Matrix Algebra

Ex 1. A man is 30 years older than his son. In 8 years he is 4 times than the son.

a) Write as equations. Reduce the augmented matrix.

Let the fathers age be y and the sons age x.

$$\begin{cases} y = x + 30 \\ y + 8 = 4 \cdot (x + 8) \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 30 \\ 1 & -4 & 24 \end{array}\right] \begin{array}{l} \\ -R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 30 \\ 0 & -3 & -6 \end{array}\right] \begin{array}{l} +R_2 \\ \cdot\left(-\frac{1}{3}\right) \end{array}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 32 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} y = 32 \\ x = 2 \end{array}$$

b) Check the solution.

$$32 = 2 + 30$$

$$32 + 8 = 40 = 4(2 + 8)$$

Ex 2. $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 7 & 2 & 8 \\ 2 & 4 & 0 & 4 \end{bmatrix}$

a) Determine A^T. $\qquad A^T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 7 & 4 \\ 2 & 2 & 0 \\ 4 & 8 & 4 \end{bmatrix}$

b) Solve $\underline{x}\,\underline{\underline{A}} = [2 \;\; 5 \;\; 2 \;\; 6]$.

We have that
$$\underline{x}\,\underline{\underline{A}} = \underline{b}$$
$$\Leftrightarrow (\underline{x}\,\underline{\underline{A}})^T = \underline{b}^T$$
$$\Leftrightarrow \underline{\underline{A}}^T \underline{x} = \underline{b}^T .$$

This implies that we can write the augmented matrix and solve for $\underline{x}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 3 & 7 & 4 & 5 \\ 2 & 2 & 0 & 2 \\ 4 & 8 & 4 & 6 \end{array}\right]\begin{array}{l} \\ -3R_1 \\ -2R_1 \\ -4R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & -2 & -2 & -1 \\ 0 & -4 & -4 & -2 \\ 0 & -4 & -4 & -2 \end{array}\right]\begin{array}{l} \\ \cdot(-\frac{1}{2}) \\ -2R_2 \\ -2R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} -3R_2 \\ \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$, then

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Ex3.

a) Solve by Gaussian elimination.

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 1 & 4 \end{array}\right]\begin{array}{l} \\ -2R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 6 \end{array}\right] \qquad \underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right] \begin{array}{c} \\ -2R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -6 \end{array}\right] \qquad \underline{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Let $\underline{\underline{A}} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $\underline{\underline{B}} = \begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix}$ be given and consider $\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{B}}$.

b) Which form does $\underline{\underline{X}}$ have? Solve the system by Gaussian elimination.

$$\underline{\underline{X}} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 3 \\ 2 & 1 & 4 & 0 \end{array}\right] \begin{array}{c} \\ -2R_1 \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & -1 & 3 \\ 0 & 3 & 6 & -6 \end{array}\right]$$

$\Rightarrow \quad x_{21} = 2, \quad x_{22} = -2 \quad \Rightarrow \quad x_{11} = 1, \quad x_{12} = 1.$

This corresponds with the expected solutions of a).

c) Justify $\underline{\underline{A}}$ is invertible, and use $\underline{\underline{A}}^{-1}$ to solve the equation.

We've seen $\underline{\underline{A}}$ is regular by earlier elimination, and $\underline{\underline{A}}$ is quadratic.

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{B}} \iff \underline{\underline{X}} = \underline{\underline{A}}^{-1}\underline{\underline{B}}$$

$$\underline{\underline{A}}^{-1} = \frac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = 3$$

$$\underline{\underline{X}} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 3 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

Ex 4. Let

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \underline{\underline{B}} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

a) Show that $\underline{\underline{BA}}$ is regular, and determine $(\underline{\underline{BA}})^{-1}$.

$$\underline{\underline{BA}} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 5 & 6 \end{bmatrix}$$

Since $\underline{\underline{BA}}$ is 2×2 and $\det(\underline{\underline{BA}}) = -6 \neq 0$ it follows that $\rho(\underline{\underline{BA}}) = 2$, so $\underline{\underline{BA}}$ is regular.

$$(\underline{\underline{BA}})^{-1} = \frac{1}{-6} \begin{bmatrix} 6 & -6 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \frac{5}{6} & -\frac{2}{3} \end{bmatrix}.$$

b) Show that $\underline{\underline{AB}}$ is not singular, and therefore $(\underline{\underline{AB}})^{-1}$ cannot be determined.

$$\underline{\underline{AB}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 6 & 7 & 8 \end{bmatrix} \end{array} \qquad v_3 = 2v_2 - v_1.$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix} \begin{matrix} \\ \\ -2R_1 \end{matrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \begin{matrix} \\ \\ -3R_2 \end{matrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(\underline{\underline{AB}}) = 2.$$

Ex5. Consider
$$\begin{cases} x_1 + x_2 + 2x_3 = 3 \\ 2x_1 - x_2 + 4x_3 = 0 \\ x_1 + 3x_2 - 2x_3 = 3 \\ -3x_1 - 2x_2 + x_3 = 0 \end{cases}$$

a) Find the RREF of the above. Determine rank of $\underline{\underline{A}}$ and $\underline{\underline{T}}$.

How many solutions are there? Compute them.

$$\underline{\underline{T}} = \begin{array}{c} \underline{\underline{A}} \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & -1 & 4 & 0 \\ 1 & 3 & -2 & 3 \\ -3 & -2 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ -2R_1 \\ -R_1 \\ +3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -3 & 0 & -6 \\ 0 & 2 & -4 & 0 \\ 0 & 1 & 7 & 9 \end{array}\right] \begin{array}{l} \\ \cdot\left(-\frac{1}{3}\right) \\ \cdot\left(\frac{1}{2}\right) \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 7 & 9 \end{array}\right] \begin{array}{l} -R_2 \\ \\ -R_2 \\ -R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 7 & 7 \end{array}\right] \begin{array}{l} -2R_3 \\ \\ \cdot\left(-\frac{1}{2}\right) \\ -7R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \rho(\underline{\underline{A}}) = \rho(\underline{\underline{T}}).$$

b) State the solution to $\underline{\underline{A}}\,\underline{x} = \underline{0}$.

It follows by a) that the only solution is $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now we consider

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 + x_5 = 2$$
$$3x_1 + 4x_2 + 5x_3 + 6x_4 - 3x_5 = 3$$

c) Find the RREF of the above. Determine rank of $\underline{\underline{A}}$ and $\underline{\underline{T}}$.

How many solutions are there? Compute them.

$$\underline{\underline{T}} = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 5 & 1 & 2 \\ 3 & 4 & 5 & 6 & -3 & 3 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ -3R_1 \end{array} \to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & -3 & -9 & 0 \\ 0 & -2 & -4 & -6 & -18 & 0 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(-\frac{1}{2}) \end{array}$$

(with $\underline{\underline{A}}$ marked above the coefficient part)

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 1 & 2 & 3 & 9 & 0 \\ 0 & 1 & 2 & 3 & 9 & 0 \end{array}\right] \begin{array}{l} -2R_2 \\ \\ -R_2 \end{array} \to \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -13 & 1 \\ 0 & 1 & 2 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t_3$, $x_4 = t_4$ and $x_5 = t_5$. $\rho(\underline{\underline{A}}) = \rho(\underline{\underline{T}})$

$$\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} 13 \\ -9 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_3, t_4, t_5 \in \mathbb{R}.$$

There are infinitely many solutions.

d) Find the solutions to the homogeneous equation.

This corresponds to the parametrized parts of the solution of c):

$$\underline{x} = t_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} 13 \\ -9 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_3, t_4, t_5 \in \mathbb{R}.$$