Multiple choice round 3

1. 16 athletes running. Three fastest receive medals. How many ways to distribute?

$_{16}P_3$

2. 8 chairs, each to be painted using one color. The colors are: red, green, blue, black and orange. How many different ways can we choose the colors?

$_{12}C_8$

3. 5 cards from a deck. How many ways for exactly one ace?

$4 \cdot {}_{48}C_4$

4. Permutations of MATHEMATICS without repeats.

$$\frac{11!}{2!2!2!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{8} = 11\cdot 10\cdot 9\cdot 7!$$

5. Bag with 7 red, 6 blue and 8 black. How many ways can you draw 3 red, 5 blue and 2 black if order doesn't matter?

$$\begin{aligned} {}_7C_3 \cdot {}_6C_5 \cdot {}_8C_2 &= \frac{7!}{3!\,4!} \cdot \frac{6!}{5!\,1!} \cdot \frac{8!}{2!\,6!} \\ &= \frac{7\cdot 6\cdot 5}{6} \cdot 6 \cdot \frac{8\cdot 7}{2} \\ &= 7^2\cdot 6\cdot 5\cdot 4 \\ &= 7^2\cdot 5\cdot 3\cdot 2^3 \end{aligned}$$

6. Die with probabilities p_i:

$$p_1 = \frac{1}{6}, \quad p_2 = \frac{3}{8}, \quad p_3 = \frac{1}{12}, \quad p_4 = \frac{1}{24}, \quad p_5 = \frac{1}{12}, \quad p_6 = \frac{1}{4}$$

What is the probability of an even number?

$$P(E) = p_2 + p_4 + p_6 = \frac{3}{8} + \frac{1}{24} + \frac{1}{4} = \frac{9}{24} + \frac{1}{24} + \frac{6}{24} = \frac{16}{24} = \frac{2}{3}$$

7. Which sequence has the characteristic equation $x^2 - 3x + 1 = 0$?

$$x^2 = 3x - 1 \implies e_n = 3e_{n-1} - e_{n-2}$$

8. Solution to $a_n = 7a_{n-1} - 12a_{n-2}$, $a_1 = 1$, $a_2 = 7$.

$$\begin{aligned} & x^2 = 7x - 12 \\ \iff\ & x^2 - 7x + 12 = 0 \\ \iff\ & x = \frac{7\pm 1}{2} = \begin{cases} 4 \\ 3 \end{cases} \end{aligned}$$

$$\begin{cases} 4u + 3v = 1 \\ 16u + 9v = 7 \end{cases} \implies 16u + 9v - 12u - 9v = 7 - 3$$

$$\iff 4u = 4$$

$$\iff u = 1$$

$$\implies 4\cdot 1 + 3v = 1 \iff v = -1$$

$$a_n = 4^n - 3^n$$

9. What is $\sum_{i=1}^{20} 3^{i-1}$? $\quad 1 + 3 + 3^2 + \dots + 3^{19} = \frac{3^{20}-1}{3-1} = \frac{1}{2}(3^{20}-1)$.

3.2.17 Choose 6 books from a list of 10 fiction and 10 non fiction.
How many ways can such a selection be made?

$$_{(20+6-1)}C_6 = {}_{25}C_6 = \frac{25!}{6!\,19!} = 177100$$

3.4.20 (a) Probability of guessing 4-digit pin?

$$P(E) = \frac{|E|}{|A|} = \frac{1}{10^4} = 0{,}0001 = 0{,}01\%$$

(b) If the 4-digits are birthday digits (MM-DD), then what's the probability of guessing the pin if we know the birthday?

$$P(E) = \frac{1}{{}_4P_4} = \frac{1}{4!} = \frac{1}{24} = 0{,}0417 = 4{,}17\%, \quad \text{if distinct.}$$

$$P(E) = \frac{1}{{}_4P_4/2} = \frac{1}{12}, \quad \text{if two digits are the same.}$$

$$P(E) = \frac{1}{{}_4P_4/{}_3P_3} = \frac{6}{24} = \frac{1}{4}, \quad \text{if three digits are the same.}$$

$$P(E) = \frac{1}{{}_4P_4/4} = \frac{4}{24} = \frac{1}{6}, \quad \text{if there are two pairs.}$$

$P(E) = 1$, if all the digits are the same, 11/11.

