More about Gauss' Divergence Theorem

Ex1. Let $h: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$h(x,y) = 1 - x^3.$$

We consider the area $0 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Let $\mathcal{F}$ be the surface over the area in question lifted up by h.

a) Parametrize $\mathcal{F}$.

$$r(u,v) = \begin{bmatrix} u \\ v \\ 1-u^3 \end{bmatrix}, \quad u \in [0,1], \quad v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Let $$V(x,y,z) = \begin{bmatrix} xz \\ x \cos y \\ 3x^2 \end{bmatrix}.$$

b) Compute $\mathrm{Flux}(V, \mathcal{F}_r)$.

$$r_u' = \begin{bmatrix} 1 \\ 0 \\ -3u^2 \end{bmatrix}, \quad r_v' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow N_{\mathcal{F}}(u,v) = \begin{bmatrix} 3u^2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \begin{bmatrix} u(1-u^3) \\ u \cos v \\ 3u^2 \end{bmatrix} \cdot \begin{bmatrix} 3u^2 \\ 0 \\ 1 \end{bmatrix} du\, dv = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 -3u^6 + 3u^3 + 3u^2 \, du\, dv$$

$$= \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \cdot \left(-\frac{3}{7} + \frac{3}{4} + 1\right) = \pi \cdot \frac{37}{28}.$$

Now let Ω be the spatial region between the (x,y)-plane and $\mathcal{F}$.

c) Parametrize Ω.

$$r(u,v,w) = \begin{bmatrix} u \\ v \\ w \cdot (1-u^3) \end{bmatrix}, \quad w \in [0,1].$$

d) Calculate the flux through Ω.

$$\mathrm{Div}(V)(x,y,z) = z - x\sin y \qquad \begin{bmatrix} xz \\ x\cos y \\ 3x^2 \end{bmatrix}$$

$$\begin{array}{ccc} r_u' & r_v' & r_w' \end{array}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3u^2w & 0 & 1-u^3 \end{vmatrix} = 1-u^3 = J_r(u,v,w)$$

$$\mathrm{Flux}(V,\partial\Omega) = \int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \left(w(1-u^3) - u\cdot\sin v\right)\cdot(1-u^3)\,du\,dv\,dw$$

$$= \int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 u^6w - 2u^3w + u^4\sin v - u\sin v + w \;du\,dv\,dw$$

$$= \int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{7}w - \frac{1}{2}w + \left(\frac{1}{5} - \frac{1}{2}\right)\sin v + w \;dv\,dw$$

$$= \int_0^1 \frac{9\pi}{14}w + 0 \;dw$$

$$= \frac{9\pi}{28}.$$

Ex2. Given two spatial regions calculate the flux of 6 vector fields through their surfaces.

The point is to see that the divergence is constant for all fields, and so

$$\mathrm{Flux}(V_i, \partial\Omega_j) = \mathrm{div}(V_i)\cdot \mathrm{Vol}(\Omega_j), \quad i=1,..,6 \text{ and } j=1,2.$$

Instead we can also let Maple do the work.

```
> k:=(u,v,w)-> <w*sin(u)*cos(v),w*sin(u)*sin(v),w*cos(u)>:
  r:=(u,v,w)-> <u*cos(v),u*sin(v),u^2+w*(1-u^2)>:
  B:=[0,Pi,-Pi,Pi,0,1]:
  C:=[0,1,-Pi,Pi,0,1]:
  k(u,v,w),r(u,v,w);
```

$$\begin{bmatrix} w\sin(u)\cos(v) \\ w\sin(u)\sin(v) \\ w\cos(u) \end{bmatrix}, \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ u^2 + w\left(-u^2+1\right) \end{bmatrix}$$

```
> V[1]:=(x,y,z)-> <1,2,3>:
  V[2]:=(x,y,z)-> <-x,y/2,-z/3>:
  V[3]:=(x,y,z)-> <x-y*z,-2*y+x*z^2,3*z+y*x^3>:
  V[4]:=(x,y,z)-> <k__1,k__2,k__3>:
  V[5]:=(x,y,z)-> <y-x^3,3*x^2*y,25+10*z>:
  V[6]:=(x,y,z)-> <2*x*z-2*x*y-z,z^3+y^2,-z^2>:
```

```
> for i from 1 to 6 do divIntGo(k,B,V[i]) od;
```

$$0$$
$$-\frac{10\pi}{9}$$
$$\frac{8\pi}{3}$$
$$0$$
$$\frac{40\pi}{3}$$
$$0$$

```
> for i from 1 to 6 do divIntGo(r,C,V[i]) od;
```

$$0$$
$$-\frac{5\pi}{12}$$
$$\pi$$
$$0$$
$$5\pi$$
$$0$$

Ex3. We have Ω_r given by $r(u,v,w) = \begin{bmatrix} u\cos v \\ u\sin v \\ w \end{bmatrix}$, $u \in [0,2]$, $v \in [0, \frac{\pi}{2}]$, $w \in [0,5]$.

a) Describe Ω_r and determine the volume.

