Extremum Investigation

Ex1. Determine all extrema for $f(x,y) = x^2y + y$. (Real-valued)

$$\nabla f(x,y) = \begin{bmatrix} 2xy \\ x^2+1 \end{bmatrix}$$

Since $x^2+1=0$ has no real solutions it follows that there is no stationary point for f, and so there is no extrema as f is defined on $\mathbb{R}^2$ (no boundary to consider).

Ex2. Show that $f(x,y) = x^2 + 4y^2 - 2x - 4y$ has one extremum.

a) Determine the point, value and assess whether it's a maximum or a minimum.

$$\nabla f(x,y) = \begin{bmatrix} 2x-2 \\ 8y-4 \end{bmatrix} \Rightarrow \nabla f(x,y) = \underline{0} \iff (x,y) = \left(1, \tfrac{1}{2}\right)$$

We've found the point, and the value is then

$$f\left(1,\tfrac{1}{2}\right) = 1 + 4\cdot\tfrac{1}{4} - 2 - 4\cdot\tfrac{1}{2} = 1+1-2-2 = -2.$$

We estabish the nature of $(1, \frac{1}{2}, -2)$ by means of the Hessian.

$$Hf(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \Rightarrow$$ $(1, \frac{1}{2}, -2)$ is a minimum as the Hessian is constant and positive for all (x,y).

b) What is the difference between an extremum and a proper extremum?

An extremum has values $\geq$ or $\leq$ all values of neighbouring points, while a proper extremum has strict inequalities.

The extremum in a) is proper. This follows from 21.17 as the eigenvalues of $Hf(x,y)$ are both positive.

Ex 3. We're given a real function

$$f(x,y) = x^3 + 2y^3 + 3xy^2 - 3x^2$$

a) Show that $A=(2,0)$, $B=(1,-1)$ and $C=(0,0)$ are stationary points of f and investigate them.

$$\nabla f(x,y) = \begin{bmatrix} 3x^2 + 3y^2 - 6x \\ 6y^2 + 6xy \end{bmatrix}$$

We now have

$$\nabla f(A) = \begin{bmatrix} 3 \cdot 2^2 - 6 \cdot 2 \\ 6 \cdot 2 \cdot 0 \end{bmatrix} = \underline{0}$$

$$\nabla f(B) = \begin{bmatrix} 3 \cdot 1 + 3 \cdot (-1)^2 - 6 \cdot 1 \\ 6 \cdot (-1)^2 + 6 \cdot 1 \cdot (-1) \end{bmatrix} = \underline{0}$$

$$\nabla f(C) = \underline{0}$$

So A, B and C are stationary points of f.

$$Hf(x,y) = \begin{bmatrix} 6x - 6 & 6y \\ 6y & 12y + 6x \end{bmatrix}$$

We determine of what type the stationary points are.

$$Hf(A) = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix} \Rightarrow A \text{ is a proper minimum.}$$

$$f(A) = -4.$$

$$Hf(B) = \begin{bmatrix} 0 & -6 \\ -6 & -6 \end{bmatrix} \Rightarrow B \text{ is a saddle point.}$$

$$Hf(C) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{No conclusion to be had.}$$

We can use f to check values around C.

$$f(C) = 0 \quad \text{and} \quad f(0,y) = 2y^3.$$

Thus C is not an extrema as $f(C) < f(0,y)$ for $y > 0$ and $f(C) > f(0,y)$ for $y < 0$.

b) Show that P_2 developed at A may be written as

$$z - c_3 = \frac{1}{2}\lambda_1(x - c_1)^2 + \frac{1}{2}\lambda_2(y - c_2)^2.$$

Which quadratic form is this, and what do the constants mean?

```
> f:= (x,y)-> x^3+2*y^3+3*x*y^2-3*x^2:
  f(x,y);
```

$$x^3 + 3xy^2 + 2y^3 - 3x^2$$

```
> mtaylor(f(x,y),[x=2,y=0],3);
```

$$3(x-2)^2 + 6y^2 - 4$$

We have $P_2(x,y) = 3(x-2)^2 + 6y^2 - 4$.

