Spatial Regions and Volume Integrals

Ex 1. Compute the integral.

$$\int_1^2\int_1^2\int_1^2 \frac{xy}{z}\, dx\, dy\, dz = \int_1^2\int_1^2 \frac{y}{z}\left[\frac{1}{2}x^2\right]_1^2 dy\, dz$$

$$= \int_1^2 \frac{1}{z}\int_1^2 2y - \frac{1}{2}y\ dy\ dz = \int_1^2 \frac{1}{z}\ \left[y^2 - \frac{1}{4}y^2\right]_1^2 dz$$

$$= \int_1^2 \frac{9}{4}\ \frac{1}{z}\ dz = \frac{9}{4}\left[\ln(z)\right]_1^2 = \frac{9}{4}\ \ln(2).$$

Ex 2. A region Ω is parametrized by

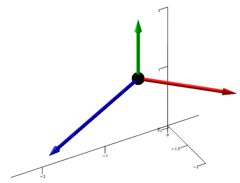
$$r(u,v,w) = \begin{bmatrix} \frac{1}{2}u^2 - v^2 \\ -uv \\ w \end{bmatrix}, \quad u,v,w \in [0,2].$$

a) Given $P = r(1,1,1)$, when we draw $r_u'(P)$, $r_v'(P)$ and $r_w'(P)$ from P, the vectors span a parallelipiped. Compute the volume.

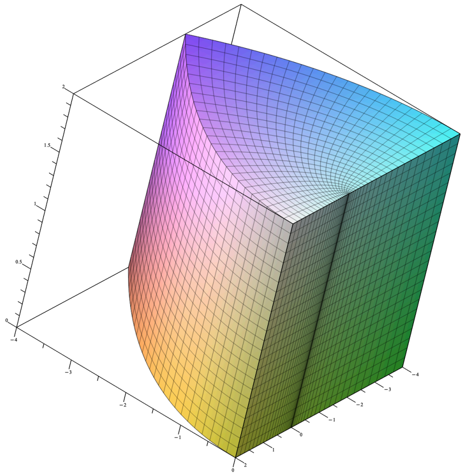
$$P = r(1,1,1) = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$

$$r_u' = \begin{bmatrix} u \\ -v \\ 0 \end{bmatrix}, \quad r_v' = \begin{bmatrix} -2v \\ -u \\ 0 \end{bmatrix}, \quad r_w' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Vol_P = \left|\det\left(\begin{bmatrix} 1 & -2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\right| = |-1-2| = 3.$$



b) Illustrate Ω.



```
u1:=plot3d(r(0,v,w),w=0..2,v=0..2):
u2:=plot3d(r(2,v,w),w=0..2,v=0..2):
v1:=plot3d(r(u,0,w),u=0..2,w=0..2):
v2:=plot3d(r(u,2,w),u=0..2,w=0..2):
w1:=plot3d(r(u,v,0),u=0..2,v=0..2):
w2:=plot3d(r(u,v,2),u=0..2,v=0..2):
display(u1,u2,v1,v2,w1,w2);
```

c) Determine $J_r(u,v,w)$.

By 25.19.

$$J_r(u,v,w) = \left| \det\left(\begin{bmatrix} u & -2v & 0 \\ -v & -u & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right| = u^2 + 2v^2 .$$

d) Compute the volume of Ω.

$$\mathrm{Vol}_\Omega = \int_\Omega 1 \, d\mu = \int_0^2 \int_0^2 \int_0^2 u^2 + 2v^2 \, du \, dv \, dw$$

$$= 2 \int_0^2 \int_0^2 u^2 + 2v^2 \, du \, dv = 2 \int_0^2 \left[\frac{1}{3} u^3 + 2v^2 u \right]_0^2 dv$$

$$= 2 \int_0^2 \frac{8}{3} + 4v^2 \, dv = 2 \left[\frac{8}{3} v + \frac{4}{3} v^3 \right]_0^2$$

$$= 2 \left(\frac{16}{3} + \frac{32}{3} \right) = 2 \cdot 16 = 32 .$$

Ex 3. A profile curve C in the xz-plane is given by

$$s(u) = \begin{bmatrix} \sin(u) \\ \cos(u) \end{bmatrix}, \quad u \in [0, \pi].$$

a) Determine a parametric representation of the surface S obtained by a 2π sweep of C about the z-axis.

