Plane Integrals and Surface Integrals

Ex 1. Compute the integral.

a) $\int_B x^2y^2 + x \, d\mu \quad , \quad B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2, -1 \le y \le 0\}$.

$$\int_B x^2y^2 + x \, d\mu = \int_{-1}^{0}\int_0^2 x^2y^2 + x \, dx\, dy$$

$$= \int_{-1}^{0} \left[\frac{y^2}{3}x^3 + \frac{1}{2}x^2 \right]_0^2 dy$$

$$= \int_{-1}^{0} \frac{8}{3}y^2 + 2 \, dy$$

$$= \left[\frac{8}{9}y^3 + 2y \right]_{-1}^{0} = -\left(-\frac{8}{9} - 2\right) = \frac{26}{9}.$$

b) $\int_B \frac{y}{1+xy} \, d\mu \quad , \quad B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$.

$$\int_B \frac{y}{1+xy} \, d\mu = \int_0^1\int_0^1 \frac{y}{1+xy} \, dx\, dy = \int_0^1 \left[\ln(1+xy) \right]_0^1 dy$$

$$= \int_0^1 \ln(1+y) \, dy = \left[(1+y)\cdot\ln(1+y) - (1+y) \right]_0^1$$

$$= 2\ln 2 - 2 + 1 = 2\ln 2 - 1.$$

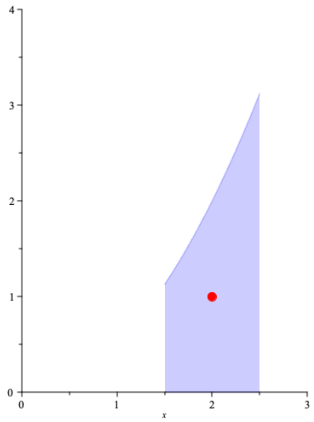
Ex 2. Let $P_0 = (2,1)$ and $B = \{(x,y) \in \mathbb{R}^2 \mid \frac{3}{2} \le x \le \frac{5}{2}, \ 0 \le y \le \frac{1}{2}x^2\}$.

a) Sketch B and parametrize. Determine u_0 and v_0 s.t. $r(u_0,v_0) = P_0$.

We set

$$r(u,v) = \begin{bmatrix} u \\ v\cdot\frac{1}{2}u^2 \end{bmatrix}, \quad u \in \left[\frac{3}{2}, \frac{5}{2}\right] \text{ and } v \in [0,1].$$

```
> f:=(x,y)->1/(x^2+y):
  r:=(u,v)-> <u,0>+v*<0,1/2*u^2>:
  points:=pointplot(<2,1>,color=red,symbol=solidcircle,symbolsize=15):
  B:=shadebetween(0, 1/2*x^2, x=3/2..5/2, color=blue,transparency=0.8):
  display(points,B,view=[0..3,0..4],scaling=constrained);
  vsolve(r(u,v)=<2,1>,{u,v});
```



$$\left\{u=2,\ v=\frac{1}{2}\right\}$$

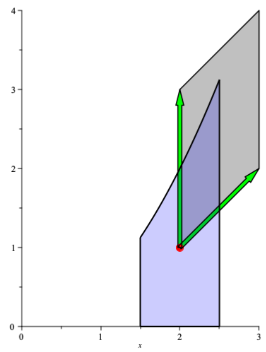
b) Illustrate again with $r_u'(u_0,v_0)$ and $r_v'(u_0,v_0)$. What is the area of the spanned parallelogram?

```
> ru:=diff(r(u,v),u):
  rv:=diff(r(u,v),v):
  rutang:=subs(u=2,v=1/2,ru);
  rvtang:=subs(u=2,v=1/2,rv);
```

$$rutang := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$rvtang := \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

```
> boundary:=implicitplot([y=0,y=1/2*x^2,x=3/2,x=5/2],x=3/2..5/2,y=0..1/2*x^2,thickness=2,color=black):
  rutangvek:=arrow(r(2,1/2),rutang,width=0.05,head_length=0.2,color=green):
  rvtangvek:=arrow(r(2,1/2),rvtang,width=0.05,head_length=0.2,color=green):
  para:=polygonplot(<<2,1>|<3,2>|<3,4>|<2,3>>,color=grey):
  display(points,B,boundary,rutangvek,rvtangvek,para,view=[0..3,0..4],scaling=constrained);
  areapara=abs(det(rutang,rvtang));
```



