105. Brúka integralkriteriið til at vísa ólíkningarnar.

(i) $\frac{\pi}{4} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{\pi}{4} + \frac{1}{2}$. Lat $f(x) = \frac{1}{x^2+1}$.

$$\int f(x)\,dx = \int \frac{1}{x^2+1}\,dx = \arctan(x), \quad \text{so}$$

$$\int_1^t f(x)\,dx = [\arctan(t)]_1^t = \arctan(t) - \arctan(1)$$
$$\to \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \text{tá } t \to \infty.$$

Per integralkriteriið fáa vit, at

$$\int_1^\infty \frac{1}{x^2+1}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \int_1^\infty \frac{1}{x^2+1}\,dx + f(1).$$

$$\iff \frac{\pi}{4} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{\pi}{4} + \frac{1}{2}.$$

(ii) $\frac{1}{8} \leq \sum_{n=2}^{\infty} \frac{1}{n^3} \leq \frac{1}{4}$. Lat $f(x) = \frac{1}{x^3}$.

$$\int f(x)\,dx = \int \frac{1}{x^3}\,dx = -\frac{1}{2x^2}, \quad \text{so}$$
$$\int_2^t f(x)\,dx = \left[-\frac{1}{2x^2}\right]_2^t = \frac{1}{8} - \frac{1}{2t^2} \to \frac{1}{8} \quad \text{tá } t \to \infty.$$

Per integralkriteriið fáa vit, at

$$\int_2^\infty \frac{1}{x^3}\,dx \leq \sum_{n=2}^{\infty} \frac{1}{n^3} \leq \int_2^\infty \frac{1}{x^3}\,dx + f(2)$$
$$\iff \frac{1}{8} \leq \sum_{n=2}^{\infty} \frac{1}{n^3} \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

144. (i) Fyri hvørji $x \in \mathbb{R}$ er rekkjan $\sum_{n=1}^{\infty} (x+1)^n$ konvergent?

Kvotientrekkjan er per 5.2 konvergent tá

$$|x+1| < 1$$
$$\iff -1 < x+1 < 1$$
$$\iff -2 < x < 0.$$

Altso er rekkjan konvergent fyri $x \in (-2,0)$.

(ii) Finn sumfunktiónina $f(x)$ hjá rekkjuni tá $x \in (-2,0)$.

Sumfunktiónin er per 5.2 givin við $f(x) = \frac{1}{1-(x+1)} = \frac{1}{x}$, $x \in (-2,0)$.

159 Er rekkjan $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+4}$ betinget konvergent, absolut konvergent ella divergent?

Vit kanna absolut konvergens.

$$\left|(-1)^n \frac{n}{n^3+4}\right| = \left|\frac{n}{n^3+4}\right| = \frac{1}{n^2+\frac{4}{n}} \leq \frac{1}{n^2}.$$

Absolutta rekkjan er lægri enn $\frac{1}{n^2}$ fyri øll $n \in \mathbb{N}$, so við samanberingskriteriið er rekkjan absolut konvergent.

162. Vís, at fyri eitthvørt $\alpha > 1$ og $N \in \mathbb{N}$ er

(i) $\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{(N+1)^\alpha} \frac{\alpha+N}{\alpha-1}$.

Við korollar 4.35 til integralkriteriið kunnu vit vurdera rekkjuna $\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha}$ við ta kontinuertu og avtakandi funktiónini $f(x) = \frac{1}{x^\alpha}$, $x \in [1, \infty)$.

$$\int \frac{1}{x^\alpha}\,dx = \int x^{-\alpha}\,dx = \frac{1}{1-\alpha} x^{1-\alpha} = -\frac{1}{\alpha-1} \frac{x}{x^\alpha}.$$

$$\int_{N+1}^{t} \frac{1}{x^\alpha}\,dx = \left[-\frac{1}{\alpha-1}\frac{x}{x^\alpha}\right]_{N+1}^{t} = \frac{1}{\alpha-1}\frac{N+1}{(N+1)^\alpha} - \frac{1}{\alpha-1}\frac{t}{t^\alpha}$$

$$\to \frac{1}{\alpha-1}\frac{N+1}{(N+1)^\alpha} \quad \text{tá } t \to \infty \text{ tí } \alpha > 1.$$

Integralið er konvergent fyri $\alpha > 1$, so vit brúka 4.35(i).

$$\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{\alpha-1}\frac{N+1}{(N+1)^\alpha} + \frac{1}{(N+1)^\alpha} = \frac{1}{\alpha-1}\frac{N+1}{(N+1)^\alpha} + \frac{\alpha-1}{(N+1)^\alpha(\alpha-1)}$$

$$= \frac{1}{\alpha-1}\frac{N+\alpha}{(N+1)^\alpha}.$$

(ii) $\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{(N+1)^{\alpha-1}} \frac{\alpha}{\alpha-1}$.

