Second-Order Differential Equations

Ex 1. a) Determine the general solution for

$$x''(t) + 2x'(t) + 5x(t) = 0, \quad t \in \mathbb{R}.$$

We just solve $\lambda^2 + 2\lambda + 5 = 0$ and plug into the solution according to proposition 18.2.

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

(18-10)

$$x(t) = c_1 e^{t} \cos(2t) + c_2 e^{t} \sin(2t), \quad t \in \mathbb{R}, \quad c_1, c_2 \text{ constants.}$$

b) Solve $x''(t) - 6x'(t) + 9x(t) = 0, \quad t \in \mathbb{R}.$

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4 \cdot 9}}{2} = 3$$

(18-11)

$$x(t) = c_1 e^{3t} + c_2 t e^{3t}, \quad c_1, c_2 \in \mathbb{R}.$$

c) Solve $x''(t) + 3x'(t) - 4x(t) = 0, \quad t \in \mathbb{R}.$

$$\lambda = \frac{-3 \pm \sqrt{(-3)^2 - 4 \cdot (-4)}}{2} = \frac{-3 \pm 5}{2} = \begin{cases} 1 \\ -4 \end{cases}$$

(18-9)

$$x(t) = c_1 e^{t} + c_2 e^{-4t}, \quad c_1, c_2 \in \mathbb{R}.$$

Ex 2. Given $x''(t) + 4x'(t) + 29x(t) = -25\sin(2t) + \frac{109}{4} e^{-\frac{1}{2}t} - 8\cos(2t), \; t \in \mathbb{R}.$

a) Use dsolve for the solution.

```
> eq:= diff(x(t),t,t)+4*diff(x(t),t)+29*x(t)=-25*sin(2*t)+109/4*exp(-1/2*t)-8*cos(2*t);
```

$$eq := \frac{d^2}{dt^2} x(t) + 4 \frac{d}{dt} x(t) + 29\, x(t) = -25 \sin(2t) + \frac{109 e^{-\frac{t}{2}}}{4} - 8\cos(2t)$$

```
> dsolve(eq,x(t));
```

$$x(t) = e^{-2t} \sin(5t)\, c_2 + e^{-2t} \cos(5t)\, c_1 - \sin(2t) + e^{-\frac{t}{2}}$$

b) Plot the solution that passes through $(0,1)$ and has the slope $-\frac{5}{2}$ at $t=0$. Then also for the slope $\frac{1}{2}$ at $t=0$.

```
> dsolve({eq,x(0)=1,D(x)(0)=-5/2},x(t));
  dsolve({eq,x(0)=1,D(x)(0)=1/2},x(t));
```

$$x(t) = -\sin(2t) + e^{-\frac{t}{2}}$$

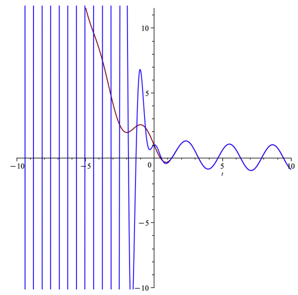
$$x(t) = \frac{3 e^{-2t} \sin(5t)}{5} - \sin(2t) + e^{-\frac{t}{2}}$$

```
> with(plots):
> with(DEtools):

> A:=DEplot(eq,x(t),t=-5..10,[[x(0)=1,D(x)(0)=-5/2]],x=-5..5):
  A1:=DEplot(eq,x(t),t=-5..10,[[x(0)=1,D(x)(0)=1/2]],x=-5..5):

> B:=plot(-sin(2*t) + exp(-t/2),t=-5..10):

> C:=plot((3*exp(-2*t)*sin(5*t))/5 - sin(2*t) + exp(-t/2),t=-10..10,view=[-10..10,-10..10],color=blue):
> display(B,C);
```



DEplot doesn't draw nice plots compared to just using the particular solutions.

Ex3. Consider the linear map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by

$$f(x(t)) = x''(t) + 3x'(t) - 4x(t).$$

a) Guess a solution to $f(x(t)) = 29 - 12t$.

The highest power of t is 1, so let's guess we have $3t$ in there to make $-12t$. Using $x(t) = 3t + b$ we get the constant terms $3 \cdot 3 - 4 \cdot b = 29 \iff b = -5$ leaving us with $x(t) = 3t - 5$.

