ONE MATHEMATICAL CAT, PLEASE!
UNDERSTANDING CALCULUS

by

DR. CAROL J.V. FISHER BURNS

TABLE OF CONTENTS

PREFACE

intended audience for this text

This text is intended to be used for a one-semester (16 week) course introducing calculus to non-mathematics and non-engineering majors. The course should meet approximately four hours per week. Alternatively, the text can be used for a more leisurely-paced two-semester course, meeting about three hours per week. At Idaho State University, the text has been used for a course that meets a general education requirement in mathematics.

neglect of the language in which mathematics is expressed

The *language* in which mathematical ideas are expressed is usually underemphasized in the standard curriculum. Emphasis is placed on *what* is said, not *how* it is said. Without an understanding of the language of mathematics, students can't read their mathematics books, and can't express mathematical ideas in a coherent way.

writing across the curriculum

Educators have stressed the importance of *writing across the curriculum*; this text directly focuses attention on writing skills. As the need arises, students are exposed to elements of the mathematics language, and are given ample exercises to practice the language while learning the mathematics concepts. Chapter 1 provides the foundational language issues on which the rest of the text builds, and hence has a flavor that is very nontraditional.

two-column format used in this text

This text is designed to be easy to use for the instructor. The two-column page format identifies the key concept discussed in almost every paragraph. Definitions, notation, theorems, examples, and exercises appear **bold–faced** in the left-hand column. Key ideas in expository paragraphs appear in *italics* in this thin column. Thus, the instructor can skim the sections and easily identify the topics presented.

how to use the two-column format

Some people using this text will choose to ignore the left-hand 'key idea' column on a first reading, using the column merely as a feature that enables easy location of results. Others will choose to read the 'key idea' phrase *before* reading the companion paragraph, as a way to help maintain focus on the central idea. (Too often, students lose the forest because of all the trees!) In addition, the two-column format provides lots of room for writing in the margins.

review material is interspersed throughout the text

Many students learn better when there is an *immediate* use for the material. For this reason, necessary review material is included in the sections where it is needed. Occasionally, review material is purposefully repeated; this saves the student 'look-up' time, and provides an opportunity for the author to give a slightly different viewpoint to already-introduced ideas.

two types of exercises:

There are two types of exercises in this text.

in-section exercises, to encourage active reading

In order to learn to read mathematics, students *must read mathematics.* Unfortunately, too many students rely on lectures alone as a source of information, and use the book solely for the exercises (and answers to exercises).

To counter this problem, the text has an abundance of 'in-section' exercises—exercises intertwined with the exposition. These exercises directly address concepts discussed in the paragraphs immediately preceding them; thus, in order to do the exercises, the student *must read the book.* For example, a student might be asked to re-write a paragraph so that it is correct for a slight modification of an idea.

The in-section exercises are designed not only to encourage *reading*, but also to encourage *active reading*—the correct way to read mathematics is with *pencil in hand*!

It is intended by the author that <u>every one</u> of the in-section exercises be attempted by <u>every</u> student.

optional Student's Solution Manual

Complete answers to the in-section exercises are available in the supplemental Complete Solution Manual.

more traditional end-of-section exercise sets

Secondly, there are traditional end-of-section exercise sets. These exercises provide reinforcement of both the calculus and language concepts discussed in the section.

Abbreviated answers to odd-numbered end-of-section problems are given at the end of the text. The supplemental Complete Solution Manual contains answers to *all* the exercises. These solutions are carefully written in *complete mathematical sentences*, to reinforce the correct writing of mathematics that is emphasized throughout the text.

♣ *symbol*

The clubsuit symbol ♣ identifies the specific part(s) of a question that must be answered by the student.

Many of the exercises contain a moderate amount of exposition. The question(s) that the student must answer are often imbedded in this exposition, and the ♣ makes them easier to spot.

★ *symbol*

More advanced material in the text is labeled with the star symbol ★. This material may not be appropriate for a first reading. However, this ★ device allows the author to say more of the complete truth without interrupting the exposition.

★★ *symbol*

Material that is labeled with ★★ is probably appropriate for the instructor's eyes only.

the 'quick quiz'

The author firmly believes that the only way to really learn mathematics is *a little bit at a time.* Two hours each day is far superior to a Saturday marathon. To gently encourage this every-day commitment to the subject, the author has found the following 'quick quiz' technique extremely successful.

At the beginning or end of each class, a very short (1–2 minute) quiz is given, over material covered in the previous lecture. The question is extremely basic. The quiz is worth 1 point; to get this point, the student must answer the question correctly, *using complete and correct mathematical sentences.* For example, a student who is asked to differentiate $f(x) = x^2$ and writes $f(x) = x^2 = 2x$ has not written a correct mathematical sentence, and will not get the point. The mistake is quickly corrected!

Any points that are accumulated on these quizzes get added on to the student's next test grade; thus, they cannot hurt the student, but can certainly help. This positive reinforcement technique has been *extremely* successful in getting students to attend class, and read over their notes before the next lecture.

Some sample 'quick quiz' questions are included at the end of each section. Solutions to the 'quick quiz' questions are given at the end of the text.

'Keywords'

Each section is concluded with a list of 'keywords'. Students studying for an exam should look through each 'keyword' list to ensure that they have not missed any important information.

$\TeX$

This text was typeset using $\TeX$ (pronounced so that it rhymes with *blecchhh*). $\TeX$ is a typesetting system that is ideally suited to books containing lots of mathematics. From the $\TeX$ output, pdf files were created to be put on the World Wide Web.

ACKNOWLEDGMENTS

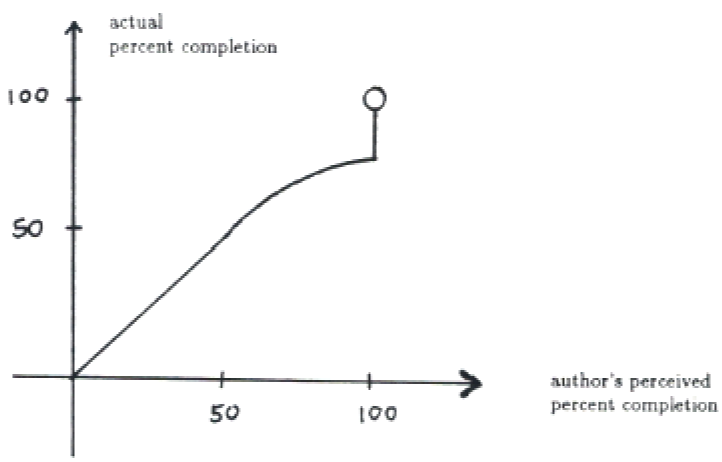
Many thanks go to Warren Esty, of Montana State University—I have benefited tremendously from his insights into student difficulties with the mathematical language. Many of his ideas are woven throughout these pages.

Thanks go also to Pat Lang, for his time and participation in class-testing of this text. Joel Blackburn is owed thanks for his extremely careful reading of the text—he found typos that had slipped past dozens of proof-readings.

Without Bob, Julia, and my family, no undertaking as great as this would have been possible.

In closing, a valuable lesson has been learned in writing this text. The relation between an author's *perceived* percent completion of the text, and the *actual* percent completion of the text, is a beautiful example of a non-function, as shown below. The hollow dot at the point $(100, 100)$ is interpreted as,

a text is never finished.



STUDY STRATEGIES
for Students of Mathematics

- Find out what material is going to be covered the next day, and skim the section *before* it's covered in class. Make note of any questions that you have, and be sure to get those questions answered during lecture.
- Use your text. Mark in it. Highlight important material. You bought the text—get your $$ worth from it.
- Ask questions in class! A question that you have is likely to be a question that others have. Be the person brave enough to ask, and your classmates will thank you for it.
- Take complete class notes, and read them over as soon as possible after the lecture (while you still remember what was said). Read actively! This means: have pencil and paper beside you, and use them. In particular, re-work all the problems that were covered in class, *without looking at your notes*. If you get stuck, you have your notes to look back at—see where you went wrong, then *close the notes and try again*. You may need to repeat this process several times, but it's worth it.

  In the re-reading process, be sure to fill in any gaps in your understanding. This way you will have a complete set of comprehensible notes when it comes time to study for the exams. Remember that the topics emphasized in class are likely to be those that your instructor feels are most important.
- Read the section in the text again, thoroughly, after the lecture.
- Do some mathematics *almost every day*. In this author's opinion, two hours each day is much better than a Saturday marathon.
- Find someone with similar study habits with whom to work. Mathematics is much more fun when you can *talk* about it with someone.
- Use index cards for important definitions, formulas, key problems; or whenever the instructor says "This is important!" Keep the stack of cards with you and flip through whenever you're waiting in line or 'on hold' on the telephone.
- If you don't understand something, seek help *immediately*. The material builds, and it will be difficult to learn new material with gaps in understanding of previous stuff.

TABLE OF SYMBOLS

SYMBOL	MEANING OF SYMBOL	PAGE # of first appearance
$=$	equal to	3
$\approx$	approximately equal to	10
$\clubsuit$	student input required	iv
$\bigstar$	advanced material	iv
$\bigstar\bigstar$	more advanced material	iv
α, β, γ, δ, ϵ, θ, λ, μ, π, ρ, τ, ϕ, ω, Γ, Δ	Greek letters	7
π	an important constant, $\pi \approx 3.14$	20
e	an important constant, $e \approx 2.72$	99
$\mathbb{R}$	set of real numbers	8
$\mathbb{C}$	set of complex numbers	19
$\mathbb{Z}$	set of integers	27
$\mathbb{Q}$	set of rational numbers	27
$>$	greater than	9
$<$	less than	9
$(x_1, x_2, \ldots, x_n)$	n-tuple	17
(a, b)	ordered pair	17, 41
$:=$	equal, by definition	18
$i := \sqrt{-1}$	imaginary number	19
$\in$	is an element of	22
$\notin$	is not an element of	22
$\{\ \ \}$	set notation	22
$\{\, x \mid \text{some property}\}$	set-builder notation	23
(a,b), $[a,b)$, $(a,b]$, (a,∞), $(-\infty,b]$	interval notation	23
$\emptyset$	the empty set	24
$\iff$	is equivalent to	30, 271
If A, then B	implication	165–167
$A \implies B$	implication, alternate notation	167
$\forall$	for all, for every	92
$\exists$	there exists	92
$!$	a unique	92
$\wedge$, $\vee$, $\neg$	synonyms for 'and', 'or', 'not'	325
m	slope of a line	49
$\lvert x\rvert$	absolute value of x	63
$\lvert x - y\rvert$	distance from x to y	116
$\lVert x\rVert$	norm symbol	141
$A \cup B$	set union	64, 82
$A \cap B$	set intersection	83
$A \subset B$	subset	83
$A - B$	set subtraction	195
$f, \quad f(x)$	function notation	60
$\mathcal{D}(f)$	domain of a function	69
$\mathcal{R}(f)$	range of a function	88
$f\colon A \to B$	function notation	78
$f+g,\ \ kf,\ \ \frac{f}{g},\ \ \sqrt{f}$	special functions	85, 86
$g \circ f$	composite function	86

f^{-1}	inverse function for f	94
e^x	the exponential function	99
$\ln x$	the natural logarithm function	100
$\displaystyle\lim_{x\to c} f(x) = l$, $\lim_{x\to c} f(x) = l$	limits (display and text style)	109, 121
as $x \to c$, $f(x) \to l$	alternate notation for limits	118
$\displaystyle\lim_{x\to c^+} f(x) = l$	right-hand limit	129
$\displaystyle\lim_{x\to c^-} f(x) = l$	left-hand limit	130
$\blacksquare$	end-of-proof marker	133
$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, 1^∞	indeterminate forms	154–158
f'	the derivative function	194
$f'(x)$	the derivative of f, at x	193
$\frac{dy}{dx}$, $\frac{dy}{dx}(c)$, $\frac{dy}{dx}\vert_{x=c}$	Leibniz notation for the derivative	204
$\frac{d}{dx}$	the $\frac{d}{dx}$ operator	204
f'', f''', $f^{(4)}$, $f^{(n)}$	higher order derivatives, prime notation	249
$\frac{d^2y}{dx^2}$, $\frac{d^ny}{dx^n}$	higher order derivatives, Leibniz notation	255
$\sqrt[n]{x}$, $\sqrt{x}$	radical notation	213
$x^{1/n}$	fractional exponent notation	216
$k!$	factorial notation	153
$\displaystyle\sum_{j=1}^{n} a_j$, $\sum_{j=1}^{n} a_j$	summation notation (display and text style)	251
$\frac{(+)(-)}{(-)}$	testing the sign of a function	284
$\times$, $\times$, $\times\!\!\times$	graphing symbols	309
$x \gg 0$	x is much greater than 0	313
$x \ll 0$	x is much less than 0	313
$f(c^+)$, $f(c^-)$	investigating f near c	335
$d(t)$, $v(t)$, $a(t)$	distance, velocity, acceleration functions	362–374
g	acceleration due to gravity	367
	vector	366
	free-body diagram	367
$\int f(x)\,dx$	indefinite integral	344
$\int_a^b f(x)\,dx$	definite integral	408
$F(x)\,\big\vert_a^b$	notation used when evaluating definite integrals	409
$\int u\,dv = uv - \int v\,du$	integration by parts formula	391
$\Vert P\Vert$	norm of a partition	419
$R(P)$	Riemann sum	420

TRUTH TABLES

A	B	A and B	A or B	$A \Rightarrow B$	$A \iff B$	not B
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	
F	F	F	F	T	T	

This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 1

ESSENTIAL PRELIMINARIES

The goal of Chapters 1 and 2 is to *review*, and to expose you to some new views of old ideas. In addition, elements of the language of mathematics are developed that will be needed throughout the course.

It is imperative that Chapter 1 be covered thoroughly, since it provides the foundational language issues on which the rest of the text builds. Instructors and students alike will probably find the flavor of Chapter 1 to be very nontraditional, because language issues are rarely emphasized in lower level mathematics courses.

1.1 The Language of Mathematics
Expressions versus Sentences

a hypothetical situation

Imagine the following scenario: you are sitting in class, and the instructor passes a small piece of paper to each student. You are told that the paper contains a paragraph on *Study Strategies for Students of Mathematics*; your job is to read it and paraphrase it. Upon glancing at the paper, however, you observe that it is written in a foreign language that you do not understand!

the importance of language

Now, is the instructor being fair? Of course not. Indeed, the instructor is probably trying to make a point. Although the *ideas* in the paragraph may be simple, there is no access to these ideas without a knowledge of the *language* in which the ideas are expressed. This situation has a very strong analogy in undergraduate mathematics courses. Students frequently have trouble understanding the ideas being presented; not because the ideas are difficult, but because they are being presented in a foreign language—the language of mathematics.

Study Strategies for Students of Mathematics

A list of *Study Strategies for Students of Mathematics* (in English) appears in this text after the *Preface*. Be sure to read both of these sections.

characteristics of the language of mathematics

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express. It is:

- precise (able to make very fine distinctions);
- concise (able to say things briefly);
- powerful (able to express complex thoughts with relative ease).

The language of mathematics can be learned. However, it requires the efforts needed to learn any foreign language.

a major goal of this text

Throughout this text, attention is paid not only to the ideas presented, but also to the language in which these ideas are expressed. Besides understanding calculus, a major goal of this course is for you to improve your skills in *reading* and *writing* mathematics. These skills can be carried with you into any setting where mathematics is used to express ideas.

complete and correct mathematical sentences

You can't learn to read without reading. *So read the text.* You can't learn to write without writing. So you will be given ample opportunities to practice writing complete and correct mathematical sentences.

We now begin our study of the language of mathematics. The ideas introduced here will be elaborated on throughout the text.

English nouns versus mathematical 'nouns'

In English, a *noun* is a word that names something. An English noun is usually a person, place, or thing; for example, *Julia*, *Idaho*, and *rat*. Note that there are conventions regarding nouns in English; for example, proper names are capitalized.

The mathematical analogue of a noun is called an *expression*.

mathematical expression

A *mathematical expression* is a name given to some mathematical object of interest. The phrase '*mathematical expression*' is usually shortened to '*expression*'.

conventions regarding the naming of mathematical 'nouns'

In mathematics, an 'object of interest' is often a number, a set, or a function. There are conventions regarding the naming of 'nouns' in mathematics, just as there are in English. For example, real numbers are usually named with lowercase letters (like a, x, t, α, β, and γ), whereas sets are usually named with capital letters (like A, B, and C). Such conventions are addressed throughout the text.

Without sets and functions, modern mathematics could not exist. Sets, and the important sets of numbers, are reviewed throughout Chapter 1. Functions are discussed in Chapter 2. The Algebra Review at the end of the current section reviews the most commonly used Greek letters, and the real numbers.

By themselves, nouns are not extremely useful. It is when nouns are used in *sentences* to express complete thoughts that things get really interesting.

sentences

A declarative English *sentence* begins with a capital letter, ends with a period, and expresses a complete thought:

Many students are a bit apprehensive of their first Calculus course.

A *mathematical sentence* must also express a complete thought. However, there are a lot of symbols (and layouts) available in the construction of mathematical sentences that are not available in the construction of English sentences.

Many students have trouble distinguishing between mathematical *expressions* and mathematical *sentences*. Exercises and examples that help you understand the difference will appear throughout the text.

how to decide if something is a sentence

A good way to decide if something is a *sentence* is to *read it out loud*, and ask yourself the question: Does it express a complete thought? If the answer is 'yes', it's a sentence.

The difference between expressions and sentences is explored in the next example.

EXAMPLE

sentences versus expressions

Problem: Classify the entries in the list below as:

- an English noun
- a mathematical expression
- a sentence

In any sentence, circle the verb. Try to fill in the blanks yourself before looking at the solutions.

(For the moment, don't worry about the *truth* of sentences. This issue is addressed in the next example.)

1.	cat	________________
2.	x	________________
3.	The word 'cat' begins with the letter 'k'.	________________
4.	$1 + 2 = 4$	________________
5.	$(a + b)^2$	________________
6.	$2x - 1 = 0$	________________
7.	The cat is black.	________________
8.	$(a + b)^2 = a^2 + 2ab + b^2$	________________
9.	$-3t < 2$	________________
10.	$y + y + y$	________________
11.	$y + y + y = 3y$	________________
12.	$(a + b)^2 = a^2 + b^2$	________________
13.	This sentence is false.	________________
14.	$x^2 < 0$	________________
15.	$1 + \sqrt{2}$	________________

Solution:

1.	cat	English noun
2.	x	mathematical expression
3.	The word 'cat' begins with the letter 'k'.	sentence
4.	$1 + 2 = 4$	sentence
5.	$(a + b)^2$	mathematical expression
6.	$2x - 1 = 0$	sentence
7.	The cat is black.	sentence
8.	$(a + b)^2 = a^2 + 2ab + b^2$	sentence
9.	$-3t < 2$	sentence
10.	$y + y + y$	mathematical expression
11.	$y + y + y = 3y$	sentence
12.	$(a + b)^2 = a^2 + b^2$	sentence
13.	This sentence is false.	sentence
14.	$x^2 < 0$	sentence
15.	$1 + \sqrt{2}$	mathematical expression

Note that sentences express a complete thought, but nouns (expressions) do not. For example, read aloud: x. *What about* x? Now read aloud: $2x - 1 = 0$. Here, a complete thought about object 'x' has been expressed.

EXAMPLE
truth of sentences

Problem: Consider the entries in the previous example that are *sentences*. Which are true? False? Are there possibilities other than true and false?

Solution:

3.	The word 'cat' begins with the letter 'k'.	FALSE
4.	$1+2=4$	FALSE
6.	$2x-1=0$	The truth of this sentence (true or false) depends on the choice of x. If x is $1/2$, then it is true. Otherwise, it is false. Sentences such as these are studied in more detail in the next section.
7.	The cat is black.	The truth of this sentence cannot be determined out of context. If the cat being referred to is indeed black, then the sentence is true. Otherwise, it is false.
8.	$(a+b)^2 = a^2+2ab+b^2$	Here, it is assumed that a and b represent numbers. Then, this sentence is (always) true: its truth does *not* depend on the numbers chosen for a and b. ♣ Why?
9.	$-3t < 2$	It is assumed that t represents a number. This sentence is sometimes true, sometimes false, depending on the number chosen for t. In sentences such as these, mathematicians are often interested in finding the choices that make the sentence TRUE.
11.	$y+y+y=3y$	TRUE, for all real numbers y.
12.	$(a+b)^2 = a^2+b^2$	The truth of this sentence depends on the choices for a and b. For example, if $a=0$ and $b=1$, then it is true (♣ check). If $a=1$ and $b=1$, then it is false (♣ check).
13.	This sentence is false.	IF this sentence is true, then it would have to be false. IF this sentence is false, then it would have to be true. So this sentence is not true, not false, and not sometimes true/sometimes false.
14.	$x^2 < 0$	It is assumed that x represents a real number. Since every real number, when squared, is nonnegative, this sentence is (always) false.

EXERCISE 1

♣ Write a few (English) sentences that discuss the difference between mathematical *expressions* and *sentences*.

(Remember that the ♣ (clubsuit) symbol means that student input is required.)

EXERCISE 2

♣ 1. In algebra, how did you go about finding the choice(s) for x that make the equation $2x - 1 = 0$ true? Do it.

♣ 2. In algebra, how did you go about finding the choice(s) for t that make the inequality $-3t < 2$ true? Do it.

♣ 3. What happens if you take the usual algebra approach, and try to 'solve' the equation $y + y + y = 3y$?

♣ 4. What are all the possible choice(s) for a and b that make the equation $(a+b)^2 = a^2 + b^2$ true? Be sure to write complete sentences in your answer.

♣ 5. What name is commonly given to English sentences that are *intentionally* false? To English sentences that are *nonintentionally* false?

EXERCISE 3

sentences versus expressions

♣ Classify each entry in the list below as: an English noun (NOUN), a mathematical expression (EXP), or a sentence (SEN).

♣ In any sentence, circle the verb.

♣ Classify the truth value of any entry that is a *sentence*: TRUE (T), FALSE (F), or SOMETIMES TRUE/SOMETIMES FALSE (ST/SF). The first one is done for you.

1.	$a + b = b + a$	SEN, T
2.	$a + b$	______
3.	$a + b = 5$	______
4.	rectangle	______
5.	Every rectangle has three sides.	______
6.	$x + (-x) \neq 0$	______
7.	$3 \leq 3$	______
8.	$y \geq y$	______
9.	$y > y + 1$	______
10.	$y > y - 1$	______
11.	Bob	______
12.	Bob has red hair.	______
13.	For all nonzero real numbers x, $x^0 = 1$.	______
14.	The distance between real numbers a and b is $b - a$.	______
15.	$a(b + c)$	______
16.	$a(b + c) = ab + ac$	______

★★

Do you really want to be reading this? Remember that the symbol ★★ means that the material is probably appropriate for the instructor's eyes only.

Experienced mathematicians tend to regard sentences with variables as implicit generalizations; thus a mathematician who reads

'The distance between real numbers a and b is $b - a$.'

will automatically interpret it as

'For all real numbers a and b, the distance between a and b is $b - a$.'

The latter sentence is, of course, false. However, at this point in the text, the student is expected to view the sentence

'The distance between real numbers a and b is $b - a$.'

as being sometimes true, sometimes false (depending on the choices made for a and b).

EXERCISE 4

♣ 1. Use the English noun 'Julia' in three sentences: one that is true, one that is false, and one whose truth cannot be determined without additional information.

♣ 2. Use the expression $x^2 + y^2$ in three mathematical sentences: one that is (always) true, one that is (always) false, and one that is sometimes true/sometimes false.

read mathematics out loud

After you write any mathematics (perhaps you have solved a homework problem) you should read it back to yourself, out loud, and be sure that

- it expresses a complete thought;
- it expresses a correct thought.

ALGEBRA REVIEW

Greek letters, the real numbers

Greek letters

Mathematicians are extremely fond of Greek letters. Here are some that are most commonly used, together with their names.

uppercase	lowercase	Name of Greek letter	Pronunciation
	α	alpha	AL-fa
	β	beta	BĀ-ta
Γ	γ	gamma	GAM-a
Δ	δ	delta	DEL-ta
	ϵ	epsilon	EP-si-lon
	θ	theta	THĀ-ta
	λ	lambda	LAM-da
	μ	mu	mew
	π	pi	pie
	ρ	rho	row
	τ	tau	Ow! with a 't' in front
	ϕ	phi	fee
	ω	omega	o-MĀ-ga

EXERCISE 5

learn the Greek letters

♣ Learn to recognize and name all the Greek letters listed in the table above. Practice writing them. Learn how to correctly pronounce each name (ask your instructor if you're uncertain). Once you think you have them mastered, test yourself by filling in the blanks below.

uppercase	lowercase	Name of Greek letter
	______	alpha
	______	beta
______	______	gamma
______	______	delta
	______	epsilon
	______	theta
	______	lambda
	______	mu
	______	pi
	______	rho
	______	tau
	______	phi
	______	omega

EXERCISE 6

$\mathbb{R}$

the real numbers

The *real numbers*, denoted by the symbol $\mathbb{R}$, can most easily be understood in terms of a number line:

$$\longleftrightarrow$$

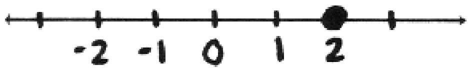
(The arrows suggest that the line extends infinitely far in both directions.) This line is a conceptually perfect picture of the real numbers in the following sense:

- Every point on this line is uniquely identified with a real number.
- Every real number is uniquely identified with a point on this line.