Vit hava $\alpha > 1$, so $\alpha + N < \alpha + \alpha N$. Hesin eginleiki og (i) geva okkum vurderingina

$$\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{(N+1)^\alpha}\frac{\alpha+N}{\alpha-1} \leq \frac{1}{(N+1)^\alpha}\frac{\alpha+\alpha N}{\alpha-1} = \frac{1}{(N+1)^\alpha}\frac{\alpha(1+N)}{\alpha-1} = \frac{1}{(N+1)^{\alpha-1}}\frac{\alpha}{\alpha-1}.$$

521. Vit hyggja at rekkjuna $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^5+1}$.

(i) Vís, at fyri $n \in \mathbb{N}$ er $\dfrac{n}{n^5+1} \leq \dfrac{1}{n^4}$.

Vit fáa beinleiðis, at $\dfrac{n}{n^5+1} = \dfrac{1}{n^4+\frac{1}{n}} \leq \dfrac{1}{n^4}$ tí $\dfrac{1}{n} > 0$ fyri $n \in \mathbb{N}$.

(ii) Vís, at rekkjan er konvergent.

Vit hava við (i), at $\frac{n}{n^5+1} \leq \frac{1}{n^4}$ fyri ~~ell~~ $n \in \mathbb{N}$, og av tí at rekkjan $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ er konvergent, so er $\sum_{n=1}^{\infty} \frac{n}{n^5+1}$ konvergent per 4.20(i).

(iii) Tað er givið, at $\displaystyle\sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \frac{1}{(N+1)^3}$ (sí 162). Finn við hesum og (i) eitt $N \in \mathbb{N}$, so at $\displaystyle\sum_{n=N+1}^{\infty} \frac{n}{n^5+1} \leq 0{,}05$.

Ólíkningin úr (i) og niðurstøðan úr (ii) letur okkum vurdera

$$\sum_{n=1}^{\infty} \frac{n}{n^5+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \frac{1}{(N+1)^3}.$$

Vit kunnu nú loysa eftir N, so at hetta síðsta er undir 0,05.

$$\frac{4}{3} \frac{1}{(N+1)^3} \leq \frac{1}{20} \iff \frac{80}{3} \leq (N+1)^3 \qquad \text{(evt. bara seta inn)}$$
$$\iff N+1 \geq \sqrt[3]{\frac{80}{3}}$$
$$\iff N \geq \sqrt[3]{\frac{80}{3}} - 1 = 1{,}9876.$$

Fyri $N=2$ er $\displaystyle\sum_{n=N+1}^{\infty} \frac{n}{n^5+1} \leq 0{,}05$.

(iv) Nýt úrslitið í (iii) at estimera $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^5+1}$ við eitt frávik á í mesta lagi 0,05.

Vit hava per (iii), at $\displaystyle\sum_{n=2+1}^{\infty} \frac{n}{n^5+1} \leq 0{,}05$, so restin er altso $\displaystyle\sum_{n=1}^{2} \frac{n}{n^5+1}$.
Vit kunnu per konvergens skriva, at

$$\sum_{n=1}^{\infty} \frac{n}{n^5+1} = \sum_{n=1}^{2} \frac{n}{n^5+1} + \sum_{n=2+1}^{\infty} \frac{n}{n^5+1}$$
$$\iff \sum_{n=1}^{\infty} \frac{n}{n^5+1} - \sum_{n=1}^{2} \frac{n}{n^5+1} = \sum_{n=2+1}^{\infty} \frac{n}{n^5+1} \leq 0{,}05.$$

Altso $\displaystyle\sum_{n=1}^{2} \frac{n}{n^5+1}$ er ein approksimatión av rekkjuni við frávik á í mesta lagi 0,05.

$$\sum_{n=1}^{2} \frac{n}{n^5+1} = \frac{1}{2} + \frac{2}{33} = \frac{33}{66} + \frac{4}{66} = \frac{37}{66}.$$