Parseval Vit kunnu brúka Parsevals setning bæði til at rokna virðið á eini røkkju og at rokna góða approksimatión í, hvussu nógvir liðir skulu nýtast í $S_N(x)$.

Setn. 6.25 Set fyri, at $f \in L^2(-\pi,\pi)$ við Fourierkoefficientarnar $\{a_n\}_{n\in\mathbb{N}}$ og $\{b_n\}_{n\in\mathbb{N}}$ ella $\{c_n\}_{n\in\mathbb{N}}$. So er

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2) = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

Dømi 6.26(ii) Lat $f(x) = x, \quad x \in [-\pi,\pi[$

Vit hava $a_n = 0 \quad \forall n \in \mathbb{N}$ og

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\left(\left[-\frac{x}{n}\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\left[\sin(nx)\right]_0^{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{n}(-1)^{n+1} + 0\right)$$

$$= \frac{2}{n}(-1)^{n+1}.$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx).$$

Vit hava, at

$$\int_{-\pi}^{\pi}|f(x)|^2\,dx = 2\int_0^{\pi}x^2\,dx = \frac{2}{3}\left[x^3\right]_0^{\pi} = \frac{2\pi^3}{3}.$$

Per Parseval:

$$\frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{1}{2}\sum_{n=1}^{\infty}\left|\frac{2}{n}(-1)^{n+1}\right|^2 = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Longleftrightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Diff. likn. skipan Vit venda nú aftur til høvuðsáhugamálið, differentiallikningaskipanir. Minst til, at ein differentiallikning á n'ta stigi

$$D_n(y) = u,$$

kann skrivast til eina skipan á 1. stigi. Her á serligum formi

$$\begin{cases} \dot{x} = Ax + bu \\ y = d^T x \end{cases}$$

Yvirførslufunktiónin fyri skipanir er

$$H(s) = -d^T(A-sI)^{-1}b, \quad \text{har} \quad \det(A-sI) \neq 0.$$

Setn. 2.21 Um $u(t) = e^{st}$, har $\det(A - sI) \neq 0$, so er ein loysn $y(t) = H(s)e^{st}$.

Um skipanin er asymptotiskt stabil, so hava øll eginvirdini negativan realpart.
Lat nú $s = in$, $n \in \mathbb{Z}$, so hevur s ikki negativan realpart, so $\det(A - sI) \neq 0$.
Specielt er fyri

ávirkan $u(t) = e^{int} \longrightarrow$ loysn $y(t) = H(in)\, e^{int}$

Superpositiónir $u(t) = \sum_{n=-N}^{N} c_n e^{int} \longrightarrow$ loysn $y(t) = \sum_{n=-N}^{N} c_n H(in) e^{int}$

Setn. 7.8 Lat $H(s)$ vera yvirførslufunktiónin hjá eini asymptotiskt stabila skipan.
Lat u vera ein 2π-periodisk stykkivíst differentiabul og kontinuert ávirkan givin við Fourierrekkjuni

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad t \in \mathbb{R}. \qquad \text{(Fouriers setningur)}$$

So hevur differentiallíkningaskipanin eina loysn givið við Fourierrekkjuni

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int}, \quad t \in \mathbb{R}.$$

Loysnin kann sjálvandi enda við at vera torfør at arbeiða við, og í veruleikanum, so er neydugt at nýta avsnitssummar. Vit hava etablerað nógva teori fyri at vera ílkædd til at gera vurderingar.

Uppskrift til Fourierrekkjumetoduna

1. Kanna at differentiallíkningaskipanin er asymptotiskt stabil.
   - Vís, at systemmatrican hevur negativan realpart fyri øll eginvirdini við at rokna, ella Routh-Hurwitz.
2. Rokna $H(s)$, $s \in D_m(H)$.
3. Finn Fourierrekkjuna hjá periodisku ávirkanini $u \sim \sum_{n=-\infty}^{\infty} c_n e^{int}$.
4. Kanna at u er stykkivís differentiabul og kontinuert.
   - Minst til at tekna og vátta at $u(-\pi^+) = u(\pi^-)$.
   - Stk.vís. diff.: ofta er skjótari at víðka f_i til $\mathbb{R}$ heldur enn $[x_i, x_{i+1}]$ til at vátta differentiabilitet.

$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$

5. Set upp loysnina $y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int}$
6. Finn $N \in \mathbb{N}$, so at

$$\left| y(t) - \sum_{n=-N}^{N} c_n H(in) e^{int} \right| \leq \varepsilon \qquad \forall t \in \mathbb{R}.$$

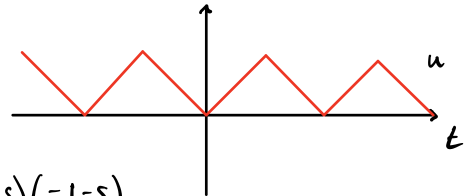
   - Typiskt kendir setningar:

     6.16, 6.17 ella integralkriteriið.

