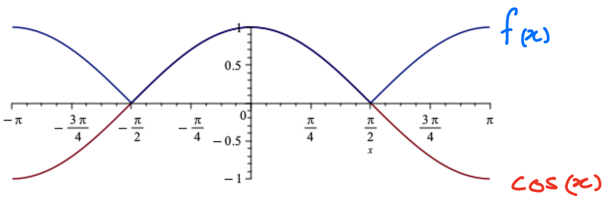
u.10. 1. *Hvat fyri 2π-periodisk funktión er stykkivís differentiabul?*

(i) $f(x) = |\cos(x)|$, $x \in [-\pi, \pi[$.

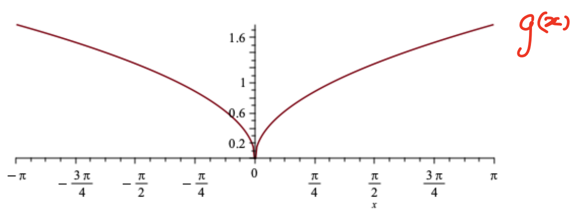


Set upp deilipunktini
$x_1 = -\pi$, $x_2 = -\frac{\pi}{2}$,
$x_3 = \frac{\pi}{2}$, $x_4 = \pi$.

Lat
$$f_1(x) = -\cos(x), \quad x \in \left[-\pi, -\tfrac{\pi}{2}\right],$$
$$f_2(x) = \cos(x), \quad x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],$$
$$f_3(x) = -\cos(x), \quad x \in \left[\tfrac{\pi}{2}, \pi\right].$$

Her er $f(x) = f_i(x)$ fyri $x \in]x_i, x_{i+1}[$, $i = 1,2,3$, og f_1, f_2 og f_3 eru allar differentiablar við kontinuertar avleiddar. Tískil er f stykkivís differentiabul.

(ii) $g(x) = \sqrt{|x|}$, $x \in [-\pi, \pi[$.



Set $x_1 = -\pi$, $x_2 = 0$, $x_3 = \pi$.

Lat $g_1(x) = \sqrt{-x}$ og $g_2(x) = \sqrt{x}$.

Nú skal $g_2(x)$ vera differentiabul á $[0, \pi]$, men $g_2'(x) = \frac{1}{2\sqrt{x}}$ er ei definerað fyri $x = 0$. Vit hava, at
$$\lim_{x \to 0^+} g_2'(x) = \infty,$$
so $g(x)$ er ikki stykkivís differentiabul.

2. (i) Vís, at Fourierrekkjan hjá 2π-periodisku funktiónini $f(x) = |x|$, $x \in [-\pi, \pi[$ er
$$f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}.$$

Funktiónin f er lika, so $b_n = 0$ $\forall n \in \mathbb{N}$ per 6.3. Vit rokna koefficientarnar a_n.
$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{1}{\pi} \left[x^2\right]_0^{\pi} = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cdot\cos(nx)\,dx = \frac{2}{\pi}\left(\left[x\cdot\frac{1}{n}\sin(nx)\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{n}\cdot\left[\frac{1}{n}\cos(nx)\right]_0^\pi = \frac{2}{n^2\pi}\left((-1)^n-1\right)$$

Vit fáa nú

$$\frac{1}{2}\cdot\pi + \sum_{n=1}^{\infty}\frac{2\cdot((-1)^n-1)}{n^2\pi}\cos(nx) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}.$$

(ii) Vís, at Fourierrekkjan konvergerar punktvíst ímóti $f(x)$ $\forall x\in\mathbb{R}$.

Vit brúka setn. 6.12 beinleiðis, tí $f(x)=|x|$ er 2π-periodisk og stykkivís differentiabul, tí $\pm x$ er $C^\infty(\mathbb{R})$. So Fourierrekkjan konvergerar ímóti $f(x)$, har sum f er kontinuert. Eftir sum, at f er kontinuert, so hava vit per 6.13, at rekkjan konvergerar ímóti $f(x)$ $\forall x\in\mathbb{R}$.

(iii) Finn N, so at $|f(x)-S_N(x)|\le\varepsilon$ við $\varepsilon=0{,}1$ og $\varepsilon=0{,}01$.

Við korollar 6.16 fáa vit estimerað.

$$\int_{-\pi}^{\pi}|f'(t)|^2\,dt = \int_{-\pi}^{\pi}1\,dt = 2\pi.$$

$$N \ge \frac{2\pi}{\pi\cdot 0{,}1^2} = 200 \quad \text{og} \quad N\ge\frac{2\pi}{\pi\cdot 0{,}01^2} = 20000.$$

(iv) Hevur Fourierrekkjan eina konvergenta majorantrekkju?

