IMO SL 2017A6

Let R be a ring, S be an abelian (additive) group, and $\iota : S \to R$ be a group homomorphism. Find all functions $f : R \to S$ such that for any $x, y \in R$,

$$f(\iota(f(x))\iota(f(y))) + f(x+y) = f(xy). \tag{*}$$

We say that $f : R \to S$ is *ι-good* if f satisfies the above functional equation. We say that f is *non-periodic ι-good* if f is ι-good and f has no non-zero period. The main goal is to find id_R-good functions, but working in generality allows better proofs. To classify ι-good functions, we reduce to non-periodic case.

Answer

Let I be a double-sided ideal of R and $[\cdot] : R \to R/I$ be the projection map. Let $\phi : R/I \to G$ be a group homomorphism such that $[\iota(\phi(x))] = x$ for all $x \in R/I$. Let $a \in Z(R/I)$ such that $a^2 = 1$. Then the function $f(x) = \phi(a(1 - [x]))$ is ι-good. These are all the ι-good functions in the following cases:

1. G is 2-torsion free and 3-torsion free.
2. $G = R$ is a simple ring of characteristic not equal to 2 and $\iota = \mathrm{id}_R$. Since $Z(R)$ is a field, this means that the good functions are 0, $x \mapsto 1 - x$, and $x \mapsto x - 1$.
3. $G = R$ is a field of characteristic 2 and $\iota = \mathrm{id}_R$. That is, the good functions are 0 and $x \mapsto x + 1$.

References

- `https://artofproblemsolving.com/community/c6h1480146p8693244`.

  Solution in AoPS by **anantmudgal09** (post #75). We follow the main step of this solution when G is 2-torsion-free.

- `https://artofproblemsolving.com/community/c6h1480146p29214012`.

  Solution in AoPS by **BlazingMuddy** (author of this project, post #176). We follow the main step of this solution when R is a field. More generally, the steps allows computing $f(x)$ when x is a unit of R.

Excellent functions

Let R be a ring and G be an abelian group. An *excellent function* from R to G is a function $f : R \to G$ such that for all $x, y \in R$,

$$f(x + y - xy) + f(1 - (x + y)) = f(1 - xy). \tag{**}$$

Excellent functions appear in the classification of non-periodic good functions. Namely, an injective non-periodic good function induces an excellent function.

Clearly, group homomorphisms are excellent functions. The main question is: when are there are no other excellent functions? We prove this in some subcases.

By plugging $y = 0$ into (**), we get $f(x) + f(1 - x) = f(1)$ for all $x \in R$. By plugging $y = 1$ instead, we get $f(1)+f(-x) = f(1-x)$, which implies $f(x)+f(1) = f(x+1)$ for all $x \in R$. Thus we also get $f(-x) = -f(x)$ for all $x \in R$ and $f(0) = 0$. By induction on $n \in \mathbb{Z}$, we get $f(n) = nf(1)$ and $f(x+n) = f(x)+nf(1)$ for all $x \in R$. By replacing (x, y) with $(-x, -y)$ and using the equalities $f(-x) = -f(x)$ and $f(x + 1) = f(x) + 1$, one gets

$$f(xy + x + y) = f(xy) + f(x + y). \tag{3.1}$$

Now we continue with more equations. The above one yields

$$f((x + 1)(y + 1)) = f(xy) + f(x + y) + f(1).$$

By induction on $n \in \mathbb{Z}$, we get that for any $x, y \in R$,

$$f((x + n)(y + n)) = f(xy) + nf(x + y) + n^2 f(1). \tag{3.2}$$

Plugging $y = 0$ and replacing x with $x - n$, we get $f(nx) = nf(x)$ for any $n \in \mathbb{Z}$ and $x \in R$. We now prove:

Lemma 0.1. *For any $a, b \in R$, we have $2f(3a + b) = 2(3f(a) + f(b))$. In particular, f must be a group homomorphism if G is 2-torsion free and 3-torsion free.*

