Stokes' Theorem and Potentials

Ex1. Consider the disc F given by

$$x^2+y^2 \leq 4 \text{ and } z=0.$$

a) Parametrize F and ∂F such that the orientations obey the right hand rule.

The disc has the center at the origin and a radius of 2, and the same applies to the circle which defines the boundary.

$$r(u,v) = \begin{bmatrix} 2\cdot u \cos v \\ 2\cdot u \sin v \\ 0 \end{bmatrix}, \quad s(v) = \begin{bmatrix} 2 \cos v \\ 2 \sin v \\ 0 \end{bmatrix},$$

with $u \in [0,1]$ and $v \in [0,2\pi]$ parametrize F and ∂F respectively.

b) Let $\underline{N}$ denote the normal of F and $\underline{T}$ the tangent of ∂F. Show that $\underline{T} \times \underline{N}$ points away from the surface along the boundary.

$$\underline{N} = \begin{bmatrix} 2 \cos v \\ 2 \sin v \\ 0 \end{bmatrix} \times \begin{bmatrix} -2u \sin v \\ 2u \cos v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4u \end{bmatrix}$$

$$\underline{T} = \begin{bmatrix} -2 \sin v \\ 2 \cos v \\ 0 \end{bmatrix} \Rightarrow \underline{T} \times \underline{N} = \begin{bmatrix} 8u \cos v \\ 8u \sin v \\ 0 \end{bmatrix}$$

Indeed $\underline{T} \times \underline{N}$ is outwardpointing, and along the boundary we have $u=1$, so for any angle v the vector is outwardpointing and of magnitude 8.

Let the vector field V be given by $V(x,y,z) = \begin{bmatrix} x^2 - y \\ -yz \\ xz \end{bmatrix}$.

c) Determine the circulation of V along ∂F by Stokes' Theorem.

$$\mathrm{Rot}(V) = \begin{bmatrix} 0 - (-y) \\ 0 - z \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} y \\ -z \\ 1 \end{bmatrix}$$

$$V(r(u,v)) \cdot \underline{N} = \begin{bmatrix} 2u \sin v \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 4u \end{bmatrix} = 4u$$

$$\int_0^{2\pi}\int_0^1 4u \, du \, dv = 2\pi \cdot 2 = 4\pi$$

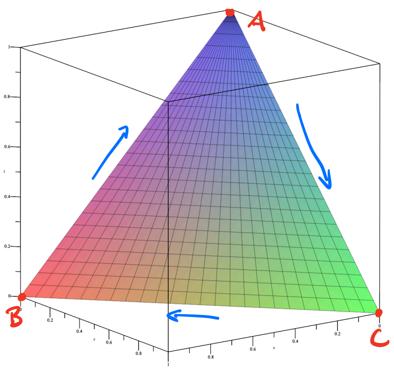
d) What happens if the parametrization doesn't follow the right had rule?

Then we get -4π.

Ex 2. The vertices $A(0,0,1)$, $B(1,0,0)$ and $C(0,1,0)$ define a triangular surface T. Let $V(x,y,z) = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$.

a) Parametrize T and plot T.

$$r(u,v) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} 1-u \\ u \\ -1 \end{bmatrix}, \quad u,v \in [0,1]$$



b) Choose a positive orientation for ∂T and show it on a figure.

We have
$$\underline{r}_u' = \begin{bmatrix} -v \\ v \\ 0 \end{bmatrix}, \quad \underline{r}_v' = \begin{bmatrix} 1-u \\ u \\ -1 \end{bmatrix} \Rightarrow \underline{N} = \begin{bmatrix} -v \\ -v \\ -v \end{bmatrix}$$

Since $\underline{N}$ points downwards a positive orientation is then ACBA.

c) Compute the circulation of V along ∂T.

$$\mathrm{Rot}(V) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathrm{Rot}(V) \cdot \underline{N} = -3v$$

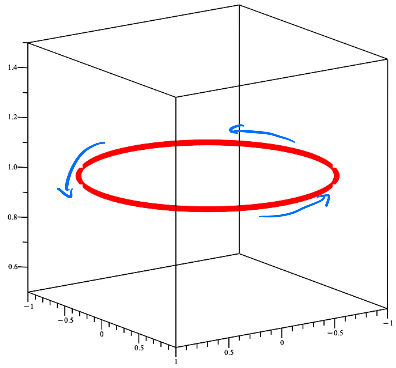
$$\int_0^1 \int_0^1 -3v \, du \, dv = \left[-\frac{3}{2} v^2 \right]_0^1 = -\frac{3}{2}.$$

