Taylor's Formulas

Ex1. a) Taylor expansion of

$$f(x) = 2\cos(x) - 2\sin(x)$$

by using the limit formula to the second degree (4.8) at $x_0 = 0$.

$f(0) = 2$, $f'(x) = -2\sin(x) - 4\cos(2x)$

$f'(0) = -4$,

$f''(x) = -2\cos(x) + 8\sin(2x)$

$f''(0) = -2$.

Thus $f(x) = 2 - 4x - x^2 + x^2 \cdot \varepsilon(x)$

b) A smooth function f of one variable fulfills that $f(2) = 1$ and $f'(2) = 1$. The second degree Taylor polynomial satisfies with $x_0 = 2$ that $P_2(1) = 1$. Determine $P_2(x)$.

$$P_2(x) = 1 + (x-2) + \frac{1}{2} \cdot f''(2)\,(x-2)^2$$

$$P_2(1) = 1 = 1 + (1-2) + \frac{1}{2} f''(2)\,(1-2)^2$$

$$= \frac{1}{2} \cdot f''(2)$$

$$\Leftrightarrow f''(2) = 2$$

So we have $P_2(x) = -1 + x + (x-2)^2$.

Ex 2. We're given

$$f(x,y) = e^{x + xy - 2y}, \quad (x,y) \in \mathbb{R}^2.$$

a) State the limit formula of second degree for f at $(x_0,y_0) = (0,0)$.

Use 21.7

$$f(0,0) = 1$$

$$f_x'(x,y) = e^{x+xy-2y} \cdot (1+y) \qquad f_x'(0,0) = 1$$

$$f_y'(x,y) = e^{x+xy-2y} \cdot (x-2) \qquad f_y'(0,0) = -2$$

$$f_{xy}''(x,y) = e^{x+xy-2y}(x-2)(1+y) + e^{x+xy-2y} \qquad f_{xy}''(0,0) = -1$$

$$f_{yx}''(x,y) = e^{x+xy-2y} \cdot (1+y)(x-2) + e^{x+xy-2y} \qquad f_{yx}''(0,0) = -1$$

$$f_{xx}''(x,y) = e^{x+xy-2y} \cdot (1+y)^2 \qquad f_{xx}''(0,0) = 1$$

$$f_{yy}''(x,y) = e^{x+xy-2y} \cdot (x-2)^2 \qquad f_{yy}''(0,0) = 4$$

$$f(x,y) = 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2 + \rho_{(0,0)}^2(x,y) \cdot \varepsilon(x,y).$$

b) Compute $\nabla f(0,0)$ and the Hessian matrix $Hf(0,0)$.

Using a) we have

$$\nabla f(0,0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad Hf(0,0) = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}.$$

c) Redo a), but on matrix form.

Using 21.8 we have

$$f(x,y) = 1 + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$+ \rho^2_{(0,0)}(x,y) \cdot \varepsilon(x,y).$$

d) Approximate f by P_2 at $(0,0)$ and Q_2 at $(1,1)$.

By a) it follows that

$$P_2(x,y) = 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2,$$

and

$$Q_2(x,y) = \frac{3}{2} - x - y + 2x^2 - xy + \frac{1}{2}y^2.$$

e) Compute P_2 and Q_2 at $\left(\frac{3}{4}, \frac{1}{2}\right)$. Compare with $f\left(\frac{3}{4}, \frac{1}{2}\right)$.

$$f\left(\frac{3}{4}, \frac{1}{2}\right) = e^{\frac{1}{8}} = 1,13315 \quad \text{(5 dec.)}$$

$$P_2\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{37}{32} = 1,15625$$

$$Q_2\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{9}{8} = 1,125$$

Evidently Q_2 approximates f better at $\left(\frac{3}{4}, \frac{1}{2}\right)$ than P_2.
The point is also closer to Q_2's expansion point.

Ex 3. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \sqrt{x^2+y^2}.$$

a) Determine P_2 for f at $(3,4)$.

```
> f:=(x,y)-> sqrt(x^2+y^2);
```

$$f := (x,y) \mapsto \sqrt{y^2+x^2}$$

```
> P2:=unapply(mtaylor(f(x,y),[x=3,y=4],3),(x,y));
```

$$P2 := (x,y) \mapsto \frac{3\cdot x}{5} + \frac{4\cdot y}{5} + \frac{8\cdot(x-3)^2}{125} - \frac{12\cdot(x-3)\cdot(y-4)}{125} + \frac{9\cdot(y-4)^2}{250}$$

```
> expand(P2(x,y));
```

$$\frac{3}{5}x + \frac{4}{5}y + \frac{8}{125}x^2 - \frac{12}{125}xy + \frac{9}{250}y^2$$

b) Determine using P_2 the diagonal length of a rectangle with sides 2.9 and 4.2.

```
> P2(2.9,4.2);
```

5.104000000

```
> f(2.9,4.2);
```

5.103920062

c) Compare with the exact value.

