# Linear maps

Ex 1. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x_1, x_2) = (x_1 - x_2, -x_1 + x_2)$$

$$g(x_1, x_2) = (-x_2, x_1^2).$$

a) Show that only one of these is linear.

Let $\alpha \in \mathbb{R}$ and $\underline{x}, \underline{y} \in \mathbb{R}^2$, then

$$\begin{aligned}
f(\alpha \underline{x} + \underline{y}) &= f(\alpha \cdot x_1 + y_1, \alpha \cdot x_2 + y_2) \\
&= \begin{bmatrix} \alpha x_1 + y_1 - (\alpha x_2 + y_2) \\ -(\alpha x_1 + y_1) + \alpha x_2 + y_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha (x_1 - x_2) + y_1 - y_2 \\ \alpha(-x_1 + x_2) - y_1 + y_2 \end{bmatrix} \\
&= \alpha \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ -y_1 + y_2 \end{bmatrix} \\
&= \alpha f(\underline{x}) + f(\underline{y}).
\end{aligned}$$

Therefore $f$ is linear by definition 12.5.

By counterexample $g$ is not linear, say $k = 2$ and $\underline{x} = (2, 0)$.

$$g(2\underline{x}) = g(4, 0) = (0, 4^2) = (0, 16),$$

whereas

$$2 \cdot g(\underline{x}) = 2\, g(2, 0) = 2(0, 2^2) = (0, 8).$$

So $g(k\underline{x}) \neq k \cdot g(\underline{x})$ for all $\underline{x} \in \mathbb{R}^2$.

b) Determine the kernel of $f$.

We solve $f(\underline{x}) = \underline{0}$.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ +R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set $x_2 = t \in \mathbb{R}$, then the solution is

$$\underline{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad t \in \mathbb{R}.$$

So $\ker(f) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

c) State the range of $f$.

$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ -1 & 1 & b_2 \end{array}\right] \begin{array}{l} \\ +R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right]$$

The range consist of vectors $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for which $b_1 + b_2 = 0$,

i.e. $$b = t \begin{bmatrix} 1 \\ -1 \end{bmatrix} , \quad t \in \mathbb{R}.$$

So $f(\mathbb{R}^2) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

Ex2. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be given by

$$f(\underline{x}) = \begin{bmatrix} x_1 + x_2 + 3x_3 + x_4 \\ 3x_1 - x_2 + 2x_3 + 4x_4 \\ 2x_1 + 2x_2 + 6x_3 + 2x_4 \end{bmatrix}.$$

a) Use theorem 12.18.2 to conclude that $f$ is linear. State the mapping matrix ${}_{\underline{e}}F_{\underline{e}}$.

The map $f$ yields an image on coordinate form ${}_{\underline{e}}\underline{y} = {}_{\underline{e}}F_{\underline{e}}\, \underline{x}$

with ${}_{\underline{e}}F_{\underline{e}} \in \mathbb{L}^{3\times 4}$. Thus $f$ is a linear map.

$$ {}_{\underline{e}}F_{\underline{e}} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & -1 & 2 & 4 \\ 2 & 2 & 6 & 2 \end{bmatrix}. $$

b) Compute the dimension of the image, and provide a basis.

$$ \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & -1 & 2 & 4 \\ 2 & 2 & 6 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 5/4 & 5/4 \\ 0 & 1 & 7/4 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Since the rank is 2 it follows that $\dim(f(\mathbb{R}^4)) = 2$. We can use the first two lin. indept. vectors as a basis for the image: $f(\mathbb{R}^4) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$. Or as a basis $\left( \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right)$.

c) Provide a kernel for $f$.

The dimension of the kernel is 2 by proposition 12.26.

The basis for the kernel can be chosen from the reduced form $\begin{bmatrix} 1 & 0 & 5/4 & 5/4 \\ 0 & 1 & 7/4 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

We get $\left( \begin{bmatrix} -5/4 \\ -7/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/4 \\ 1/4 \\ 0 \\ 1 \end{bmatrix} \right)$ since these vectors span $\ker(f)$.

d) Does $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ belong to $f(\mathbb{R}^4)$?