3.4.33 3 balls at random from 7 red and 5 black.

(a) 3 red:

$$\frac{{}_7C_3}{{}_{12}C_3} = \frac{\frac{7!}{3!\,4!}}{\frac{12!}{3!\,9!}} = \frac{7!\,9!}{4!\,12!} = \frac{7\cdot6\cdot5}{12\cdot11\cdot10} = \frac{7}{44}$$

(b) at least 2 black: exactly one black has the probability

$$\frac{5\cdot{}_7C_2}{{}_{12}C_3} = \frac{5\cdot\frac{7!}{2!\,5!}}{\frac{12!}{3!\,9!}} = \frac{7\cdot6\cdot5\cdot3\cdot2\cdot1}{12\cdot11\cdot10\cdot2\cdot1} = \frac{21}{44}$$

$$1 - \frac{7}{44} - \frac{21}{44} = \frac{16}{44} = \frac{4}{11}$$

(c) at most 2 balls are black: exactly 2 black balls

$$\frac{7\cdot{}_5C_2}{{}_{12}C_3} = \frac{7\,\frac{5!}{2!\,3!}}{\frac{12!}{3!\,9!}} = \frac{7\cdot5\cdot4\cdot3}{12\cdot11\cdot10} = \frac{7}{22}$$

$$\frac{7}{44} + \frac{21}{44} + \frac{7}{22} = \frac{42}{44} = \frac{21}{22}$$

(d) at least one ball is red:

$$\frac{21}{22}$$

3.4.36 Each day 5 secretaries draw numbers to determine their order of breaks.

(a) Probability that today's order is the same as yesterday?

$$\frac{1}{{}_5P_5} = \frac{1}{5!} = \frac{1}{120}$$

(b) Four have the same order: $\frac{1}{120}$

(c) At least one has the same position: none with same position $4! + 2\cdot{}_5C_2 = 24 + 20 = 44$.

$$1 - \frac{44}{120} = \frac{76}{120} = \frac{19}{30} = 63{,}33\%$$

4.1.5 List the elements given $A = \{a, b\}$ and $B = \{4,5,6\}$.

(a) $A \times B = \{ (x,y) \in A \times B \mid x \in A \wedge y \in B \} = \{ (a,4), (a,5), (a,6), (b,4), (b,5), (b,6) \}$

(b) $B \times A = \{ (4,a), (4,b), (5,a)\ (5,b), (6,a), (6,b) \}$

4.2.3 $A = \mathbb{Z}_+$ and $a \, R \, b \iff 2a \leq b+1$. Which pairs belong?

(a) $(2,2)$: $2 \cdot 2 > 2+1$, $(2,2) \notin R$.

(b) $(3,2) \notin R$ by (a).

(c) $(6,15) \in R$: $6 \cdot 2 \leq 15+1$

(d) $(1,1) \in R$: $2 \cdot 1 \leq 1+1$

(e) $(15,6) \notin R$: $2 \cdot 15 > 6+1$

(f) $(n,n) \in R \iff 2 \cdot n \leq n+1 \iff n \leq 1$.

