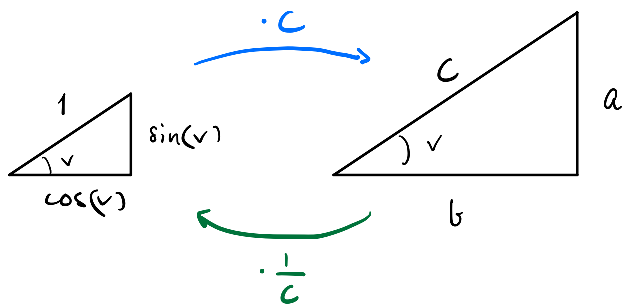
Trig, powers, exponentials and inverse functions

Ex1. $$\frac{2^{-1}\cdot 2^{4}\cdot (2^{3})^{-2}}{2^{-5}} = \frac{2^{4}\cdot 2^{5}}{2\cdot 2^{6}} = 2^{2} = 4$$

Ex2.
a)

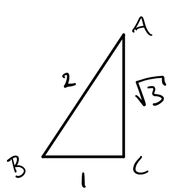


Proportional: dividing the arbitrary triangle by c in turn yield

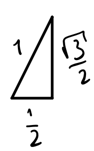
$$\cos(v) = \frac{b}{c} \quad \text{and} \quad \sin(v) = \frac{a}{c}$$

A scaling factor c or $\frac{1}{c}$ allways exist for similar objects.

b)



The angles of the triangle are well known, but let's scale by $\frac{1}{2}$, so we have:



These coordinates yield

$\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$.

c) We have $\cos^2(v)+\sin^2(v) = 1^2$, which follows from the Pythagorean theorem, i.e. hypotenuse has length 1, so the sides are by definition $\cos(v)$ and $\sin(v)$.

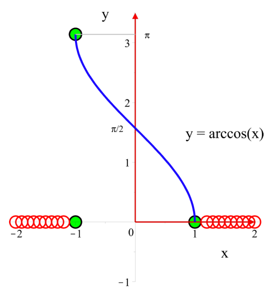
Ex 3.

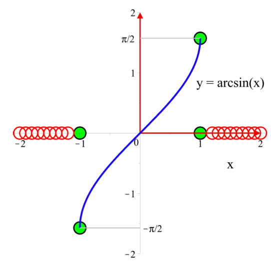
a) Compute (Remember domains! Figure 3.9)

$$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\arcsin(1) = \frac{\pi}{2}$$





Let $A = \{x \in \mathbb{R} \mid x \in [0, 2\pi]\}$ and $B = \{x \in \mathbb{R} \mid x \in [-\pi, \pi]\}$.

b) Solve for sets A, B and $\mathbb{R}$.

$$\cos(x) = \frac{1}{2} \iff x = \frac{\pi}{3} \vee x = \frac{5\pi}{3}, \quad x \in A.$$

$$x = \frac{\pi}{3} \vee x = -\frac{\pi}{3}, \quad x \in B.$$

$$x = \frac{\pi}{3} + 2p\pi \vee x = -\frac{\pi}{3} + 2p\pi, \quad p \in \mathbb{Z},\ x \in \mathbb{R}.$$

c)

$$\sin(x) = -\frac{\sqrt{3}}{2} \iff x = \frac{4\pi}{3} \vee x = \frac{5\pi}{3}, \quad x \in A.$$

$$x = -\frac{2\pi}{3} \vee x = -\frac{\pi}{3}, \quad x \in B.$$

$$x = \frac{4\pi}{3} + 2p\pi \vee x = \frac{5\pi}{3} + 2p\pi, \quad p \in \mathbb{Z},\ x \in \mathbb{R}.$$

d)

$$e^{i \cdot v} = \frac{1}{2} - \frac{\sqrt{3}}{2} i \quad \text{for sets } A \text{ and } B.$$

$$\iff v = \frac{5\pi}{3}, \quad v \in A$$

$$v = -\frac{\pi}{3}, \quad v \in B.$$

Ex 4. Write as a power.

$$3^2 \cdot 3^3 = 3^5, \qquad (5^8)^{-2} = 5^{-16},$$

$$3^2 \cdot 3^{-5} = 3^{-3}, \qquad \frac{4^{1,3}}{4^{2,3}} = 4^{-1},$$

$$\left(\frac{1}{2}\right)^5 \cdot 6^5 = 3^5, \qquad \frac{5^3}{0{,}5^3} = 10^3.$$

Ex 5. Let $z = 1+i$ and compute

$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\sin(v) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos(v) \Rightarrow \operatorname{Arg}(z) = \frac{\pi}{4}.$$

Use the result to determine moduli and Arg for:

z^2, z^5, z^8 and z^{-10}.

$$|z^2| = |z|^2 = \sqrt{2}^2 = 2, \quad \operatorname{Arg}(z^2) = \operatorname{Arg}(z) + \operatorname{Arg}(z) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

$$|z^5| = \sqrt{2}^5 = 4\sqrt{2}, \quad \operatorname{Arg}(z^5) = \left[\frac{5\pi}{4}\right] = -\frac{3\pi}{4}$$

$$|z^8| = \sqrt{2}^8 = 16, \quad \operatorname{Arg}(z^8) = \left[\frac{8\pi}{4}\right] = [2\pi] = 0$$

$$|z^{-10}| = \sqrt{2}^{-10} = \frac{1}{32}, \quad \operatorname{Arg}(z^{-10}) = \left[-10\frac{\pi}{4}\right] = -\frac{\pi}{2}$$

Provide the rectangular form of the numbers above.

$$z^2 = 2i, \qquad z^5 = 4\sqrt{2} \cdot \left(\cos\left(-\frac{3\pi}{4}\right) + i \cdot \sin\left(-\frac{3\pi}{4}\right)\right)$$

