Lineerar skipanir

Lineer avmyndan leggur upp til, at vit eisini kunnu leita fram, hvat ið fer hvar.
Ein avmyndan $\underline{\underline{A}}$ úr $\mathbb{R}^2$ í $\mathbb{R}^2$ kann fyri sovitt senda $\underline{u}$ í $\underline{b}$, men um $\underline{b}$ er ein givin vektorur, hvat var $\underline{u}$ so?

Skipanin er einføld $\quad \underline{\underline{A}}\,\underline{u} = \underline{b}$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \text{sum matrica og vektorar}$$

$$\Leftrightarrow u_1 \underline{a}_1 + u_2 \underline{a}_2 = \underline{b} \qquad \text{fald við skalarar}$$

$$\Leftrightarrow u_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + u_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \text{vektorar}$$

$$\Leftrightarrow \begin{cases} u_1 a_{11} + u_2 a_{12} = b_1 \\ u_2 a_{21} + u_2 a_{22} = b_2 \end{cases}$$

Vit hava greitt samsvar millum matricur og líkningar, tað er høast alt orsøkin til at vit hava matricur. Vit vilja systematiskt loysa líkningaskipanir, meðan vit arbeiða við matricur!

Dømi 5.1 $\quad \underline{\underline{A}} = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ og $\underline{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Líkningin ljóðar

$\underline{\underline{A}}\,\underline{u} = \underline{b}$, og tann ókendi vit loysa eftir er $\underline{u}$.

$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, her er $u_1 = 1$ og $u_2 = \frac{1}{2}$ eitt upplagt val, tí so er $1 \cdot u_1 + 6 \cdot u_2 = 4$. Vit fáa eisini, at $2u_1 + 4u_2 = 4$, so loysnin vísir seg at vera røtt!

Tankin er eins og við vanligar líkningar $y = ax + b$. Tú roknar y givið x, men tú loysir x givið y.
Vit rokna $\underline{b}$ givið $\underline{u}$, men vit loysa $\underline{u}$ givið $\underline{b}$.
Skiljast sum, at avmyndan er rein útrokning, men meiri tankavirksemi er í at finna aftur, hvat ið avmyndast til eitt ávíst stað.

Consistent Líkningaskipanin er consistent um *minst ein loysn* $\underline{u}$ *finst*. Rúmið av loysnum hevur 3 møguleikar: 1) Ein loysn $\underline{u}$. Her er $|A| \neq 0$, so matricen hevur fullan rank (non-singular). 2) Eingin loysn er, so skipanin er inconsistent. 3) Óendaliga nógvar loysnir eru.

Cramer's rule Ein loysn finst via determinantar. Um $|A|=0$, so nýtast vit at kanna 2) ella 3).
Um $|A|\neq 0$, so er loysnin $\underline{u}$ hjá $\underline{\underline{A}}\,\underline{u}=\underline{b}$ givin við

$$u_1 = \frac{|b, a_2|}{|A|} \quad \text{og} \quad u_2 = \frac{|a_1, b|}{|A|}$$

Dømi 5.2 $\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ og $\underline{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

$$\left.\begin{aligned} u_1 &= \frac{\begin{vmatrix} 4 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{24-16}{12-4} = \frac{8}{8} = 1 \\ u_2 &= \frac{\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{8-4}{8} = \frac{4}{8} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \underline{u} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

Hetta er smart, men generaliserar ikki væl til hægri dimensiónir, tí determinantar skulu roknast í heilum.

Gauss eliminatión Í smærri systemum kemur hetta ikki til sín rætt, men yvirhøvur er Gauss eliminatión "bread and butter".

Skipanin

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \underline{u} = \underline{b}$$

er serlig, tí seinna raðið gevur $u_2 a_{22} = b_2 \Leftrightarrow u_2 = \frac{b_2}{a_{22}}$, altso vit eru nærstan í mál, og vit fáa

$$\begin{aligned} & u_1 a_{11} + u_2 a_{12} = b_1 \\ \Leftrightarrow\ & u_1 a_{11} + \frac{b_2}{a_{22}} \cdot a_{12} = b_1 \\ \Leftrightarrow\ & u_1 = \left(b_1 - \frac{b_2}{a_{22}} a_{12}\right) / a_{11} \end{aligned}$$

Tá vit bakka eitt rað og substituera inn, so er tað "back substitution".
Vit fara at seta hesa støðu upp fyri at loysa skipanir. Diagonalelement: pivots.

Shears $\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ og $\underline{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Vit kunnu velja shear matricur at flyta $\underline{a}_1$ í ein vektor, sum er parallelur við $\underline{e}_1$. $S_1 = \begin{bmatrix} 1 & 0 \\ -\frac{v_2}{v_1} & 1 \end{bmatrix}$, $S_2 = \begin{bmatrix} 1 & -\frac{v_1}{v_2} \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \underline{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \underline{u} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \underline{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Altso $4u_2 = 2 \Leftrightarrow u_2 = \frac{1}{2}$ og so fæst $2u_1 + 4u_2 = 4$

$$\begin{aligned} &\Leftrightarrow 2u_1 + 4 \cdot \tfrac{1}{2} = 4 \Leftrightarrow 2u_1 + 2 = 4 \\ &\Leftrightarrow u_1 = 1. \end{aligned}$$

Dømi 5.3 $\begin{bmatrix} -1 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, vel $S_1 = \begin{bmatrix} 1 & 0 \\ \frac{-2}{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Við operatiónum: totalmatrican er

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 2 & 2 & 2 \end{array}\right] \cdot(-1) \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 2 \end{array}\right] -2R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 10 & 2 \end{array}\right] \qquad u_2 = \frac{2}{10} = \frac{1}{5}$$

$$u_1 - 4 \cdot \frac{1}{5} = 0 \iff u_1 = \frac{4}{5}$$

Unsolvable $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \underline{a}_1 = \frac{1}{2} \underline{a}_2$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 1 \end{array}\right] -2R_1 \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right] \qquad 0 \cdot u_1 + 0 \cdot u_2 \neq -1$$

Underdetermined Skipanin $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \underline{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ er consistent, men umbødar veruliga bert eina likning!