It is important to realize that a particular real number *may have lots of names*: for example,

$$2,\ 3^2-7,\ \frac{4}{2},\ 5-3,\ \frac{2\pi}{\pi},\ \text{and}\ \frac{-11.4}{-5.7}$$

are all names for the unique number shown below:



The name that we choose to use for a number depends on what we are doing. For example, we will see that when talking about the slope of a line, the 'names' $\frac{2}{1}$ or $\frac{-4}{-2}$ are often more useful than the shorter name 2.

♣ 1. Give ten more 'names' for the number 5.

♣ 2. Give three more 'names' for the number 2.3 .

EXERCISE 7

positive
negative

'negative' versus 'minus'

The real numbers to the right of zero on the number line are called *positive*. For example, the numbers 3, $\frac{1017}{23}$, 0.00023 and 10^{52} are positive. The real numbers to the left of zero are called *negative*. The number 0 (read this as 'zero', *not* 'oh') is not positive or negative.

The symbol '$-$' is read differently depending upon the context. If the symbol '$-$' is being used to denote a negative number, it is read as *negative*:

'-3' is read as *negative three*.

If the symbol $-$ is being used to denote the operation of subtraction, it is read as *minus*:

'$3-5$' is read as *three minus five*.

Here's an example that uses both:

'$3-(-5)$' is read as *three minus negative five*.

♣ 1. How would you read '$-4-(-3)$'?

♣ 2. What is the least positive number that can be represented on your hand-held calculator? Call it L. What do you get when you use your calculator to compute $L/2$?

♣ 3. What is the greatest positive number that can be represented on your hand-held calculator? Call it G. What happens when you use your calculator to compute $(G+1)-G$?

EXERCISE 8
inequality symbols
$>$, $<$

One extremely nice property of the real numbers is that they are *ordered*. This means the following: given any two real numbers, either

- they are *equal*, or
- one of the numbers lies further to the left on the real number line.

When a number n lies to the left of a number m, we write

$$n < m$$

and read this as

n is less than m.

Never read this as 'n is smaller than m'. Why not? Well, consider the numbers -5 and 3. Certainly -5 lies to the left of 3, so that $-5 < 3$ (read as *negative five is less than three*). But would you really want to say that -5 is 'smaller than' 3?

♣ 1. Comment. That is, why might you be uncomfortable saying that -5 is 'smaller' than 3?

Similarly, when n lies to the right of m, we write

$$n > m$$

and read this as

n is greater than m.

Again, *never read this as* 'n is bigger than m'.

♣ 2. Find numbers n and m for which $n > m$ is true, but you would feel uncomfortable saying that n is 'bigger' than m.

♣ 3. Read the following sentences out loud, and determine if they are TRUE or FALSE:

a) $-3 < 2$
b) $\frac{2}{5} > \frac{3}{7}$
c) $-3 > -7$

EXERCISE 9
reread the section

♣ Reread this section. You will need to read each section in this text *at least twice* to fully understand the material. Also, don't expect to read mathematics the same way that you read English. You're probably used to measuring reading rates in units of 'pages per hour'. Mathematics is read in units of 'hours per page'.

QUICK QUIZ
sample questions

1 What is the mathematical analogue of an English noun?

2 In English, a noun is usually a person, place, or thing. List three common types of mathematical 'nouns'.

3 Use the mathematical expression x in a sentence that is always true.

4 Circle the entries that are *sentences*:

$\frac{2}{y} - 1$ $\qquad$ $\sqrt{x} > 2$ $\qquad$ $4 - 3 = 7$

KEYWORDS
for this section

English noun, mathematical expression, sentences, Greek letters, the real numbers, $\mathbb{R}$, positive, negative, minus, inequality symbols.

END-OF-SECTION EXERCISES

Classify each entry in the list below as: an expression (EXP), or a sentence (SEN).

In any sentence, circle the verb.

Classify the truth value of any entry that is a *sentence*: TRUE (T), FALSE (F), or SOMETIMES TRUE/SOMETIMES FALSE (ST/SF).

NOTE: The symbol '$\approx$' means '*is approximately equal to*'.

1. $\frac{1}{3}$
2. π
3. $\frac{1}{3} = 0.\overline{3}$
4. $\frac{1}{3} = 0.3\overline{3}$
5. $\frac{1}{3} = 0.33$
6. $\frac{1}{3} = 0.33333$
7. $\frac{1}{3} \approx 0.33$
8. $\frac{1}{3} \approx 0.333333$
9. $x^2 > 0$
10. $y^2 > 0$
11. $x^2 \geq 0$
12. $y^2 \geq 0$
13. $(-3)(-5)$
14. $(-3) + (-5)$
15. $-5 < -3$
16. $-3 < -5$
17. $|t| > 0$ (Need help with absolute values? You might want to skip ahead to the Algebra Review in Section 2.1)
18. $|x| > 0$
19. $|t| \geq 0$
20. $|x| \geq 0$
21. $|t| < 0$
22. $|x| < 0$
23. $|3 - \pi| = \pi - 3$
24. $|\pi - 3| = \pi - 3$
25. $|t| = t$
26. $|t| = -t$
27. $\frac{x}{y} \div \frac{z}{w}$
28. $\frac{x}{y} \cdot \frac{w}{z}$
29. $\frac{x}{y} \div \frac{z}{w} = \frac{xw}{yz}$
30. $\frac{x}{y} \cdot \frac{w}{z} = \frac{xw}{yz}$
31. $a(bc) = (ab)c$
32. $a + (b + c) = (a + b) + c$
33. $3x^2 = (3x)^2$
34. $(2 \cdot 3)^2 = 2 \cdot 3^2$
35. $\sqrt{(-3)^2} = -3$
36. $\sqrt{(-3)^2} = 3$

END-OF-SECTION EXERCISES

(continued)

37 The sentence below is TRUE. What name does this result usually go by? (Dust off your algebra book.)

For all real numbers a and b, $a + b = b + a$.

38 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a and b, $ab = ba$.

39 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a, b, and c, $a(b + c) = ab + ac$.

40 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a, b, and c, $a(bc) = (ab)c$.

41 To mathematicians, subtraction is just a special kind of addition, since for all real numbers x and y,

$$x - y = x + (-y) \ .$$

That is, to subtract a number is the same as to add the opposite of the number.

♣ What is this result telling you if $x = 1$ and $y = 3$? How about if $x = 1$ and $y = -3$?

42 (Refer to the previous exercise.) To mathematicians, division is just a special kind of multiplication. WHY? Be sure to answer in a complete sentence.

43 (Importance of ASSOCIATIVE laws) The word ‘associative’ has the same root as the English words *sociable* and *associate*. These English words have to do with groups (e.g., a *sociable* person is one who enjoys being in a group of people). Thus it should not be surprising that ‘associative’ laws in mathematics have to do with grouping.

The *associative law of multiplication* states: for all real numbers x, y and z, $x(yz) = (xy)z$. Thus, the grouping of numbers in a product is irrelevant. It is *because of this property* that we are able to write xyz, with no parentheses.

♣ Using complete sentences, comment on why the associative law makes expressions like xyz unambiguous.

44 (Refer to the previous exercise.) What property of the real numbers allows us to write things like $a + b + c$ with no ambiguity?

1.2 The Role of Variables

variables

In this section, a name is given to mathematical sentences that are 'sometimes true, sometimes false'—they are called *conditional* sentences. The truth of such sentences depends on the choices that are made for the objects that are allowed to *vary* in the sentence. In mathematics, an object that is allowed to *vary* is appropriately called a *variable*. Variables play a very important role in mathematics, and are the focus of the current section.

sentences come in several flavors

As discussed in the previous section, sentences come in several flavors. There are:

true

TRUE sentences, such as:

- $\pi + (-\pi) = 0$
- $t^2 \geq 0$
- $(-3)^2 = 9$
- $x + x + x = 3x$

Note that the sentence $x+x+x = 3x$ is true for *any* real number x. A sentence with verb '=' is called an *equation*.

false

FALSE sentences, such as:

- $5^0 = 5$
- $\sqrt{(-7)^2} = -7$
- $(3 + 2)^2 = 3^2 + 2^2$
- $x^2 < 0$

Note that the sentence $x^2 < 0$ is false for *all* real numbers x. A sentence with verb $<$, $\leq$, $>$, or $\leq$ is called an *inequality*. The prefix 'in' is commonly used to mean 'not'; as in the English words *inept*, *insane*, and *insecure*. Hence, **in***equality* means, roughly, *NOT equal*.

Mathematicians *hate* to see false sentences written down, except perhaps in a book on logic!

conditional

CONDITIONAL sentences, such as:

- $x = 3$
- $x^2 + 2x + 1 > 0$
- $\sqrt{2x + 1} \neq 5$
- $y = 2x + 4$

This is a very interesting type of sentence. By definition, a *conditional sentence* is one that is sometimes true and sometimes false.

For example, the equation $x = 3$ is true when x is 3 and false when x is not 3. Thus, the *condition* of the sentence depends on the value(s) of the variable(s) involved.

EXERCISE 1
classifying sentences

♣ Classify the following equations as (always) true, (always) false, or conditional.

♣ For those that are conditional, can you say when they are true? False?

1. $2 \cdot 4^2 = 32$ (See Algebra Review—exponents.)
2. $2x^3 = 2 \cdot x \cdot x \cdot x$
3. $x - 3 = 0$
4. $\sqrt{9} = -3$
5. $x + y = 4$
6. $\sqrt{(-9)^2} = -9$
7. $|x| > 0$

'place holders'

When working with conditional sentences, the concept of 'place holder' becomes important. To illustrate this point, consider a familiar example.

solving an equation

Recall from algebra that to *solve an equation* like

$$2x + 4 = 10$$

means to find the number that makes it true. (You learned in algebra that equations of this type have only one solution.) To accomplish this, a number (here denoted by 'x') must be found that has the following property: when it is doubled (multiplied by 2), and 4 is added to it, the result is 10.

Instead of pulling out 'standard' algebra techniques right now, just stop and think. First, what number, when added to 4, yields 10? The number 6. Thus, twice the desired number x must equal 6. Therefore, x must equal 3.

Now if you are asked to solve the equation $2t + 4 = 10$, you should recognize that it has already been done. The only difference is that the letter 't' is used as a place holder, instead of 'x'.

EXERCISE 2
solving simple equations mentally

♣ 1. Without writing anything down, solve the equation $2x - 8 = 6$.

♣ 2. Now solve the equation $(x - 2)^2 = 9$ mentally.

The notion of 'place holder' is much too imprecise for the language of mathematics. To formalize this notion, the concept of *variable* is introduced.

The word *set* appears in the next definition. For now, just think of a *set* as a collection of things (like numbers). Sets will be discussed in more detail in the next section.

DEFINITION
variable
universal set

A *variable* is a symbol (often a letter) that is used to represent a member of a specified set.

This 'specified set' is referred to by mathematicians as the *universal set*.

Thus, the *universal set* gives the objects (often numbers) that we are allowed to draw on for a particular variable.

symbols traditionally used to denote variables

The letters x, y, and t are commonly used in elementary mathematics courses to denote variables, with universal set $\mathbb{R}$.

the role of definitions in mathematics

Definitions are extremely important in mathematics. In order to communicate effectively, people must agree on the meanings of certains words and phrases. English occasionally fails in this respect. Consider the following conversation in a car at a noisy intersection:

Carol: "Turn left!"

Bob: "I didn't hear you! Left?"

Carol: "Right!"

Question: Which way will Bob turn? It depends on how Bob interprets the word 'right'. If he interprets it as the opposite of 'left', he will turn right. If he interprets it as 'correct', he will turn left.

Such ambiguity is not tolerated in mathematics. By *defining* words and phrases, mathematicians assure that everyone agrees on their meaning.

EXERCISE 3
ambiguity in English

♣ Come up with another example of an English word or phrase that is ambiguous, and where this could cause communication problems.

EXAMPLE
same equation, different universal sets

To illustrate the roles that variables and universal sets play in solving equations, consider the equation:

$$x^2 = -1$$

You are asked to solve this equation. There is only one variable, x. What is the universal set? You had better find out, because it will affect your answer.

If the universal set is $\mathbb{R}$, then you must find all *real numbers* which, when squared, equal -1. There are none. So in this case, you would say that there are no solutions to the equation.

Suppose, however, that the universal set is $\mathbb{C}$ (the complex numbers). (See Algebra Review—complex numbers.) In $\mathbb{C}$, there are *two* numbers that make the equation $x^2 = -1$ true; i and $-i$. So in this case two solutions are obtained.

EXERCISE 4
seeing how the universal set affects the solutions of an equation

♣ Solve the equation $x^4 = 1$.

1. First, let $\mathbb{R}$ be the universal set. How many solutions do you get?
2. Next, let $\mathbb{C}$ be the universal set. How many solutions do you get?
3. Finally, let the integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ be the universal set. How many solutions do you get? (See Algebra Review—integers.)

EXERCISE 5

♣ Solve the equation $x^2 = 2$ taking the universal set to be:

1. $\mathbb{R}$
2. the rational numbers (See Algebra Review—rational and irrational numbers.)
3. the irrational numbers
4. the integers

EXERCISE 6

theorem

the Fundamental Theorem of Algebra

In mathematics, the word *theorem* is used to denote a result that is both *true* and *important*.

♣ Look up the *Fundamental Theorem of Algebra* in an algebra book. What is it saying? In particular, it tells us that the complex numbers $\mathbb{C}$ are 'nicer' than the real numbers $\mathbb{R}$ in one way. What way is this?

three main uses of variables

Variables are frequently used:

- **in mathematical expressions to denote quantities that are allowed to change (vary)**

For example, let A denote the area of a circle that has radius r. Then, $A = \pi r^2$. Here, r and A are variables, because they vary from circle to circle. The symbol π, on the other hand, is *not* a variable. It *never* changes. It is called a *constant*.

- **in equations and inequalities to denote a quantity that is initially unknown, but that one would like to know**

For example, in the equation $2x - 3 = 1$, the value of x that makes this true is (initially) unknown. One goal in algebra is to find the appropriate number.

- **to state a general principle**

For example, the sentence

$$\text{For all real numbers } x \text{ and } y,\ x + y = y + x.$$

should be recognized as a precise statement of the *Commutative Law Of Addition*. To 'commute' means to change places; say, to go from home to work and then back again. All commutative laws have the same theme: a 'changing of places' of the objects involved does not affect the final answer.

DEFINITION

constant

A *constant* is a quantity that *does not vary* (that is, remains the same—constant) during some discourse.

symbols traditionally used to denote constants

All specific real numbers are constants: like 5, 0, $\sqrt{2}$ and π.

Symbols are frequently used to denote constants. This can be confusing, because symbols are also used to denote variables. However, there are some strong mathematical *conventions* that exist when naming constants. No reasonable person would use x or y to denote a constant, because these letters are too often used to denote variables.

Frequently, the 'earlier' letters in alphabets, like a, A, α or b, B, β or c, C, γ are used to denote constants.

EXAMPLE

types of equations

$2x + \sqrt{x} - 5 = 0$ is an equation in one variable, x.

$x^2 - x = \sqrt{3}y - 7$ is an equation in two variables, x and y.

$x + \frac{y}{z+x+y} = 4$ is an equation in three variables, x, y, and z.

variables vs. constants

The equation $Ax + By + C = 0$, where A and B are not both zero, is an equation in two variables, x and y. Here, convention dictates that A, B and C are *constants*, not variables. If we genuinely wanted A, B and C to be treated as variables, this would need to be stated explicitly.

The equations that can be expressed in this way $Ax + By + C = 0$ form a very important class of equations called the *linear equations in two variables*. There is one equation for each choice of the constants A, B, and C. For example, $2x+3y-5 = 0$ is a linear equation in the two variables x and y. So is $x-y = \pi$, since it can be written as $x - y - \pi = 0$.

EXERCISE 7

♣ 1. Give an example of a linear equation in the variables w and z.

♣ 2. Give an example of an inequality in the variable x, where this variable appears three times.

♣ 3. A book defines a *quadratic equation* as one that can be written in the form:

$$\alpha x^2 + \beta x + \gamma = 0 \ , \quad \alpha \neq 0$$

According to the conventions in mathematics, what should you assume are variables? What are constants?

solving equations

Now that variables, constants, and equations have been discussed, it's time to talk more precisely about *the solutions of equations*.

DEFINITION

solution of an equation in one variable

satisfy

A *solution of an equation in one variable* is a number (from the universal set) which, when substituted for the variable, makes the equation into a true statement. Such a number is said to *satisfy* the equation.

In this course, the universal set is $\mathbb{R}$, unless otherwise specified. Thus, we will be looking only for REAL NUMBER solutions.

EXAMPLE

- The number 3 is a solution of the equation $x = 3$. Are there any others?
- The numbers 2 and -2 both satisfy the equation $y^2 = 4$. Are there any others?
- The number $\sqrt{3}$ is a solution of the equation $t^2 + 2 = 5$. Are there any others? (Yes, $-\sqrt{3}$.)
- The equation $x^2 = -4$ has no solutions (in the real numbers).

DEFINITION

solution of an equation in two variables

A *solution of an equation in two variables* is a *pair* of numbers which, when substituted for the variables, makes the equation into a true statement.

EXAMPLE

The choices $x = 2$ and $y = 2$ give a (single) solution of the equation $x + y = 4$. Note that this choice of *two* numbers yields only *one* solution; it is incorrect to say '$x = 2$, $y = 2$ **are** solutions of $x + y = 4$'.

The pair $x = 1$, $y = 3$ is another solution. So is $x = 1.1$, $y = 2.9$. Any guesses as to how many solutions there are? (ANS: an infinite number!)

n-tuple
ordered pair

To talk about equations in 2 or more variables, the concept of *n-tuple* is used. An *n-tuple* is an ordered list of n numbers. By convention, the n numbers are separated by commas, and enclosed in parentheses (,).

For example, $(1, 2)$ is a 2-tuple, more commonly known as an *ordered pair*.

The 5-tuple $(1, 2, 3, 4, 5)$ is different from the 5-tuple $(5, 4, 3, 2, 1)$ since the order is important.

DEFINITION

solution of an equation in n variables

A *solution of an equation in n variables* is an n-tuple of numbers which, when substituted for the n variables, makes the equation into a true statement.

A convenient way to denote a typical n-tuple is:

$$(x_1, x_2, x_3, \ldots, x_n)$$

Note that the subscript on the variable tells the position in the list. Thus, x_3 denotes the third number in this list.

EXERCISE 8

Consider the equation in four variables $x_1 + 2x_2 + 3x_3 + x_4 = 0$.

♣ 1. What are the four variables in this equation?

♣ 2. What is meant by *a solution* to this equation?

♣ 3. Find a solution to this equation (at least one solution should be obvious).

♣ 4. Find another solution. Any guesses as to how many solutions there are?

finding ALL the solutions

One is usually interested in finding *all* the solutions of a given equation. To discuss this collection of solutions precisely, the concept of *set* will first be reviewed. This is the subject of the next section.

ALGEBRA REVIEW

exponents, complex numbers, integers, rational & irrational numbers

strength of operations

Recall from algebra that the expression -3^2 means $(-1)\cdot(3^2)$. That is, take 3, square it, and *then* multiply by -1. This is a consequence of the mathematical *conventions* regarding order of operations. Mathematicians have agreed that when no order is specified (say with parentheses), then the *strongest* operations will act first. This agreement is usually referred to as the *order of operations*. But what is the 'strength' of the various operations?

addition and subtraction have equal 'strength'

Start with addition. Since subtraction is a special kind of addition,

$$a - b := a + (-b)\ ,$$

both addition and subtraction have equal strength. The symbol ':=' just used emphasizes that the equality is *by definition*.

multiplication is a sort of 'super-addition'

Multiplication is a sort of 'super-addition'. For example, $3\cdot 4 = 4+4+4 = 3+3+3+3$. Thus, multiplication is 'stronger than' addition. Since division is a special kind of multiplication,

$$\frac{a}{b} := a\cdot\frac{1}{b}\ ,$$

both multiplication and division have equal strength.

exponentiation is a sort of 'super-multiplication'

Exponentiation is a sort of 'super-multiplication'. For example, $2^3 = 2\cdot 2\cdot 2$. Thus, exponentiation is 'stronger than' multiplication.

A sentence that students sometimes use to help them remember the conventions about order of operations is:

Please **E**xcuse **M**y **D**ear **A**unt **S**ally.

'P' stands for **P**arentheses, 'E' for **E**xponents, 'M' and 'D' for **M**ultiplications and **D**ivisions, 'A' and 'S' for **A**dditions and **S**ubtractions.

exponents are short-sighted

With these conventions, the exponentiation must be done before the multiplication in the expression -3^2. One way to remember that -3^2 means $(-1)\cdot(3^2)$ is that *exponents are extremely short-sighted*. When the exponent 2 'looks down' in -3^2, all it 'sees' is a 3, so this is what gets squared. However, when the exponent 2 'looks down' in $(-3)^2$, it sees a group, and in that group is a -3. So this is what gets squared.

$x^0 = 1$ for $x \neq 0$

By definition, $x^0 = 1$ for all real numbers x except 0. The expression 0^0 is undefined, just as division by zero is undefined. Thus,

- $5^0 = 1$,
- $\pi^0 = 1$, and
- $(-\sqrt{7})^0 = 1$.

♣ What is -7^0? $(-7)^0$? x^0? (Careful!)

Why is 0^0 undefined?

Why is 0^0 undefined? Here's one reason. Think of every real number as corresponding to a drawer in a (very large) filing cabinet. Thus, 2, $5-3$, and $\frac{2\pi}{\pi}$ all go into the same drawer. There's also a drawer to accomodate things that are 'not defined', like $\frac{2}{0}$.

Mathematicians want the sentence

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

to *always be true*. That is, for *any* real number x, the names $\frac{x^3}{x^3}$, x^{3-3}, and x^0 should all go into the same 'drawer'. Since $\frac{x^3}{x^3}$ is not defined when x is zero, we also want x^0 to be undefined when x is zero.

The Freshman's Dream

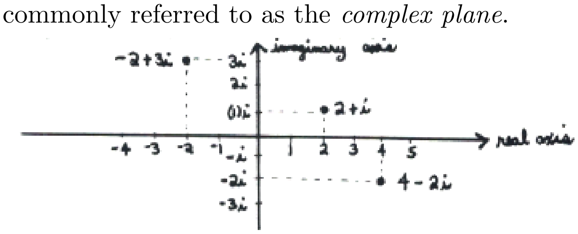
The 'Freshman's Dream' is that $(a+b)^2$ is equal to a^2+b^2. Unfortunately, this is *but* a dream. Take, for example, $a=1$ and $b=1$. Then $(a+b)^2 = (1+1)^2 = 4$ but $a^2+b^2 = 1^2+1^2 = 2$. The correct expression is:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

the number $i := \sqrt{-1}$

If we are only allowed to use real numbers, the equation $x^2 = -1$ has no solution. (♣ Why not?) To accomodate situations such as this, the real numbers can be 'extended' to a larger number system, by defining *one* new number. This 'new' number is usually denoted by the letter i, and is defined by $i := \sqrt{-1}$. That is, i is a number which, when squared, equals -1.

By introducing this single new number i, and using the scheme illustrated below, we are given access to a whole *plane* of numbers, called the *complex numbers*. This plane is commonly referred to as the *complex plane*.



the complex numbers, $\mathbb{C}$

More precisely, the *complex numbers*, denoted by $\mathbb{C}$, are numbers that can be expressed in the form

$$a + bi \ ,$$

where a and b are real numbers, and $i := \sqrt{-1}$. Remember that the symbol ':=' is used to emphasize that this is the *definition* of i.

The complex numbers include all the real numbers (just take $b = 0$), in addition to many numbers that are *not* real.

