Multiple choice round 2

1. $\forall a \in A \, \exists b \in B: b \mid a^2$.

For every a in A there exist a b in B, such that b divides the square of a.

2. Truth table $(\sim p) \wedge (q \vee r)$

FFTTTF

3. Contrapositive: If I get 3 projects approved and get enough points in the tests, then I will pass the course.

$p \wedge q \Rightarrow r$, so $\sim r \Rightarrow \sim p \vee \sim q$.

4. Negation: All dogs smell and destroy the furniture.

$\forall d: p \wedge q$, so $\exists d: \sim p \vee \sim q$.

5. Negation: $\forall a \in A \, \exists b \in B$: $a+b$ is odd $\Rightarrow$ $a+b^2$ is even.

$\exists a \in A \, \forall b \in B$: $a+b$ is odd $\wedge$ $a+b^2$ is odd. Thm 2 p. 67.

6. $p: \sqrt{2} \in \mathbb{Q}$, $q: \gcd(14, 41) = 1$, $r: 4! = 10$.

$(p \vee \sim q) \Rightarrow r$, $F \Rightarrow F$ is true.

7. Determine the tautology for

A stone cannot fly
Mum Kathy can fly
$\therefore$ Mum Kathy is not a stone

$((p \Rightarrow \sim q) \wedge (r \Rightarrow q)) \Rightarrow (r \Rightarrow \sim p)$.

8. $(A \cup B) \cap (A \cup \bar{B}) = A$.

9. $1+2+3+\cdots+15 = 120$.

3.1.9 (a) $${}_nP_{n-1} = \frac{n!}{(n-(n-1))!} = n!$$

(b) $${}_nP_{n-2} = \frac{n!}{(n-(n-2))!} = \frac{n!}{2}$$

(c) $${}_{n+1}P_{n-1} = \frac{(n+1)!}{((n+1)-(n-1))!} = \frac{(n+1)!}{2}$$

3.1.19 Arrangements of BOUGHT if the vowels are kept together.

We have $5!$ arrangements with OU and for UO, so $2\cdot 5! = 240$.

3.1.33 Zeros at the end of $12!$, $26!$ and $53!$

Search for prime factors 2 and 5, since these make 10.

$$12! = 1\cdot \underset{2}{2}\cdot 3\cdot \underset{2^2}{4}\cdot \underset{5}{5}\cdot \underset{2}{6}\cdot 7\cdot \underset{2^3}{8}\cdot 9\cdot \underset{2\ 5}{10}\cdot 11\cdot \underset{2}{12}$$

We have 2^9 and 5^2, which implies the number of zeros is 2.

Since powers of 2 are plentifull, we need only determine the powers of 5. For $26!$ we have 5, 10, 15, 20 and $25 = 5^2$, so 6 zeros.
For $53!$ we have the additional 30, 35, 40, 45, $50 = 2\cdot 5^2$, so 12 zeros.

For a different approach/refinement:

$$12! : \quad \left\lfloor \frac{12}{5} \right\rfloor + \left\lfloor \frac{12}{5^2} \right\rfloor + \cdots = 2 + 0 + \cdots = 2$$

$$26! : \quad \left\lfloor \frac{26}{5} \right\rfloor + \left\lfloor \frac{26}{5^2} \right\rfloor + \left\lfloor \frac{26}{5^3} \right\rfloor + \cdots = 5 + 1 + 0 + \cdots = 6$$

$$53! : \quad \left\lfloor \frac{53}{5} \right\rfloor + \left\lfloor \frac{53}{5^2} \right\rfloor + \left\lfloor \frac{53}{5^3} \right\rfloor + \cdots = 10 + 2 + 0 + \cdots = 12$$

3.1.34 The number of zeros at the end of $n!$ is the sum $\sum_{i=1}^{n} \left\lfloor \frac{n}{5^i} \right\rfloor$.
This follows from our assertions in 3.1.33.

3.2.2 (a) $${}_nC_{n-1} = \frac{n!}{(n-1)!\,(n-(n-1))!} = n$$

(b) $${}_nC_{n-2} = \frac{n!}{(n-2)!\,(n-(n-2))!} = \frac{n(n-1)}{2}$$

(c) $${}_{n+1}C_{n-1} = \frac{(n+1)!}{(n-1)!\,((n+1)-(n-1))!} = \frac{(n+1)\,n}{2}$$

3.5.4 $d_n = n \cdot d_{n-1}$, $d_1 = 2$.