Let's equate P_2 to z and rewrite.

$$z = 3(x-2)^2 + 6y^2 - 4$$

$$\iff \quad z + 4 = \frac{1}{2} \cdot 6 (x-2)^2 + \frac{1}{2} \cdot 12 (y-0)^2$$

$$\iff \quad z + 4 = \frac{(x-2)^2}{\sqrt{\frac{1}{3}}^{\,2}} + \frac{(y-0)^2}{\sqrt{\frac{1}{6}}^{\,2}}$$

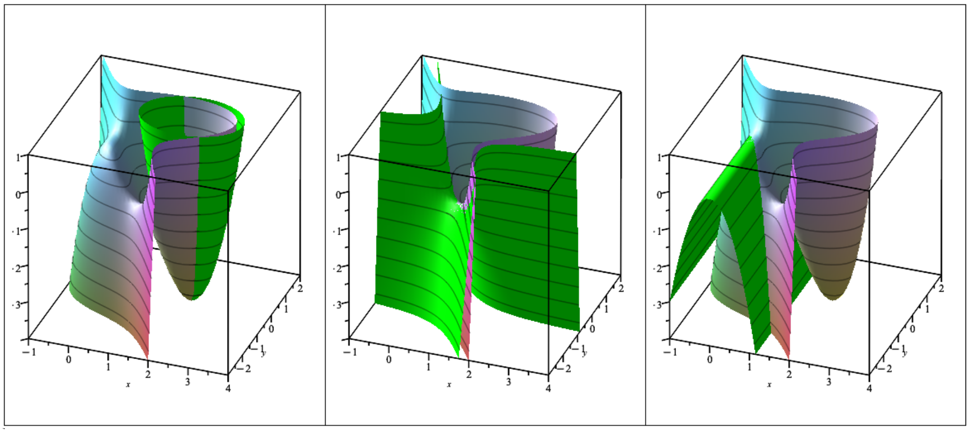
From the table in section 22.3 this amounts to an elliptic paraboloid, which is upwards facing. The constants yield the top point

$$T = (c_1, c_2, c_3) = (2, 0, -4).$$

c) Draw the graph of f and P_2 at A, B, C. Discuss whether the eigenvalues of each Hessian reveals the nature of the shape of the quadratic surfaces.

```
> plotf:=plot3d(f(x,y),x=-1..4,y=-3..2,style=patchcontour,orientation=[-70,60]):
  plotP2A:=plot3d(P2A(x,y),x=-1..4,y=-3..2,style=surface,color=green,style=patchcontour):
  plotP2B:=plot3d(P2B(x,y),x=-1..4,y=-3..2,style=surface,color=green,style=patchcontour):
  plotP2C:=plot3d(P2C(x,y),x=-1..4,y=-3..2,style=surface,color=green,style=patchcontour):
```

```
>  plotA:=display(plotf,plotP2A,view=-4..1):
   plotB:=display(plotf,plotP2B,view=-4..1):
   plotC:=display(plotf,plotP2C,view=-4..1):
  display(<plotA|plotB|plotC>);
```



The eigenvalues indeed reveal the shape. When developed at B we have eigenvalues of opposite signs, which gives hyperbolicity, in this case a hyperbolic paraboloid. Around C one eigenvalue is zero, and we get a parabolic cylinder.

Ex 4. Let $f(x,y) = xy(2-x-y)+1$ be real valued. Let M denote the region $[0,1]\times[0,1]$ in the (x,y)-plane.

a) Determine stationary points on M.

$$\nabla f(x,y) = \begin{bmatrix} y(2-2x-y) \\ x(2-x-2y) \end{bmatrix}$$

We have $(0,0)$, $(2,0)$, $(0,2)$ or $\left(\frac{2}{3},\frac{2}{3}\right)$ by

$$\begin{cases} 2y - 2xy - y^2 = 0 \\ 2x - 2xy - x^2 = 0 \end{cases} \iff \begin{cases} y(2-y) = 2xy \\ x(2-x) = 2xy \end{cases}$$

$$\Rightarrow \quad y(2-y) - x(2-x) = 0 \quad \Rightarrow \quad x = y$$

Plug into above eq. $\Rightarrow \quad -y^2 + 2y = 2y^2$

$$\iff -3y^2 + 2y = 0$$

$$\iff y(-3y+2) = 0$$

$$\iff y = 0 \lor y = \frac{2}{3}.$$

b) Determine global max/min of f on M along with values.