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$ er ein konvergent majorantrekkja,

tí $|\cos((2n-1)x)|\le 1$ $\forall x\in\mathbb{R}$.

(v) Rokna Fourierkoefficientarnar hjá f á kompleksan form.

Við lemma 6.22 eru koefficientarnir c_n givnir við

$$c_0 = \frac{1}{2}a_0 = \frac{\pi}{2},\quad c_n = \frac{1}{2}a_n = \frac{1}{n^2\pi}\left((-1)^n-1\right),$$

$$c_{-n} = \frac{1}{2}a_n = \frac{1}{n^2\pi}\left((-1)^n-1\right).$$

$$\Rightarrow\quad c_{\pm(2n-1)} = \frac{-2}{(2n-1)^2\pi},\quad \text{meðan}\quad c_{\pm 2n}=0.$$

(vi) Rokna virðið á $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Tá $x=0$, so er $f(0)=0$, tí hava vit fyri Fourierrekkjuna

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\cdot 0)}{(2n-1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Leftrightarrow \quad -\frac{\pi}{2} \cdot \left(-\frac{\pi}{4}\right) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$

3. Lat f vera 2π-periodisk og givin við

$$f(t) = \begin{cases} \sin(t), & 0 < t \leq \pi, \\ 0, & \pi < t \leq 2\pi. \end{cases}$$

(i) Vís, at Fourierrekkjan konvergerar ímóti $f(t)$ $\forall t \in \mathbb{R}$.

Set $t_1 = 0$, $t_2 = \pi$, $t_3 = 2\pi$ og $f_1(t) = \sin(t)$, $t \in [0,\pi]$, $f_2(t) = 0$, $t \in [\pi, 2\pi]$.

Legg til merkis, at $f_1(0) = f_1(\pi) = 0 = f_2(0) = f_2(2\pi)$. Tískil er f kontinuert, tí $f_1, f_2 \in C^\infty(\mathbb{R})$, og stykkivis differentiabul. Per korollar 6.13, so konvergerar Fourierrekkjan uniformt ímóti f $\forall t \in \mathbb{R}$.

(ii) Brúka korollar 6.16 at finna N, so at $|f(t) - S_N(t)| \leq 0{,}1$ $\forall t \in \mathbb{R}$.

$$\int_{-\pi}^{\pi} |f'(t)|^2 \, dt = \int_0^{\pi} \cos(t)^2 \, dt = \frac{1}{2} \left[\cos(t)\sin(t) + t \right]_0^{\pi} = \frac{\pi}{2}$$

$$\therefore \quad N \geq \frac{\frac{\pi}{2}}{\pi \cdot 0{,}1^2} = 50 .$$

(iii) Fourierrekkjan hjá f er $\frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos(2nt)$.
Vís, at

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} .$$

Fyri $t=0$ er $f(0)=0$, so vit innseta í Fourierrekkjuna.

$$0 = f(0) = \frac{1}{\pi} + \frac{1}{2}\sin(0) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos(2n\cdot 0)$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

$$\Leftrightarrow \quad -\frac{1}{\pi}\cdot\left(-\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

(iv) Vís, at $\displaystyle\sum_{n=N+1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{4N+2}$.

Vit hava, at $\displaystyle\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

So $$\sum_{n=1}^{N} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N+1}\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{1}{2} - \frac{1}{4N+2}.$$

Vit hava nú

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \Leftrightarrow \sum_{n=1}^{N} \frac{1}{(2n-1)(2n+1)} + \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{1}{2} - \frac{1}{4N+2} + \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

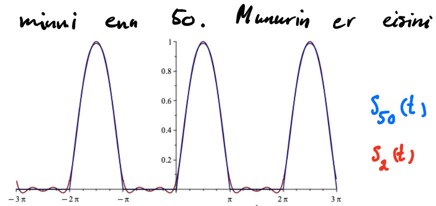
$$\Leftrightarrow \quad \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{4N+2}.$$

(v) Brúka 6.17 og (iv) til at finna N, so at $|f(t) - S_N(t)| \le 0{,}1 \quad \forall t \in \mathbb{R}$.

$$|f(t) - S_N(t)| \le \sum_{n=N+1}^{\infty} \frac{2}{\pi} \frac{1}{(2n-1)(2n+1)} = \frac{2}{\pi}\frac{1}{4N+2} = \frac{1}{(2N+1)\pi}$$

So $N \geq 2$. Her brúka vit altso ikki a_0, b_1 og $a_{2n-1} = b_{n+1} = 0$ til at avgera N, so at munurin er minni enn 0,1.

