Symmetric Matrices

Ex1. In R^5 we have $\underline{a} = (-2, 0, 2, 2, -2)$ and $\underline{b} = (1, 2, -1, -1, 1)$.

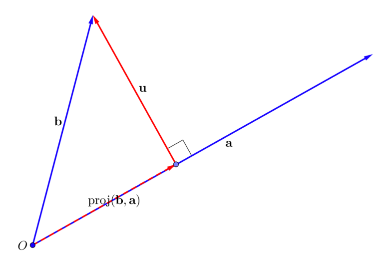
a) Determine the length of $\underline{a}$ and $\underline{b}$ as well as the angle between them.

$$|\underline{a}| = \sqrt{(-2)^2 + 0^2 + 2^2 + 2^2 + (-2)^2} = \sqrt{16} = 4.$$

$$|\underline{b}| = \sqrt{1^2 + 2^2 + (-1)^2 + (-1)^2 + 1^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\cos(v) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{-8}{4 \cdot 2\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow v = \frac{3\pi}{4}.$$

b) Determine $\underline{u} = \underline{b} - \text{proj}(\underline{b}, \underline{a})$ and show that $\underline{u} \perp \underline{a}$.



$$\text{proj}(\underline{b}, \underline{a}) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|^2}\,\underline{a} = \frac{-8}{16}\,\underline{a}$$

$$= -\frac{1}{2}\,\underline{a} = (-1, 0, 1, 1, -1)$$

$$\underline{u} = \underline{b} - \text{proj}(\underline{b}, \underline{a}) = (0, 2, 0, 0, 0)$$

It follows from $\underline{a} \cdot \underline{u} = 0$ that $\underline{a} \perp \underline{u}$.

Ex2.

a) Do the vectors $\underline{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$ constitute a ONB for $\mathbb{R}^3$?

The vectors are all orthogonal, since $\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_1 \cdot \underline{v}_3 = \underline{v}_2 \cdot \underline{v}_3 = 0$.

We need only check $|\underline{v}_i| = 1$ for one $i \in \{1, 2, 3\}$ as the coordinates have the same values.

$$|\underline{v}_1| = \sqrt{\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2} = \sqrt{\tfrac{1}{9} + \tfrac{4}{9} + \tfrac{4}{9}} = \sqrt{1} = 1.$$

As we have 3 lin. indept. and indeed ortonormal vectors, then these constitute an ONB for $\mathbb{R}^3$.

b) Create an ONB for $\mathbb{R}^3$, which includes $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$.

The vector clearly has length equal to 1 as we recognise the coordinates for the $\frac{\pi}{4}$ rad point on the unit circle in the plane. We complete the ONB with

$$\left(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right), (0,0,1)\right).$$

Ex3. Determine solutions to $x_1 + x_2 + x_3 = 0$, and explain that this is a subspace of $\mathbb{R}^3$. Provide an ONB for the solution space.

Let $x_2 = t_2$ and $x_3 = t_3$ be free variables. Then any solution is of the type

$$\underline{x} = t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t_2, t_3 \in \mathbb{R}.$$

The set of solutions corresponds to the subspace

$$\operatorname{span}\{(-1,1,0), (-1,0,1)\}.$$

We can construct an ONB starting with $\underline{u}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Using Gram-Schmidt we have

$$\underline{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \underline{u}_1\right) \underline{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \underline{u}_2 = \frac{\underline{w}_2}{|\underline{w}_2|} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{\sqrt{6}}{3} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

The vectors $(\underline{u}_1, \underline{u}_2)$ are an ONB for the solution space.

Ex 4. Are the matrices orthogonal?

$\underline{\underline{A}} = \begin{bmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$ is not orthogonal, since the vectors are not orthonormal: $\sqrt{(1/2)^2 + (1/2)^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \neq 1.$

$\underline{\underline{B}} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ is not orthogonal, by the argument above.

$\underline{\underline{C}} = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ Observe that

$$\underline{\underline{C}}\,\underline{\underline{C}}^T = \frac{1}{25} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By proposition 15.15 $\underline{\underline{C}}$ is an orthogonal matrix.

$\underline{\underline{D}} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ We try the same argument again.

$$\underline{\underline{D}}\,\underline{\underline{D}}^T = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

By proposition 15.15 $\underline{\underline{D}}$ is an orthogonal matrix.

