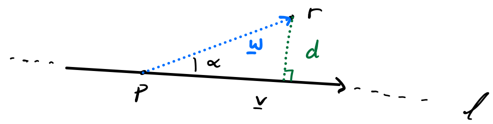
3D interactions Vit hyggja at, hvussu fundamental skap spæla saman.

- punkt
- linjur
- plan

Punkt - Punkt Pythagoras.

Punkt - Linja Longdin úr punkt til linju sóu vit í 3.6.2. Vit eru noydd at brúka parametriseraða linju í 3D.

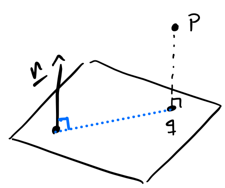


$$d = \|\underline{w}\| \cdot \sin\alpha \quad , \quad \sin\alpha = \sqrt{1-\cos^2\alpha} \quad \text{og} \quad \cos\alpha = \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\|\underline{w}\|}$$

$$\Rightarrow \quad d = \|\underline{w}\| \cdot \sqrt{1-\left(\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|}\right)^2}$$

Punkt - plan Hetta minnir um 3.7, men vit vísa hví longdin úr punkt p til plan P kann roknast eins og í 8.4.

Lat P hava implicitta likning



$$\underline{n}\cdot x + c = 0 \quad \text{tvs.} \quad \underline{n}\cdot(x-\hat{q}) = 0$$

ella

$$Ax_1 + Bx_2 + Cx_3 + D = 0$$

Linjan úr q til p er paralell við $\underline{n}$, so vit kunnu skriva

$$p = q + t\,\underline{n} \iff q = p - t\,\underline{n}$$

Eftirsum at q liggur í P, so er

$$\underline{n}\cdot q + c = 0 \iff \underline{n}\cdot(p - t\underline{n}) + c = 0$$

$$\iff \underline{n}\cdot p - \underline{n}\cdot t\underline{n} + c = 0 \iff -t\,\underline{n}\cdot\underline{n} = -c - n\cdot p$$

$$\iff t = \frac{c + \underline{n}\cdot p}{n\cdot\underline{n}}$$

, um $\underline{n}$ er normaliseradur

$$t = c + \underline{n}\cdot p.$$

Dømi 2021

$P: \; 8x_1 - 4x_2 + x_3 - 1 = 0$ og $p = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Her er $\underline{n} = \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$, so vit

$$\underline{n} \cdot p = \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 8 - 8 + 1 = 1$$

$$\underline{n} \cdot \underline{n} = 64 + 16 + 1 = 81$$

fáa vit $t = \dfrac{-1 + 1}{81} = 0$. Altso frástøðan er 0.

Lat $q = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$, so fáa vit $\underline{n} \cdot q = -8$, og

$$t = \frac{-1 - 8}{81} = \frac{-9}{81} = -\frac{1}{9}$$

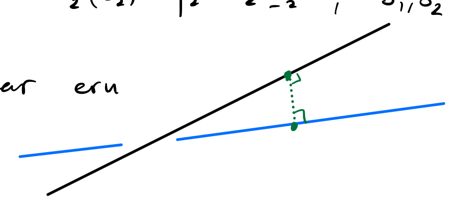
Nú er frástøðan $\frac{1}{9} \cdot \|\underline{n}\| = 1$.

linja - linja

Ofta møtast linjur ikki í 3D, tá sigast tær at vera skeivar í mun til hvønn annan (skew).
Lat l_1 og l_2 vera givnar við

$$x_1(s_1) = p_1 + s_1 \underline{v}_1 \quad \text{og} \quad x_2(s_2) = p_2 + s_2 \underline{v}_2, \quad s_1, s_2 \in \mathbb{R}.$$

Punktini x_1 og x_2, har linjurnar eru tættast geva vektorar, sum eru ortogonalir við $\underline{v}_1$ og $\underline{v}_2$:



$$\begin{cases} (x_2 - x_1) \cdot \underline{v}_1 = 0 \\ (x_2 - x_1) \cdot \underline{v}_2 = 0 \end{cases}$$

Skriva vit x_1 og x_2 heilt út við teirri parametrisering fáa vit

$$\begin{cases} (p_2 + s_2 \underline{v}_2 - (p_1 + s_1 \underline{v}_1)) \cdot \underline{v}_1 = 0 \\ (p_2 + s_2 \underline{v}_2 - (p_1 + s_1 \underline{v}_1)) \cdot \underline{v}_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (p_2 - p_1)\cdot \underline{v}_1 + s_2 \underline{v}_2 \cdot \underline{v}_1 - s_1 \underline{v}_1 \cdot \underline{v}_1 = 0 \\ (p_2 - p_1)\cdot \underline{v}_2 + s_2 \underline{v}_2 \cdot \underline{v}_2 - s_1 \underline{v}_1 \cdot \underline{v}_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (p_2 - p_1)\cdot \underline{v}_1 = s_1 \underline{v}_1 \cdot \underline{v}_1 - s_2 \underline{v}_2 \cdot \underline{v}_1 \\ (p_2 - p_1)\cdot \underline{v}_2 = s_1 \underline{v}_1 \cdot \underline{v}_2 - s_2 \underline{v}_2 \cdot \underline{v}_2 \end{cases}$$

Her eru bert s_1 og s_2 ókendar støddir, so vit loysa líkningaskipanina sum vant.