Electrical engineers use the complex numbers in studying current flow, and by so doing are able to eliminate a great deal of the drudgery involved in analyzing sinusoidal circuits. However, electrical engineers have to give $\sqrt{-1}$ a different name, because i is already used to denote current. So they define $j := \sqrt{-1}$.

integers

The *integers* are the numbers:

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The dots ' ... ' indicate that the established pattern is to be repeated *ad infinitum*. The symbols i, j, k, m and n are often used to denote integers. For example, when a mathematician says *For all* $n > 1$, that mathematician is allowing the possibility that n is 2 or 100 or 2017, but not 5.2.

rational numbers

The *rational numbers* are those real numbers that can be expressed as a ratio of integers, with non-zero denominator. The word 'ratio' imbedded in 'rational' should help you remember this definition. Thus, $\frac{2}{3}$ is a rational number.

Is 5 rational? Certainly: it can be written as $5 = \frac{5}{1} = \frac{-10}{-2} = \dots$. Note that the symbols 5, $\frac{5}{1}$ and $\frac{-10}{-2}$ all represent the unique real number 5, but the last two are more convenient names to use when determining that 5 is a rational number.

irrational numbers

The *irrational numbers* are those real numbers that are *not* rational. Thus, they cannot be expressed as a ratio of integers. The prefix 'irr' is commonly used in English to negate: consider *irregular*, *irreconcilable*, *irrelevant*, and *irresponsible*. You might recall from algebra that irrational numbers can alternately be described as the real numbers having infinite, non-repeating decimal expansions. So—we can't write an irrational number as a ratio of integers, and we can't write it in decimal form. How should we discuss it? Answer: give it a symbolic name (like π, or e, or $\sqrt{2}$)!

the irrational number π

The most familiar irrational number is π. The most common *approximations* to π are:

$$\pi \approx \frac{22}{7} \quad \text{and} \quad \pi \approx 3.14159$$

Precisely, π is the ratio of the circumference to diameter of *any* circle. Here's a great elementary school classroom exercise: have students bring in a circular object. Give them each a piece of string and a ruler. Have them measure the circumference of the circle and its diameter, then walk to the class computer and compute:

$$\frac{\text{circumference}}{\text{diameter}}$$

Within measuring error, each student will get a number close to 3.1!

QUICK QUIZ

sample questions

1. According to normal conventions in mathematics, what are the variables in the equation
$$Ax^2 + Bxy + Cy^2 = 0\ ?$$
What are the constants?
2. Solve $x^2 = 3$ taking the universal set to be $\mathbb{R}$. Then, take the universal set to be the integers.
3. What does is mean to 'solve' an equation? (Answer in English.) Give three solutions of the equation $x + y = 4$.
4. Classify each of the sentences
$$x^2 \ge 0\ , \quad x > 0$$
as (always) true, (always) false, or conditional.
5. List two of the three main uses for variables.

KEYWORDS
for this section

Conditional sentences, equation, inequality, variable, universal set, constant, complex numbers, integers, rational numbers, irrational numbers, solution of an equation in 1, 2, *and* n *variables, satisfying an equation,* n*-tuple, definition, theorem.*

You should know what letters are commonly used to denote variables and constants. You should also know what letters are commonly used to denote integers.

END-OF-SECTION EXERCISES

Answer ALL ODD numbers except Item No. 19

♣ Classify each entry in the list below as: an expression (EXP), or a sentence (SEN).

♣ Classify the truth value of any entry that is a *sentence*: (always) TRUE (T), (always) FALSE (F), or CONDITIONAL (C).

1. π
2. $\pi > 3$
3. $\pi = 3.14$
4. $i^2 + 1 = 0$
5. $\pi \approx 3.14$
6. A common rational approximation to π is $\frac{22}{7}$.
7. π is expressable as a ratio of integers.
8. The number i satisfies $y^2 = -1$.
9. The equation $3x^2 + 2x - \sqrt{x} = 0$ has three variables.
10. In the equation $ax^2 + bx + c = 0$, the variables are a, b, c and x.
11. In this course, the universal set is assumed to be the real numbers.
12. The 3-tuple $(1, 2, 3)$ is identical to the 3-tuple $(2, 1, 3)$.
13. $x^0 - 1$
14. $(y - z)^2 = y^2 - z^2$

♣ Solve each equation, taking the universal set to be:

a) $\mathbb{R}$

b) the rational numbers

c) the integers

15. $x^3 - 1 = 0$
16. $x^2 = 7$
17. $(x - 1)(x + \pi)(2x - 3) = 0$
18. $x(x^2 - 4)(x^2 - 2) = 0$
19. The Fundamental Theorem of Algebra tells us that the equation $x^3 = 1$ must have three solutions in $\mathbb{C}$.

    a) Plot the points 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ in the complex plane.

    b) Show that all these complex numbers lies on the circle of radius 1, centered at the origin. (Hint: use Pythagorean's Theorem.)

    c) Show that all these complex numbers satisfy the equation $x^3 = 1$.

    d) Solve $x^3 - 1 = 0$ with universal set $\mathbb{C}$.

1.3 Sets and Set Notation

Introduction

In algebra, the word *set* appears when discussing the *solution set* of an equation. There are many other places where sets are important. For example, sets will be used to simplify our discussion of graphing equations and functions. The focus of this section is sets, and the notation used in connection with sets.

DEFINITION

sets

well-defined

A *set* is a well-defined collection of objects. *Well-defined* means that, given any object, either the object *is* in the set, or *isn't* in the set.

EXAMPLE

a non-set

"*The collection of some people*" is not a set. It is not well-defined. To see this, observe that we cannot definitively answer the question: Is 'Carol' in this collection?

EXAMPLE

a set

"*The collection of all irrational numbers with the digit* 5 *in the* 10^{-2013} *slot of their decimal expansion*" is a set. Call it S. Given any number, either it is in S or it isn't. For example, 3 isn't in S, because 3 isn't irrational. Is π in S? This author doesn't know. But either it *does* have a digit 5 in the appropriate slot, or it doesn't. No other choices are possible.

Thus, to qualify as a set, one need only be certain that *any* object is either in the collection, or not. It's not necessary to know which of these two situations occurs.

NOTATION

$\in$, $\notin$

The sentence $x \in S$ means *x is an element of S*. It can also be read as:

x is a member of S
x belongs to S
x is in S

The sentence $x \notin S$ means *x is NOT an element of S*.

♣ What is the 'verb' in the sentence $x \in S$?

NOTATION

set notation

roster (list) method

The members of a set are often separated by commas, and enclosed in braces $\{\ \}$. That is, the elements are listed; this is called the *roster* or *list* method.

If the set contains many elements, then it is often convenient to use *dots* to continue an established pattern. This is illustrated by the following examples:

EXAMPLE

- The set $\{a, b, c\}$ contains 3 elements, a, b, and c. Roster notation is particularly useful when a set contains a small finite number of elements.
- The set of *counting numbers* is $\{1, 2, 3, \dots\}$. The dots indicate that the established pattern continues ad infinitum.
- The set $\{1, 2, 3, \dots, 100\}$ enumerates the counting numbers between 1 and 100, inclusive.
- Let $S = \{\ 1,\ \{1\},\ \{1, \{1\}\}\ \}$. Here, S is a set containing (among other things) sets. There are three elements in S: 1, $\{1\}$, and $\{1, \{1\}\}$. Thus it is correct to say: $1 \in S$, $\{1\} \in S$, and $\{1, \{1\}\} \in S$.

EXERCISE 1

♣ Let $S = \{2,\ \pi,\ \{2, \pi\},\ \{2\}\}$. How many elements does S have? What are they?

NOTATION

set-builder notation

Even more important for large (usually infinite) sets is the following *set-builder notation*:

Let $\mathcal{U}$ denote a universal set. The notation

$$\{x \in \mathcal{U} \mid \text{some property that } x \text{ is to satisfy}\}$$

is *extremely* useful in many cases where roster notation fails. Here, the vertical bar ‘ | ’ is read as *such that* or *with the property that*. The set includes all elements from the universal set that satisfy the stated property.

If the universal set is understood, one can more simply write:

$$\{x \mid \text{some property that } x \text{ is to satisfy}\}$$

EXAMPLE

For example,

$$\{x \mid x \text{ is a counting number and } x \geq 4\}$$

is read as *the set of all x such that x is a counting number and x is greater than or equal to* 4.

Another way to denote this set would be $\{4, 5, 6, \ldots\}$. Yet another way would be $\{y \mid y \text{ is an integer and } y \geq 4\}$. Thus, we see that a given set can be expressed in different ways.

EXERCISE 2

♣ 1. Describe the set $\{-3, -2, -1, 0, 1, 2, 3\}$ in two different ways.

♣ 2. Describe the set $\{\ldots, -1, 0, 1, 2\}$ in two different ways.

EXAMPLE

The set $\{N \mid N \text{ is a name that begins with C}\}$ is read as *the set of all N with the property that N is a name that begins with C*. Here, the universal set is understood to be the set of all possible names. Thus, the name *Carol* is an element of this set, but *Karol* is not. Observe that it would be extremely difficult to describe this set without set-builder notation.

NOTATION

interval notation

(,); endpoints are not included

[,]; endpoints are included

Interval notation is a particularly convenient way to denote intervals of real numbers.

Recall that the symbol := means *equals, by definition*. Define

$$\begin{aligned}
(a, b) &:= \{x \in \mathbb{R} \mid a < x < b\} \\
[a, b) &:= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &:= \{x \in \mathbb{R} \mid a < x \leq b\} \\
(a, \infty) &:= \{x \in \mathbb{R} \mid x > a\} \\
(-\infty, b] &:= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned}$$

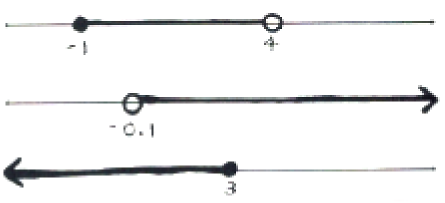
Other combinations are possible. Compound inequalities like $a < x < b$ are investigated in the Algebra Review at the end of this section, and the mathematical word ‘and’ is introduced.

Note how convenient set-builder notation is for these definitions. Observe that:

- *parentheses* (,) are used when an endpoint is *not* to be included
- *brackets* [,] are used when an endpoint *is* to be included
- The symbol ∞ is always used with parentheses. This is because ∞ is not a real number. It's more of an idea: given *any* real number, another real number can always be found that is greater.

EXERCISE 3

♣ Use interval notation to describe the sets shown below. (A solid dot indicates that an endpoint is included; a hollow dot indicates that an endpoint is excluded.)



(a,b) has two different meanings

The careful reader will observe that the notation (a,b) can be used to denote an *interval*, or an *ordered pair*. Context will determine which interpretation is correct.

NOTATION

empty set

$\emptyset$

There is exactly one set containing no elements. It is called the *empty set*, and is denoted by either $\{\ \}$ or $\emptyset$.

Computer scientists use the symbol $\emptyset$ for the number zero, to distinguish it from the capital letter 'oh'. So if you are communicating with a computer scientist, it is probably better to use $\{\ \}$ to denote the empty set.

capital letters are used to denote sets

Capital letters, like A, $\mathcal{B}$, S and Γ, are commonly used to denote sets.

Now, we are ready to define the *solution set* of an equation.

DEFINITION

solution set of an equation

solving an equation

The *solution set* of an equation is the set of all its solutions. To *solve an equation* means to find its solution set (i.e., find all solutions.)

EXAMPLE

solution sets

the quadratic formula

- The solution set of $x^2 = 4$ is $\{2, -2\}$.
- The solution set of $ax^2 + bx + c = 0$ for $a \neq 0$ is:

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

  You learned this when you studied the *quadratic formula* in algebra.

  ♣ What is (are) the variable(s) in the equation $ax^2 + bx + c = 0$? Judging by the solution set given above, what is the universal set?

- The solution set of $x - 1 = 0$ is $\{1\}$.
- The solution set of $x = 0$ is $\{0\}$.

EXERCISE 4

♣ 1. Based on the definition of the *solution set of an equation*, write a precise definition for the *solution set of an inequality*.

♣ 2. Solve the following simple inequalities in one variable. Where possible, use interval notation for the solution sets.

a) $x > 4$

b) $y^2 \ge 0$

c) $t^2 < 0$

d) $|x| \le 1$

♣ 3. How many solutions do inequalities usually have?

Now comes the $100 question: How do we go about *finding* the solution set of a given equation (or inequality)? This is the topic of the next section.

the mathematical word 'and'

Be sure to read the algebra review that follows, since the precise meaning of the mathematical word 'and' is introduced.

ALGEBRA REVIEW

compound inequalities, mathematical word 'and', integers, rational numbers

compound inequalities

A sentence like $a < x < b$ that uses more than one inequality symbol is called a *compound inequality*.

The compound inequality $a < x < b$ is really just a shorthand for two simple inequalities, connected by the mathematical word 'AND':

$$a < x \quad \text{AND} \quad x < b$$

Thus, to truly understand compound inequalities, one must understand the mathematical word 'and'.

the mathematical word 'and'

Let P and Q denote mathematical sentences that are either true or false. For example, P might be the true sentence '$3 > 1$'. Similarly, Q could be the false sentence '$5 < 5$'. The mathematical word 'AND' gives a way to *combine* the sentences P and Q into a 'bigger' mathematical sentence.

By definition, a mathematical sentence of the form

$$P \quad \text{AND} \quad Q$$

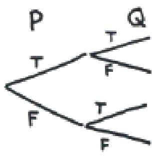
is true exactly when P is true and Q is true.

For example, the mathematical sentence '$1 < 3$ and $3 < 5$' is true, since both $1 < 3$ and $3 < 5$ are true.

However, the sentence '$3 < 3$ and $3 < 5$' is false, because $3 < 3$ is false.

truth table for 'P and Q'

There is a more precise way to discuss the mathematical sentence 'P and Q'. Note that the truth value (true or false) of this sentence depends on the truth values of P and Q. P can be true or false. Q can be true or false. Together, there are four possible combinations of truth values, which are summarized in the truth table below.



P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

This truth table shows that:

- When P is true and Q is true, the sentence 'P and Q' is true.
- When P is true and Q is false, the sentence 'P and Q' is false.
- When P is false and Q is true, the sentence 'P and Q' is false.
- When P is false and Q is false, the sentence 'P and Q' is false.

EXERCISE 5

Determine the truth value (T or F) of the following sentences:

♣ 1. $\pi > 3$ and $|2| = 2$

♣ 2. $|3 - \pi| > 0$ and $-3^2 = -9$

♣ 3. $3 \le 3$ and $-2 < -4$

♣ 4. $1 \in (1, 3)$ and $1 \in [1, 3)$

♣ 5. $1 \in \{x \mid x < 1\}$ and $1 \in \{\{1\}\}$

Determine the value(s) of x for which each of the following sentences is true; false. Where possible, use interval notation to express your answer.

♣ 6. $x > 0$ and $x > 2$

♣ 7. $x > 0$ and $x < 2$

♣ 8. $x > 0$ and $x < -2$

return to compound inequalities

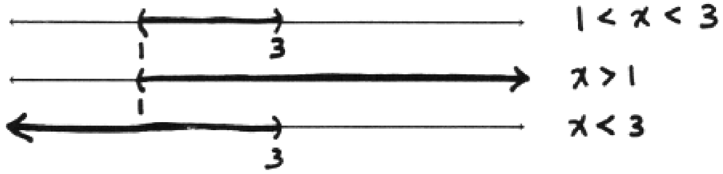
With the precise meaning of 'and' now in hand, we can return to a study of compound inequalities.

For what values of x will the mathematical sentence

$$a < x \text{ AND } x < b$$

be true? Only for values of x for which both $a < x$ ($x > a$) is true, and $x < b$ is true. That is, only for the values of x which are greater than a, and (at the same time) less than b.

The sketches below illustrate this construction for the compound inequality $1 < x < 3$.



EXERCISE 6

♣ 1. Make a similar sketch to explain the compound inequality:

$$2 \le x < 5$$

♣ 2. Discuss the meaning of the compound inequality $3 < x < 1$. Are there any choices for x that make this true? Does it make sense to write $a < x < b$ if a is greater than b?

EXERCISE 7

$\mathbb{Z}$
integers

The symbol $\mathbb{Z}$ is frequently used to denote the set of integers, so we can write

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The German word for 'numbers' is 'zahlen', which could explain the choice of the letter $\mathbb{Z}$.

♣ What are the positive integers? negative? nonnegative? nonpositive? Answer using correct set notation. (Hint: The word *nonnegative* means *not negative*. Thus, using interval notation, the nonnegative real numbers are $[0, \infty)$. Figure out what *nonpositive* means.)

EXERCISE 8

$\mathbb{Q}$ *(for 'Quotient')*
rational numbers

The symbol $\mathbb{Q}$ is frequently used to denote the set of rational numbers (since they are *Quotients* of integers). By long division, an equivalent characterization of $\mathbb{Q}$ is the set of all real numbers with finite or infinite repeating decimal representations. For example, $\frac{1}{7} = 0.\overline{142857}$ and $\frac{2}{5} = 0.4$.

♣ 1. Using long division, find the decimal representations of $\frac{1}{6}$, $\frac{2}{7}$ and $\frac{1}{25}$.

A rational number is in *reduced form* if there are no factors common to both numerator and denominator. For example, $\frac{6}{8}$ is not in reduced form, since:

$$\frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4}$$

A rational number in reduced form can be expressed as a *finite* decimal only if the denominator has no factors other than 2's and 5's. For example, $\frac{7}{400}$ has a finite expansion since:

$$\frac{7}{400} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} \cdot \frac{5 \cdot 5}{5 \cdot 5} = \frac{175}{10^4} = 0.0175$$

♣ 2. Why is it that fractions with only 2's and 5's downstairs can be written as finite decimals? (Study the previous example.)

♣ 3. Decide (without using your calculator) if $\frac{3}{120}$ has a finite decimal expansion.

♣ 4. Decide if $\frac{41}{333}$ has a finite decimal expansion.

♣ 5. Decide if $\frac{10}{81}$ has a finite decimal expansion.

QUICK QUIZ
sample questions

1. T or F: the set $\{1, 2, \{3, 4\}\}$ has 4 members.
2. T or F: $1 \in (1, 3)$.
3. T or F: $3 \in \{t \in \mathbb{R} \mid 2 < t < 5\}$
4. T or F: $3 > 3$ and $3 \le 3$
5. T or F: $\frac{3}{105}$ has a finite decimal expansion.

KEYWORDS *for this section*	*Sets, well-defined, roster method, set-builder notation, interval notation, empty set, solution set of an equation, solving an equation, quadratic formula, nonnegative, nonpositive, compound inequality, mathematical word 'and'.* You should know the symbols $\in$, $\notin$, $\{\ \}$, $\emptyset$, $:=$, $\mathbb{Z}$, and $\mathbb{Q}$. You should know what types of letters are commonly used to denote sets.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\{3\}$
2. $\{1,2,3\}$
3. $1 \in \{1,2,3\}$
4. $0 \in [0, \frac{1}{2})$
5. $0 \in (0, \frac{1}{2})$
6. $\frac{1}{2} \in [0, \frac{5}{9})$
7. $x \in S$
8. $1 \notin \{1,2,3\}$
9. $x \in \{1,2,3\}$
10. $1 \in \{t \,|\, t \le 1\}$
11. $\{x \,|\, x \ge 1\}$
12. $1 \in \{\ \{1\}\ ,\ \{1,\{1\}\}\ \}$
13. $\{1\} \in \{\ \{1\}\ ,\ \{1,\{1\}\}\ \}$
14. $x > 1$ and $x < 1$
15. $y \ge 1$ and $y \le 1$
16. $x \le 3$ and $x > 5$
17. $|x| \ge 0$ and $x^2 \ge 0$
18. $|x| > 0$ and $x^2 \ge 0$
19. The set $\{\ \{1\}\ ,\ \{1,\{2\}\}\ \}$ has two elements.
20. The set $\{\ \{a\}\ ,\ \{b,c\}\ \}$ has three elements.
21. The number $\frac{3}{7}$ has a finite decimal expansion.
22. The number $\frac{7}{35}$ has a finite decimal expansion.

1.4 Mathematical Equivalence

Introduction

In this section, the idea of *mathematical equivalence* is introduced. Whereas the '=' sign gives us a way to compare mathematical *expressions*, the idea of 'being equivalent' gives us a way to compare mathematical *sentences*.

a motivating example

For motivation, consider the two mathematical sentences: '$2x - 3 = 0$' and '$x = \frac{3}{2}$'. They certainly *look* different. But in one very important way, they are the same: no matter what real number is chosen for the variable x, these two sentences *always have the same truth values*.

For example, choose x to be $\frac{3}{2}$.

Substitution into '$2x - 3 = 0$' yields the TRUE sentence '$2(\frac{3}{2}) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the TRUE sentence '$\frac{3}{2} = \frac{3}{2}$'.

Next, choose x to be, say, 5.

Substitution into '$2x - 3 = 0$' yields the FALSE sentence '$2(5) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the FALSE sentence '$5 = \frac{3}{2}$'.

No matter what real number is chosen for x, these two sentences will ALWAYS have the same truth values. Indeed, '$2x - 3 = 0$' is true when x is $\frac{3}{2}$, and false otherwise. Also, '$x = \frac{3}{2}$' is true when x is $\frac{3}{2}$, and false otherwise.

sentences that always have the same truth values can be used interchangeably

When two mathematical sentences always have the same truth values, then they can be used *interchangeably*, and a mathematician will use whichever sentence is easiest for a given situation.

The mathematical verb that is used to compare sentences with the same truth values is: 'is equivalent to'. Thus, it is correct to say that '$2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$'.

the 'implied domain' of a sentence

Sentences naturally have a largest set of 'choices' for which the sentence is defined.

For example, the sentence '$\frac{1}{x} = 1$' is only defined for nonzero real numbers, since division by zero is undefined.

The sentence '$\sqrt{x} = 3$' is only defined for nonnegative real numbers.

The largest set of choices for which a sentence is defined will be referred to as the '*implied domain*' of the sentence, or more simply, the '*domain*' of the sentence. (Something is 'implied' if it is not explicitly stated, but merely understood.) Remember that, in this course, we are only 'choosing from' the real numbers. This idea is explored further in the next example.

EXAMPLE

finding 'implied domains' of sentences

Problem: Find the implied domain for each of the following sentences:

1. $\frac{1}{x(y-1)} = 2$
2. $\sqrt{x} > 0$
3. $\sqrt[3]{x} = -2$
4. $ax + by + c = 0$

Solution:

1. The expression '$\frac{1}{x(y-1)}$' is not defined if x is 0, or if $y = 1$. Thus, the implied domain of the sentence is $\{(x, y) \,|\, x \neq 0 \text{ and } y \neq 1\}$.

2. The expression '$\sqrt{x}$' is only defined for nonnegative real numbers x. Thus, the implied domain of the sentence is $\{x \,|\, x \geq 0\}$.

3. The expression '$\sqrt[3]{x}$' makes sense for all real numbers x. The implied domain of the sentence is $\mathbb{R}$.

4. In the sentence '$ax + by + c = 0$', convention dictates that only the x and y are variables; a, b and c are constants. This sentence is defined for all real numbers x and y; thus, the implied domain is $\{(x, y) \,|\, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

EXERCISE 1

finding implied domains

Find the implied domain for each sentence. Write your answers using correct set notation.

♣ 1. $\frac{x}{y} = 1$

♣ 2. $ax + b = 0$

♣ 3. $\sqrt{x^2} - 4 = 0$

♣ 4. $\sqrt{x^3} = 2$

♣ 5. $\sqrt[4]{xy} - x = y - 5$. (Here, you may want to merely *shade* the allowable choices for (x, y) in the xy-plane.)

DEFINITION

equivalent sentences

Two mathematical sentences (with the same domains) are *equivalent* if they always have precisely the same truth values. That is, no matter what choice of variable(s) is made from the domain, if one sentence is true, so is the other; and if one sentence is false, so is the other.

EXAMPLE

equivalent sentences

The sentences $x + 1 = 0$ and $x = -1$ are equivalent. Each sentence has domain $\mathbb{R}$, because each is defined for all real numbers.

The sentence '$x + 1 = 0$' is true exactly when x is -1, and false otherwise.

The sentence '$x = -1$' is true exactly when x is -1, and false otherwise.

the symbol '$\iff$'

The symbol '$\iff$' is used by mathematicians to say that two sentences are equivalent. Thus, the sentence

$$x + 1 = 0 \quad \iff \quad x = -1$$

(read as '$x + 1 = 0$' is equivalent to '$x = -1$') means that the two component sentences being compared *always have the same truth values.*

★★

Experienced mathematicians realize that the sentence

$$x + 1 = 0 \iff x = -1$$

is really an implicit generalization:

$$\text{For all } x,\ x + 1 = 0 \iff x = -1\ .$$

The quantifier 'For all' is addressed later on in the text.

