Systems of Linear Differential equations

Ex 1. a) Given $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2 t \end{bmatrix}$, $t \in \mathbb{R}$,

1. Find eigenvalues and spaces, and use this to state the general solution.

Let $\underset{=}{A}$ denote the coefficient matrix, then

$$\det(\underset{=}{A} - \lambda \underset{=}{I}) = (1-\lambda)(-1-\lambda) - 8$$
$$= \lambda^2 - 9.$$

Thus the eigenvalues are ± 3.

$$\lambda = -3: \quad \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$$\lambda = 3: \quad \begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix} \underline{u} = \underline{0} \iff \underline{u} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

So the eigenspaces are

$$E_{-3} = \operatorname{span}\{(-2,1)\},$$
$$E_3 = \operatorname{span}\{(4,1)\}.$$

The general solution for some constants $c_1, c_2 \in \mathbb{R}$ is given by

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

This follows from 17.4.

2. Find the solution that satisfies $x_1(0) = 0$ and $x_2(0) = 3$.

We plug in and solve for c_1 and c_2.

$$c_1 e^{-3\cdot 0}\begin{bmatrix}-2\\1\end{bmatrix} + c_2 e^{3\cdot 0}\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix}-2c_1\\c_1\end{bmatrix} + \begin{bmatrix}4c_2\\c_2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix}-2 & 4\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\Leftrightarrow \quad c_1 = 2 \ \wedge \ c_2 = 1$$

The solution amounts to

$$\underline{x}(t) = 2e^{-3t}\begin{bmatrix}-2\\1\end{bmatrix} + e^{3t}\begin{bmatrix}4\\1\end{bmatrix},$$

where

$$x_1(t) = -4e^{-3t} + 4e^{3t}, \quad \text{and}$$

$$x_2(t) = 2e^{-3t} + e^{3t}.$$

b) We're given $\begin{bmatrix}x_1'(t)\\x_2'(t)\end{bmatrix} = \begin{bmatrix}2 & -5\\1 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$, $t \in \mathbb{R}$.

1. Eigenvalues, vectors and general complex solution.

```
> Eigenvectors(A,output=list);
```

$$\left[\left[I, 1, \left\{\begin{bmatrix}2+I\\1\end{bmatrix}\right\}\right], \left[-I, 1, \left\{\begin{bmatrix}2-I\\1\end{bmatrix}\right\}\right]\right]$$

We have $\lambda_1 = i$ with $\underline{v}_1 = \begin{bmatrix}2+i\\1\end{bmatrix}$ and $\lambda_2 = -i$ with $\underline{v}_2 = \begin{bmatrix}2-i\\1\end{bmatrix}$.

$E_i = \text{span}\{\underline{v}_1\}$ and $E_{-i} = \text{span}\{\underline{v}_2\}$.

By proposition 17.2 the complex solution is

$$\underline{x}(t) = c_1 e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}, \quad t \in \mathbb{R},$$

where $c_1, c_2 \in \mathbb{C}$ are constants.

2. State the general real solution.

We'll use 17.5. For this we select $\lambda = i$ and $\underline{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\underline{u}_1(t) = \cos(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\underline{u}_2(t) = \sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now the solution is given by

$$\underline{x}(t) = k_1 \underline{u}_1(t) + k_2 \underline{u}_2(t), \quad t \in \mathbb{R},$$

for constants $k_1, k_2 \in \mathbb{R}$.

3. Find the solution for which $x_1(0) = 0$ and $x_2(0) = 3$.

After plugging in we have

$$k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow k_1 = 3 \wedge k_2 = -6.$$

This leaves

$$\underline{x}(t) = 3 \left(\cos(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - 6 \left(\sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

so that

$$x_1(t) = -15 \sin(t) \quad \text{and} \quad x_2(t) = 3\cos(t) - 6\sin(t).$$

Complex and real solutions agree for real initial conditions.

c) Repeat for $\underline{x}'(t) = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \underline{x}(t)$.

1.

```
> Eigenvectors(A,output=list);
```

$$\left[\left[0, 2, \left\{\left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]\right\}\right]\right]$$

We have $\lambda = 0$ and $\underline{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where $am(\lambda) = 2 > gm(\lambda) = 1$.

$E_0 = span\{\underline{v}\}$.