Dømi 2019

$$l_1(s_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{og} \quad l_2(s_2) = \begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix} + s_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \quad s_1, s_2 \in \mathbb{R}.$$

eq1: $$\begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = s_1 \cdot 2 - s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \quad 0 = 2s_1 + s_2$$

eq2: $$\begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = s_1 \cdot (-1) - s_2 \cdot 14$$

$$\Leftrightarrow \quad 27 = -s_1 - 14 s_2 \qquad \begin{bmatrix} 2 & 1 \\ -1 & -14 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 27 \end{bmatrix}$$

Við eq1 er $s_2 = -2s_1$, so

$$-s_1 - 14\cdot(-2s_1) = 27 \Leftrightarrow -s_1 + 28 s_1 = 27 \Leftrightarrow s_1 = 1$$

og

$$s_2 = -2 \cdot 1 = -2$$

$$l_1(1) = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

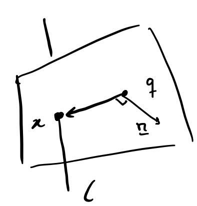
$$l_2(-2) = \begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix} - 2 \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Tvs. frástøðan er 0.

Linja - plan

Ray-tracing

Sum vant brúka vit $x = p + t\underline{v}$ í



$$(x - q)\cdot \underline{n} = 0$$

$$\Leftrightarrow (p + t\underline{v} - q)\cdot \underline{n} = 0 \Leftrightarrow t\,\underline{v}\cdot\underline{n} + (p - q)\cdot \underline{n} = 0$$

$$\Leftrightarrow t = \frac{(q - p)\cdot \underline{n}}{\underline{v}\cdot\underline{n}} \Rightarrow x = p + \frac{(q-p)\cdot\underline{n}}{\underline{v}\cdot\underline{n}}\,\underline{v}$$

Dømi 2019

$$l(t) = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{og} \quad P: \; x_1 - 2x_2 + 4x_3 = 5$$

$$\underline{n} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \text{og} \quad q = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}. \qquad q - p = \begin{bmatrix} 5-3 \\ -(-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Parametriskt: $\quad x = q + u_1 \underline{r}_1 + u_2 \underline{r}_2 \quad$ og $\quad x = p + t\underline{v}$

$$\Rightarrow \quad p + t\underline{v} = q + u_1 \underline{r}_1 + u_2 \underline{r}_2$$

$$\Leftrightarrow \quad p - q = u_1 \underline{r}_1 + u_2 \underline{r}_2 - t\underline{v}$$

$$\Leftrightarrow \quad \begin{bmatrix} p - q \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & -v \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ t \end{bmatrix}$$

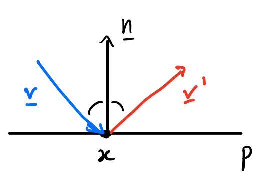
Til avmarkaðar tríkantar

$$x = p_1 + u_1(p_2 - p_1) + u_2(p_3 - p_1)$$

gera vit á sama hátt, og barycentrisku koordinatini eru positiv, um linjan skerir inni í tríkantinum.

$$(u_1, u_2, 1 - u_1 - u_2).$$

Reflection



Lat $\underline{v}, \underline{v}'$ og $\underline{n}$ vera normeraðir.

$$-\underline{v} \cdot \underline{n} = \underline{v}' \cdot \underline{n} \quad \text{og} \quad c\,\underline{n} = \underline{v}' - \underline{v}$$

fyri $c \in \mathbb{R}$. So vit fáa

$$-\underline{v} \cdot \underline{n} = (\underline{v} + c\,\underline{n}) \cdot \underline{n} \Leftrightarrow -\underline{v} \cdot \underline{n} = \underline{v} \cdot \underline{n} + c\,\underline{n} \cdot \underline{n}$$

$$\Leftrightarrow \quad c = -2\,\underline{v} \cdot \underline{n}.$$

$$\Rightarrow \quad \underline{v}' = \underline{v} - 2(\underline{v} \cdot \underline{n})\underline{n} = \underline{v} - 2(\underline{n}^T \underline{v})\underline{n} = \underline{v} - 2\,\underline{n}\,\underline{n}^T \underline{v} = \underbrace{(\underline{\underline{I}} - 2\underline{n}\underline{n}^T)}_{H}\underline{v}$$

Dømi 2019

P: $\frac{3}{5}x_1 - \frac{4}{5}x_3 = 0$ og $\underline{v} = \begin{bmatrix}4\\-2\\3\end{bmatrix}$.

