the integrals.

a) $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} u\cdot\cos(u+v)\,du\,dv$$

$$\int_0^{\frac{\pi}{2}} u\cdot\cos(u+v)\,du = \Big[u\cdot\sin(u+v)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin(u+v)\,du$$

$$= \frac{\pi}{2}\cdot\sin\left(v+\frac{\pi}{2}\right) - \Big[-\cos(u+v)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\cdot\sin\left(v+\frac{\pi}{2}\right) + \cos\left(v+\frac{\pi}{2}\right) - \cos(v)$$

$$= \frac{\pi}{2}\cos(v) - \cos(v) - \sin(v)$$

$$= \left(\frac{\pi}{2}-1\right)\cos(v) - \sin(v).$$

$$\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-1\right)\cos(v) - \sin(v)\,dv$$

$$= \left(\frac{\pi}{2}-1\right)\Big[\sin(v)\Big]_0^{\frac{\pi}{2}} - \Big[-\cos(v)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 1 + (-1) = \frac{\pi}{2} - 2.$$

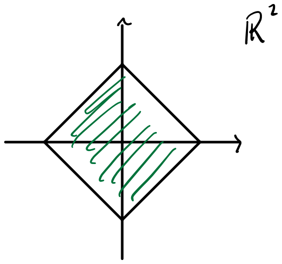
b) $$\int_0^1\int_0^1 \frac{v}{(uv+1)^2}\,du\,dv \quad , \quad t = uv+1 \Rightarrow dt = v\,du$$

$$= \int_0^1\int_1^{v+1} \frac{1}{t^2}\,dt\,dv = \int_0^1 \left[-\frac{1}{t}\right]_1^{v+1} dv$$

$$= \int_0^1 -\frac{1}{v+1} + 1\,dv = \Big[-\ln(v+1)+v\Big]_0^1 = 1 - \ln(2).$$

Ex2. $\int_B 2xy \, d\mu$ where $B = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, 0 \leq y, x+y \leq 1\}$.

a) Sketch and parametrize B.



$$r(u,v) = \begin{bmatrix} u \\ v(1-u) \end{bmatrix}, \quad u \in [0,1], \quad v \in [0,1].$$

since $x+y \leq 1 \iff y \leq 1-x$.

b) Determine the Jacobian.

$$\underline{\underline{J}} = \begin{bmatrix} 1 & 0 \\ -v & 1-u \end{bmatrix} \Rightarrow J_r(u,v) = \det(\underline{\underline{J}}) = 1-u.$$

c) Compute the integral $\int_B 2xy \, d\mu$.

$$\int_B 2xy \, d\mu = \int_0^1 \int_0^1 2 \cdot u \cdot v(1-u) \cdot (1-u) \, du \, dv$$

$$= \int_0^1 \int_0^1 2 \cdot u \cdot v \cdot (1 - 2u + u^2) \, du \, dv$$

$$= \int_0^1 \int_0^1 2vu^3 - 4vu^2 + 2vu \quad du \, dv$$

$$= \int_0^1 \left[\frac{1}{2} v u^4 - \frac{4}{3} v u^3 + v u^2 \right]_0^1 dv$$

$$= \int_0^1 \frac{1}{2} v - \frac{4}{3} v + v \quad dv$$

$$= \int_0^1 \frac{1}{6} v \, dv = \frac{1}{12} \left[v^2 \right]_0^1 = \frac{1}{12}$$

Ex 3. Given $h(x,y) = \sqrt{3}\, y$ and $M = \{ (x,y) \in \mathbb{R} \mid x \in [0,1],\ y \in [0,2] \}$.

Compute $\int_G xyz \, d\mu$, where G is the graph of h.

Firstly we have $r(u,v) = \begin{bmatrix} u \\ v \\ \sqrt{3}\, v \end{bmatrix}$, $u \in [0,1]$, $v \in [0,2]$.

$$r_u'(u,v) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_v'(u,v) = \begin{bmatrix} 0 \\ 1 \\ \sqrt{3} \end{bmatrix} \Rightarrow r_u' \times r_v' = \begin{bmatrix} 0 \\ -\sqrt{3} \\ 1 \end{bmatrix}$$

Thus $J_r(u,v) = |r_u' \times r_v'| = 2$. Now we get

$$\int_G xyz \, d\mu = \int_0^2 \int_0^1 u \cdot v \cdot \sqrt{3} v \cdot 2 \, du \, dv$$

$$= \int_0^2 2\sqrt{3}\, v^2 \left[\tfrac{1}{2} u^2 \right]_0^1 dv$$

$$= \int_0^2 \sqrt{3}\, v^2 \, dv$$

$$= \left[\frac{\sqrt{3}}{3} v^3 \right]_0^2 = \frac{8\sqrt{3}}{3}.$$

Ex 4. A parabola segment is given by $z = \frac{x^2}{4}$, $x \in [0,2]$.

a) Explain why $r(u) = \left(u, 0, \frac{u^2}{4} \right)$, $u \in [0,2]$

is a parametric representation of the parabola K.

The curve is in the xz-plane, so as $y = 0$ the second coordinate is 0.

b) A surface F is determined by rotating K 2π-radians around the z-axis.

$$R_z = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ so for $v \in (0, 2\pi]$ we can rotate K.

$$R_z(v)\ r(u) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ u^2/4 \end{bmatrix}$$

$$= \begin{bmatrix} u \cdot \cos v \\ u \cdot \sin v \\ u^2/4 \end{bmatrix}, \quad u \in [0,2], \quad v \in [0, 2\pi].$$

c) For $f(x,y) = x^2 + y^2$ Compute $\int_F f \, d\mu$.