The precise definition of the connective ' $\iff$ ' is given by the following truth table:

A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

determining if two sentences are equivalent, by comparing their solution sets

Suppose that two sentences have the same domains. If these two sentences have the same solution sets, then they must be TRUE at exactly the same time. Therefore, they must also be FALSE at exactly the same time.

This observation allows us to determine if two sentences (with the same domain) are *equivalent* by comparing their solution sets.

EXAMPLE

sentences that are NOT equivalent

The equations $x^2 = 9$ and $x = 3$ are *not* equivalent. (Remember that sentences using the verb '=' are called *equations*.) They both have domain $\mathbb{R}$, since both are defined for all real numbers. However, the first has solution set $\{3, -3\}$, whereas the second has solution set $\{3\}$.

That is, choose x to be -3. For this choice, the sentence '$x^2 = 9$' becomes '$(-3)^2 = 9$', which is TRUE; whereas the sentence '$x = 3$' becomes '$-3 = 3$', which is FALSE. The sentences do NOT always have the same truth value. They CANNOT be used interchangeably.

showing that two sentences are NOT equivalent

The previous example points out that to show that two sentences are NOT equivalent, we need only exhibit ONE choice of variable(s) for which the sentences have different truth values.

For example, the equations '$x = 0$' and '$x(x - 1) = 0$' are NOT equivalent. Choosing x to be 1, the first sentence is false, but the second is true.

EXAMPLE

sentences with different domains

The sentences '$\frac{1}{x} = 1$' and '$x = 1$' have different domains. The first is defined only for nonzero real numbers; the second for *all* real numbers.

However, these sentences ARE very much alike: the first is undefined when x is 0, true when x is 1, and false otherwise. The second is true when x is 1, and false otherwise. As long as we restrict ourselves to choices for x for which BOTH sentences make sense, then they do act exactly the same.

For now, however, we will only compare sentences that have the SAME domains.

EXERCISE 2

Check that each sentence in a given pair has the same implied domain. Then, decide if the sentences are equivalent, or not.

- ♣ 1. $2x - 4 = 0$, $3x - 6 = 0$
- ♣ 2. $x^2 - 16 = 0$, $x = 4$
- ♣ 3. $|x| = 3$, $x = 3$
- ♣ 4. $|x| = 3$, '$x = 3$ and $x = -3$' (Careful!)
- ♣ 5. $x > 0$, $2x > 0$
- ♣ 6. $x > 0$, $-x < 0$
- ♣ 7. $x^2 \geq 0$, $x \geq 0$
- ♣ 8. $1 < x \leq 3$, '$x > 1$ and $x \leq 3$'

EXAMPLE

solving by inspection

The equations $2x + 3 = 0$ and $x = -\frac{3}{2}$ are equivalent; they have the same domains, and the same solution sets. Note that the second equation is 'simpler' than the first, in the sense that *it can be solved 'by inspection'*. That is, looking at the equation $x = -\frac{3}{2}$, it is *immediate* what makes it true: there is only one real number that is equal to $-\frac{3}{2}$.

EXERCISE 3

Which of the following sentences (if any) would you say can be solved '*by inspection*'?

- ♣ 1. $7x - 3 = 0$
- ♣ 2. $x^2 \geq 0$
- ♣ 3. $(x + 1)^2 \geq 0$
- ♣ 4. $x = -0.2$
- ♣ 5. $x < 0$
- ♣ 6. $x^3 < 0$

solving an equation

The goal in solving an equation is to transform the original (harder) equation into an *equivalent* one that can be solved easily. However, in this "transforming" process, we must be sure that we do *not* change the solution set! Thus we are interested in answering the question: *What can be done to an equation that does not alter its solution set*? The next two theorems answer this question:

THEOREM (Form A)

Let a, b and c be real numbers. Then:

$$a = b \iff a + c = b + c$$

If $c \neq 0$, then:

$$a = b \iff ac = bc$$

THEOREM (Form B)

Adding the same number to both sides of an equation does not change its solution set.

Subtracting the same number from both sides of an equation does not change its solution set.

Multiplying both sides of an equation by any nonzero number does not change its solution set.

Dividing both sides of an equation by any nonzero number does not change its solution set.

interpreting this theorem

Both of these theorems say exactly the same thing. Form A is the way a mathematician would give the answer, and this is the form that would appear in most textbooks.

Form B is the *translation* of Form A that an instructor makes so that the students can understand it. Most beginning students have absolutely no idea what Form A is saying, because they don't understand the language in which it is expressed.

Most students will have relatively few problems with Form B, because it tells them *what they can do*. "You can add the same number to both sides of an equation" gives the student something that they can *do*.

On the contrary, Form A is just a statement of *fact*. Most students don't recognize Form A as telling them what they can *do*. But it does. *Facts can tell you what to do, once you learn to make the correct translation.*

good reference material

Warren W. Esty of Montana State University has written a delightful text entitled **The Language of Mathematics**, that directly addresses student difficulties with the language of mathematics. It is highly recommended reading for students of mathematics at all levels.

how to translate Form A

So how is the student to translate Form A?

The first sentence says *Let a, b and c be real numbers.* This sentence tells the reader that the universal set for the variables a, b and c is $\mathbb{R}$: that is, until otherwise notified, when the reader sees the symbols a, b and c, they are assumed to represent real numbers.

CAUTION: Just because the symbol a is different from the symbol b does not mean that our choice for a must be different than the choice for b! The variable a can be any real number; the variable b can be any real number. If desired, we can choose both to be, say, 2.

Next comes a statement that two sentences are equivalent: $a = b$ is equivalent to $a + c = b + c$. This says that, *no matter what real numbers are represented by a, b and c*, the sentence $a = b$ will have the same truth value as the sentence $a + c = b + c$.

For example, suppose that we make the choices $a = 3$, $b = 3$ and $c = 4.2$. In this case the sentence '$a = b$' becomes '$3 = 3$', which is true. The sentence '$a + c = b + c$' becomes '$3 + 4.2 = 3 + 4.2$', which is also true.

As a second example, suppose that $a = 3$, $b = 2$ and $c = -1$. Then the sentence '$a = b$' becomes '$3 = 2$', which is false. The sentence '$a + c = b + c$' becomes '$3 + (-1) = 2 + (-1)$', which is also false.

Now we can rephrase the sentence:

$$a = b \quad \iff \quad a + c = b + c$$

(Note that this entire display is a mathematical sentence, which compares the 'smaller' mathematical sentences $a = b$ and $a + c = b + c$, telling us that they always have the same truth value, and hence can be used interchangeably.)

This FACT tells us that we never change the truth of an equation by adding the same number to both sides. In other words, adding the same number to both sides of an equation doesn't change its solution set.

subtraction is a special kind of addition

But what about subtraction? Does the sentence

$$a = b \iff a + c = b + c$$

also tell us that we can *subtract* the same number from both sides of an equation? Of course! Subtraction is just a special kind of addition:

$$x - y := x + (-y)$$

To subtract a number means to add its opposite. That is, to subtract y means to add $-y$. Notice that the left-hand side '$x - y$' of the sentence illustrates a pattern; the right-hand side '$x + (-y)$' tells us what to *do* with this pattern.

By using the language of mathematics, we get two for the price of one: *one* mathematical sentence has told us that both adding and subtracting the same number to (from) both sides of an equation doesn't change its solution set.

EXERCISE 4

♣ 1. What is a *theorem*?

♣ 2. Discuss the meaning of this mathematical theorem:
For all real numbers x, y and z:

$$x = y \iff x + z = y + z$$

If we choose $x = 3$, $y = 4$ and $z = 5$, what is this theorem telling us?

♣ 3. Does this 'theorem' make sense:
For all real numbers a, b and c:

$$x = y \iff x + z = y + z$$

Why or why not?

Continuing our translation of Form A, we come to the sentence:
If c is not equal to zero, then:

$$a = b \iff ac = bc$$

The first part of the sentence informs us that the universal set for c has changed: now, whenever the reader sees the variable c, is it assumed to be a *nonzero* real number. But as long as c is nonzero, the sentences '$a = b$' and '$ac = bc$' will always have the same truth values. In particular, they're both true at exactly the same times. Thus, multiplying both sides of an equation by a nonzero number won't change its solution set.

EXAMPLE

For example, take $a = b = 7$ and $c = -2$. Then the sentence '$a = b$' becomes '$7 = 7$', which is true. The sentence '$ac = bc$' becomes '$(7)(-2) = (7)(-2)$', which is also true.

If we take $a = 0$, $b = -7$ and $c = 3$, then the sentence '$a = b$' becomes '$0 = -7$', which is false. Also, '$ac = bc$' becomes '$(0)(3) = (-7)(3)$', which is also false.

division is a special kind of multiplication

But what about division? Providing $c \neq 0$, does the sentence

$$a = b \iff ac = bc$$

tell us that we can *divide* both sides of an equation by the same nonzero number? Of course! Division is just a special type of multiplication: for $y \neq 0$,

$$\frac{x}{y} := x \cdot \frac{1}{y}$$

To divide by y means to multiply by the reciprocal of y. Again, we get two for one.

EXERCISE 5

Consider this theorem:

For all real numbers a, b, and c:

$$a < b \iff a + c < b + c$$

♣ 1. What is this theorem telling you that you can *do*? Answer in English.

♣ 2. What is the theorem telling you when $a = 1$, $b = 2$ and $c = 3$?

♣ 3. What is the theorem telling you when $a = 2$, $b = 1$ and $c = 3$?

♣ 4. How might an algebra instructor 'translate' this theorem for the students?

EXERCISE 6

Consider this theorem:

For all real numbers a and b, and for $c < 0$:

$$a < b \iff ac > bc$$

♣ 1. What is this theorem telling you that you can *do*?

♣ 2. How might an algebra instructor 'translate' this theorem for the students?

♣ 3. Write a theorem, the way a mathematician would, that tells you that multiplying both sides of an inequality (using '$<$') by a positive number gives an equivalent inequality in the same direction.

EXAMPLE

The next example is *extremely important*. It illustrates the basic procedure used in solving a (simple) equation. It should seem trivial to you—but make sure you understand *why* you're doing what you're doing.

Problem: Solve the equation $-3x + 5 = 2$.

Solution: Find an *equivalent equation* that can be solved by inspection. That is, transform the original equation into an equivalent one of the form $x =$ *some number*.

$-3x + 5 = 2$	Begin with the original equation.
$(-3x + 5) - 5 = 2 - 5$	Isolate the x term by subtracting 5 from both sides. By the theorem, this does not change the solution set.
$\dfrac{-3x}{-3} = \dfrac{-3}{-3}$	Divide both sides by -3. By the theorem, this does not change the solution set.
$x = 1$	The result is an equivalent equation that can be solved by inspection.
	Since the solution sets of $x = 1$ and $-3x + 5 = 2$ are the same, the solution set of $-3x + 5 = 2$ is $\{1\}$.

$\iff$ *is implicit*

In practice, the solution to this problem would be written down as follows, merely as a list of equations:

$$\begin{aligned} -3x + 5 &= 2 \\ -3x &= -3 \\ x &= 1 \end{aligned}$$

Each line is a complete mathematical sentence. However, the sentences aren't put together into a cohesive paragraph. The reader is left to *guess* what the connection is between, say, $-3x = -3$ and $x = 1$ (they are equivalent).

NOTATION

$\iff$ *is explicit*

A much better way to write down the solution is:

$$-3x + 5 = 2 \iff -3x = -3 \iff x = 1$$

or

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \\ &\iff \quad x = 1 \end{aligned}$$

Now, the relationship between the component sentences is clear: they are equivalent. Since we know what makes $x = 1$ true, we also know what makes $-3x + 5 = 2$ true. Here, the sentences have been combined into a cohesive mathematical paragraph.

The latter form

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \\ &\iff \quad x = 1 \end{aligned}$$

is particularly nice, because the original equation $-3x+5 = 2$ stands out at the top of the left-hand column, and the much simpler equivalent equation $x = 1$ stands out at the bottom of the right-hand column.

This form can be 'annotated' easily as follows:

$$\begin{aligned} -3x + 5 = 2 \quad &\iff \quad -3x = -3 \qquad \text{(subtract 5)} \\ &\iff \quad x = 1 \qquad\quad\;\; \text{(divide by } -3\text{)} \end{aligned} \tag{*}$$

YOU WILL BE EXPECTED TO WRITE COMPLETE AND CORRECT MATHEMATICAL PARAGRAPHS IN THIS COURSE.

EXERCISE 7

♣ Solve the equation $3x - 7 = 1$. Be sure to write a complete mathematical paragraph. Tell what you're doing in each step of your solution. Use the form illustrated in (*).

EXERCISE 8

♣ Solve the inequality $3x - 7 < 1$. Be sure to write a complete mathematical paragraph. Tell what you're doing in each step of your solution. Use the form illustrated in (*).

INCORRECT notation

It is *incorrect* and completely unacceptable to write:

$$\begin{aligned} -3x + 5 = 2 &= -3x = -3 \\ &= x = -1 \end{aligned}$$

Taken literally, this says that 2 is equal to -3 which is equal to -1. Absurd.

Remember:

- The equals sign '$=$' is used to compare *expressions*.
- The 'is equivalent to' sign '$\iff$' is used to compare *sentences*.

What goes wrong if c is zero?

Note that multiplying by zero can change the truth value of a sentence. For example, the equation $3 = 5$ is false, but the equation $3 \cdot 0 = 5 \cdot 0$ is true. Therefore, *multiplying both sides of an equation by zero does not necessarily yield an equivalent equation*, and therefore is not allowed.

EXAMPLE
adding a solution

For example, consider the equation $x = 2$. It has solution set $\{2\}$. Multiplying both sides by x yields the new equation $x^2 = 2x$, which has solution set $\{0, 2\}$. Thus, the equations

$$x = 2 \quad \text{and} \quad x^2 = 2x$$

are *NOT* equivalent, since they have different solution sets.

What happened? Well, as long as x is nonzero, multiplication by x is just multiplication by a nonzero number, which doesn't alter the solution set. But when x is zero, multiplication by x took us from the false statement $0 = 2$ to the true statement $0 = 0$.

Thus, *multiplying by a variable expression may ADD a solution.* Adding a solution isn't really too serious, providing that you check your 'potential' solutions into the original equation at the final step.

EXAMPLE
losing a solution

More serious is this next situation. Begin with $x^2 = 2x$ and divide both sides by x, yielding the new equation $x = 2$. The equations are not equivalent and a solution has been *lost*. If we separately investigate the situation when x is zero, it will be discovered that 0 is also a solution of the original equation. If we neglect to do this, the solution 0 is lost forever.

EXERCISE 9
contradiction

♣ Solve the equation $3(x + 2) = (3x + 1) + 4$. Show that this equation is equivalent to the equation $6 = 5$, which is never true. This type of equation, which is always false, is sometimes referred to as a *contradiction*.

EXERCISE 10
identity

♣ Solve the equation $2x - (7 - x) = x + 1 - 2(4 - x)$. Show that this equation is equivalent to the equation $0 = 0$, which is true for all values of x. This type of equation is sometimes referred to as an *identity*.

QUICK QUIZ
sample questions

1. T or F: the sentences '$x = 2$' and '$x^2 = 4$' are equivalent. Justify your answer.
2. Write a theorem, the way a mathematician would, that says that adding the same number to both sides of an equation does not change its solution set.
3. Write a theorem, the way a mathematician would, that says that adding the same number to both sides of an inequality (using '$>$') does not change its solution set.
4. What is the 'implied domain' of the sentence '$\frac{1}{(x-3)y} = 2$'? Use correct set notation for your answer.
5. Fill in the blank: the goal in solving an equation is to transform it into an ________________ one that can be easily solved.
6. Fill in the blanks:

   The '$=$' sign is used to compare ________________.

   The '$\iff$' sign is used to compare ________________.

KEYWORDS
for this section

Equivalent sentences, implied domain for a sentence, solving by inspection, general approach for solving an equation, subtraction is a special kind of addition, division is a special kind of multiplication, multiplying by a variable, dividing by a variable, contradiction, identity.

You should be familiar with the notation $\iff$ and be able to use it correctly when solving equations and inequalities.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C). The first one is done for you.

1. $x = 3 \iff 2x - 6 = 0$; SEN, T. Both sentences have the same implied domain, and the same solution sets.
2. $x = 4$
3. $x = 4 \iff 4 - x = 0$
4. $\frac{1}{x} = 3 \iff \frac{1}{3x} = 1$
5. $|y| = 0 \iff y = 0$
6. $|y| = 1 \iff y = 1$
7. y^3
8. $y^3 = 8 \iff y = 2$
9. $y^3 + 8 = 0 \iff y + 2 = 0$
10. $x(x-1) = 0 \iff x - 1 = 0$

♣ Using the theorems in this section, solve the following equations/inequalities. BE SURE TO WRITE COMPLETE AND CORRECT MATHEMATICAL SENTENCES.

11. $5x - 7 = 3$
12. $5 - 3y = 9$
13. $3x < x - 11$
14. $3t + 7 \ge -2$

1.5 Graphs

graphs

The word 'graph' always refers to some pictorial representation of information. Graphs are particularly helpful when there is a large amount of information (often infinite) to be understood.

In this section, we study graphs. The section is rather long, but most of the material should be review.

the graph of a sentence

The *graph of a sentence* (equation/inequality) is just a *picture of its solution set.* More correctly, the phrase usually refers to a *partial* picture of the solution set. The tool used to show this 'picture' depends on the nature of the solution set; whether it is a collection of *numbers*, or *pairs of numbers*, or, say, *triples of numbers.*

graphing sentences in one variable; use the real number line

Suppose you are asked to graph the equation in one variable, $x = 2$. For any such equation in one variable, the solution set is a collection of *numbers*, and the real number line can be used to display these numbers. Here, the solution set is $\{2\}$, since 2 is the only real number that is equal to 2. The graph is very uninteresting: one dot (at 2) on the real number line.

The graph of $x = 2$, viewed as an equation in one variable

You'll rarely be asked to graph simple equations in one variable like this: these types of equations usually have a small finite number of solutions, and a picture is not needed to understand this information.

EXERCISE 1

graphing sentences in one variable

Graph the following sentences in one variable:

♣ 1. $3x = x + x + x$

♣ 2. $x^2 < 0$

♣ 3. $x^2 \le 0$

♣ 4. $(x-3)(x+1) = 0$ (see Algebra Review—zero factor law)

♣ 5. $2x - 1 = 7$

♣ 6. $x^2 - 4x = 5$ (see Algebra Review—zero factor law)

♣ 7. $|x| = 2$

♣ 8. $|t| < 2$ and $t \ge 1$

graphing sentences in two variables; use the rectangular coordinate system

The graphs of sentences in two variables are usually much more interesting, because such sentences usually have an infinite number of members in their solution set.

For example, consider the two-variable equation $y = x$. Remember that a solution of this equation consists of a pair of numbers—a choice for x and a choice for y—that makes the equation true. Once a choice is made for x, the choice for y is uniquely determined, since y must equal x.

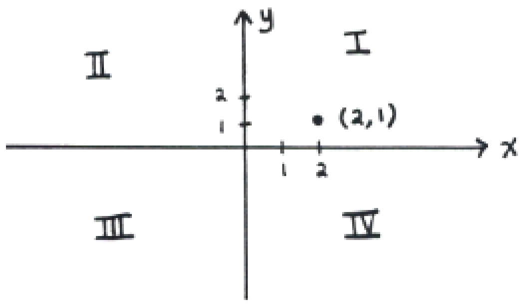
Here are some of the ordered pairs in the solution set of this equation: $(0,0)$, (π,π), $(-\frac{2}{3},-\frac{2}{3})$, and $(-2017.1,-2017.1)$. It is of course impossible to list everything in the solution set, but the solution set *can* be described compactly using set-builder notation. The solution set of the equation $y=x$ is:

$$\{(x,y) \mid y=x\} = \{(x,x) \mid x \in \mathbb{R}\}$$

The graph of this equation is then a (partial) 'picture' of all these ordered pairs. But how can we 'picture' an ordered pair? Answer: by using the *rectangular coordinate system*, discussed next.

DEFINITION
the rectangular coordinate system
origin
x-axis
y-axis

The device used for graphically representing ordered pairs is called the *rectangular coordinate system*, also commonly referred to as the *Cartesian plane* (named after the French mathematician René Descartes, 1596–1650). It is the plane formed by two intersecting perpendicular lines. Their point of intersection is called the *origin.* By convention, the horizontal line is called the *x-axis* and the vertical line the *y-axis*, with positive directions to the right and up, respectively. (Thus, yet another name commonly used for the Cartesian plane is the *xy-plane.*)



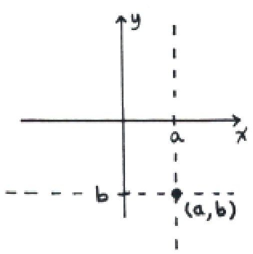
quadrant I
quadrant II
quadrant III
quadrant IV

The xy-plane is naturally divided into four quadrants, which are numbered as follows:

- quadrant I $= \{(x,y) \mid x>0 \text{ and } y>0\}$
- quadrant II $= \{(x,y) \mid x<0 \text{ and } y>0\}$
- quadrant III $= \{(x,y) \mid x<0 \text{ and } y<0\}$
- quadrant IV $= \{(x,y) \mid x>0 \text{ and } y<0\}$

Observe that the axes are not part of any quadrant.

every ordered pair corresponds to a unique point in the xy-plane



Every ordered pair (a,b) corresponds to a unique point in the xy-plane in the following way:

- go to 'a' on the x-axis; draw a vertical line through this point;
- go to 'b' on the y-axis; draw a horizontal line through this point;
- the unique point where these two lines intersect is the point associated with the ordered pair (a,b).

Here's a slightly less precise, but perhaps easier way to find the point (a,b): start at the origin, move 'a' units in the x-direction, then 'b' units in the y-direction. (If a is positive, move to the right; if a is negative, move to the left. If b is positive, move up; if b is negative, move down.)

♣ Could we say this instead? *Start at the origin, move 'b' units in the y-direction, then 'a' units in the x direction.*

we will freely interchange the words 'point' and 'ordered pair'

Observe that every ordered pair corresponds to precisely one point in the plane; and every point in the plane corresponds to precisely one ordered pair. Thus, we can freely interchange the words 'ordered pair' and 'point' without confusion.

*Why the name '**ordered** pair'?*

NOTE: If $x \neq y$, then (x, y) is a different point than (y, x). Hence the name '**ordered** pair'!

EXERCISE 2

♣ 1. Plot the points $(1,3)$, $(-1,-5)$, $(0,-\pi)$, $(-\sqrt{2},0)$ on a rectangular coordinate system. What quadrant (if any) is each point in?

♣ 2. Plot several points in the solution set of the equation $y = x$. See a pattern forming?

DEFINITION

graph of a sentence in 2 variables

The *graph* of a sentence in 2 variables is a (partial) picture of its solution set; that is, it is the set of all ordered pairs (x, y) that satisfy the sentence, displayed on a rectangular coordinate system.

A graph portrays an infinite number of solution points in an organized, easy-to-analyze fashion. In many cases, one of the variables is allowed to range over an infinite interval of real numbers, so that it is impossible to show the entire graph. In such instances, one usually shows a representative part of the graph, or enough of the graph to capture everything of interest. This is illustrated in the next examples.

EXAMPLE

graph of $x = 2$, viewed as an equation in two variables, $x + 0y = 2$

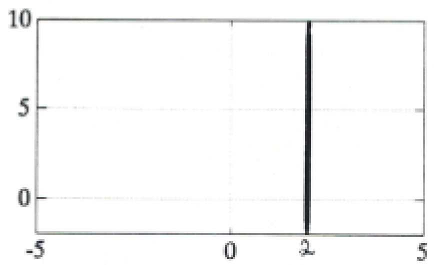
Suppose you are asked to graph the equation $x = 2$. Out of context, you should rightfully be confused: are you to treat this as an equation in one variable (x), or two variables (x and y, say $x + 0y = 2$)? The answer is important. As was seen earlier, if $x = 2$ is treated as an equation in one variable, the graph is boring—a single point at 2 on the real number line. However, treated as an equation in two variables, its graph is:

$$\{(x,y) \mid x = 2,\ y \in \mathbb{R}\} = \{(2,y) \mid y \in \mathbb{R}\}$$

Thus, the graph is the vertical line shown below.

Note: A comma is sometimes used in mathematics to mean the mathematical word 'AND'. Thus:

$$\{(x,y) \mid x = 2,\ y \in \mathbb{R}\} = \{(x,y) \mid x = 2 \ \text{ AND } \ y \in \mathbb{R}\}$$



The graph of $x = 2$, viewed as an equation in two variables.

