Taylor polynomials and Maple

Ex 1. a) Find first and second degree Taylor polynomials expanded at $x_0 = 0$.

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

1. $f(x) = e^x, \quad x \in \mathbb{R}$ $\qquad f'(x) = e^x, \quad e^0 = 1$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

2. $f(x) = \cos(x), \quad x \in \mathbb{R}$ $\qquad f'(x) = -\sin(x), \quad -\sin(0) = 0$

$$f''(x) = -\cos(x), \quad -\cos(0) = -1$$

$$P_1(x) = 1$$

$$P_2(x) = 1 - \frac{x^2}{2}$$

3. $f(x) = e^{\sin(x)}, \quad x \in \mathbb{R}$ $\qquad f'(x) = \cos(x) \cdot e^{\sin(x)}, \quad f'(0) = 1$

$$f''(x) = -\sin(x) \cdot e^{\sin(x)} + \cos^2(x) \cdot e^{\sin(x)}, \quad f''(0) = 1$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

b) Let $x_0 = 1$ and $f(x) = \frac{1}{x}, \quad x > 0$.

$$f'(x) = -\frac{1}{x^2}, \quad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3}, \quad f''(1) = 2$$

$$P_1(x) = 1 - (x - 1) = 2 - x$$

$$P_2(x) = 1 - (x-1) + \frac{2}{2}(x-1)^2$$
$$= 2 - x + x^2 - 2x + 1 = x^2 - 3x + 3$$

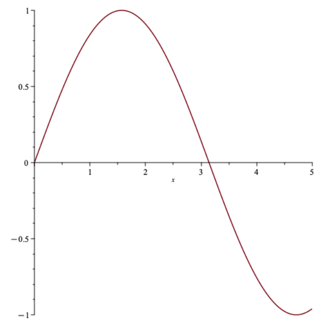
Ex. 2 Try different prompts with Maple.

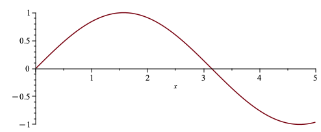
```
> 2+2;
```
4

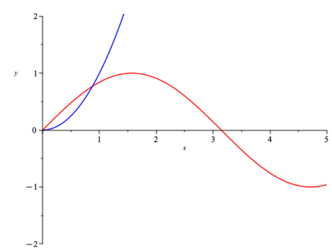
Simple calculations

```
> diff(sin(x),x);
```
$\cos(x)$

Function taking 2 inputs

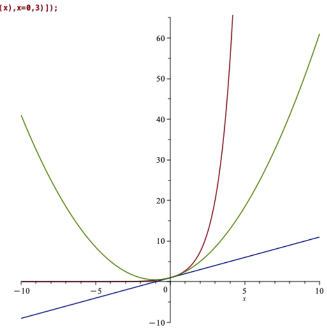
```
> plot(sin(x),x=0..5);
```


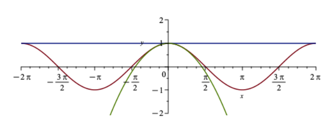
```
> plot(sin(x),x=0..5,scaling=constrained);
```


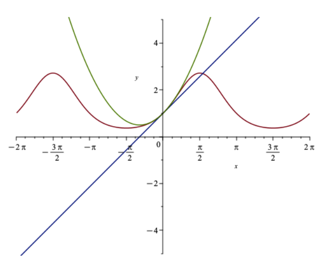
```
> plot([sin(x),x^2],x=0..5,y=-2..2,color=[red,blue],scaling=constrained);
```


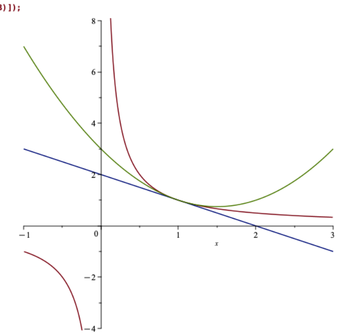
Plotting has rather many options. For now it's plenty to plot several graphs or using display to show several graphs together.

Ex 3. Plot the functions from Ex1.

```
> plot([exp(x),mtaylor(exp(x),x=0,2),mtaylor(exp(x),x=0,3)]);
```


```
> plot([cos(x),mtaylor(cos(x),x=0,2),mtaylor(cos(x),x=0,3)],x=-2*Pi..2*Pi,y=-2..2,scaling=constrained);
```


```
> plot([exp(sin(x)),mtaylor(exp(sin(x)),x=0,2),mtaylor(exp(sin(x)),x=0,3)],x=-2*Pi..2*Pi,y=-5..5,scaling=constrained);
```


```
> plot([1/x,mtaylor(1/x,x=1,2),mtaylor(1/x,x=1,3)]);
```