Ex3. Let
$$V(x,y,z) = \begin{bmatrix} y e^{xy} + z^2 \\ x e^{xy} + z^2 + x \\ 2x^2 + 2y^2 \end{bmatrix},$$

and let $K = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \ z = 1 \}$.

a) Sketch K and draw a positive orientation.

```
> V:=(x,y,z)-><y*exp(x*y)+z^2, x*exp(x*y)+z^2+x, 2*x^2+2*y^2>:
  rK:=u-> <cos(u),sin(u),1>:
  curve:=spacecurve([cos(u),sin(u),1],u=0..2*Pi,color=red,thickness=10);
```

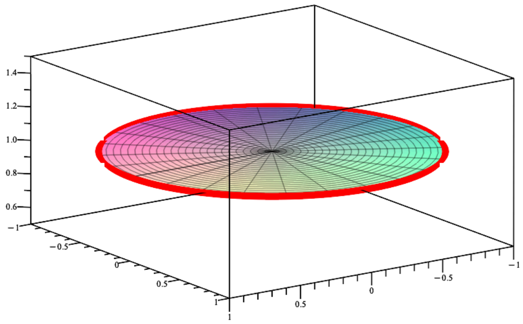


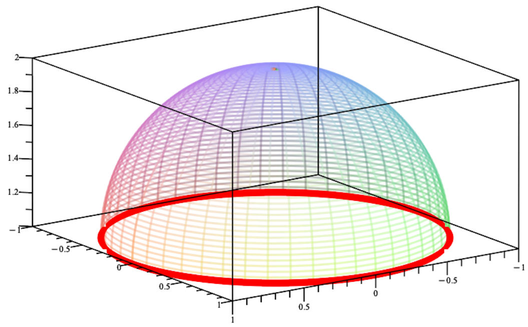
Just use the standard orientation of counterclockwise.

b) Choose two surfaces F_1 and F_2 that have K as their boundary.

Parametrize.
$$r_1(u,v) = \begin{bmatrix} u \cdot \cos v \\ u \cdot \sin v \\ 1 \end{bmatrix}, \quad u \in [0,1] \text{ and } v \in [0, 2\pi].$$

$$r_2(u,v) = \begin{bmatrix} \sin u \cdot \cos v \\ \sin u \cdot \sin v \\ \cos(u) + 1 \end{bmatrix}, \quad u \in \left[0, \tfrac{\pi}{2}\right] \text{ and } v \in [0, 2\pi).$$





c) Is there a difference in the circulation of V along ∂F_1 and ∂F_2?

No, this will be evident by computation as well.

d) Compute circulations.

$$\mathrm{Rot}(V) = \begin{bmatrix} 4y - 2z \\ 2z - 4x \\ 1 \end{bmatrix}$$

$$\underline{N}_1 = \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} \times \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$$

$$\underline{N}_2 = \begin{bmatrix} \cos u \cdot \cos v \\ \cos u \cdot \sin v \\ -\sin u \end{bmatrix} \times \begin{bmatrix} -\sin u \cdot \sin v \\ \sin u \cdot \cos v \\ 0 \end{bmatrix} = \begin{bmatrix} \sin^2 u \cdot \cos v \\ \sin^2 u \cdot \sin v \\ \cos u \cdot \sin u \end{bmatrix}$$

For F_1 we get the integrand

$$\begin{bmatrix} 4 \cdot u \cdot \sin v - 2 \\ 2 - 4u \cdot \cos v \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} = u$$

$$\Rightarrow \int_0^{2\pi} \int_0^1 u \, du \, dv = 2\pi \cdot \frac{1}{2} = \pi.$$

For F_2 we get the integrand

$$\begin{bmatrix} 4\cdot\sin u\cdot\sin v - 2(\cos u+1) \\ 2(\cos u+1) - 4\cdot\sin u\cdot\cos v \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sin^2 u\cdot\cos v \\ \sin^2 u\cdot\sin v \\ \cos u\cdot\sin u \end{bmatrix}$$

```
> prik(rotV(vop(r2(u,v))),N2(u,v));
  simplify(%);
  normal(%);
```

$$(4\sin(u)\sin(v) - 2 - 2\cos(u))\sin(u)^2\cos(v) + (2 + 2\cos(u) - 4\sin(u)\cos(v))\sin(u)^2\sin(v) + \cos(u)\sin(u)$$

$$-2\sin(u)\left((\cos(v)-\sin(v))(1+\cos(u))\sin(u) - \frac{\cos(u)}{2}\right)$$

$$\sin(u)\,(2\sin(u)\sin(v)\cos(u) - 2\sin(u)\cos(u)\cos(v) + 2\sin(u)\sin(v) - 2\sin(u)\cos(v) + \cos(u))$$

Not quite worth it by hand.

```
> int(prik(rotV(vop(r1(u,v))),N1(u,v)),[u=0..1,v=0..2*Pi]);
```

$$\pi$$

```
> int(prik(rotV(vop(r2(u,v))),N2(u,v)),[u=0..Pi/2,v=0..2*Pi]);
```

$$\pi$$

d) Why is it better suited using Stokes' than the tangential line integral?

