Multiple choice round

1. $A = \{1,2,4\}$, $B = \{2,3\}$ og $C = \{1,2\}$.

$A \cap (B \cup C) = A \cap \{1,2,3\} = \{1,2\}$.

2. Find $\gcd(42,18)$

$$\left.\begin{aligned} 42 &= 2 \cdot 3 \cdot 7 \\ 18 &= 2 \cdot 3^2 \end{aligned}\right\} \Rightarrow \gcd(42,18) = 2 \cdot 3 = 6.$$

3. Find $\operatorname{lcm}(24,15)$

$$\left.\begin{aligned} 24 &= 2^3 \cdot 3 \\ 15 &= 3 \cdot 5 \end{aligned}\right\} \Rightarrow \operatorname{lcm}(24,15) = 2^3 \cdot 3 \cdot 5 = 120.$$

4. Find $\gcd(8085, 7623)$

$$8085 = 1 \cdot 7623 + 462$$
$$7623 = 16 \cdot 462 + 231$$
$$462 = 2 \cdot 231$$

$\gcd(8085, 7623) = 231$.

5. $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. 200 in base 3.

$$200 = 66 \cdot 3 + 2$$
$$66 = 22 \cdot 3 + 0$$
$$22 = 7 \cdot 3 + 1$$
$$7 = 2 \cdot 3 + 1$$
$$2 = 0 \cdot 3 + 2$$

$(21102)_3$

7. Which number is even: $(101)_{13} = 1 + 0 + 13^2 = 70$

Remember to count the number of even/odd entries.

8.
$$\begin{array}{r} (1232)_4 \\ +\ (3120)_4 \\ \hline (11012)_4 \end{array}$$

9. Prime factorization:

$$\begin{aligned} 495 &= 5 \cdot 99 \\ &= 3 \cdot 5 \cdot 33 \\ &= 3^2 \cdot 5 \cdot 11 \end{aligned}$$

2.4.21 Let $P(n)$: n^2+n is odd for $n \in \mathbb{Z}_+$.

(a) Prove that $P(k) \Rightarrow P(k+1)$ is a tautology.

Assume that $P(k)$ is true for fixed $k \geq 1$, then

$$\begin{aligned} (k+1)^2 + (k+1) &= k^2+2k+1+k+1 \\ &= k^2+k+2k+2 \\ &= k^2+k+2(k+1). \end{aligned}$$

Thus $P(k+1)$ is true, since $2 | (k^2+k)$ by assumption, and $2 | 2(k+1)$.
Hence $P(k) \Rightarrow P(k+1)$ is a tautology.

(b) Is $P(n)$ true for all n?

Not true, for instance $P(1)$: 1^2+1 is odd, which is false.
Indeed for odd n we have n^2 is odd, so n^2+n is even. Furthermore for even n we have n^2 is even, so n^2+n is even. So $P(n)$ is false for all n.

2.4.22 Explain the flaw. For $z \neq 0$, $z^n = 1$, $n \geq 0$.

Basis step: $P(0)$: $z^0 = 1$ is true by definition.
Induction step: $z^{k+1} = \frac{z^k}{z^{k-1}} \cdot z^k = \frac{1}{1} \cdot 1 = 1$.

In the induction step both $P(k)$ and $P(k-1)$ are assumed to be true, which is not the case.

2.4.23 Explain the flaw in the argument. $P(n)$: all trucks are the same color.

The induction step doesn't hold for $k=1$.

3.1.5 A fair six-sided die is tossed four times. How many different sequences are there?

Each toss is independent with 6 possible outcomes. Thus 6^4, see Thm. 3.

3.1.8 (a) ${}_4P_4 = \frac{4!}{(4-4)!} = 4! = 24.$

(b) ${}_6P_5 = \frac{6!}{(6-5)!} = 6! = 720.$

(c) ${}_7P_2 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = 7 \cdot 6 = 42.$

3.2.1 (a) $_7C_7 = \frac{7!}{7!(7-7)!} = 1.$

(b) $_7C_4 = \frac{7!}{4!(7-4)!} = \frac{7!}{4!\,3!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 7\cdot 5 = 35.$

(c) $_{16}C_5 = \frac{16!}{5!(16-5)!} = \frac{16!}{5!\,11!} = \frac{16\cdot 15\cdot 14\cdot 13\cdot 12}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 16\cdot 7\cdot 13\cdot 3 = 4368.$

3.2.5 How many ways may a six card hand be drawn from 52 cards?

$$_{52}C_6 = \frac{52!}{6!\,(52-6)!} = \frac{52!}{6!\,46!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= 13\cdot 17\cdot 5\cdot 49\cdot 8\cdot 47$$

$$= 20\,358\,520.$$

3.1.18 Different permutations of GROUP.

All the letters are distinct, so we have the permutation of 5 elements.

$$_5P_5 = 5! = 120.$$

3.2.8 How many different eight-card hands with five red cards and three black cards can be dealt from 52 cards?

$$_{26}C_5 \cdot {}_{26}C_3 = \frac{26!}{5!(26-5)!} \quad \frac{26!}{3!(26-3)!}$$

$$= \frac{26\cdot 25\cdot 24\cdot 23\cdot 22}{5\cdot 4\cdot 3\cdot 2\cdot 1} \quad \frac{26\cdot 25\cdot 24}{3\cdot 2\cdot 1}$$

$$= 26\cdot 5\cdot 23\cdot 22 \cdot 26\cdot 25\cdot 4$$

$$= 171\,028\,000.$$

3.1.20 Find the number of distinguishable permutations of BOOLEAN.

See Thm. 5: $\frac{7!}{2!} = 7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520.$