Fourierrekkjur

Seinast knýttu vit eina Fourierrekkju til 2π-periodiskar funktiónir $f \in L^2(-\pi,\pi)$.

Reel FR.
$$f \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

har
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \qquad n = 0,1,2,\dots$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, \qquad n = 1,2,\dots$$

Ymsar reglur eru, so at vit kunnu sleppa avstað við at rokna minni. Sjálv Fourierrekkjan er uppbygt av líka og ólíka partum.

Høvuðsúrslitið seinast var Fouriers' setningur (6.12): f 2π-per. og stykkivís diff., so
(i) Um f er kontinuert í x, so konvergerar FR. ímóti $f(x)$.
(ii) Um f er diskontinuert í x, so konvergerar FR ímóti $\frac{f(x^+)+f(x^-)}{2}$.

Dømi

Lat $f(x) = x^2$, $x \in [-\pi,\pi[$. Her hava vit $x_1 = -\pi$ og $x_2 = \pi$ við $f_1(x) = x^2$ fyri $x \in [x_1,x_2]$. Funktiónin $f_1(x)$ er differentiabul við kontinuerta avleidda funktión $f_1'(x) = 2x$ fyri øll $x \in [x_1,x_2]$. Umframt tað er $f(x) = f_1(x)$, $x \in]x_1,x_2[$. So $f(x) = x^2$ er stykkivís differentiabul.

Lat
$$f(x) = \begin{cases} 0, & x \in]-\pi,0], \\ \ln(x), & x \in]0,\pi]. \end{cases}$$

Set $x_1 = -\pi$, $x_2 = 0$ og $x_3 = \pi$, og lat $f_1 : [x_1,x_2] \to \mathbb{C}$ og $f_2 : [x_2,x_3] \to \mathbb{C}$ vera
$$f_1(x) = 0 \quad \text{og} \quad f_2(x) = \ln(x).$$

Nú er $f_2'(x) = \frac{1}{x}$ ei definerað í $x_2 = 0$, so f er ikki stykkivís differentiabul.

Dømi 6.8 $f(x) = x^2$, $x \in [-\pi,\pi[$, 2π-periodisk.

Funktiónin er líka, so $b_n = 0$ $\forall n \in \mathbb{N}$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\left[x^2\,\frac{1}{n}\sin(nx)\right]_0^{\pi} - \int_0^{\pi} 2x\,\frac{1}{n}\sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(\left[2x\,\frac{1}{n^2}\cos(nx)\right]_0^{\pi} - \int_0^{\pi} 2\,\frac{1}{n^2}\cos(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(2\pi\,\frac{1}{n^2}\cos(n\pi) - 0\right)$$

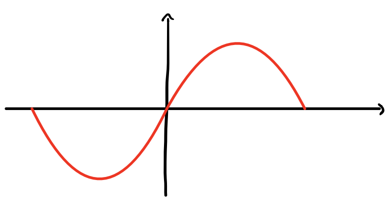
$$= \frac{4\cos(n\pi)}{n^2}, \qquad \cos(n\pi) = (-1)^n,\ n\in\mathbb{N}.$$

$$\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = (-1)^n$$

$$= 4\,\frac{(-1)^n}{n^2}$$

$$f \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

Dømi Lat $f(t) = \frac{\pi}{8}(\pi t - t^2)$, $t \in [0,\pi]$



$$f \sim \sum_{n=1}^{\infty}\frac{\sin((2n-1)t)}{(2n-1)^3}$$

$$|f(x) - S_N(x)| \le \varepsilon$$

6.16 $$\int_{-\pi}^{\pi}|f'(t)|^2\,dt = 2\int_0^{\pi}\left(-\frac{\pi}{4}t + \frac{\pi^2}{8}\right)^2 dt = 2\int_0^{\pi}\left(\frac{\pi^2}{16}t^2 - \frac{\pi^3}{16}t + \frac{\pi^4}{64}\right)dt$$

$$= 2\left[\frac{\pi^2}{48}t^3 - \frac{\pi^3}{32}t^2 + \frac{\pi^4}{64}t\right]_0^{\pi} = 2\left(\frac{\pi^5}{48} - \frac{\pi^5}{32} + \frac{\pi^5}{64}\right)$$

$$= \frac{\pi^5}{24} - \frac{\pi^5}{16} + \frac{\pi^5}{32} = \frac{\pi^5}{96}.$$

$$N \ge \frac{\frac{\pi^5}{96}}{\pi\cdot\varepsilon^2} = \frac{\pi^4}{96\,\varepsilon^2} \approx 1{,}015\cdot\frac{1}{\varepsilon^2}$$

$\varepsilon = 0{,}1$: $N \ge 102$

$\varepsilon = 0{,}01$: $N \ge 10150$

Men rekkjan er ólíka, so N refererar til allar liðirnar, men vit rokna bert ólíka liðir saman. Tað merkir 5000+ liðir skulu roknast.