$areapara = 2$

c) Determine $J_r(u,v)$ and compute $\int_B \frac{1}{x^2+y}\, d\mu$

$$r_u' = \begin{bmatrix} 1 \\ vu \end{bmatrix}, \quad r_v' = \begin{bmatrix} 0 \\ \frac{1}{2}u^2 \end{bmatrix}, \quad |\det([r_u' \ r_v'])| = \frac{1}{2}u^2$$

$$\int_{\mathcal{B}} \frac{1}{x^2+y}\, d\mu = \int_0^1 \int_{3/2}^{5/2} \frac{1}{u^2 + v\cdot\frac{1}{2}u^2} \cdot \frac{1}{2}u^2 \, du\, dv$$

$$= \int_0^1 \int_{3/2}^{5/2} \frac{1}{2+v}\, du\, dv$$

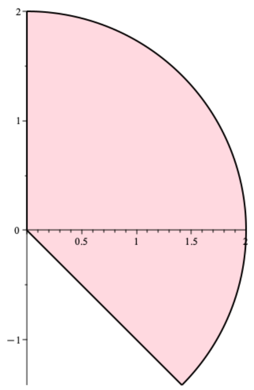
$$= \int_0^1 \frac{1}{2+v}\, dv \cdot \overbrace{\left(\frac{5}{2} - \frac{3}{2}\right)}^{=1}$$

$$= \Big[\ln(2+v)\Big]_0^1 = \ln 3 - \ln 2 .$$

Ex 3. Let $f(x,y) = x^2 - y^2$ and $\mathcal{B} = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \le \rho \le a,\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}\}$.

a), b) Sketch and compute the area of $\mathcal{B}$ by rudimentary methods, and then by integration.

```
> f:=(x,y)->x^2-y^2:
  B:=plot({[2*cos(v),2*sin(v),v=-Pi/4..Pi/2],[x,-x,x=0..sqrt(2)]},color=pink,filled=true):
  boundary:=plot({[2*cos(v),2*sin(v),v=-Pi/4..Pi/2],[x,-x,x=0..sqrt(2)],[0,y,y=0..2]},color=black,thickness=2):
  display(B,boundary,scaling=constrained);
  area=3/8*Pi*a^2;
```



$area = \frac{3\pi a^2}{8}$

Let $r(u,v) = u\cdot a\cdot \begin{bmatrix} \cos v \\ \sin v \end{bmatrix}$, $u \in [0,1]$ and $v \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$.

$$r_u' = \begin{bmatrix} a\cdot\cos v \\ a\cdot \sin v \end{bmatrix}, \quad r_v' = \begin{bmatrix} -ua\sin v \\ ua\cos v \end{bmatrix}, \quad J_r(u,v) = u\cdot a^2 .$$

The area is given by

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 u \cdot a^2 \, du \, dv = \left(\frac{\pi}{2} - \left(-\frac{\pi}{4}\right)\right) \cdot \left[\frac{1}{2} u^2 a^2\right]_0^1$$

$$= \frac{3\pi}{4} \cdot \frac{a^2}{2} = \frac{3\pi a^2}{8} .$$

c) Determine $\int_B f(x,y) \, d\mu$.

$$\int_B f(x,y) \, d\mu = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \left((u \cdot a \cdot \cos v)^2 - (u \cdot a \cdot \sin v)^2\right) u \cdot a^2 \, du \, dv$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 u^3 a^4 \left(2 \cos^2 v - 1\right) du \, dv$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \left(2 \cos^2 v - 1\right) \frac{a^4}{4} \, dv$$

$$= \Big[\sin(v) \cdot \cos(v) \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \frac{a^4}{4}$$

$$= -\left(-\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) \cdot \frac{a^4}{4}$$

$$= \frac{a^4}{8} .$$

Ex4. Let F_r be given by $r(u,v) = \begin{bmatrix} u \cos v \\ u \sin v \\ v \end{bmatrix}$, $u \in [0,2]$, $v \in [0, 2\pi]$.

a) Find $P_0 = (0, 1, \frac{\pi}{2})$ with r.

$$r\left(1, \frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 1 \\ \frac{\pi}{2} \end{bmatrix} .$$

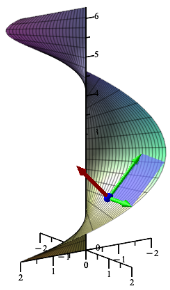
b) Parametrize the parallelogram spanned by $r_u'(1, \frac{\pi}{2})$ and $r_v'(1, \frac{\pi}{2})$.

```
> ru:=diff(r(u,v),u):
  rv:=diff(r(u,v),v):
  rutang:=subs(u=1,v=Pi/2,ru):
  rvtang:=subs(u=1,v=Pi/2,rv):
  Parameter:=Po+s*rutang+t*rvtang;
```

