The Riemann Integral and Its Use on Curves

Ex 1. Classic antiderivatives that must be known by heart.

```
> [int(x^n,x),
  int(1/x,x),
  int(ln(x),x),
  int(1/(1+x^2),x),
  int(cos(x),x),
  int(sin(x),x),
  int(exp(x),x)];
  kontrol=
  [seq(diff(%[i],x),i=1..7)];
```

$$\left[\frac{x^{n+1}}{n+1}, \ln(x), x\ln(x) - x, \arctan(x), \sin(x), -\cos(x), e^x\right]$$

$$kontrol = \left[\frac{x^{n+1}}{x}, \frac{1}{x}, \ln(x), \frac{1}{x^2+1}, \cos(x), \sin(x), e^x\right]$$

Ex 2. Integration rules to master.

a)

```
> [int(x^0,x),int(x^(-1),x)];
  int(x^(p/q),x);
  [int(1/(a*x+b),x),
  int(cos(a*x+b),x),
  int(sin(a*x+b),x),
  int(exp(a*x+b),x)]
```

$$[x, \ln(x)]$$

$$\frac{q\,x^{1+\frac{p}{q}}}{p+q}$$

$$\left[\frac{\ln(a\,x+b)}{a}, \frac{\sin(a\,x+b)}{a}, -\frac{\cos(a\,x+b)}{a}, \frac{e^{a\,x+b}}{a}\right]$$

b) How would you most efficiently integrate $\frac{1}{x^k}$, $k>0$, $x\neq 0$?

The power rule is quite efficient.

$$\int \frac{1}{x^k}\,dx = \int x^{-k}\,dx = \frac{1}{k+1}\,x^{-k+1} = \frac{1}{(k+1)\,x^{k-1}}$$

Ex 3. Compute the indefinite integral

$$\int 5\cos(x+1) - \sin(5x) + \frac{2}{x-3} - 7\,dx\,, \quad x>3.$$

Integration is linear, so we get

$$\int 5\cos(x+1) - \sin(5x) + \frac{2}{x-3} - 7\,dx$$

$$= 5\sin(x+1) + \frac{1}{5}\cos(5x) + 2\ln(x-3) - 7x + k,\ k\in\mathbb{R}.$$

Ex 4. Decide if convergent / divergent, and state the limit for the following sequences.

$a_n = \dfrac{1}{n}$, this sequence is convergent and $\lim\limits_{n\to\infty} a_n = 0$.

$$b_n = \frac{n-1}{2n} = \frac{1 - \frac{1}{n}}{2} \to \frac{1}{2} \quad \text{for } n \to \infty.$$

$c_n = \dfrac{n}{1000}$, this sequence is divergent.

$$d_n = \frac{4n^2 + 16}{8 - 3n^2} = \frac{4 + \frac{16}{n^2}}{\frac{8}{n^2} - 3} \to -\frac{4}{3} \quad \text{for } n \to \infty.$$

Quick Maple check:

```
> [limit(1/n,n=infinity),
  limit((n-1)/(2*n),n=infinity),
  limit(n/1000,n=infinity),
  limit((4*n^2+16)/(8-3*n^2),n=infinity)];
```

$$\left[0, \frac{1}{2}, \infty, -\frac{4}{3}\right]$$

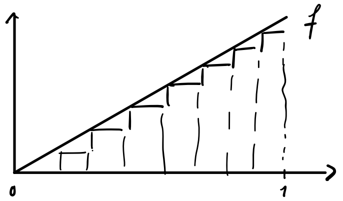
Ex 5. Looking at left sums. The following is given

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n) \qquad (*)$$

a) State a left sum for

$$f(x) = x \quad , \quad x \in [0,1],$$

where $[0,1]$ is subdivided into n segments of equal length.



Height $f\left(\frac{i}{n}\right)$, $i = 0, 1, \ldots, n-1$

Width $\frac{1}{n}$

The left sum is

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right) = \frac{1}{n} \cdot \left(\frac{0}{n} + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right)$$

$$\overset{(*)}{=} \frac{n}{2} \left(\frac{0}{n^2} + \frac{n-1}{n^2}\right) = \frac{n^2 - n}{2n^2} = \frac{n-1}{2n}$$

$$\to \frac{1}{2} \quad \text{for } n \to \infty.$$

b) What is $\int_0^1 x\,dx$?

From a) we found the limit for $n \to \infty$ to be $\frac{1}{2}$.

c) Repeat for $f(x) = 3x+1$, $x \in [0,1]$.

