Integration Techniques

Ex1.

a) Find antiderivatives.

$$\int x^3\,dx = \frac{1}{4}x^4$$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = -\frac{1}{2x^2}$$

$$\int \sin\left(3x - \frac{\pi}{2}\right) dx = -\frac{1}{3}\cos\left(3x - \frac{\pi}{2}\right)$$

b) Compute the definite integrals using a).

$$\int_0^1 x^3\,dx = \frac{1}{4}\left[x^4\right]_0^1 = \frac{1}{4}$$

$$\int_1^2 \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^2 = -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}$$

$$\int_{-\pi/2}^{0} \left(\sin 3x - \frac{\pi}{2}\right) dx = \left[-\frac{1}{3}\cos\left(3x - \frac{\pi}{2}\right)\right]_{-\pi/2}^{0} = 0 - \left(-\frac{1}{3}\cdot 1\right) = \frac{1}{3}$$

Ex 2.

a) Antiderivative of $x\cos x$ and check if correct.

$$\int x\cos x\,dx = x\cdot\sin x - \int \sin x\,dx$$

$$= x\cdot\sin x + \cos x$$

$$(x\cdot\sin x + \cos x)' = \sin x + x\cdot\cos x - \sin x = x\cdot\cos x.$$

b) Determine the indefinite integral.

$$\int t e^t \, dt = t e^t - e^t + k \quad , \quad k \in \mathbb{R}.$$

c) Antiderivative of $x^2 \ln x$, $x > 0$.

$$\int x^2 \ln x \, dx = \frac{1}{3} x^3 \cdot \ln x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{3} x^3 \cdot \ln x - \frac{1}{9} x^3 .$$

d) Solve $x'(t) - 2 \cdot x(t) = 3t$ with the general solution formula.

See Thm. 16.16.

$p(t) = -2$, $q(t) = 3t$ and $P(t) = -2t$.

$$x(t) = e^{2t} \int e^{-2t} \cdot 3t \, dt + c \cdot e^{2t} \quad , \quad c \in \mathbb{R}.$$

$$\int e^{-2t} \cdot t \, dt = -\frac{1}{2} t \cdot e^{-2t} + \frac{1}{2} \int e^{-2t} \, dt$$

$$= -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t}$$

$$\Rightarrow x(t) = -\frac{3}{2} t - \frac{3}{4} + c e^{2t} .$$

Ex 3.

a) Antiderivctive of $h(x) = x e^{x^2}$.

$$\int x e^{x^2} \, dx = \frac{1}{2} \int e^t \, dt$$

$$= \frac{1}{2} e^t = \frac{1}{2} e^{x^2} .$$

$t = x^2$, so $\frac{dt}{dx} = 2x$ and $dt = 2x \, dx$.

b) Indefinite integral

$$\int \frac{x}{x^2+1}\,dx \qquad\qquad t = x^2+1, \text{ so } \frac{dt}{dx} = 2x$$

and $dt = 2x\,dx$

$$= \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln(t) + k = \frac{1}{2}\ln(x^2+1) + k, \quad k \in \mathbb{R}.$$

c) Compute $\displaystyle\int_0^{\pi} \frac{\sin(x)}{3-\cos(x)}\,dx$ $\qquad t = 3-\cos(x)$

$dt = \sin(x)\,dx$

$$= \int_{3-\cos(0)}^{3-\cos(\pi)} \frac{1}{t}\,dt$$

$$= \Big[\ln(t)\Big]_2^4 = \ln(4) - \ln(2) = \ln(2)$$

Ex 4.

a) A curve is given as a segment of the graph of $\ln(x)$:

$$K = \left\{ (x,y) \in \mathbb{R}^2 \mid y = \ln(x),\ x \in [1, 2\sqrt{2}] \right\}.$$

State a parametric representation and determine the Jacobian.

$$\underline{r}(u) = \begin{bmatrix} u \\ \ln(u) \end{bmatrix}, \qquad u \in [1, 2\sqrt{2}]$$

$$J(u) = |r'(u)| = \sqrt{1+\frac{1}{u^2}} = \frac{1}{u}\sqrt{u^2+1}.$$

b) Compute $\int_K x^2 \, d\mu$. $\qquad f(x) = x^2 \Rightarrow f(\underline{r}(u)) = u^2$.

$$\int_1^{2\sqrt{2}} f(\underline{r}(u)) \cdot J(u) \, du = \int_1^{2\sqrt{2}} u^2 \frac{1}{u} \sqrt{u^2+1} \, du$$

$$= \int_1^{2\sqrt{2}} u \sqrt{u^2+1} \, du \qquad t = u^2+1$$

$$dt = 2u \, du$$

$$= \frac{1}{2} \int_{1^2+1}^{(2\sqrt{2})^2+1} \sqrt{t} \, dt$$

$$= \frac{1}{2} \left[\frac{2}{3} t^{3/2} \right]_2^9 = \frac{1}{3} \cdot \left(27 - 2\sqrt{2} \right)$$

$$= 9 - \frac{2\sqrt{2}}{3}.$$

Ex 5. The Gherkin is described by

$$x = f(z) = \frac{1}{2} \sqrt{-z^2 + 2z + 3}, \quad z \in [0,3].$$

a) Calculate the volume.

$$V = \pi \cdot \int_0^3 f(z)^2 \, dz = \frac{\pi}{4} \int_0^3 -z^2 + 2z + 3 \, dz$$

$$= \frac{\pi}{4} \cdot \left[-\frac{1}{3} z^3 + z^2 + 3z \right]_0^3 = \frac{\pi}{4} \cdot 9 = \frac{9\pi}{4}.$$

b) Determine the area of the region under the curve.

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> f:=z-> 1/2*sqrt(-z^2+2*z+3);
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$$f := z \mapsto \frac{\sqrt{-z^2 + 2 \cdot z + 3}}{2}$$

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> int(f(z),z);
  int(f(z),z=0..3);
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$$-\frac{(-2z+2)\sqrt{-z^2+2z+3}}{8} + \arcsin\left(\frac{z}{2} - \frac{1}{2}\right)$$

$$\frac{\sqrt{3}}{4} + \frac{2\pi}{3}$$