101. (i) $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\cdots$

Rekkjan svarar til $\sum_{n=1}^{\infty}\frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} \geq \frac{1}{3}\int_1^{\infty}\frac{1}{x}\,dx$.

Intergralið divergerar og tí ger rekkjan eisini.

(ii) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n \leq \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$.

Seinna rekkjan konvergerar, so per samanbering konvergerar $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$.

$$S_N = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^N} = \sum_{n=1}^{N}\left(\frac{1}{3}\right)^n$$

$$\left(1-\frac{1}{3}\right)S_N = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{1}{3^{n+1}}\right) = \frac{1}{3}-\frac{1}{3^{N+1}}$$

$$\Rightarrow S_N = \frac{\frac{1}{3}-\frac{1}{3^{N+1}}}{1-\frac{1}{3}} \to \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad \text{tá } N\to\infty.$$

(iii) $\frac{1}{101}+\frac{2}{201}+\frac{3}{301}+\cdots = \sum_{n=1}^{\infty}\frac{n}{100n+1}$, $a_n = \frac{n}{100n+1} = \frac{1}{100+\frac{1}{n}} \to \frac{1}{100}$ tá $n\to\infty$.

Per n'te leds kriteriet er rekkjan divergent.

(iv) $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots = \sum_{n=0}^{\infty}\frac{1}{10^n} = 1+\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$.

Rekkjan er konvergent per samanbering við $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$.

$$1+\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n = 1+\frac{1}{9} = \frac{10}{9},$$

tí

$$\left(1-\frac{1}{10}\right)\sum_{n=0}^{N}\left(\frac{1}{10}\right)^n = \sum_{n=0}^{N}\left(\frac{1}{10^n}-\frac{1}{10^{n+1}}\right) = 1-\frac{1}{10^{N+1}}$$

$$\Rightarrow S_N = \frac{1-\frac{1}{10^{N+1}}}{1-\frac{1}{10}} \to \frac{1}{\frac{9}{10}} = \frac{10}{9} \quad \text{tá } N\to\infty.$$

(v) $\frac{\log(2)}{2}+\frac{\log(3)}{3}+\frac{\log(4)}{4}+\cdots = \sum_{n=2}^{\infty}\frac{\log(n)}{n} \geq \sum_{n=3}^{\infty}\frac{\log(n)}{n} \geq \sum_{n=3}^{\infty}\frac{1}{n}$.

Divergent per samanbering.

(vi) $\frac{1}{1}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[4]{4}}+\cdots = \sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{n}} \geq \sum_{n=1}^{\infty}\frac{1}{n}$. Divergent per samanbering.

u4. 6. $\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}$

(1) Rokna S_1, S_2 og S_3.

$S_1 = \frac{2\cdot 1+1}{1^4+2\cdot 1^3+1^2} = \frac{3}{4}$. $\quad S_2 = \frac{3}{4} + \frac{5}{16+16+4} = \frac{3}{4} + \frac{5}{36} = \frac{33}{36} = \frac{8}{9}$

$S_3 = \frac{8}{9} + \frac{7}{81+54+9} = \frac{8}{9} + \frac{7}{144} = \frac{15}{16}$.

(2) Vís, at $\frac{2n+1}{n^4+2n^3+n^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$.

$$\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{2n+1}{n^4+2n^3+n^2}.$$

(3) Vís, at $S_N = 1 - \frac{1}{(N+1)^2}$.

$$S_N = \sum_{n=1}^{N} \frac{2n+1}{n^4+2n^3+n^2} \overset{(ii)}{=} \sum_{n=1}^{N} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right)$$

$$= 1 - \frac{1}{(N+1)^2}.$$

(4) Vís, at $\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}$ er konvergent og rokna summin.

$$S_N = 1 - \frac{1}{(N+1)^2} \to 1 \quad \text{tá} \quad N \to \infty.$$

Rekkjan er tískil konvergent per definitión við summin

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2} = 1.$$

u5. 1. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

(1) Hvat sigur n'te lids kriteriið?

$$a_n = \ln\left(1+\frac{1}{n}\right) \to \ln(1) = 0 \quad \text{tá} \quad n \to \infty.$$

(c) Eingin niðurstøða.

(2) Givið er, at $S_N = \ln(N+1)$ fyri $N \in \mathbb{N}$. Hvat sigur S_N okkum?

(b) Rekkjan er divergent, tí $S_N \to \infty$ tá $N \to \infty$.

2. Vit vita, at $\sum_{n=1}^{\infty} \frac{1}{n^2}$ er konvergent, og at $\sum_{n=1}^{\infty} \frac{1}{n}$ er divergent. Nýt samanberings kriteriið ella ekvivalentkriteriið til at avgera um rekkjurnar eru konvergentar.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$. Vit hava, at $n^3+1 > n^2$ fyri øll $n \geq 1$, so at $\frac{1}{n^3+1} < \frac{1}{n^2}$ fyri øll $n \geq 1$.

Per 4.20 er $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ konvergent, tí $\sum_{n=1}^{\infty} \frac{1}{n^2}$ er konvergent.

(2) $\sum_{n=1}^{\infty} \frac{1}{n+\sin(n)}$. Vit vísa ekvivalens.

$$\frac{\frac{1}{n}}{\frac{1}{n+\sin(n)}} = \frac{n+\sin(n)}{n} = \frac{1+\frac{\sin(n)}{n}}{1} \to 1 \quad \text{tá } n \to \infty.$$

Rekkjan er ekvivalent við $\sum_{n=1}^{\infty} \frac{1}{n}$ og er tí divergent per 4.24.

(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, her er $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ fyri øll $n \geq 1$, so per 4.20 er rekkjan divergent.

3. Kanna konvergens/divergens og rokna sum.

(1) $\sum_{n=1}^{\infty} \frac{n+3}{n+2}$. $a_n = \frac{n+3}{n+2} = \frac{1+\frac{3}{n}}{1+\frac{2}{n}} \to 1$ tá $n \to \infty$. Divergent per 4.19.

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{4}$, sí 101 ella kvotientrekkjur.

(3) $\sum_{n=1}^{\infty} (1+(-1)^n)$. $a_n = 1+(-1)^n = \begin{cases} 0 & n \text{ ólika} \\ 2 & n \text{ lika} \end{cases}$. Divergent per 4.19.

(4) $\sum_{n=1}^{\infty} (\sqrt{5}-1)^n$. $a_n = (\sqrt{5}-1)^n > (\sqrt{4}-1)^n = 1$. Divergent per 4.19.

(5) $\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}$. $a_n = \frac{1}{n^2+7n+12} < \frac{1}{n^2}$. Per 4.20 er rekkjan konvergent.

$$\frac{1}{(n+3)(n+4)} = \frac{(n+4)-(n+3)}{(n+3)(n+4)} = \frac{1}{n+3} - \frac{1}{n+4}$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n+3} - \frac{1}{n+4} = \frac{1}{4} - \frac{1}{N+4} \to \frac{1}{4} \quad \text{tá } N \to \infty.$$

4. Brúka 4.30 at vísa $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ er konvergent.

$a_n = \frac{1}{(2n)!} > 0$ fyri øll $n \in \mathbb{N}$ og

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}}\right| = \frac{(2n)!}{(2n+2)!} = \frac{(2n)\cdot(2n-1)\cdot(2n-2)\cdots 1}{(2n+2)(2n+1)(2n)(2n-1)\cdots 1} = \frac{1}{(2n+2)(2n+1)} \to 0 \quad \text{tá } n \to \infty.$$

Per 4.30(i) er rekkjan konvergent.