$d_1 = 2$
$d_2 = 2 \cdot d_1 = 2 \cdot 2 = 4$
$d_3 = 3 \cdot d_2 = 3 \cdot 4 = 12$
$d_4 = 4 \cdot d_3 = 4 \cdot 12 = 48$ The relation is not linear homogeneous.

3.5.5 $e_n = 5e_{n-1} + 3$, $e_1 = 1$.

$e_1 = 1$
$e_2 = 5 \cdot e_1 + 3 = 8$
$e_3 = 5 \cdot e_2 + 3 = 43$
$e_4 = 5 \cdot e_3 + 3 = 218$ The relation is not linear homogeneous.

3.5.18 $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 2$, $a_2 = 6$.

This is linear homogeneous of degree 2. The associated polynomial has the following characteristic equation.

$$x^2 = 4x + 5 \iff x^2 - 4x - 5 = 0$$
$$\iff x = \frac{4 \pm \sqrt{36}}{2} = \begin{cases} 5 \\ -1 \end{cases}$$

Let $s_1 = 5$ and $s_2 = -1$, then we solve u and v by use of $a_1 = 2$ and $a_2 = 6$.

$$\begin{cases} u \cdot 5 + v \cdot (-1) = 2 \\ u \cdot 5^2 + v \cdot (-1)^2 = 6 \end{cases} \iff \begin{cases} 5u - v = 2 \\ 25u + v = 6 \end{cases}$$

$$\Rightarrow 30u = 8 \iff u = \frac{8}{30} = \frac{4}{15}$$

$$\Rightarrow 5 \cdot \frac{4}{15} - v = 2 \iff v = -2 + \frac{4}{3} = -\frac{2}{3} = -\frac{10}{15}$$

Thus $a_n = \frac{4}{15} 5^n - \frac{10}{15} (-1)^n$

3.5.19 $b_n = -3b_{n-1} - 2b_{n-2}$, $b_1 = -2$, $b_2 = 4$

$$x^2 = -3x - 2 \iff x^2 + 3x + 2 = 0$$
$$\iff x = \frac{-3 \pm 1}{2} = \begin{cases} -1 \\ -2 \end{cases}$$

$$\begin{cases} -u - 2v = -2 \\ u + 4v = 4 \end{cases} \Rightarrow 2v = 2 \iff v = 1$$
$$\Rightarrow u + 4 \cdot 1 = 4 \iff u = 0.$$ So $b_n = (-2)^n$

3.5.20 $\quad c_n = -6c_{n-1} - 9c_{n-2}, \quad c_1 = 2{,}5, \quad c_2 = 4{,}7$

$$x^2 = -6x - 9 \Leftrightarrow x^2 + 6x + 9 = 0$$
$$\Leftrightarrow (x+3)^2 = 0$$
$$\Leftrightarrow x = -3.$$

$$c_n = u(-3)^n + vn(-3)^n : \quad \begin{cases} -3u - 3v = 2{,}5 \\ 9u + 18v = 4{,}7 \end{cases} \Leftrightarrow \begin{cases} -9u - 9v = 7{,}5 \\ 9u + 18v = 4{,}7 \end{cases}$$

$$\Rightarrow 9v = 12{,}2 = \frac{122}{10} \Leftrightarrow v = \frac{122}{90}$$
$$\Rightarrow -3u - 3\cdot\frac{122}{90} = \frac{5}{2}$$
$$\Leftrightarrow u + \frac{122}{90} = -\frac{5}{6}$$
$$\Leftrightarrow u = -\frac{75}{90} - \frac{122}{90} = -\frac{197}{90}$$

$$c_n = -\frac{197}{90}\cdot(-3)^n + \frac{122}{90}\cdot n\cdot(-3)^n$$

3.5.22 $\quad e_n = 2e_{n-2}, \quad e_1 = \sqrt{2}, \quad e_2 = 6$

$$x^2 = 0 + 2 \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm\sqrt{2}.$$

$$\begin{cases} \sqrt{2}u - \sqrt{2}v = \sqrt{2} \\ 2u + 2v = 6 \end{cases} \Rightarrow 4u = 8 \Leftrightarrow u = 2$$
$$\Rightarrow 2\cdot 2 + 2v = 6$$
$$\Leftrightarrow 2v = 2$$
$$\Leftrightarrow v = 1$$

$$c_n = 2(\sqrt{2})^n + (-\sqrt{2})^n.$$