No, since row 3 is 2 times row 1, then elimination leads to $R_3 - 2 \cdot R_1$ with all zeros, but $3-2=1$ on the right hand side. The augmented system is inconsistent.

e) Solve $f(\underline{x}) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

After appropriate elimination we get

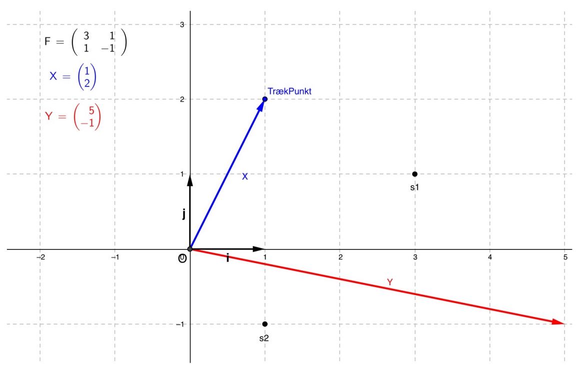
$$\left[\begin{array}{cccc|c} 1 & 0 & 5/4 & 5/4 & 1 \\ 0 & 1 & 7/4 & -1/4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is $\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \ker(f)$.

Ex 3. 1. Check calculation.

a)
$$\underline{\underline{F}}\,\underline{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underline{y}$$

2. Change $\underline{\underline{F}} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. Then find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.




3. Find the image of $\underline{i}$ and $\underline{j}$. Does this fit with the numbers in $\underline{\underline{F}}$?

Obviously $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b) New sheet.

1. What happens when moving $\underline{x}$ around?

We get a scalar multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

2. Compute $\det(\underline{\underline{F}})$ and give a basis for the image space.

$\det(\underline{\underline{F}}) = 0$, and $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ is a basis.

3. Dimension of the kernel? Determine equation for the line containing the kernel.

The kernel is 1-dimensional by proposition 12.26.

It follows that the line $y = -\frac{1}{2}x$ is the kernel.

So $F\left(t\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \underline{0}$.

c) New sheet.

1. $\underline{x}$ is bound to a line. Follow $\underline{y}$ as $\underline{x}$ varies.

Looks like $\underline{y}$ follows a different line.

2. Displace the line segment in a parallel manner and repeat. What happens to the image?

$\underline{\underline{F}}$ maps lines to lines and it preserves parallel lines as well.

d) New sheet.

1. Parabola : F rotates and slightly scales the parabola.

Ex 4. Another Geogebra sheet.

a)

1. How should $\underline{\underline{F}}$ be changed so that it maps to mirror images over the coordinate axes?

$$\underline{\underline{F}}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \underline{\underline{F}}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Test $\underline{\underline{F}} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

This yields a scaling in the vertical direction wrt. $k$.

3. Test $\underline{\underline{F}} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

This yields a scaling in the horizontal direction wrt. $k$.

4. Describe the red house when $\underline{\underline{F}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

The house is 3 times as wide and 2 times as tall as the blue house.

5. What is special about diagonal matrices?

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ scales by precisely a along the first axis and by b along the second axis. This literally expands/contracts the plane.

Ex 5. We have $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by

a)

$$_{e}\underline{\underline{F}}_{e} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

The kernel is 1-dimensional. Find a basis for $f(V)$.

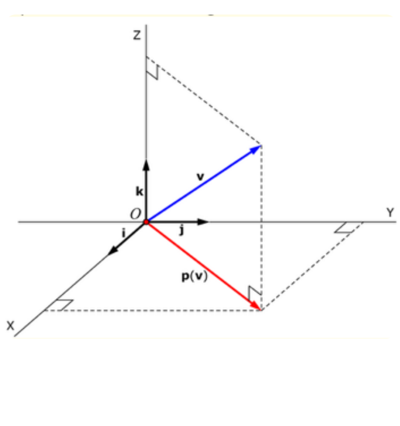
We pick two lin. indept. vectors: $\left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$.

b) Let p be the map that project orthogonally into the xy-plane.

Show p is linear. Determine $_{e}\underline{\underline{P}}_{e}$.

Determine basis for ker(p) and p(V).

Also check 12.26 is satisfied.




