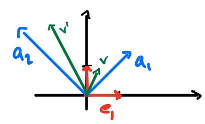
Linear maps Vit sóu við "Target boxes", hvussu vit kunnu avmynda úr $[0,1]\times[0,1]$ til nýggjar kassar. Einhvør vektorur $\underline{v}\in\mathbb{R}^2$ kann skrivast

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1\underline{e}_1 + v_2\underline{e}_2 .$$

Vilja vit avmynda yvir í eina aðra skipan, altso senda $\underline{v}$ til $\underline{v}'$, so eru aðrir vektorar, sum kunnu umboða $\underline{v}'$.

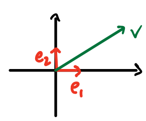


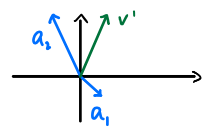
$$\underline{v}' = v_1\underline{a}_1 + v_2\underline{a}_2 .$$

Her er v' givin í somu koordinat sum áður, men vit hava skift úr eini $[e_1, e_2]$-skipan til eina $[a_1, a_2]$-skipan.

Dømi Lat $\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ og $\underline{a}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Fyri ein vektor $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ í $[e_1, e_2]$-skipanini so er

$$\underline{v}' = 3\,\underline{a}_1 + 2\,\underline{a}_2 = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$





$v' = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ í $[a_1, a_2]$, men

$v' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ í $[e_1, e_2]$.

Matrix Slagið av multiplikation omanfyri systematisera vit við at finna uppá matricur. Vit skriva nú vektorarnar

$$\underline{a}_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} \quad \text{og} \quad \underline{a}_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} .$$

Matrican $\underline{\underline{A}}$, sum umboðar avmyndanina er

$$\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

og hendan skal avmynda vektorar á fylgjandi hátt:

$$\underline{v}' = \underline{\underline{A}}\,\underline{v}, \quad \text{har}$$

$$\underline{\underline{A}}\,\underline{v} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left[v_1\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} + v_2\begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}\right] = \begin{bmatrix} v_1 a_{11} + v_2 a_{21} \\ v_1 a_{12} + v_2 a_{22} \end{bmatrix} .$$

Dømi Lat $\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ og $\underline{a}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Við $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ fáa vit, at

$$\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \quad \text{so} \quad \underline{v}' = \underline{\underline{A}}\,\underline{v} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 - 1\cdot 2 \\ -1\cdot 3 + 3\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Dømi Við $\underline{\underline{A}} = \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$ og $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$:

		3
		2
0	-1	-2
2	4	14

$$\underline{v}' = \begin{bmatrix} -2 \\ 14 \end{bmatrix}$$

Notatión $[\underline{e}_1, \underline{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ og hendan eitur identitetsmatricam (hjá $\mathbb{R}^2$).

Additión Lat A og B vera tvær 2x2 matricur. Additión er definerað koordinatvíst.

$$\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} = \underline{\underline{(A+B)}}$$

Dimensiónin má altso vera eins. Additión er kommutativ og distributiv.

$$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}, \quad \underline{\underline{A}}\underline{v} + \underline{\underline{B}}\underline{v} = (\underline{\underline{A}} + \underline{\underline{B}})\underline{v} \quad \text{og} \quad \underline{\underline{A}}(\underline{u} + \underline{v}) = \underline{\underline{A}}\underline{u} + \underline{\underline{A}}\underline{v}.$$

Umbýta vit rað og súlu í eini matricu $\underline{\underline{A}}$, so fáa vit transponeraðu matricuna $\underline{\underline{A}}^T$.

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix} \Rightarrow \underline{\underline{A}}^T = \begin{bmatrix} 1 & -7 \\ 5 & 2 \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \underline{v}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Evt. lektiur at vísa hetta. Vit fáa $(A+B)^T = A^T + B^T$, $A^{T^T} = A$ og $[cA]^T = cA^T$.
Ein matrica, har $A = A^T$ sigst at vera symmetrisk.

Rank Matricur fáa tillutað ein rank alt eftir óheftni av vektorunum, sum matrican er gjørd burturúr.

$\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2]$: í 2x2 førinum er full rank, um $\underline{a}_1$ og $\underline{a}_2$ eru lin. óheftir. Tvs. rank er 2. Um $\underline{a}_1 = k\underline{a}_2$ so er $\text{rank}(A) = 1$, og vit siga eisini, at matrican er singulær. Bert $\underline{\underline{0}}$ hevur rank 0.

Fortelur okkum, um dimensiónin av tí avmyndaða fellur.

Ein $n \times m$ matrica kann avmynda vektorar í einum m-dim. rúmi.
Matricur eru lineerar:

$$\underline{\underline{A}}(a\underline{u} + b\underline{v}) = a\underline{\underline{A}}\underline{u} + b\underline{\underline{A}}\underline{v}$$

Dømi Lat $\underline{\underline{A}} = \begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix}$, $\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ og $\underline{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. So er

$$\underline{\underline{A}}(3\underline{u} + 2\underline{v}) = \begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} \right) = \begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 73 \\ 7 \end{bmatrix}.$$

$$3\underline{\underline{A}}\underline{u} + 2\underline{\underline{A}}\underline{v} = 3\begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 11 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 73 \\ 7 \end{bmatrix}.$$

Scaling

Matricur broyta uppá allar vektorar, men hvussu $[\underline{e}_1, \underline{e}_2]$ broytist sigur okkum beinleiðis hvørja skipan vit enda í.