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right) = \frac{1}{n} \cdot \sum_{i=0}^{n-1} 3 \cdot \frac{i}{n} + \frac{1}{n} \cdot \sum_{i=0}^{n-1} 1$$

$$= 3 \cdot \frac{n-1}{2n} + 1$$

$$= 3\left(\frac{1}{2} - \frac{1}{2n}\right) + 1$$

$$= \frac{5}{2} - \frac{3}{2n} \to \frac{5}{2} \quad \text{for } n \to \infty.$$

Ex 6.

a) Compute

$$\int_0^1 \frac{1}{1+u^2}\,du = \left[\arctan(u)\right]_0^1 = \frac{\pi}{4}$$

b) Compute

1. $\int_1^2 \int_0^1 \frac{e^{2u}}{v}\,du\,dv$ and 2. $\int_0^{\frac{\pi}{2}} \int_0^1 v \cdot \cos(uv)\,du\,dv$.

1. $\int \frac{e^{2u}}{v}\, du = \frac{1}{v} \int e^{2u}\, du = \frac{1}{2v} e^{2u}$

$\Rightarrow \int_0^1 \frac{e^{2u}}{v}\, du = \left[\frac{1}{2v} e^{2u} \right]_0^1 = \frac{e^2}{2v} - \frac{1}{2v}$.

$\int_1^2 \frac{e^2}{2v} - \frac{1}{2v}\, dv = (e^2 - 1) \cdot \left[\frac{\ln(2v)}{2} \right]_1^2 = \frac{(e^2-1)}{2} \cdot \ln(2)$

2 $\int_0^1 v \cdot \cos(uv)\, du = \left[\sin(uv) \right]_0^1 = \sin(v)$

$\int_0^{\pi/2} \sin(v)\, dv = \left[-\cos(v) \right]_0^{\pi/2} = 0 - (-1) = 1$.

c) Compute $\int_0^1 \int_0^1 \int_0^1 24 x^3 y^2 z \, dx\, dy\, dz$

$= \int_0^1 \int_0^1 \left[6x^4 y^2 z \right]_0^1 dy\, dz$

$= \int_0^1 \int_0^1 6 y^2 z \; dy\, dz = \int_0^1 \left[2y z \right]_0^1 dz$

$= \int_0^1 2z\, dz = \left[z^2 \right]_0^1 = 1$.

Ex 7. Consider $\underline{r}(u) = \begin{bmatrix} 2u^2 \\ u^3 \end{bmatrix}$, $u \in [0, 2]$.

a) Difference between a tangent and tangent vector.

A tangent vector omits a direction for a tangent.

b) Determine the tangent vector of $\underline{r}(u)$ at $(2,1)$ and its length.

Note that $r(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so the point is on the curve.

$$r'(u) = \begin{bmatrix} 4u \\ 3u^2 \end{bmatrix}, \quad \text{so} \quad r'(1) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ and } |r'(1)| = 5.$$

c) Plot the curve and tangent vector.

```
> r:=u-> <2*u^2,u^3>:
  diff(r(u),u);
  dr:=unapply(%,u):
  punkt=r(1);
  tangentvektorur=dr(1);
```

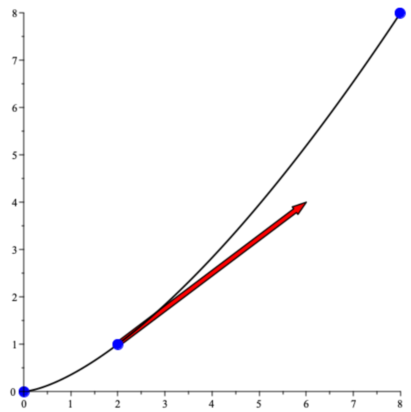
$$\begin{bmatrix} 4u \\ 3u^2 \end{bmatrix}$$

$$punkt = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$tangentvektorur = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

```
> Vlongd=sqrt(prik(dr(1),dr(1)));
  curve:=plot([vop(r(u)),u=0..2],thickness=2,color=black):
  tangentv:=arrow(r(1),dr(1),width = .1, head_length = .3,color=
  red):
  points:=pointplot({r(0),r(1),r(2)},symbol=solidcircle,
  symbolsize=18,color=blue):
  display(curve,tangentv,points);
```

$$Vlongd = 5$$



d) How long is the curve for $u \in [0,1]$ and $u \in [1,2]$?

