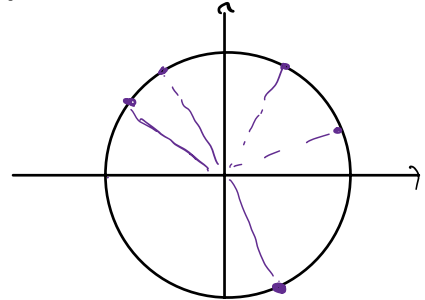


## Complex Numbers in Polar form

Ex 1.  $\sin\left(\frac{3\pi}{2}\right) = -1$

Ex 2.a) Radian values of angles in degrees:

Degrees	Radians
30	$\frac{\pi}{6}$
60	$\frac{\pi}{3}$
120	$\frac{2\pi}{3}$
135	$\frac{2\pi}{3} + \frac{\pi}{12} = \frac{3\pi}{4}$
300	$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

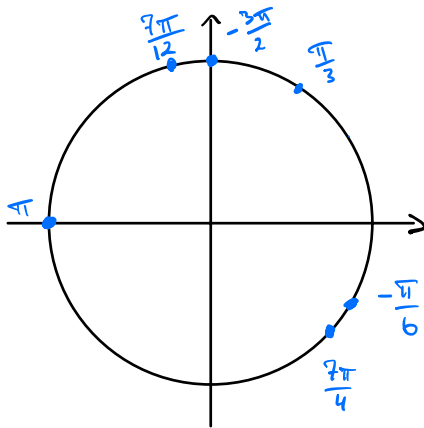


$$180 - 135 = 45 \dots$$

$$360 - 300 = 60$$

b) Point and degrees corresponding to

$$\pi, \frac{\pi}{3}, -\frac{\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{12}, -\frac{3\pi}{2}, \frac{7\pi}{4}$$



$$\frac{\pi}{6} \text{ is } 30^\circ, \text{ so } \frac{\pi}{12} \text{ is } 15^\circ$$

Radians	Degrees
$\pi$	180
$\frac{\pi}{3}$	60
$-\frac{\pi}{6}$	-30 or 330
$\frac{7\pi}{12}$	105
$-\frac{3\pi}{2}$	-270 or 90
$\frac{7\pi}{4}$	315 or -45

Ex 3. Sine and Cosine

a) Use the triangle to compute  $\sin(\frac{\pi}{4})$  and  $\cos(\frac{\pi}{4})$ .

Since  $x=y$  it follows that

$$\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4})$$

by definition. Additionally the Pythagorean theorem yields

$$\cos^2(\frac{\pi}{4}) + \sin^2(\frac{\pi}{4}) = 1, \text{ let's use } x \text{ as unknown.}$$

$$\Leftrightarrow 2x^2 = 1$$

$$\Leftrightarrow x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \quad (\text{Note positive solution!})$$

b) Compute solutions to pairs via

$$\cos(p \frac{\pi}{4}) \text{ and } \sin(p \frac{\pi}{4})$$

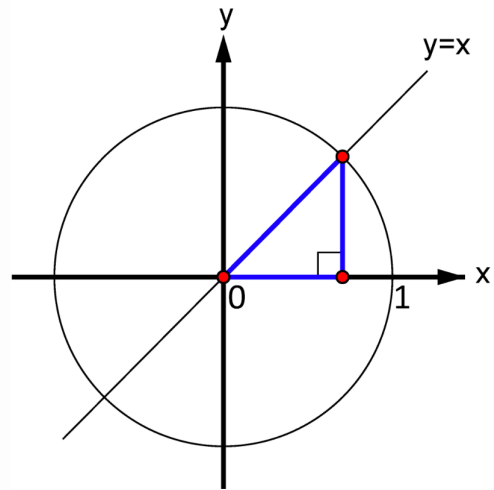
for  $p = 3, 5, 7, -1, -3, -5, -7$  via symmetry.

$$p=3, -5: \cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$p=5, -3: \cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$$

$$p=7, -1: \cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\frac{7\pi}{4}) = -\frac{\sqrt{2}}{2}$$

$$p=1, -7: \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$



c) Given  $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$  and  $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ,

draw  $(\cos \frac{\pi}{6}, \sin \frac{\pi}{6})$  and

find similar variations for

$p = 2, 4, 5, 7, 8, 10, 11$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

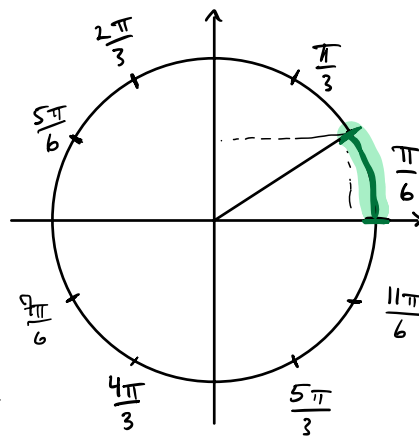
$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}, \quad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