Lat $\underline{\underline{A}} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, so er $\underline{\underline{A}}\,\underline{e}_1 = \begin{bmatrix} a_{11} \\ 0 \end{bmatrix}$ og $\underline{\underline{A}}\, e_2 = \begin{bmatrix} 0 \\ a_{22} \end{bmatrix}$, so

tølini a_{11} og a_{22} er ein skalering av skipanini, og vit velja frítt, um vit varðveita lutføll við $a_{11} = a_{22}$ ella ei.

Reflection

Lat nú $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so at $\underline{\underline{A}}\,\underline{e}_1 = \underline{e}_1$, men $\underline{\underline{A}}\,\underline{e}_2 = -e_2$, so er skipanin vend á høvdið. Altso spegling um fyrstu á! Líknandi er

$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ spegling um seinnu ás, og

$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ er spegling um báðar ásirnar (kann skrivast sum $AB = C$).

Her er C eisini ein rotatión.

Vit kunnu eisini spegla koordinatini við at umbýta hesi, so $\underline{\underline{A}}\,\underline{e}_1 = \underline{e}_2$ og $\underline{\underline{A}}\,\underline{e}_2 = \underline{e}_1$.

Tá er $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Í mun til skalering er areal (støddin) varðveitt undir reflectiónir.

Rotatión

Matricur, sum roterar α stig skrivast $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. Hesar hoyra til kategriina "rigid body motions".

Shears

Gera rektangul til parallelogram, altso flyta horisontalt/vertikalt.

Eftir fyrstu á flyta vit við d_1: $A = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$.

Eftir seinnu ás: $A = \begin{bmatrix} 1 & 0 \\ d_2 & 1 \end{bmatrix}$.

Projection

Serlig shears senda niður eina dimensión:

$$\begin{bmatrix} 1 & 0 \\ -\frac{v_2}{v_1} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 + v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -\frac{v_1}{v_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 - v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}.$$

Hetta kann sum so gerast sum projections matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}.$$

Projectión á ein ávísan vektor u kann konstruerast. Vit gera $\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2]$, har $\underline{a}_1 = \frac{u \cdot e_1}{\|u\|^2} u = u_1 \underline{u}$, $\underline{a}_2 = \frac{u \cdot e_2}{\|u\|^2} u = u_2 \underline{u}$.

Eisini skrivað

$$A = u u^T = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} [u_1 \quad u_2] = \left[u_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, u_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right]$$

Dømi 4.4

$\underline{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, so er projectiónsmatrican $\underline{\underline{A}}$ givin við

$$\underline{\underline{A}} = \left[\frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Legg til merkis, at um $\underline{\underline{A}}\underline{u} = \underline{v}$, so er $\underline{\underline{A}}\underline{v} = \underline{v}$.

Determinant

Vit hava stutt nevnt, at onkrar matricur avmynda, so at støddin broytist. Determinantar kunnu brúkast til at avgera í hvussu stóran mun.

2D

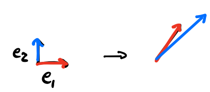
Fyri $\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2]$ so senda vit $e_1 \to a_1$ og $e_2 \to a_2$, so at eindarkvadratið í staðin broytist til eitt parallelogram útspent av a_1 og a_2 (full rank). Støddin á hesum er givið við determinantinum

$$|A| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$

Úrslitið kann vera negativt, um a_1, a_2 býta um uppá e_1, e_2.
Positiva virðið í $\mathbb{R}^2$ er arealið á parallelogramminum.

Dømi

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 1 \cdot 4 - 3 \cdot 2 = -2$$



Vit hava, at $|ca_1, a_2| = c|A|$, $|ca_1, ca_2| = c^2 |A|$

$$|a_1, a_2| = -|a_2, a_1|.$$

Matrix multiplikatión

Samansetan av fleiri matricum hevur egnu kompositiónsreglu, sum fylgjur matrix-vektor regluni. Lat

$$A = [a_1, a_2] \quad \text{og} \quad B = [b_1, b_2]$$

So definera vit AB at vera

$$C = AB = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \left[\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix} \right]$$

$$= \left[\begin{bmatrix} a_{11} b_{11} + a_{21} b_{12} \\ a_{12} b_{11} + a_{22} b_{12} \end{bmatrix}, \begin{bmatrix} a_{11} b_{21} + a_{21} b_{22} \\ a_{12} b_{21} + a_{22} b_{22} \end{bmatrix} \right] = [\underline{c}_1, \underline{c}_2].$$

Hetta samsvarar við $\underline{\underline{A}}(\underline{\underline{B}}\,\underline{v}) = (\underline{\underline{AB}})\underline{v}$.

Tað passar vanliga ikki at matrix produktið er kommutativt, so $AB \neq BA$ vanliga. Hetta er í lagi fyri t.d. rotatiónir.