Proceed with 17.7.

$$(\underline{\underline{A}} - 0 \cdot \underline{\underline{I}})\, \underline{b} = \underline{v} \iff \underline{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \cdot \underline{v}, \quad t \in \mathbb{R}.$$

$$\underline{u}_1(t) = e^{0 \cdot t}\, \underline{v} \quad \text{and} \quad \underline{u}_2(t) = t e^{0 \cdot t}\, \underline{v} + e^{0 \cdot t}\, \underline{b}$$

$$\Rightarrow \quad \underline{x}(t) = c_1 \underline{u}_1(t) + c_2 \underline{u}_2(t)$$

$$= c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \left(t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -c_1 - c_2 t \\ 2c_1 + c_2(2t-1) \end{bmatrix}, \quad t \in \mathbb{R}.$$

2. Find the solution with $x_1(0) = 0$ and $x_2(0) = 3$.

$$\begin{bmatrix} -c_1 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \iff c_1 = 0 \wedge c_2 = -3.$$

So we have

$$x_1(t) = 3t \quad \text{and} \quad x_2(t) = -6t + 3.$$

Ex 2. We're given the following

```
> A:=<<1,1,3>|<1,3,1>|<3,1,1>>;
```

$$A := \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

```
> Eigenvectors(A,output=list);
```

$$\left[\left[2, 1, \left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}\right], \left[5, 1, \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right], \left[-2, 1, \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}\right]\right]$$

a) How do the three diff. eq. look in normal form?

$$x_1'(t) = x_1(t) + x_2(t) + 3x_3(t)$$

$$x_2'(t) = x_1(t) + 3x_2(t) + x_3(t)$$

$$x_3'(t) = 3x_1(t) + x_2(t) + x_3(t)$$

b) State the general solution both on matrix form and ordinary.

$$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R},$$

for constants $c_1, c_2, c_3 \in \mathbb{R}$.

$$x_1(t) = c_1 e^{2t} + c_2 e^{5t} - c_3 e^{-2t},$$

$$x_2(t) = -2c_1 e^{2t} + c_2 e^{5t},$$

$$x_3(t) = c_1 e^{2t} + c_2 e^{5t} + c_3 e^{-2t}.$$

Ex 3. Let $f : (C^\infty(\mathbb{R}, \mathbb{C}))^2 \to (C^\infty(\mathbb{R}, \mathbb{C}))^2$ be given by

$$f(\underline{x}(t)) = \underline{x}'(t) - \underline{\underline{A}}\,\underline{x}(t) \quad , \quad t \in \mathbb{R}.$$

a) Show that f is linear.

Let $\underline{x}$ and $\underline{y}$ be smooth complex-valued vectorfunctions, and α a scalar. Then

$$\begin{aligned} f(\alpha \underline{x} + \underline{y}) &= (\alpha \underline{x} + \underline{y})' - \underline{\underline{A}}(\alpha \underline{x} + \underline{y}) \\ &= \alpha \underline{x}' + \underline{y}' - \alpha \underline{\underline{A}}\,\underline{x} - \underline{\underline{A}}\,\underline{y} \\ &= \alpha(\underline{x}' - \underline{\underline{A}}\,\underline{x}) + \underline{y}' - \underline{\underline{A}}\,\underline{y} \\ &= \alpha f(\underline{x}) + f(\underline{y}). \end{aligned}$$

Thus f is linear. We used linearity of the derivative and $\underline{\underline{A}}$ along the way.

b) Explain that a system as in Ex 1. can be viewed as a homogeneous vector equation of the type

$$f(\underline{x}(t)) = \underline{0}.$$

If we denote the coefficient matrix as $\underline{\underline{A}}$, then each is of the type

$$\underline{x}'(t) = \underline{\underline{A}}\,\underline{x}(t) \iff \underline{x}'(t) - \underline{\underline{A}}\,\underline{x}(t) = \underline{0} \iff f(\underline{x}(t)) = \underline{0},$$

for some $\underline{\underline{A}}$.

c) How can the structural theorem be applied to $f(\underline{x}(t)) = \underline{q}(t)$?

There is one particular inhomogeneous solution, so $L_{inhom} = \underline{x}_0(t) + L_{hom}$.

Ex4. a) Find a general solution to

$$\underline{x}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \underline{x}(t), \quad t \in \mathbb{R}.$$

We have

```
> Eigenvectors(A,output=list);
```

$$\left[\left[-2, 1, \left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}\right], \left[1, 1, \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right]\right]$$

so it follows that

$$\underline{x}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R},$$

for constants $c_1, c_2 \in \mathbb{R}$.

b) Guess a solution to $\quad \underline{x}'(t) = \underline{A}\,\underline{x}(t) + \begin{bmatrix} 0 \\ -2t \end{bmatrix}$.