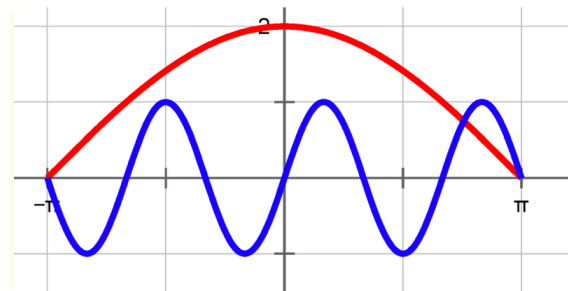
$$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$$



d)  $x \mapsto A \cos(bx)$

$x \mapsto C \sin(dx)$

Determine the coefficients from the graphs.



— cosine since  $\neq 0$  for  $x=0$ . Amplitude  $A=2$  and angular frequency  $b = \frac{1}{2}$ . (Half a period)

— sine since  $= 0$  for  $x=0$ . Amplitude  $C=1$  and angular frequency  $d=3$ . (Three periods)

#### Ex 4. Polar coordinates

a) Given  $z_0 = 1 + i\sqrt{3}$ ,  $z_1 = -1 + i\sqrt{3}$ ,  $z_2 = -1 - i\sqrt{3}$  and  $z_3 = 1 - i\sqrt{3}$

1. State the circle centered at 0, which the numbers lie on.

$$\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$C = \{z \in \mathbb{C} \mid |z| = 2\}$$

2. Compute  $\arg(z_0)$  and state the principal argument of  $z_1$ ,  $z_2$  and  $z_3$ . Additionally what are the polar coordinates?

Firstly  $z_0 = 1 + i\sqrt{3}$  with  $|z_0| = 2$ . Let's denote the angle by  $v$ , then

$$\cos(v) = \frac{1}{2} \quad \text{and} \quad \sin(v) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \arg(z_0) = \frac{\pi}{3} + p \cdot 2\pi, \quad p \in \mathbb{Z}.$$

See  $\arg$  vs.  $\text{Arg}$  in note (1-17) following definition 1.27.

$$\text{Arg}(z_0) = \frac{2\pi}{3}, \quad \text{Arg}(z_1) = -\frac{2\pi}{3}, \quad \text{Arg}(z_2) = -\frac{\pi}{3}$$

since  $\text{Arg}(z)$  is the angle for which  $\arg(z) \in ]-\pi, \pi]$ .

By definition 1.26 the polar coordinates are

$$(2, \frac{\pi}{3}), \quad (2, \frac{2\pi}{3}), \quad (2, -\frac{2\pi}{3}), \quad (2, -\frac{\pi}{3})$$

b) For  $z = 2 - 2i$  a student gets the argument  $\frac{\pi}{4}$ .

Clearly the student didn't remember to evaluate both for sine and cosine. The argument  $\frac{\pi}{4}$  satisfies

$\cos(\frac{\pi}{4}) = \frac{2}{2\sqrt{2}}$ , however this is not the case for

$$\sin(\frac{\pi}{4}) \neq \frac{-2}{2\sqrt{2}} = \sin(-\frac{\pi}{4})$$

c)  $|-2 + 2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$

$$\text{Arg}(-2 + 2i) = \frac{3\pi}{4}$$

$$|-\frac{1}{6} + \frac{1}{2\sqrt{3}}i| = \sqrt{\left(-\frac{1}{6}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2} = \sqrt{\frac{1}{36} + \frac{3}{36}} = \frac{2}{6} = \frac{1}{3}$$

$$\cos(v) = \frac{-\frac{1}{6}}{\frac{1}{3}} = -\frac{1}{2} \quad \text{Arg}\left(-\frac{1}{6} + \frac{1}{2\sqrt{3}}i\right) = \frac{2\pi}{3}$$

