Complex Numbers

Ex 1. a) $\frac{1}{3} + \frac{1}{2} - \frac{1}{12} = \frac{3}{4}$

$$\left(\frac{2}{6} + \frac{3}{6} - \frac{1}{12} = \frac{10}{12} - \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \right)$$

Ex 2. a)

$$z = 3(i-10) - 5(7-2i) - i(3i-5) + 3i(i-5)$$

$$= 3i - 30 - 35 + 10i + 3 + 5i - 3 - 15i$$

$$= -65 + 3i$$

b) $a = 5 - i(3-i) + 6i \ , \quad b = -5 - 4(-2i + 1)$

$$z = a + ib = 5 - i(3-i) + 6i + i\left(-5 - 4(-2i+1)\right)$$

$$= 5 - 3i - 1 + 6i - 5i - 8 - 4i$$

$$= -4 - 6i$$

Ex 3. a)

$$\frac{-2+3i}{i} = \frac{2i+3}{1} = 3 + 2i$$

$$\mathrm{Re}\left(\frac{-2+3i}{i}\right) = 3 \ , \quad \mathrm{Im}\left(\frac{-2+3i}{i}\right) = 2$$

b)

$$\frac{3}{5} - \frac{3-2i}{2+i} = \frac{3}{5} - \frac{(3-2i)(2-i)}{4+1}$$

$$= \frac{3}{5} - \frac{6 - 3i - 4i - 2}{5} = \frac{3}{5} - \frac{4}{5} + \frac{7}{5}i$$

$$= -\frac{1}{5} + \frac{7}{5}i \qquad \mathrm{Re}\left(\frac{3}{5} - \frac{3-2i}{2+i}\right) = -\frac{1}{5}$$

$$\mathrm{Im}\left(\frac{3}{5} - \frac{3-2i}{2+i}\right) = \frac{7}{5}$$

c) Let $b = 5$, $c = \frac{6}{7}$ and $d = \frac{2}{3}$

$$c + d = \frac{18}{21} + \frac{14}{21} = \frac{32}{21}$$

$$d \cdot b = \frac{2}{3} \cdot 5 = \frac{10}{3}$$

$$\frac{b}{d} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$$

$$\frac{d}{c} = \frac{\frac{2}{3}}{\frac{6}{7}} = \frac{14}{18} = \frac{7}{9}$$

d) Let $k = 1 + \sqrt{3}\,i$, $n = 5i$, $m = 1 + i$, $s = 3 + 4i$

$$\frac{m}{n} = \frac{1+i}{5i} = \frac{(1+i)\cdot(-5i)}{25} = \frac{5 - 5i}{25} = \frac{1}{5} - \frac{1}{5}i$$

$$\frac{k}{s} = \frac{1 + \sqrt{3}\,i}{3 + 4i} = \frac{(1 + \sqrt{3}\,i)(3 - 4i)}{9 + 16} = \frac{3 - 4i + 3\sqrt{3}\,i + 4\sqrt{3}}{25}$$

$$= \frac{3 + 4\sqrt{3}}{25} + \frac{3\sqrt{3} - 4}{25}i$$

$$\frac{1}{m} + s = \frac{1}{1+i} + 3 + 4i = \frac{1-i}{2} + 3 + 4i$$

$$= \frac{7}{2} + \frac{7}{2}i$$

Ex 4. a) $(u+v)^2 + (u-v)^2 = u^2 + 2uv + v^2 + u^2 - 2uv + v^2$

$= 2u^2 + 2v^2$

b) $\dfrac{u^2-v^2}{u+v} + \dfrac{v^2-u^2}{v-u} = \dfrac{(u+v)(u-v)}{u+v} + \dfrac{(v+u)(v-u)}{v-u}$

$= u - v + v + u$

$= 2u$

c) $(3+5i)(3+5i) = 9 - 25 + 30i = -16 + 30i$

$(3i+5)(3i-5) = -9 - 25 = -34$

$\dfrac{3-4i}{3+4i} = \dfrac{(3-4i)^2}{9+16} = \dfrac{9-16-24i}{25} = -\dfrac{7}{25} - \dfrac{24}{25}i$

d) Prove that $z \cdot \bar{z} = |z|^2$

Firstly we have for $z = a+ib$

$|z|^2 = \sqrt{a^2+b^2}^{\,2} = a^2+b^2$. (def. of modulus)

Now it follows by computation that

$z \cdot \bar{z} = (a+ib)\cdot(a-ib) = a^2+b^2$

Ex 5. a) Solve $(1-i)z + 1 = 2+i$

$\Leftrightarrow \quad z = \dfrac{2+i-1}{1-i} = \dfrac{(1+i)^2}{2} = \dfrac{2i}{2} = i$

$\left[(1-i)i + 1 = i + 1 + 1 = 2 + i\right]$

b) $(x+2i)(x-2i)(x-5)=0$ Zero product rule

$x=-2i \lor x=2i \lor x=5$

c) $x^4-x^3+4x^2-4x=0$, roots $0, 1, 2i, -2i$

$\Leftrightarrow x(x^3-x^2+4x-4)=0 \Rightarrow x=0$ is a root.