Ex 5. Let
$$\underline{\underline{A}} = \begin{bmatrix} 0 & -a & 0 & a \\ a & 0 & a & 0 \\ 0 & -a & 0 & -a \\ -a & 0 & a & 0 \end{bmatrix}.$$

a) Determine for which a the matrix $\underline{\underline{A}}$ is orthogonal.

$$\underline{\underline{A}}\,\underline{\underline{A}}^T = \begin{bmatrix} 0 & -a & 0 & a \\ a & 0 & a & 0 \\ 0 & -a & 0 & -a \\ -a & 0 & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & 0 & -a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ a & 0 & -a & 0 \end{bmatrix} = \begin{bmatrix} 2a^2 & 0 & 0 & 0 \\ 0 & 2a^2 & 0 & 0 \\ 0 & 0 & 2a^2 & 0 \\ 0 & 0 & 0 & 2a^2 \end{bmatrix}.$$

It follows that $\underline{\underline{A}}$ is orthogonal if and only if
$$2a^2 = 1 \iff a = \pm \frac{1}{\sqrt{2}}.$$

b) Determine the values of a for which $\underline{\underline{A}}$ is special-orthogonal ($\det \underline{\underline{A}} = 1$).

$$\det(\underline{\underline{A}}) = -a \cdot \det\left(\begin{bmatrix} -a & 0 & a \\ -a & 0 & -a \\ 0 & a & 0 \end{bmatrix}\right) - a \cdot \det\left(\begin{bmatrix} -a & 0 & a \\ 0 & a & 0 \\ -a & 0 & -a \end{bmatrix}\right)$$
$$= -a(-a^3 - a^3) - (-a) \cdot (a^3 + a^3)$$
$$= 2a^4 + 2a^4 = 4a^4.$$

Thus $\underline{\underline{A}}$ is special-orthogonal for $a = \pm \frac{1}{\sqrt{2}}$ as then
$$4 \cdot \left(\pm \frac{1}{\sqrt{2}}\right)^4 = 4 \cdot \frac{1}{\sqrt{2}^4} = 4 \cdot \frac{1}{4} = 1.$$

Ex 6. Let $\underline{\underline{A}} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ with $a, b, c \in \mathbb{R}$. Note that $\underline{\underline{A}}$ is symmetric.

a) Show that $\underline{\underline{A}}$ has real eigenvalues.

This is always the case when symmetric.
$$P(\lambda) = (a - \lambda)(b - \lambda) - c^2 = \lambda^2 + (-a - b)\lambda + ab - c^2.$$

Then we have

$$d = (-a-b)^2 - 4 \cdot (ab - c^2)$$
$$= a^2 + 2ab + b^2 - 4ab + 4c^2$$
$$= (a-b)^2 + 4c^2 \geq 0,$$

so that $P(\lambda)$ has 2 real roots with multiplicity. Thus a symmetric matrix has real eigenvalues.

b) Show that if $\underline{\underline{A}}$ is not a diagonal matrix, then there are two distinct eigenvalues.

Since $d = (a-b)^2 + 4c^2$ only evaluates to zero when $c = 0$ and $a = b$, then there are distinct eigenvalues when $c \neq 0$.

Ex 7. Given $\underline{\underline{A}} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$ find an orthogonal matrix $\underline{\underline{Q}}$ such that

$$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \Lambda .$$

First we find the eigenvectors, which can diagonalize $\underline{\underline{A}}$. Then we create an ONB from them to get $\underline{\underline{Q}}$.

$$P(\lambda) = (-2-\lambda)^3 - 1 - 1 - (-2-\lambda) - (-2-\lambda) - (-2-\lambda)$$
$$= -\lambda^3 - 6\lambda^2 - 12\lambda - 8 - 2 + 6 + 3\lambda$$
$$= -\lambda^3 - 6\lambda^2 - 9\lambda - 4$$
$$= (1+\lambda)^2 (-4-\lambda)$$

```
> Eigenvectors(A,output=list);
```

$$\left[\left[-1, 2, \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}\right], \left[-4, 1, \left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}\right]\right]$$

Now let $\underline{u}_1 = (1,0,1)$, $\underline{u}_2 = (1,1,0)$ and $\underline{u}_3 = (-1,1,1)$.

We set $\underline{v}_1 = \dfrac{\underline{u}_1}{|\underline{u}_1|} = \dfrac{1}{\sqrt{2}}\,\underline{u}_1$. Then compute

$$\underline{w}_2 = \underline{u}_2 - (\underline{u}_2 \cdot \underline{v}_1)\,\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\Rightarrow \underline{v}_2 = \frac{\sqrt{6}}{3} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\underline{w}_3 = \underline{u}_3 - (\underline{u}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{u}_3 \cdot \underline{v}_2)\,\underline{v}_2$$
$$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - 0 - 0 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Let $\underline{\underline{Q}}$ be given by $\begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 & -\sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \end{bmatrix}$, then we have $\underline{\underline{\Lambda}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ where $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}}$.

