Determinants and vector-geometry

Ex1. $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

a) Compute $\det(\underline{\underline{A}})$ by expansion.

Let's take row 3 and use the rule of Sarrus on 3x3 matrices.

$$\det(\underline{\underline{A}}) = (-1)^{3+1} \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 2 & 0 \end{bmatrix}\right) + (-1)^{3+2} \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 0 \end{bmatrix}\right)$$

$$= 4 + 4 - 2 - (4 - 2 - 8)$$

$$= 12$$

b) Determine $\det(\underline{\underline{A}})$ by triangulation.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{matrix} \\ \\ -R_1 \\ -R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{2}R_2 \\ -\frac{1}{2}R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = 1 \cdot 2 \cdot (-2) \cdot (-3) = 12$$

Ex2.

a) Let $P(x) = -x^6 + x^5 + x^4 - x^3$. Factorize, find roots and determine mult.

$$P(x) = -x^3 (x^3 - x^2 - x + 1)$$

$$= -x^3 (x+1)(x-1)^2$$

root	alg. mult.
$x=0$	3
$x=1$	2
$x=-1$	1

b) Given $\underline{\underline{A}} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & 0 & a^2 & a^3 \\ 1 & a & a & a^3 \\ 1 & a & a^2 & a \end{bmatrix}$, $a \in \mathbb{R}$, determine $\det(\underline{\underline{A}})$.

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> A:=<1,a,a^2,a^3; 1,0,a^2,a^3; 1,a,a,a^3; 1,a,a^2,a>;
  Determinant(A);
  solve(%=0,a);
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$$A := \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & 0 & a^2 & a^3 \\ 1 & a & a & a^3 \\ 1 & a & a^2 & a \end{bmatrix}$$

$$-a^6 + a^5 + a^4 - a^3$$

$$0, 0, 0, -1, 1, 1$$

We see that this corresponds to $P(a)$.

c) For which values a is $\underline{\underline{A}}$ a singular matrix?

Values for which $\det(\underline{\underline{A}}) = 0$, i.e. $a = -1, 0, 1$.

d), e) Find the rank of $\underline{\underline{A}}$ for $a \in \{-4, -3, \dots, 4\}$ or $\mathbb{R}$.

For $a \in \mathbb{R} \setminus \{-1, 0, 1\}$ we find $\rho(\underline{\underline{A}}) = 4$, as it is regular.

For $a = -1$ we have $\rho(\underline{\underline{A}}) = 4 - 1 = 3$, since the mult. is 1.

For $a = 0$ we have $\rho(\underline{\underline{A}}) = 4 - 3 = 1$, — " — 3.

For $a = 1$ we have $\rho(\underline{\underline{A}}) = 4 - 2 = 2$, — " — 2.

f) Find the solution to $\underline{\underline{A}}\,\underline{x} = \underline{0}$ for all $a \in \mathbb{R}$.

By subtracting row one from each subsequent row clearly $\underline{0}$ is the only solution for $a \in \mathbb{R} \setminus \{-1, 0, 1\}$.

$$\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & 0 & a^2 & a^3 \\ 1 & a & a & a^3 \\ 1 & a & a^2 & a \end{bmatrix} \to \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & -a & 0 & 0 \\ 0 & 0 & a-a^2 & 0 \\ 0 & 0 & 0 & a-a^3 \end{bmatrix}$$

$a = -1$:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

$a = 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x = t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_2, t_3, t_4 \in \mathbb{R}.$$

$a = 1$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x = t_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_3, t_4 \in \mathbb{R}.$$

Ex 3. $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$.

a) Show A and B are regular by using determinants.
Conclude AB is regular.

$$\det(A) = 2 - 3 = -1 \neq 0$$
$$\det(B) = 1 - 0 = 1 \neq 0$$
$$\Rightarrow \det(AB) = \det(A) \cdot \det(B) = -1$$

All three are regular and thus invertible.

b) Compute.

$$AC = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

See 9.2 for 2x2 inverse.

$$BD = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$DC = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 4 & -14 \end{bmatrix}$$

c) Find A^{-1} and B^{-1}. $A^{-1} = C$ and $B^{-1} = D$.

d) Find $(\underline{\underline{AB}})^{-1}$.

$$(\underline{\underline{AB}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1} = \underline{\underline{DC}} = \begin{bmatrix} -1 & 3 \\ 4 & -14 \end{bmatrix}.$$

Ex 4. $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 4 \\ 0 & 2 & 1 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 7 & 9 \\ 1 & 1 & 2 \end{bmatrix}$.

a) Compute determinants with Maple.

