IMO SL 2017A6

Let R be a ring, S be an abelian (additive) group, and $\iota : S \to R$ be a group homomorphism. Find all functions $f : R \to S$ such that for any $x, y \in R$,

$$f(\iota(f(x))\iota(f(y))) + f(x+y) = f(xy). \tag{*}$$

We say that $f : R \to S$ is *ι-good* if f satisfies the above functional equation. The main goal is actually to find only id_R-good functions. However, the given framework makes the problem slightly more tractable, even in the $\iota = \mathrm{id}_R$ case.

We say that f is *non-periodic ι-good* if f is ι-good and f has no non-zero period. We say that f is *reduced ι-good* if f is non-periodic ι-good and $\iota f(0) = 1$. To classify ι-good functions, we reduce to non-periodic and reduced cases.

1 Basic results

Here, we collect basic results regarding ι-good functions, mainly those obtained by plug-and-chug. By plugging $x = y = 0$ into (*), we get $f(\iota f(0)^2) = 0$. By plugging $y = 1$ into (*) and comparing, we get that $f(a) = f(b)$ implies $f(a+1) = f(b+1)$. By induction, we get that for any $n \in \mathbb{N}$ and $a, b \in R$,

$$f(a) = f(b) \implies f(a+n) = f(b+n).$$

Plugging $y = n$ into (*), we also get that $f(a) = f(b)$ implies $f(na) = f(nb)$ for any $n \in \mathbb{N}$. We now show:

Theorem 1.1. *For any $c \in R$ such that $f(c) = 0$, we have $f(x+c) = f(x+1)$ for any $x \in R$. In particular, $f(1) = 0$, and if f is non-periodic, then $f(c) = 0 \iff c = 1$.*

Proof. We first show that $f(c^2) = f(c^3) = 0$. First, for any $d \in R$ such that $f(d) = 0$, plugging $y = d$ into (*) yields

$$f(0) + f(x+d) = f(xd) \quad \forall x \in R.$$

Now we have $f(c^2) = f(0) + f(2c)$ and $f(c^3) = f(0) + f(c^2 + c) = 2f(0) + f(2c+1)$. Thus our goal reduces to showing that $f(2c) = -f(0)$ and $f(2c+1) = -2f(0)$. Since $f(c+1) = -f(0)$, the former yields $f(2c) = f(c+1)$, and so

$$f(2c+1) = f(c+2) = f(2c) - f(0) = -f(0) - f(0) = -2f(0).$$

We now show that $f(2c) = -f(0)$. Since $f(c) = f(\iota(f(0))^2) = 0$, it suffices to show that $f(2\iota(f(0))^2) = -f(0)$. Indeed, we have $f(0) + f(c+1) = f(c) = 0$, so $f(c+1) = -f(0)$. Plugging $x = 0$ and $y = c+1$ into (*) yields $f(-\iota(f(0))^2) = 2f(0)$. Plugging $x = 0$ and $y = -\iota(f(0))^2$ into (*) yields $f(2\iota(f(0))^2) = -f(0)$.

We now go back to the main goal. We write down $f(xc^4)$ in two ways:

$$f(xc^4) = f(0) + f(xc^2 + c^2) = f(0) + f((x+1)c^2) = 2f(0) + f(x+1+c^2),$$
$$f(xc^4) = f(0) + f(xc + c^3) = f(0) + f((x+c^2)c) = 2f(0) + f(x+c+c^2).$$

Replacing x with $x - c^2$, we are done. $\square$

Plugging $y = 1$ into (*) yields $f(0) + f(x+1) = f(x)$ or $f(x+1) = f(x) - f(0)$ for all $x \in R$. Next, for any $x \in R$, we have $f(\iota(f(0))\iota(f(x))) = f(0) - f(x)$. Replacing x with $\iota(f(0))\iota(f(x))$, we get

$$f(\iota(f(0))^2 - \iota(f(0))\iota(f(x))) = f(\iota(f(0))\iota(f(x) - f(0))) = f(x).$$