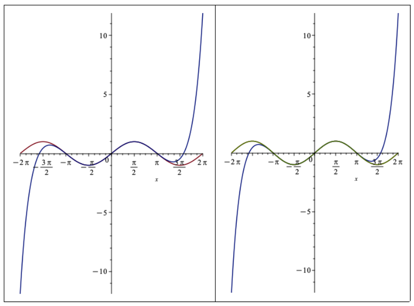
Ex 4. Approximate $f(x) = \sin(x)$ and see how well P_9 does expanded at $x_0 = 0$.

```
> f:=x-> sin(x):
  P9:= unapply(mtaylor(f(x),x=0,10),x);
  P17:= unapply(mtaylor(f(x),x=0,18),x);
```

$$P9 := x \mapsto x - \frac{1}{6}\cdot x^3 + \frac{1}{120}\cdot x^5 - \frac{1}{5040}\cdot x^7 + \frac{1}{362880}\cdot x^9$$

$$P17 := x \mapsto x - \frac{1}{6}\cdot x^3 + \frac{1}{120}\cdot x^5 - \frac{1}{5040}\cdot x^7 + \frac{1}{362880}\cdot x^9 - \frac{1}{39916800}\cdot x^{11} + \frac{1}{6227020800}\cdot x^{13} - \frac{1}{1307674368000}\cdot x^{15} + \frac{1}{355687428096000}\cdot x^{17}$$

```
> A:=plot([f(x),P9(x)]):
  B:=plot([f(x),P9(x),P17(x)]):
  display(<A|B>,);
```



We just use mtaylor.

P_9 is quite good around 0, but falls off around $\pm \frac{3\pi}{2}$. We see P_{17} approximates well on the entirety of $[-2\pi, 2\pi]$.

Ex 5. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt{2x-1}$.

a) Since $Dm(\sqrt{x}) = [0, \infty[$ we have

$$\sqrt{2x-1} = 0 \iff 2x - 1 = 0$$
$$\iff x = \frac{1}{2},$$

and so $Dm(f) = [\frac{1}{2}, \infty[$.

b) Determine P_3 of f at $x_0 = 1$.

$$f'(x) = \frac{1}{2\sqrt{2x-1}} \cdot 2 = \frac{1}{\sqrt{2x-1}}, \quad f'(1) = 1$$

$$f''(x) = \left((2x-1)^{-\frac{1}{2}}\right)' = -\frac{1}{2}(2x-1)^{-\frac{3}{2}} \cdot 2$$
$$= -\frac{1}{(2x-1)^{3/2}}, \quad f''(1) = -1$$

$$f'''(x) = \left(-(2x-1)^{-\frac{3}{2}}\right)' = \frac{3}{2}(2x-1)^{-5/2} \cdot 2$$

$$= \frac{3}{(2x-1)^{5/2}} \quad , \qquad f'''(1) = 3$$

$$P_3(x) = 1 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{3}{6}(x-1)^3$$

$$= x - \frac{1}{2}(x^2-2x+1) + \frac{1}{2}(x^3-3x^2+3x-1)$$

$$= x - \frac{1}{2}x^2 + x - \frac{1}{2} + \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{2}$$

$$= -1 + \frac{7}{2}x - 2x^2 + \frac{1}{2}x^3$$

c) Determine R_3 and that the error at $x = \frac{3}{2}$ is at most $\frac{5}{2^7}$.

Using lemma 4.5 we get

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4 \quad , \quad \xi \in \,]1, x[\,.$$

$$f^{(4)}(x) = -\frac{15}{(2x-1)^{7/2}} \qquad \text{(just follow the pattern)}$$

Thus we get

$$R_3(x) = -\frac{15}{4!(2\xi-1)^{7/2}}(x-1)^4 = -\frac{5}{8}\frac{1}{(2\xi-1)^{7/2}}(x-1)^4.$$

The error is at most

$$\left|R_3\left(\frac{3}{2}\right)\right| = \left|-\frac{5}{8}\frac{1}{(2\xi-1)^{7/2}}\left(\frac{3}{2}-1\right)^4\right| \leq \frac{5}{8}\frac{1}{(2\cdot 1-1)^{7/2}}\left(\frac{1}{2}\right)^4$$

$$= \frac{5}{8} \cdot 1 \cdot \frac{1}{2^4} = \frac{5}{2^7}.$$

Ex 6. Past exercises to complete with Maple.

a)
```
> I^2;
  I^3;
  (-I)^4;
  (-I)^5;
```
$$-1$$
$$-I$$
$$1$$
$$-I$$

b)
```
> (-2+3*I)/I
```
$$3+2I$$

c)
```
> w:=1-I:
  abs(w);
  argument(w);
  abs(exp(w));
  argument(exp(w));
```
$$\sqrt{2}$$
$$-\frac{\pi}{4}$$
$$e$$
$$-1$$

d)
```
> exp(I*Pi/2);
  3*exp(1+Pi*I);
```
$$I$$
$$-3e$$

e)
```
> solve(z^2+(2+2*I)*z-2*I=0,z);
```
$$-1-I+\sqrt{2}+I\sqrt{2},\ -1-I-\sqrt{2}-I\sqrt{2}$$

f)
```
> diff(t^2+I*sin(t),t);
  diff(1+I*t^5,t);
  diff(t^5-I,t);
  diff(3*exp(I*t),t);
  diff(I*exp(2*t+3*I*t),t);
```
$$2t+I\cos(t)$$
$$5It^4$$
$$5t^4$$
$$3Ie^{It}$$
$$(-3+2I)e^{2t+3It}$$

Ex 7. Let $f(x) = 2\cos(x) + i\sin(2x)$, $x \in \mathbb{R}$.

a) Determine P_3 for f expanded at $x_0 = 0$.

