Linear systems of equations

Ex1. Solve some systems by hand.

a)
$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -2R_2 \\ +2R_3 \\ \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Alternatively just backwards substitute:

$$x_3 = 2 \Rightarrow x_2 = -1 + 2 \cdot 2 = 3$$

$$\Rightarrow x_1 = 2 - 2 \cdot 3 + 4 \cdot 2 = 4$$

b)
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -4R_1 \end{array} \longrightarrow \left[\begin{array}{ccccc} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 4 & 8 & -1 & 5 \end{array}\right] \begin{array}{l} \\ \\ -4R_2 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} +R_3 \\ \\ \cdot(-1) \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

c)

$$\left[\begin{array}{cc|c} i & -2 & -i \\ 1 & 1+i & 1 \end{array}\right] \begin{array}{l} \cdot(-i) \\ \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 2i & -1 \\ 1 & 1+i & 1 \end{array}\right] \begin{array}{l} \\ -R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 2i & -1 \\ 0 & 1-i & 2 \end{array}\right] \begin{array}{l} \\ \cdot(1+i) \end{array}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2i & -1 \\ 0 & 2 & 2+2i \end{array}\right] \begin{array}{l} -R_2\cdot i \\ \cdot\frac{1}{2} \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1-2i \\ 0 & 1 & 1+i \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} 1-2i \\ 1+i \end{bmatrix}$$

d)

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 1 & 4 & 8 & 9 \end{array}\right] \begin{array}{l} \\ \\ -R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 6 & 7 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

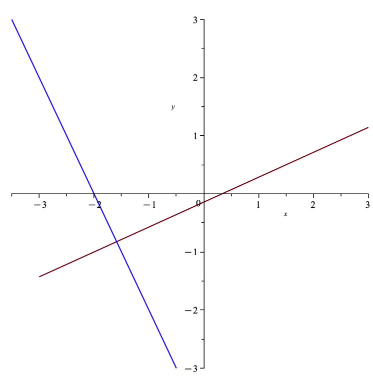
$\Rightarrow$ There is no solution such that $0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 1$!

Ex 2. Run some Maple.

```
> restart;
> with(LinearAlgebra):
> A:=<3,-7;-2,-1>:
  b:=<1,4>:
> LinearSolve(A,b);
```

$$\begin{bmatrix} -\frac{27}{17} \\ -\frac{14}{17} \end{bmatrix}$$

```
> with(plots):
> linje1:=implicitplot(3*x-7*y=1,x=-3..3,y=-3..3):
  linje2:=implicitplot(-2*x-y=4,x=-5..3,y=-3..3,color=blue):
  display(linje1,linje2);
```



```
> T:=<1,0,-1,1,0;
   1,1,1,1,1;
   4,4,4,3,5>;
```

$$T := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 3 & 5 \end{bmatrix}$$

```
> T1:= RowOperation(T , [2,1],-1);
  T2:= RowOperation(T1, [3,1],-4);
  T3:= RowOperation(T2, [3,2],-4);
  T4:= RowOperation(T3, 3,-1);
  trapT:= RowOperation(T4, [1,3],-1);
```

$$T1 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 4 & 4 & 4 & 3 & 5 \end{bmatrix}$$

$$T2 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 4 & 8 & -1 & 5 \end{bmatrix}$$

$$T3 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$T4 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$trapT := \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

```
> trapT2:= ReducedRowEchelonForm(T);
```

$$trapT2 := \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

```
> LinearSolve(T,free=t);
```

$$\begin{bmatrix} t_3 + 1 \\ -2 t_3 + 1 \\ t_3 \\ -1 \end{bmatrix}$$

The methods are similar, though the solution is extracted by using LinearSolve. We can readily do this from the system as well with little effort.