Let's try $x_2(t) = at + b$.

$$a = -2(at+b) - 2t \iff a = (-2a-2)t - 2b$$

$$\iff a = -1 \ \wedge \ b = \tfrac{1}{2}$$

This leaves $x_2(t) = -t + \frac{1}{2}$. We use this and guess $x_1(t) = at + b$.

$$a = (at+b) + (-t + \tfrac{1}{2}) + 0 \iff a = (a-1)t + (b + \tfrac{1}{2})$$

$$\iff a = 1 \ \wedge \ b = \tfrac{1}{2}$$

So $\quad \underline{x}_0(t) = \begin{bmatrix} t + \frac{1}{2} \\ -t + \frac{1}{2} \end{bmatrix}$ and the general solution is

$$\underline{x}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t + \frac{1}{2} \\ -t + \frac{1}{2} \end{bmatrix}, \quad t \in \mathbb{R},$$

for $(c_1, c_2) \in \mathbb{R}^2$.

Ex 5. A linear system is given by

$$\frac{d}{dx} x_1(t) = \frac{1}{2} x_1(t) - x_2(t) + \cos(4t)$$

$$\frac{d}{dx} x_2(t) = \frac{3}{2} x_1(t) - 2 x_2(t) - 1$$

a) Write on matrix form and get eigenvalues/vectors.

$$\underline{x}'(t) = \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{3}{2} & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} \cos(4t) \\ -1 \end{bmatrix}$$

```
> Eigenvectors(A,output=list);
```

$$\left[\left[-1, 1, \left\{\left[\begin{array}{c} \frac{2}{3} \\ 1 \end{array}\right]\right\}\right], \left[-\frac{1}{2}, 1, \left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right\}\right]\right]$$

We have $\lambda_1 = -1$ with $\underline{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\lambda_2 = -\frac{1}{2}$ with $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b) Find the solution with dsolve.

```
> deq:=diff(x__1(t),t)=1/2*x__1(t)-x__2(t)+cos(4*t),diff(x__2(t),t)=3/2*x__1(t)-2*x__2(t)-1;
```

$$deq := \frac{d}{dt} x_1(t) = \frac{x_1(t)}{2} - x_2(t) + \cos(4t),\ \frac{d}{dt} x_2(t) = \frac{3 x_1(t)}{2} - 2 x_2(t) - 1$$

```
> dsolve([deq]);
```

$$\left\{x_1(t) = -\frac{4 e^{-t} c_1}{3} + \frac{296 \sin(4t)}{1105} - \frac{28 \cos(4t)}{1105} + e^{-\frac{t}{2}} c_2 + 2,\ x_2(t) = -2 e^{-t} c_1 + 1 - \frac{93 \cos(4t)}{1105} + \frac{36 \sin(4t)}{1105} + e^{-\frac{t}{2}} c_2\right\}$$

Here the homogeneous solution is given in eigenvalues and vectors

$$x_1(t) = -\frac{4}{3} c_1 e^{-t} + c_2 e^{-\frac{1}{2}t},$$

$$x_2(t) = -2 c_1 e^{-t} + c_2 e^{-\frac{1}{2}t},$$

and the inhomogeneous part has the solution

$$\underline{x}_0(t) = \begin{bmatrix} \frac{296}{1105} \sin(4t) - \frac{28}{1105} \cos(4t) \\ \frac{36}{1105} \sin(4t) - \frac{93}{1105} \cos(4t) + 1 \end{bmatrix} + \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