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad 0 = 0 \text{ passar trivielt.}$$

Homogen system Skipanir av slagnum $\underline{\underline{A}}\underline{u} = \underline{0}$ (vit hava hugt uppá inhomogen).
Triviel loysn er $\underline{u} = \underline{0}$. Ikki triviel loysn:

Les $\underline{\underline{A}}\underline{u} = \underline{0}$: "hvat fyri vektorur er avmyndar $\underline{\underline{A}}$ í $\underline{0}$?"

$$u_1 \underline{a}_1 + u_2 \underline{a}_2 = \underline{0} \iff u_1 \underline{a}_1 = u_2 \underline{a}_2$$

Her eru $\underline{a}_1$ og $\underline{a}_2$ parallelir! So A hevur rank 1.
Um $\underline{\underline{A}}\underline{u} = \underline{0}$ bert hevur triviella loysn, so eru $\underline{a}_1$ og $\underline{a}_2$ lineert óheftir, so $|A| \neq 0$.
Enn betur er A invertibel, altso øvugta avmyndanin A^{-1} er til, so at

$$AA^{-1} = I = A^{-1}A.$$

Inverse

Fyri 2D er skjótasti mátin at minnast formilin

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vit leita eftir eina matricu, so at

$$\underline{\underline{A}}\,\underline{u} = \underline{b} \iff \underline{\underline{B}}\,\underline{\underline{A}}\,\underline{u} = \underline{\underline{B}}\,\underline{b} \iff \underline{\underline{I}}\,\underline{u} = \underline{\underline{B}}\,\underline{b}$$
$$\iff \underline{u} = \underline{\underline{B}}\,\underline{b}$$

Hendan matrican B er A^{-1}.
Vit kunnu brúka shear matricur at rudda upp, og síðani skalera niður.

Dømi

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \underline{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \underline{u} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\iff \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \underline{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \iff \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \underline{u} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\iff \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \underline{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \iff \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \underline{u} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\iff \underline{\underline{I}}\,\underline{u} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$, men nú eru tað matricurnar eg vil kenna.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 6 & 0 & 1 \end{array}\right] \cdot \frac{1}{2} \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 6 & 0 & 1 \end{array}\right] - R_1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 4 & -\frac{1}{2} & 1 \end{array}\right] \cdot \frac{1}{4} \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right] - 2R_2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right]$$

Annars eru ortogonalmatricur, tær við ortogonalar vektorar $\underline{a}_1$ og $\underline{a}_2$ og $\|\underline{a}_1\| = \|\underline{a}_2\| = 1$, lættar at invertera. Tá er $A^{-1} = A^T$.

Dømi 5.8 $\quad A = \begin{bmatrix} -1 & 4 \\ 2 & 2 \end{bmatrix} \qquad |A| = -1\cdot 2 - 2\cdot 4 = -10 \neq 0$, so A^{-1} finst.

$$\left[\begin{array}{cc|cc} -1 & 4 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]\begin{array}{l} \\ +2R_1 \end{array} \xrightarrow{\cdot(-1)} \left[\begin{array}{cc|cc} 1 & -4 & -1 & 0 \\ 0 & 10 & 2 & 1 \end{array}\right]\begin{array}{l} \\ \cdot\frac{1}{10} \end{array}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -4 & -1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{10} \end{array}\right]\begin{array}{l} +4R_2 \\ \\ \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{10} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

Formil $\quad A^{-1} = -\frac{1}{10}\begin{bmatrix} 2 & -4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix}$

Vektorpar $\quad$ Við vektorpørum $[\underline{u}_1, \underline{u}_2]$ og $[\underline{b}_1, \underline{b}_2]$ kunnu vit leita fram $\underline{\underline{A}}$, so at

$$\underline{\underline{A}}\,\underline{u}_1 = \underline{b}_1 \quad \text{og} \quad \underline{\underline{A}}\,\underline{u}_2 = \underline{b}_2$$

$$\begin{aligned} & \underline{\underline{A}}\,[\underline{u}_1, \underline{u}_2] = [\underline{b}_1, \underline{b}_2] \\ \Leftrightarrow\quad & \underline{\underline{A}}\,\underline{\underline{U}} = \underline{\underline{B}} \\ \Leftrightarrow\quad & \underline{\underline{A}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} = \underline{\underline{B}}\,\underline{\underline{U}}^{-1} \\ \Leftrightarrow\quad & \underline{\underline{A}} = \underline{\underline{B}}\,\underline{\underline{U}}^{-1} \end{aligned}$$

Dømi 5.9 $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \underline{v}_1' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ og $\underline{v}_2' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\underline{\underline{V}}^{-1}: \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -R_1 \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right]\begin{array}{l} \\ \cdot\frac{1}{2} \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]\begin{array}{l} +R_2 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \Rightarrow \underline{\underline{V}}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$