Vector Spaces

Ex 1. State the zero vector and dimension of the following vector spaces.

1. $\mathbb{R}^4$ has $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\dim(\mathbb{R}^4) = 4$.

2. $\mathbb{C}^4$ has $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\dim(\mathbb{C}^4) = 4$.

3. $C^0([0,1])$ has $f(x) = 0$ for all $x \in [0,1]$ and $\dim\left(C^0([0,1])\right) = \infty$.

4. $\mathbb{R}^{4\times 2}$ has $\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\dim\left(\mathbb{R}^{4\times 2}\right) = 8$.

5. $P_4(\mathbb{R})$ has $f(x) = 0$ for all $x \in \mathbb{R}$ and $\dim\left(P_4(\mathbb{R})\right) = 5$.

Ex 2. Determine whether the systems are lin. dependent or independent.
If dependent, then write one as a combination.

1. $(1,2,1,0),\ (2,7,3,1),\ (3,12,5,2) \in \mathbb{R}^4$.

$$-\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 5 \\ 2 \end{bmatrix}$$

The set of vectors is lin. dept.

2. $(1,i),\ (1+i,-1+i) \in \mathbb{C}^2$.

$$(1+i)\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+i \end{bmatrix}$$

The set of vectors is lin. dept.

3. $1+2x+3x^2+x^3,\quad 2+5x-x^2+x^3,\quad -3+2x-4x^2-2x^3 \in P_3(\mathbb{R})$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & -1 & 1 \\ -3 & 2 & -4 & -2 \end{bmatrix} \begin{matrix} \\ -2R_1 \\ +3R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -7 & -1 \\ 0 & 8 & 5 & 1 \end{bmatrix} \begin{matrix} \\ \\ -8R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -7 & -1 \\ 0 & 0 & 61 & 9 \end{bmatrix}$$

The set of vectors is lin. indept.

4. $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 & -2 \\ 3 & 3 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}$.

$$3\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -2 \\ 3 & 3 & 2 \end{bmatrix}$$

The set of vectors is lin. dept.

Ex 3. a) Given $((1,2,3), (-1,0,2), (1,6,a))$, which value of a must be avoided in order for the vectors to be a basis for $\mathbb{R}^3$? See method 11.42.

$$\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 6 \\ 3 & 2 & a \end{bmatrix}\right) = 0 \iff -18+4-(12-2a) = 0$$

$$\iff a = 13.$$

The value $a=13$ is to be avoided. For $a \in \mathbb{R}\setminus\{13\}$ the vectors span $\mathbb{R}^3$, and there are only 3 of them.

b) Let $\underline{a}_1 = (1,-1,2,1)$, $\underline{a}_2 = (0,1,1,3)$, $\underline{a}_3 = (1,-2,2,-1)$, $\underline{a}_4 = (0,1,-1,3)$ and $\underline{v} = (1,-2,2,-3)$.

Prove that $(\underline{a}_1, \dots, \underline{a}_4)$ is a basis for $\mathbb{R}^4$, and compute ${}_a\underline{v}$.

$$\det\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 \\ 2 & 1 & 2 & -1 \\ 1 & 3 & -1 & 3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 3 & -1 & 3 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & 3 \end{bmatrix}\right)$$

$$= -6+6-1-(-6+1-6) \quad - 3 - 1 + 6 - (1+3+6)$$

$$= 10 - 8 = 2$$

The vectors are lin. indept. and there are 4 of them corresponding to $\dim(\mathbb{R}^4)$, so they are a basis.

The linear combination that yields ${}_a\underline{v}$ is

$$2\underline{a}_1 - \underline{a}_2 - \underline{a}_3 - \underline{a}_4 = {}_a\underline{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

Ex 4.

a) We have a basis for $P_2(\mathbb{R})$ given by vectors

$$P_1(x) = 1 + x^2, \quad P_2(x) = -1 - x - 3x^2, \quad P_3(x) = 6 + x + 5x^2.$$

Determine wrt. p the vectors

$$Q_1(x) = 3 + 2x + 7x^2, \quad Q_2(x) = 2 + x + 4x^2, \quad Q_3(x) = 5 + 2x^2.$$

$$Q_1(x) = P_1(x) - 2P_2(x) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$Q_2(x) = P_1(x) - P_2(x) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$Q_3(x) = P_2(x) + P_3(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{array}{cccc} P_1 & P_2 & P_3 & Q \\ a\cdot\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + & b\cdot\begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} + & c\cdot\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix} = & \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \end{array}$$

b) Same question only in reverse.

Given

$$Q_1(x) = 3 + 2x + 7x^2, \quad Q_2(x) = 2 + x + 4x^2, \quad Q_3(x) = 5 + 2x^2.$$

with coordinates wrt. P

$$(1,-2,0), \quad (1,-1,0) \text{ and } (0,1,1)$$

determine the basis vectors.

We have

$$\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 1 & 0 \\ 7 & 4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ +2R_1 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -R_2 \\ -R_3 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 1 & 0 \\ 7 & 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 6 \\ 0 & -1 & 1 \\ 1 & -3 & 5 \end{bmatrix}$$

Thus $P_1(x) = 1 + x^2$, $P_2(x) = -1 - x - 3x^2$ and $P_3(x) = 6 + x + 5x^2$.

Ex 5.

a) Consider G3. Do subspaces with dimension 0, 1, 2, 3 or 4 exist?

The zero vector is a 0 dimensional subspace of G3.
One dimensional subspaces are any vector on the same line.
Vectors one the same plane represent a 2 dimensional subspace.
G3 is a 3 dimensional subspace of itself. No 4 dimensional subspace.

b) Is $\{a\cos(x) + b\sin(x) \mid a, b \in \mathbb{R}\}$ a subspace of $C^0(\mathbb{R})$?

