1.1.2 $A = \{x \mid x \in \mathbb{R} \wedge x < 6\}$. Check if x is real and if $x < 6$.

(a) $3 \in A$ is true. (b) $6 \in A$ is false. (c) $5 \notin A$ is false. (d) $8 \notin A$ is false. (e) $-8 \in A$ is true.

(f) $3.4 \notin A$ is false.

1.1.11 Which of these are the empty set?

(a) $\{x \mid x \in \mathbb{R} \wedge x^2 - 1 = 0\}$ contains $x = 1$, so not empty.

(b) $\{x \mid x \in \mathbb{R} \wedge x^2 + 1 = 0\} = \emptyset$ since there does not exist a real number x, such that $x^2 = -1$.

(c) $\{x \mid x \in \mathbb{R} \wedge x^2 = -9\} = \emptyset$ as above mutatis mutandis.

(d) $\{x \mid x \in \mathbb{R} \wedge x = 2x+1\}$ is nonempty, since it contains $x = 2x+1 \iff x = -1$.

(e) $\{x \mid x \in \mathbb{R} \wedge x = x+1\} = \emptyset$ since $x = x+1 \iff 0 = 1$, i.e. no solution $x \in \mathbb{R}$.

1.1.12 All subsets of $\{a, b\}$.

$\emptyset = \{\}, \{a\}, \{b\}, \{a, b\}$.

1.1.28 (a) If $A = \{3, 7, 2\}$, find $\mathcal{P}(A)$.

$\mathcal{P}(A) = \{\emptyset, \{2\}, \{3\}, \{7\}, \{2,3\}, \{2,7\}, \{3,7\}, \{2,3,7\}\}$.

(b) $|A| = 3$.

(c) $|\mathcal{P}(A)| = 8$.

1.2.5 Let $U = \{1, 2, 3, \ldots, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$

$C = \{x \in \mathbb{Z}_+ \mid x^2 \leq 16\}$ and $D = \{7, 8\}$.

(a) $A \cup B = \{1, 2, 4, 5, 6, 8, 9\}$. Union and intersection

(b) $A \cup C = \{1, 2, 3, 4, 6, 8\}$.

(c) $A \cup D = \{1, 2, 4, 6, 7, 8\}$.

(d) $B \cup C = \{1, 2, 3, 4, 5, 9\}$.

(e) $A \cap C = \{1, 2, 4\}$.

(f) $A \cap D = \{8\}$.

(g) $B \cap C = \{2, 4\}$.

(h) $C \cap D = \emptyset$.

1.2.9 Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, f, g\}$, $B = \{a, e\}$ and $C = \{b, h\}$.

(a) $\bar{A} = \{b, d, e, h\}$. Complements: $\bar{A} = U - A$.

(b) $\bar{B} = \{b, c, d, f, g, h\}$.

(c) $\overline{A \cup B} = \{b, d, h\}$.

(d) $\overline{A \cap B} = \{b, c, d, e, f, g, h\}$.

(e) $\bar{U} = \emptyset$.

(f) $A - B = \{c, f, g\}$.

1.2.15 Let $|A| = 6$, $|B| = 8$, $|C| = 6$, $|A \cup B \cup C| = 11$, $|A \cap B| = 3$, $|A \cap C| = 2$ and $|B \cap C| = 5$.

Find $|A \cap B \cap C|$.

Apply theorem 3: $11 = 6 + 8 + 6 - 3 - 2 - 5 + |A \cap B \cap C|$

$$\Rightarrow |A \cap B \cap C| = 1.$$

1.2.28 100 people:

37	$\|F\|$
33	$\|V\|$
9	$\|F \cap C\|$
12	$\|V \cap C\|$
10	$\|F \cap V\|$
12	$\|\overline{F \cup V} \cap C\| = \|\bar{F} \cap \bar{V} \cap C\|$
3	$\|F \cap V \cap C\|$

Those who eat cheese: $|F \cap C| + |V \cap C| = 19$

$$|C| = |\overline{F \cup V} \cap C| + |(F \cup V) \cap C| - |F \cap V \cap C| = 12 + 19 - 3 = 30.$$

Those who want none of the servings are $|\overline{F \cap V \cap C}|$. Theorem 3 gives us

$$|F \cup V \cup C| = 37 + 33 + 30 - 10 - 12 - 9 + 3 = 72.$$

So 72 people eat at least one offering, while $100 - 72 = 28$ people don't partake.

$$|\overline{F \cap V \cap C}| = 28.$$

1.3.8 $b_n = 3n^2 + 2n - 6$.