As shown above P_2 overshoots slightly. At this order of magnitude it could be irrelevant, but this entirely depends on the need for accuracy.

Ex 4. We're given

$$\underline{\underline{A}} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

a) State $\underline{\underline{Q}}$ and $\underline{\underline{\Lambda}}$, such that $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$ or $\underline{\underline{\Lambda}} = \underline{\underline{Q}}^T\underline{\underline{A}}\,\underline{\underline{Q}}$.

```
> A:=<-2,1,1; 1,-2,-1; 1,-1,-2>;
```

$$A := \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

```
> Eigenvectors(A,output=list);
```

$$\left[\left[-4, 1, \left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}\right], \left[-1, 2, \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}\right]\right]$$

```
> q1:=1/sqrt(3)*<-1,1,1>:
  q2:=1/sqrt(2)*<1,0,1>:
  q3:=kryds(q1,q2):
> Q:=<q1|q2|q3>;
  L:=<-4,0,0; 0,-1,0; 0,0,-1>;
```

$$Q := \begin{bmatrix} -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}\sqrt{2}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}\sqrt{2}}{6} \end{bmatrix}$$

$$L := \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> Transpose(Q).A.Q;
```

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now consider

$$f(x,y,z) = -2x^2 - 2y^2 - 2z^2 + 2xy + 2xz - 2yz + 2x + y + z + 5 .$$

b) State the quadratic form $k(x,y,z)$. Reduce it and write in a basis without mixed terms.

$$k(x,y,z) = -2x^2 - 2y^2 - 2z^2 + 2xy + 2xz - 2yz$$
$$= [x \ \ y \ \ z] \ \underline{\underline{A}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

If we write this w.r.t. the ortonormalbasis of $\underline{\underline{Q}}$, then

$$k(\tilde{x}, \tilde{y}, \tilde{z}) = -4\tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = [\tilde{x} \ \ \tilde{y} \ \ \tilde{z}] \ \underline{\underline{\Lambda}} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} .$$

c) State the ordinary ONB for $\mathbb{R}^3$ in which f has no mixed terms. Determine this expression of f.

We'll continue with the ONB supplied by Q, and write the linear terms as

$$[2 \ \ 1 \ \ 1] \ \underline{\underline{Q}} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \frac{3\sqrt{2}}{2} \tilde{y} + \frac{\sqrt{6}}{2} z$$

Thus $f(x,y,z) = f(\tilde{x},\tilde{y},\tilde{z})$

$$= -4\tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 + \frac{3\sqrt{2}}{2}\tilde{y} + \frac{\sqrt{6}}{2}z + 5$$

Ex5. We're given

$$f(x,y) = x^3 - 3x^2 + y^3 - 3y^2, \quad (x,y) \in \mathbb{R}^2,$$

and the set

$$A = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2,\ -2 \le y \le 2\}.$$

a) Consider M to be an elevated surface copy of A.

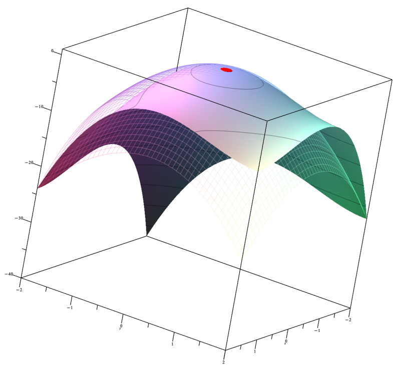
Illustrate in Maple.

The contour lines represent M. Note the approximating polynomial.

```
> f:=(x,y) -> x^3-3*x^2+y^3-3*y^2:
  f(x,y);
```

$$x^3 + y^3 - 3x^2 - 3y^2$$

```
> graph:=plot3d(f(x,y),x=-2..2,y=-2..2,style=patchcontour):
  M:=plot3d(mtaylor(f(x,y),[x=0,y=0],3),x=-2..2,y=-2..2,style=wireframe):
  p1:=plottools[sphere]([0,0,f(0,0)],0.1,style=patchnogrid,color=red):
  display(graph,M,p1,orientation=[60,70,30],view=-40..1);
```



b) What is the largest value f attains along the boundary?

These are functions of one variable, so we just check all sides.

We find that $f(0,2) = f(2,0) = -4$ is th maximum along the boundary by the following investigation

