Diagonalization by Similarity Transformation

Ex 1.

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 4 & 4 \\ 6 & 6 & 6 \\ -6 & -7 & -7 \end{bmatrix} \text{ defines } f: \mathbb{R}^3 \to \mathbb{R}^3.$$

a) Easiest way to check wether $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

are eigenvectors of f? Show this.

$$\underline{\underline{A}}\,\underline{v}_1 = \begin{bmatrix} 3-4 \\ 6-6 \\ -6+7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 \cdot \underline{v}_1 \quad , \text{ so } \lambda_1 = -1.$$

$$\underline{\underline{A}}\,\underline{v}_2 = \begin{bmatrix} 4-4 \\ 6-6 \\ -7+7 \end{bmatrix} = \underline{0} = 0 \cdot \underline{v}_2 \quad , \text{ so } \lambda_2 = 0.$$

$$\underline{\underline{A}}\,\underline{v}_3 = \begin{bmatrix} 3+8-8 \\ 6+12-12 \\ -6-14+14 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix} = 3 \cdot \underline{v}_3, \quad \text{so } \lambda_3 = 3.$$

b) How can we easiest argue that $\underline{v}_1, \underline{v}_2$ and $\underline{v}_3$ are lin. indept.?

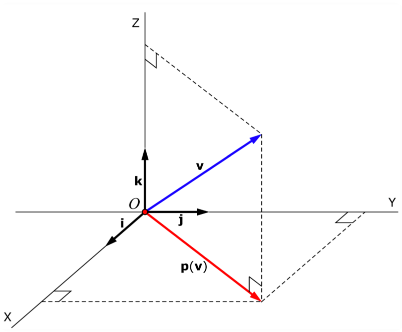
The eigenvalues are distinct, so they are lin. indept. by proposition 13.11.1.

c) How can we easiest show that $f(\mathbb{R}^3) = \text{span}\{\underline{v}_1, \underline{v}_3\}$?

The eigenvectors form an eigenbasis, and $f(\mathbb{R}^3)$ is spanned by the image of the basis vectors.

$$f(\mathbb{R}^3) = \text{span}\{f(\underline{v}_1), f(\underline{v}_2), f(\underline{v}_3)\} = \text{span}\{\underline{v}_1, \underline{v}_3\}.$$

Ex2.



a) Determine the eigenvalues and spaces of the projection p onto XY.

We have $\lambda_1 = 1$ with $am(1) = gm(1) = 2$, while $\lambda_2 = 0$ with $am(0) = gm(0) = 1$.

$$E_1 = \text{span}\{(1,0,0),(0,1,0)\},$$

$$E_0 = \text{span}\{(0,0,1)\}.$$

b) Choose two different eigenbases, and determine the diagonal matrix for p in the new bases.

If we use the standard basis, then p has the matrix

$$\underline{\underline{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any lin. indept. vectors in the XY-plane will do for $\lambda_1 = 1$, and say $-\underline{k}$ for $\lambda_2 = 0$. For vectors $(1,1,0)$, $(1,-1,0)$ and $(0,0,-1)$ we also receive the diagonal matrix $\underline{\underline{P}}$.

Ex3. The following output is given about a map f with matrix $\underline{\underline{A}}$.

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A:=<16,18,-24|-13,-15,24|-2,-2,4>:
Eigenvectors(A,output=list)
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$$\left[\left[4, 1, \left\{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right\}\right], \left[3, 1, \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}\right], \left[-2, 1, \left\{\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}\right]\right]$$

a) State $\underline{\underline{A}}$.

$$\underline{\underline{A}} = \begin{bmatrix} 16 & -13 & -2 \\ 18 & -15 & -2 \\ -24 & 24 & 4 \end{bmatrix}.$$

b) State eigen information of $\underline{\underline{A}}$.

$$\lambda_1 = 4, \quad am(4) = gm(4) = 1 \quad \text{and} \quad \underline{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = 3, \quad am(3) = gm(3) = 1 \quad \text{and} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_3 = -2, \quad am(-2) = gm(-2) = 1 \quad \text{and} \quad \underline{v}_3 = \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}.$$

c) Provide a basis of eigenvectors of f.