Proof. By replacing x with $2x$ and using $n = 2$ in (3.2), we get that for any $x, y \in R$,

$$2f((x + 1)(y + 2)) = 2(f(xy) + f(2x + y) + 2f(1)).$$

By symmetry, for any $x, y \in R$,

$$2f((x + 2)(y + 1)) = 2(f(xy) + f(x + 2y) + 2f(1)).$$

We can now write $2f((x + 3)(y + 3))$ in two ways:

$$2(f(xy) + 3f(x + y) + 9f(1)) = 2f((x + 3)(y + 3)) = 2(f(xy) + f(2x + y) + f(x + 2y) + 9f(1)).$$

After some cancellations, we simplify to

$$6f(x + y) = 2(f(2x + y) + f(x + 2y)).$$

The lemma follows by change of variables $(x, y) = (2a + b, -(a + b))$ and slight algebraic manipulation. □

Some further directions. The set of excellent functions, say $E(R, G)$, form a group under addition. Thus, we can consider the quotient $Q(R, G) = E(R, G)/ \operatorname{Hom}(R, G)$. The main goal can be rephrased as $Q(R, G) = 0$ for all R and G.

To prove $Q(R, G) = 0$ by first principles, we would check that $f(x+y) = f(x)+f(y)$ for any $x, y \in E(R, G)$. Now we fix x and y, and consider the evaluation homomorphism $\phi : \mathbb{Z}[X, Y] \to R$ given by $X \mapsto x$ and $Y \mapsto y$. Then one can check that $f \circ \phi \in E(\mathbb{Z}[X, Y], G)$, and if f is a group homomorphism, then $f(x + y) = f(x) + f(y)$. Thus, the goal reduces to proving $Q(\mathbb{Z}[X, Y], G) = 0$ for all groups G. We can still make further reductions.

The first lemma says that $Q(R, G)$ is 6-torsion for any R and G. Thus we can write $Q(R, G) = Q_2(R, G) \times Q_3(R, G)$, where $Q_p(R, G)$ is the set of p-torsion elements for $p = 2, 3$. When R is free as an abelian group, e.g. $R = \mathbb{Z}[X, Y]$, we can reduce even further. For $p = 2, 3$, by shifting with a group homomorphism, each element of $Q_p(R, G)$ has a representative that sends a fixed $\mathbb{Z}$-basis of R, say $\mathcal{B}$, to zero. Then f is p-torsion as an element of $E(R, G)$, so in fact $f \in E(R, G[p])$, where $G[p]$ is the set of p-torsion elements of G. If f is non-zero, then we can choose an appropriate $\mathbb{F}_p$-linear functional $\alpha : G[p] \to \mathbb{F}_p$ such that $\alpha \circ f \in E(R, \mathbb{F}_p)$ is non-zero either. Since it still sends the basis $\mathcal{B}$ to zero, $\alpha \circ f$ is not a group homomorphism, which means $Q(R, \mathbb{F}_3) \neq 0$. This means that the goal reduces entirely to two equalities: $Q_2(\mathbb{Z}[X, Y], \mathbb{F}_2) = 0$ and $Q_3(\mathbb{Z}[X, Y], \mathbb{F}_3) = 0$.

If $p = 3$, then the second lemma says more. For any ring R and $f \in E(R, \mathbb{F}_3)$ and $a, b \in R$, we have $f(3a + b) = 3f(a) + f(b) = f(b)$. In particular, f naturally induces an excellent function in $E(R/(3), \mathbb{F}_3)$. That is, we get $E(R, \mathbb{F}_3) \equiv E(R/(3), \mathbb{F}_3)$. Thus, the equality $Q_3(\mathbb{Z}[X, Y], \mathbb{F}_3) = 0$ reduces further to $Q_3(\mathbb{F}_3[X, Y], \mathbb{F}_3) = 0$.

Basic results

Here, we collect basic results regarding ι-good functions, mainly those obtained by plug-and-chug. An important observation is that if $f(a) = f(b)$ and $f(c) = f(d)$, then we have $f(a+c) = f(b+d) \iff f(ac) = f(bd)$. Taking $c = d$ gives $f(a+c) = f(b+c) \iff f(ac) = f(bc) \iff f(ca) = f(cb)$ whenever $f(a) = f(b)$. Taking $c = d = 1$, we get that $f(a) = f(b)$ implies $f(a + 1) = f(b + 1)$. By induction on $n \in \mathbb{N}$, we get that $f(a) = f(b)$ implies $f(a + n) = f(b + n)$. We also get that $f(a) = f(b)$ implies $f(na) = f(nb)$.

