# More about Gradient Vector Fields

Ex 1. Let

$$V = \begin{bmatrix} x+z \\ -y-z \\ x-y \end{bmatrix}.$$

a) Compute $\mathrm{Curl}(V)(x,y,z)$ and justify that $V$ is a gradient vector field.

$$\mathrm{Curl}(V)(x,y,z) = \begin{bmatrix} -1-(-1) \\ 1-1 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By proposition 27.14 a vector is a gradient vector field if and only if the curl is $\underline{0}$. Thus $V$ is a gradient vector field.

b) Determine the indefinete integral of $V$ by the stair method.

$$r_1(u) = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix},\ u \in [0,x], \quad r_2(u) = \begin{bmatrix} x \\ u \\ 0 \end{bmatrix},\ u \in [0,y], \quad r_3(u) = \begin{bmatrix} x \\ y \\ u \end{bmatrix},\ u \in [0,z]$$

$$\int_T V \cdot \underline{e}\, d\mu = \int_0^x V(r_1(u)) \cdot r_1'(u)\, du + \int_0^y V(r_2(u)) \cdot r_2'(u)\, du + \int_0^z V(r_3(u)) \cdot r_3'(u)\, du$$

$$= \int_0^x u\, du + \int_0^y -u\, du + \int_0^z x-y\, du$$

$$= \frac{1}{2}x^2 - \frac{1}{2}y^2 + xz - yz + k, \quad k \in \mathbb{R}.$$

Ex 2. Compute the divergence and curl at $(1,1,1)$ for

$$V(x,y,z) = \begin{bmatrix} -yx \\ xy^2 \\ xyz \end{bmatrix}.$$

$$\mathrm{Div}(V)(x,y,z) = -y + 2xy + xy = -y + 3xy.$$

$$\mathrm{Div}(V)(1,1,1) = -1 + 3\cdot 1\cdot 1 = 2.$$

$$\mathrm{Curl}(V)(x,y,z) = \begin{bmatrix} xz - 0 \\ 0 - yz \\ y^2 + x \end{bmatrix} = \begin{bmatrix} xz \\ -yz \\ y^2 + x \end{bmatrix}.$$

$$\mathrm{curl}(V)(1,1,1) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Ex 3. Let

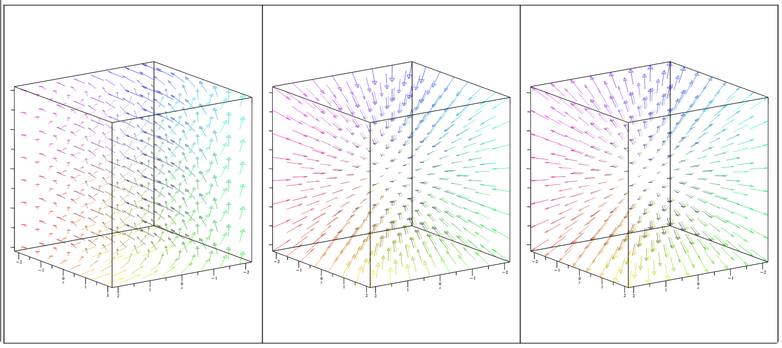
$$V(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}, \quad W(x,y,z) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}, \quad U(x,y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

a) Try to guess the behaviors.

$V$ is a rotation field, $W$ is an implosion and $U$ is an explosion.

b) Predict the look and then plot.

```
> V:=(x,y,z)-> <-y,x,1>:
  W:=(x,y,z)-> <-x,-y,-z>:
  U:=(x,y,z)-> <x,y,z>:
> p1:=fieldplot3d(V(x,y,z),x=-2..2,y=-2..2,z=-2..2,arrows=slim):
  p2:=fieldplot3d(W(x,y,z),x=-2..2,y=-2..2,z=-2..2,arrows=slim):
  p3:=fieldplot3d(U(x,y,z),x=-2..2,y=-2..2,z=-2..2,arrows=slim):
> display(<p1|p2|p3>);
```



c) Compute divergence and curl.

```
> div(V),div(W),div(U);
  rot(V)(x,y,z),rot(W)(x,y,z),rot(U)(x,y,z);
```

$$(x, y, z) \mapsto 0,\ (x, y, z) \mapsto -3,\ (x, y, z) \mapsto 3$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

d) Which of V, W and U are gradient vector fields.