Ex8. We are given $\underline{\underline{A}} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & -2 \end{bmatrix}$ and $\underline{\underline{B}} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

a) Show that $\underline{\underline{A}}$ has the eigenvalue -4 as well as another with am. of 2.

```
> A:=<1,-1,2; -1,1,2; 2,2,-2>:
  B:=<2,-1,-1; -1,2,-1; -1,-1,2>:
> Determinant(A-s*IdentityMatrix(3)):
  solve(%=0,s);
```

$$-4, 2, 2$$

b) Determine the eigenvector corresponding to -4.

```
> LinearSolve(A-(-4)*IdentityMatrix(3),<0,0,0>,free=t);
```

$$\begin{bmatrix} t_2 \\ t_2 \\ -2\,t_2 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ will do.

c) Find eigenvalues for $\underline{\underline{B}}$.

```
> Eigenvectors(B,output=list);
```

$$\left[\left[0, 1, \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right], \left[3, 2, \left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}\right]\right]$$

d) Find an eigenvector corresponding to the root 0.

The root has am(0) = 1, and an eigenvector is $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

e) Show the eigenvectors of b) and d) are orthogonal.

$$\begin{bmatrix}1\\1\\-2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 1+1-2 = 0 \Rightarrow \underline{v}_1 \perp \underline{v}_2 .$$

f) Determine an orthogonal matrix Q that diagonalizes both $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

$$\underline{v}_3 = \begin{bmatrix}1\\1\\-2\end{bmatrix} \times \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\-3\\0\end{bmatrix}$$ so we normalize.

$$\frac{\underline{v}_1}{|\underline{v}_1|} = \frac{\sqrt{6}}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix}, \quad \frac{v_2}{|\underline{v}_2|} = \frac{\sqrt{3}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \frac{v_3}{|v_3|} = \frac{1}{\sqrt{18}}\begin{bmatrix}3\\-3\\0\end{bmatrix}.$$

So let $\underline{\underline{Q}}$ be given by

$$\begin{bmatrix}\sqrt{6}/6 & \sqrt{3}/3 & \sqrt{2}/2\\ \sqrt{6}/6 & \sqrt{3}/3 & -\sqrt{2}/2\\ -\sqrt{6}/3 & \sqrt{3}/3 & 0\end{bmatrix}.$$

Now we have

$$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \begin{bmatrix}-4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} \text{ and } \underline{\underline{Q}}^T \underline{\underline{B}}\, \underline{\underline{Q}} = \begin{bmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

Ex 9. Let $\underline{v}_1 = (1,1,1,1)$, $\underline{v}_2 = (3,1,1,3)$, $\underline{v}_3 = (2,0,-2,4)$ and $\underline{v}_4 = (1,1,-1,3)$.

A subspace $\mathcal{U} \subseteq \mathbb{R}$ is determined by $\mathcal{U} = \operatorname{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$.

a) Show $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is a basis for $\mathcal{U}$, and find a lin. comb. for $\underline{v}_4$.

$$\begin{array}{c}\begin{bmatrix} 1 & 3 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 4 & 3 \end{bmatrix}\end{array} \begin{array}{l} \\ -R_1 \\ -R_1 \\ -R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{array}{l} \\ \cdot\left(-\frac{1}{2}\right) \\ +R_2 \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{array}{l} -3R_2 \\ \\ \cdot\left(-\frac{1}{2}\right) \\ +R_3 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} +R_3 \\ -R_3 \\ \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system has 3 lin. indept. vectors, which contite a basis for $\mathcal{U}$, i.e. $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$. The vector $\underline{v}_4$ is given by

$$\underline{v}_4 = 2\underline{v}_1 - \underline{v}_2 + \underline{v}_3 .$$

b) State an ONB for $\mathcal{U}$.

Just some Gram-Schmidt again. Let's try with Maple.

```
> v1:=<1,1,1,1>:
  v2:=<3,1,1,3>:
  v3:=<2,0,-2,4>:
> GramSchmidt([v1,v2,v3],normalized);
```

$$\left[\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right]$$

c) Determine the orthogonal complement in $\mathbb{R}^4$ to $\mathcal{U}$.

We need a vector that is orthogonal wrt. the ONB found in b).

The vector $(-1,-1,1,1)$ is sufficient, so

$$\mathcal{U}^{\perp} = \operatorname{span}\{(-1,-1,1,1)\}.$$