This is just a quarter turn of a cylinder.

$$\mathrm{Vol}(\Omega_r) = \frac{1}{4} \cdot \pi \cdot 2^2 \cdot 5 = 5\pi.$$

b) Determine the flux through the surface of vector fields U and V given $\mathrm{Div}(U) = \pi$ and $\mathrm{Div}(V) = yz$.

Firstly $\mathrm{Flux}(U, \partial\Omega_r) = \pi \cdot 5\pi = 5\pi^2$.

Second we have

$$\begin{array}{ccc} r_u' & r_v' & r_w' \end{array}$$
$$\begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \cdot \cos^2 v + u \cdot \sin^2 v = u = J_r(u,v,w)$$

$$\mathrm{Flux}(V, \partial\Omega_r) = \int_0^5 \int_0^{\frac{\pi}{2}} \int_0^2 u \cdot \sin v \cdot w \cdot u \; du\, dv\, dw$$

$$= \int_0^5 \int_0^{\frac{\pi}{2}} \frac{8}{3} \sin v \cdot w \; dv \, dw$$

$$= \int_0^5 \frac{8}{3} w \cdot 1 \, dw = \frac{8}{6} \cdot 25 = \frac{100}{3}.$$

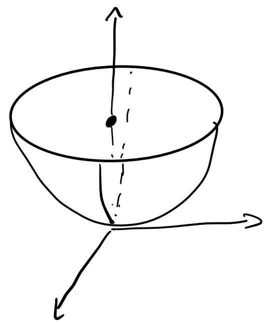
Ex4. Let

$$V(x,y,z) = \begin{bmatrix} e^y + \cos(yz) \\ e^z + \sin(xz) \\ x^2 z^2 \end{bmatrix} \quad \text{and } \mathcal{F}: x^2+y^2+z^2-4z=0, \; z \le 2.$$

a) Sketch $\mathcal{F}$.

$$x^2+y^2+z^2-4z=0 \iff x^2+y^2+(z-2)^2 = 2^2, \quad z \le 2.$$

This is the lower half of a sphere of radius 2 and a center at $(0,0,2)$.



b) Compute $\int_\Omega \mathrm{Div}(V) \, d\mu$, where the normal is outward-pointing.

$$k(u,v,w) = w \cdot \begin{bmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u + 2 \end{bmatrix}, \quad u \in [\tfrac{\pi}{2}, \pi], \; v \in [0, 2\pi], \; w \in [0,2].$$

```
> V:=(x,y,z)-> <exp(y)+cos(y*z),exp(z)+sin(x*z),x^2*z^2>:
  k:=(u,v,w)-> <w*sin(u)*cos(v),w*sin(u)*sin(v),w*cos(u)+2>:
  k(u,v,w);
  divIntGo(k,[Pi/2,Pi,0,2*Pi,0,2],V);
```

$$\begin{bmatrix} w \sin(u) \cos(v) \\ w \sin(u) \sin(v) \\ w \cos(u) + 2 \end{bmatrix}$$

$$\frac{176\,\pi}{15}$$

c) Parametrize the lid.

$$r(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ 2 \end{bmatrix}, \quad u\in[0,2], \quad v\in[0,2\pi].$$

d) Compute the flux through the disk.

```
> r:=(u,v)-> <u*cos(v),u*sin(v),2>:
fluxIntGo(r,[0,2,0,2*Pi],V);
```

16π

e) Now state the flux through $\mathcal{F}$.

$$\frac{176\pi}{15} - 16\pi = -\frac{64\pi}{15}$$

Ex5. Compute the flux given

$$V(x,y,z) = \begin{bmatrix} x^3 + xy^2 \\ 4yz^2 - 2x^2y \\ -z^3 \end{bmatrix} \quad \text{and} \quad \Omega = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le a^2\}.$$

We have $J_r(u,v,w) = w^2\cdot\sin u$ and $\mathrm{Div}(V)(x,y,z) = x^2+y^2+z^2$.

$$\int_0^a\int_0^{2\pi}\int_0^{\pi} w^2 \cdot w^2\cdot\sin u \; du\, dv\, dw$$

$$= 2\pi\cdot\int_0^a w^4\cdot\Big[-\cos u\Big]_0^{\pi}\, dw$$

$$= 2\pi\cdot 2\,\Big[\tfrac{1}{5}w^5\Big]_0^a = \frac{4\pi}{5}\cdot a^5.$$

Ex6. Let $V(x,y,z) = \begin{bmatrix} 2x \\ 3y \\ -z \end{bmatrix}$ and $\Omega = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1\right\}$.

Determine the flux. This is an ellipsoid and $\mathrm{Div}(V) = 4$, so

$$\mathrm{Flux}(V,\partial\Omega) = \mathrm{Div}(V)\cdot\mathrm{Vol}(\Omega) = 4\cdot\frac{4}{3}\cdot\pi\cdot a\cdot b\cdot c = \frac{16}{3}\pi abc.$$