Linear first order differential equations

Ex.1 General solution to diff. eq.

a) Let $x'(t) - 2x(t) = e^t$, $t \in \mathbb{R}$. Compute the solution.

$p(t) = -2$, $P(t) = -2t$

$q(t) = e^t$

Now we plug into the formula (proposition 16.16).

$$x(t) = e^{-(-2t)} \int e^{-2t} \cdot e^t \, dt + c e^{-(-2t)}, \quad c \in \mathbb{R}$$
$$= e^{2t} \int e^{-t} \, dt + c e^{2t}$$
$$= e^{2t} \cdot (-e^{-t}) + c e^{2t}$$
$$= -e^t + c e^{2t}.$$

b) Compute the solution that passes through $(0,1)$.

$$-e^0 + c e^{2 \cdot 0} = 1 \iff c = 2$$

$$\Rightarrow \quad x(t) = -e^t + 2e^{2t}.$$

c) Let $x'(t) + \frac{1}{t} x(t) = -2t^2$, $t > 0$. Compute the solution.

$$x(t) = e^{-\ln(t)} \int e^{\ln(t)} \cdot (-2t^2) \, dt + c e^{-\ln(t)}$$
$$= \frac{1}{t} \int -2t^3 \, dt + c \cdot \frac{1}{t}$$
$$= \frac{1}{t} \left(-\frac{1}{2} t^4\right) + c \cdot \frac{1}{t}$$
$$= -\frac{1}{2} t^3 + c \cdot \frac{1}{t}, \quad c \in \mathbb{R}.$$

d) Compute the solution that passes through $(1,-1)$.

$$-\frac{1}{2}\cdot 1^3 + c\cdot\frac{1}{1} = -1 \iff c = \frac{1}{2}$$

$$\Rightarrow x(t) = -\frac{1}{2}t^3 + \frac{1}{2t}$$

e) For $c \in \mathbb{C}\setminus\{0\}$ we consider $z'(t) - c\,z(t) = 2$. Compute the solution.

$$z(t) = e^{-(-ct)}\int e^{-ct}\cdot 2\,dt + k e^{-(-ct)}$$

$$= e^{ct}\left(-\frac{2}{c}e^{-ct}\right) + k e^{ct}$$

$$= -\frac{2}{c} + k e^{ct}, \quad k \in \mathbb{C} \quad \text{(see remark 16.17)}$$

f) Determine for $c = -1+i$ the solution $z(t)$ for which $z(0) = i$.

$$-\frac{2}{-1+i} + k\cdot e^{(-1+i)\cdot 0} = i \iff k = i + \frac{2}{-1+i}$$

$$k = i + \frac{2}{-1+i} = i + \frac{-2-2i}{2} = -1$$

$$\Rightarrow z(t) = 1 + i - e^{(i-1)t}.$$

Ex2. The structural theorem.

a) A map $f: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is given by

$$f(x(t)) = x'(t).$$

Show that f is linear and determine $\ker(f)$.

Compute the solution to

$$f(x(t)) = \sin(t),$$

and then conclude a general solution.

Linearity can be concluded from

$$f(\alpha\, x(t) + y(t)) = (\alpha\, x(t) + y(t))' = \alpha\, x'(t) + y'(t)$$
$$= \alpha\, f(x(t)) + f(y(t))$$

for $\alpha \in \mathbb{R}$ and $x, y \in C^\infty(\mathbb{R})$.

The kernel is all functions $x \in C^\infty(\mathbb{R})$ for which $x(t) = c$ for some constant $c \in \mathbb{R}$. This follows since $c' = 0$.

Now we note that $(-\cos(t))' = \sin(t)$, which constitutes a solution to $f(x(t)) = \sin(t)$.

Lastly the structure theorem yields the general solutions

$$x(t) = -\cos(t) + c \quad , \qquad c \in \mathbb{R}.$$

b) A map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is given by

$$f(x(t)) = x'(t) - x(t)$$

Show that f is linear and determine $\ker(f)$.