     Approksimatión í effekt: 7.12.

Dømi 7.9

$$\begin{cases} \dot{x} = \begin{pmatrix} -4 & 0 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u \\ y = (1 \quad 3)\, x \end{cases}$$



$u(t) = |t| \ , \quad t \in [-\pi, \pi[.$

$$A = \begin{pmatrix} -4 & 0 \\ 1 & -1 \end{pmatrix} \qquad \det(A - sI) = (-4-s)(-1-s)$$

Røturnar eru $s = -4$ og $s = -1$. Realparturin er negativur, so systemið er asymptodiskt stabilt.

$$H(s) = -(1 \quad 3)\, \frac{1}{(-4-s)(-1-s)} \begin{pmatrix} -1-s & 0 \\ -1 & -4-s \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= -(1 \quad 3)\, \frac{1}{(-4-s)(-1-s)} \begin{pmatrix} -4-4s \\ -4 \end{pmatrix}$$

$$= \frac{1}{(-4-s)(-1-s)} \left(4 + 4s + 12 \right)$$

$$= \frac{4(4+s)}{-(4+s)(-1-s)} = \frac{4}{s+1} \ , \quad s \notin \{-4, -1\}.$$

Vit roknaðu Fourierkoefficientarnar at vera

$$b_n = 0 \quad \forall n \in \mathbb{N} \quad \text{og} \quad a_n = \begin{cases} \pi & , n = 0 \\ 0 & , n \text{ lika} \\ -\frac{4}{\pi} \frac{1}{n^2} & , n \text{ ólika} \end{cases}$$

Funktiónin u er 2π-periodisk stykkivis differentiabul og kontinuert, so

$$u(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \cos\left((2n-1)t\right).$$

Vit hava nú

$$c_n = \begin{cases} \frac{\pi}{2} & , n = 0 \\ 0 & , n \text{ lika} \\ -\frac{2}{\pi} \frac{1}{n^2} & , n \text{ ólika} , \end{cases}$$

so

$$u(t) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{\substack{n \in \mathbb{Z} \\ n \text{ ólika}}} \frac{1}{n^2} e^{int} .$$

Loysnin er nú

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int} = \sum_{n=-\infty}^{\infty} \frac{4 c_n}{1+in} e^{int} .$$

Vit skriva tiskil, at

$$y(t) = c_0 H(0) + \sum_{\substack{n\in\mathbb{Z}\\ n \text{ ólika}}} c_n H(in) e^{int} = \frac{\pi}{2}\cdot 4 - \sum_{\substack{n\in\mathbb{Z}\\ n \text{ ólika}}} \frac{1}{n^2} \frac{4}{1+in} e^{int}.$$

$$= 2\pi - \frac{8}{\pi} \sum_{\substack{n\in\mathbb{Z}\\ n \text{ ólika}}} \frac{1}{n^2} \frac{1}{1+in} e^{int}.$$

Def. 7.10 Effektin av einari 2π-periodiska funktión f er

$$\mathcal{P}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, dt.$$

Typisk roknast við Fourierkoefficientarnar tí vit hava Parsevals setning.

Lemma 7.11 (i) $\mathcal{P}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2$, (ii) $\mathcal{P}(S_N) = \sum_{n=-N}^{N} |c_n|^2$.

Setn. 7.12 Fyri $0<\delta<1$ er (i)

$$\frac{\mathcal{P}(S_N)}{\mathcal{P}(f)} \geq \delta \iff \sum_{|n|>N} |c_n|^2 \leq (1-\delta)\,\mathcal{P}(f).$$

(ii)

$$\frac{\mathcal{P}(S_N)}{\mathcal{P}(f)} \geq \delta \iff \sum_{n=N+1}^{\infty} (|a_n|^2 + |b_n|^2) \leq 2(1-\delta)\,\mathcal{P}(f).$$

Dømi 7.13 $f(t) = t$, $t \in [-\pi, \pi[$. Vit roknaðu $a_n = 0 \;\forall n \in \mathbb{N}_0$ og $b_n = \frac{2}{n}(-1)^{n+1}$.

Vit avgera hvat $N \in \mathbb{N}$ skal vera, so at $S_N(x)$ hevur 90 % av effektini hjá f.

$$\mathcal{P}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |t|^2 dt = \frac{1}{\pi} \int_0^{\pi} t^2 dt = \frac{\pi^2}{3}.$$

Stytt $\mathcal{P}(S_N)$ við $\mathcal{P}(f)$, so vit hava.

$$\frac{1}{2} \frac{3}{\pi^2} \sum_{n=1}^{N} \left|\frac{2}{n}(-1)^{n+1}\right|^2 = \frac{6}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \geq 0{,}9 \quad \text{tá} \quad N \geq 6.$$