Affin avmyndan í 3D

Vit hava framvegis, at $x' = \underline{A}(x - o) + p$, ella $x' = \underline{A}x + p$ er ein affin avmyndan av x yvir í x'. Her er $\underline{A}$ ein 3×3 matrica, so at $[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ skipanin verður broytt svarandi til $\underline{A} = [\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3]$. Punktið p er ein translatión av origo, altso vit velja hvar vit ynskja at avseta vektorin $\underline{A}x$.

1. Affinar avmyndanir varðveita lutføll, eins og í ch. 6.
2. Parallel plan eru parallel eftir affina avmyndan, eins og linjur í ch. 6.
3. Plan sum skerast enda í plan sum skerast. Skeringslinjan eftir avmyndan er sama sum at avmynda upprunaligu skeringslinjuna.
4. Barycentriskar kombinatiónir eru invariantar undir affina avmyndan.

$$x = c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4$$

har $c_1 + c_2 + c_3 + c_4 = 1$, so er

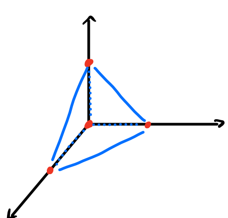
$$x' = c_1 p_1' + c_2 p_2' + c_3 p_3' + c_4 p_4' .$$

Eitt punkt í einum tetraeder (fýraflatningur við fýra tríhantaðum síðum) kann tískil lýsast við koordinat í mun til hornini.

Flyta punkt Beinleiðis translatión uttan aðra ávirkan skrivast

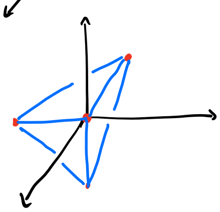
$$x' = Ix + p \qquad \text{við} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Dømi 10.1



Vit avmynda $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

yvir í $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$



Uttan avmyndan kunnu vit brúka barycentrisk koordinat á $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$x = -2 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad x' = -2 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Givið koordinatir p_1, p_2, p_3, p_4 og p_1', p_2', p_3', p_4', so kunnu vit finna $\underline{\underline{A}}$.
Vel eitt punkt p, t.d. p_1 og konstruera vektorarnar

$$v_2 = p_2 - p_1, \quad v_3 = p_3 - p_1, \quad v_4 = p_4 - p_1$$

og

$$v_2' = p_2' - p_1', \quad v_3' = p_3' - p_1', \quad v_4' = p_4' - p_1'$$

Vit hava, at $x' = \underline{\underline{A}}(x - p_1) + p_1$ (vit translatera til origo og út!).
So er

$$\underline{\underline{A}}\,\underline{v}_2 = \underline{v}_2', \quad \underline{\underline{A}}\,\underline{v}_3 = \underline{v}_3', \quad \underline{\underline{A}}\,\underline{v}_4 = \underline{v}_4'$$

$$\Rightarrow \quad \underline{\underline{A}}\left[\underline{v}_2 \;\; \underline{v}_3 \;\; \underline{v}_4\right] = \left[\underline{v}_2' \;\; \underline{v}_3' \;\; \underline{v}_4'\right]$$

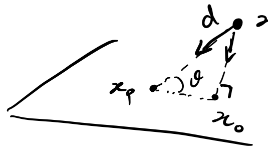
$$\Leftrightarrow \quad \underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}'$$

$$\Leftrightarrow \quad \underline{\underline{A}} = \underline{\underline{V}}'\,\underline{\underline{V}}^{-1}$$

Til dømi 10.1 sæst at $\underline{\underline{A}} = -\underline{\underline{I}}$

Parallelar projektiónir

Vit fara nú at projektera á ein flata í ein ávísan rætning.

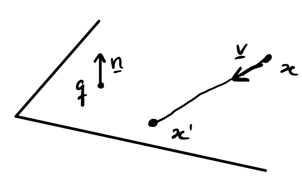


x_0 svarar til ortogonal projektión.
x_p er projektión parallel við flatan.

Vinkulin θ vísir á, um talan er um oblique projektión.

Ein flati er givin við punkt q og normal $\underline{n}$. Um x projekterast á x' í flatanum, so vil

$$(x' - q)\cdot \underline{n} = 0.$$



Hartil er ein linja úr x, sum skerir planið í x'. Lat projektiónsrætningin vera $\underline{v}$.

$$x' = x + t\underline{v}$$

Nú er

$$(x' - q)\cdot \underline{n} = 0$$

$$\Leftrightarrow \quad (x + t\underline{v} - q)\cdot \underline{n} = 0$$

$$\Leftrightarrow \quad (x - q)\cdot \underline{n} + t\underline{v}\cdot \underline{n} = 0$$

$$\Leftrightarrow \quad t = \frac{(q - x)\cdot \underline{n}}{\underline{v}\cdot \underline{n}}, \qquad \underline{v}\cdot\underline{n} = 0 \Leftrightarrow \underline{v} \perp \underline{n}$$

$$\Leftrightarrow P \parallel \underline{v}$$

Skeringspunktið er altso

$$x' = x + \frac{(q - x)\cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v} \qquad (*)$$

Dømi 2019

P: $2x_1 + 2x_2 - x_3 + 2 = 0$, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ og $\underline{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.

Hvat er projektiónin x' av x á P eftir $\underline{v}$?

Tá $x_1 = x_2 = 0$, so er $-x_3 + 2 = 0 \iff x_3 = 2$, so $q = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

$$\underline{n} = \frac{1}{\sqrt{2^2 + 2^2 + (-1)^2}} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

$$x' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-5/3}{-1/3} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -10 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ -4 \end{bmatrix}$$

(*) Hetta er eitt fínt útrykk, men lat okkum kanna, at tað er affint.