Again by 27.14 we have that W and U are gradient vector fields.

Ex 4. Let $f(x,y,z) = \cos(xyz)$ and $V(x,y,z) = \nabla f(x,y,z)$.

A curve $K$ is the straight line from $(\pi, \frac{1}{2}, 0)$ to $(\frac{1}{2}, \pi, -1)$.

a) Compute $\int_K V \cdot \underline{e}\, d\mu$.

We have the antiderivative $f$, so this amounts to

$$f(\tfrac{1}{2}, \pi, -1) - f(\pi, \tfrac{1}{2}, 0) = \cos(-\tfrac{\pi}{2}) - \cos(0) = -1.$$

See 27.10.

Now let $V(x,y,z) = \nabla(x^2 + yz)$ and $K$ be given by $r(u) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ \sin(2u) \end{bmatrix}$, $u \in [0, 2\pi]$.

b) Compute $\int_K V \cdot \underline{e}\, d\mu$.

To be clear we start and end at the same point, so 27.12 tells us the circulation is zero. Note that it is a requirement that V is a gradient vector field for this to be the case.

Ex 5. Let $C_1: x^2+y^2=1$ and $C_2: (x-1)^2+(y-1)^2=1$, and let

$$V(x,y,z) = \begin{bmatrix} x^2+y^2 \\ x\cdot y \end{bmatrix}.$$

a) Compute the tangential line integral of V along the paths counter clockwise.

$$r_1(u) = \begin{bmatrix} \cos(u) \\ \sin(u) \end{bmatrix}, \quad u\in[0,2\pi].$$

$$r_2(u) = \begin{bmatrix} \cos(u)+1 \\ \sin(u)+1 \end{bmatrix}, \quad u\in[0,2\pi].$$

$$\int_{C_1} V\cdot \underline{e}\, d\mu = \int_0^{2\pi} \begin{bmatrix} \cos^2 u + \sin^2 u \\ \cos u \cdot \sin u \end{bmatrix} \cdot \begin{bmatrix} -\sin u \\ \cos u \end{bmatrix} du$$

$$= \int_0^{2\pi} -\sin u + \cos^2 u\cdot \sin u \quad du$$

$$= \int_0^{2\pi} -\sin u + \int_0^{2\pi} \cos^2 u\cdot\sin u \; du \quad , \quad \begin{aligned} t &= \cos u \\ dt &= -\sin u\, du \end{aligned}$$

$$= \Big[\cos u\Big]_0^{2\pi} - \int_1^1 t^2\, dt$$

$$= 0.$$

$$\int -\sin u + \cos^2 u\cdot\sin u \; du$$

$$= \cos u - \frac{1}{3}\cos^3 u$$

$$\int_{C_2} V \cdot \underline{e}\, d\mu = \int_0^{2\pi} \begin{bmatrix} (\cos u + 1)^2 + (\sin u + 1)^2 \\ (\cos u + 1) \cdot (\sin u + 1) \end{bmatrix} \cdot \begin{bmatrix} -\sin u \\ \cos u \end{bmatrix} du$$

```
> V:=(x,y)-> <x^2+y^2,x*y>:
  V(x,y);
```

$$\begin{bmatrix} x^2+y^2 \\ x\,y \end{bmatrix}$$

```
> r:= u -> <cos(u)+1,sin(u)+1>:
  r(u);
```

$$\begin{bmatrix} \cos(u)+1 \\ \sin(u)+1 \end{bmatrix}$$

```
> dr:=diff(r(u),u);
```

$$dr := \begin{bmatrix} -\sin(u) \\ \cos(u) \end{bmatrix}$$

```
> prik(V(vop(r(u))),dr);
  simplify(%);
```

$$-\left((\sin(u)+1)^2 + (\cos(u)+1)^2\right)\sin(u) + (\cos(u)+1)\,(\sin(u)+1)\,(\sin(u)+1)\cos(u)$$

$$(\sin(u)+3)\cos(u)^2 + (-\sin(u)+1)\cos(u) - 3\sin(u) - 2$$

```
> int(prik(V(vop(r(u))),dr),u=0..2*Pi);
```

$$-\pi$$

b) Now compute in the clockwise direction.