We need to check the boundary, but on M we have

$$\left(\frac{2}{3},\frac{2}{3}, f\left(\frac{2}{3},\frac{2}{3}\right)\right) = \left(\frac{2}{3},\frac{2}{3},\frac{35}{27}\right).$$

```
> VectorCalculus[Hessian](f(x,y),[x,y]=[2/3,2/3],determinant);
```

$$\begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}, \frac{4}{3}$$

Thus we have a proper local maximum.

Along the boundary we have corners yielding 1, as well as along $f(x,0)$ and $f(0,y)$. We find maximal values along $f(x,1)$ and $f(1,y)$ to be $\frac{5}{4}$ at $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$.

```
> f(0,0),f(1,0),f(0,1),f(1,1);
```

1, 1, 1, 1

```
> diff(f(x,0),x);
  diff(f(x,1),x);
  diff(f(0,y),y);
  diff(f(1,y),y);
```

$$0$$
$$1-2x$$
$$0$$
$$1-2y$$

```
> solve(diff(f(x,1),x)=0,x):f(%,1);
  solve(diff(f(1,y),y)=0,y):f(1,%);
```

$$\frac{5}{4}$$
$$\frac{5}{4}$$

Therefore we have a global maximum on M, which is $\left(\frac{2}{3},\frac{2}{3},\frac{35}{27}\right)$.

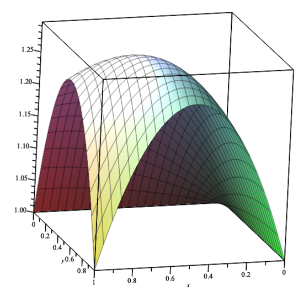
The global minimum value is 1 obtained for $(x,0)$ with $0\le x\le 1$ and $(0,y)$ with $0\le y\le 1$.

c) Determine the range of f on M.

Since M is connected this is just $[1,\frac{35}{27}]$.

d) Plot f.

```
> plot3d(f(x,y),x=0..1,y=0..1);
```



Ex 5. Let $f(x,y) = x^2 - 3y^2 - 3xy$ and $M = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 1\}$.

a) Explain that f has a max/min on M and determine these.

Since M is connected and f is continuous in both variables by proposition 21.10.

$$\nabla f(x,y) = \begin{bmatrix} 2x - 3y \\ -6y - 3x \end{bmatrix} \Rightarrow (0,0) \text{ is the only stationary point}$$

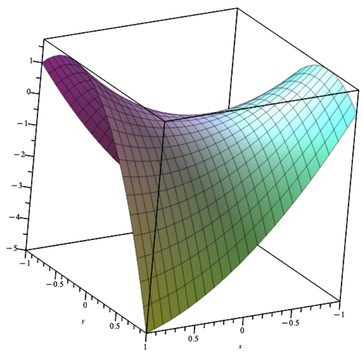
```
> f:=(x,y)-> x^2-3*y^2-3*x*y:
  f(x,y);
```

$$x^2 - 3xy - 3y^2$$

```
> f(0,0);
```

$$0$$

```
> plot3d(f(x,y),x=-1..1,y=-1..1);
```



The boundary points are on the unit circle, so let

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \end{bmatrix}, \quad u \in [0, 2\pi].$$

Then we have

```
> g:=unapply(f(cos(u),sin(u)),u):
  g(u);
```

$$\cos(u)^2 - 3\sin(u)^2 - 3\sin(u)\cos(u)$$

```
> diff(g(u),u)=0;
  sol:=solve(%,u);
```

$$-8\sin(u)\cos(u) - 3\cos(u)^2 + 3\sin(u)^2 = 0$$

$$sol := \arctan(3), \arctan(3) - \pi, -\arctan\left(\frac{1}{3}\right), -\arctan\left(\frac{1}{3}\right) + \pi$$

```
> for i from 1 to 4 do
  (cos(sol[i]),sin(sol[i]),f(cos(sol[i]),sin(sol[i])));
  od;
```