Using b) we have $v = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$.

d) Determine a mapping matrix for f wrt. the basis v.

Let $\underline{\underline{V}} = [\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3]$, then the new matrix $\underline{\underline{\Lambda}}$ is given by

$$\underline{\underline{\Lambda}} = \underline{\underline{V}}^{-1} \underline{\underline{A}} \, \underline{\underline{V}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

e) State $\underline{\underline{V}}$ and $\underline{\underline{\Lambda}}$ such that $\underline{\underline{\Lambda}} = \underline{\underline{V}}^{-1} \underline{\underline{A}} \, \underline{\underline{V}}$.

This is done in d).

Ex4. Given $\underline{\underline{A}} = \begin{bmatrix} 9 & -6 \\ 8 & -7 \end{bmatrix}$ investigate whether we may diagonalize $\underline{\underline{A}}$.

a) If possible then do so.

$$P(\lambda) = 0 \iff (9-\lambda)(-7-\lambda) + 48 = 0$$

$$\iff \lambda^2 - 2\lambda - 15 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2} = \begin{cases} 5 \\ -3 \end{cases}$$

Associated eigenvectors are then solved for.

$$\lambda = 5: \begin{bmatrix} 4 & -6 \\ 8 & -12 \end{bmatrix} \underline{u} = \underline{0} \iff \underline{u} = t \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$$\lambda = -3: \begin{bmatrix} 12 & -6 \\ 8 & -4 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Let $\underline{\underline{V}} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, then $\underline{\underline{V}}^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

Now we have $\underline{\underline{\Lambda}} = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ as expected by the following.

$$\underline{\underline{V}}^{-1} \underline{\underline{A}} \, \underline{\underline{V}} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 9 & -6 \\ 8 & -7 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 15 & -3 \\ 10 & -6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 \\ 0 & -12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}.$$

b) Do the same for $\underline{\underline{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

$$P(\lambda) = (2-\lambda)(1-\lambda)^2 - (2-\lambda) = (2-\lambda)\left((1-\lambda)^2 - 1\right)$$

$$= (2-\lambda)(\lambda^2 - 2\lambda) = (2-\lambda)\lambda(\lambda-2) = -\lambda(2-\lambda)^2$$

$$\Rightarrow \quad P(\lambda) = 0 \iff \lambda = 2 \vee \lambda = 0$$

Note $am(2) = 2$.

$$\lambda = 0: \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$$\lambda = 2: \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \underline{u} = \underline{0} \iff \underline{u} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \vee \underline{u} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then we can diagonalize with

$$\underline{\underline{V}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \text{ and we have } \underline{\underline{\Lambda}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) Do the same for $\underline{\underline{C}} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$P(\lambda) = (3-\lambda)^2(2-\lambda) - (2-\lambda) = (2-\lambda)(\lambda^2 - 6\lambda + 9 - 1)$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 8) = (2-\lambda)(2-\lambda)(4-\lambda)$$

$$\Rightarrow \quad P(\lambda) = 0 \iff \lambda = 2 \vee \lambda = 4.$$

Again $am(2) = 2$.

$$\lambda = 2: \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \iff \underline{v} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

However, now $gm(2) = 1 \neq 2$, so there is no diagonalization of $\underline{\underline{C}}$.

Ex5. Given $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\underline{\underline{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

a) Show $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are similar.

For both we get $P(\lambda) = \lambda^2 + 1$ which has roots $\lambda = \pm i$.

$$\lambda_A = i: \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \underline{u} = \underline{0} \iff \underline{u}_A = t\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

This implies for $\lambda_A = -i$ we get $\underline{v}_A = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

Now onwards with $\underset{=}{B}$, and we get the mirrored relation

$$\lambda = i \Rightarrow \underline{u}_B = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = -i \Rightarrow \underline{v}_B = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

up to scalar multiples.