Ex 3. Maple

a)

```
> restart;
> with(LinearAlgebra):
> A:= <1,2,2;0,1,3;1,4,8>:
  b:=<6,3,12>:
  T:=<1,2,2,6;0,1,3,3;1,4,8,12>:
```

b)

```
> T;
  T1:= RowOperation(T , [3,1],-1);
  T2:= RowOperation(T1 , [3,2],-2);
  T3:= RowOperation(T2 , [1,2],-2);
```

$$\begin{bmatrix} 1 & 2 & 2 & 6 \\ 0 & 1 & 3 & 3 \\ 1 & 4 & 8 & 12 \end{bmatrix}$$

$$T1 := \begin{bmatrix} 1 & 2 & 2 & 6 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 6 & 6 \end{bmatrix}$$

$$T2 := \begin{bmatrix} 1 & 2 & 2 & 6 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T3 := \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> ReducedRowEchelonForm(T);
```

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> LinearSolve(T,free=t);
```

$$\begin{bmatrix} 4 t_3 \\ 3 - 3 t_3 \\ t_3 \end{bmatrix}$$

c) The solution to the system follows from b)

$$\underline{x} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

d) Pros and cons: row operations are tedious, but are genuine practice in solving equations.

ReducedRowEchelonForm simply skips all the steps, but we can still tell whether there are solutions/inconsistent system.

LinearSolve just yields the answer, sometimes we need to need to see the problem as well.

Ex4. Let $\underline{\underline{A}}\,\underline{x} = \underline{0}$ and $\underline{\underline{A}}\,\underline{x} = \underline{b}$ be our homogeneous and
a) inhomogeneous systems. Assume $\underline{x}_1 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ is a solution to $\underline{\underline{A}}\,\underline{x} = \underline{0}$ and $\underline{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$.

Is $\underline{y}_0 = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$ a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$? $\underline{y}_0 = \underline{x}_1 + \underline{x}_0$, so yes (6.37)

Is $\underline{z}_0 = \begin{bmatrix} 2 \\ 9 \\ 8 \end{bmatrix}$ a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$? $\underline{z}_0 = \underline{x}_0 + \underline{y}_0$, so no (6.37)

Is the difference between two solutions of $\underline{\underline{A}}\,\underline{x} = \underline{b}$ again a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$? $\underline{x}_0 - \underline{y}_0$ no (6.37), though homogeneous solution.

We want to use proposition 6.35 and 6.37. We can do this without calculating, but it is more apparent why the answers unfurl as they do by using linearity.

First we have $\underline{y}_0 = \underline{x}_1 + \underline{x}_0$, and so

$$\underline{\underline{A}}\,\underline{y}_0 = \underline{\underline{A}}(\underline{x}_1 + \underline{x}_0) = \underline{\underline{A}}\,\underline{x}_1 + \underline{\underline{A}}\,\underline{x}_0 = 0 + \underline{b} = \underline{b}.$$

Thus $\underline{y}_0$ is a solution to the inhomogeneous system.

$$\underline{\underline{A}}\,\underline{z}_0 = \underline{\underline{A}}(\underline{x}_0 + \underline{y}_0) = \underline{\underline{A}}\,\underline{x}_0 + \underline{\underline{A}}\,\underline{y}_0 = \underline{b} + \underline{b} = 2\underline{b}.$$

So $\underline{z}_0$ is not a solution. The difference of any two inhomogeneous solutions is a homogeneous solution:

$$\underline{\underline{A}}(\underline{x}_0 - \underline{y}_0) = \underline{\underline{A}}\,\underline{x}_0 - \underline{\underline{A}}\,\underline{y}_0 = \underline{b} - \underline{b} = \underline{0}.$$

b)

Describe what $\rho(\underline{\underline{T}})$ means for the structure of the solution set.

Let's assume we don't have $\rho(\underline{\underline{A}}) < \rho(\underline{\underline{T}})$, since that yields an inconsistent system, i.e. no solution.

Let there be n variables (unknown). If $\rho(\underline{\underline{T}}) = n$, then one solution exists.

Now if $\rho(\underline{\underline{T}}) < n$, then more solutions exist, which generally is written on a standard parameterform.

In both cases we have a particular solution $\underline{x}_0$ if the system is inhomogeneous, and a homogeneous solution, such that we have either

$$\underline{x}_0 + \underline{0} \quad \text{or} \quad \underline{x}_0 + t_1 \underline{v}_1 + t_2 \underline{v}_2 + \cdots + t_{n-k} \underline{v}_{n-k} .$$

Ex 5.

a) Find for $a \in \mathbb{R}$ the solutions to

$$\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -aR_1 \end{array} \to \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & 1-a \end{array}\right] \begin{array}{l} \\ \\ +R_2 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right]$$

Check $a=1$, since we want to use $\frac{1}{a-1}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

$t_1, t_2 \in \mathbb{R}$.

