Linear systems Tankin er at kunna loysa líkningaskipani systematiskt. Tað er møguligt, at ein líkningaskipan ikki hevur eina loysn, altso er inconsistent.

Dømi Ein líkningaskipan verður sum oftast umset til matrix form.

$$\begin{cases} 3u_1 - 2u_2 - 10u_3 + u_4 = 0 \\ u_1 - u_3 = 4 \\ u_1 + u_2 - 3u_3 + 3u_4 = 1 \\ u_2 + 2u_4 = -4 \end{cases}$$

Koefficientarnir hoyrandi til $u_1, \dots, u_4$ svarar til sjálva matricuna, har vit skulu minnast til forteikn og tølini 0 og 1.

$$\begin{bmatrix} 3 & -2 & -10 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -3 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ -4 \end{bmatrix}$$

Generellar skipanir skrivast tískil

$$\begin{cases} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n = b_1 \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n = b_2 \\ \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n = b_n \end{cases} \qquad \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

So vit fáa $\underline{\underline{A}}\,\underline{u} = \underline{b}$. Set fyri, at $\underline{\underline{A}}$ er $n \times n$.
Um vektorarnir $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ eru lineert óheftir, so finst ein loysn $\underline{u}$ til líkningaskipanina. Hetta svarar til, at vektorarnir útspenna eitt n-dimensionelt volumen mát.

Gauss elimination Vit hava brúkt shears og rekkjuoperatiónir til at gera $\underline{\underline{A}}$ til eina ovara trikantsmatricu. Síðani substituera vit "aftur eftir" fyri at loysa. Tað ber eisini til at bliva við at redusera $\underline{\underline{A}}$ til eina eindarmatricu, og tá nevnist tað reducud row echelon form. Í tí føri er skipanin loyst, so at substitutión ikki er neyðug.

Dømi 12.2 Loysnin hjá $\underline{\underline{A}}\,\underline{u} = \underline{b}$ skilja vit sum vektorin $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$. So við

$$\begin{bmatrix} 2 & -2 & 0 \\ 4 & 0 & -2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

leita vit eftir u_1, u_2 og u_3, so at vit fáa úrslitavektorin $\underline{b}$.
Total matrican:

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 4 \\ 4 & 0 & -2 & -2 \\ 4 & 2 & -4 & 0 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ -2R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 4 \\ 0 & 4 & -2 & -10 \\ 0 & 6 & -4 & -8 \end{array}\right] \begin{array}{l} \\ \\ -\frac{3}{2}R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 4 \\ 0 & 4 & -2 & -10 \\ 0 & 0 & -1 & 7 \end{array}\right] \Rightarrow u_3 = -7$$

$$\begin{aligned} & 4u_2 - 2 \cdot (-7) = -10 \\ \Leftrightarrow\ & 4u_2 = -24 \\ \Leftrightarrow\ & u_2 = -6 \end{aligned} \qquad \begin{aligned} & 2u_1 - 2 \cdot (-6) = 4 \\ \Leftrightarrow\ & 2u_1 = -8 \\ \Leftrightarrow\ & u_1 = -4 \end{aligned}$$

Loysnin er altso $\underline{u} = \begin{bmatrix} -4 \\ -6 \\ -7 \end{bmatrix}$.

Legg til merkis, at hetta ikki er sama mannagongd sum í bókini. Valið at pivotera, so at a_{ii} er størst nýtist ikki, men tað kann vera praktiskt!
Algoritman við pivotering er givin á s. 251.

Dømi Vit fremja Gauss elimination við pivotering.