By taking $c = d = -1$ and replacing a and b with $a + 1$ and $b + 1$, we get that $f(a) = f(b)$ implies $f(-(a + 1)) = f(-(b + 1))$. Then $f(a) = f(b)$ implies $f(-a) = f(-b)$, and in fact we get more:

$$f(a) = f(b) \iff f(a + 1) = f(b + 1) \iff f(-a) = f(-b).$$

By induction, we get $f(a) = f(b) \iff f(a + n) = f(b + n)$ for any $n \in \mathbb{Z}$.

By plugging $x = y = 0$ into (*), we get $f(\iota(f(0))^2) = 0$. We now show:

Theorem 0.2. *For any $c \in R$ such that $f(c) = 0$, we have $f(x+c) = f(x+1)$ for any $x \in R$. In particular, $f(1) = 0$, and if f is non-periodic, then $f(c) = 0 \iff c = 1$ and so $\iota(f(0))^2 = 1$.*

Proof. We first show that $f(c^2) = f(c^3) = 0$. First, for any $d \in R$ such that $f(d) = 0$, plugging $y = d$ into (*) yields

$$f(0) + f(x + d) = f(xd) \quad \forall x \in R.$$

Now we have $f(c^2) = f(0) + f(2c)$ and $f(c^3) = f(0) + f(c^2 + c) = 2f(0) + f(2c + 1)$. Thus our goal reduces to showing that $f(2c) = -f(0)$ and $f(2c + 1) = -2f(0)$. Since $f(c + 1) = -f(0)$, the former yields $f(2c) = f(c + 1)$, and so

$$f(2c + 1) = f(c + 2) = f(2c) - f(0) = -f(0) - f(0) = -2f(0).$$

We now show that $f(2c) = -f(0)$. Since $f(c) = f(\iota(f(0))^2) = 0$, it suffices to show that $f(2\iota(f(0))^2) = -f(0)$. Indeed, we have $f(0)+f(c+1) = f(c) = 0$, so $f(c+1) = -f(0)$. Plugging $x = 0$ and $y = c+1$ into (*) yields $f(-\iota(f(0))^2) = 2f(0)$. Plugging $x = 0$ and $y = -\iota(f(0))^2$ into (*) yields $f(2\iota(f(0))^2) = -f(0)$.

We now go back to the main goal. We write down $f(xc^4)$ in two ways:

$$f(xc^4) = f(0) + f(xc^2 + c^2) = f(0) + f((x + 1)c^2) = 2f(0) + f(x + 1 + c^2),$$

$$f(xc^4) = f(0) + f(xc + c^3) = f(0) + f((x + c^2)c) = 2f(0) + f(x + c + c^2).$$

Replacing x with $x - c^2$, we are done. ☐

Plugging $y = 1$ into (*) yields $f(0) + f(x + 1) = f(x)$ or $f(x + 1) = f(x) - f(0)$ for all $x \in R$. Next, for any $x \in R$, we have $f(\iota(f(0))\iota(f(x))) = f(0) - f(x)$. Replacing x with $\iota(f(0))\iota(f(x))$, we get

$$f(\iota(f(0))^2 - \iota(f(0))\iota(f(x))) = f(\iota(f(0))\iota(f(x) - f(0))) = f(x).$$

By the previous lemma,

$$f(\iota(f(0))^2 - \iota(f(0))\iota(f(x))) = f(1 - \iota(f(0))\iota(f(x))) = f(-\iota(f(0))\iota(f(x))) - f(0),$$

and thus

$$f(-\iota(f(0))\iota(f(x))) = f(x) + f(0) = f(x-1) \quad \forall x \in R. \tag{1.1}$$