Nær brúka vit at $\|\underline{n}\| = 1$?

$$x' = x + \frac{(q - x)\cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v} = \underline{\underline{I}} x - \frac{x \cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v} + \frac{q \cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v}$$

$$(x \cdot \underline{n})\, \underline{v} = (\underline{n}^T x)\, \underline{v} = \underline{v}\, (\underline{n}^T x) = (\underline{v}\, \underline{n}^T) x$$

$$\Rightarrow x' = \underline{\underline{I}} x - \frac{\underline{v}\, \underline{n}^T}{\underline{v}\ \underline{n}} x + \frac{q \cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v} = \overbrace{\left(\underline{\underline{I}} - \frac{\underline{v}\, \underline{n}^T}{\underline{v} \cdot \underline{n}}\right)}^{\underline{\underline{A}}} x + \overbrace{\frac{q \cdot \underline{n}}{\underline{v} \cdot \underline{n}} \underline{v}}^{p}$$

Sum vant er projektiónin idempotent, so at

$$\underline{\underline{A}}\left(\underline{\underline{A}}x+p\right)+p = Ax+p$$

Prógv.

$$\underline{\underline{A}}^2 = \left(\underline{\underline{I}} - \frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}}\right)\left(\underline{\underline{I}} - \frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}}\right) = \underline{\underline{I}}^2 - 2\frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}} + \left(\frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}}\right)^2$$

$$= \underline{\underline{A}} - \frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}} + \frac{\underline{v}(\underline{n}^T\underline{v})\underline{n}^T}{(\underline{v}\cdot\underline{n})^2} = \underline{\underline{A}}$$

$$\underline{\underline{A}}\left(\underline{\underline{A}}x+p\right)+p = \underline{\underline{A}}^2x + \underline{\underline{A}}p + p = \underline{\underline{A}}x + \underline{\underline{A}}p + p$$

$$\underline{\underline{A}}p = \underline{\underline{A}}\frac{q\cdot\underline{n}}{\underline{v}\cdot\underline{n}}\underline{v} = \left(\underline{\underline{I}} - \frac{\underline{v}\,\underline{n}^T}{\underline{v}\cdot\underline{n}}\right)\underbrace{\frac{q\cdot\underline{n}}{\underline{v}\cdot\underline{n}}}_{\alpha}\underline{v}$$

$$= \alpha\underline{\underline{I}}\,\underline{v} - \alpha\underline{v}\left(\frac{\underline{n}^T\underline{v}}{\underline{v}\cdot\underline{n}}\right) = 0$$

Dømi 10.3 P: $x_1 + x_2 + x_3 - 1 = 0$, $x = \begin{bmatrix}3\\2\\4\end{bmatrix}$ og $\underline{v} = \begin{bmatrix}0\\0\\-1\end{bmatrix}$

Her er $q = \begin{bmatrix}0\\0\\1\end{bmatrix}$ eitt val.

$$\underline{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$x' = \begin{bmatrix}3\\2\\4\end{bmatrix} + \frac{\left(\begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}3\\2\\4\end{bmatrix}\right)\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$= \begin{bmatrix}3\\2\\4\end{bmatrix} + \frac{-3-2-3}{-1}\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}3\\2\\4\end{bmatrix} + 8\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}3\\2\\-4\end{bmatrix}$$

Perspective projection

Vit kunnu við projektiónini $x' = x + \frac{(q-x)\cdot\underline{n}}{\underline{v}\cdot\underline{n}}\underline{v}$ fremja eina projektión í mun til perspektiv. Hetta gera vit við at velja perspektivið fyri hvørt x heldur enn at brúka ein fastan rætning $\underline{v}$ til at projektera. Um vit vilja hava Perspektivu projektiónina av x á P ímóti origo, so er $\underline{v} = -x$.
Vilja vit projicera á y, so skriva vit $\underline{v} = y - x$

Tískil fáa vit eina nýggja projektión fyri hvørt x.

$$x' = x + \frac{(q - x)\cdot \underline{n}}{x \cdot \underline{n}} x = x + \frac{q \cdot \underline{n}}{x \cdot \underline{n}} x - \frac{x \cdot \underline{n}}{x \cdot \underline{n}} x$$

$$= \frac{q \cdot \underline{n}}{x \cdot \underline{n}} x$$

Dømi 2021 $P: x_1 + x_2 + x_3 - 1 = 0 \quad , \quad x = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$

Perspektiv projektión av x á P ímóti origo.

$$x' = \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{1 + 3 + 2} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/2 \\ 1/3 \end{bmatrix}$$

Minst til at hetta ikki kann vera affint, tí lutfell millum punkt broytast fyri at gera perspektiv tekningar í 2D.

Dømi 10.5   Lat P vera $x_3 = 1$, so kunnu vit velja $q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
$\underline{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so $q \cdot \underline{n} = 1$ og $x \cdot \underline{n} = x_3$.

Perspektiv projektiónin er altso $x' = \frac{1}{x_3} x$.

Lat $\underline{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$

$$\Rightarrow \quad \underline{x}_1' = \frac{1}{4} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Her var $\underline{x}_2 = \frac{1}{2} \underline{x}_1 + \frac{1}{2} \underline{x}_3$, men $\underline{x}_2' = \frac{2}{3} \underline{x}_1' + \frac{1}{3} \underline{x}_3'$.

$\mathbb{R}^4$ Tað ber til hjá okkum at gera $x' = \underline{\underline{A}} x + p$ til homogena skipan

$$x' = \underline{\underline{M}} x \quad , \quad \text{har} \quad \underline{\underline{M}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & p_1 \\ a_{21} & a_{22} & a_{23} & p_2 \\ a_{31} & a_{32} & a_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Stytt so $x_4 = 1$.