4.2.4 $A = \{ a, b, c, d \}$, $B = \{1,2,3\}$

$R = \{ (a,1), (a,2), (b,1), (c,2), (d,1) \}$

$\mathrm{Dom}(R) = A$, $\mathrm{Ran}(R) = \{1,2\}$

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

4.2.5 $A = \{$daisy, rose, violet, daffodil, peony$\}$

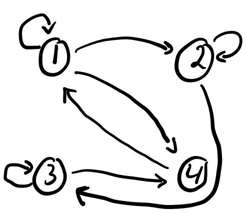
$B = \{$red, white, purple, yellow, blue, pink, orange$\}$

$R = \{$(daisy, red), (violet, pink), (rose, purple), (daffodil, white)$\}$

$\mathrm{Dom}(R) = A \setminus \{\text{peony}\}$, $\mathrm{Ran}(R) = \{\text{red, pink, purple, white}\}$

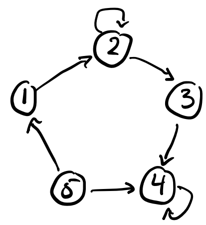
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2.23 $A = \{1,2,3,4\}$, $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.



$R = \{ (1,1), (1,2), (1,4), (2,2), (2,3),$
$(3,3), (3,4), (4,1) \}.$

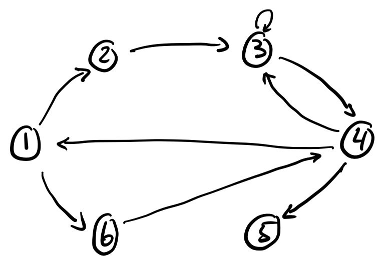
4.2.25



$R = \{(1,2), (2,2), (2,3), (3,4), (4,4), (5,1), (5,4)\}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4.3.1-8



1. Paths of length 1: (1,2), (1,6), (2,3), (3,3), (3,4), (4,3), (4,1), (4,5), (6,4)

2(a) Paths of length 2 from 2: (2,3,3), (2,3,4)

(b) all length 2: (1,2,3), (1,6,4), (2,3,3), (2,3,4), (3,3,3), (3,3,4), (3,4,3),
(3,4,1), (3,4,5), (4,3,3), (4,3,4), (4,1,2), (4,1,6),
(6,4,1), (6,4,3), (6,4,5)

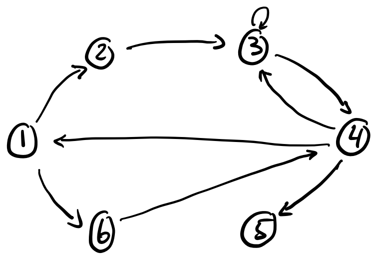
3(a) Same but for 3: (3,3,3,3), (3,3,3,4), (3,3,4,3), (3,3,4,1), (3,3,4,5), (3,4,1,2),
(3,4,1,6), (3,4,3,3), (3,4,3,4)

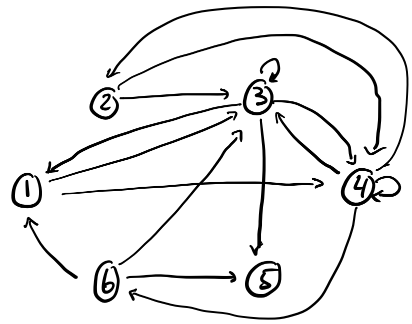
3(b) (3,3,3,3), (3,3,3,4), (3,3,4,3), (3,3,4,1), (3,3,4,5), (3,4,1,2),
(3,4,1,6), (3,4,3,3), (3,4,3,4),
(1,2,3,3), (1,2,3,4), (1,6,4,1), (1,6,4,3), (1,6,4,5),
(2,3,3,3), (2,3,4,1), (2,3,4,3), (2,3,4,5),
(4,1,2,3), (4,1,6,5), (4,3,3,3), (4,3,3,4), (4,3,4,1), (4,3,4,5),
(6,4,1,2), (6,4,1,6), (6,4,3,3), (6,4,3,4)

4. Cycle starting at 2: (2,3,4,1,2)

5. —"— 6 : (6,4,1,6)

6. Digraph of R^2





7. $$M_{R^2} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

8(a) Let $A = \{1, \ldots, 6\}$

$R^\infty = \{(i,j) \in A \setminus \{5\} \times A\}$

8(b)

$$M_{R^\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$