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> restart;
> with(LinearAlgebra):
> A:=<1,2,3;3,-2,4;0,2,1>:
  B:=<4,2,1;0,7,9;1,1,2>:
> Determinant(A);
  Determinant(B);
                                        2
                                        31
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b) Compute. $\det(\underline{\underline{A}}^7) = \det(\underline{\underline{A}})^7 = 2^7 = 128$

$\det(\underline{\underline{A}}^T\underline{\underline{B}}) = \det(\underline{\underline{A}})\cdot\det(\underline{\underline{B}}) = 2\cdot 31 = 62$

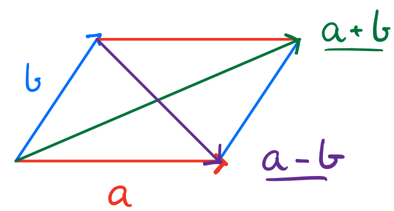
c) Show $\underline{\underline{A}}$ has an inverse and compute.

Since $\det(\underline{\underline{A}}) = 2 \neq 0$, then $\underline{\underline{A}}$ is regular and has an inverse.

$$\det(\underline{\underline{A}}^{-1}) = \frac{1}{2}, \quad \det(\underline{\underline{A}}^{-7}) = \frac{1}{2^7} = \frac{1}{128}$$

Ex 5.

a) Draw two vectors $\underline{a}$ and $\underline{b}$ and construct their sum and difference.



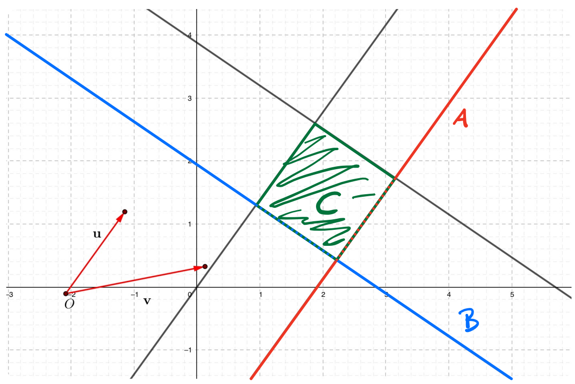
b) Now try scaling a vector $\underline{c}$.

c) Construct in Geogebra

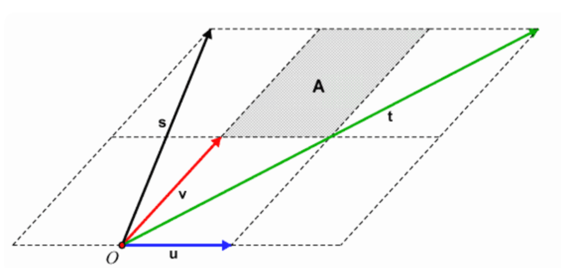
$$A = \{ P \mid \vec{OP} = \underline{v} + t\underline{u},\ t \in \mathbb{R} \}$$

$$B = \{ P \mid \vec{OP} = \underline{v} + t(\underline{u} - \underline{v}),\ t \in \mathbb{R} \}$$

$$C = \{ P \mid \vec{OP} = \underline{v} + s\,\underline{u} + t(\underline{u} - \underline{v}),\ t \in \mathbb{R} \}$$



Ex 6.



a) State $\underline{s}$ as a lin. comb. of $\underline{u}$ and $\underline{v}$.

$$\underline{s} = -\underline{u} + 2\underline{v}$$

$$\underline{t} = 2\underline{u} + 2\underline{v}$$

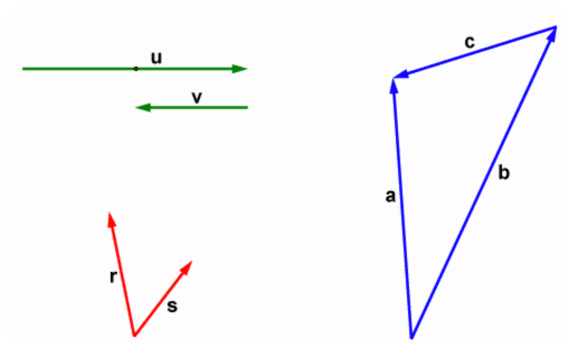
b) Show that $\underline{v} = \frac{1}{3}\underline{s} + \frac{1}{6}\underline{t}$.