$$\int_{C_1} V \cdot \underline{e}\, d\mu = 0 \quad \text{and} \quad \int_{C_2} V \cdot \underline{e}\, d\mu = \pi .$$

```
> r:=u-> <cos(u),-sin(u)>:
  dr:=diff(r(u),u):
  int(prik(V(vop(r(u))),dr),u=0..2*Pi);
```

$$0$$

```
> r:=u-> <cos(u)+1,-sin(u)+1>:
  dr:=diff(r(u),u):
  int(prik(V(vop(r(u))),dr),u=0..2*Pi);
```

$$\pi$$

c) Is $V$ a gradient vector field?

No, as demonstrated along the closed curve $C_2$ the integral is non-zero. See 27.12.

Ex 6. Let

$$V(x,y,z) = \begin{bmatrix} 5x - 4z \\ -2x - y \\ 2x - z \end{bmatrix}$$

and $\mathcal{A} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; -\frac{1}{2} \le x \le \frac{1}{2},\ 1 \le y \le 2,\ -\frac{1}{2} \le z \le \frac{1}{2} \right\}$.

a) Determine the flow curve $r(t)$ of $V$ for arbitrary point $r(0) = (x, y, z)$ in $A$.

```
> V:=(x,y,z)-> <5*x-4*z,-2*x-y+4*z,2*x-z>:
  V(x,y,z);
```

$$\begin{bmatrix} 5x - 4z \\ -2x - y + 4z \\ 2x - z \end{bmatrix}$$

```
> l1:=diff(x(t),t)=5*x(t)-4*z(t);
  l2:=diff(y(t),t)=-2*x(t)-y(t)+4*z(t);
  l3:=diff(z(t),t)=2*x(t)-z(t);
```

$$l1 := \frac{d}{dt} x(t) = 5\,x(t) - 4\,z(t)$$

$$l2 := \frac{d}{dt} y(t) = -2\,x(t) - y(t) + 4\,z(t)$$

$$l3 := \frac{d}{dt} z(t) = 2\,x(t) - z(t)$$

```
> dsolve({l1,l2,l3,x(0)=x,y(0)=y,z(0)=z},{x(t),y(t),z(t)});
  r:=unapply(<rhs(%[1]),rhs(%[2]),rhs(%[3])>,(x,y,z,t)):
  r(x,y,z,t);
```

$$\left\{x(t) = (2\,x - 2\,z)\,e^{3\,t} + (-x + 2\,z)\,e^{t}, y(t) = (-x + 2\,z)\,e^{t} + (x - 2\,z + y)\,e^{-t}, z(t) = \frac{(2\,x - 2\,z)\,e^{3\,t}}{2} + (-x + 2\,z)\,e^{t}\right\}$$

$$\begin{bmatrix} (2\,x - 2\,z)\,e^{3\,t} + (-x + 2\,z)\,e^{t} \\ (-x + 2\,z)\,e^{t} + (x - 2\,z + y)\,e^{-t} \\ \frac{(2\,x - 2\,z)\,e^{3\,t}}{2} + (-x + 2\,z)\,e^{t} \end{bmatrix}$$

b) Provide an expression $\mathrm{Vol}(t)$ for the volume of $A$ as a function of $t$.

```
> VectorCalculus[Jacobian](r(x,y,z,t),[x,y,z]):
  Determinant(%):
  Jacobi:=simplify(%);
```

$$Jacobi := e^{3\,t}$$

```
> int(Jacobi,x=-1/2..1/2,y=1..2,z=-1/2..1/2);
```

$$e^{3\,t}$$

$\mathrm{Vol}(t) = e^{3t}$.

c) How large is the volume at $t=1$?

$\mathrm{Vol}(1) = e^3$.

d) What is $\dfrac{\mathrm{Vol}'(0)}{\mathrm{Vol}(0)}$ as well as $\mathrm{Div}(V)(x,y,z)$?

$$\frac{\mathrm{Vol}'(0)}{\mathrm{Vol}(0)} = \frac{3}{1} = 3 \quad \text{and} \quad \mathrm{Div}(V)(x,y,z) = 5 - 1 - 1 = 3.$$