Then with $\underset{=}{V} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and $\underset{=}{U} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ we have similarity.

$$\underset{=}{\Lambda} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \underset{=}{V}^{-1} \underset{=}{A} \underset{=}{V} = \underset{=}{U}^{-1} \underset{=}{B} \underset{=}{U}.$$

b) Find $\underset{=}{M}$ such that $\underset{=}{B} = \underset{=}{M}^{-1} \underset{=}{A} \underset{=}{M}$.

$$\underset{=}{U}^{-1} \underset{=}{B} \underset{=}{U} = \underset{=}{V}^{-1} \underset{=}{A} \underset{=}{V} \Leftrightarrow \underset{=}{B} = \underset{=}{U} \underset{=}{V}^{-1} \underset{=}{A} \underset{=}{V} \underset{=}{U}^{-1}$$

$$= \underset{=}{M}^{-1} \underset{=}{A} \underset{=}{M}$$

Here $\underset{=}{M} = \underset{=}{V} \underset{=}{U}^{-1}$ and $\underset{=}{M}^{-1} = (\underset{=}{V} \underset{=}{U}^{-1})^{-1}$.

$$\underset{=}{M} = \underset{=}{V} \underset{=}{U}^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \left(\frac{-1}{2i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \right)$$

$$= -\frac{1}{2i} \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

c) Now consider $\underset{=}{A}$ as the matrix for $f: \mathbb{R}^2 \to \mathbb{R}^2$. Determine a basis m of $\mathbb{R}^2$ such that f is represented by $\underset{=}{B}$.

The basis is given by the matrix $\underset{=}{M}$, so $m = ((-1, 0), (0, 1))$.

Ex 6. Given

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 5 & 1 & 1 \end{bmatrix}.$$

a) Compute eigenvalues and spaces.

$$P(\lambda) = (3-\lambda)(1-\lambda)^2 + (3-\lambda) = (3-\lambda)(\lambda^2 - 2\lambda + 2)$$

$\lambda_1 = 3$.

$$\lambda^2 - 2\lambda + 2 = 0 \iff \lambda = \frac{2 \pm 2i}{2} = \begin{cases} 1+i \\ 1-i \end{cases}$$

For λ_1 we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 5 & 1 & -2 \end{bmatrix} \underline{v}_1 = \underline{0} \iff \underline{v}_1 = t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$$\lambda = 1-i: \begin{bmatrix} 2+i & 0 & 0 \\ 0 & i & -1 \\ 5 & 1 & i \end{bmatrix} \underline{v}_2 = \underline{0} \iff \underline{v}_2 = t \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \quad t \in \mathbb{R}.$$

It follows that $\underline{v}_3 = t \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}, \quad t \in \mathbb{R}.$

Now we have $E_3 = \operatorname{span}\{\underline{v}_1\}$, $E_{1-i} = \operatorname{span}\{\underline{v}_2\}$ and $E_{1+i} = \operatorname{span}\{\underline{v}_3\}$.

b) Diagonalize $\underline{\underline{A}}$.

Let $\underline{\underline{V}}$ be the matrix $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & i & -i \end{bmatrix}$, then $\underline{\underline{\Lambda}}$ is the diagonal matrix

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1+i \end{bmatrix} = \underline{\underline{V}}^{-1} \underline{\underline{A}}\, \underline{\underline{V}}$$

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> A:=<3,0,0; 0,1,-1; 5,1,1>;
  V:=<1,-1,2|0,1,I|0,1,-I>;
  'Lambda'=MatrixInverse(V).C.V;
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$$A := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 5 & 1 & 1 \end{bmatrix}$$

$$V := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & I & -I \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1-I & 0 \\ 0 & 0 & 1+I \end{bmatrix}$$