We need the Jacobian to determine the line integral.

$$J(u) = \sqrt{r'(u) \cdot r'(u)} = \sqrt{16u^2 + 9u^4} = u\sqrt{9u^2 + 16}$$

$$\int_0^1 J(u)\,du = \frac{61}{27} \approx 2.26 \quad \text{and} \quad \int_1^2 J(u)\,du \approx 9.26$$

```
> assume(u>0):interface(showassumed=0):
  sqrt(prik(dr(u),dr(u))):
  Jacobi:=simplify(%);
  F:=unapply(int(Jacobi,u),u);
```

$$Jacobi := u\sqrt{9u^2+16}$$

$$F := u \mapsto \frac{(9 \cdot u^2+16)^{3/2}}{27}$$

```
> F(1)-F(0);
  simplify(%);
  evalf(%);
```

$$\frac{25\sqrt{25}}{27} - \frac{16\sqrt{16}}{27}$$

$$\frac{61}{27}$$

$$2.259259259$$

```
> F(2)-F(1);
  simplify(%);
  evalf(%);
```

$$\frac{52\sqrt{52}}{27} - \frac{25\sqrt{25}}{27}$$

$$\frac{104\sqrt{13}}{27} - \frac{125}{27}$$

$$9.258419730$$

Ex 8. We're given $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + (y-1)^2 = 4 \text{ and } z = 1\}$.

a) State the centre and radius. Parametrize the circle. Determine the Jacobian.

We have the centre $(0,1,1)$ and radius of 2.

$$\underline{r}(u) = 2\begin{bmatrix}\cos(u)\\ \sin(u)\\ 0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad u \in [0, 2\pi].$$

Now $\underline{r}'(u) = 2\begin{bmatrix}-\sin(u)\\ \cos(u)\\ 0\end{bmatrix}$, so we get the Jacobian

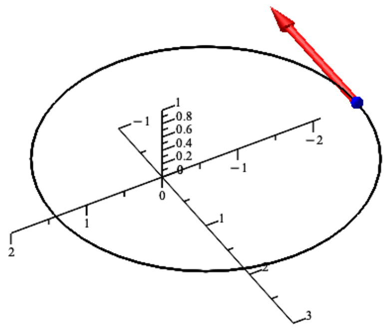
$$J(u) = \sqrt{r'(u) \cdot r'(u)} = 2\sqrt{\sin^2 u + \cos^2 u} = 2.$$

```
> r:=u-> <2*cos(u),2*sin(u)+1,1>:
  diff(r(u),u);
  dr:=unapply(%,u):
  sqrt(prik(dr(u),dr(u))):
  Jacobi:=simplify(%);
```

$$\begin{bmatrix}-2\sin(u)\\ 2\cos(u)\\ 0\end{bmatrix}$$

$$Jacobi := 2$$

```
> tangentv:=arrow(r(Pi),dr(Pi),color=red):
curve:=plot3d(r(u),u=-Pi..Pi,thickness=2,color=black):
points:=pointplot3d({r(Pi)},symbol=solidcircle,symbolsize=18,
color=blue):
display(curve,tangentv,points,axes=normal,scaling=constrained,
view=0..1,orientation=[60,50,0]);
```



c) Given $f(x,y,z) = x^2+y^2+z^2$ determine $f(\underline{r}(u))$, and then compute

$$\int_C f \, d\mu \, .$$

$$
\begin{aligned}
f(\underline{r}(u)) &= (2\cos u)^2 + (2\sin u + 1)^2 + 1 \\
&= 4\cos^2 u + 4\sin^2 u + 4\sin u + 1 + 1 \\
&= 4\sin u + 6
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{2\pi} f(\underline{r}(u)) \cdot J(u) \, du &= \int_0^{2\pi} (4\sin u + 6) \cdot 2 \, du \\
&= \left[-8\cos u + 12u \right]_0^{2\pi} \\
&= 24\pi \, .
\end{aligned}
$$

d) Does the integral depend on the choice of parametrization?

No.

e) Does the integral depend on the location of the circle?

In general absolutely yes.

Ex 9. Mid-point sums. $\{(u,v) \mid v = u^2;\ u \in [0,1]\}$.

We just use the Pythagorean Theorem on lines approximating curve segments. Sum over these segments to approach the true curve length for $\delta u \to 0$.

$$\sum_{i=1}^{n} \sqrt{\delta u^2 + (u_{i+1}^2 - u_i^2)^2} = \sum_{i=1}^{n} \sqrt{\delta u^2 + (u_{i+1} - u_i)^2 (u_{i+1} + u_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + (2u_i + \delta u)^2}\ \delta u \qquad \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

b) Why is the above a mid-point sum for

$$f(u) = \sqrt{1+4u^2}\ ?$$

Determine the arc length with Maple.

For a mid-point $m_i \in [u_i, u_{i+1}]$

$$\sum_{i=1}^{n} \sqrt{1 + (2u_i + \delta u)^2}\ \delta u = \sum_{i=1}^{n} \sqrt{1 + 4 \cdot m_i^2}\ \delta u$$

```
> int(sqrt(1+4*x^2),x=0..1);
  evalf(%);
```

$\sqrt{1+4 \cdot u^2}$

$$\frac{\sqrt{5}}{2} + \frac{\operatorname{arcsinh}(2)}{4}$$

1.478942857