We use proposition 11.47. Clearly $a \cdot \cos(x) + b \cdot \sin(x) \in C^0(\mathbb{R})$.

Let $f_1(x)$ and $f_2(x)$ be function of the above nature, then

$$f_1(x) + f_2(x) = (a_1 + a_2)\cos(x) + (b_1 + b_2)\sin(x) \in C^0(\mathbb{R})$$

and

$$c \cdot f_1(x) = c a_1 \cos(x) + c a_2 \sin(x) \in C^0(\mathbb{R}).$$

Since the stability requirements are met, then we indeed have a subspace of $C^0(\mathbb{R})$.

c) Is $\{(x_1, x_2, x_3, x_4) \mid x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 0\}$ a subspace of $\mathbb{R}^4$?

No, just observe

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \notin \{(x_1, x_2, x_3, x_4) \mid x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 0\}.$$

d) Subset of $P_2(\mathbb{R})$ with the root 1 in $P_2(\mathbb{R})$. If a subspace, then determine a basis.

For f, g with 1 as a root, then $f + g \in P_2(\mathbb{R})$, and

$$(\alpha \cdot f + g)(1) = \alpha \cdot f(1) + g(1) = 0, \text{ so } 11.47 \text{ is satisfied.}$$

A basis for the subspace is the vectors

$$1 - x^2 \quad \text{and} \quad x - x^2.$$

e) Subset of $P_2(\mathbb{R})$ with double in $P_2(\mathbb{R})$.

No this isn't closed under addition.

Ex 6. a) Explain why the solution is a subspace of $\mathbb{R}^5$. State dimension and basis.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 2 & 0 \\ 2 & 3 & 1 & 3 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ -2R_1 \end{array} \to \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 2 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ \\ -R_2 \end{array}$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & -4 & 3 & -3 & 0 \\ 0 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = t_1 \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_1, t_2, t_3 \in \mathbb{R}.$$

$$V = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$$

The set V is lin. indept. set of vectors in $\mathbb{R}^5$.

Linear combinations are solutions, hence V spans a subspace.

It follows that $\dim(V) = 3$ and as defined V is a basis.

b) Show that $\underline{a}_1 = (1,0,1,0,1,0)$ and $\underline{a}_2 = (0,1,1,1,1,-1)$ span the same subspace of $\mathbb{R}^6$ as

$\underline{b}_1 = (4,-5,-1,-5,-1,5)$ and $\underline{b}_2 = (-3,2,-1,2,-1,-2)$.

$\underline{b}_1 = 4\underline{a}_1 - 5\underline{a}_2$ and $\underline{b}_2 = -3\underline{a}_1 + 2\underline{a}_2$

Also $\underline{a}_1$ and $\underline{a}_2$ are clearly lin. indept. by observing entry 1 and 2.

Ex 7.

a) "Meditation" about: *A matrix is a vector is a vector is a matrix.*

Sure a matrix is a vector, see example 11.26 for basis vectors of $\mathbb{R}^{2\times 3}$.

b) Show linear independence of vectors.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Just follow example 11.39 and write out as traditional vectors.

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> v1:=<1,0,0,0,-2,0,0,0,3>:
  v2:=<0,-3,0,0,2,0,0,-1,0>:
  v3:=<0,0,1,0,-2,0,3,0,0>:
  v4:=<0,0,0,-1,2,-3,0,0,0>:
  A:=<v1|v2|v3|v4>:
> GaussianElimination(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) *We consider the subspace $U \subset \mathbb{R}^{3\times 3}$ spanned by the above vectors.*

Show $\begin{bmatrix} 2 & -3 & 2 \\ -3 & 8 & -9 \\ -6 & -1 & 6 \end{bmatrix} \in U$ *for some basis and determine coords.*

Let $u = (\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)$ be the *above basis in its respective* order.

$$2\,\underline{u}_1 + \underline{u}_2 - 2\,\underline{u}_2 + 3\,\underline{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{yields} \quad \begin{bmatrix} 2 & -3 & 2 \\ -3 & 8 & -9 \\ -6 & -1 & 6 \end{bmatrix} \text{ in } u.$$

d) Find $\underline{v} \in \mathbb{R}^{3\times 3}$ such that $\underline{v} \notin U$.

One such choice is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Ex 8. In $P_2(\mathbb{R})$ we're given

$$P_1(x) = 1 - 3x + 2x^2, \quad P_2(x) = 1 + x + 4x^2, \quad P_3(x) = 1 - 7x.$$

a) Show $(P_1(x), P_2(x))$ is a basis for $\operatorname{span}\{P_1(x), P_2(x), P_3(x)\}$.

$P_1 \neq c \cdot P_2$ so these are lin. indept. Further

$$P_3(x) = 2P_1(x) - P_2(x)$$

so $(P_1(x), P_2(x))$ indeed is a basis for $\operatorname{span}\{P_1(x), P_2(x), P_3(x)\}$.

b) Check if $Q_1(x) = 1 + 5x + 9x^2$ and $Q_2(x) = 3 - x + 10x^2$ belong to $\operatorname{span}\{P_1(x), P_2(x), P_3(x)\}$, and if so determine coordinates in $(P_1(x), P_2(x))$.

$$\begin{array}{ccc} P_1 & P_2 & Q_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -3 & 1 & 5 \\ 2 & 4 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 7 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array}\right]$$

No solution, so $Q_1(x)$ is outside of the given span.

$$Q_2(x) = P_1(x) + 2P_2(x) \quad \text{so in coords} \quad {}_P Q_2(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

c) State the simplest basis for $\operatorname{span}\{P_1(x), P_2(x), P_3(x), Q_1(x), Q_2(x)\}$.

This is the monomial basis $(1, x, x^2)$.