```
> f(x,-2);
  solve(diff(f(x,-2),x)=0,x);
  f(0,-2);
  f(2,-2);
```

$$x^3 - 3x^2 - 20$$
$$0, 2$$
$$-20$$
$$-24$$

```
> f(x,2);
  solve(diff(f(x,2),x)=0,x);
  f(0,2);
  f(2,2);
```

$$x^3 - 3x^2 - 4$$
$$0, 2$$
$$-4$$
$$-8$$

```
> f(-2,y);
  solve(diff(f(-2,y),y)=0,y);
  f(-2,0);
  f(-2,2);
```

$$y^3 - 3y^2 - 20$$
$$0, 2$$
$$-20$$
$$-24$$

```
> f(2,y);
  solve(diff(f(2,y),y)=0,y);
  f(2,0);
  f(2,2);
```

$$y^3 - 3y^2 - 4$$
$$0, 2$$
$$-4$$
$$-8$$

```
> f(-2,-2); f(-2,2); f(2,-2); f(2,2);
```

$$-40$$
$$-24$$
$$-24$$
$$-8$$

c) Compute the normal vector for the tangent plane at

$$R = (0, 0, f(0,0)) = (0, 0, 0).$$

Justify based on this that $(0,0)$ is a stationary point.

$$z = 0 \implies \underline{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The derivatives $f_x'(0,0)$ and $f_y'(0,0)$ are necessarily 0 (see 20.13), and so $\nabla f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which tells us we have a stationary point.

d) Discuss maximum, local maximum and proper local maximum.

A maximum is generally the point at which the largest value is attained (globally). A local maximum is largest in a sufficiently small neighbourhood, $f(x,y) \le f(x_0,y_0)$.

If $f(x,y) < f(x_0,y_0)$ in a neighbourhood, then $f(x_0,y_0)$ is a proper maximal value. Def. 21.15.

e) 1. Plot P_2 with f. We did this above.

2. If the observation is correct, then $f(x,y)$ is negative in a neighbourhood of $(0,0)$. Show this with the limit formula.

$$f(x,y) = \frac{1}{2}\left(-6 \cdot x^2 - 6 \cdot y^2\right) + \rho^2_{(0,0)}(x,y) \cdot \varepsilon(x,y)$$

$$= -3\left(x^2+y^2\right) + \rho^2_{(0,0)}(x,y) \cdot \varepsilon(x,y)$$

$$= \left(-3 + \varepsilon(x,y)\right)\left(x^2+y^2\right)$$

For $(x,y) \to 0$ we have $\varepsilon(x,y) \to 0$, hence the above is negative around $(0,0)$, and we have a proper local maximum.

f) Try again with $(2,2)$.

$$f(x,y) = 3(x-2)^2 + 3(y-2)^2 - 8 + \rho^2_{(2,2)}(x-2,y-2) \cdot \varepsilon(x-2,y-2)$$

$$= \left(3 + \varepsilon(x-2,y-2)\right)\left((x-2)^2 + (y-2)^2\right) - 8$$

This approaches $3 - 8$, which is negative but greater than -8 around $(2,2)$ (remember $f(2,2) = -8$), so we have a proper local minimum.

g) Compute ∇f and find stationary points.

$$\nabla f(x,y) = \begin{bmatrix} 3x^2 - 6x \\ 3y^2 - 6y \end{bmatrix}, \quad \nabla f = \underline{0} \iff (x,y) = (0,0), (0,2), (2,0), (2,2).$$

h) What is the global maximum?

We have cleared the boundary in b), so from attained values we may assert the global maximum to be

$$f(0,0) = 0.$$

Ex 6. Let $f \in C^\infty(\mathbb{R}^2)$ and

$$f(x,0) = e^x \quad \text{and} \quad f_y'(x,y) = 2y \cdot f(x,y).$$

a) Find P_2 of f at $(0,0)$.

Apply 21.5

$$f(0,0) = 1, \quad f_x'(0,0) = 1, \quad f_{xx}''(0,0) = 1$$

$$f_y'(0,0) = 0 \qquad f_{xy}''(0,0) = f_{yx}''(0,0) = 2 \cdot 0 \cdot f_x'(0,0) = 0$$

$$f_{yy}''(x,y) = 2 \cdot f(x,y) + 2y \cdot f_y'(x,y) = 2f(x,y) + 4y^2 \cdot f(x,y)$$

$$f_{yy}''(0,0) = 2$$

$$P_2(x,y) = 1 + x + \frac{1}{2}x^2 + y^2$$