$$\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}, -\frac{7}{2}$$

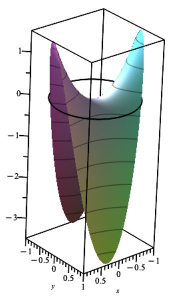
$$-\frac{\sqrt{10}}{10}, -\frac{3\sqrt{10}}{10}, -\frac{7}{2}$$

$$\frac{3\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}, \frac{3}{2}$$

$$-\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, \frac{3}{2}$$

So the minimum is $-\frac{7}{2}$ and the maximum is $\frac{3}{2}$ in the points given above.

```
> F:=plot3d(f(x,y),x=-1..1,y=-sqrt(1-x^2)..sqrt(1-x^2),style= patchcontour,scaling=constrained,axes=boxed,projection=0.8):
> C:=spacecurve([cos(t),sin(t),0],t=0..2*Pi,scaling=constrained, axes=boxed,thickness=2,projection=0.8,color=black):
> display(F,C);
```



Ex6. Let $f(x,y,z) = \sin(x^2+y^2+z^2-1) - x^2+y^2-z^2$ and consider

$$K = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 \leq 1\}.$$

a) Show that $(0,0,0)$ is the only stationary point in the interior of K, and investigate whether it is an extrema.

$$\nabla f(x,y,z) = \begin{bmatrix} 2x\left(\cos(x^2+y^2+z^2-1) - 1\right) \\ 2y\left(\cos(x^2+y^2+z^2-1) + 1\right) \\ 2z\left(\cos(x^2+y^2+z^2-1) - 1\right) \end{bmatrix}$$

$$2x\left(\cos(x^2+y^2+z^2-1) - 1\right) = 0$$

$$\Leftrightarrow \quad x = 0 \;\vee\; \cos(x^2+y^2+z^2-1) - 1 = 0$$

$$\cos(x^2+y^2+z^2-1) - 1 = 0$$

$$\Leftrightarrow \quad \cos(x^2+y^2+z^2-1) = 1$$

$$\Leftrightarrow \quad x^2+y^2+z^2 = 1$$

But this implies (x,y,z) is not an interior, in fact on the boundary.

A similar argument yields $(x,y,z)=(0,0,0)$, mutatis mutandis.

```
> VectorCalculus[Hessian](f(x,y,z),[x,y,z]=[0,0,0]);
```

$$\begin{bmatrix} 2\cos(1)-2 & 0 & 0 \\ 0 & 2\cos(1)+2 & 0 \\ 0 & 0 & 2\cos(1)-2 \end{bmatrix}$$

Since the signs are alternating we don't have an extrema in $(0,0,0)$.

b) State the global max/min on K as well as the points.

Along the boundary we get

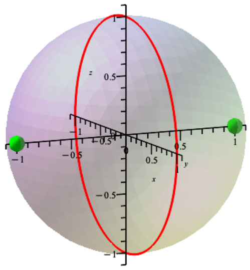
$$f(x,y,z) = -x^2+y^2-z^2, \quad x^2+y^2+z^2=1 \iff x^2+z^2=1-y^2$$
$$= 2y^2-1$$

For $y\in[-1,1]$ the minimal value is -1 for $y=0$ and the maximal value is 1 at the endpoints $y=\pm 1$.

So the minimum is attained on the unit circle $x^2+z^2=1$ in the y-plane while the maximum is in $(0,1,0)$ and $(0,-1,0)$.

The values are -1 and 1.

```
> K:=implicitplot3d(x^2+y^2+z^2=1,x=-1.1..1.1,y=-1.1..1.1,z=-1.1..1.1,transparency=0.7,style=patchnogrid,grid=[20,20,20]):
  C:=spacecurve(<cos(u),0,sin(u)>,u=-Pi..Pi,thickness=3,color= red):
  P:=pointplot3d({[0,-1,0],[0,1,0]},symbol=solidsphere,color= green,symbolsize=30):
> display(K,C,P,axes=normal,orientation=[-25,80,0]);
```



c) Determine the range of f on K.

This is just $[-1,1]$.