Compute the solution to

$$f(x(t)) = 5 \quad ,$$

and then conclude a general solution.

Let $\alpha \in \mathbb{R}$ and $x, y \in C^\infty(\mathbb{R})$, then

$$f(\alpha\, x(t) + y(t)) = (\alpha x + y)' - (\alpha x + y) = \alpha x' + y' - \alpha x - y$$
$$= \alpha\,(x' - x) + y' - y$$
$$= \alpha\, f(x(t)) + f(y(t)).$$

Thus we conclude linearity.

The kernel satisfies $x'(t) - x(t) = 0$

$$\Leftrightarrow \quad x'(t) = x(t).$$

Only the exponential function has this property. Thus the kernel consist of functions $c \cdot e^t$, $c \in \mathbb{R}$.

A simple solution to $f(x(t)) = 5$ is $x(t) = 5$. Thus the general solution is

$$x(t) = 5 + c \cdot e^t, \qquad c \in \mathbb{R}.$$

c) A map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is given by

$$f(x(t)) = x'(t) + 2x(t)$$

Show that f is linear and determine $\ker(f)$.

Compute the solution to

$$f(x(t)) = 2t,$$

and then conclude a general solution.

$$f(\alpha x(t) + y(t)) = (\alpha x + y)' + 2(\alpha x + y) = \alpha x' + y' + \alpha \cdot 2x + 2y$$
$$= \alpha(x' + 2x) + y' + 2y = \alpha f(x(t)) + f(y(t)).$$

The kernel has solutions for which

$$x'(t) = -2x(t) \Leftrightarrow x(t) = ce^{-2t}, \quad c \in \mathbb{R}.$$

Let's guess $at + b$ and plug in.

$$(at+b)' + 2(at+b) = 2t$$

$$\Leftrightarrow \quad a + 2at + 2b = 2t \quad \Rightarrow \quad a = 1 \quad \Rightarrow \quad b = -\tfrac{1}{2}.$$

So a solution is $x(t) = t - \frac{1}{2}$. The general solution is

$$x(t) = t - \tfrac{1}{2} + ce^{-2t}.$$

d) State the diff. eq. in Leibniz notation.

1. $$\frac{d}{dt}\, x(t) = \sin(t)$$

2. $$\frac{d}{dt}\, x(t) - x(t) = 5$$

3. $$\frac{d}{dt}\, x(t) + 2\, x(t) = 2t$$

Ex3. Given $$\frac{d}{dt}\, x(t) + \cos(t)\cdot x(t) = \cos(t), \quad t \in \mathbb{R}.$$

a) Solve the diff. eq. by the general formula.

$$x(t) = e^{-\sin(t)} \int e^{\sin(t)} \cdot \cos(t)\, dt + c\, e^{-\sin(t)}, \quad c \in \mathbb{R}.$$

$$= e^{-\sin(t)} \cdot e^{\sin(t)} + c\, e^{-\sin(t)}$$

$$= 1 + c\, e^{-\sin(t)}.$$

With Maple we get,

```
> 'x(t)'=exp(-sin(t))*Int(exp(sin(t))*cos(t),t)+c*exp(-sin(t));
  exp(-sin(t))*int(exp(sin(t))*cos(t),t)+c*exp(-sin(t));
  'x(t)'=simplify(%);
```

$$x(t) = e^{-\sin(t)} \left(\int e^{\sin(t)} \cos(t)\, dt \right) + c\, e^{-\sin(t)}$$

$$e^{-\sin(t)}\, e^{\sin(t)} + c\, e^{-\sin(t)}$$

$$x(t) = 1 + c\, e^{-\sin(t)}$$

and by dsolve we get the following

```
> dsolve(diff(x(t),t)+cos(t)*x(t)=cos(t),x(t));
```

$$x(t) = 1 + e^{-\sin(t)}\, c_1$$

b) Solve instead by the structural theorem.

Firstly just guess $x(t) = 1$, since $p(t) = \cos(t) = q(t)$.