EXAMPLE

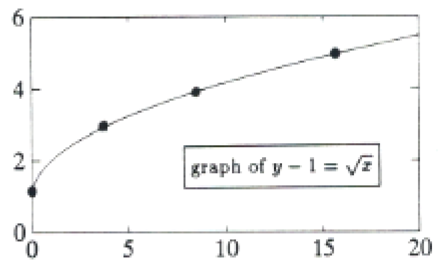
graphing a simple equation in 2 variables

Problem: Graph the equation $y - 1 = \sqrt{x}$.

Solution: Observe that here, y can easily be solved for x by adding 1 to both sides and obtaining the equivalent equation $y = \sqrt{x} + 1$. Thus, it is easy to choose (allowable) values for x, and compute the corresponding value of y:

x	$y = 1 + \sqrt{x}$
0	1
4	3
9	4
16	5
25	6

Note that x was chosen so that the corresponding y values were easy to compute. Plotting these points appears to illustrate a pattern; the graph is completed by drawing a smooth curve through the sample points. (Does this last step make you a bit uneasy? More on this in a moment.) Observe that the resulting graph is only a partial picture of all the ordered pairs that make the equation true.



EXERCISE 3 ♣ Graph the equation $x - 1 = \sqrt{y}$. Compare your graph with the one in the previous example.

EXAMPLE

graphing an inequality in 2 variables

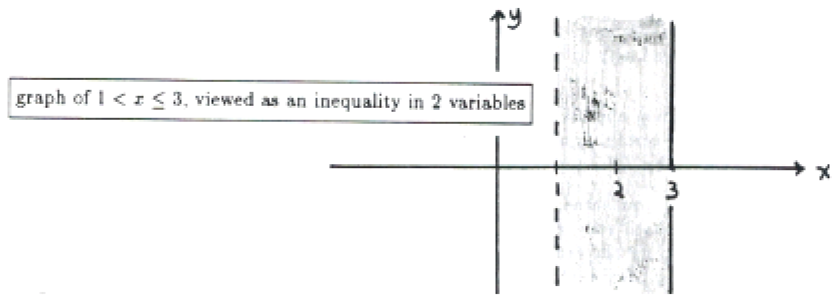
Problem: Graph the compound inequality $1 < x \leq 3$, viewed as an inequality in 2 variables (say, $1 < x + 0y \leq 3$).

Solution: The solution set is:

$$\{(x, y) \,|\, 1 < x \leq 3, \quad y \in \mathbb{R}\}$$

Thus, we seek all points with x values between 1 and 3 (not including 1, including 3). The points can have *any* y-values.

The graph is shown below. Note that a *solid* line means that points are to be included; a *dashed* line means that points are *not* to be included.



EXAMPLE

graphing an inequality in 2 variables

Problem: Graph the inequality $y > x$.

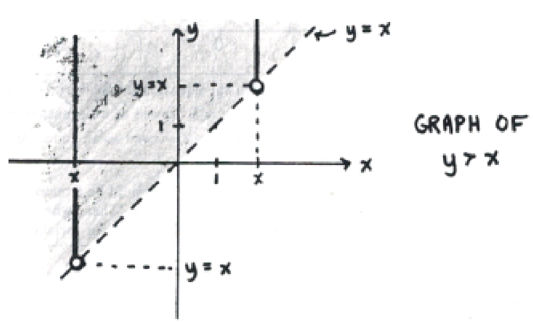
Solution: The solution set is:

$$\{(x, y) \,|\, x \in \mathbb{R}, \; y > x\}$$

Thus, we seek all points (x, y) with the property that their y-value is *greater than* their x-value. For example, the point $(1, 1.1)$ satisfies this property, but the point $(-3, -3.1)$ does not.

The easiest way to obtain this graph is to first graph the *boundary*, $y = x$. We don't want these points, whose y-values *equal* their x-values, so the line $y = x$ is dashed.

Now, choose any value of x. Corresponding to this x-value, we want all points with y-values *greater than* x. Thus, we want the points that lie *above* the line. Letting x vary over all real numbers, the desired graph consists of all the points above the line $y = x$.



general scheme for graphing an equation in 2 variables

Given an equation in 2 variables, our basic goal is to gain a good understanding of what the solution pairs look like. If we are able to solve the equation for one of the variables, say y, in terms of the remaining variable—that is, get it in the form

$$y = < \text{stuff involving } x >$$

then solution points are easily generated: merely choose allowable values for x, and calculate the corresponding values of y.

conjecture

If these points are plotted, a pattern may be displayed, leading to a conjecture (educated guess) about the form of the graph. However, *plotting points is extremely inefficient and not foolproof*, and should only be used in connection with other methods. Plotting several points, however, is always a good way to begin: it can help to confirm one's belief about the nature of a graph, or catch mistakes.

classifying an equation

One common approach to graphing an equation is to *classify the equation* as being of a certain type. Then, use information about this known type to graph the equation. The approach is illustrated in the next example.

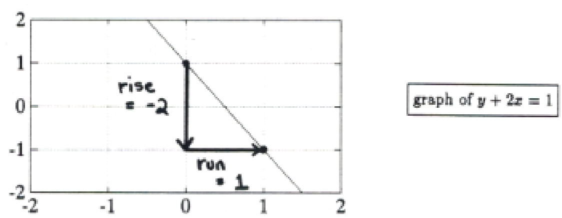
EXAMPLE

lines;

$ax + by = c$

In algebra, you learned that every equation of the form $ax+by = c$, when a and b are not both zero, graphs as a line in the xy-plane. This class of equations (one for every allowable choice of a, b and c) is known as *the linear equations in x and y*. Once an equation has been identified as linear, only two points need to be plotted to obtain the graph. Or, one can rewrite the equation in an equivalent form that is easier to work with.

For example, consider the equation $y+2x = 1$. This is equivalent to $y = -2x+1$, which is now in slope-intercept form, $y = mx + b$ (see Algebra Review—lines). Thus, one can 'read off' that the line crosses the y-axis at 1, and has a slope of $-2 = \frac{-2}{1} = \frac{\text{rise}}{\text{run}}$. Using this information, the line is easily graphed.



EXERCISE 4

♣ 1. Graph the equation in two variables, $x - 3y + 5 = 0$.

♣ 2. Graph $y = 3$, viewed as an equation in two variables.

♣ 3. Graph $|y| > 2$, viewed as an equation in two variables.

♣ 4. Graph $y < -x$.

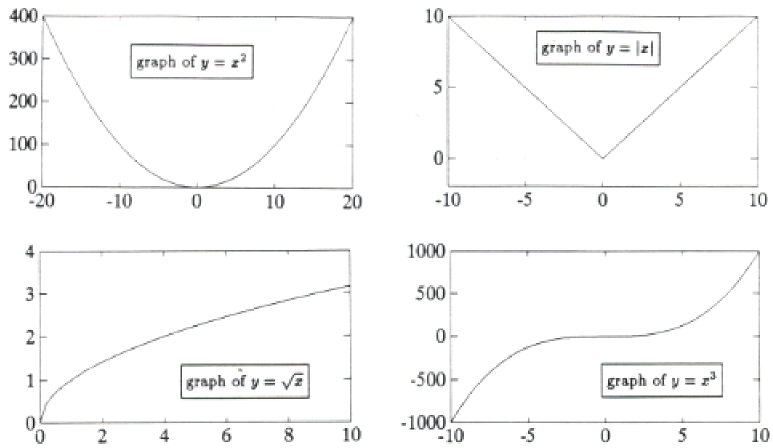
♣ 5. Think about what would be a reasonable way to 'picture' ordered triples (x, y, z) of real numbers. Then, what would the graph of $x = 2$ look like, viewed as an equation in three variables, $x + 0y + 0z = 2$?

using calculus to graph

The techniques of calculus give us extremely powerful tools for graphing many equations in 2 variables. These techniques will be discussed later on in this text.

common graphs that you should know

You should be familiar with all the following common graphs:



common variations on these graphs

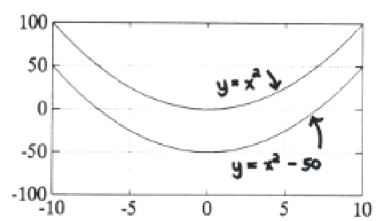
Once the shapes of these basic graphs are known, it is easy to obtain some slight modifications. For example, consider the graph of $y = x^2 - 50$. One must form a picture of its solution set:

$$\{(x, y) \mid y = x^2 - 50\} = \{(x, x^2 - 50) \mid x \in \mathbb{R}\}$$

How does this relate to the graph of $y = x^2$, whose solution set is

$$\{(x, x^2) \mid x \in \mathbb{R}\}\ ?$$

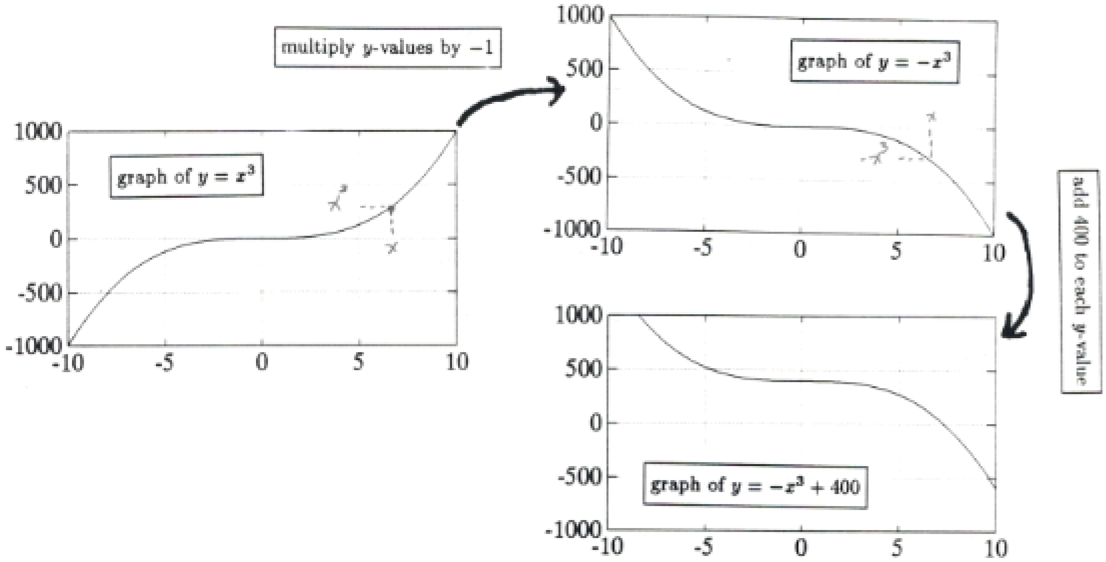
Each y-value has been reduced by 50; hence the graph of $y = x^2$ must be shifted down 50 units to obtain the graph of $y = x^2 - 50$.



EXAMPLE

‘building up’ a graph from simpler pieces

The sequence of graphs below illustrates how one can easily obtain the graph of $y = -x^3 + 400$.



EXERCISE 5	Graph the following equations, by building them up from simpler pieces. ♣ 1. $y = -x^2 + 1$ ♣ 2. $\|x\| + y = 3$ ♣ 3. $\sqrt{x} - 4 + y = 0$

ALGEBRA REVIEW

zero factor law, mathematical word ‘or’, lines

THEOREM *the zero factor law*	THEOREM (The Zero Factor Law). For all real numbers a and b: $ab = 0 \iff a = 0 \text{ or } b = 0$

translation of the ‘zero factor law’

Here is how an algebra instructor might translate this theorem for students: *Whenever a product of real numbers equals zero, at least one of the factors must be zero.*

★ The theorem actually says more than this, but the instructor is paraphrasing the most useful part of the result.

One goal of this course is that *you* become able to do the ‘translating’ yourself. To this end, you must first understand the meaning of the mathematical word ‘or’.

the mathematical word 'or'

By definition, a mathematical sentence of the form

$$A \text{ or } B$$

is true when A is true, or B is true, or BOTH A and B are true.

This information is summarized in the truth table below:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE

truth of 'or' sentences

For example, the mathematical sentence

$$2 = 1 \text{ or } 3 = 2 + 1$$

is true because '$3 = 2 + 1$' is true. The mathematical sentence

$$\sqrt{9} = 3 \text{ or } -3^2 = 9$$

is true because '$\sqrt{9} = 3$' is true (even though '$-3^2 = 9$' is false). The mathematical sentence

$$(-3)^2 = -9 \text{ or } \sqrt{(-4)^2} = -4$$

is false because both component sentences are false.

The sentence '$x = 3$ or $x = 5$' is conditional. Its solution set is $\{3, 5\}$. For all other choices of x, it is false.

CAUTION: the English word 'or' versus the mathematical word 'or'

CAUTION: there's a slight difference in the English and mathematical uses of the word 'or'.

If you say to a friend, "I'm going to study math or English tonight," you probably mean that you'll study math, or English, but NOT BOTH.

However, when a mathematician makes a true statement 'A or B', this means that A is true, or B is true, or BOTH A and B are true.

EXERCISE 6

the mathematical word 'or'

Classify each sentence as (always) TRUE, (always) FALSE, or CONDITIONAL:

♣ 1. $1 < 1$ or $1 > 1$

♣ 2. $1 \le 1$ or $1 > 1$

♣ 3. $|x| > 0$ or $|x| = 0$

♣ 4. $x = 3$ or $x = -3$

♣ 5. $\sqrt{t^2} = t$ or $\sqrt{t^2} = -t$

return to the 'zero factor law'

Now, return to the zero factor law. What is it telling us? The theorem compares two mathematical sentences: the sentence '$ab = 0$' and the sentence '$a = 0$ or $b = 0$'. Since these two sentences are equivalent, they always have the same truth values. Therefore, the two sentences can be used interchangeably.

For example, choosing $a = 3$ and $b = 0$, the equation '$ab = 0$' becomes '$3 \cdot 0 = 0$', which is true; and the sentence '$a = 0$ or $b = 0$' becomes '$3 = 0$ or $0 = 0$', which is also true.

♣ What is the theorem telling us if $a = 1$ and $b = 2$?

typical use of the zero factor law

Here's how the zero factor law is typically used. Suppose you are asked to solve the equation $x^2 - x - 6 = 0$. This equation cannot be solved by inspection. So it is transformed into an *equivalent* equation that *can* be solved by inspection, as follows. First, factor the left-hand side, and then use the zero factor law:

$$\begin{aligned} x^2 - x - 6 = 0 &\iff (x-3)(x+2) = 0 \\ &\iff x - 3 = 0 \ \text{ or } \ x + 2 = 0 \\ &\iff x = 3 \ \text{ or } \ x = -2 \end{aligned}$$

Here, the zero factor law was applied with 'a' equal to '$x - 3$' and 'b' equal to '$x + 2$'. The last equation can be solved by inspection, and has solution set $\{3, -2\}$. Therefore, the equation $x^2 - x - 6 = 0$ also has solution set $\{3, -2\}$ (check this).

EXERCISE 7

using the zero factor law

Solve the following sentences. Use the zero factor law. Be sure to write complete mathematical sentences.

♣ 1. $x(5x - 3) = 0$

♣ 2. $x^2 - x = 12$

♣ 3. $(3x - 2)^2 - 16 = 0$

lines

Every line in the plane is uniquely determined by two distinct (different) points on the line. And every two distinct points uniquely determine a line. The information on lines included here should be a review; it is merely included for your convenience.

horizontal and vertical lines

The points on a horizontal line all have the same y-values; hence horizontal lines are all of the form $y = k$, for a real number k.

The points on a vertical line all have the same x-values; hence vertical lines are all of the form $x = k$, for a real number k.

EXERCISE 8

Graph each sentence. Interpret each as a sentence in two variables, x and y.

♣ 1. $3x = 4$

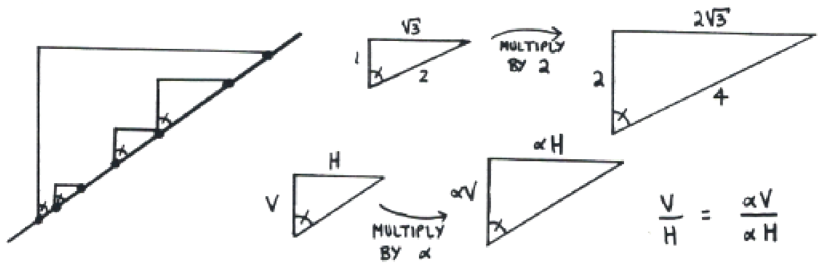
♣ 2. $5 = 4 - y$

♣ 3. $x = 4$ or $y = -1$

♣ 4. $x = 4$ and $y = -1$

non-vertical, non-horizontal lines

Non-vertical, non-horizontal lines have a beautiful property: no matter what two points are chosen on the line, the right triangles formed using the line as the hypotenuse *all have the same angles* (see the sketch below). Such triangles are called *similar triangles*. By appropriately 'magnifying' a triangle (that is, by multiplying *each side* by the same number), a triangle can be made to coincide with any similar triangle. This is illustrated in the sketch below.



slope of a line

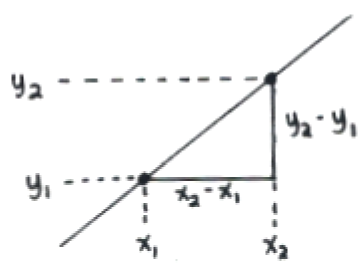
An important consequence of this fact is that the ratio of 'rise' (vertical travel) over 'run' (horizontal travel) in traveling from one point to another on the line, does NOT depend on which two points are used! This ratio is called the SLOPE OF THE LINE.

a convenient formula for the slope

Letting (x_1, y_1) and (x_2, y_2) be any two distinct points on a non-vertical line, the slope m of the line is found by:

$$m := \frac{y_2 - y_1}{x_2 - x_1}$$

Note that the slope of a horizontal line is zero; the slope of a vertical line is undefined. ♣ WHY?



EXAMPLE
finding the slopes of lines

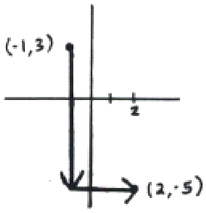
Problem: Find the slope of the line through the points $(-1, 3)$ and $(2, -5)$.

Solution #1. Letting $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (2, -5)$, we get:

$$m = \frac{-5 - 3}{2 - (-1)} = \frac{-8}{3} = -\frac{8}{3}$$

Solution #2. Letting $(x_1, y_1) = (2, -5)$ and $(x_2, y_2) = (-1, 3)$, we get:

$$m = \frac{3 - (-5)}{-1 - 2} = \frac{8}{-3} = -\frac{8}{3}$$

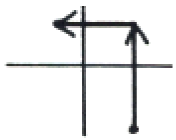


Solution #3. Traveling from the point $(-1, 3)$ to the point $(2, -5)$ via the rules 'rise first, then run', we obtain:

$$m = \frac{\text{'down 8'}}{\text{'right 3'}} = \frac{-8}{3} = -\frac{8}{3}$$

Solution #4. Traveling from the point $(2, -5)$ to the point $(-1, 3)$ via the rules 'rise first, then run', we obtain:

$$m = \frac{\text{'up 8'}}{\text{'left 3'}} = \frac{8}{-3} = -\frac{8}{3}$$



EXERCISE 9
finding slopes of lines

Find the slope of the line through each pair of points. If the slope is undefined, so state. Use several different approaches.

♣ 1. $(3, -2)$, $(-1, 5)$

♣ 2. $(a, 3)$, $(a, -1)$ (Here, a is any real number.)

♣ 3. $(-2, b)$, $(3, b)$ (Here, b is any real number.)

equations of lines;

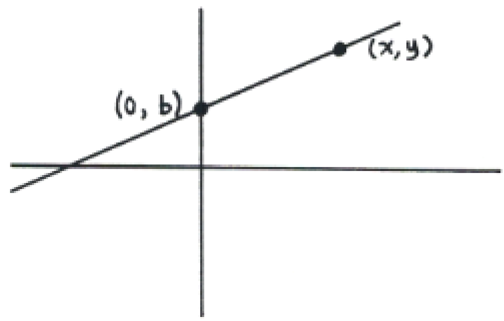
$y = mx + b$;

slope-intercept form

Every non-vertical line must cross the y-axis at exactly one point; call this point $(0, b)$. Let m denote the slope of the line. Then, if (x, y) is ANY other point on the line, then $x \neq 0$ (♣ Why?), so:

$$\begin{aligned} m = \frac{y-b}{x-0} \quad &\iff \quad y - b = mx \\ &\iff \quad y = mx + b \end{aligned}$$

That is, the solution set of the equation $y = mx + b$ is precisely the points on the line with slope m, that crosses the y-axis at b. The equation $y = mx + b$ is thus appropriately called the *slope-y-intercept form of a line.*



standard form of a line;

$ax + by = c$

If you stop and think a moment, you'll see that every equation of the form $ax + by = c$ (when a and b are not both zero), graphs as a line in the plane.

The set of all equations that can be written in the form $ax + by = c$, where a and b are not both zero, are called the *linear equations in 2 variables.* This is certainly a reasonable name, due to the previous observation!

EXAMPLE
graphing a line

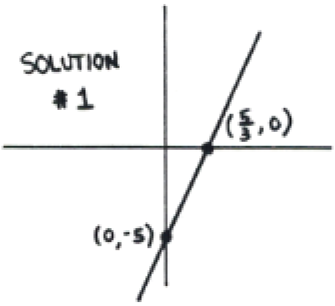
Problem: Graph $3x - y = 5$.

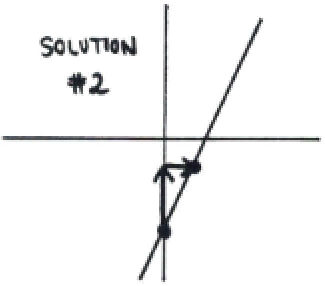
Solution #1: Recognize that the equation is linear in x and y; thus it graphs as a line. Plot ANY two points to graph the line! (So, choose two EASY points!)

When $x = 0$, we have $3(0) - y = 5 \iff y = -5$. Thus, the point $(0, -5)$ is on the graph.

When $y = 0$, we have $3x - 0 = 5 \iff x = \frac{5}{3}$. Thus, the point $(\frac{5}{3}, 0)$ is on the graph.

Plot these two points, and sketch the line through them.





Solution #2: Put the equation in $y = mx + b$ form, and read off b and m:

$$\begin{aligned} 3x - y = 5 &\iff -y = 5 - 3x \\ &\iff y = 3x - 5 \end{aligned}$$

The line crosses the y-axis at -5, and has a slope of 3:

$$m = 3 = \frac{3}{1} = \frac{\text{rise}}{\text{run}}$$

Sketch the line.

EXERCISE 10
graphing

Graph the following sentences. Take two different approaches in each case.

♣ 1. $-5x = y + 4$

♣ 2. $x + y = 1$ or $x = 1$ or $y = 1$

QUICK QUIZ
sample questions

1. Graph the equation $x = 3$, viewed as an equation in 1 variable; viewed as an equation in 2 variables.
2. Graph the inequality $x < 3$, viewed as an inequality in 1 variable; viewed as an inequality in 2 variables.
3. Graph the equation $y - x^2 + 1 = 0$.
4. Graph the inequality $y \le 2x$.
5. TRUE or FALSE: '$3 < 3$ or $3 \ge 3$'.
6. Find the value(s) of x for which the following sentence is TRUE; show them on a number line: '$x \ge 3$ or $x < -1$'.

KEYWORDS *for this section*	*Graphs, graph of a sentence, graphing sentences in one variable, graphing sentences in two variables, rectangular coordinate system (origin, x-axis, y-axis, quadrants I, II, III, IV), the zero factor law, the mathematical word 'or', conjecture, linear equations in 2 variables, graphing lines.*

END-OF-SECTION EXERCISES

Graph each of the following sentences in one variable. (Show the solution sets on a number line.)

1. $x = \pi$
2. $x - 3 = 0$
3. $|x| = 2$
4. $|x| \geq 2$
5. $3x < -2$
6. $2x - 5 > 3$
7. $x = 0$ or $|x| = 1$
8. $x = 1$ and $|x| = 1$
9. $x = 1$ or $|x| = 1$
10. $2x = 1$ and $2x \neq 1$
11. $|3x + 1| = 7$
12. $|3x + 1| < 7$

Graph each of the following sentences in two variables (x and y). (Show the solution sets in the xy-plane.)

13. $x + y = 2$
14. $y - 4x = -3$
15. $x = 1$ or $y = -2$
16. $x = 1$ and $y = -2$
17. $|y| = 1$
18. $|x| = 2$
19. $|x + y| = 1$ (Hint: For $a \geq 0$, $|z| = a \iff z = a$ or $z = -a$.)
20. $|x + y| < 1$ (Hint: For $a > 0$, $|z| < a \iff -a < z < a$.)

