ub. 1. (1) $\sum\limits_{n=1}^{\infty} \frac{\sin(n)+\cos(n)}{n^2}$, $\left|\frac{\sin(n)+\cos(n)}{n^2}\right| \leq \frac{2}{n^2}$ $\forall n \in \mathbb{N}$, og $2 \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ er konvergent.

Per 4.20(i) er $\sum\limits_{n=1}^{\infty} \left|\frac{\sin(n)+\cos(n)}{n^2}\right|$ konvergent og tí er $\sum\limits_{n=1}^{\infty} \frac{\sin(n)+\cos(n)}{n^2}$ absolut konvergent.

(2) $\sum\limits_{n=1}^{\infty} \cos(n\pi) \frac{2}{n+5} = \sum\limits_{n=1}^{\infty} (-1)^n \frac{2}{n+5}$, $\left|(-1)^n \frac{2}{n+5}\right| = \frac{2}{n+5} \geq \frac{2}{6n}$ $\forall n \in \mathbb{N}$.

Rekkjan $\frac{1}{3} \sum\limits_{n=1}^{\infty} \frac{1}{n}$ er divergent, so rekkjan er ikki absolut konvergent. Fylgjan $\frac{2}{n+5} > 0$ $\forall n \in \mathbb{N}$ og er monotont fallandi, tí $\frac{2}{n+5} \geq \frac{2}{(n+1)+5}$.

Umfram tað er $\lim\limits_{n \to \infty} \frac{2}{n+5} = 0$. Per 4.38 er rekkjan tí konvergent, so vit konkludera treytaðan konvergens.

(3) $\sum\limits_{n=1}^{\infty} (-2)^n \frac{1}{n^2+7}$, $\left|(-2)^n \frac{1}{n^2+7}\right| = \frac{2^n}{n^2+7} \to \infty$ tá $n \to \infty$. Tískil gongur $(-2)^n \frac{1}{n^2+7}$ ikki ímóti 0 tá $n \to \infty$, so rekkjan er divergent per 4.19.

2. Vís, at $\sum\limits_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)}$ er divergent við integralkriteriið.

Funktiónin $f(x) = \frac{1}{x \ln(x)}$, $x \geq 2$, er kontinuert og fallandi.

$$\int_2^t \frac{1}{x \ln x}\,dx = \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\,du \qquad \begin{aligned} u &= \ln(x) \\ du &= \tfrac{1}{x}\,dx \end{aligned}$$

$$= \Big[\ln(u)\Big]_{\ln(2)}^{\ln(t)}$$

$$= \ln(\ln(t)) - \ln(\ln(2))$$

$$\to \infty \quad \text{tá} \quad t \to \infty.$$

Per 4.33(ii) er $\sum\limits_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)}$ divergent.

3. (1) Vís, at $\sum\limits_{n=1}^{\infty} n e^{-n^2}$ er konvergent við integralkriteriið.

Funktiónin $f(x) = \frac{x}{e^{x^2}}$ er kontinuert og fallandi á $[1, \infty)$.

$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1-2x^2)\,e^{-x^2} < 0$ tá $x > \frac{1}{\sqrt{2}}$.

$$\int_1^t x e^{-x^2}\,dx = \int_1^{t^2} \frac{e^{-u}}{2}\,du \qquad \begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}$$

$$= \left[-\frac{e^{-u}}{2}\right]_1^{t^2}$$

$$= \frac{1}{2e} - \frac{e^{-t^2}}{2} \to \frac{1}{2e} \quad \text{tá} \quad t \to \infty.$$

Per 4.33(i) er $\sum\limits_{n=1}^{\infty} n e^{-n^2}$ konvergent.

(2) Finn $N \in \mathbb{N}$, so at S_N approksimerar summin við einum fráviki í mesta lagi 0,001.

Vit loysa $f(N+1) \leq 0{,}001$.

$$(N+1) \cdot e^{-(N+1)^2} \leq 0{,}001$$

$$\iff \quad N \geq 1{,}8185.$$

Altso $N \geq 2$, so er $S_2 + \int_2^\infty x e^{-x^2} dx \approx \sum_{n=1}^\infty n e^{-n^2}$ per korollar 4.35 (ii)

(3) Vit fáa $S_2 + \int_3^\infty x e^{-x^2} dx = e^{-1} + 2e^{-4} + \frac{1}{2} e^{-9} = 0{,}40457.$

4. Finn eina approksimatión av $\sum_{n=1}^\infty \frac{1}{n^4}$ við frávik á í mesta lagi 0,02.

Per uppgávu 162 er $S_N \leq \frac{1}{(N+1)^{4-1}} \frac{4}{4-1} = \frac{4}{3} \frac{1}{(N+1)^3}$. Vit loysa N.

$$\frac{4}{3} \frac{1}{(N+1)^3} \leq \frac{1}{50} \iff \frac{4 \cdot 50}{3} \leq (N+1)^3$$

$$\iff N+1 \geq \sqrt[3]{\frac{200}{3}}$$

$$\iff N \geq \sqrt[3]{\frac{200}{3}} - 1 = 3{,}0548.$$

Altso $N \geq 4$, so er $S_4 \approx \sum_{n=1}^\infty \frac{1}{n^4}$.

$$S_4 = \sum_{n=1}^4 \frac{1}{n^4} = \frac{22369}{20736} = 1{,}07875.$$

5. $\sum_{n=1}^\infty (-1)^{n-1} \frac{1}{n}$ er konvergent.

(1) Finn N, so at S_N er høgst 0,01 frá rekkjuni.

Tað er givið, at $b_n = \frac{1}{n}$, $n \geq 1$, er positiv og monotont fallandi umframt, at $b_n \to 0$ tá $n \to \infty$. Per 4.38 fáa vit vurderað

$$b_{N+1} = \frac{1}{N+1} \leq 0{,}01$$

$$\iff 99 \leq N.$$

So $S_{99} \approx \sum_{n=1}^\infty (-1)^{n-1} b_n$.

(2) Finn S_N. Vit hava, at $S_{99} = 0{,}69817$.

(3) Sama men við $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^n}$.

$$b_{N+1} = \frac{1}{(N+1)^{N+1}} \leq 0{,}01$$

$$\Leftrightarrow \quad N \geq 2{,}5973.$$

So $S_3 \approx \sum_{n=1}^{3} (-1)^{n-1} \frac{1}{n^n}$. Vit hava, at $S_3 = 0{,}78704$.

(4) Konvergensurin á $\frac{1}{n^n}$ er nógv skjótari enn hjá $\frac{1}{n}$, so rekkjan í (3) leggur nógv minni liðir afturat, meðan $\frac{1}{n}$ konvergerar seint.