Eigen Things

The basics: Lat $\underline{\underline{A}}$ vera ein $n \times n$ matrica, so hevur hendan fastar rætningar, fixed directions, t.v.s. vektorar $\underline{r}$ sum eru ein skalering λ av sær sjálvum undir avmyndan við $\underline{\underline{A}}$.

Vit skriva altso
$$\underline{\underline{A}}\,\underline{r} = \lambda\,\underline{r}$$
$$\Leftrightarrow \underline{\underline{A}}\,\underline{r} - \lambda\underline{r} = \underline{0}$$
$$\Leftrightarrow (\underline{\underline{A}} - \lambda\underline{\underline{I}})\,\underline{r} = \underline{0}$$

Loysnin $\underline{r} = \underline{0}$ er triviel og sigur einki um geometriina hjá $\underline{\underline{A}}$. Vit leita eftir loysnum við $\underline{r} \neq \underline{0}$.
Í so fall finst ein loysn tá determinanturin er 0, so at
$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0.$$

Vit fáa altso karakteristiska polynomið á n'ta stigi, har røturnar eru eginvirðini.

Dømi 15.1 Lat $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, so er $p(\lambda) = \det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 3-\lambda & 1 & 0 \\ 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$.

Hetta er ein ovara tríhantsmatrica, so
$$p(\lambda) = (1-\lambda)(3-\lambda)(4-\lambda)(2-\lambda),$$
har røturnar eru $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 2$ og $\lambda_4 = 1$.

Vit sóu, at determinanturin er varðveittur undir Gauss elimination, men so er ikki við eginvirðini.
Ein generel $n \times n$ matrica hevur $p(\lambda)$ á n'ta stigi, og eingin beinleiðis formil finst til at loysa røtur í polynom við stig $n \geq 5$. Aðrar metodur kunnu brúkast, men typiskt er at loysa numeriskt við CAS amboð.

Minst til at $p(0) = \det(\underline{\underline{A}}) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$.

Dømi 15.3 Vit finna eginvektorar við innsetan av okkurn eginvirði og loysa.

$$\lambda_1 = 4: \quad \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \underline{r}_1 = \underline{0} \quad \Leftrightarrow \quad \underline{r}_1 = \begin{bmatrix} 1/3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3 \qquad \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \underline{r}_2 = \underline{0} \iff \underline{r}_2 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Umfram hetta fáa vit

$$\underline{r}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix} \quad \text{og} \quad \underline{r}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Dømi 15.4 Lat $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ so er $\lambda_1 = \lambda_2 = 2$ og $\lambda_3 = 1$.

Fyri λ_1 og λ_2: $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{r}_1 = \underline{0} \iff \underline{r}_1 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 1$: $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{r}_3 = \underline{0} \iff \underline{r}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Vit fáa tvær fastar rætningar. Eginvirðið 2 sigst at hava algebraiskan multiplicitet á 2 og geometriskan multiplicitet á 1.

Rotatiónsmatricur eru dømi, har bert ein rætningur er fastur.

Dømi 15.6 Symmetriskar matricur hava reel eginvirði og hava eigendekompositión.
Lat

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (3-\lambda)^3 - (3-\lambda) = 0, \quad x = 3 - \lambda$$

$$\iff x^3 - x = 0$$
$$\iff x(x^2 - 1) = 0$$
$$\iff \lambda_1 = 4, \ \lambda_2 = 3, \ \lambda_3 = 2.$$

$$\lambda_1 = 4: \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \underline{r}_1 = \underline{0} \iff \underline{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\lambda_2 = 3: \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{r}_2 = \underline{0} \iff \underline{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2: \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underline{r}_3 = \underline{0} \iff \underline{r}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$$

Við

$$\Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{og} \quad R = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

so er $\underline{\underline{A}} = R \Lambda R^T$.

Projektións matricur eru eitt sindur lættari, tí tær senda minst eina dimensión burtur, so tá er minst eitt eginvirði 0.

Trace

Vit definera nú sporið á eini matricu $\underline{\underline{A}}$. Hetta kallast trace.

$$\begin{aligned} tr(\underline{\underline{A}}) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ &= a_{11} + a_{22} + \cdots + a_{nn} \end{aligned}$$

Til tíðir ber til at brúka $tr(\underline{\underline{A}})$ og $det(\underline{\underline{A}})$ til at avgera eginvirðir!

Fyri 2×2 :
$$\lambda_{1,2} = \frac{tr(\underline{\underline{A}}) \pm \sqrt{tr(\underline{\underline{A}})^2 - 4\, tr(\underline{\underline{A}})}}{2}$$

The Power Method