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 4 \\ 4 & 0 & -2 & -2 \\ 4 & 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 0 & -2 & -2 \\ 2 & -2 & 0 & 4 \\ 4 & 2 & -4 & 0 \end{array}\right] \begin{array}{l} \\ -\frac{2}{4}R_1 \\ -\frac{4}{4}R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 4 & 0 & -2 & -2 \\ 0 & -2 & 1 & 5 \\ 0 & 2 & -2 & 2 \end{array}\right] \begin{array}{l} \\ \\ -\frac{2}{-2}R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 4 & 0 & -2 & -2 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & -1 & 7 \end{array}\right]$$

$$u_3 = \frac{7}{-1} = -7, \qquad u_2 = \frac{1}{-2}\left(5 - 1 \cdot (-7)\right) = -\frac{1}{2} \cdot 12 = -6$$

$$u_1 = \frac{1}{4} \cdot \left(-2 - 0 \cdot (-6) - (-2) \cdot (-7)\right) = \frac{1}{4} \cdot (-2 - 0 - 14) = -4$$

$$\underline{u} = \begin{bmatrix} -4 \\ -6 \\ -7 \end{bmatrix}$$

Homogen System

Vit kunnu lættliga hava meiri enn eina loysn við undirdetermineraðar skipanir. Ein loysn kann tá parametriserast við fríum variablum.

Dømi 12.4

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \underline{u} = \underline{0} \iff \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u} = \underline{0}$$

Loysn: $t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}.$

Inversar matricur

Tann inversa matrican hjá $\underline{\underline{A}}$ er $\underline{\underline{A}}^{-1}$ og hevur eginleikan

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}} = \underline{\underline{A}}^{-1}\underline{\underline{A}}$$

Her stendur, at $\underline{\underline{A}}\,\underline{a}_1^{-1} = \underline{e}_1$, $\underline{\underline{A}}\,\underline{a}_2^{-1} = \underline{e}_2$ og $\underline{\underline{A}}\,\underline{a}_3^{-1} = \underline{e}_3$. So vit kunnu Gauss eliminera fram til loysnirnar. Vanliga er roknað í einum.

Dømi 12.6

$$\left[\begin{array}{ccc|ccc} -4 & 2 & -1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 & 1 & 0 \\ -5 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \cdot (-\tfrac{1}{4}) \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & -1/4 & 0 & 0 \\ -3 & 1 & 0 & 0 & 1 & 0 \\ -5 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ +3R_1 \\ +5R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & -1/4 & 0 & 0 \\ 0 & -1/2 & 3/4 & -3/4 & 1 & 0 \\ 0 & -1/2 & 1/4 & -5/4 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \cdot(-2) \\ -R_2 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 1/4 & -1/4 & 0 & 0 \\ 0 & 1 & -3/2 & 3/2 & -2 & 0 \\ 0 & 0 & -1/2 & -1/2 & -1 & 1 \end{array}\right] \begin{array}{l} -\tfrac{1}{4}R_3 \\ +\tfrac{3}{2}R_3 \\ \cdot(-2) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right] \begin{array}{l} +\tfrac{1}{2}R_2 \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

LU

Háttalagið við shears riggar væl til dekompositión av $\underline{\underline{A}}$ í niðara og ovara tríkantsmatricur. Eftir Gauss elimination er U funnin, tvs.

$$S_{n-1} S_{n-2} \ldots S_2 S_1 \underline{\underline{A}} = U \iff A = \underbrace{S_1^{-1} S_2^{-1} \ldots S_{n-2}^{-1} S_{n-1}^{-1}}_{\underline{\underline{L}}} \underline{\underline{U}}$$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{22} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Determinant Vit hava sæð nøkrar metodur til smærri skipanir

- Sarrus' rule
- Cofactor expansion (expansion of minors)
- Eginvirðir

Við Gauss elimination til eina matricu $\underline{U}$ gevur produktið eftir diagonalinum determinantin. Um vit brúka pivotering, so skifta vit fortekn fyri hvørja pivotering.

