Mapping matrices and change of basis

Ex 1. Let $\underline{\underline{F}} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{bmatrix}$.

a) Calculate $\underline{\underline{F}}\,\underline{u}_i$, $i = 1,2,3$.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1+2 \\ 3+6 \\ -1-2-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1-2+4-1 \\ -3+6-3 \\ 1-4+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $\underline{u}_2, \underline{u}_3 \in \ker(f)$.

b) State whether $\underline{b} = \begin{bmatrix} 2 \\ 9 \\ -5 \end{bmatrix}$ is in the image $f(\mathbb{R}^4)$.

Since $f(\underline{u}_1) = \underline{b}$, then $\underline{b} \in f(\mathbb{R}^4)$.

c) Compute the dimension of $f(\mathbb{R}^4)$.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{bmatrix} \begin{matrix} \\ -3R_1 \\ +R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \begin{matrix} \\ \cdot\left(-\frac{1}{3}\right) \\ +R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\rho(F) = 2$ it follows that $\dim(f(\mathbb{R}^4)) = 2$.

d) State $\dim(\ker(f))$.

By proposition 12.26 we have $\dim(\ker(f)) = 4 - 2 = 2$.

f) State a basis for the kernel of f.

We just need two lin. indept. vectors in the kernel.

One such basis is $(\underline{u}_2, \underline{u}_3)$ as they are lin. indept. and both are in $\ker(f)$ by a).

g) State the solution to $f(\underline{x}) = \underline{b} = \begin{bmatrix} 2 \\ 9 \\ -5 \end{bmatrix}$.

By a) the solution is $\underline{x} = \underline{u}_1$.

Ex 2. Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with coordinate matrix $\underline{\underline{F}}$.

It is given that

$$\mathrm{rref}(\underline{\underline{F}}) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

a) Read off the basis of $\ker(f)$. Also state $\dim(f(\mathbb{R}^3))$.

Let $x_3 = t$, then a basis for $\ker(f)$ is given by the vector $\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ and the kernel is the line $t\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$.

By 12.26 $\dim(f(\mathbb{R}^3)) = 3 - 1 = 2$. (or read $\rho(F) = 2$).

b) Can we determine a basis for $f(\mathbb{R}^3)$?

Not with the given information. The reduced form holds no information on which vectors were initially involved.

Ex 3. A new basis $a = (\underline{a}_1, \underline{a}_2)$ for $\mathbb{R}^2$ is given by

$$\underline{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{a}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

a) State ${}_e\underline{\underline{M}}_a$. Given ${}_a\underline{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ compute ${}_e\underline{v}$.

$${}_e\underline{\underline{M}}_a = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \quad \text{and thus} \quad {}_e\underline{v} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

b) State ${}_a\underline{\underline{M}}_e$. Given ${}_e\underline{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ compute ${}_a\underline{v}$.

$${}_a\underline{\underline{M}}_e = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \quad \text{and thus} \quad {}_a\underline{v} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Note the inverse of $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\underline{\underline{A}}^{-1} = \dfrac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

c) Let f be linear and given by ${}_e\underline{\underline{F}}_e = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}.$

Determine ${}_a\underline{\underline{F}}_a$.

$$\begin{aligned} {}_a\underline{\underline{F}}_a &= {}_a\underline{\underline{M}}_e \; {}_e\underline{\underline{F}}_e \; {}_e\underline{\underline{M}}_a \\ &= \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

d) Let ${}_a\underline{v} = \begin{bmatrix} m \\ n \end{bmatrix}$ and compute $f(\underline{v})$ wrt. basis a.

$${}_a\underline{\underline{F}}_a \; {}_a\underline{v} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m+n \\ n \end{bmatrix}.$$

Ex 4. Basis $a = (\underline{a}_1, \underline{a}_2)$ and $c = (\underline{c}_1, \underline{c}_2, \underline{c}_3)$. A linear map f is given for which

$$f(\underline{a}_1) = \underline{c}_1 - 2\underline{c}_2 + \underline{c}_3$$

$$f(\underline{a}_2) = -2\underline{c}_1 + 4\underline{c}_2 - 2\underline{c}_3$$

a) State ${}_c\underline{\underline{F}}_a$ and compute $f(3\underline{a}_1 - \underline{a}_2)$.