Doing differentials first may simplify. Also consider that many functions have no antiderivative, so it's not necessarily a good idea to pack them into exponentials.

Ex 4. A cylinder is given by $(x+1)^2+y^2=1$. A plane equation is given by $z=2-x$, and

$$V(x,y,z) = \begin{bmatrix} y \\ z \\ x \end{bmatrix}.$$

a) Parametrize the curve given by the intersection of the cylinder and plane.

$$r_K(u) = \begin{bmatrix} \cos u + 1 \\ \sin u \\ 1 - \cos u \end{bmatrix}, \quad u \in [0, 2\pi].$$

b) Determine the circulation of V along K without use of Stokes'.

$$\begin{bmatrix} \sin u \\ 1-\cos u \\ \cos u + 1 \end{bmatrix} \cdot \begin{bmatrix} -\sin u \\ \cos u \\ \sin u \end{bmatrix} = -\sin^2 u + \cos u - \cos^2 u + \cos u \cdot \sin u + \sin u$$

$$= \cos u + \sin u + \cos u \cdot \sin u - 1$$

$$\int_0^{2\pi} \cos u + \sin u + \cos u \cdot \sin u - 1 \; du$$

$$= \left[\sin u \quad - \cos u \quad + \quad \frac{1}{2} \sin^2 u \right]_0^{2\pi} \quad - \quad 2\pi \quad = \quad -2\pi .$$

c) Parametrize the disc F with boundary K.

$$r_F(u,v) = \begin{bmatrix} u \cdot \cos v + 1 \\ u \cdot \sin v \\ 1 - \cos v \cdot u \end{bmatrix}, \qquad u \in [0,1], \; v \in [0, 2\pi]$$

d) Compute the circulation again.

$$Rot(V) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad \underline{N} = \begin{bmatrix} \cos v \\ \sin v \\ -\cos v \end{bmatrix} \times \begin{bmatrix} -u \cdot \sin v \\ u \cdot \cos v \\ u \cdot \sin v \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ u \end{bmatrix}$$

$$\int_0^{2\pi} \int_0^1 -2u \; du \, dv = 2\pi \cdot (-1) = -2\pi .$$

Ex5. Let $V(x,y,z) = (y^2, \; x - 2xz, \; -xy)$ and

$$F = \{ (x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{a^2 - x^2 - y^2} \text{ and } x^2 + y^2 \leq a^2 \}.$$

a) Explain why F is a hemisphere and sketch.

Since we have real numbers it follows that $z \geq 0$, and from

$$z = \sqrt{a^2 - x^2 - y^2} \implies x^2 + y^2 + z^2 = a^2$$

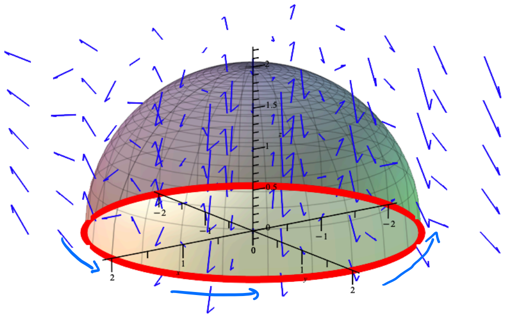
clearly a sphere presents itself, so it's the upper hemisphere.

```
> V:=(x,y,z)-><y^2,x-2*x*z,-x*y>:
  assume(a>0);interface(showassumed=0):
  r:=(u,v)-><u*cos(v),u*sin(v),sqrt(a^2-u^2)>:
  Jacobian(r(u,v),[u,v]):
  kryds(Column(%,1),Column(%,2)):
  N:=simplify(%);
```

```
> rboundary:=v->r(a,v):
```

```
> rotfield:=fieldplot3d(Curl(V)(x,y,z),x=-2..2,y=-2..2,z=0..2,thickness=2,color=blue,grid=[5,5,5]):
  surf:=plot3d([u*cos(v),u*sin(v),sqrt(4-u^2)],u=0..2,v=0..2*Pi,axes=normal,transparency=0.5,projection=0.8,scaling=constrained):
```

```
> curve:=spacecurve(<2*cos(v),2*sin(v),0>,v=0..2*Pi,scaling=constrained,axes=normal,color=red,thickness=10):
  display(rotfield,surf,curve);
```