$$Parameter := \begin{bmatrix} -t \\ s+1 \\ t+\frac{\pi}{2} \end{bmatrix}$$

c) Illustrate.

```
> N:=simplify(kryds(ru,rv)):
  No:=subs(u=1,v=Pi/2,N):
  rutangvek:=arrow(Po,rutang,width=0.1,head_length=0.3,color=green):
  rvtangvek:=arrow(Po,rvtang,width=0.1,head_length=0.3,color=green):
  Nvek:=arrow(Po,No,width=0.1,head_length=0.3,color=red):
  surface1:=plot3d(r(u,v),u=0..2,v=0..2*Pi):
  para:=plot3d(Parameter,s=0..1,t=0..1,style=patchnogrid,color=blue,transparency=0.3):
  points:=plot3d({Po},x=0..4,y=1..5,style=point,symbol=solidcircle,symbolsize=20,color=blue):
  display(surface1,points,rutangvek,rvtangvek,Nvek,para,scaling=constrained,view=0..2*Pi,axes=normal);
```



d) Compute the area of the parallelogram.

```
> areapara=len(kryds(rutang,rvtang));
  areapara=simplify(len(No));
```

$$areapara = \sqrt{2}$$
$$areapara = \sqrt{2}$$

e) Difference in Jacobians.

A relative change in area in the plane and on a surface is described by tangent vectors. However, we compute the area in the plane by means of the determinant, and along a surface the area is given by the length of the normal. Note that the latter still describes an area, so there exists a basis for which this Jacobian can be boiled down to a determinant for a fixed point.

f) Compute $J_r(u,v)$ and the area of F_r.

```
> sqrt(prik(N,N)):
  Jacobi:=simplify(%);
  int(int(Jacobi,u=0..2),v=0..2*Pi);
  evalf(%);
```

$$Jacobi := \sqrt{u^2+1}$$
$$2\sqrt{5}\,\pi + \operatorname{arcsinh}(2)\,\pi$$
$$18.58494406$$

Ex5. Let C_1 be given by

$$\text{Directrix} = \{x,y \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \text{ and } z \in [0,1].$$

a) Parametrize C_1.

$$r(u,v) = \begin{bmatrix} \cos u + 1 \\ \sin u \\ v \end{bmatrix}, \quad u \in [0, 2\pi], \quad v \in [0,1].$$

b) Compute $\int_{C_1} x + yz \, d\mu$.

```
> f:=(x,y)->x+y*z:
  N:=kryds(diff(r(u,v),u),diff(r(u,v),v)):
  sqrt(prik(N,N)):
  Jacobi:=simplify(%);
  int(int(f(vop(r(u,v))),u=0..2*Pi),v=0..1);
```

$$Jacobi := 1$$
$$2\pi$$

Let C_2 be given by

$$\text{Directrix} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x^2+y^2 = 1\} \text{ and } z = [0, \tfrac{1}{2} + x^2].$$

c) Parametrize C_2, state its Jacobian and compute the area.

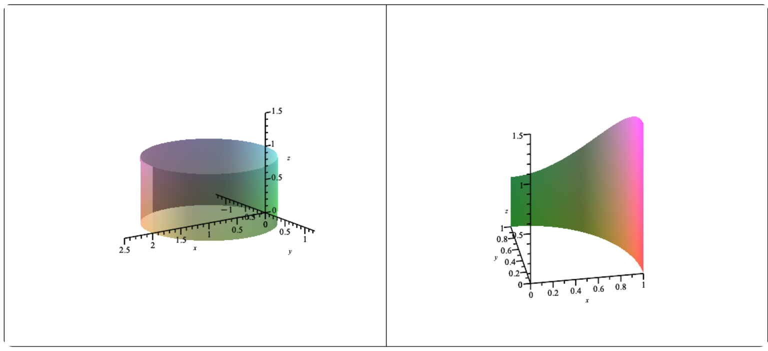
$$r(u,v) = \begin{bmatrix} \cos u \\ \sin u \\ v\left(\frac{1}{2} + \cos^2 u\right) \end{bmatrix}, \quad u \in [0, 2\pi], \quad v \in [0,1].$$

```
> r2:=(u,v)-><cos(u),sin(u),v*(1/2+cos(u)^2)>:
  r2(u,v):
  N2:=kryds(diff(r2(u,v),u),diff(r2(u,v),v)):
  prik(N2,N2):
  simplify(%);
  Jacobi2:=cos(u)^2 + 1/2;
  int(int(Jacobi2,u=0..Pi/2),v=0..1);
```

$$\frac{\left(2\cos(u)^2+1\right)^2}{4}$$

$$Jacobi2 := \frac{1}{2}+\cos(u)^2$$

$$\frac{\pi}{2}$$



Ex 6. Let $B=\{(x,y)\in\mathbb{R}^2 \mid 1\le x\le 2,\ 0\le y\le x^3\}$.

a) What is the mass of B?

$$r(u,v)=\begin{bmatrix} u \\ v\cdot u^3 \end{bmatrix}, \quad u\in[1,2], \quad v\in[0,1].$$

$$r_u' = \begin{bmatrix} 1 \\ 3vu^2 \end{bmatrix}, \quad r_v' = \begin{bmatrix} 0 \\ u^3 \end{bmatrix}, \quad J_r(u,v) = u^3.$$

$$\int_0^1\int_1^2 u^3\,du\,dv = \left[\frac{1}{4}u^4\right]_1^2 = \frac{16}{4}-\frac{1}{4} = \frac{15}{4}.$$

b) What are the units of the mass function f?