This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 2

FUNCTIONS

Without *functions*, modern mathematics would not exist. Since functions and function notation are so deeply imbedded in the mathematical language, *students must understand functions in order to understand mathematics.* Functions are the subject of this chapter.

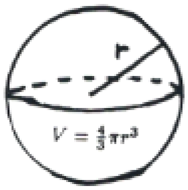
2.1 Functions and Function Notation

Introduction

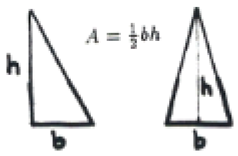
inputs and outputs

In many common relationships between variables, there are natural input/output roles assumed by the variables.

For example, in the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere, one naturally thinks of 'inputting' a radius r into the formula, and 'outputting' the volume of the sphere with that radius.



Although the radius is 'naturally' viewed as the input when the formula is written in the form $V = \frac{4}{3}\pi r^3$, this need not always be the case. Suppose, for example, that spherical containers are being designed to hold various amounts of liquid. Given a desired volume, it is necessary to determine the radius of the sphere that will yield that volume. In this case, one can solve for r, yielding the equivalent equation $r = \sqrt[3]{\frac{3V}{4\pi}}$. In this formulation of the equation, the volume V is now viewed as the input, and the radius r as the output.



As a second example, in the formula $A = \frac{1}{2}bh$ for the area of a triangle, one naturally thinks of 'inputting' both the base b and height h of the triangle, and 'outputting' the area of a triangle with that base and height. However, the equivalent equation $h = \frac{2A}{b}$ views the area and base as inputs, and the height as an output.

EXERCISE 1

♣ 1. Consider the formula $A = \pi r^2$ for the area of a circle. What is naturally viewed as the input? Output? Rewrite the equation so that the radius is the 'natural' output.

♣ 2. Come up with a common relationship between variables (different from the example above) that has two inputs.

♣ 3. Come up with a common relationship between variables that has three inputs.

functions

The language of mathematics provides a very precise tool for discussing this kind of input-output relationship between variables: functions. Functions, and the notation used in connection with functions, is the topic of this section.

function, informal definition

A *function* is a special relationship between variables, where to each choice of input there corresponds *a unique* output. In this case, we say that 'the output *is a function of* the input'.

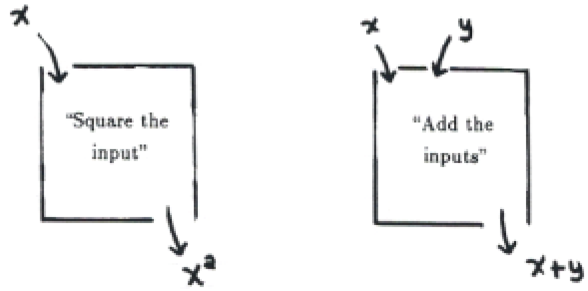
The most important word in this informal definition of function is the word 'unique'. The examples in this section should clarify the importance of the uniqueness of the output.

For example, in the formula $V = \frac{4}{3}\pi r^3$, to each value of input r there corresponds a unique volume V. We say that *V is a function of r.*

In the formula $A = \frac{1}{2}bh$, to each choice of base b and height h there corresponds a unique area A. We say that *A is a function of b and h.* (Here, we can view the input as an ordered pair of numbers, (b, h).)

functions as ‘black boxes’

It is often helpful to think of a function in terms of a ‘black box’. The input goes in the top of the box. The box itself is the ‘rule’ that does something to the input. The output drops out the bottom of the box.



EXERCISE 2

♣ 1. In the first box pictured above, what will the output be if the input is 5? -5? x? x^2? $x+h$?

♣ 2. In the second box pictured above, what will the output be if the inputs are 3 and 4? What if the inputs are x^2 and y^2? What if the inputs are t and t?

What should you think when you hear the phrase: ‘y is a function of x’?

When you hear the phrase *y is a function of x*, you can roughly think: *y depends on x*. More precisely, think: *y is an output that is uniquely determined by the input x.*

functions often arise naturally from equations

In the previous chapter, we studied equations. Many (but not all) equations describe a function relationship between their variables. The form in which the equation is written often leads to a choice of input/output roles for the variables.

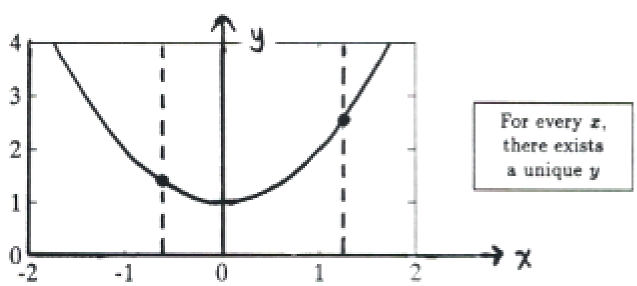
obtaining a function by solving for y in terms of x

For example, suppose that an equation in the variables x and y is such that we are able to solve the equation for y in terms of x, thus obtaining an (equivalent) equation:

$$y = < \textit{some formula involving } x >$$

In this form, y is naturally viewed as an output that depends on the input x. That is, once a value is chosen for x, we can plug it into the formula, and obtain the unique corresponding value of y. So, y is a function of x.

For example, consider the equation $y - x^2 = 1$. This is equivalent to $y = x^2 + 1$; once an input x is chosen, the unique output y is found by squaring x, then adding 1. So, y is a function of x. The graph of $y = x^2 + 1$ is shown below.

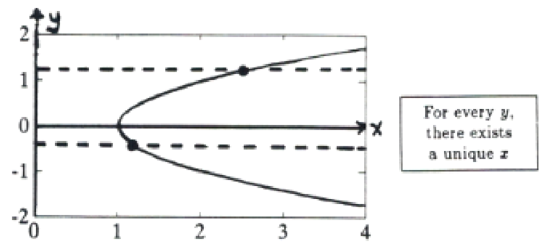


vertical line test;
y is a function of x

There is a nice graphical way to see that every input x corresponds to a unique output y. Imagine a vertical line sweeping through all the x-values. No matter what x-value we 'stop' this vertical line at, it hits exactly one point on the graph—the unique y-value associated with that x-value.

horizontal line test;
x is a function of y

Now reverse the roles of x and y in the previous example. That is, consider the equation $x - y^2 = 1$, which is equivalent to $x = y^2 + 1$, and has the graph shown below. Associated to each 'input' y there is a unique 'output' x, so in this case, x is a function of y.



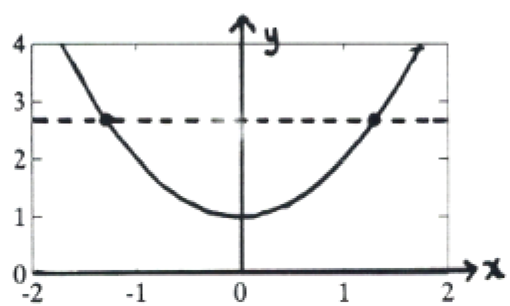
How can we graphically check that x is a function of y? We must check that each allowable y-value is associated with a *unique* x-value. To do this, imagine a horizontal line sweeping through the graph, checking each y-value. If this horizontal line never hits the graph at more than one point, then x is a function of y.

EXAMPLE
y is a function of x
but
x is not a function of y

Once more consider the equation $y - x^2 = 1$. Suppose it is desired to view x as the 'output'; in this case, we are motivated to solve for x (see the Algebra Review, this section) giving:

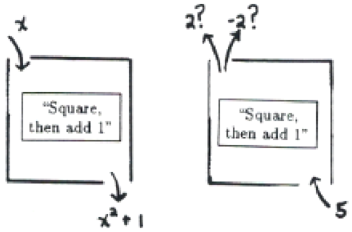
$$\begin{aligned} y - x^2 = 1 &\iff x^2 = y - 1 && \text{(add } x^2\text{, subtract 1, rearrange)} \\ &\iff x = \pm\sqrt{y-1} && \text{(take square roots, correctly!)} \end{aligned}$$

Now, corresponding to an allowable 'input' y, we obtain *two* 'outputs': $x = +\sqrt{y-1}$ and $x = -\sqrt{y-1}$. In particular, we do not obtain a *unique* output value. Thus, although y is a function of x in this equation; x is not a function of y. Note that the graph of $y - x^2 = 1$ does not pass a horizontal line test.

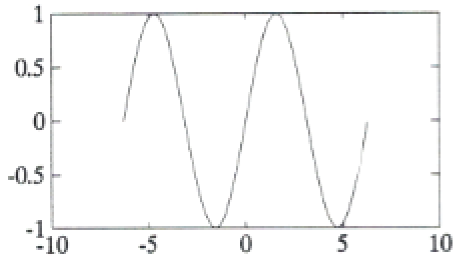


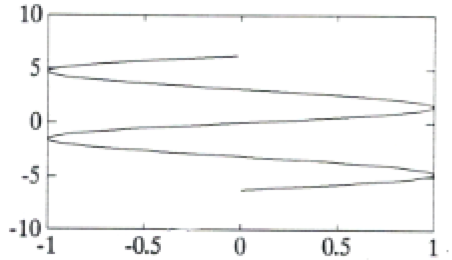
'black box' interpretation of the previous example

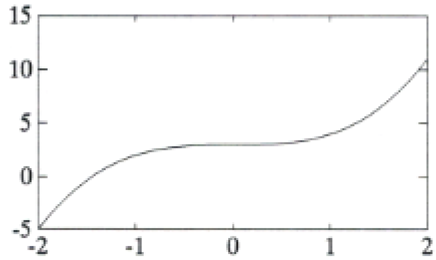
It may be helpful to interpret this example in terms of a 'black box'. Since y is a function of x, we can drop any input into the 'square, then add 1' box, and a unique output will drop out the bottom. However, since x is not a function of y, we can *not* necessarily reverse this process. That is, suppose we pick up the number 5 from the output pile. Can we put it in the box (backwards) to determine where it came from? The answer is no: both 2 and -2 gave rise to the output 5.



EXERCISE 3 ♣ Consider the graphs shown below. Which ones describe y as a function of x? Which ones describe x as a function of y?







EXAMPLE

an equation in x and y with no function relationships between the variables

Consider the equation $x^2 + y^2 = 9$. Solving for y yields:

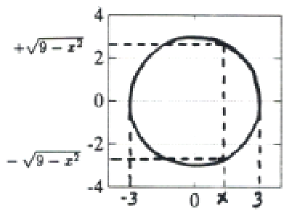
$$
\begin{aligned}
x^2 + y^2 = 9 &\iff y^2 = 9 - x^2 && \text{(subtract } x^2\text{)}\\
&\iff y = \pm\sqrt{9 - x^2} && \text{(take square roots, correctly!)}
\end{aligned}
$$

What are the allowable values to choose for x? The expression under the square root must be nonnegative:

$$
\begin{aligned}
9 - x^2 \ge 0 &\iff 9 \ge x^2 && \text{(add } x^2 \text{ to both sides)}\\
&\iff x^2 \le 9 && \text{(rearrange)}\\
&\iff |x| \le 3 && \text{(take square roots—correctly!)}\\
&\iff -3 \le x \le 3 && \text{(solve the abs. value inequality)}
\end{aligned}
$$

Given an appropriate value of x ($x \in [-3, 3]$), we get *two* associated values of y: $+\sqrt{9 - x^2}$ and $-\sqrt{9 - x^2}$. So in this case, y is not a function of x. The graph of $x^2 + y^2 = 9$ is shown below. Observe that it fails the vertical line test.

The exercise below completes this example.



EXERCISE 4

♣ 1. Solve the equation $x^2 + y^2 = 9$ for x. Be sure to write a complete mathematical paragraph.

♣ 2. What are the allowable values for y? Be sure to write a complete mathematical paragraph when finding them.

♣ 3. Is x a function of y?

★
global vs. local

★ At most points on the graph of $x^2 + y^2 = 9$, y is *locally* a function of x, in the following sense. Let (a, b) be a point on the graph with $a \ne 3$ and $a \ne -3$. Then, there exists an interval I containing a such when the graph is restricted to x-values in this interval, y is a function of x. This observation becomes important in the section on *implicit differentiation*.

EXAMPLE

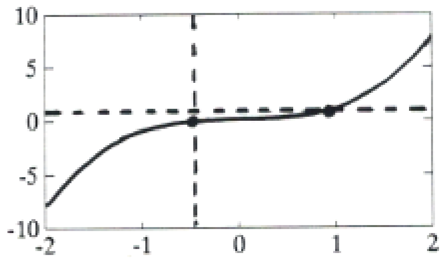
y is a function of x and x is a function of y

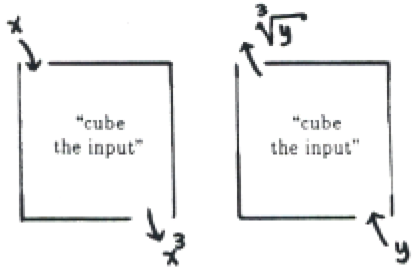
Consider the equation $y = x^3$. Given an input x, there is a unique corresponding output y, obtained by cubing x. So, y is a function of x.

If we choose to view y as the input, we can rewrite the equation in the equivalent form $x = \sqrt[3]{y}$. Now, given an 'input' y, there is a unique corresponding 'output' x, obtained by taking the cube root of y. So, x is also a function of y.

The graph of $y = x^3 \iff x = \sqrt[3]{y}$ is shown below. Observe that it passes both a vertical line test *and* a horizontal line test.

Here's the 'black box' interpretation of this example. When an input is dropped in the top of the 'cube' box, a unique output drops out the bottom. If we pick up a number from the output pile, we can use the box 'backwards' to obtain the unique input from which it came. That is, associated to every input is a unique output; and associated to every output is a unique input. This type of relationship between x and y is particularly nice.





EXERCISE 5

Consider the equation $x = |y|$.

♣ 1. Graph this equation.

♣ 2. Is y a function of x? Why or why not?

♣ 3. Is x a function of y? Why or why not?

EXERCISE 6

Consider the equation $y = 3$, viewed as an equation in two variables, x and y.

♣ 1. Graph this equation.

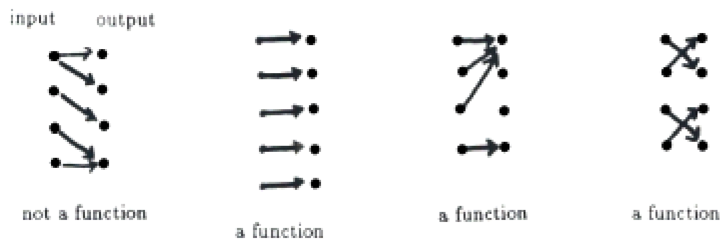
♣ 2. Is y a function of x? Why or why not?

♣ 3. Is x a function of y? Why or why not?

♣ 4. What 'black box' would you associate with the equation $y = 3$? In particular, what is the 'rule' that the black box performs in this case?

mapping diagrams

It is sometimes helpful to view functions/non-functions in terms of mapping diagrams, as illustrated below.



notation for functions

Next, we introduce an *extremely important* notation used in connection with functions.

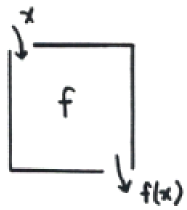
FUNCTION NOTATION

x, the input
f, the rule
$f(x)$, the output corresponding to the input x

In a function relationship between variables, once an input is chosen, there is a unique corresponding output. It is convenient to give a name to the output that illustrates its relationship to the input.

Here's how it's done: if the input is x, the output is called $f(x)$ (read as 'f *of* x'). Letters other than f are possible. The letter f is merely common because it is the first letter in the word 'function'.

The sketch below illustrates the relationship between the function f, the input x, and the output $f(x)$.



Until you become an expert on functions, it is important that you distinguish between the *function* f (the 'rule'), and the output $f(x)$ that comes from the input x. Unfortunately, f and $f(x)$ are often used synonymously, leading to confusion.

Correct: *The function f is the squaring function.*
Incorrect: *The function $f(x)$ is the squaring function.*

naming functions

The function that takes an input and squares it can be described in function notation in any of the following ways:

$$f(x) = x^2 \quad \text{or} \quad g(x) = x^2 \quad \text{or} \quad h(x) = x^2 \quad \text{or} \quad S(x) = x^2$$

It is often good to choose a letter name for the function that helps to describe the function; in this sense, perhaps '$S(x) = x^2$' is good, because 'S' is the first letter in the word 'square'.

If $S(x) = x^2$, then what is $S(3)$? Answer: $S(3) = 3^2 = 9$. '$S(3)$' is the name given to the unique output when 3 is the input.

What is $S(x+y)$? Answer: $S(x+y) = (x+y)^2$. Be sure to write a complete sentence for the answer: don't just say '$(x+y)^2$'.

What is $S(x^2)$? Answer: $S(x^2) = (x^2)^2 = x^4$.

What is $S(\square)$? Answer: $S(\square) = (\square)^2$. (Fill in the box with any input you want!)

dummy variables

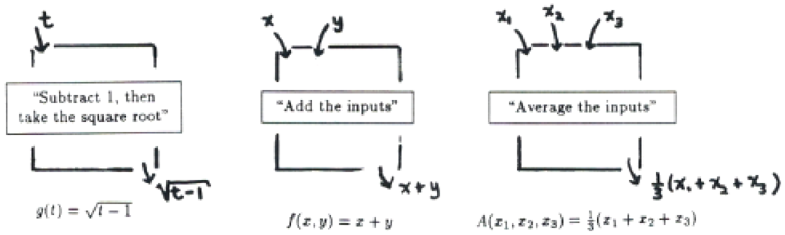
Here are some more ways that the squaring function can be described:

$$S(y) = y^2 \quad \text{or} \quad S(t) = t^2 \quad \text{or} \quad S(\alpha) = \alpha^2 \quad \text{or} \quad S(\omega) = \omega^2$$

The variable in parentheses after the function name represents a typical input; you may give any name you want to this input. Then, the formula on the right-hand side tells you what the function (the 'rule') does to this input. Since lots of different names can be used to express the same information, this input variable is called a *dummy variable*. Try to choose an appropriate name for the dummy variable. One common name is 'x'. However, if the inputs represent time values, 't' is probably more appropriate.

EXAMPLE

The 'black boxes' below illustrate the use of function notation.

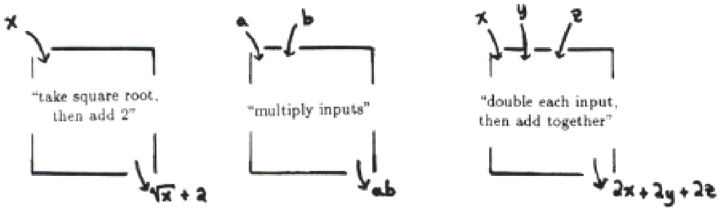


EXERCISE 7

♣ 1. Describe, in function notation, the rule: 'take a number, double it, then add 3.' Express this same rule in three different ways. What is the output if $x + y$ is the input?

♣ 2. Describe, in function notation, the rule: 'take 2 numbers and average them'. Do this in three different ways. What is the output if x and $5x$ are the inputs?

EXERCISE 8 ♣ Write function notation that corresponds to the ‘black boxes’ shown below.



function of 1 variable
function of 2 variables

A function like $f(x) = x^2 - 1$ that takes one input is called a *function of one variable.*

A function like $f(x, y) = 2x - 3y$ that takes two inputs is called a *function of two variables.*

♣ What do you suspect a function like $f(x, y, z) = x + y + z$, that takes three inputs, is called?

practice with function notation

Here’s some practice with function notation:

Consider the function g given by $g(x) = x^2 + 3$. Then:

$$\begin{aligned}
g(0) &= 0^2 + 3 = 3 \\
g(3) &= 3^2 + 3 = 12 \\
g(-3) &= (-3)^2 + 3 = 12 \\
g(x+h) &= (x+h)^2 + 3 = x^2 + 2xh + h^2 + 3 \\
g(x^2) &= (x^2)^2 + 3 = x^4 + 3 \\
g(g(t)) &= g(t^2 + 3) = (t^2 + 3)^2 + 3 \ \ \text{(inside out)} \\
g(g(t)) &= (g(t))^2 + 3 = (t^2 + 3)^2 + 3 \ \text{(outside in)}
\end{aligned}$$

The last two lines illustrate two different correct paths leading to the same result.

Now consider the function g given by $g(x, y) = x^3 + y$. Then:

$$\begin{aligned}
g(0, 0) &= 0^3 + 0 = 0 \\
g(-1, 2) &= (-1)^3 + 2 = 1 \\
g(2, -1) &= 2^3 + (-1) = 7 \\
g(a, b) &= a^3 + b \\
g(y, x) &= y^3 + x \\
g(f(x), x) &= (f(x))^3 + x \\
g\big(g(a, b), g(c, d)\big) &= (g(a, b))^3 + g(c, d) = (a^3 + b)^3 + (c^3 + d)
\end{aligned}$$

EXERCISE 9

Consider the function d given by $d(x) = \frac{f(x+h)-f(x)}{h}$. Here, f is a function of one variable, and h is a constant.

Find the following. Write complete mathematical sentences.

♣ 1. $d(0)$

♣ 2. $d(y)$

♣ 3. $d(x+h)$

♣ 4. $d(x^2)$

Now, define D by $D(x,h) = \frac{f(x+h)-f(x)}{h}$. Find the following:

♣ 5. $D(0,1)$

♣ 6. $D(y,k)$

♣ 7. $D(x+\epsilon,k)$

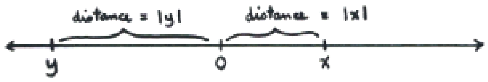
♣ 8. $D(x+h,2h)$

ALGEBRA REVIEW

absolute value, geometric definition, set union $\cup$

absolute value

Let x denote any real number. The *absolute value of* x, denoted $|x|$, is its distance from 0 on the number line.



EXAMPLE

solving a simple absolute value equation

Problem: Solve the absolute value equation $|x| = 3$.

Solution: Think of this as follows: We seek all numbers x whose distance from 0 on the number line is equal to 3. There are two such numbers, 3 and -3. Thus, the solution set of the equation $|x| = 3$ is $\{3, -3\}$. One commonly writes:

$$|x| = 3 \quad \iff \quad x = \pm 3$$

where '$x = \pm 3$' is read as 'x *equals plus or minus* 3', and means '$x = 3$ or $x = -3$'.

EXERCISE 10

Consider the sentence:

$$x = 3 \text{ or } x = -3 \qquad (*)$$

♣ 1. Let x be 3. For this choice of x, is (*) true or false? (If necessary, review the mathematical meaning of the word 'or'.)

♣ 2. What is the solution set of (*)?

♣ 3. Let x be 4. For this choice of x, is (*) true or false?

EXAMPLE

solving a simple absolute value inequality

Solve the absolute value inequality $|y| > 2$. Here, we seek all real numbers y whose distance from 0 is greater than 2. Since we can walk both *to the right* and *to the left* on the number line, the solution set has two pieces, and can be expressed in several different ways:

solution set of $|y| > 2 =$ 

$$
\begin{aligned}
&= (2,\infty) \cup (-\infty,-2) \qquad \text{(the symbol `$\cup$' is discussed below)}\\
&= \{y \mid y > 2 \text{ or } y < -2\}\\
&= \{y \mid y > 2\} \cup \{y \mid y < -2\}
\end{aligned}
$$

$\cup$, set union

Here, the symbol $\cup$ has been used to denote the operation of *set union*. For sets A and B, a new set $A \cup B$ (read as 'A union B') is formed by 'throwing together' all the elements of both sets. Precisely:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

For example, if $A = \{1,2,3\}$ and $B = \{3,4,5\}$ then $A \cup B = \{1,2,3,4,5\}$.

EXERCISE 11

Write the following sets in as simple a way as possible:

♣ 1. $(-1,0) \cup [0,2)$

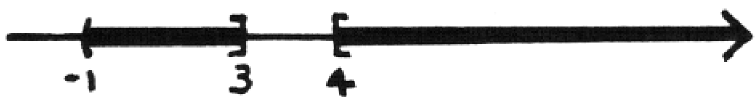
♣ 2. $\{x \mid x \text{ is rational}\} \cup \{x \mid x \text{ is irrational}\}$

♣ 3. $\{-1,2,100\} \cup \mathbb{Z}$

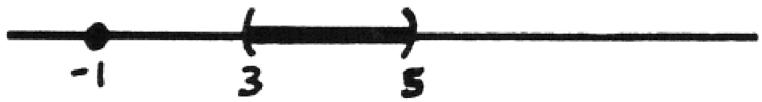
♣ 4. $\{t \mid t \geq 0\} \cup \{x \mid x < 0\}$

Write the following sets, using correct set notation and the $\cup$ symbol (if appropriate).