Linearity: let $\alpha \in \mathbb{R}$ and $\underline{u}, \underline{v} \in V$.

$$p(\alpha \underline{u} + \underline{v}) = p\left( \begin{bmatrix} \alpha u_1 + v_1 \\ \alpha u_2 + v_2 \\ \alpha u_3 + v_3 \end{bmatrix} \right) = \begin{bmatrix} \alpha u_1 + v_1 \\ \alpha u_1 + v_2 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \alpha p(\underline{u}) + p(\underline{v}).$$

$$
{}_{e}\underline{\underline{P}}_{e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

The basis for $\ker(p)$ is $\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ and the basis for $p(V)$ is $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

We see that $\dim(V) = 3 = 1 + 2 = \dim(\ker(p)) + \dim(p(V))$.

Ex 6. We want to mirror across the line $y = x$.

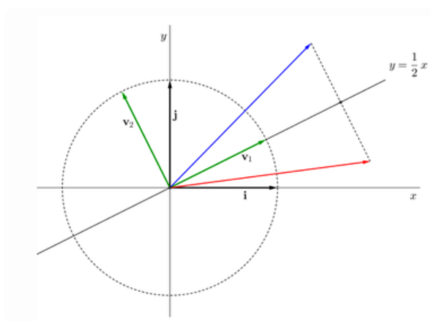
a) Determine $s(\underline{i})$ and $s(\underline{j})$ and write out ${}_{e}\underline{\underline{S}}_{e}$. Use this to express the mirror image of an arbitrary vector $\underline{u}$.

We want to flip coordinat, so $s(\underline{i}) = \underline{j}$ and $s(\underline{j}) = \underline{i}$. Thus it follows that ${}_{e}\underline{\underline{S}}_{e} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The image of an arbitrary vector is then

$$
{}_{e}\underline{\underline{S}}_{e}\,\underline{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}.
$$

We will now work towards the following reflection




b) Determine ${}_{v}\underline{\underline{R}}_{v}$, the reflection in basis v.

We need only flip $\underline{v}_2$ in this setting, so ${}_{v}\underline{\underline{R}}_{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

c) Determine ${}_e\underline{\underline{R}}_e$ and let $\underline{u} \in \mathbb{R}^2$ be arbitrary. Determine an expression of the mirror image of $\underline{u}$ in the line $y = \frac{1}{2}x$.

We have

$$ {}_e\underline{\underline{R}}_e = {}_e\underline{\underline{M}}_v \, {}_v\underline{\underline{R}}_v \, {}_vM_e $$

so we need the coordinates for $\underline{v}_1$ and $\underline{v}_2$.

Firstly $\underline{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and so $\underline{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. This follows as $\underline{v}_1 \perp \underline{v}_2$ and they are both unit length.

Now define

$$ {}_e\underline{\underline{M}}_v = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}. $$

We now determine ${}_v\underline{\underline{M}}_e = {}_e\underline{\underline{M}}_v^{-1}$. Note that $\det({}_e\underline{\underline{M}}_v) = 1$.

$$ {}_v\underline{\underline{M}}_e = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}. $$

Therefore we get

$$ {}_e\underline{\underline{R}}_e = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} $$

$$ = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} $$

$$ = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}. $$

Image of $\underline{u}$:

$$ {}_e\underline{\underline{R}}_e \, \underline{u} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3u_1 + 4u_2 \\ 4u_1 - 3u_2 \end{bmatrix}. $$

Ex 7. Let $\underline{a}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix}3\\7\end{bmatrix}$, $\underline{c}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $\underline{c}_2 = \begin{bmatrix}2\\3\\1\end{bmatrix}$ and $\underline{c}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(\underline{a}_1) = \underline{c}_1 + \underline{c}_2 - 3\underline{c}_3 \quad \text{and} \quad f(\underline{a}_2) = \underline{c}_1 - \underline{c}_2 - 2\underline{c}_3.$$

a) Show $(\underline{a}_1, \underline{a}_2)$ is a basis for $\mathbb{R}^2$ and $(\underline{c}_1, \underline{c}_2, \underline{c}_3)$ is a basis for $\mathbb{R}^3$.

$\det\left(\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\right) = 7 - 6 = 1 \neq 0$ so $\underline{a}_1$ and $\underline{a}_2$ are lin. indept. and thus span $\mathbb{R}^2$.

