Próvtøka 13. mai 2017

Opg. 1. (i) $\sum_{n=1}^{\infty} (-1)^n \frac{3}{2n^2+n}$

Vi hava, at $\sum_{n=1}^{\infty} \left|(-1)^n \frac{3}{2n^2+n}\right| = \sum_{n=1}^{\infty} \frac{3}{2n^2+n}$ og $\frac{3}{2n^2+n} \leq \frac{1}{n^2}$ $\forall n \in \mathbb{N}$.
Per samanbering við konvergentu rekkjuni $\sum \frac{1}{n^2}$ er rekkjan absolut konvergent.

(ii) $\sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{n!}\right) x^n$

Her eru $\sum_{n=0}^{\infty} (-x)^n + \sum_{n=0}^{\infty} \frac{x^n}{n!}$ tvær konvergentar rekkjur fyri $\rho = 1$ og $\rho = \infty$. Teirri summur er altso konvergent fyri $\rho = 1$.

(iii) $y'' + 4y' + 4y = 0$, $P(\lambda) = (\lambda + 2)^2$.

Her er -2 dupultrót í $P(\lambda)$, so $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$, $c_1, c_2 \in \mathbb{R}$.

(iv) $P(\lambda) = \lambda^3 + 2\lambda^2 + (2-c)\lambda + 2c$, $c \in \mathbb{R}$.

Við Routh-Hurwitz: $2-c > 0$ og $2c > 0$, so $c > 0$ og $2 > c$.
Altso $0 < c < 2$. Determinanturin er

$$\det \begin{pmatrix} 2 & 2c \\ 1 & 2-c \end{pmatrix} = 4 - 2c - 2c = 4 - 4c > 0 \iff c < 1.$$

So $0 < c < 1$.

(v) $\sum_{n=1}^{\infty} (2-x)^n$ er konvergent, um $|2-x| < 1 \iff -1 < 2-x < 1$
$\iff -3 < -x < -1$
$\iff 3 > x > 1$.

(vi) Funktiónin $f(x) = 1 - \sin^2\left(\frac{x}{2}\right)$ hevur Fourierkoefficientar

$b_n = 0 \quad \forall n \in \mathbb{N}$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} \cos^2\left(\frac{x}{2}\right) dx = \frac{2}{\pi} \left[\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) + \frac{x}{2}\right]_0^{\pi} = 1.$$

$a_1 = \frac{1}{2}$ og $a_n = 0$ fyri $n \geq 2$. (Expand).

Opg. 2 $\qquad \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = u(t) \quad , \quad P(\lambda) = (\lambda^2+1)(\lambda+1)$

(i) Loysn við $\lambda = \pm i$ og $\lambda = -1$. Við 1.15

$$y(t) = c_1 e^{it} + c_2 e^{-it} + c_3 e^{-t} \quad , \quad c_1, c_2, c_3 \in \mathbb{C}.$$

$$y(t) = k_1 \cos(t) + k_2 \sin(t) + k_3 e^{-t} \quad , \quad k_1, k_2, k_3 \in \mathbb{R}.$$

(ii) $H(s)$ við $u(t) = e^{st}$.

Við (1.20) : $\quad H(s) = \dfrac{1}{(s^2+1)(s+1)} \quad , \quad s \notin \{\pm i, -1\}.$

(iii) Stationert svar, 1.27, hjá $u(t) = \cos(2t)$.

$$y(t) = Re\left(H(2i) \cdot e^{2it}\right) = Re\left(\left(-\tfrac{1}{15} + \tfrac{2}{15} i\right) \cdot \left(\cos(2t) + i \sin(2t)\right)\right)$$

$$= -\frac{1}{15}\cos(2t) - \frac{2}{15}\sin(2t).$$

(iv) Partikuler loysn til $u(t) = e^{-t}$.

Yvirførslufunktiónin er ikki definerað í $s=-1$, so vit gita. Funktiónin $y(t) = e^{-t}$ er ein homogen loysn, so vit gita $y(t) = cte^{-t}$.

$$\mathcal{D}_3(y) = u(t) \iff \left(3ce^{-t} - cte^{-t}\right) + \left(-2ce^{-t} + cte^{-t}\right) + \left(ce^{-t} - cte^{-t}\right) + cte^{-t} = e^{-t}$$

$$\iff \quad 3c - 2c + c = 1$$

$$\iff \quad c = \tfrac{1}{2}.$$

$$y(t) = \tfrac{1}{2} t e^{-t}.$$

(v) Skriva reellu loysnina til $u(t) = e^{-t} + \cos(2t)$.