This is kg/m^2 in the plane.

c) What is the center of mass?

$$x_c = \frac{4}{15} \cdot \int_0^1 \int_1^2 u \cdot u^3 \, du \, dv = \frac{4}{15} \left[\frac{1}{5} u^5 \right]_1^2 = \frac{4}{15} \left(\frac{32}{5} - \frac{1}{5} \right)$$

$$= \frac{4}{15} \cdot \frac{31}{5} = \frac{124}{75}.$$

$$y_c = \frac{4}{15} \int_0^1 \int_1^2 v \cdot u^3 \cdot u^3 \, du \, dv = \frac{4}{15} \int_0^1 \left[\frac{v}{7} \cdot u^7 \right]_1^2 dv$$

$$= \frac{4}{15} \int_0^1 \frac{127}{7} v \, dv = \frac{4}{15} \cdot \frac{127}{14} = \frac{254}{105}.$$

Center of mass: $\left(\frac{124}{75}, \frac{254}{105} \right)$.

d) Get the center of mass assuming $f(x,y) = x^2$.

Let's use Maple to avoid arthritis.

```
> with(Integrator9):
> f1:=(x,y)->1:
  f2:=(x,y)->x^2:
> r1:=(u,v)-><u,v*u^3>:
  B1:=[1,2,0,1]:
  mass1=planIntGo(r1,B1,f1);
  center1=planCmGo(r1, B1, f1);
  mass2=planIntGo(r1,B1,f2);
  center2=planCmGo(r1, B1, f2);
```

$$mass1 = \frac{15}{4}$$

$$center1 = \left[\frac{124}{75}, \frac{254}{105} \right]$$

$$mass2 = \frac{21}{2}$$

$$center2 = \left[\frac{254}{147}, \frac{73}{27} \right]$$

```
> assume(a>0):interface(showassumed=0):
  r2:=(u,v)-><u*cos(v),u*sin(v)>:
  B2:=[0,a,-Pi/4,Pi/2]:
  mass1= planIntGo(r2, B2, f1);
  center1=planCmGo(r2, B2, f1);
  mass2= planIntGo(r2, B2, f2);
  center2=planCmGo(r2, B2, f2);
```

$$mass1 = \frac{3 a^2 \pi}{8}$$

$$center1 = \left[\frac{4 a \left(2 + \sqrt{2}\right)}{9 \pi}, \frac{4 \sqrt{2} \, a}{9 \pi} \right]$$

$$mass2 = \frac{a^4 (2 + 3 \pi)}{32}$$

$$center2 = \left[\frac{8 a \left(5 \sqrt{2} + 8\right)}{30 + 45 \pi}, \frac{8 \sqrt{2} \, a}{30 + 45 \pi} \right]$$

e) and f) Redo for B as in Ex3.

See above image to the right.

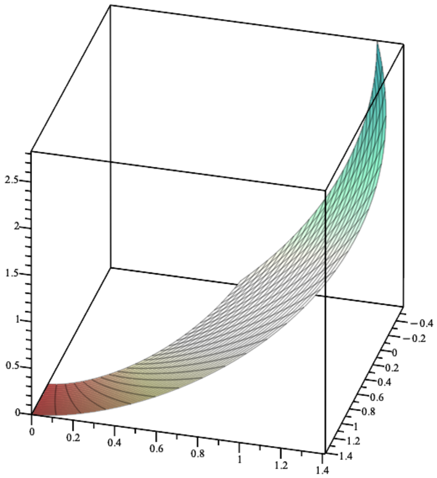
Ex 7. Let F be parametrized by

$$r(u,v) = \begin{bmatrix} \sqrt{u} \cos v \\ \sqrt{u} \sin v \\ v^{3/2} \end{bmatrix}, \quad u \in [1,2], \quad v \in [0, u].$$

a) State a reparametrization so that the region of integration is rectangular.

Let

$$r(u,v) = \begin{bmatrix} \sqrt{u} \cos(uv) \\ \sqrt{u} \sin(uv) \\ (uv)^{3/2} \end{bmatrix}, \quad u \in [1,2], \quad v \in [0,1].$$



b) Compute $\int_F x^2 + y^2 \, d\mu$.

```
> B:=[1,2,0,1]:
  f:=(x,y,z)->x^2+y^2:
  fladeIntGo(r,B,f);
  evalf(%);
```

$$-\frac{28}{81} + \frac{91\sqrt{13}}{162}$$

$$1.679661519$$