$\det\left(\begin{bmatrix}1 & 2 & 1\\2 & 3 & 2\\2 & 1 & 1\end{bmatrix}\right) = 3 + 8 + 2 - 6 - 2 - 4 = 13 - 12 = 1$ and

so the vectors span $\mathbb{R}^3$.

b) Determine ${}_c\underline{\underline{F}}_a$.

$${}_c\underline{\underline{F}}_a = \begin{bmatrix}1 & 1\\1 & -1\\-3 & -2\end{bmatrix}$$

c) State ${}_e\underline{\underline{F}}_a$.

$${}_e\underline{\underline{F}}_a = {}_e\underline{\underline{M}}_c\,{}_c\underline{\underline{F}}_a = \begin{bmatrix}1 & 2 & 1\\2 & 3 & 2\\2 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\\-3 & -2\end{bmatrix} = \begin{bmatrix}0 & -3\\-1 & -5\\0 & -1\end{bmatrix}.$$

d) State ${}_c\underline{\underline{F}}_e$.

$${}_c\underline{\underline{F}}_e = {}_c\underline{\underline{F}}_a\,{}_a\underline{\underline{M}}_e = {}_c\underline{\underline{F}}_a\,{}_e\underline{\underline{M}}_a^{-1} = \begin{bmatrix}1 & 1\\1 & -1\\-3 & -2\end{bmatrix}\begin{bmatrix}7 & -3\\-2 & 1\end{bmatrix} = \begin{bmatrix}5 & -2\\9 & -4\\-17 & 7\end{bmatrix}.$$

e) State ${}_e\underline{\underline{F}}_e$.

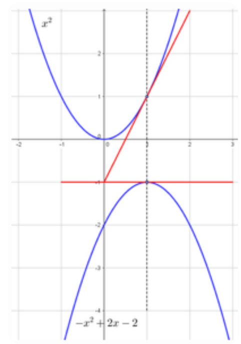
$$ {}_e\underline{\underline{F}}_e = {}_e\underline{\underline{M}}_c \; {}_c\underline{\underline{F}}_a \; {}_a\underline{\underline{M}}_e $$

$$ = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & -3 \\ -1 & -5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 3 & -2 \\ 2 & -1 \end{bmatrix}. $$

Ex8. Let $f: P_2(\mathbb{R}) \to \mathbb{R}$ be given by $f(P(x)) = P'(1)$.

a) Compute $f(x^2)$ and $f(-x^2+2x-2)$. Formulate what $f$ does. Does it match the figure?




$$ f(x^2) = 2\cdot 1 = 2 $$

$$ f(-x^2+2x-2) = -2\cdot 1 + 2 = 0 $$

The map $f$ determines the slope of the tangent of $P(x)$ for $x=1$. This does indeed line up with the figure.

b) Show that $f$ is linear.

Let $\alpha \in \mathbb{R}$ and $P, Q \in P_2(\mathbb{R})$, then

$$ f(\alpha P(x) + Q(x)) = (\alpha P(1) + Q(1))' = \alpha P'(1) + Q'(1) $$

$$ = \alpha f(P(x)) + f(Q(x)). $$

So $f$ is a linear map.

c) One of the polynomials is in $\ker(f)$, which one? Determine a basis for $\ker(f)$.

Since $f(-x^2+2x+2)=0$ it follows that $-x^2+2x-2 \in \ker(f)$.

Any constant maps to zero, and multiples of $x^2-2x$. So a basis is given by $(1, x^2-2x)$.

d) Explain that $f(P_2(\mathbb{R}))$ is equal to $\mathbb{R}$.

Clearly $b\cdot x \in P_2(\mathbb{R})$ and $f(bx)=b$. This is true for any $b \in \mathbb{R}$. So given an arbitrary $c \in \mathbb{R}$, we can map $c\cdot x \in P_2(\mathbb{R})$ to $c \in \mathbb{R}$. As $c$ was arbitrary it follows that $f(P_2(\mathbb{R}))=\mathbb{R}$.

e) Let $g: P_2(\mathbb{R}) \to \mathbb{R}$ be given by $g(P(x)) = P'(0)+1$.

Formulate what $g$ does and prove it is not linear.

The map $g$ evaluates the slope of $P(x)$ at $x=0$, and then adds 1 to the slope.

Let's see if the multiplication requirement fails.

$$2\cdot g(P(x)) = 2\cdot P'(0)+2$$

$$g(2\cdot P(x)) = (2\cdot P(0))'+1 = 2P'(0)+1$$

Clearly $g$ is not linear.