Dømi 12.11

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad \det(\underline{\underline{A}}) = 2 \cdot (-1) \cdot 2 \cdot 5 = -20$$

Sarrus er bert til 3x3, so vit kunnu royna cofactor expansión.
Vel síðstu soylu:

$$\det(\underline{\underline{A}}) = (-1)^{1+4} \cdot 4 \cdot \begin{vmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} + (-1)^{2+4} \cdot 3 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} + (-1)^{3+4} \cdot 0 \begin{vmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix} + (-1)^{4+4} \cdot 5 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 5 \cdot (-4) = -20$$

Cramer's rule Vit kunnu loysa eina líkning $\underline{\underline{A}}\,\underline{u} = \underline{b}$ við determinantar. Vektorurin $\underline{b}$ skal permuterast ígjøgnum $\underline{\underline{A}}$.

Dømi 12.12

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \quad \det(\underline{\underline{A}}) = 4 \quad \text{og} \quad \underline{b} = \begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}.$$

Her er

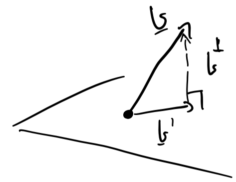
$$u_1 = \frac{\begin{vmatrix} 6 & 2 & 0 \\ 9 & 1 & 2 \\ 4 & 1 & 1 \end{vmatrix}}{4}, \quad u_2 = \frac{\begin{vmatrix} 2 & 6 & 0 \\ 1 & 9 & 2 \\ 2 & 4 & 1 \end{vmatrix}}{4}, \quad u_3 = \frac{\begin{vmatrix} 2 & 2 & 6 \\ 1 & 1 & 9 \\ 2 & 1 & 4 \end{vmatrix}}{4}$$

Least squares Vónandi hava øll sæð ella gjørt eina roynd í at approksimera eina punktmongd lineert. Tað er typiskt amboð til at gera niðurstøðu um eina heild.
Besta approksimatiónin er tann, sum er tættast við øll punktini samstundis.
Hvussu minimera vit so hesa frástøðu?

$$\underline{\underline{A}}\,\underline{u} = \underline{b}, \quad \text{har } \underline{\underline{A}} \text{ er } n \times 2 \text{ og } \underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}, \ b \text{ er } n \times 1$$

$\underline{\underline{A}}$ hevur rank 2 og vit kunnu hava n-dim., so vit nýtast ikki altíð hava eina loysn! Vit finna ta loysnina, sum er tættast við!
Approximatión: $\underline{\underline{A}}\,\underline{u} = \underline{b}'$, har $\underline{b}'$ er í column space hjá $\underline{\underline{A}}$.

So finst $\underline{b}^{\perp}$, so at



$$\underline{b} = \underline{b}' + \underline{b}^{\perp}$$

Her er $\underline{a}_1^T \underline{b}^{\perp} = 0$ og $\underline{a}_2^T \underline{b}^{\perp} = 0$,
so

$$\underline{\underline{A}}^T \underline{b}^{\perp} = \underline{0}.$$

Substituera inn $\underline{b} - \underline{b}'$, so

$$\underline{\underline{A}}^T(\underline{b} - \underline{b}') = 0 \iff \underline{\underline{A}}^T(\underline{b} - \underline{\underline{A}}\,\underline{u}) = 0$$

$$\iff \underline{\underline{A}}^T\underline{b} - \underline{\underline{A}}^T\underline{\underline{A}}\,\underline{u} = 0$$

$$\iff \underbrace{\underline{\underline{A}}^T\underline{\underline{A}}}_{\text{square}}\,\underline{u} = \underline{\underline{A}}^T\underline{b}$$

Her er loysnin sum hevur minst frávik $\|\underline{\underline{A}}\,\underline{u} - \underline{b}\|^2$, ti vit nýttu ortogonalu samansetingina $\underline{b}' + \underline{b}^{\perp}$ og $\|\underline{b}^{\perp}\|$ er styttsta leið til $\underline{b}$!

Loysnin er nú

$$\underline{\underline{A}}^T\underline{\underline{A}}\,\underline{u} = \underline{\underline{A}}^T\underline{b}$$

$$\iff \underline{u} = (\underline{\underline{A}}^T\underline{\underline{A}})^{-1}\underline{\underline{A}}^T\underline{b}$$