Per (i), (iii), (iv) og superpositión

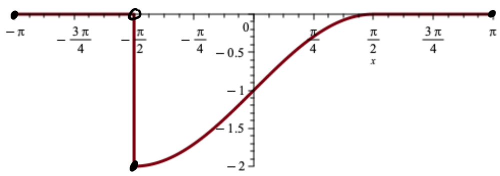
$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 e^{-t} - \frac{1}{15}\cos(2t) - \frac{2}{15}\sin(2t) + \frac{1}{2} t e^{-t},$$

$$c_1, c_2, c_3 \in \mathbb{R}.$$

Opg. 3.

$$f(x) = \begin{cases} \sin(x) - 1 , & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 , & x \in [-\pi, -\frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi] . \end{cases}$$

(i) Tekna grafin hjá f á $[-\pi,\pi]$.



(ii) Finn summin hjá Fourierrekkjuni hjá f í $x=-\pi$, $x=-\frac{\pi}{2}$, $x=0$, $x=\frac{\pi}{2}$ og $x=\pi$.

Her er $\sin(x)-1$ differentiabul á $\mathbb{R}$ við kontinuerta avleidda funktión $\cos(x)$ á $\mathbb{R}$. Tí er f bæði 2π-periodisk og stykkivís differentiabul. Per Fouriers setning er f eins við Fourierrekkjuni, har f er kontinuert, so rekkjan samsvarar við

$$f(-\pi) = f(\pi) = f(\tfrac{\pi}{2}) = 0$$

$$f(0) = -1$$

Í $x = -\frac{\pi}{2}$ konvergerar Fourierekkjan ímóti $\dfrac{f(-\frac{\pi}{2}^+) + f(-\frac{\pi}{2}^-)}{2} = -1.$

(iii) Finn Fourierkoefficientarnar hjá f.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x) - 1\,dx = \frac{1}{\pi}\left[-\cos(x) - x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -1.$$

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin(x)-1)\cos(nx)\,dx = -\frac{2\sin(\frac{n\pi}{2})}{\pi n} = \begin{cases} 0 , & n \text{ lika} \\ \frac{2}{\pi n} , & n = 4k-1 \\ -\frac{2}{\pi n} , & n = 4k+1 \end{cases} \quad k \in \mathbb{N}$$

$$a_1 = -\frac{2}{\pi} , \quad b_1 = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin(x)-1)\sin(nx)\,dx = -\frac{2\cos(\frac{n\pi}{2})\,n}{\pi(n^2-1)} = \begin{cases} 0 , & n>1 \text{ ólika} \\ \frac{2n}{\pi(n^2-1)} , & n = 4k+2 \\ -\frac{2n}{\pi(n^2-1)} , & n = 4k \end{cases} \quad k \in \mathbb{N}.$$

Opg. 4 $\qquad t \dfrac{d^2y}{dt^2} + y = t^2$

(i) Finn y_p við at gita.

Lat $y(t) = at^2+bt+c$, so er við innsetan

$$t(2a) + at^2+bt+c = t^2$$

$$\Leftrightarrow \quad at^2 + (2a+b)t + c = t^2$$

$$\Leftrightarrow \quad a=1, \quad b=-2 \quad \text{og} \quad c=0$$

Altso er $y_p(t) = t^2 - 2t$.

(ii) Lat $y(t) = \sum_{n=0}^{\infty} a_n t^n$ vera loysn. Avger rekursiónsformul fyri a_n.

$$t\left(\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2}\right) + \sum_{n=0}^{\infty} a_n t^n = \sum_{n=2}^{\infty} a_n n(n-1)t^{n-1} + a_0 + \sum_{n=1}^{\infty} a_n t^n$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_{n+1}(n+1)n + a_n\right)t^n = 0$$

$$\overset{5.21}{\Leftrightarrow} \quad a_0 = 0 \quad \text{og} \quad a_{n+1}(n+1)n + a_n = 0$$

$$\Leftrightarrow \quad a_{n+1} = -\frac{a_n}{n^2+n}, \quad n \in \mathbb{N}.$$