In 3D we have

$$s(u) = \begin{bmatrix} \sin(u) \\ 0 \\ \cos(u) \end{bmatrix}, \quad u \in [0, \pi].$$

Apply R_z to $s(u)$ to obtain $r(u,v)$.

$$R_z \, s(u) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(u) \\ 0 \\ \cos(u) \end{bmatrix} = \begin{bmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}$$

where $u \in [0, \pi]$ and $v \in [0, 2\pi]$.

b) Show that points on S satisfy $x^2 + y^2 + z^2 = 1$. Which object is this?

This is the unit sphere (not solid).

$$(\sin u \cos v)^2 + (\sin u \sin v)^2 + \cos^2 u$$

$$= \sin^2 u \cdot \cos^2 v + \sin^2 u \cdot \sin^2 v + \cos^2 u$$

$$= \sin^2 u \cdot (\cos^2 v + \sin^2 v) + \cos^2 u$$

$$= \sin^2 u + \cos^2 u = 1.$$

c) Compute the area of S.

First we can use $\quad 4 \cdot \pi \cdot r^2 = 4\pi.$

Second option is to use the Jacobian.

$$r_u' = \begin{bmatrix} \cos u \cdot \cos v \\ \cos u \cdot \sin v \\ -\sin u \end{bmatrix}, \quad r_v' = \begin{bmatrix} -\sin u \cdot \sin v \\ \sin u \cdot \cos v \\ 0 \end{bmatrix}.$$

$$r_u' \times r_v' = \begin{bmatrix} \sin^2 u \cdot \cos v \\ \sin^2 u \cdot \sin v \\ \cos u \cdot \sin u \end{bmatrix}$$

$$\Rightarrow J_r(u,v) = \sqrt{(\sin^2 u \cdot \cos v)^2 + (\sin^2 u \cdot \sin v)^2 + (\cos u \cdot \sin u)^2}$$

$$= \sqrt{\sin^4 u \cdot (\cos^2 v + \sin^2 v) + \cos^2 u \cdot \sin^2 u}$$

$$= \sqrt{\sin^2 u \cdot (\sin^2 u + \cos^2 u)}$$

$$= \sin u$$

$$\text{Area}_S = \int_0^{2\pi} \int_0^{\pi} \sin u \; du \, dv = 2\pi \cdot \left[-\cos u \right]_0^{\pi}$$

$$= 2\pi \cdot (1 - (-1)) = 4\pi.$$

Given a profile M with parametric representation

$$s(u,v) = \begin{bmatrix} u \cdot \sin v \\ 0 \\ u \cdot \cos v \end{bmatrix}, \quad u \in [0,1], \quad v \in [0,\pi].$$

d) Rotate M by 2π about the z-axis. Which object is this?

This is a rotation of a half disk, so we get a solid unit sphere.

$$r(u,v,w) = \begin{bmatrix} u \cdot \sin v \cdot \cos w \\ u \cdot \sin v \cdot \sin w \\ u \cdot \cos v \end{bmatrix}, \quad u \in [0,1], \; v \in [0,\pi], \; w \in [0,2\pi].$$

e) Determine $\int_K (z+1)\, d\mu$.

$$r_u' = \begin{bmatrix} \sin v \cdot \cos w \\ \sin v \cdot \sin w \\ \cos v \end{bmatrix}, \quad r_v' = \begin{bmatrix} u \cdot \cos v \cdot \cos w \\ u \cdot \cos v \cdot \sin w \\ -u \cdot \sin v \end{bmatrix}, \quad r_w' = \begin{bmatrix} -u \cdot \sin v \cdot \sin w \\ u \cdot \sin v \cdot \cos w \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det([r_u' \ r_v' \ r_w']) &= u^2 \cdot \cos^2 v \cdot \sin v \cdot \cos^2 w \\ &+ u^2 \cdot \sin^3 v \cdot \sin^2 w \\ &+ u^2 \cdot \cos^2 v \cdot \sin v \cdot \sin^2 w \\ &+ u^2 \cdot \sin^3 v \cdot \cos^2 w \\ &= u^2 \cdot (\cos^2 v \cdot \sin v + \sin^3 v) \\ &= u^2 \cdot \sin v \end{aligned}$$

$$\begin{aligned} \int_K z+1 \, d\mu &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (u \cdot \cos v + 1) \cdot u^2 \cdot \sin v \, du\, dv\, dw \\ &= 2\pi \int_0^{\pi}\int_0^1 u^3 \sin v \cdot \cos v + u^2 \sin v \, du\, dv \\ &= 2\pi \int_0^{\pi} \left[\frac{1}{4} u^4 \sin v \cdot \cos v + \frac{1}{3} u^3 \sin v \right]_0^1 dv \\ &= 2\pi \int_0^{\pi} \frac{1}{4} \sin v \cdot \cos v + \frac{1}{3} \sin v \, dv \\ &= 2\pi \cdot \left[\frac{1}{8} \sin^2 v - \frac{1}{3} \cos v \right]_0^{\pi} \\ &= 2\pi \left(-\frac{1}{3} \cdot (-1) - \left(-\frac{1}{3} \cdot 1\right)\right) \\ &= 2\pi \cdot \frac{2}{3} = \frac{4}{3}\pi. \end{aligned}$$