$b_1 = 3 \cdot 1^2 + 2 \cdot 1 - 6 = -1$

$b_2 = 3 \cdot 2^2 + 2 \cdot 2 - 6 = 10$

$b_3 = 3 \cdot 3^2 + 2 \cdot 3 - 6 = 27$

$b_4 = 3 \cdot 4^2 + 2 \cdot 4 - 6 = 50$

1.3.13 $e_1 = 0$, $e_n = e_{n-1} - 2$

$e_1 = 0$

$e_2 = e_1 - 2 = 0 - 2 = -2$

$e_3 = e_2 - 2 = -2 - 2 = -4$

$e_4 = e_3 - 2 = -4 - 2 = -6$

1.3.14 $f_1 = 4, \quad f_n = n \cdot f_{n-1}$

$$f_1 = 4$$
$$f_2 = 2 \cdot f_1 = 2 \cdot 4 = 8$$
$$f_3 = 3 \cdot f_2 = 3 \cdot 8 = 24$$
$$f_4 = 4 \cdot f_3 = 4 \cdot 24 = 96$$

1.3.15 $1, 3, 5, 7, \ldots$ Recursive: $a_1 = 1, \quad a_n = a_{n-1} + 2$

Explicit: $a_n = 2n - 1$

1.3.16 $0, 3, 8, 15, 24, 35, \ldots$ Recursive: $a_1 = 0, \quad a_n = a_{n-1} + 2n - 1$

Explicit: $a_n = n^2 - 1$

1.4.1 $m = 20, \quad n = 3$ write as $qn + r$ with $0 \leq r < n$.

$$20 = 6 \cdot 3 + 2$$

1.4.3 $m = 3, \quad n = 22$ $\quad 3 = 0 \cdot 22 + 3$

1.4.5 Write as powers of primes (prime factorization)

(a) $828 = 2 \cdot 414 = 2^2 \cdot 207 = 2^2 \cdot 3 \cdot 69 = 2^2 \cdot 3^2 \cdot 23$

(b) $1666 = 2 \cdot 833 = 2 \cdot 7 \cdot 119 = 2 \cdot 7^2 \cdot 17$

(c) $1781 = 13 \cdot 137$

(d) $1125 = 5 \cdot 225 = 5^2 \cdot 45 = 5^3 \cdot 9 = 3^2 \cdot 5^3$

(e) 107

1.4.6 Find $\gcd(a,b) = d$, and write $d = sa + tb$.

$a = 60, \quad b = 100$

$$100 = 1 \cdot 60 + 40$$
$$60 = 1 \cdot 40 + 20$$
$$40 = 2 \cdot 20 + 0$$

Hence $d = 20$ and

$$20 = 60 - 40 = 60 - (100 - 60)$$
$$= 2 \cdot 60 - 1 \cdot 100$$

where $s = 2$ and $t = -1$

1.4.7 $a = 45, \quad b = 33$

$$45 = 1 \cdot 33 + 12$$
$$33 = 2 \cdot 12 + 9$$
$$12 = 1 \cdot 9 + 3$$
$$9 = 3 \cdot 3 + 0$$

Thus $d = 3$ and

$$3 = 12 - 9 = 12 - (33 - 2\cdot 12)$$
$$= 3\cdot 12 - 33 = 3(45 - 33) - 33$$
$$= 3\cdot 45 - 4\cdot 33$$

$s = 3$, $t = -4$.

1.4.10 $\begin{aligned} 72 &= 2^3\cdot 3^2 \\ 108 &= 2^2\cdot 3^3 \end{aligned} \Rightarrow \operatorname{lcm}(72,108) = 2^3\cdot 3^3 = 8\cdot 27 = 216.$

1.4.11 $\begin{aligned} 150 &= 2\cdot 3\cdot 5^2 \\ 70 &= 2\cdot 5\cdot 7 \end{aligned} \Rightarrow \operatorname{lcm}(70,150) = 2\cdot 3\cdot 5^2\cdot 7 = 1050.$

1.4.26 Show that if $\gcd(a,c) = 1$ and $c \mid ab$, then $c \mid b$.

Assuming $\gcd(a,c) = 1$, then there are integers s and t such that $sa + tc = 1$ by theorem 4. Thus $sab + tcb = b$, and if $c \mid ab$ we have $c \mid b$.

1.4.27 Show that if $\gcd(a,c) = 1$, $a \mid m$ and $c \mid m$, then $ac \mid m$.

There is an integer b such that $m = a\cdot b$, and since $c \mid m$, then $c \mid ab$. Now with 1.4.26 it follows that $c \mid b$ since $\gcd(a,c) = 1$. Thus there is an integer n such that $b = c\cdot n$, which gives us $m = ab = acn \Rightarrow ac \mid m$.

1.4.29 Show that $\gcd(ca, cb) = c\cdot\gcd(a,b)$.

Let $d = \gcd(a,b)$, then $d \mid a$ and $d \mid b$. There exist integers m and n such that $a = d\cdot m$ and $b = d\cdot n$. Multiply through by c, then

$$cdm = ca \quad \text{and} \quad cdn = cb.$$

Therefore $c\cdot d = c\cdot\gcd(a,b) \mid ca$ and $c\cdot\gcd(a,b) \mid cb$, so $c\cdot\gcd(a,b)$ is a common divisor.
Theorem 4 yields integers s and t such that $\gcd(a,b) = sa + tb$, so we have

$$c\cdot\gcd(a,b) = sca + tbc.$$

Any common divisor of ca and bc divides $c\cdot\gcd(a,b)$. But then $c\cdot\gcd(a,b)$ is indeed the greatest common divisor of $\gcd(ca,cb)$, i.e.

$$c\cdot\gcd(a,b) = \gcd(ca,cb).$$