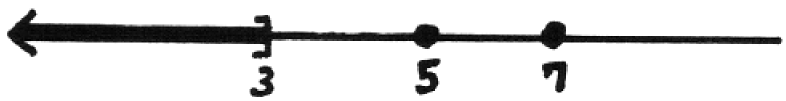
♣ 5.



♣ 6.



♣ 7.



EXERCISE 12

Solve the following absolute value equations/inequalities. Be sure to write complete and correct mathematical sentences. Show the solution sets on a number line.

♣ 1. $|x| < 2$ (Hint: What numbers have a distance from 0 that is less than 2?)

♣ 2. $|t| \geq 3$

♣ 3. $5 - |x| = 1$

♣ 4. $2 - |t| < -3$

Write an absolute value equation or inequality whose solution set is the set of numbers shown:

♣ 5. 

♣ 6. 

♣ 7. 

$|x| = \sqrt{x^2}$

Here is a characterization of the absolute value that is particularly useful in many situations:

$$\text{For all real numbers } x,\ \ |x| = \sqrt{x^2}\,.$$

For example, $\sqrt{(-5)^2} = |-5| = 5$, *not* -5! (Remember what $\sqrt{x^2}$ represents; the *nonnegative* number which, when squared, yields x^2.)

taking the square root of both sides of an equation

We have learned that adding the same number to both sides of an equation does not change its solution set; and multiplying both sides by any nonzero number doesn't change its solution set. How about taking the square root of both sides? Answer: *Providing you take the square root correctly, you WILL get an equivalent equation.*

take the square root incorrectly; lose a solution

Unfortunately, many students *don't* take the square root correctly, and write down things like this:

$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = 3$$

This student has *lost a solution.* The only time that $\sqrt{x^2}$ equals x, is if x happens to be nonnegative. The original equation $x^2 = 9$ has solution set $\{3, -3\}$; the final equation $x = 3$ only has solution 3.

taking the square root correctly

If the student had instead used the fact that is correct for *all* real numbers, $\sqrt{x^2} = |x|$, the solution would not be lost. Writing a complete mathematical sentence this time, we get:

$$\begin{aligned} x^2 = 9 \quad &\iff \quad \sqrt{x^2} = \sqrt{9} \\ &\iff \quad |x| = 3 \\ &\iff \quad x = \pm 3 \end{aligned}$$

It is conventional to leave out the step '$\sqrt{x^2} = \sqrt{9}$', and go directly to '$|x| = 3$'.

The next theorem makes this idea precise. As you read it, be thinking: what do these facts tell me that I can DO?

THEOREM

taking square roots of equations and inequalities

For all real numbers z, and for $a \ge 0$:

$$\begin{aligned} z^2 = a &\iff |z| = \sqrt{a} &&\iff z = \pm\sqrt{a} \\ z^2 > a &\iff |z| > \sqrt{a} &&\iff z > \sqrt{a} \text{ or } z < -\sqrt{a} \\ z^2 < a &\iff |z| < \sqrt{a} &&\iff -\sqrt{a} < z < \sqrt{a} \end{aligned}$$

here's how an instructor might translate this theorem

This is the way an instructor might 'translate' (part of) this theorem for students: 'You can take the square root of both sides of an equation $z^2 = a$ providing that you use the correct formula for $\sqrt{z^2}$; that is, $\sqrt{z^2} = |z|$.'

Here's a typical use of this theorem, where 'z' is '$x - 1$':

Problem: Solve $(x-1)^2 = 5$.

Solution:

$$\begin{aligned} (x-1)^2 = 5 &\iff |x-1| = \sqrt{5} \\ &\iff x - 1 = \pm\sqrt{5} \\ &\iff x = 1 \pm \sqrt{5} \end{aligned}$$

The solutions of '$(x-1)^2 = 5$' are $1 \pm \sqrt{5}$. ♣ Check!

EXERCISE 13

Use the previous theorem to solve the following equations/inequalities. Be sure to write down complete mathematical sentences.

♣ 1. $t^2 = 7$

♣ 2. $(2t-5)^2 = 3$

♣ 3. $x^2 < 4$

♣ 4. $x^2 + 6x + 9 > 4$ (Hint: Factor the left-hand side.)

♣ 5. $(|t| - 2)^2 < 1$

EXERCISE 14

♣ 1. Write a theorem, the way a mathematician would, that says how to go about solving an inequality of the form $z^2 \ge a$ (for $a \ge 0$). Then, use your theorem to solve $(2x-1)^2 \ge 3$. Show your solution set on a number line.

♣ 2. Write a theorem, the way a mathematician would, that says how to go about solving an inequality of the form $z^2 \le a$ (for $a \ge 0$). Then, use your theorem to solve $(2x-1)^2 \le 3$. Show the solution set on a number line. Compare your answer with the previous problem—do you believe your result?

★

An astute student may have noticed that the previous theorem does not apply to a sentence like $y^2 = 9 - x^2$, since the right-hand side is not nonegative for all values of x. Indeed, the sentences '$y^2 = 9 - x^2$' and '$y = \pm\sqrt{9 - x^2}$' have different implied domains.

However, if x and y are any real numbers that make $y^2 = 9 - x^2$ TRUE, then $9 - x^2$ must be nonnegative (since it equals y^2, which is nonnegative). Then, it must also be true that $y = \pm\sqrt{9 - x^2}$.

And, if x and y are any real numbers that make $y = \pm\sqrt{9 - x^2}$ TRUE, then, the sentence $y^2 = 9 - x^2$ must also be true.

Thus, the sentences $y^2 = 9 - x^2$ and $y = \pm\sqrt{9 - x^2}$ do indeed have identical graphs.

EXERCISE 15

♣ 1. Solve the equation $y^2 - x^2 = 1$ for y in terms of x. Be sure to write a complete and correct mathematical sentence.

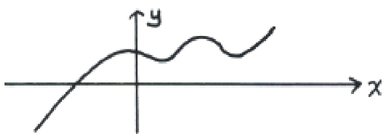
♣ 2. Solve the equation $y^2 - x^2 = 1$ for x in terms of y. Be sure to write a complete and correct mathematical sentence.

♣ 3. If (x, y) is a pair of real numbers that makes the sentence '$y^2 - x^2 = 1$' true, what (if anything) can be said about y? (Hint: Look at your solution to (2).)

QUICK QUIZ

sample questions

1. In the graph shown, is y a function of x? Is x a function of y? Justify your answers.



2. Solve the equation $x^2 - y + 1 = 0$ for y. Be sure to write a complete mathematical sentence. Is y a function of x?
3. Solve the equation $x^2 - y + 1 = 0$ for x. Be sure to write a complete mathematical sentence. Is x a function of y?
4. Describe, in function notation, the rule: 'take a number, halve it, subtract 3, then square the result'.
5. Let $g(x) = 2x^2 - 1$. Find $g(-1)$; find $g(x^2)$.

KEYWORDS *for this section*	*Inputs, outputs, function, 'black box' view of functions, vertical line test, horizontal line test, absolute value of* x*, solving absolute value equations, solving absolute value inequalities, solving equations of the form* $z^2 = a$*, solving inequalities of the form* $z^2 < a$*,* $z^2 > a$*,* $z^2 \le a$*,* $z^2 \ge a$*, set union* $(A \cup B)$*,* $\lvert x\rvert = \sqrt{x^2}$*, mapping diagrams, function notation,* f *versus* $f(x)$*, dummy variable, function of* n *variables.*

END-OF-SECTION EXERCISES

more practice with function notation

Find the indicated function values.

1. $f(x) = x^3 - 1: \quad f(0), \quad f(1), \quad f(-1), \quad f(t), \quad f(f(2))$
2. $g(x) = -x^4 + x: \quad g(x+h), \quad g(-x), \quad g(-1)$
3. $f(x) = |x|: \quad f(-2), \quad f(t), \quad f(-t), \quad f(x^2)$
4. $g(x) = |x-2|: \quad g(-x), \quad g(|t|), \quad g(\sqrt{2}), \quad g(x+2)$
5. $h(x) = \frac{1}{x}: \quad h(-x), \quad h(h(x)), \quad h(\frac{1}{x}), \quad h(x+\Delta x), \quad h(|x|)$
6. $h(x) = \sqrt{x^2-1}: \quad h(t), \quad h(x+\Delta x), \quad h(-x), \quad h(1)$
7. $h(x,y) = x^2 + y^2 - 1: \quad h(1,1), \quad h(x,x), \quad h(y,x), \quad h(x+\Delta x, y+\Delta y)$
8. $h(x,y) = \frac{1}{x(y-1)}: \quad h(0,0), \quad h(y,x), \quad h(x^2,y), \quad h(x,y^2)$

2.2 Graphs of Functions

Introduction

Associated with every function is a *set* called *the domain of the function.* This set influences what the graph of the function looks like.

DEFINITION

domain of f, $\mathcal{D}(f)$

The set of inputs to a function f is called the *domain of f*, and denoted by $\mathcal{D}(f)$.

the domain convention

The *domain convention* says the following: if the domain of a function is not explicitly specified, then it is assumed to be all inputs for which the function makes sense. Things to watch for:

- division by zero is not allowed
- numbers under even roots ($\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc.) must be nonnegative
- 0^0 is not defined

There is a very convenient notation for functions, to be discussed later on in this section, that explicitly shows the domain. This is useful when we want to take the domain to be different than the set dictated by the domain convention.

EXAMPLE

using the domain convention, function of one variable

Consider the function given by $f(x) = \frac{\sqrt{x+1}}{x+2}$. No domain is specified for the function, so the *domain convention* is used to determine the domain. The expression under the radical must be nonnegative, and $x+2$ cannot equal zero. Thus:

$$\begin{aligned}\mathcal{D}(f) &= \{x \mid x+1 \geq 0 \text{ and } x+2 \neq 0\}\\ &= \{x \mid x \geq -1 \text{ and } x \neq -2\}\\ &= \{x \mid x \geq -1\} \qquad (*)\end{aligned}$$

Note that if $x \geq -1$ is true, then automatically $x \neq -2$ is true. So in this case:

$$(\, x \geq -1 \text{ and } x \neq -2\,) \iff x \geq -1$$

By the *domain convention*, the domain of $f(x) = \frac{\sqrt{x+1}}{x+2}$ is (using interval notation) $[-1, \infty)$.

another use of the '=' sign; equality of sets

In the example above, the answer was written down using a *complete mathematical sentence.* The '=' signs used in (*) are for *equality of sets*: two sets are equal when they have the same members. For example, $A = \{1, 2, 3\}$ and $B = \{2, 3, 1\}$ are equal sets. The order in which the elements are listed is unimportant.

So far, you have seen the equal sign ('=') used in two different contexts: equality of *numbers* (or, expressions representing numbers), and equality of *sets.* You must be able to recognize, from context, the proper interpretation of an '=' sign.

EXAMPLE

using the domain convention, function of two variables

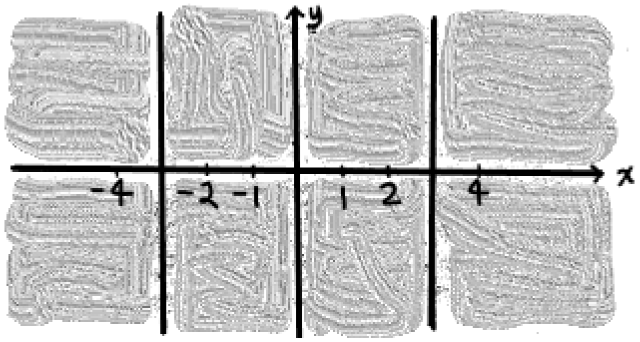
Consider the function given by $g(x,y) = \frac{1}{x^2-9} + \frac{1}{xy}$. If no domain is specified, we need only exclude inputs (x,y) for which the function does not make sense. The function is not defined if $x^2 - 9 = 0$; also, it is not defined if $xy = 0$. So the following points must be *excluded*:

$$\begin{aligned} \{(x,y) \mid x^2 - 9 = 0\} &= \{(x,y) \mid x^2 = 9\} \\ &= \{(x,y) \mid x = 3 \text{ or } x = -3\} \\ &= \{(3,y) \mid y \in \mathbb{R}\} \cup \{(-3,y) \mid y \in \mathbb{R}\} \end{aligned}$$

and

$$\begin{aligned} \{(x,y) \mid xy = 0\} &= \{(x,y) \mid x = 0 \text{ or } y = 0\} \\ &= \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,0) \mid x \in \mathbb{R}\} \end{aligned}$$

The domain of the function g is the portion of the xy-plane that remains after the necessary points are excluded. This domain is shaded below.



EXERCISE 1

interpreting a mathematical sentence

Analyze the mathematical sentence that appears in the example above:

$$\begin{aligned} \{(x,y) \mid x^2 - 9 = 0\} &= \{(x,y) \mid x^2 = 9\} && \text{(line 1)} \\ &= \{(x,y) \mid x = 3 \text{ or } x = -3\} && \text{(line 2)} \\ &= \{(3,y) \mid y \in \mathbb{R}\} \cup \{(-3,y) \mid y \in \mathbb{R}\} && \text{(line 3)} \end{aligned}$$

The lines have been numbered for easy reference.

♣ 1. Seven equals signs appear in this mathematical sentence. Which are being used for equality of numbers? Which are being used for equality of sets?

♣ 2. What allows the replacement of $x^2 = 9$ by ($x = 3$ or $x = -3$) in going from line 1 to line 2?

♣ 3. What does the symbol $\cup$ mean in line 3?

♣ 4. Why is line 2 equal to line 3?

EXERCISE 2
finding domains

Use the *domain convention* to find the domains of the following functions. Be sure to write *complete mathematical sentences.* Show the domains on a number line, or in the xy plane, whichever is appropriate.

♣ 1. $f(x) = \frac{\sqrt{x-1}}{x+2}$

♣ 2. $g(x) = \frac{\sqrt[3]{x-1}}{x^2-4}$

♣ 3. $h(x,y) = \frac{\sqrt{x}}{x+y}$

Now, we are in a position to define the *graph of a function.* First, the *graph of a function of one variable*:

DEFINITION
the graph of a function of one variable

Let f be a function of one variable. Then:

$$\text{the graph of } f = \{(x, f(x)) \mid x \in \mathcal{D}(f)\}$$

That is, the graph of f consists of points of the form (input, output), where x is the input, and $f(x)$ is the corresponding output.

In other words, the graph of f is the same as the graph of the equation $y = f(x)$. Merely plot the function values $f(x)$ as the y-values, and proceed as earlier.

slight abuse of notation

Actually, most people think of the *graph of* f as a (partial) *picture* of the set $\{(x, f(x) \mid x \in \mathcal{D}(f)\}$. Since there is such a close association between the set of points, and the *picture* of the set of points, this should cause no confusion.

EXAMPLE
graphing a function of one variable

Problem: Graph the function defined by $f(x) = \frac{1}{x-1}$.

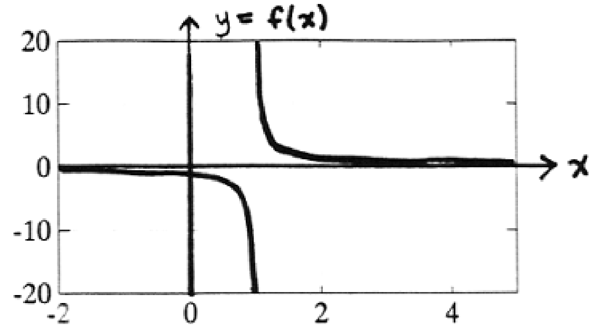
Solution: By the domain convention:

$$\mathcal{D}(f) = \{x \mid x - 1 \neq 0\} = \{x \mid x \neq 1\}$$

Then:

$$\begin{aligned}\text{the graph of } f &= \{(x, f(x)) \mid x \in \mathcal{D}(f)\} \\ &= \{\big(x, \frac{1}{x-1}\big) \mid x \neq 1\}\end{aligned}$$

The graph is shown below. It is of course impossible to show *all* points of the form $(x, f(x))$, since x is allowed to take on values in $(-\infty, 1) \cup (1, \infty)$. It is customary to show the interesting part(s) of the graph. A mathematician looking at this graph would assume that the pattern displayed near the graph boundaries would continue ad infinitum.



EXAMPLE
a 'punctured' graph

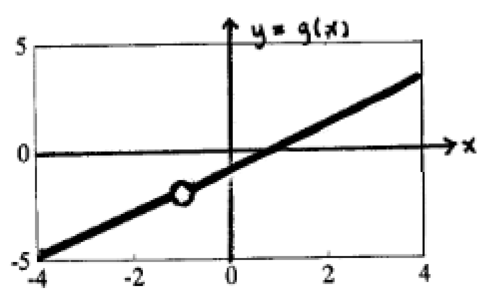
Problem: Graph $g(t) = \frac{t^2-1}{t+1}$.

Solution: Note that t cannot equal -1, for this would produce division by zero. But in the same breath you must notice that the numerator is *also* zero when $t = -1$. Since -1 is a zero of $t^2 - 1$, this means that $(t - (-1))$ is a factor of $t^2 - 1$ (see the Algebra Review, this section). Indeed, factoring yields $t^2 - 1 = (t + 1)(t - 1)$. For values of t different from -1, the function has a simpler expression:

$$\frac{t^2 - 1}{t + 1} = \frac{(t + 1)(t - 1)}{t + 1} \overset{t \neq -1}{=} t - 1$$

Note that the expressions $\frac{t^2-1}{t+1}$ and $t - 1$ are NOT exactly the same! They *are* the same a great deal of the time; whenever t is not -1. But when t is -1, they act differently: $\frac{t^2-1}{t+1}$ is not defined, but $t - 1$ is perfectly well defined, and equals $(-1) - 1 = -2$.

The graph of g is shown below.



EXAMPLE
more on 'puncturing' a graph

Problem: Write a formula for a function whose graph is the same as the graph of $f(x) = x^2 + 2$, but is 'punctured' where $x = 3$.

Solution: $P(x) = (x^2 + 2) \cdot \frac{x-3}{x-3} = \frac{x^3-3x^2+2x-6}{x-3}$

EXERCISE 3

graphing functions of one variable

Graph the following functions:

♣ 1. $f(x) = \sqrt{x} - 2$

♣ 2. $g(t) = 2|t| - 1$

♣ 3. $h(\omega) = \frac{\omega^2+\omega-6}{\omega+3}$

DEFINITION

the graph of a function of two variables

Let f be a function of two variables. Then:

$$\text{the graph of } f = \{(x, y, f(x, y)) \mid (x, y) \in \mathcal{D}(f)\}$$

Since the graph of a function of two variables is a set of points *in space*, it is more difficult to draw. We will restrict ourselves to graphing only functions of one variable.

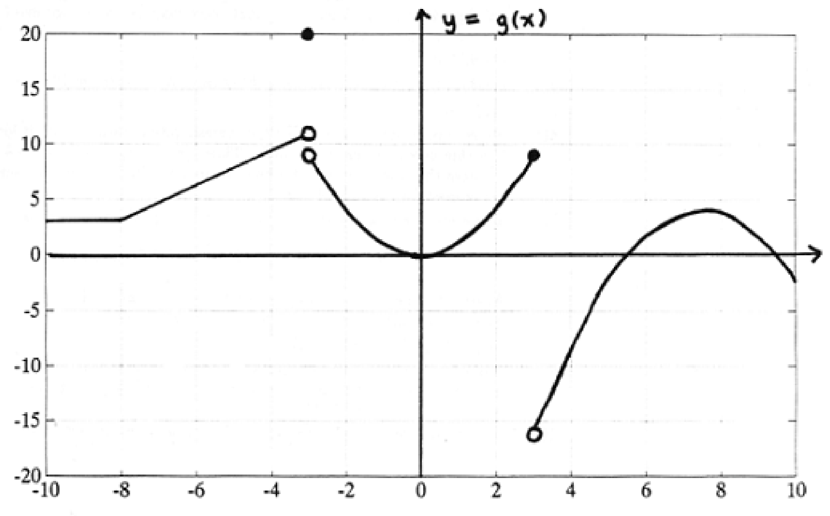
EXERCISE 4

♣ What do you suppose is the definition of *the graph of a function of three variables*?

reading information off a graph

Often, you will be *given* the graph of a function and asked to read information off the graph.

For example, consider the graph of a function g, shown below.



questions about the graph

Notice first that the graph is labeled $y = g(x)$. This tells you that the y values on the graph are the function values from a function g.

You could be asked the following questions:

1) How can you confirm that this is indeed the graph of a *function*?
2) What is $g(-10)$?
3) What is $g(-9.2)$?
4) What is $g(-3)$?
5) What is $g(0)$?
6) Find: $\{x \mid g(x) = 10\}$
7) Find: $\{x \mid g(x) = 3\}$
8) Find: $\{x \mid g(x) \geq 0\}$
9) Based on this graph, what would you suspect that $g(-11)$ is?
10) Based on this graph, what would you suspect that $g(20)$ is?

the answers

When answering these questions, be sure to write *complete mathematical sentences.*

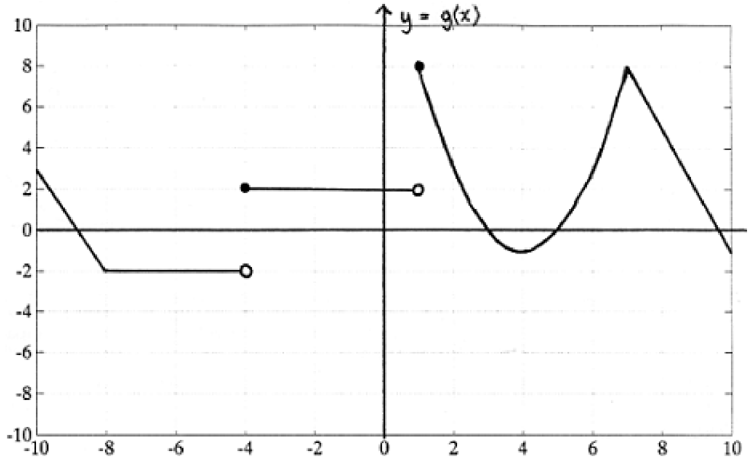
1) The graph passes the vertical line test. Every input has associated to it a *unique* output.
2) $g(-10) = 3$. **Don't just give the answer as: 3!** When reading information off a graph, it may be necessary to use your judgment and estimate. Perhaps it would be better to say $g(-10) \approx 3$; here, the symbol '$\approx$' means 'is approximately equal to'. Most people just use the '$=$' sign, with the understanding that there may be some error involved in reading off the graph.
3) $g(-9.2) = 3$
4) $g(-3) = 20$. Note that *the dot is filled in at the y value of* 20.
5) $g(0) = 0$
6) There is only one x value where the corresponding y value is 10; it looks like this occurs when $x \approx -3.5$. Thus, $\{x \mid g(x) = 10\} = \{-3.5\}$. Observe that the number -3.5 must go in a *set*, because the '$=$' sign is being used for equality of sets.
7) There are lots of x-values with corresponding y-values equal to 3:

$$\{x \mid g(x) = 3\} = [-10, -8] \cup \{-1.7, 1.7, 6.5, 8.5\}$$

Again, it is necessary to approximate. Recall that $[-10, -8]$ is a *set*. The union symbol $\cup$ is used to 'join together' the two sets.

8) $\{x \mid g(x) \geq 0\} = [-10, 3] \cup [5.5, 9.5]$
9) Assuming that the 'interesting' part of the graph is shown, and that the patterns indicated on the boundaries of the graph would continue, one would estimate that $g(-11) = 3$.
10) This author would estimate that $g(20)$ is a large negative number.

EXERCISE 5 Answer the following questions about the graph of the function g shown below. It may be necessary to estimate.



♣ 1. How can you confirm that this is indeed the graph of a *function*?

♣ 2. What is $g(-10)$?

♣ 3. What is $g(-6.4)$?

♣ 4. What is $g(-4)$?

♣ 5. What is $g(1)$?

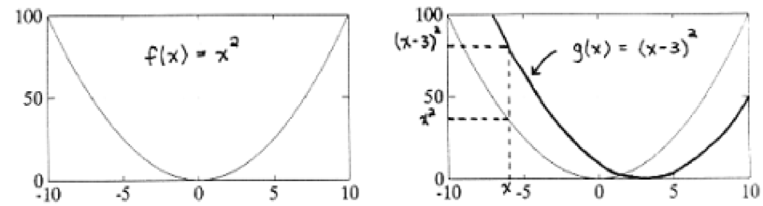
♣ 6. Find: $\{x \mid g(x) = 8\}$

♣ 7. Find: $\{x \mid g(x) = -4\}$

♣ 8. Find: $\{x \mid g(x) \leq 0\}$

shifting graphs left and right

Consider the function f given by $f(x) = x^2$. We can define a new function g that uses the function f, by $g(x) := f(x-3) = (x-3)^2$. The graphs of f and g are shown below. Remember that the symbol ':=' is used when it is desired to emphasize that something is being *defined*.