The homogeneous solution is $x(t) = c\, e^{-\sin(t)}$, and so the general solution is $x(t) = 1 + c\, e^{-\sin(t)}$.

Ex4. Superposition

a) Guess: $x'(t) + x(t) = 2\cos(t)$, $t \in \mathbb{R}$.

Say we use $a\sin(t) + b\cos(t)$, then we get

$$(a\sin(t) + b\cos(t))' + a\sin(t) + b\cos(t)$$

$$= a\cos(t) - b\sin(t) + a\sin(t) + b\cos(t).$$

So $a = b = 1$ is a solution, i.e. $\sin(t) + \cos(t)$.

b) Guess: $x'(t) + x(t) = t^2 - 1$, $t \in \mathbb{R}$.

Let's try $at^2 + bt + c$, then

$$(at^2 + bt + c)' + at^2 + bt + c = 2at + b + at^2 + bt + c$$

$$= at^2 + (2a+b)t + (b+c)$$

It follows that $a = 1$, $2a + b = 0$ and $b + c = -1$, thus a solution is $t^2 - 2t + 1$.

c) Solve $x'(t) + x(t) = 2\cos(t) + t^2 - 1$, $t \in \mathbb{R}$.

The homogeneous solution is ce^{-t}, and by superposition the general solution follows by a) and b) namely

$$x(t) = ce^{-t} + \sin(t) + \cos(t) + t^2 - 2t + 1, \quad c \in \mathbb{R},\ t \in \mathbb{R}.$$

Ex5. We're given $x'(t) + \frac{1}{7}x(t) = 3 - 2\cos(t)$.

a) Get the solution with dsolve.

b) Get solution through $(0,0)$.

c) Plot. Make some plots for different initial conditions a)

```
> deq:= diff(x(t),t)+1/7*x(t)=3-2*cos(t);
  dsolve(deq,x(t));
```

$$deq := \frac{\mathrm{d}}{\mathrm{d}t}\, x(t) + \frac{x(t)}{7} = 3 - 2\cos(t)$$

$$x(t) = 21 - \frac{7\cos(t)}{25} - \frac{49\sin(t)}{25} + \mathrm{e}^{-\frac{t}{7}}\, c_1$$

b)

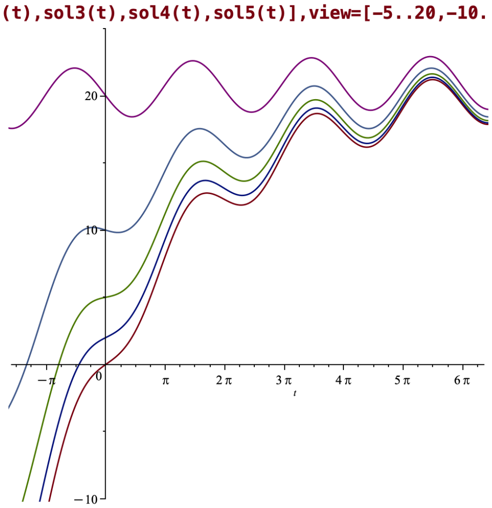
```
> dsolve({x(0)=0,deq},x(t));
  sol1:=unapply(rhs(%),t);
```

$$x(t) = 21 - \frac{7\cos(t)}{25} - \frac{49\sin(t)}{25} - \frac{518\,\mathrm{e}^{-\frac{t}{7}}}{25}$$

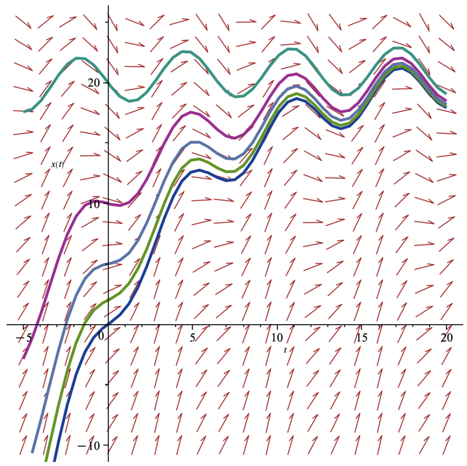
$$sol1 := t \mapsto 21 - \frac{7\cdot\cos(t)}{25} - \frac{49\cdot\sin(t)}{25} - \frac{518\cdot\mathrm{e}^{-\frac{t}{7}}}{25}$$