Now the general solution is by 1c)

$$x(t) = c_1 e^{t} + c_2 e^{-4t} + 3t - 5 \quad , \quad c_1, c_2 \in \mathbb{R}.$$

b) Guess a solution to $f(x(t)) = \cos(t)$.

Let's guess $x(t) = a\cos(t) + b\sin(t)$.

$$f(x(t)) = -a\cos(t) - b\sin(t) + 3(-a\sin(t) + b\cos(t)) - 4(a\cos(t) + b\sin(t))$$

$$= (-a + 3b - 4a)\cos(t) + (-b - 3a - 4b)\sin(t)$$

$$= \cos(t) \iff a = -\frac{5}{34} \text{ and } b = \frac{3}{34}.$$

$$x(t) = c_1 e^{t} + c_2 e^{-4t} - \frac{5}{34}\cos(t) + \frac{3}{34}\sin(t).$$

c) Solve $f(x(t)) = 29 - 12t + \cos(t)$.

Apply superposition:

$$x(t) = c_1 e^{t} + c_2 e^{-4t} - \frac{5}{34}\cos(t) + \frac{3}{34}\sin(t) + 3t - 5.$$

Given $v = (\cos(t), \sin(t), e^t, t, 1)$, $t \in \mathbb{R}$, is a lin. indept. set of vectors consider f restricted to $U \subseteq C^{\infty}(\mathbb{R})$ that has basis v.

d) Show $f(U)$ is a subspace in U, and determine ${}_{v}\underline{F}_{v}$.

The image $f(U)$ is spanned by the basis v, so clearly $f(U)$ is a subspace of U as f is linear.

$$\begin{array}{c} \\ {}_{v}\underline{F}_{v} = \end{array} \begin{array}{c} \begin{array}{ccccc} \cos(t) & \sin(t) & e^t & t & 1 \end{array} \\ \begin{bmatrix} -5 & 3 & 0 & 0 & 0 \\ -3 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 3 & -4 \end{bmatrix} \end{array}$$

e) State the vector of $q(t) = \cos(t) + 29 - 12t$, and find all solutions in U to $f(x(t)) = q(t)$.

$${}_{v}\underline{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -12 \\ 29 \end{bmatrix} \text{ and } x(t) = c_1 e^{t} - \frac{5}{34}\cos(t) + \frac{3}{34}\sin(t) + 3t - 5.$$

f) Does a solution exist in U with $x(0)=0$ and $x'(0)=1$?

```
> dsolve({diff(x(t),t,t)+3*diff(x(t),t)-4*x(t)=29-12*t+cos(t),x(0)=0,D(x)(0)=1},x(t));
```

$$x(t) = \frac{123\,e^{-4t}}{85} + \frac{37\,e^{t}}{10} + 3t - 5 + \frac{3\sin(t)}{34} - \frac{5\cos(t)}{34}$$

There's no solution in U, since it contains e^{-4t}.

Ex 4. Modelling with Maple sheet (Hooke's law).

Ex 5. We have $x''(t) + a_1 x'(t) + a_0 x(t) = q(t)$, $t \in \mathbb{R}$.

It is claimed that $x_1(t) = \sin(t)$ and $x_2(t) = \frac{1}{2}\sin(2t)$ are both solutions. Prove that this is false.

By the existence and uniqueness theorem the initial conditions (t_0, x_0, v_0) have a unique solution. Let $t_0 = 0$, then $x_1(0) = 0 = x_2(0)$ as well as $x_1'(0) = 1 = x_2'(0)$. Since all initial conditions are equal for $t_0 = 0$ it follows from uniqueness that x_1 and x_2 can't both be a solution to the equation.

Ex 6. Given $x''(t) + a_1 x'(t) + a_0 x(t) = q(t)$, $t \in \mathbb{R}$, and solutions

$$x_1(t) = \sin(t) + 2e^{t}$$

$$x_2(t) = \sin(t) + e^{t} - e^{-t}.$$

a) Determine the general solution.

The general solution is $x(t) = c_1 e^{t} + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$.

b) Determine the general solution to the inhomogeneous equation.

$$x(t) = c_1 e^{t} + c_2 e^{-t} + \sin(t).$$

c) Determine a_0, a_1 and $q(t)$.