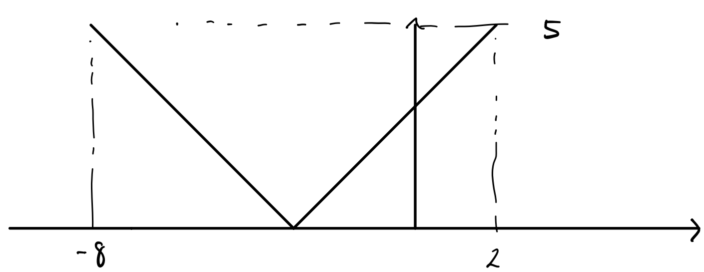
Need only test $x^3-x^2+4x-4=0$

$x=1$: $1^3-1^2+4\cdot1-4=0$

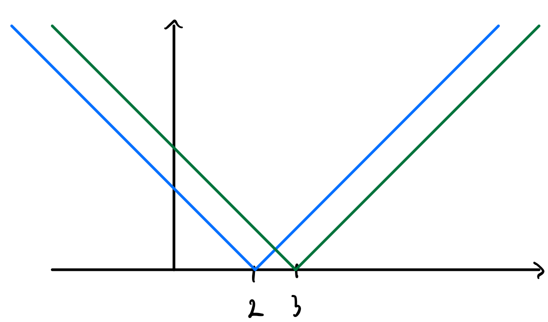
$x=2i$: $(2i)^3-(2i)^2+4\cdot2i-4=-8i+4+8i-4=0$

$x=-2i$: $(-2i)^3-(-2i)^2+4\cdot(-2i)-4=8i+4-8i-4=0$

Ex 6. a) $|x+3|=5 \Leftrightarrow x=2 \lor x=-8$



b) $|x-2|=|3-x|$, note $|3-x|=|x-3|$



$x=\frac{5}{2}$

$x-2=3-x$

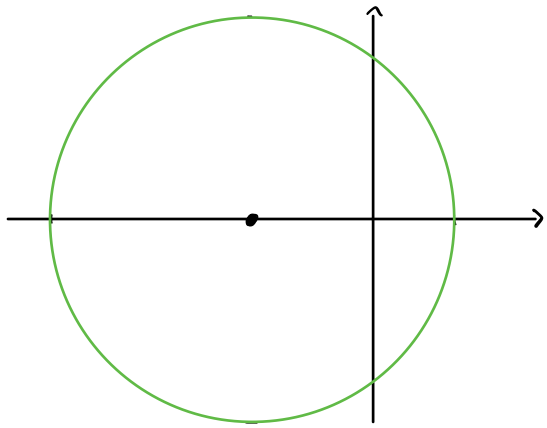
$\Leftrightarrow 2x=5$

$\Leftrightarrow x=\frac{5}{2}$

c) Complex solutions, $z = a+ib$

$|z+3| = 5$

$\{z \in \mathbb{C} \mid |z+3| = 5\}$



$|z-2| = |3-z|$ $\quad z = \frac{5}{2} + ib$ for all $b \in \mathbb{R}$.

Ex 7. a) Let $A = \{n \in \mathbb{N} \mid n = m^2 \text{ where } m \in \{1,2,3,4,5\}\}$

and $B = \{n \in \mathbb{N} \mid n = 2m-1 \text{ where } m \in \{1,2,3,4,5\}\}$.

Then $A = \{1, 4, 9, 16, 25\}$ and

$B = \{1, 3, 5, 7, 9\}$.

We have

$$A \cap B = \{1, 9\}$$

$$A \cup B = \{1, 3, 4, 5, 7, 9, 16, 25\}$$

b) Let $C = \{n \in \mathbb{N} \mid n = 2m \text{ where } m \in \mathbb{N}\}$ and

$D = \{n \in \mathbb{N} \mid n = 3m \text{ where } m \in \mathbb{N}\}$.

$$C \cap D = \{n \in \mathbb{N} \mid n = 6m \text{ where } m \in \mathbb{N}\}$$

$$C \cup D = \{n \in \mathbb{N} \mid n = 2m \vee n = 3(2m-1) \text{ where } m \in \mathbb{N}\}$$

c) The set $\mathbb{R} \setminus \mathbb{Q}$ is the reals with no rationals, i.e. the set of irrational numbers.

The set $\mathbb{C} \setminus \mathbb{R}$ is purely imaginary numbers, i.e. $z = a + ib$ with $a = 0$ and $b \in \mathbb{R}$.