Firstly

$${}_c\underline{\underline{F}}_a = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & -2 \end{bmatrix}.$$

Second we get

$$f(3\underline{a}_1 - \underline{a}_2) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ 5 \end{bmatrix}.$$

b) Which of the vectors $\underline{a}_1 + 2\underline{a}_2$ and $2\underline{a}_1 + \underline{a}_2$ belong to $\ker(f)$?

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector $2\underline{a}_1 + \underline{a}_2$ is in $\ker(f)$.

d) Which of the vectors $\underline{c}_1 - 2\underline{c}_2 + \underline{c}_3$ and $2\underline{c}_1 - \underline{c}_2 + 2\underline{c}_3$ belong to $f(V)$?

Since $f(\underline{a}_1) = \underline{c}_1 - 2\underline{c}_2 + \underline{c}_3$, then this is in the image of f.

We check the second vector.

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -2 & 4 & -1 \\ 1 & -2 & 2 \end{array}\right] \begin{array}{l} \\ +2R_1 \\ -R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

This system is inconsistent, and so $2\underline{c}_1 - \underline{c}_2 + 2\underline{c}_3 \notin f(V)$.

e) State a basis for the range of f.

Let's just go with $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. There is a 1-dim. kernel, and so the image is 1-dim. by 12.26. It's also quite clear from ${}_{\varepsilon}\underline{\underline{F}}_{a}$, since $-2 \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$.

Ex 5. Two bases given for $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively:

$$v = (\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right),$$

$$w = (\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

a) Show that these are indeed bases.

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \right) = 1 \qquad \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = 1$$

The vectors are lin. indept., so v is a basis for $\mathbb{R}^3$ and w is a basis for $\mathbb{R}^4$.

b) Let $f: \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$f(\underline{v}_1) = \underline{w}_1 + \underline{w}_2,$$
$$f(\underline{v}_2) = \underline{w}_2 + \underline{w}_3,$$
$$f(\underline{v}_3) = \underline{w}_3 + \underline{w}_4.$$

State ${}_{w}\underline{\underline{F}}_{v}$.

$${}_{w}\underline{\underline{F}}_{v} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) Determine ${}_e\underline{\underline{F}}_e$.

$$ {}_e\underline{\underline{F}}_e = {}_e\underline{\underline{M}}_w \, {}_w\underline{\underline{F}}_v \, {}_v\underline{\underline{M}}_e = {}_e\underline{\underline{M}}_w \, {}_w\underline{\underline{F}}_v \left({}_e\underline{\underline{M}}_v\right)^{-1} $$

$$ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 14 & -6 & 2 \\ 13 & -6 & 2 \\ 14 & -7 & 2 \\ 8 & -4 & 1 \end{bmatrix} $$

Ex 6. Let

$$ f(\underline{e}_1) = \underline{c}_1 + \underline{c}_2 + \underline{c}_3 + \underline{c}_4 \quad \text{and} $$

$$ f(\underline{e}_2) = \underline{c}_1 - 3\,\underline{c}_2 + 7\,\underline{c}_4 . $$

a) Determine ${}_c\underline{\underline{F}}_e$.

$$ {}_c\underline{\underline{F}}_e = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ 1 & 0 \\ 1 & 7 \end{bmatrix} . $$

b) Solve $f(\underline{x}) = 5\,\underline{c}_1 + 3\,\underline{c}_2 - 3\,\underline{c}_3 + 17\,\underline{c}_4$.

$$ \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -3 & 3 \\ 1 & 0 & -3 \\ 1 & 7 & 17 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] $$

$$ {}_e\underline{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{or} \quad \underline{x} = 3\,\underline{e}_1 + 2\,\underline{e}_2 . $$