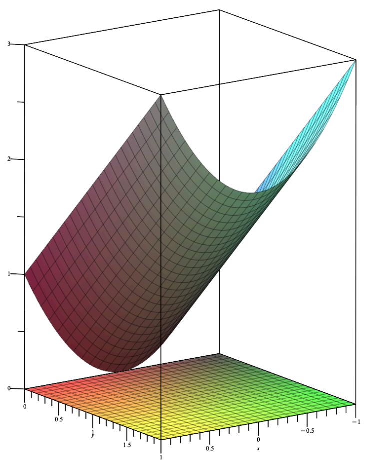
Ex 4. Let $h(x,y) = x^2 + y$.

a) Given

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x \in [-1,1] \text{ and } y \in (0,2] \}$$

we consider the space B between A and h. State a parametric representation of B and plot both A and h.

$$r(u,v,w) = \begin{bmatrix} u \\ v \\ w(u^2+v) \end{bmatrix}, \quad u \in [-1,1],\ v \in [0,2],\ w \in [0,1].$$



```
> h:=(x,y)-> x^2+y;
```

$h := (x,y) \mapsto x^2 + y$

```
> plot3d({h(x,y),<x,y,0>},x=-1..1,y=0..2,scaling=constrained);
```

b) Determine Jacobian and compute $\int_B x^2 - y \, d\mu$.

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2uw & vw & u^2+v \end{bmatrix} \right) = u^2 + v$$

$$\int_B x^2 - y \, d\mu = \int_0^1 \int_0^2 \int_{-1}^1 (u^2 - v) \cdot (u^2 + v) \, du \, dv \, dw$$

$$= \int_0^2 \int_{-1}^1 u^4 - v^2 \, du \, dv = \int_0^2 \left[\frac{1}{5} u^5 - v^2 \cdot u \right]_{-1}^1 dv$$

$$= \int_0^2 \frac{2}{5} - 2v^2 \, dv = \left[\frac{2}{5} v - \frac{2}{3} v^3 \right]_0^2$$

$$= \frac{4}{5} - \frac{16}{3} = \frac{12}{15} - \frac{80}{15} = -\frac{68}{15}.$$

c) Let C be the unit disk after translation by $(1,0)$. Compute the volume D between C and h.

First C has coordinates $\begin{bmatrix} u \cdot \cos v + 1 \\ u \cdot \sin v \end{bmatrix}$, $u \in [0,1]$, $v \in [0, \frac{\pi}{2}]$.

Then
$$r(u,v,w) = \begin{bmatrix} u \cdot \cos v + 1 \\ u \cdot \sin v \\ w \cdot ((u \cdot \cos v + 1)^2 + u \cdot \sin v) \end{bmatrix}, \quad w \in [0,1].$$

```
> r:=(u,v,w)->&lt;x(u,v),y(u,v),w*h(x(u,v),y(u,v))&gt;: r(u,v,w);
```

$$\begin{bmatrix} u\cos(v)+1 \\ u\sin(v) \\ w\left((u\cos(v)+1)^2 + u\sin(v)\right) \end{bmatrix}$$

```
> J:=VectorCalculus[Jacobian](r(u,v,w),[u,v,w]);
```

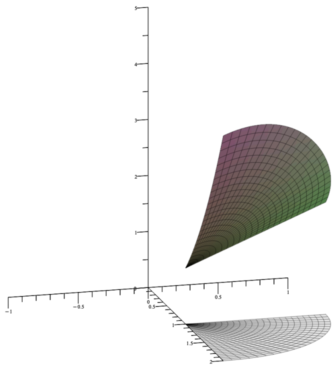
$$J := \begin{bmatrix} \cos(v) & -u\sin(v) & 0 \\ \sin(v) & u\cos(v) & 0 \\ w\left(2(u\cos(v)+1)\cos(v)+\sin(v)\right) & w\left(-2(u\cos(v)+1)u\sin(v)+u\cos(v)\right) & (u\cos(v)+1)^2+u\sin(v) \end{bmatrix}$$

```
> Determinant(J):
  simplify(%);
```