c)

```
> sol2:=unapply(rhs(dsolve({x(0)=2,deq},x(t))),t):
  sol3:=unapply(rhs(dsolve({x(0)=5,deq},x(t))),t):
  sol4:=unapply(rhs(dsolve({x(0)=10,deq},x(t))),t):
  sol5:=unapply(rhs(dsolve({x(0)=20,deq},x(t))),t):
> plot([sol1(t),sol2(t),sol3(t),sol4(t),sol5(t)],view=[-5..20,-10..25]);
```




```
> with(DEtools):
> DEplot(deq,x(t),t=-5..20,x=-10..25,[x(0)=0,x(0)=2,x(0)=5,x(0)=10,x(0)=20])
```



Ex6. Follow the instructions in the provided maple sheet.

Ex7. Let $f: (C^\infty(\mathbb{R}), \mathbb{C}) \to (C^\infty(\mathbb{R}), \mathbb{C})$ be given by

$$f(z(t)) = z''(t) + z(t).$$

a) Explain that $U = \{e^{it}, e^{-it}\}$ is a 2-dim. subspace of $\ker(f)$.

Fistly $\quad (e^{it})'' + e^{it} = i^2 e^{it} + e^{it} = 0$ and

$$(e^{-it})'' + e^{-it} = i^2 e^{-it} + e^{-it} = 0,$$

so $e^{it}, e^{-it} \in \ker(f)$. The vectors can be shown to be lin. indept. by $c_1 e^{it} + c_2 e^{-it} = 0$ for say $t = 0$ and $t = \frac{\pi}{2}$.

We get the equations

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 \cdot i - c_2 i = 0$$

$$\Rightarrow c_1 = -c_2 \quad \Rightarrow \quad -2c_2 i = 0$$

Therefore $c_1 = c_2 = 0$ is the only solution. Thus U is a 2-dim. subspace of $\ker(f)$.

b) A real function $z_0(t) \in U$ satisfies $z(0) = 1$ and $z'(0) = 0$. Determine $z_0(t)$.

$$z''(t) + z(t) = 0 \quad \text{has the solution } \cos(t).$$

Note that for $z_0(t) = \cos(t)$ we have $z_0(0) = 1$ and

$z_0'(0) = -\sin(0) = 0$.

Ex8. Let $U \subseteq C^\infty(\mathbb{R})$ spanned by $\cos(t), \sin(t), e^t$.

a) Show that $\cos(t), \sin(t), e^t$ constitutes a basis for U.

We need to show there's only one trivial solution to

$$c_1 \cos(t) + c_2 \sin(t) + c_3 e^t = 0,$$

namely $c_1 = c_2 = c_3 = 0$. Let's make equations with $t = 0, \frac{\pi}{2}, \pi$.

$$\begin{cases} c_1 + 0 + c_3 = 0 \\ 0 + c_2 + c_3 e^{\frac{\pi}{2}} = 0 \\ -c_1 + 0 + c_3 e^{\pi} = 0 \end{cases} \iff \begin{cases} c_1 + 0 + c_3 = 0 \\ 0 + c_2 + c_3 e^{\frac{\pi}{2}} = 0 \\ c_3 e^{\pi} + c_3 = 0 \end{cases}$$

By adding the first eq to the third it follows $c_3 = 0$. Since $c_3 = 0$ it forces the solution with $c_1 = c_2 = c_3 = 0$. The vectors therefore constitu a basis.

b) Let $f(x(t)) = x'(t) + 2x(t)$. Show f maps U onto itself.

$$f(\cos(t)) = -\sin(t) + 2 \cdot \cos(t) \in U$$

$$f(\sin(t)) = \cos(t) + 2 \cdot \sin(t) \in U$$

$$f(e^t) = e^t + 2e^t = 3e^t \in U$$

Clearly f maps U to U.

c) State F wrt. the basis of U.