$$\frac{1}{3}(-\underline{u} + 2\underline{v}) + \frac{1}{6}(2\underline{u} + 2\underline{v})$$
$$= -\frac{1}{3}\underline{u} + \frac{2}{3}\underline{v} + \frac{1}{3}\underline{u} + \frac{1}{3}\underline{v}$$
$$= \underline{v}.$$

c) Determine a, b, c and d such that the area A is described by
$$A = \{ P \mid \vec{OP} = x\underline{u} + y\underline{v},\ x \in [a,b] \text{ and } y \in [c,d] \}.$$

Let $x \in [0,1]$ and $y \in [1,2]$, then the parametric representation is satisfactory.

Ex 7. Determine lin. dependence or independence.

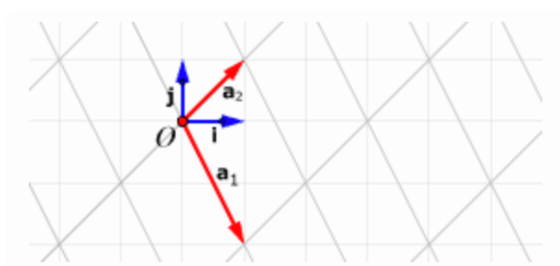


$\underline{v} = -\frac{1}{2}\underline{u} \iff \underline{v} + \frac{1}{2}\underline{u} = \underline{0}$

$\underline{r}$ and $\underline{s}$ are not parallel, so this pair is lin. indept.

$\underline{a} = \underline{b} + \underline{c} \iff \underline{a} - \underline{b} - \underline{c} = \underline{0}$

Ex 8.



1. ${}_e\underline{u} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ wrt. the standard basis.

${}_a\underline{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or $2\underline{a}_1 + 3\underline{a}_2$.

2. ${}_a\underline{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ determine in standard basis.

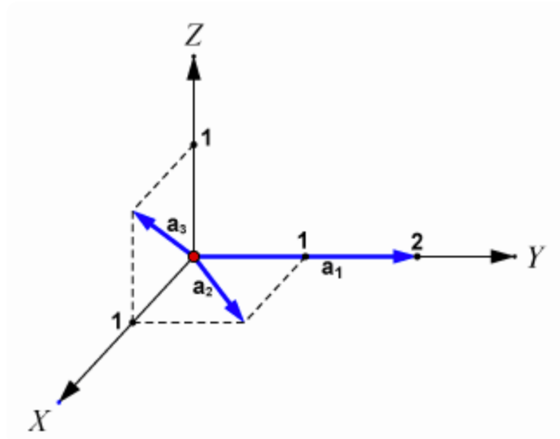
${}_e\underline{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ or $-3\underline{i} + 0\underline{j}$.

Ex 9.

a) Determine the matrix

$[\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3]$,

and conclude that the vectors constitute a basis.



$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(\underline{\underline{A}}) = -2 \neq 0$. The matrix has full rank, so the vectors are linearly independent.

b) Given $_a\underline{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $_a\underline{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $_a\underline{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ compute these in $e = (\underline{i}, \underline{j}, \underline{k})$.

$$\underline{\underline{A}}\, {}_a\underline{u} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad \underline{\underline{A}}\, {}_a\underline{v} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{\underline{A}}\, {}_a\underline{w} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

c) A plane is given $\alpha: x + 2y - 2z = -1$.

Give a parametric representation of α. (this is in $a = (\underline{a}_1, \underline{a}_2, \underline{a}_3)$!)

Pick 3 solutions to get vectors.

$(-1, 0, 0)$, $(0, -\frac{1}{2}, 0)$ and $(0, 0, \frac{1}{2})$

$${}_a\alpha: \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}. \quad \text{(scaled the vectors)}$$

d) Change α to be in standard basis.

We could convert the entire thing, but let's do it one vector at a time.

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{A}}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad \underline{\underline{A}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \quad \underline{\underline{A}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$${}_e\alpha : \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}.$$

e) Find an equation for ${}_e\alpha$.

We get the normal and insert the given point.

$$\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix}$$

$${}_e\alpha : \quad -3x - (y-(-2)) + 7z = 0$$

$$\iff \quad -3x - y + 7z = 2$$