6.17 $$|f(x) - S_N(x)| \le \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \le \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)^3}.$$

Nýt $\sum_{n=1}^{\infty} \frac{1}{n^3}$, hon konvergerar og er majorant.

$$\sum_{n=N+1}^{\infty} \frac{1}{(2n-1)^3} \le \int_{N+1}^{\infty} \frac{1}{(2x-1)^3}\, dx + \frac{1}{(2N+1)^3} = \frac{1}{4(2N-1)^2} + \frac{1}{(2N+1)^3}$$

$\varepsilon = 0{,}1$: $N \ge 2$.

$\varepsilon = 0{,}01$: $N \ge 4$.

Kompleks FR. Vit hava $e^{i\theta} = \cos\theta + i\sin\theta$ og $e^{-i\theta} = \cos\theta - i\sin\theta$.

Loysa vit eftir $\cos\theta$ og $\sin\theta$, so hava vit

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Lemma 6.20 Fourierrekkjan hevur avsnitssum

$$S_N(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) = \sum_{n=-N}^{N} c_n e^{inx},$$

har $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}.$$

Pf. Vit fáa, at

$$\begin{aligned} a_n \cos(nx) + b_n \sin(nx) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\, dx \; \frac{e^{inx} + e^{-inx}}{2} \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \; \frac{e^{inx} - e^{-inx}}{2i} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos(nx) - i\sin(nx)\right) dx \; e^{inx} \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos(nx) + i \sin(nx)\right) dx \; e^{-inx} \\ &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx\right) e^{inx} + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx}\, dx\right) e^{-inx}. \end{aligned}$$

Nú stendur $a_n \cos(nx) + b_n \sin(nx) = c_n e^{inx} + c_{-n} e^{-inx}$, og vit fáa eisini, at

$$\frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = c_0.$$

Tískil er
$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n\cos(nx) + b_n\sin(nx)$$
$$= c_0 + \sum_{n=1}^{N} c_n e^{inx} + c_{-n}e^{-inx}$$
$$= \sum_{n=-N}^{N} c_n e^{inx}. \qquad \square$$

Def. 6.21 Fourierrøkkjan hjá f á kompleksum formi skrivast
$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Lemma 6.22 Umsetan av koefficientum:

(i) $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$.

(ii) $a_0 = 2c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$.

Fylgir nærstan beinleiðis: $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^0\,dx = \frac{1}{2}a_0$

$n>0$:
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos(nx) - i\sin(nx))\,dx$$
$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - i\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right)$$
$$= \frac{1}{2}(a_n - ib_n).$$

$n>0$:
$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos(nx) + i\sin(nx))\,dx$$
$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx + i\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right)$$
$$= \frac{1}{2}(a_n + ib_n).$$

Dømi 6.24
$$f(x) = \begin{cases} -1, & x \in]-\pi, 0[\\ 0, & x = 0 \\ 1, & x =]0,\pi[\\ 0, & x = \pi \end{cases} \qquad a_n = 0 \quad \forall n \in \mathbb{N}_0 \quad \text{og} \quad b_n = \begin{cases} 0, & n \text{ líka}, \\ \frac{4}{n\pi}, & n \text{ ólíka}. \end{cases}$$

Nú er $c_n = 0$ fyri øll líka n. Fyri ólíka n er
$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left(0 - i\frac{4}{n\pi}\right) = -\frac{2i}{n\pi}$$
$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2}\left(0 + i\frac{4}{(-n)\pi}\right) = -\frac{2i}{n\pi}$$
$$f \sim -\frac{2i}{\pi}\sum_{n\in\mathbb{Z}, n=2k-1} \frac{1}{n}e^{inx}.$$