Let's use Maple, since we know differentiation works the same for complex functions.

```
> f:=x-> 2*cos(x)+I*sin(2*x):
> P3:= unapply(mtaylor(f(x),x=0,4),x);
```
$$P3 := x \mapsto 2 + 2\cdot I\cdot x - x^2 - \frac{4\cdot I\cdot x^3}{3}$$

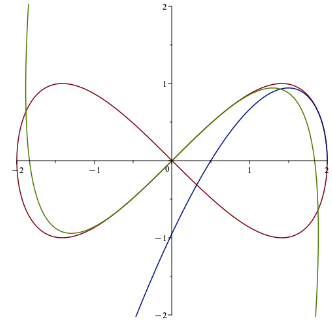
b) Expand Q_3 at $x_1 = \frac{\pi}{2}$.

```
> Q3:= unapply(mtaylor(f(x),x=Pi/2,4),x);
  expand(Q3(x));
```
$$Q3 := x \mapsto -(2+2\cdot I)\cdot\left(x-\frac{\pi}{2}\right) + \left(\frac{1}{3}+\frac{4\cdot I}{3}\right)\cdot\left(x-\frac{\pi}{2}\right)^3$$
$$-2x - 2Ix + \pi + I\pi + \frac{x^3}{3} + \frac{4Ix^3}{3} - \frac{\pi x^2}{2} - 2I\pi x^2 + \frac{x\pi^2}{4} + Ix\pi^2 - \frac{\pi^3}{24} - \frac{I\pi^3}{6}$$

c) Why is P_3 smarter to use than Q_3 for approximating $f(1)$?

It has a significantly simpler form. Though we can see that Q_3 performs better at $x=1$, it is hard labour unless we delegate to Maple:

```
> complexplot([f(x),P3(x),Q3(x)],x=0..2*Pi,view=[-2..2,-2..2])
```




```
> f(1);evalf(%);
  P3(1);evalf(%);
  Q3(1);evalf(%);
```

$$2\cos(1) + I\sin(2)$$
$$1.080604612 + 0.9092974268\,I$$
$$1 + \frac{2\,I}{3}$$
$$1. + 0.6666666667\,I$$
$$-(2+2\,I)\left(1-\frac{\pi}{2}\right) + \left(\frac{1}{3}+\frac{4\,I}{3}\right)\left(1-\frac{\pi}{2}\right)^3$$
$$1.079602566 + 0.8936323010\,I$$

Ex 8. Let $f(x) = \ln(1+x)$.

a) State the limit formula (4.8) for f at $x_0 = 0$ of degrees 1, 2 and 3.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

deg 1: $f(x) = x + x \cdot \varepsilon(x)$

deg 2: $f(x) = x - \frac{1}{2}x^2 + x^2 \cdot \varepsilon(x)$

deg 3: $f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3 \cdot \varepsilon(x)$

b) Which result from a) can't be used to determine

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} \;?$$

What is the limit?

First degree is insufficient: $\lim_{x\to 0} \frac{x + x\cdot\varepsilon(x) - x}{x^2} = \lim_{x\to 0} \frac{\varepsilon(x)}{x}$

Second degree and above is fine:

$$\lim_{x\to 0} \frac{x - \frac{1}{2}x^2 + x^2\varepsilon(x) - x}{x^2} = \lim_{x\to 0}\left(-\frac{1}{2} + \varepsilon(x)\right) = -\frac{1}{2}.$$

c) Compute

$$\lim_{x \to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$$

We need to have a degree 3 expansion to deal with the denominator.

		$x=0$
Numerator:	$xe^x + x - 2e^x + 2$	0
1	$e^x + xe^x + 1 - 2e^x$	0
2	$e^x + e^x + xe^x - 2e^x = xe^x$	0
3	$e^x + xe^x$	1

Thus $x(e^x+1)-2(e^x-1) = \frac{1}{6}x^3 + x^3 \cdot \varepsilon(x)$,

as all lower degree terms vanish.

$$\lim_{x \to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3} = \lim_{x \to 0} \frac{\frac{1}{6}x^3 + x^3 \cdot \varepsilon(x)}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{6} + \varepsilon(x) \right) = \frac{1}{6}.$$

d) Use $f(x) = P_1(x) + R_1(x)$ to compute b) more gracefully.

$$\lim_{x \to 0} \frac{\ln(1+x) - x}{x^2} = \lim_{x \to 0} \frac{P_1(x) + R_1(x) - x}{x^2}$$

$$= \lim_{x \to 0} \frac{x - \frac{1}{2(1+\xi)^2} \cdot x^2 - x}{x^2} = \lim_{x \to 0} -\frac{1}{2(1+\xi)^2} = -\frac{1}{2}.$$

Since $\xi \in]0,x[$ it follos that $\xi \to 0$ as $x \to 0$.