$$N := \begin{bmatrix} \dfrac{u^2\cos(v)}{\sqrt{a^2-u^2}} \\ \dfrac{u^2\sin(v)}{\sqrt{a^2-u^2}} \\ u \end{bmatrix}$$

b) Determine the flux of the curl of V through $\mathcal{F}$.

```
> int(prik(V(vop(rboundary(v))),diff(rboundary(v),v)),v=0..2*Pi);
```

$$\pi a^2$$

Ex 6. Let

$$W^*(x,y,z) = -(x,y,z) \times \int_0^1 u \cdot V(ux, uy, uz)\, du,$$

then V is divergence free iff $\mathrm{curl}(W^*) = V$.

a) Let

$$U(x,y,z) = \begin{bmatrix} xz \\ yz \\ -z^2 \end{bmatrix} \quad \text{and} \quad V(x,y,z) = \begin{bmatrix} 4x^2 \\ 0 \\ 0 \end{bmatrix}.$$

Determine W^* associated to U and V.

$$W^*_U = -\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} \int_0^1 u \cdot (u\cdot x \cdot u \cdot z)\, du \\ \int_0^1 u \cdot (u \cdot y \cdot u \cdot z)\, du \\ \int_0^1 u \cdot -(u\cdot z)^2\, du \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} \frac{1}{4} xz \\ \frac{1}{4} yz \\ -\frac{1}{4} z^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} yz^2 \\ -\frac{1}{2} xz^2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathrm{Curl}(W^*_U) = \begin{bmatrix} xz \\ yz \\ -z^2 \end{bmatrix} = U.$$

$$W_V^* = -\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} \int_0^1 u \cdot (4 \cdot (ux)^2)\, du \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} x^2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -x^2 z \\ x^2 y \end{bmatrix} \quad \text{and} \quad \mathrm{Curl}(W_V^*) = \begin{bmatrix} x^2 - x^2 \\ -2xy \\ -2xz \end{bmatrix} = \begin{bmatrix} 0 \\ -2xy \\ -2xz \end{bmatrix} \neq V.$$

b) Compute the divergence of U and V.

We know that U is divergence free, so

$$\mathrm{div}(U) = 0.$$

$$\mathrm{div}(V) = 8x.$$

Ex7. Let $U(x,y,z) = (xz, yz, -z^2)$ and

$$F = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z-1)^2 = 1 \text{ and } z \geq 1\}.$$

a) Describe and sketch.

This is the upper hemisphere of radius 1 and center $(0,0,1)$.

See the analogue in ex.5.

b) Use Stokes' to determine the flux.

```
> r:=u-> <cos(u),sin(u),1>:
> prik(W1(vop(r(u))),diff(r(u),u)):
  integrand:=simplify(%);
  int(%,u=0..2*Pi);
```

$$integrand := -\frac{1}{2}$$

$$-\pi$$

c) What are the advantages to using Stokes'?

Everything is just simpler by stepping down in dimension to only a simple curve.

Ex 8. Let
$$V(x,y,z) = \begin{bmatrix} z \cdot \cos(yz) \\ x \cdot \cos(xz) \\ y \cdot \cos(xy) \end{bmatrix}$$
and let K be the square curve connecting $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(0,1,0)$ with the same orientation.

a) Show that
$$W(x,y,z) = \begin{bmatrix} \sin(xz) \\ \sin(xy) \\ \sin(yz) \end{bmatrix}$$
is a potential of V.
$$\mathrm{rot}(W) = \begin{bmatrix} \cos(yz) \cdot z \\ \cos(xz) \cdot x \\ \cos(xy) \cdot y \end{bmatrix} = V.$$

b) Determine the flux of V through a surface with K as its boundary by using the circulation of W.

```
> integrand[1]:=prik(W(u,0,0),<1,0,0>):
  integrand[2]:=prik(W(1,u,0),<0,1,0>):
  integrand[3]:=prik(W(u,1,0),<-1,0,0>):
  integrand[4]:=prik(W(0,u,0),<0,-1,0>):
> for i from 1 to 4 do
  k[i]:=int(integrand[i],u=0..1) od:
  add(k[i],i=1..4);
```
$$1 - \cos(1)$$

c) Explain how the above will also be found with the flux formula, then compute.

The flux through arbitrary surfaces will be identical as V has a vector potential.

```
> r:=(u,v)-><u,v,0>:
  Jacobian(r(u,v),[u,v]):
  N:=kryds(Column(%,1),Column(%,2)):
  integrand:=prik(V(vop(r(u,v))),N):
  int(integrand,[u=0..1,v=0..1]);
```
$$1 - \cos(1)$$