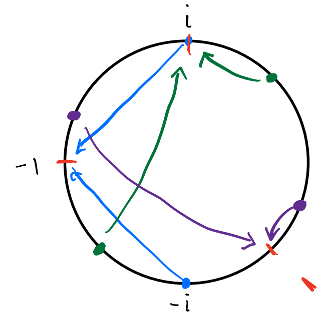
$$= 4\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = -4-4i,$$

$$z^8 = 16,$$

$$z^{-10} = -\frac{1}{32} i.$$

Ex6. Binomial equations

a) Solve the equations.



$$z^2 = -4 \iff z = \pm 2i$$

$$z^2 = i \iff z = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

$$z^2 = 1 - i \iff$$

$$z = 2^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{8}\right) - i \sin\left(\frac{\pi}{8}\right)\right) = 2^{\frac{1}{4}}\left(\frac{1}{2}\sqrt{2+\sqrt{2}} - i \cdot \frac{1}{2}\sqrt{2-\sqrt{2}}\right)$$

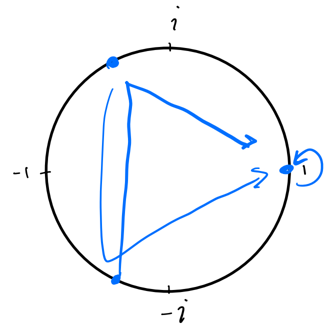
$$\vee \quad z = 2^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{8}\right) + i \sin\left(\frac{7\pi}{8}\right)\right)$$

$$= 2^{\frac{1}{4}}\left(-\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right)\right)$$

$\boxed{\text{sine is odd and cosine is even}}$

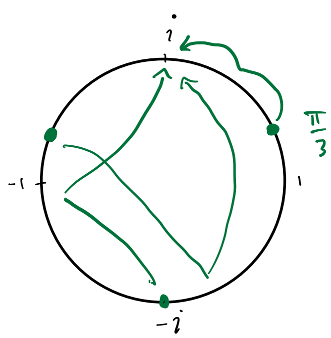
b)

$$z^3 = 1 \iff z = 1 \ \vee \ z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$



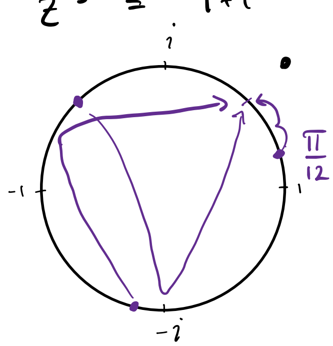
$$z^3 = -i \iff z = \frac{\sqrt{3}}{2} + \frac{1}{2} i \ \vee \ z = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$$

$$\vee \quad z = -i$$



$$z^3 = 1 + i \iff z = 2^{\frac{1}{6}} e^{\frac{\pi}{12} i} \ \vee \ z = 2^{\frac{1}{6}} e^{-\frac{7\pi}{12} i}$$

$$\vee \quad z = 2^{\frac{1}{6}} e^{\frac{3\pi}{4} i}$$



$$|z^3| = \sqrt{2} = 2^{\frac{1}{2}} \Rightarrow |z| = 2^{\frac{1}{6}}.$$

$$\left(\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}\sqrt{2+\sqrt{3}}, \ \sin\left(\frac{\pi}{12}\right) = \frac{1}{2}\sqrt{2-\sqrt{3}}\right)$$

Ex 7.

a) Let $f(t) = b \cdot a^t$, $t \in \mathbb{R}$. Determine a assuming 20% growth.

Since 1 corresponds to no change, i.e. remain at 100%, then $a = 1{,}2$ is 20% growth. Sometimes $a = 1 + r$.

r is the rate

b) Let $f(t) = 2^t$ and $h(t) = 0{,}5^{2-t}$, $t \in \mathbb{R}$.

Determine growth rate and percent growth.

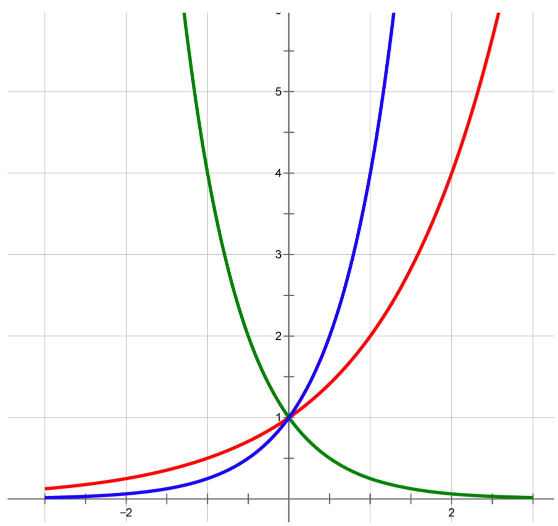
$$2 = 1 + r \iff r = 1, \quad 100\%.$$

Let's simplify $0{,}5^{2-t} = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{\left(\frac{1}{2}\right)^t} = \frac{1}{4} \cdot 2^t.$

Now h has the same growth as f.

c) State the base, and write on the form

$$x \mapsto e^{kx}, \quad x \in \mathbb{R}.$$



$$2^x = e^{\ln(2)x}$$

$a = 2$, $k = \ln(2)$

$$4^x = e^{\ln(4)x} = e^{2\ln(2)x}$$

$a = 4$, $k = \ln(4) = 2\ln(2)$

$$\left(\frac{1}{4}\right)^x = e^{\ln(\frac{1}{4})x} = e^{-\ln(4)x}$$

$$= e^{-2\ln(2)x}$$

$a = \frac{1}{4}$, $k = \ln\left(\frac{1}{4}\right) = -2\ln(2).$