For $a \neq 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & (a+2)(1-a) & 1-a \end{array}\right] \begin{array}{l} \\ \cdot \frac{1}{a-1} \\ \cdot \frac{1}{1-a} \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & a+2 & 1 \end{array}\right]$$

Check $a = -2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no solution.

For $a \neq 1, -2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & a+2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \cdot \frac{1}{a+2} \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{a+2} \end{array}\right] \begin{array}{l} \\ +R_3 \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & 0 & \frac{1}{a+2} \\ 0 & 0 & 1 & \frac{1}{a+2} \end{array}\right] \begin{array}{l} -R_2 - aR_3 \\ \\ \\ \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{a+2} \\ 0 & 1 & 0 & \frac{1}{a+2} \\ 0 & 0 & 1 & \frac{1}{a+2} \end{array}\right]$$

The solution is $\underline{x} = \left(\frac{1}{a+2}, \frac{1}{a+2}, \frac{1}{a+2}\right)$ for $a \in \mathbb{R} \setminus \{-2, 1\}$.

b) Check with Maple.

```
> restart;
> with(LinearAlgebra):
> T:=<a,1,1,1;1,a,1,1;1,1,a,1>;
```

$$T := \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{bmatrix}$$

```
> LinearSolve(T);
  LinearSolve(subs(a=1,T));
  LinearSolve(subs(a=-2,T));
```

$$\begin{bmatrix} \frac{1}{a+2} \\ \frac{1}{a+2} \\ \frac{1}{a+2} \end{bmatrix}$$

$$\begin{bmatrix} 1 - \_t7_2 - \_t7_3 \\ \_t7_2 \\ \_t7_3 \end{bmatrix}$$

```
Error, (in LinearAlgebra:-LinearSolve) inconsistent system
```

The point is that Maple doesn't consider $a = 1, -2$ on its own. We can ask it to.

Ex 6.

a) What should you consider before multiplying matrices?

For $\underline{\underline{A}}$ being $m \times n$ and $\underline{\underline{B}}$ being $k \times l$ we check that $n = k$, otherwise we can't multiply $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

b) Why is $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ in general?

Well suppose $n = k$ as in a), we can not necessarily claim $m = l$, and so $\underline{\underline{A}}\,\underline{\underline{B}}$ makes sense, but $\underline{\underline{B}}\,\underline{\underline{A}}$ doesn't.

c) If same dimensions are given for $\underline{\underline{A}}$ and $\underline{\underline{B}}$ will $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$?

No we saw this in example 7.12. If we think of these matrices as rotations and reflections, then counter examples are abundant with no required computation.

d) Compute

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 - 1 - 6 \\ 3 + 2 + 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

e) Let $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and $\underline{\underline{B}} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and compute some stuff.

```
> A:=<1,1,2;1,2,-1>:
  B:=<0,-1,-1;1,2,1>:
> 2*A-3*B;
```

$$\begin{bmatrix} 2 & 5 & 7 \\ -1 & -2 & -5 \end{bmatrix}$$

```
> 2*Transpose(A)-3*Transpose(B);
```

$$\begin{bmatrix} 2 & -1 \\ 5 & -2 \\ 7 & -5 \end{bmatrix}$$

```
> 2*A-3*Transpose(B);
Error, (in rtable/Sum) invalid input: dimensions do not match: Matrix(1 .. 2,1 .. 3) cannot be added to M
> A.B;
Error, (in LinearAlgebra:-Multiply) first matrix column dimension (3) <> second matrix row dimension (2)
```

```
> A.Transpose(B);
```

$$\begin{bmatrix} -3 & 5 \\ -1 & 4 \end{bmatrix}$$

```
> B.Transpose(A);
```

$$\begin{bmatrix} -3 & -1 \\ 5 & 4 \end{bmatrix}$$

```
> Transpose(B).A;
```

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

```
> Transpose(A).B
```

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -1 & -4 & -3 \end{bmatrix}$$

Ex 7. Solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 2 & -3 & -1 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 4 \\ -1 & 2 & -3 & -1 & 2 \end{array}\right] \begin{matrix} \\ +R_1 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 4 \\ 0 & 2 & -4 & 0 & 6 \end{array}\right] \begin{matrix} \\ \cdot \frac{1}{2} \end{matrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & -2 & 0 & 3 \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}.$$

free variables