Quite unnecessary to check sine, but let's show it's  $\frac{\sqrt{3}}{2}$ :

$$\sin(v) = \frac{\frac{1}{2\sqrt{3}}}{\frac{1}{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

d) Rectangular form

$$(4, -\pi) = -4, \quad (2, \frac{4\pi}{3}) = -1 - i\sqrt{3}$$

$$(6, \frac{21\pi}{4}) = (6, \frac{5\pi}{4}) = -3\sqrt{2} - 3\sqrt{2}i$$

Ex 5. a)  $e^{\frac{\pi}{2}i} = i$  and  $3e^{1+\pi i} = -3e$

b) Convert  $z_0 = 1+i\sqrt{3}$ ,  $z_1 = -\sqrt{3}+i$ ,  $z_2 = -1-i\sqrt{3}$ ,  $z_3 = \sqrt{3}-i$  to exponential form.

$$z_0 = 2e^{\frac{\pi}{3}i}, \quad z_1 = 2e^{\frac{5\pi}{6}i}, \quad z_2 = 2e^{-\frac{2\pi}{3}i}, \quad z_3 = 2e^{-\frac{\pi}{6}i}$$

Show that the binomial equation

$$z^4 = w,$$

where  $w$  is a complex scalar, exists to which all four numbers are a solution.

$$z_0^4 = z_1^4 = z_2^4 = z_3^4 = 16e^{-\frac{2\pi}{3}i}$$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \text{and} \quad \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

For  $w = a+ib$  we have

$$a = -\frac{1}{2} \cdot 16 = -8 \quad \text{and} \quad b = -\frac{\sqrt{3}}{2} \cdot 16 = -8\sqrt{3}.$$

Thus the rectangular form is

$$w = -8 - 8\sqrt{3}i.$$

Ex 6. a) Let  $w = 1-i$  and compute

$$|w| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \text{Arg}(w) = -\frac{\pi}{4}$$

$$|e^w| = |e \cdot e^{-i}| = |e| \cdot |e^{-i}| = e$$

$$\text{Arg}(e^w) = -1$$

b) Given  $w_1 = 1$ ,  $w_2 = e$ ,  $w_3 = i$  and  $w_4 = 2i$  determine all solutions to  $e^z = w_n$  for  $n = 1, \dots, 4$ .

$$e^z = 1 \Leftrightarrow z = 2p\pi i, \quad p \in \mathbb{Z}$$

$$e^z = e \Leftrightarrow z = 1 + 2p\pi i, \quad p \in \mathbb{Z}$$

$$e^z = i \Leftrightarrow z = \left(\frac{\pi}{2} + 2p\pi\right)i, \quad p \in \mathbb{Z}$$

$$e^z = 2i \Leftrightarrow z = \ln(2) + \left(\frac{\pi}{2} + 2p\pi\right)i, \quad p \in \mathbb{Z}$$

Determine all solutions to

$$(e^z - 1) \cdot (e^z - i) = 0$$

$$\Leftrightarrow z = 2p\pi i \quad \vee \quad z = \left(\frac{\pi}{2} + 2p\pi\right)i, \quad p \in \mathbb{Z}.$$

c) Show that  $e^z \neq 0$  for all  $z \in \mathbb{C}$ .

Let  $z = a + ib$  for  $a, b \in \mathbb{R}$ , then

$$\begin{aligned} e^{a+ib} &= e^a (\cos(b) + i \sin(b)) \\ &= e^a \cos(b) + i e^a \sin(b). \end{aligned}$$

Note that the result is zero only if both real and imaginary parts are zero. However, for given  $b \in \mathbb{R}$  there is no value for which both  $\cos(b) = 0 = \sin(b)$ .

Thus at least one of the terms must remain non-zero, i.e.  $e^z \neq 0$  for all  $z \in \mathbb{C}$  as  $e^a \neq 0$  for all  $a \in \mathbb{R}$ .

Ex 7. Draw complex numbers in Geogebra

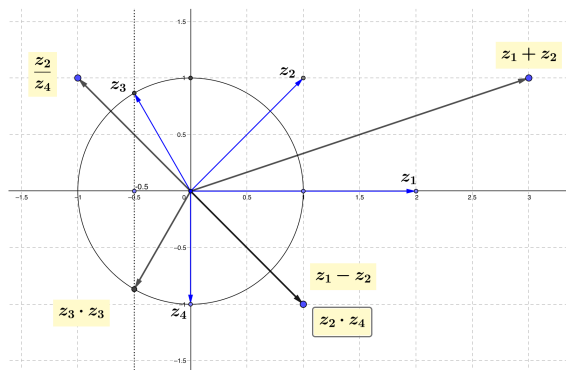
$$z_1 + z_4 = 2 + 1 + i = 3 + i$$

$$z_1 - z_2 = 2 - 1 - i = 1 - i$$

$$z_3 \cdot z_3 = \left(e^{\frac{2\pi i}{3}}\right)^2 = e^{\frac{4\pi i}{3}}$$

$$z_2 \cdot z_4 = \sqrt{2}e^{\frac{\pi i}{4}} \cdot e^{\frac{6\pi i}{4}} = \sqrt{2}e^{\frac{7\pi i}{4}} = \sqrt{2}e^{-\frac{\pi i}{4}}$$