Let's call the basis $u = (\cos(t), \sin(t), e^t)$. Read from b).

$$_u F_u = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

d) Use the matrix to find solutions in U to

$$x'(t) + 2x(t) = -\sin(t) + 3e^t, \quad t \in \mathbb{R}.$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} \cdot\frac{1}{2} \\ +\frac{1}{2}R_1 \\ \cdot\frac{1}{3} \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 5/2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{2}R_2 \\ \cdot 2/5 \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow x(t) = \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + e^t.$$

Ex 9. We have the diff. eq.

1. $x'(t) + t \cdot x(t) \cdot (1 + x(t)) = 0$
2. $x'(t) + t^2 \cdot x(t) = 0$
3. $x'(t) + x(t) = t^2$
4. $x'(t) + (x(t))^2 = t$
5. $x'(t) + t^3 \cdot x(t) = 0$
6. $x'(t) + e^{x(t)} = 1$
7. $(x'(t))^2 + x(t) = 0$

a) Three are linear, which?

See the definition: 2, 3, 5

b) Solve these with Maple "simulated manually".

```
> 'x(t)'=exp(-1/3*t^3)*int(exp(1/3*t^3)*0,t)+c*exp(-1/3*t^3);
```

$$x(t) = c\,e^{-\frac{t^3}{3}}$$

```
> exp(-t)*int(exp(t)*t^2,t)+c*exp(-t);
  'x(t)'=simplify(%);
```

$$e^{-t}(t^2 - 2t + 2)\,e^t + c\,e^{-t}$$

$$x(t) = t^2 - 2t + 2 + c\,e^{-t}$$

```
> 'x(t)'=exp(-1/4*t^4)*int(exp(1/4*t^4)*0,t)+c*exp(-1/4*t^4);
```

$$x(t) = c\,e^{-\frac{t^4}{4}}$$

c) Find a solution to the other diff. eq. with Maple.

```
> eq1:= diff(x(t),t)+t*x(t)*(1+x(t))=0:
  eq2:= diff(x(t),t)+t^2*x(t)=0:
  eq3:= diff(x(t),t)+x(t)=t^2:
  eq4:= diff(x(t),t)+(x(t))^2=t:
  eq5:= diff(x(t),t)+t^3*x(t)=0:
  eq6:= diff(x(t),t)+exp(x(t))=1:
  eq7:= (diff(x(t),t))^2+x(t)=0:
> dsolve(eq1,x(t));
  dsolve(eq2,x(t));
  dsolve(eq3,x(t));
  dsolve(eq4,x(t));
  dsolve(eq5,x(t));
  dsolve(eq6,x(t));
  dsolve(eq7,x(t));
```

$$x(t) = \frac{1}{-1 + e^{\frac{t^2}{2}} c_1}$$

$$x(t) = c_1 e^{-\frac{t^3}{3}}$$

$$x(t) = t^2 - 2t + 2 + e^{-t} c_1$$

$$x(t) = \frac{c_1 \operatorname{AiryAi}(1, t) + \operatorname{AiryBi}(1, t)}{c_1 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)}$$

$$x(t) = c_1 e^{-\frac{t^4}{4}}$$

$$x(t) = t - \ln\left(-1 + e^{t + c_1}\right) + c_1$$

$$x(t) = 0, x(t) = -\frac{1}{4} t^2 + \frac{1}{2} t c_1 - \frac{1}{4} c_1^2$$

d) Experiment with some solutions: plot for different c.

```
> plot([1/(-1 + exp(t^2/2)),exp(-t^3/3),t^2 - 2*t + 2 + exp(-t),(AiryAi(1, t) +
  AiryBi(1, t))/(AiryAi(t) + AiryBi(t)),exp(-t^4/4),t - ln(-1 + exp(t)),-1/4*t^2 +
  1/2*t - 1/4],view=[-10..10,-10..10])
```