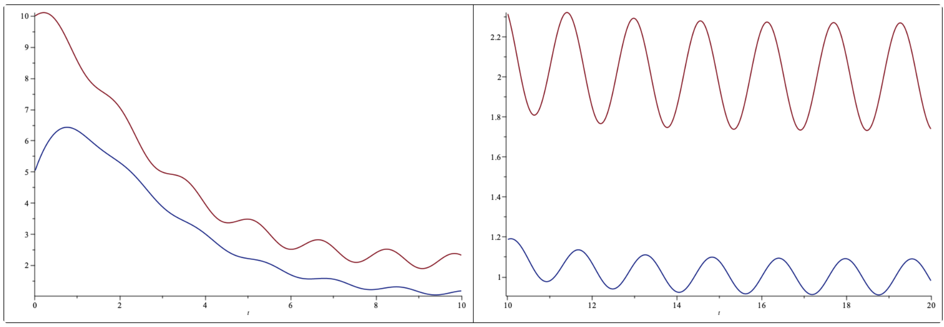
c) Plot the solution for which $x_1(0) = 10$ and $x_2(0) = 5$, first for $t \in [0,10]$ and then $t \in [10,20]$.

```
> dsolve([deq,ics]);
```

$$\left\{x_1(t) = -\frac{134\,e^{-t}}{17} + \frac{296\sin(4t)}{1105} - \frac{28\cos(4t)}{1105} + \frac{1034\,e^{-\frac{t}{2}}}{65} + 2,\ x_2(t) = -\frac{201\,e^{-t}}{17} + 1 - \frac{93\cos(4t)}{1105} + \frac{36\sin(4t)}{1105} + \frac{1034\,e^{-\frac{t}{2}}}{65}\right\}$$

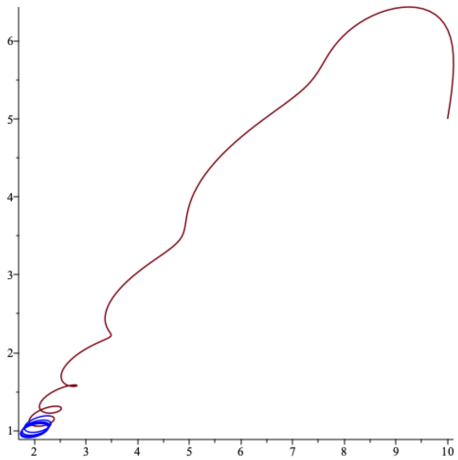
First separately

```
> H:=plot([-(134*exp(-t))/17 + (296*sin(4*t))/1105 - (28*cos(4*t))/1105 + (1034*exp(-t/2))/65 + 2,-(201*exp(-t))/17 + 1 - (93*cos(4*t))
  /1105 + (36*sin(4*t))/1105 + (1034*exp(-t/2))/65],t=0..10):
  K:=plot([-(134*exp(-t))/17 + (296*sin(4*t))/1105 - (28*cos(4*t))/1105 + (1034*exp(-t/2))/65 + 2,-(201*exp(-t))/17 + 1 - (93*cos(4*t))
  /1105 + (36*sin(4*t))/1105 + (1034*exp(-t/2))/65],t=10..20):
  display(<H|K>);
```



As a joint motion we have

```
> H1:=plot([-(134*exp(-t))/17 + (296*sin(4*t))/1105 - (28*cos(4*t))/1105 + (1034*exp(-t/2))/65 + 2,-(201*exp(-t))/17 + 1 - (93*cos(4*t))
  /1105 + (36*sin(4*t))/1105 + (1034*exp(-t/2))/65,t=0..10]):
  K1:=plot([-(134*exp(-t))/17 + (296*sin(4*t))/1105 - (28*cos(4*t))/1105 + (1034*exp(-t/2))/65 + 2,-(201*exp(-t))/17 + 1 - (93*cos(4*t))
  /1105 + (36*sin(4*t))/1105 + (1034*exp(-t/2))/65,t=10..20],color=blue):
  display(H1,K1);
```



Ex 6. We're given the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2 t \end{bmatrix}, \quad t \in \mathbb{R}.$$

a) Explain that for initial conditions (t_0, a_0, b_0) there is exactly one solution $(x_1(t), x_2(t))$ for which $x_1(t_0) = a_0$ and $x_2(t_0) = b_0$.

By theorem 17.11 this is the case. Let's check. We found the general solution

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

So we have the assumption that

$$\begin{bmatrix} -2c_1 e^{-3t_0} + 4 c_2 e^{3t_0} \\ c_1 e^{-3t_0} + c_2 e^{3t_0} \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$\iff \begin{bmatrix} -2e^{-3t_0} & 4e^{3t_0} \\ e^{-3t_0} & e^{3t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

Here we have a regular matrix (easily reduced), and as such only one solution (c_1, c_2) exists for which the system yields (a_0, b_0).

b) For arbitrary (t_0, a_0, b_0) consider whether a solution always exists.

This is indeed true if we follow the logic above in a), but in general it follows from theorem 17.11.