$$(1+\cos(v)^2 u^2 + (2\cos(v)+\sin(v))u)u$$

```
> int(Determinant(J),u=0..1,v=0..Pi/2,w=0..1);
```

$$1 + \frac{5\pi}{16}$$



Ex 5. a) Parametrize the triangle with vertices

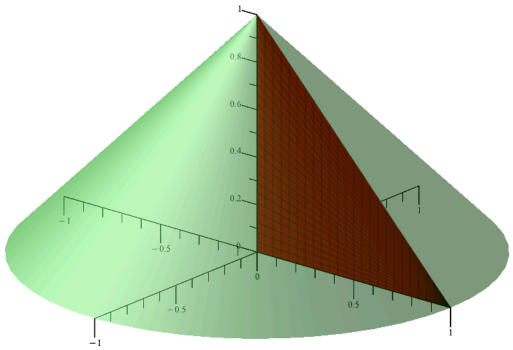
$$(0,0,0),\ (1,0,0),\ (0,0,1).$$

Set

$$s(u,v) = \begin{bmatrix} u \\ 0 \\ v(1-u) \end{bmatrix}, \quad u, v \in [0,1].$$

b) Parametrize the solid surface of revolution Ω.

This produces a cone



$$r(u,v,w) = \begin{bmatrix} u \cdot \cos w \\ u \cdot \sin w \\ v(1-u) \end{bmatrix}$$

with $w \in [0, 2\pi]$.

c) Compute the volume.

```
> VectorCalculus[Jacobian](r(u,v,w),[u,v,w]);
```

$$\begin{bmatrix} \cos(w) & 0 & -\sin(w)\,u \\ \sin(w) & 0 & \cos(w)\,u \\ -v & 1-u & 0 \end{bmatrix}$$

```
> Determinant(%);
  simplify(%);
```

$$\cos(w)^2 u^2 + \sin(w)^2 u^2 - \cos(w)^2 u - \sin(w)^2 u$$

$$u\,(u-1)$$

The above is negative for this interval, so let $J_r(u,v,w) = -u(u-1)$.

$$\int_\Omega 1 \, d\mu = \int_0^{2\pi}\int_0^1\int_0^1 -u^2 + u \; du\, dv\, dw$$

$$= 2\pi \cdot \left[-\frac{1}{3}u^3 + \frac{1}{2}u^2 \right]_0^1 = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}.$$

Ex 6. A sphere F is given by

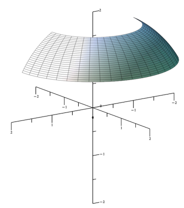
$$r(u,v) = \begin{bmatrix} R \cdot \sin u \cdot \cos v \\ R \cdot \sin u \cdot \sin v \\ R \cdot \cos u \end{bmatrix}, \quad u \in [a,b], \ v \in [c,d],$$

where $R \geq 0$, $0 \leq a \leq b \leq \pi$ and $0 \leq c \leq d \leq 2\pi$.

a) What do the parameters mean?

R is the radius of the sphere. The rang of u specifies the range of the shell starting from the positive end of the z-axis to the negative. Essentially how much of a circle is used when revolving. The range of v specifies how large an angle we revolve around the z-axis.

```
> R:=2:
  a:=Pi/6:
  b:=Pi/3:
  c:=0:
  d:=Pi:
> plot3d(r(u,v),u=a..b,v=c..d,scaling=constrained,view=[-2..2,-2..2,-2..2],axes=normal);
```



b) Compute the surface area of F.

```
> R:='R':a:='a':b:='b':c:='c':d:='d':
> ru:=diff~(r(u,v),u):
  rv:=diff~(r(u,v),v):
  kryds(ru,rv):
  N:=simplify(%);
```

$$N := \begin{bmatrix} R^2 \sin(u)^2 \cos(v) \\ R^2 \sin(u)^2 \sin(v) \\ R^2 \cos(u) \sin(u) \end{bmatrix}$$

```
> Norm(N,2);
  Jacobi:=simplify(%)assuming u>0,v>0,u<Pi,R::real;
```

$$\sqrt{|R|^4 |\sin(u)|^4 |\cos(v)|^2 + |R|^4 |\sin(u)|^4 |\sin(v)|^2 + |R|^4 |\cos(u) \sin(u)|^2}$$

$$Jacobi := \sin(u) R^2$$

```
> int(Jacobi,u=a..b,v=c..d);
```

$$R^2 (\cos(a) - \cos(b)) (d - c)$$

Ex 7.

$$r(u,v,w) = \begin{bmatrix} u \cdot \sin v \cdot \cos w \\ u \cdot \sin v \cdot \sin w \\ u \cdot \cos v \end{bmatrix}, \quad u \in [a,b],\ v \in [c,d],\ w \in [e,f].$$

a) Discuss the meaning of the parameters.