$(\lambda - 1)(\lambda + 1) = x^2 - 1 \Rightarrow a_1 = 0$ and $a_0 = -1$. (Much easier than solution below)

```
> x1:=t->sin(t)+2*exp(t);
  x2:=t->sin(t)+exp(t)-exp(-t);
  x3:=t->sin(t);
  x4:=t->exp(t)+exp(-t);
```

$$x1 := t \mapsto \sin(t) + 2 \cdot e^t$$
$$x2 := t \mapsto \sin(t) + e^t - e^{-t}$$
$$x3 := t \mapsto \sin(t)$$
$$x4 := t \mapsto e^t + e^{-t}$$

```
> l1:=diff(x1(t),t,t)+a1*diff(x1(t),t)+a0*x1(t)=q(t);
  l2:=diff(x2(t),t,t)+a1*diff(x2(t),t)+a0*x2(t)=q(t);
  l3:=diff(x3(t),t,t)+a1*diff(x3(t),t)+a0*x3(t)=q(t);
  l4:=diff(x4(t),t,t)+a1*diff(x4(t),t)+a0*x4(t)=0;
```

$$l1 := -\sin(t) + 2\,e^t + a1\left(\cos(t) + 2\,e^t\right) + a0\left(\sin(t) + 2\,e^t\right) = q(t)$$
$$l2 := -\sin(t) + e^t - e^{-t} + a1\left(\cos(t) + e^t + e^{-t}\right) + a0\left(\sin(t) + e^t - e^{-t}\right) = q(t)$$
$$l3 := -\sin(t) + a1\cos(t) + a0\sin(t) = q(t)$$
$$l4 := e^t + e^{-t} + a1\left(e^t - e^{-t}\right) + a0\left(e^t + e^{-t}\right) = 0$$

```
> solve({l1,l2,l3,l4},{a0,a1,q(t)});
```

$$\{a0 = -1,\, a1 = 0,\, q(t) = -2\sin(t)\}$$

```
> eq:=diff(x(t),t,t)-x(t)=-2*sin(t);
  dsolve(eq,x(t));
```

$$eq := \frac{d^2}{dt^2}x(t) - x(t) = -2\sin(t)$$
$$x(t) = c_2 e^t + e^{-t} c_1 + \sin(t)$$

Ex 7. a) Solve $x''(t) - 2x'(t) - 3x(t) = 10e^{(-1+2i)t}$, $t \in \mathbb{R}$.

Guess the function $ce^{(-1+2i)t}$.

```
> f:=t->c*exp((-1+2*I)*t):
> diff(f(t),t,t)-2*diff(f(t),t)-3*f(t)=10*exp((-1+2*I)*t);
```

$$-(1+8\,\mathrm{I})\,c\,e^{(-1+2\,\mathrm{I})\,t} - 3\,c\,e^{(-1+2\,\mathrm{I})\,t} = 10\,e^{(-1+2\,\mathrm{I})\,t}$$

```
> solve(%,c);
```

$$-\frac{1}{2} + \mathrm{I}$$

So let $x(t) = \left(-\frac{1}{2} + i\right)e^{(-1+2i)t}$.

b) Solve $x''(t) - 2x'(t) - 3x(t) = 10e^{-t}\cos(2t) = \mathrm{Re}\left(10e^{(-1+2i)t}\right)$.

$$x(t) = \mathrm{Re}\left(\left(-\frac{1}{2} + i\right)e^{(-1+2i)t}\right)$$
$$= -\frac{1}{2}e^{-t}\cos(2t) - e^{-t}\sin(2t).$$

c) Solve $x''(t) - 2x'(t) - 3x(t) = 10e^{-t}\sin(2t) = \mathrm{Im}\left(10e^{(-1+2i)t}\right)$.

$$x(t) = \mathrm{Im}\left(\left(-\frac{1}{2} + i\right)e^{(-1+2i)t}\right)$$
$$= -\frac{1}{2}e^{-t}\sin(2t) + e^{-t}\cos(2t).$$

Ex 8. All solutions are given by

$$x(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + 5t^3 + t^2 + 12t + 7, \quad c_1, c_2 \in \mathbb{R}.$$

State the differential equation.

From the homogeneous part we have the roots $-1 \pm 2i$.

Since

$$(\lambda - (-1+2i)) \cdot (\lambda - (-1-2i)) = \lambda^2 + 2\lambda + 5$$

we have $x''(t) + 2x'(t) + 5x(t) = 0$.

Drop $5t^3 + t^2 + 12t + 7$ through the equation to get $q(t)$.

$$x''(t) + 2x'(t) + 5x(t) = 25t^3 + 35t^2 + 94t + 61.$$