Let's find the Jacobian first.

$$r_u' = \begin{bmatrix} \cos v \\ \sin v \\ u/2 \end{bmatrix}, \quad r_v' = \begin{bmatrix} -u \cdot \sin v \\ u \cdot \cos v \\ 0 \end{bmatrix} \Rightarrow r_u' \times r_v' = \begin{bmatrix} -u^2/2 \cdot \cos v \\ -u^2/2 \cdot \sin v \\ u \end{bmatrix}$$

$$J_r(u,v) = \sqrt{\left(-\frac{u^2}{2} \cdot \cos v\right)^2 + \left(-\frac{u^2}{2} \cdot \sin v\right)^2 + u^2}$$

$$= \sqrt{\frac{u^4}{4} \cdot (\cos^2 v + \sin^2 v) + u^2} = u \cdot \sqrt{\frac{u^2}{4} + 1}$$

$$\int_F f \, d\mu = \int_0^{2\pi} \int_0^2 \left((u \cdot \cos v)^2 + (u \cdot \sin v)^2 \right) \cdot u \cdot \sqrt{\frac{u^2}{4} + 1} \, du \, dv$$

$$= \int_0^{2\pi}\int_0^2 u^3 \cdot \sqrt{\frac{u^2}{4}+1}\ du\ dv$$

$$= \int_0^2 2\pi\ u^3 \cdot \sqrt{\frac{u^2}{4}+1}\ du$$

$$= \pi \int_0^2 u^3\sqrt{u^2+4}\ du$$

Let $t^2 = u^2+4$, then $\dfrac{dt}{du} = \dfrac{u}{\sqrt{u^2+4}} = \dfrac{u}{\sqrt{t^2}}$

$$\pi \int_0^2 u^3\sqrt{u^2+4}\ du = \pi \int_{\sqrt{4}}^{\sqrt{8}} u^3\sqrt{t^2} \cdot \frac{\sqrt{t^2}}{u}\ dt$$

$$= \pi \int_{\sqrt{4}}^{\sqrt{8}} (t^2-4)\ t^2\ dt = \pi \int_{\sqrt{4}}^{\sqrt{8}} t^4 - 4t^2\ dt$$

$$= \pi \left[\frac{1}{5}t^5 - \frac{4}{3}t^3\right]_{\sqrt{4}}^{\sqrt{8}}$$

$$= \pi \left(\frac{1}{5}\cdot 64\cdot\sqrt{8} - \frac{4}{3}\cdot 8\cdot\sqrt{8} - \frac{1}{5}\cdot 32 + \frac{4}{3}\cdot 8\right)$$

$$= \pi \left(\frac{64}{15} + \frac{64\sqrt{2}}{15}\right)$$

$$= 64\pi \cdot \frac{1+\sqrt{2}}{15}.$$

Ex 5. Let $h(x,y) = 2 - x^2 - y^2$ and

$$F = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1],\ y \in [0,2],\ z = h(x,y)\},$$

$$G = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 2,\ z = h(x,y)\}.$$

a) $\displaystyle\int_F \sqrt{9-4z}\, d\mu$.

Let $r(u,v) = \begin{bmatrix} u \\ v \\ 2-u^2-v^2 \end{bmatrix}$, $u \in [0,1]$, $v \in [0,2]$.

$r_u' = \begin{bmatrix} 1 \\ 0 \\ -2u \end{bmatrix}$, $r_v' = \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix}$, $r_u' \times r_v' = \begin{bmatrix} 2u \\ 2v \\ 1 \end{bmatrix}$.

Thus $J_r(u,v) = \sqrt{(2u)^2 + (2v)^2 + 1^2} = \sqrt{1 + 4u^2 + 4v^2}$.

$$\int_F \sqrt{9-4z}\, d\mu = \int_0^2 \int_0^1 \sqrt{9 - 4\cdot(2-u^2-v^2)} \cdot \sqrt{1+4u^2+4v^2}\, du\, dv$$

$$= \int_0^2 \int_0^1 1 + 4u^2 + 4v^2\, du\, dv = \int_0^2 \left[u + \tfrac{4}{3}u^3 + 4v^2 u\right]_0^1 dv$$

$$= \int_0^2 \frac{7}{3} + 4v^2\, dv = \left[\frac{7}{3}v + \frac{4}{3}v^3\right]_0^2 = \frac{46}{3}.$$

b) $\displaystyle\int_G \sqrt{9-4z}\, d\mu$.

We have $r(u,v) = \begin{bmatrix} u\cdot\cos v \\ u\cdot\sin v \\ 2-u^2 \end{bmatrix}$, $u \in [0,\sqrt{2}]$, $v \in [0,2\pi]$.

$r_u' = \begin{bmatrix} \cos v \\ \sin v \\ -2u \end{bmatrix}$, $r_v' = \begin{bmatrix} -u\cdot\sin v \\ u\cdot\cos v \\ 0 \end{bmatrix}$, $r_u' \times r_v' = \begin{bmatrix} 2u^2\cdot\cos v \\ 2u^2\cdot\sin v \\ u \end{bmatrix}$.

Thus

$$J_r(u,v) = \sqrt{4u^4 + u^2} = u\sqrt{4u^2+1}.$$

$$\int_G \sqrt{9-4z}\, d\mu = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{9-4\cdot(2-u^2)}\; u\sqrt{4u^2+1}\; du\, dv$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} u\cdot(4u^2+1)\; du\, dv$$

$$= \int_0^{2\pi} \left[u^4 + \tfrac{1}{2}u^2 \right]_0^{\sqrt{2}} dv$$

$$= \int_0^{2\pi} 4+1\; dv$$

$$= 5\cdot 2\pi = 10\pi.$$