See ex. 6 (a)

b) What can be done to get a solid sphere?

Just set $a = 0$.

Let A be the region determined by the following choice of values:

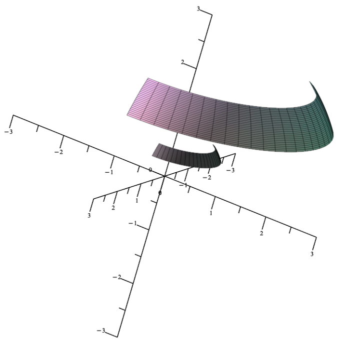
$$a = 1,\ b = 3,\ c = \frac{\pi}{4},\ d = \frac{\pi}{3},\ e = 0,\ f = \frac{3\pi}{4}$$

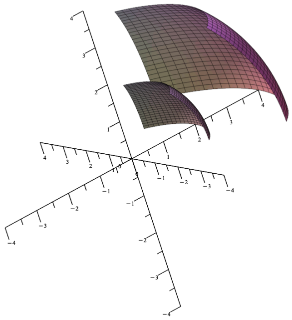
and B the region determined by the choice:

$$a = 2,\ b = 4,\ c = \frac{\pi}{4},\ d = \frac{\pi}{2},\ e = -\frac{\pi}{4},\ f = \frac{\pi}{4}.$$

c) Describe A, B and $A \cap B$. Compute the volumes.

A and B are the volume between these two surfaces:

A : 

B : 

Note $A \cap B$ is of course the overlap of the two above.

For the volumes just adjust according to the given volume.

```
> a:=1:
  b:=3:
  c:=Pi/4:
  d:=Pi/3:
  e:=0:
  f:=3*Pi/4:
> r:=(u,v,w)-> <u*sin(v)*cos(w),u*sin(v)*sin(w),u*cos(v)>: r(u,v,w);
```

$$\begin{bmatrix} u\sin(v)\cos(w) \\ u\sin(v)\sin(w) \\ u\cos(v) \end{bmatrix}$$

```
> VectorCalculus[Jacobian](r(u,v,w),[u,v,w]);
  Determinant(%):
  Jacobi:=simplify(%);
```

$$\begin{bmatrix} \sin(v)\cos(w) & u\cos(v)\cos(w) & -u\sin(v)\sin(w) \\ \sin(v)\sin(w) & u\cos(v)\sin(w) & u\sin(v)\cos(w) \\ \cos(v) & -u\sin(v) & 0 \end{bmatrix}$$

$$Jacobi := u^2\sin(v)$$

```
> int(Jacobi,u=a..b,v=c..d,w=e..f):
  simplify(%);
```

$$\frac{13\left(\sqrt{2}-1\right)\pi}{4}$$

$$Vol_A = \frac{13(\sqrt{2}-1)\pi}{4}$$

$$Vol_B = \frac{14\sqrt{2}\,\pi}{3}$$

$$Vol_{A\cap B} = \frac{19(\sqrt{2}-1)\pi}{24}$$

d) State $Vol_{A\cup B}$.

This amounts to $Vol_A + Vol_B - Vol_{A\cap B} = \dfrac{57\sqrt{2}\,\pi}{8} - \dfrac{59\pi}{24}$.

e) Compute $\int_A x\, d\Omega$, $\int_B x\, d\Omega$ and $\int_{A\cap B} x\, d\Omega$.

```
> int((u*sin(v)*cos(w))*Jacobi,u=a..b,v=c..d,w=e..f):
  simplify(%);
```

$$\frac{5\left(6+\pi-3\sqrt{3}\right)\sqrt{2}}{12}$$

```
> int((u*sin(v)*cos(w))*Jacobi,u=a..b,v=c..d,w=e..f):
  simplify(%);
```

$$\frac{15\left(2+\pi\right)\sqrt{2}}{2}$$

```
> int((u*sin(v)*cos(w))*Jacobi,u=a..b,v=c..d,w=e..f):
  simplify(%);
```

$$\frac{65\left(6+\pi-3\sqrt{3}\right)\sqrt{2}}{192}$$