Reflekteraði rætningurin: $\underline{n} = \begin{bmatrix}3/5\\0\\-4/5\end{bmatrix}$

$$n n^T = \begin{bmatrix}3/5\\0\\-4/5\end{bmatrix}\begin{bmatrix}3/5 & 0 & -4/5\end{bmatrix} = \begin{bmatrix}9/25 & 0 & -12/25\\0 & 0 & 0\\-12/25 & 0 & 16/25\end{bmatrix}$$

$$\underline{v}' = \underline{\underline{H}}\,\underline{v} = \begin{bmatrix}7/25 & 0 & 24/25\\0 & 1 & 0\\24/25 & 0 & -7/25\end{bmatrix}\begin{bmatrix}4\\-2\\3\end{bmatrix} = \begin{bmatrix}\frac{28+72}{25}\\-2\\\frac{96-21}{25}\end{bmatrix} = \begin{bmatrix}4\\-2\\3\end{bmatrix}$$

$$\underline{v}\cdot\underline{n} = \frac{12}{5} - \frac{12}{5} = 0 \Rightarrow \underline{v} \parallel P \ !$$

Plan-plan   Skering millum 3 plan kann vera eitt punkt, meðan tað skeringin eisini kann vera ein heil linja.

Vit loysa líkningaskipanina $\begin{cases}\underline{n}_1 x + d_1 = 0\\ \underline{n}_2 x + d_2 = 0\\ \underline{n}_3 x + d_3 = 0\end{cases}$

ella ekvivalent $\begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-d_1\\-d_2\\-d_3\end{bmatrix}$

Dømi 2019

$$\left[\begin{array}{ccc|c}2 & 1 & -2 & 3\\2 & 2 & 1 & 1\\3 & 2 & 0 & 2\end{array}\right]\cdot\tfrac{1}{2} \longrightarrow \left[\begin{array}{ccc|c}1 & \frac{1}{2} & -1 & \frac{3}{2}\\3 & 2 & 0 & 2\\2 & 2 & 1 & 1\end{array}\right]\begin{array}{l}\\-3R_1\\-2R_1\end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c}1 & \frac{1}{2} & -2 & \frac{3}{2}\\0 & \frac{1}{2} & 3 & -\frac{5}{2}\\0 & 1 & 3 & -2\end{array}\right]\begin{array}{l}\\\cdot 2\\-R_2\end{array} \rightarrow \left[\begin{array}{ccc|c}1 & \frac{1}{2} & -2 & 3/2\\0 & 1 & 6 & -5\\0 & 0 & -3 & 3\end{array}\right]$$

$$\Rightarrow x_3 = -1 \quad \Rightarrow x_2 - 6 = -5 \Leftrightarrow x_2 = 1$$

$$\Rightarrow x_1 + \frac{1}{2} + 2 = \frac{3}{2} \Leftrightarrow x_1 = 0$$

$$x = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Um tvey plan møtast í eini linju, so er linjan ortogonal við normalvektorarnar hjá planini. Tí linjan er í bæði planini!
So er

$$l: \quad x(t) = p + t\underline{v}, \quad \text{har} \quad \underline{v} = \underline{n}_1 \wedge \underline{n}_2$$

Definera nú planið $\underline{v} \cdot x = 0$, so vil skeringspunktið millum tey trý planini svara til p. So er linjan parametriserað líðugt.

Gram-Schmidt Vit kunnu framleiða eitt ortonormalsystem við at fylgja Gram-Schmidt algoritmuni.
Hetta ger tað lætt at fáa lokalar koordinatskipanir.
Givið $\underline{v}_1, \underline{v}_2, \underline{v}_3$, sum eru lineert óheftir, so ynskja vit ortonormalar vektorar $\underline{b}_1, \underline{b}_2, \underline{b}_3$.

Stig 1. Normalisera $\underline{v}_1$: $\quad \underline{b}_1 = \dfrac{\underline{v}_1}{\|\cdot\|}, \quad V_1 = \text{span}\{\underline{v}_1\}$.

Stig 2. Ger $\underline{b}_2$ úr partinum hjá $\underline{v}_2$, sum er ortogonalur á $\underline{b}_1$.

$$\underline{b}_2 = \frac{\underline{v}_2 - \text{proj}_{V_1}\underline{v}_2}{\|\cdot\|} = \frac{\underline{v}_2 - (\underline{v}_2 \cdot \underline{b}_1)\underline{b}_1}{\|\cdot\|}, \quad V_{12} = \text{span}\{\underline{v}_1, \underline{v}_2\}.$$

Stig 3

$$\underline{b}_3 = \frac{\underline{v}_3 - \text{proj}_{V_{12}}\underline{v}_3}{\|\cdot\|} = \frac{\underline{v}_3 - (\underline{v}_3 \cdot \underline{b}_1)\underline{b}_1 - (\underline{v}_3 \cdot \underline{b}_2)\underline{b}_2}{\|\cdot\|}$$

Dømi 2019

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{b}_1 = \underline{v}_1, \quad \underline{b}_2 = \frac{\underline{v}_2 - (\underline{v}_2 \cdot \underline{b}_1)\underline{b}_1}{\|\cdot\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\|\cdot\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}{\|\cdot\|} = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\underline{b}_3 = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 \cdot \underline{b}_2}{\|\cdot\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\|\cdot\|} = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$