Ex 7. Let $f(x,y) = e^{x^2+y^2} - 4xy$.

a) Find all stationary points of f.

```
> diff(f(x,y),x),diff(f(x,y),y);
```

$$2x e^{x^2+y^2} - 4y,\ 2y e^{x^2+y^2} - 4x$$

```
> solve([%[1]=0,%[2]=0],[x,y]);
```

$$[[x=0, y=0], [x = RootOf(2\_Z^2 - \ln(2)), y = RootOf(2\_Z^2 - \ln(2))], [x = -RootOf(2\_Z^2 - \ln(2) - I\pi), y = RootOf(2\_Z^2 - \ln(2) - I\pi)]]$$

```
> map(allvalues,%);
```

$$\left[[x=0,y=0], \left[x=\frac{\sqrt{2}\sqrt{\ln(2)}}{2}, y=\frac{\sqrt{2}\sqrt{\ln(2)}}{2}\right], \left[x=-\frac{\sqrt{2}\sqrt{\ln(2)}}{2}, y=-\frac{\sqrt{2}\sqrt{\ln(2)}}{2}\right], \left[x=-\frac{\sqrt{2I\pi+2\ln(2)}}{2}, y=\frac{\sqrt{2I\pi+2\ln(2)}}{2}\right], \left[x=\frac{\sqrt{2I\pi+2\ln(2)}}{2}, y=-\frac{\sqrt{2I\pi+2\ln(2)}}{2}\right]\right]$$

The real valued stationary points are

$$A = (0,0)$$

$$B = \left(\frac{1}{\sqrt{2}}\sqrt{\ln 2},\ \frac{1}{\sqrt{2}}\sqrt{\ln 2}\right)$$

$$C = \left(-\frac{1}{\sqrt{2}}\sqrt{\ln 2},\ -\frac{1}{\sqrt{2}}\sqrt{\ln 2}\right)$$

b) Find all extrema.

We need only check the stationary points.

```
> A:=(0,0):
  B:=(sqrt(2)*sqrt(ln(2))/2,sqrt(2)*sqrt(ln(2))/2):
  C:=(-sqrt(2)*sqrt(ln(2))/2,-sqrt(2)*sqrt(ln(2))/2):
```

```
> Hf:=unapply(VectorCalculus[Hessian](f(x,y),[x,y]),(x,y)):
  Hf(x,y);
```

$$\begin{bmatrix} 2e^{x^2+y^2} + 4x^2 e^{x^2+y^2} & 4xy e^{x^2+y^2} - 4 \\ 4xy e^{x^2+y^2} - 4 & 2e^{x^2+y^2} + 4y^2 e^{x^2+y^2} \end{bmatrix}$$

```
> Eigenvalues(Hf(A));
```

$$\begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

```
> Eigenvalues(Hf(B)):
  simplify(%);
```

$$\begin{bmatrix} 8 \\ 8\ln(2) \end{bmatrix}$$

```
> Eigenvalues(Hf(C)):
  simplify(%);
```

$$\begin{bmatrix} 8 \\ 8\ln(2) \end{bmatrix}$$

It follows from proposition 21.17 that A is a saddle point and both B and C are proper local minima.

```
> f(A),f(B),f(C);
  evalf(%);
```

$$1, 2 - 2\ln(2), 2 - 2\ln(2)$$

$$1., 0.613705639, 0.613705639$$

c) Decide whether f has a global max/min, and state values if they exist.

There is no global maximum, since for $r = x^2+y^2$ we have

$$f(x,y) = e^{x^2+y^2} - 4xy = e^{r^2} - 4xy$$
$$\to \infty \quad \text{for } r \to \infty.$$

This is justified as $x \cdot y \leq |x|\,|y| \leq x^2+y^2$. Thus we also have global minima B and C with the value $2 - 2\ln 2$.

d) State the range of f.

The range is $[2 - 2\ln 2, \infty[$.

```
> p1:=plot3d(f(x,y),x=-1..1,y=-1..1,axes=none):
> p2:=plot3d(<r*cos(v),r*sin(v),simplify(f(r*cos(v),r*sin(v)))>,r=0..2,v=0..2*Pi,axes=normal):
> display(<p1|p2>);
```