(vi) Samanber N úr (ii) og (v). Ger plot av avsnitssummunum.
Her er 2 nakað nógv minni enn 50. Munurin er eisini størur í síðsta enda.



4. Finn c_n fyri $f(x) = x$, $x \in [-\pi, \pi[$.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \left[\frac{(inx+1)\, e^{-inx}}{n^2} \right]_{-\pi}^{\pi} = \frac{(-1)^n i}{n}, \quad n = 1, 2, \ldots$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = 0.$$

5.
$$f(t) = \begin{cases} 1, & t \in [0, \frac{T}{8}], \\ 0, & t \in]\frac{T}{8}, \frac{7T}{8}[, \\ 1, & t \in [\frac{7T}{8}, T]. \end{cases}$$

f er T-periodisk.

(i) Finn Fourierrekkjuna.

Vit hava, at $f(-t) = f(T-t) = f(t)$, so f er lika.

$$a_0 = \frac{4}{T} \int_0^{\frac{T}{2}} f(t)\, dt = \frac{4}{T} \int_0^{\frac{T}{8}} 1\, dt = \frac{1}{2}.$$

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \cos(n\omega t)\, dt = \frac{4}{T} \int_0^{\frac{T}{8}} \cos(n\omega t)\, dt$$

$$\boxed{\omega = 2\pi \cdot f = 2\pi \cdot \frac{1}{T}}$$

$$= \frac{4}{Tn\omega} \left[\sin(n\omega t) \right]_0^{\frac{T}{8}} = \frac{2}{n\pi} \sin\left(\frac{n\pi}{4}\right).$$

Her er $b_n = 0 \; \forall n \in \mathbb{N}$. Vit skriva formin á a_n uppá 8 mátar (vinklar).

$$a_{8n+1} = \frac{\sqrt{2}}{(8n+1)\pi}, \quad a_{8n+2} = \frac{2}{(8n+2)\pi}, \quad a_{8n+3} = \frac{\sqrt{2}}{(8n+3)\pi},$$

$$a_{8n+4} = 0, \quad a_{8n+5} = \frac{-\sqrt{2}}{(8n+5)\pi}, \quad a_{8n+6} = \frac{-2}{(8n+6)\pi},$$

$$a_{8n+7} = \frac{-\sqrt{2}}{(8n+7)\pi}, \quad a_{8n+8} = 0.$$

Fourierrekkjan er tískil givin við

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$$

$$= \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \Big(\frac{\sqrt{2}}{8n+1} \cos((8n+1)\omega t) + \frac{2}{8n+2} \cos((8n+2)\omega t)$$

$$+ \frac{\sqrt{2}}{8n+3} \cos((8n+3)\omega t) - \frac{\sqrt{2}}{8n+5} \cos((8n+5)\omega t)$$

$$- \frac{2}{8n+6} \cos((8n+6)\omega t) - \frac{\sqrt{2}}{8n+7} \cos((8n+7)\omega t) \Big).$$

(ii) Finn summin fyri $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Fourierrekkjan í $t = \frac{T}{8}$ gevur $\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2}$ per 6.12, tí f er T-periodisk og stykkivís differentiabul.

Samstundis er
$$\cos\left((8n+1)\omega \cdot \tfrac{T}{8}\right) = \cos\left((8n+7)\omega \cdot \tfrac{T}{8}\right) = \frac{1}{\sqrt{2}},$$
$$\cos\left((8n+2)\omega \cdot \tfrac{T}{8}\right) = \cos\left((8n+6)\omega \cdot \tfrac{T}{8}\right) = 0,$$
$$\cos\left((8n+3)\omega \cdot \tfrac{T}{8}\right) = \cos\left((8n+5)\omega \cdot \tfrac{T}{8}\right) = -\frac{1}{\sqrt{2}}.$$

Vit fáa, at

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \left(\frac{\sqrt{2}}{8n+1} \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8n+3} \cdot \left(-\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{8n+5} \cdot \left(-\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{8n+7} \frac{1}{\sqrt{2}} \right)$$

$$= \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{8n+1} - \frac{1}{8n+3} + \frac{1}{8n+5} - \frac{1}{8n+7} \right)$$

$$= \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}.$$