By the previous lemma,

$$f(\iota(f(0))^2 - \iota(f(0))\iota(f(x))) = f(1 - \iota(f(0))\iota(f(x))) = f(-\iota(f(0))\iota(f(x))) - f(0),$$

so $f(-\iota(f(0))\iota(f(x))) = f(x) + f(0) = f(x-1)$. Next, by plugging $y = -1$ into (*), we get

$$f(\iota(f(x))\iota(f(-1))) + f(x) + f(0) = f(-x) \quad \forall x \in R.$$

More importantly, if $f(a) = f(b)$, then the above yields $f(-a) = f(-b)$. As a result, we get $f(0) - f(x) = f(\iota(f(0))\iota(f(x))) = f(1-x)$, or

$$f(x) + f(1-x) = f(0) \quad \forall x \in R.$$

2 Period congruence: from general to non-periodic

Here, we relate general good functions with non-periodic good functions. The main result is as follows.

Theorem 2.1. *The ι-good functions are in an explicit bijection with the disjoint union of non-periodic $\phi\iota$-good functions across all double-sided ideals $I \subseteq R$, where $\phi : R \to R/I$ is the projection map.*

More explicitly, the bijection is given by the following two results. We will omit $\circ$ for this section at some equations for readability purposes.

Theorem 2.2. *Let I be a two-sided ideal of R and $\phi : R \to R/I$ be the projection map. Let $\iota : R \to S$ be a function, and let $f : R/I \to S$ be $\phi\iota$-good function. Then $f \circ \phi : R \to S$ is an ι-good function.*

Proof. By (*) on f, for any $x, y \in R/I$,

$$f(\phi\iota f(x)\ \phi\iota f(y)) + f(x+y) = f(xy).$$

Now replace x and y with $\phi(x)$ and $\phi(y)$ with $x, y \in R$:

$$f(\phi\iota f\phi(x)\ \phi\iota f\phi(y)) + f(\phi(x) + \phi(y)) = f(\phi(x)\phi(y)).$$

Rearranging, we get that for any $x, y \in R$,

$$f\phi(\iota f\phi(x)\ \iota f\phi(y)) + f\phi(x+y) = f\phi(xy).$$

This proves that $f \circ \phi$ is an ι-good function. $\square$

Theorem 2.3. *Let $f : R \to S$ be an ι-good function, and define*

$$I_f := \{c \in R : \forall x \in R, f(x+c) = f(x)\}.$$

Then I_f is a double-sided ideal of R.

Furthermore, let $\phi : R \to R/I_f$ to be the usual projection map. Then the reduction $\tilde{f} : R/I \to S$ of f is a non-periodic $\phi \circ \iota$-good function.

Proof. Clearly, I_f is a monoid under addition. Since R is a ring, it is group under addition. So, it remains to show that $cy, yc \in I_f$ for any $c \in I_f$ and $y \in R$.

Comparing (*) using $y = c$ and $y = 0$ gives $f(xc) = f(0)$ for all $x \in R$. Then replacing y with yc yields

$$f(x) + f(\iota f(x)\iota f(0)) = f(0) = f(xyc) = f(x + yc) + f(\iota f(x)\iota f(yc)) = f(x + yc) + f(\iota f(x)\iota f(0)),$$

so $f(x + yc) = f(x)$ for all $x, y \in R$. That is, $yc \in I_f$ for any $y \in R$. Similarly, we also get $cy \in I_f$ for any $y \in R$. This shows that I_f is a double-sided ideal.

Now write $f = \tilde{f} \circ \phi$. Plugging into (*), we get that for any $x, y \in R$,

$$\begin{gathered}\tilde{f}\phi(\iota\tilde{f}\phi(x)\iota\tilde{f}\phi(y)) + \tilde{f}\phi(x + y) = \tilde{f}\phi(xy) \\ \tilde{f}(\phi\iota\tilde{f}\phi(x)\phi\iota\tilde{f}\phi(y)) + \tilde{f}(\phi(x) + \phi(y)) = \tilde{f}(\phi(x)\phi(y)).\end{gathered}$$

Since ϕ is surjective, this implies that $\tilde{f}$ is $\phi\iota$-good. Finally, one can check from the definition of I_f that $\tilde{f}$ is non-periodic. $\square$

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