$$\frac{z_2}{z_4} = \sqrt{2}e^{-\frac{5\pi i}{4}} = \sqrt{2}e^{\frac{3\pi i}{4}}$$



Ex 8. Double angles

a)  $\sin(2v) = 2 \sin(v) \cos(v)$  ,  $\cos(2v) = \cos^2(v) - \sin^2(v)$

Use this to determine  $\cos\left(\frac{\pi}{8}\right)$  and  $\sin\left(\frac{\pi}{12}\right)$ .

Note that  $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$  can be written as  $\cos\left(2\frac{\pi}{8}\right)$ .

Thus in turn we have

$$\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$$

$$\Leftrightarrow \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2} + \sin^2\left(\frac{\pi}{8}\right)$$

$$\Leftrightarrow \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2} + 1 - \cos^2\left(\frac{\pi}{8}\right)$$

$$\Leftrightarrow \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{4} + \frac{2}{4}$$

$$\Leftrightarrow \cos\left(\frac{\pi}{8}\right) = \frac{1}{2} \sqrt{2 + \sqrt{2}}$$

Similarly for  $\sin\left(\frac{\pi}{12}\right)$  we use  $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \cos\left(2\frac{\pi}{12}\right)$ .



$$\cos^2\left(\frac{\pi}{12}\right) - \sin^2\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \sin^2\left(\frac{\pi}{12}\right) = \cos^2\left(\frac{\pi}{12}\right) - \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \sin^2\left(\frac{\pi}{12}\right) = 1 - \sin^2\left(\frac{\pi}{12}\right) - \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \sin\left(\frac{\pi}{12}\right) = \frac{1}{2} \sqrt{2 - \sqrt{3}}$$

(Alternate form:  $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$ )

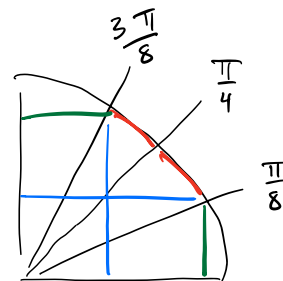
b) Use a) to find  $\sin\left(\frac{\pi}{8}\right)$ ,  $\cos\left(\frac{3\pi}{8}\right)$  and  $\sin\left(\frac{3\pi}{8}\right)$ .

$$\begin{aligned} \sin\left(\frac{\pi}{8}\right) &= \sqrt{1 - \cos^2\left(\frac{\pi}{8}\right)} = \sqrt{1 - \left(\frac{1}{2} \sqrt{2 + \sqrt{2}}\right)^2} \\ &= \sqrt{1 - \frac{1}{2} - \frac{\sqrt{2}}{4}} = \frac{1}{2} \sqrt{2 - \sqrt{2}} \end{aligned}$$

By symmetry

$$\cos\left(\frac{3\pi}{8}\right) = \frac{1}{2} \sqrt{2 - \sqrt{2}}$$

$$\sin\left(\frac{3\pi}{8}\right) = \frac{1}{2} \sqrt{2 + \sqrt{2}}$$



c) Similarly find interesting angles of the form  $\frac{p\pi}{12}$ .

$$\begin{aligned} \cos\left(\frac{\pi}{12}\right) &= \sqrt{1 - \left(\frac{1}{2} \sqrt{2 - \sqrt{3}}\right)^2} = \sqrt{1 - \frac{1}{2} + \frac{\sqrt{3}}{4}} \\ &= \frac{1}{2} \sqrt{2 + \sqrt{3}} \end{aligned}$$

$$\Rightarrow \cos\left(\frac{5\pi}{12}\right) = \frac{1}{2} \sqrt{2 - \sqrt{3}} \quad \text{and} \quad \sin\left(\frac{5\pi}{12}\right) = \frac{1}{2} \sqrt{2 + \sqrt{3}}$$

We note that only  $p=1$  and  $5$  are interesting, since  $p=2,3,4$  and  $6$  are known ( $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$  and  $\frac{\pi}{2}$ ).