Since $f(a) = f(b)$ implies $f(-a) = f(-b)$, we get

$$f(0) - f(x) = f(\iota(f(0))\iota(f(x))) = f(1-x). \tag{1.2}$$

Then we get $f(x) + f(1-x) = f(0)$, and thus

$$f(x) + f(-x) = 2f(0) \quad \forall x \in R. \tag{1.3}$$

Lemma 0.3. *Suppose that f is non-periodic ι-good. Then for any $x \in R$, we have $\iota(f(0))\iota(f(x)) = \iota(f(x)) = \iota(f(0))$.*

Proof. Plug $(\iota(f(1-x))\iota(f(0)), \iota(f(0))\iota(f(x)))$ in place of (x, y) in (*). Using $\iota(f(0))^2 = 1$ and (1.2), we get

$$f(\iota(f(x))\iota(f(1-x))) + f(\iota(f(1-x))\iota(f(0)) + \iota(f(0))\iota(f(x))) = f(\iota(f(1-x))\iota(f(x))).$$

On the other hand, (*) with $x = 1 - y$ gives $f(\iota(f(1-y))\iota(f(y))) = f(y - y^2)$ for all $y \in R$. Simplifying gives

$$f(\iota(f(1-x))\iota(f(0)) + \iota(f(0))\iota(f(x))) = 0 \iff \iota(f(1-x))\iota(f(0)) + \iota(f(0))\iota(f(x)) = 1.$$

Since $f(x) + f(1-x) = f(0)$, adding $\iota(f(x))\iota(f(0))$ to the left hand side gives

$$\iota(f(0))^2 + \iota(f(x))\iota(f(0)) = \iota(f(0))\iota(f(x)) + 1.$$

The lemma is proved, since $\iota(f(0))^2 = 1$. $\square$

Finally, we classify injective non-periodic ι-good functions. This involves the theory of excellent functions.

Lemma 0.4. *Suppose that all excellent functions $R \to G$ are group homomorphisms.*[1] *Suppose that f is non-periodic ι-good and injective. Then there exists $a \in Z(R)$ with $a^2 = 1$ and a group homomorphism $\phi : R \to G$ with $\iota \circ \phi = \mathrm{id}_R$ such that $f(x) = \phi(a(1-x))$ for all $x \in R$.*

Proof. Take $a = \iota(f(0))$; we proved $a^2 = 1$. By (1.2) and injectivity, we get $a\iota(f(x)) = 1 - x$ and thus $\iota(f(x)) = a(1-x)$ for all $x \in R$. The previous lemma then yields $ax = xa$ for all $x \in R$, and thus $a \in Z(R)$. Using the equation $\iota(f(x)) = a(1-x)$, (*) simplifies to

$$f((1-x)(1-y)) + f(x+y) = f(xy).$$

Then the function $x \mapsto f(1-x)$ is excellent, and thus is a group homomorphism, say ρ. Finally, choosing $\phi(x) = \rho(ax) = f(1-ax)$ works. $\square$

Period congruence: from general to non-periodic

Here, we relate general good functions with non-periodic good functions. The main result is as follows.

Theorem 0.5. *The ι-good functions are in an explicit bijection with the disjoint union of non-periodic $\phi\iota$-good functions across all double-sided ideals $I \subsetneq R$, where $\phi : R \to R/I$ is the projection map.*

[1] That is, $Q(R, G) = 0$, using the notation from the excellent function section.

More explicitly, the bijection is given by the following two results. We will omit $\circ$ for this section at some equations for readability purposes.

Theorem 0.6. *Let I be a two-sided ideal of R and $\phi : R \to R/I$ be the projection map. Let $\iota : S \to R$ be a function, and let $f : R/I \to S$ be $\phi\iota$-good function. Then $f \circ \phi : R \to S$ is an ι-good function.*

Proof. By (*) on f, for any $x, y \in R/I$,

$$f(\phi\iota f(x)\ \phi\iota f(y)) + f(x+y) = f(xy).$$

Now replace x and y with $\phi(x)$ and $\phi(y)$ with $x, y \in R$:

$$f(\phi\iota f\phi(x)\ \phi\iota f\phi(y)) + f(\phi(x) + \phi(y)) = f(\phi(x)\phi(y)).$$

Rearranging, we get that for any $x, y \in R$,

$$f\phi(\iota f\phi(x)\ \iota f\phi(y)) + f\phi(x+y) = f\phi(xy).$$

This proves that $f \circ \phi$ is an ι-good function. $\square$

Theorem 0.7. *Let $f : R \to S$ be an ι-good function, and define*

$$I_f := \{c \in R : \forall x \in R, f(x+c) = f(x)\}.$$

Then I_f is a double-sided ideal of R.

Furthermore, let $\phi : R \to R/I_f$ to be the usual projection map. Then the reduction $\tilde{f} : R/I_f \to S$ of f is a non-periodic $\phi \circ \iota$-good function.

Proof. Clearly, I_f is a monoid under addition. Since R is a ring, it is group under addition. So, it remains to show that $cy, yc \in I_f$ for any $c \in I_f$ and $y \in R$.

Comparing (*) using $y = c$ and $y = 0$ gives $f(xc) = f(0)$ for all $x \in R$. Then replacing y with yc yields

$$f(x) + f(\iota f(x)\iota f(0)) = f(0) = f(xyc) = f(x+yc) + f(\iota f(x)\iota f(yc)) = f(x+yc) + f(\iota f(x)\iota f(0)),$$

so $f(x+yc) = f(x)$ for all $x, y \in R$. That is, $yc \in I_f$ for any $y \in R$. Similarly, we also get $cy \in I_f$ for any $y \in R$. This shows that I_f is a double-sided ideal.

Now write $f = \tilde{f} \circ \phi$. Plugging into (*), we get that for any $x, y \in R$,

$$\begin{aligned}\tilde{f}\phi(\iota\tilde{f}\phi(x)\iota\tilde{f}\phi(y)) + \tilde{f}\phi(x+y) &= \tilde{f}\phi(xy)\\ \tilde{f}(\phi\iota\tilde{f}\phi(x)\phi\iota\tilde{f}\phi(y)) + \tilde{f}(\phi(x)+\phi(y)) &= \tilde{f}(\phi(x)\phi(y)).\end{aligned}$$

Since ϕ is surjective, this implies that $\tilde{f}$ is $\phi\iota$-good. Finally, one can check from the definition of I_f that $\tilde{f}$ is non-periodic. $\square$

Solution for subcases

Here we prove lemmas that allow us to solve the problem for the desired subcases. By Lemma 0.4, most of the work reduces to showing that f is injective.

Lemma 0.8. *If G is 2-torsion free, then every non-periodic ι-good function is injective.*

Proof. Let $a, b \in R$ such that $f(a) = f(b)$. Since $f(a) = f(b)$, $f(b) = f(a)$, and $f(a + b) = f(b + a)$, we get $f(ab) = f(ba)$. Then we have $f(-ab) = f(-ba)$ and $f(-a) = f(-b)$. This implies $f(a - b) = f(b - a)$.

Finally, substitute $x = a - b$ into (1.3). We get $2f(a - b) = 2f(0)$. Since G is 2-torsion free, we get

$$f(a - b) = f(0) \iff f(a - b + 1) = 0 \iff a - b + 1 = 1 \iff a = b.$$

This proves that f is injective. $\square$

Corollary 0.9. *Suppose that G is 2-torsion free and 3-torsion free. Then all non-periodic ι-good functions are of form $x \mapsto \phi(a(1 - x))$ for some $a \in Z(R)$ with $a^2 = 1$ and group homomorphism $\phi : R \to G$ with $\iota \circ \phi = \mathrm{id}_R$. All ι-good functions are of form $x \mapsto x \mapsto \phi(a(1 - [x]))$ for some $a \in Z(R/I)$ with $a^2 = 1$ and group homomorphism $\phi : R/I \to G$ with $[\iota(\phi(x))] = x$ for all $x \in R/I$.*

Combined with Lemma 0.1, we are done if G is 2-torsion free and 3-torsion free. Removing the 3-torsion free assumption on Lemma 0.1 would also mean removing the 3-torsion assumption here.

Subcase: Simple ring