Note that the graph of g is the same as the graph of f, except shifted three units *to the right*. This is often confusing to students: they feel that since we evaluated f at x *minus* 3, the graph ought to shift to the *left*. Let's investigate what's really happening here.

FACT:
shifting a function to the right

FACT: Let f be a function, and let $c > 0$. Define a new function g by:

$$g(x) := f(x-c)$$

Then, the graph of g is the same as the graph of f, except shifted c units to the right.

REASONING:

REASONING: The first question that needs to be answered is: "What is the domain of g?" For $g(x)$ to make sense, f must know how to act on $x - c$. Thus, $\mathcal{D}(g) = \{x \mid x - c \in \mathcal{D}(f)\}$. Then:

$$\begin{aligned}
\text{the graph of } g &= \{(x, g(x)) \mid x \in \mathcal{D}(g)\} && \text{(defn. of the graph of } g\text{)} \\
&= \{(x, f(x-c)) \mid x - c \in \mathcal{D}(f)\} && \text{(defn. of } g(x)\text{)} \\
&= \{(z+c, f(z)) \mid z \in \mathcal{D}(f)\} && \text{(define } z = x - c \text{, so } x = z + c\text{)} \\
&= \{(x+c, f(x)) \mid x \in \mathcal{D}(f)\} && \text{(change dummy variable)}
\end{aligned}$$

you must understand every line of this sentence

It is important that you understand every line of this mathematical sentence.

In particular, what happened in going from line 2 to line 3? It was desired to have a *single* variable as the input to f, instead of $x - c$. This was accomplished by *defining* z to be $x - c$. (Instead of the name 'z', we could have used ω, or t, or s, or ... .) Then, everything was rewritten in terms of z.

What happened in going from line 3 to line 4? Not really anything! Students are often more comfortable working with the variable x than the variable z; this alone was the motivation for changing the name of the dummy variable.

Be sure to understand that the x in line 1 *has nothing to do with* the x in, say, line 4. In line 1, x represents a typical element of the domain of g. In line 4, x represents a typical element of the domain of f.

It is important that you understand every line of this mathematical sentence. The next two exercises check your understanding.

EXERCISE 6

♣ Let f be a function, and let $c > 0$. Define a new function g by $g(x) := f(x+c)$. Prove that the graph of g is the same as the graph of f, except shifted c units *to the left*. Be sure to write complete mathematical sentence(s).

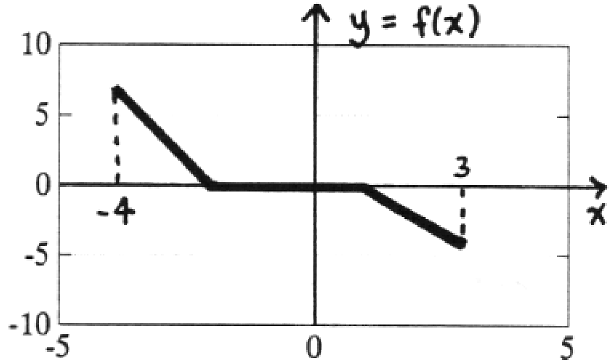
EXERCISE 7

The graph of a function f is shown below.

♣ 1. What is the domain of f?

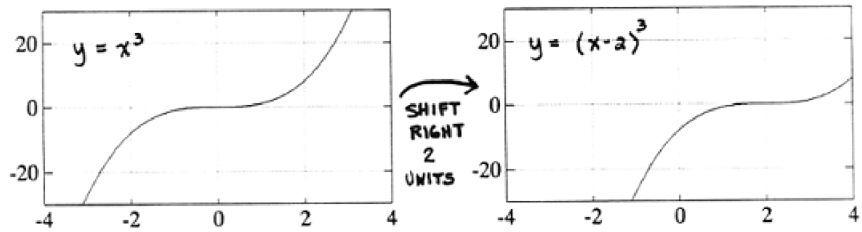
♣ 2. Define a new function h by $h(x) := f(x-2)$. What is the domain of h? Graph h.

♣ 3. Define a new function g by $g(x) := f(x+3)$. What is the domain of g? Graph g.

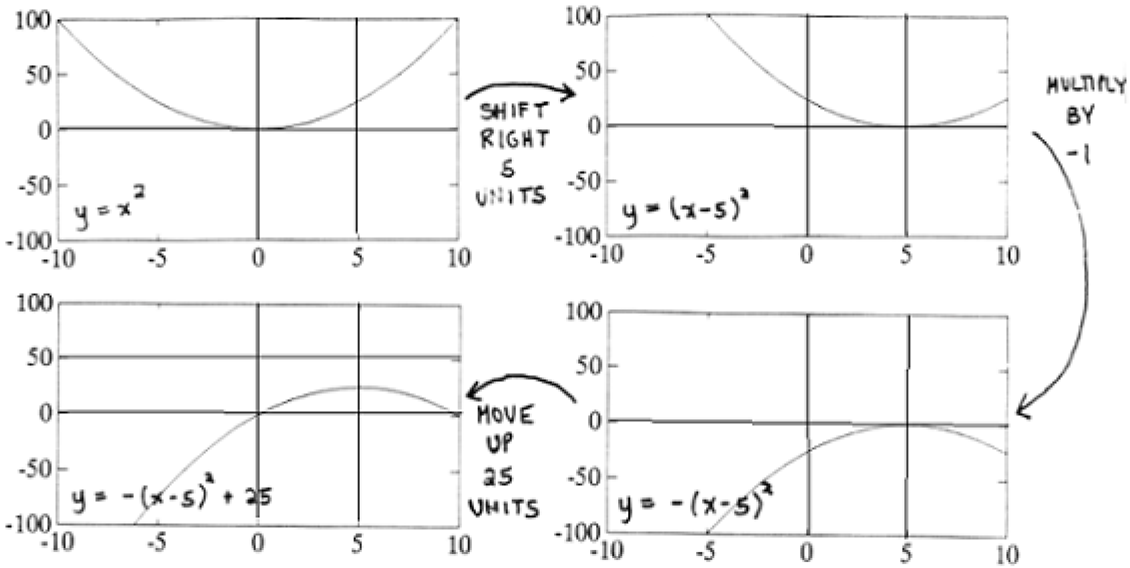


more on building graphs from simpler pieces

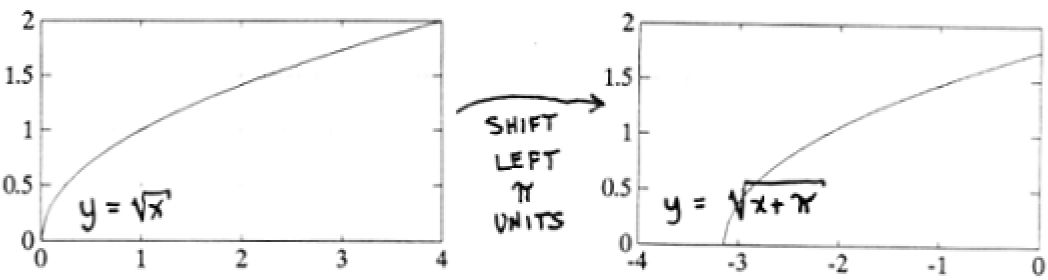
The following sketches illustrate how graphs can often be 'built up' from simpler pieces. This technique was investigated in an earlier section; some slightly more complicated examples appear here.



Graphing $f(x) = (x-2)^3$



Graphing $f(x) = -(x-5)^2 + 25$



Graphing $f(x) = \sqrt{x+\pi}$

convenient function notation that explicitly shows the domain

$f\colon A \to B$

Occasionally, one wants a function to have a domain that is *different* from its 'natural domain', that is, the domain dictated by the domain convention. In particular, if you are using a computer to graph a function, you must often restrict yourself to a set much smaller than the natural domain.

The function notation

$$f\colon A \to B$$

is particularly convenient in such cases. Read $f\colon A \to B$ as 'f, from A to B'. Here is the meaning of each part of this symbol.

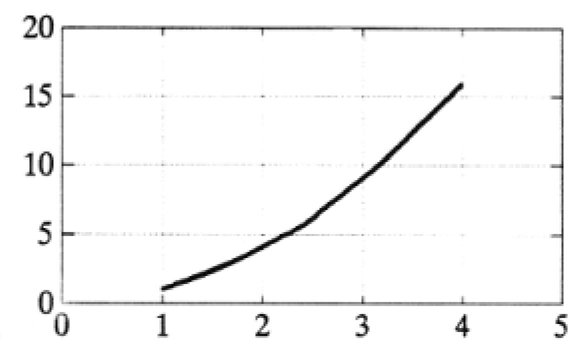
explaining the notation $f\colon A \to B$

- The first letter that appears (here, f) is the *name of the function.*
- The colon ': ' separates the function name from the rest of the symbol.
- The first letter after the colon (here, A) is the *domain of the function.* Thus, A is a *set* that contains the inputs to f.
- The arrow '$\to$' suggests that the inputs from A are being 'sent to' outputs in B.
- The last letter (here, B) can be thought of as the *output set.* It is used to answer the question: "What sort of outputs do we get when f acts on the elements from A?" In this course, the outputs of our functions will always be real numbers, so we can always let B be the real numbers, $\mathbb{R}$.
- Note that the notation $f\colon A \to B$, by itself, does *not* tell the rule that f uses to go from the inputs in A to the outputs in B. Thus, this notation *must be accompanied by a rule*, as illustrated in the examples below.

EXAMPLE

Graph: $f\colon [1,4] \to \mathbb{R},\ f(x) = x^2$

The 'natural domain' of the function f given by the rule $f(x) = x^2$ would be all real numbers, since all real numbers can be squared. Here, we want to instead take the domain to be $[1,4]$. Thus, the graph is:

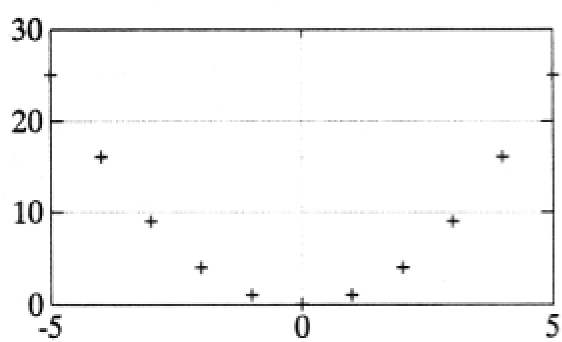


EXAMPLE

Graph the function g defined by:

$$g\colon \mathbb{Z} \to \mathbb{R},\ g(x) = x^2$$

The (partial) graph of g is shown below.



EXERCISE 8	Graph the following functions: ♣ 1. $f\colon [-2,1] \to \mathbb{R},\ \ f(x) = x^3$ ♣ 2. $g\colon [-2,-1] \cup [1,2] \to \mathbb{R},\ \ g(x) = x^2$ ♣ 3. $h\colon \{-3,-2,-1,0,1,2,3\} \to \mathbb{R},\ \ h(x) = x+1$

ALGEBRA REVIEW

factoring

To *factor an expression* means to take a *sum* (things added) and write it as a *product* (things multiplied).

For example, $x^2 + 2x - 3 = (x-1)(x+3)$. These two expressions are equal for all real numbers x. The expression $(x-1)(x+3)$ is said to be in *factored form.* The process of going from the sum $x^2 + 2x - 3$ to the product $(x-1)(x+3)$ is the process of *factoring.* You studied lots of techniques for factoring in algebra.

polynomial

Recall that a *polynomial* (in one variable x) is a sum of terms, where each term is of the form ax^n. Here, a is any real number, and $n \in \{0,1,2,3,\dots\}$. For example, $2x^3 - x^2 - 2x + 1$ is a polynomial (note that 1 can be written as $1 \cdot x^0$).

zero (root) of a polynomial

A *zero* (or *root*) of a polynomial is a *number that makes the polynomial equal to zero.* For example, 1 is a zero of $P(x) := 2x^3 - x^2 - 2x + 1$, because:

$$P(1) = 2(1)^3 - (1)^2 - 2(1) + 1 = 2 - 1 - 2 + 1 = 0$$

Also, $1/2$ is a root because $P(1/2) = 0$. (♣ Check this.)

relationship between the zeros and factors of a polynomial

There is a fundamental relationship between the zeros of a polynomial, and the factors of the polynomial. Let $P(x)$ denote any polynomial in x.

- If r is a zero of P (so that $P(r) = 0$), then $x - r$ is a factor of P.
- And, if $x - r$ is a factor of P, then r is a zero of P.

We can sometimes use this fact, together with long division, to help us factor polynomials.

EXAMPLE

For example, we saw that 1 is a root of $2x^3 - x^2 - 2x + 1$. Thus, $x - 1$ is a factor. That is, $x - 1$ must 'go into' $2x^3 - x^2 - 2x + 1$ evenly. Do a long division:

$$
\begin{array}{r|l}
 & \ \ 2x^2 + x - 1 \\
\hline
x-1 & \ \ 2x^3 - x^2 - 2x + 1 \\
 & -(2x^3 - 2x^2) \\
\cline{2-2}
 & \qquad\ \ x^2 - 2x + 1 \\
 & \qquad -(x^2 - x) \\
\cline{2-2}
 & \qquad\qquad\ -x + 1 \\
 & \qquad\qquad\ -x + 1 \\
\cline{2-2}
 & \qquad\qquad\qquad\ 0
\end{array}
$$

Now we know that $2x^3 - x^2 - 2x + 1 = (x-1)(2x^2 + x - 1)$. (♣ Finish factoring this polynomial.)

EXERCISE 9
using a zero to factor a polynomial

Check that the given number is a zero of the polynomial. Then, use this zero to get a factor. Do a long division to get another factor. Factor each polynomial as completely as possible.

♣ 1. $P(x) = x^3 + x^2 - 9x - 9;\ -1$

♣ 2. $P(x) = 2x^3 - 3x^2 - 11x + 6;\ 3$

QUICK QUIZ
sample questions

1. Use the domain convention to find the domain of $f(x) = \frac{\sqrt{2x-1}}{x^2-9}$. Be sure to write a complete mathematical sentence.
2. True or False: $\{1,2,3\} = \{3,2,1\}$. In this sentence, the '=' sign is being used for equality of ___________ .
3. What is the graph of a function f of one variable? Be sure to answer in a complete mathematical sentence.
4. Graph $f(t) = 2\sqrt{t-3}$ by building it up from 'simpler pieces'. What is $\mathcal{D}(f)$?
5. Verify that -1 is a root of $P(x) = x^4 - 2x^2 + 1$. From this information, get a factor of P. Then, use long division to get a second factor.

KEYWORDS
for this section

Domain of a function f, domain convention, two uses for the '=' sign, the graph of a function of one variable, the graph of a function of two variables, zero (root) of a polynomial, factoring polynomials, shifting graphs left and right, building graphs of functions up from simpler pieces, the $f\colon A \to B$ function notation.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\{x \mid x+1 \ge 0\}$
2. $\mathcal{D}(f) = \{x \mid x+1 \ge 0\}$
3. $(x \ge 2 \text{ and } x \ne 1) \iff x \ge 2$
4. $(x \ge 2 \text{ and } x \ne 3) \iff x \ge 2$
5. If P is a polynomial in x, and $P(-3) = 0$, then $x+3$ is a factor of P.
6. $\{x \mid x = 3\} = \{y \mid y = 3\}$
7. $\{x \mid x = 3\} = \{y \mid y - 3 = 0\}$
8. Define f by $f(x) = \frac{\sqrt{x-1}}{x+2}$.
9. The graph of g is $\{(t, g(t)) \mid t \in \mathcal{D}(g)\}$.
10. $f\colon [-3,1] \to \mathbb{R}$

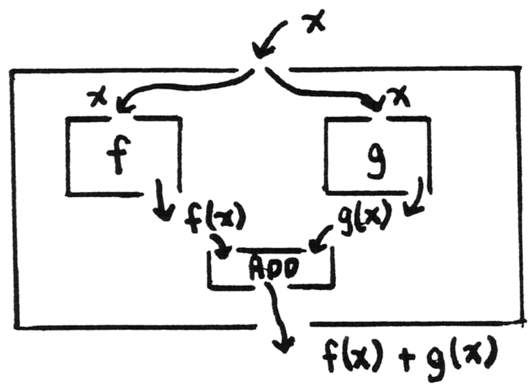
Graph the following functions.

11. $f\colon [-1,1] \to \mathbb{R},\ \ f(x) = (x+1)^3$
12. $g\colon (-\infty,4] \to \mathbb{R},\ \ g(t) = 3|t-2| - 1$
13. $h\colon \{1,4,9,16,25\} \to \mathbb{R},\ \ h(t) = \sqrt{t}$
14. $f\ :\ \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \to \mathbb{R},\ \ f(x) = \frac{1}{x}$

2.3 Composite Functions

using 'old' functions to define 'new' functions

There are many ways that functions can be 'combined' to form new functions. For example, the sketch below illustrates how functions f and g can be combined to form a 'sum' function.



After a review of some set operations, several very simple combinations of functions are discussed. Then, a very important type of combination—composition of functions—is investigated. A good understanding of function composition is necessary to understand the Chain Rule in Chapter 4.

Why are set operations being reviewed?

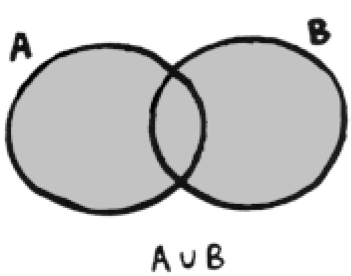
Every time that functions are combined to get new functions, one must determine how the domain of the new function is obtained from the domains of the old functions. Since the domain of a function is a *set*, finding this 'new' domain involves combining sets in various ways. The examples in this section will be greatly simplified if we first review some operations that have to do with combining sets.

set operation;

$A \cup B$,
A union B

We have seen that the *union of sets A and B* is defined by:

$$A \cup B := \{x \,|\, x \in A \text{ or } x \in B\}$$



This sentence is read as, *the set 'A union B' is defined as the set of all x with the property that x is in A, or x is in B.* The word 'or' is being used in a mathematical sense. This definition tells how an element gets in the 'new' set $A \cup B$; it must be an element for which the mathematical sentence

$$x \in A \text{ or } x \in B$$

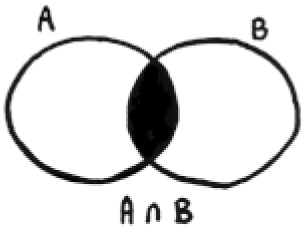
is true. When is this sentence true? By definition of the mathematical word 'or', it is true if $x \in A$ is true, or if $x \in B$ is true, or if both $x \in A$ and $x \in B$ are true.

So, this FACT is telling us that to form $A \cup B$ from sets A and B, we merely put in everything from A (the things that make $x \in A$ true) and everything from B (the things that make $x \in B$ true). Note that FACTS CAN TELL YOU WHAT TO DO.

set operation;
$A \cap B$,
A intersect B

There is another useful set operation called *set intersection*, defined as follows: Let A and B be sets. Define a new set $A \cap B$ (read as 'A intersect B') by:

$$A \cap B := \{x \,|\, x \in A \text{ and } x \in B\}$$



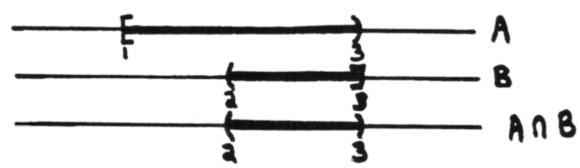
This definition says that for an element x to be in the set $A \cap B$, x must make the sentence

$$x \in A \;\text{ and } x \in B$$

true. The word 'and' is being used in the mathematical sense. When is this sentence true? By definition of the mathematical word 'and', it is true only when *both* $x \in A$ and $x \in B$ are true. So, the only elements that get into $A \cap B$ are those that are in *both* of the sets A and B.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3, 4\}$.

As a second example, if $A = [1, 3)$ and $B = (2, 3]$ then $A \cap B = (2, 3)$. You should be able to tell, from context, that the parentheses and brackets denote *interval notation* here.



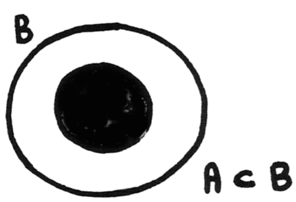
comparing sets;
$A \subset B$,
A is a subset of B
or
A is contained in B

Sometimes it is useful to know that one set is *contained* in another set. The *subset* symbol '$\subset$' is used in this situation. For sets A and B, the sentence $A \subset B$ means that everything in A is also in B. The sentence $A \subset B$ is read as '*A is a subset of B*' or '*A is contained in B*'.

EXAMPLES
using subset notation correctly

For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$ then the sentence $A \subset B$ is true. The sentence can also be written as $\{1, 2, 3\} \subset \{1, 2, 3, 4\}$.

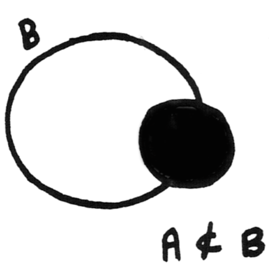
If A is *any* set, then the sentence $A \subset A$ is true. This is because any element of A (the set to the left of the $\subset$ symbol) is an element of A (the set to the right of the $\subset$ symbol).



The sentence $\{0, 1\} \subset [0, 1]$ is true; this is because $0 \in [0, 1]$ and $1 \in [0, 1]$. Note that the $\subset$ symbol is used to compare *sets*, so the symbols to the left and right of $\subset$ must be *sets*. However, the $\in$ symbol must have an *element* on the left and a *set* on the right.

If $A = \{0, 1, 2\}$ and $B = \{1, 2\}$, then the sentence $A \subset B$ is NOT true. This is because 0 is an element of A, but 0 is not an element of B.

The sentence $\{0, 1\} \subset (0, 1]$ is NOT true. This is because $0 \notin (0, 1]$.



Suppose that A is a set with element a, and B is a set with element b. Then all the following sentences are true, and illustrate the correct use of the symbols $\subset$ and $\in$:

- $a \in A$
- $\{a\} \subset A$
- $a \in A \cup B$
- $\{a, b\} \subset A \cup B$

EXERCISE 1

practice with set operations

♣ 1. True or False:

a) If A and B are sets, then $A \cup B$ is a set.

b) If C and D are sets, then $C \cap D$ is a set.

c) If A and B are sets, then $A \subset B$ is a set.

d) $\{0, 2, 4\} \subset \mathbb{Z}$

e) $\mathbb{Q} \subset \mathbb{Z}$

f) $\mathbb{Z} \subset \mathbb{Q}$

g) $\{0\} \in \{0, 1, 2\}$ (Be careful!)

♣ 2. Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{x \in \mathbb{Z} \,|\, x \geq 3\}$. Find the following sets:

a) $A \cup B$

b) $A \cap B$

c) $\mathbb{R} \cap A$

d) $\mathbb{Z} \cap B$

e) $\mathbb{Q} \cap (A \cup B)$

f) $(0, 6] \cap A$

♣ 3. For the sets A and B defined above, are the following sentences true or false?

a) $A \subset B$; Why or why not?

b) $B \subset A$; Why or why not?

♣ 4. True or False: For all sets C and D, the mathematical sentence

$$C \subset D \text{ or } D \subset C$$

is true.

♣ 5. True or False: For all sets C and D, the mathematical sentence

$$C \subset C \cup D$$

is true.

functions can take all kinds of inputs and give all kinds of outputs

In general, functions can take all kinds of inputs, and give all kinds of outputs. For example, one could define a function f that takes a NAME as an input, and gives the first letter of the name as the output. For this function, then, $f(Carol) = C$ and $f(Robert) = R$.

In more advanced mathematics courses, many important functions take *other* functions as inputs, and give functions as outputs!

ASSUMPTION IN THIS TEXT

all functions are assumed to be of this type: $f\colon \mathcal{D}(f) \to \mathbb{R}$, with $\mathcal{D}(f) \subset \mathbb{R}$

In this course, the functions that we deal with primarily are those that take a single input (a real number), and give a single output (a real number). For ease of notation, then, *henceforward in this text, unless otherwise stated, functions are ASSUMED to be functions of one variable, where both the input and output are real numbers.* That is, all functions are assumed to be of the form

$$f\colon \mathcal{D}(f) \to \mathbb{R} \ ,$$

where $\mathcal{D}(f) \subset \mathbb{R}$.

Now we are in a position to begin talking precisely about combining functions to get new functions.

the sum function
$(f+g)(x) := f(x)+g(x)$

Consider functions f and g. Define a new function, with name $f+g$, by the following rule:

$$(f+g)(x) := f(x)+g(x)$$

What does this function $f+g$ DO? Answer:

- it takes an input x
- it lets f act on x to get $f(x)$
- it lets g act on x to get $g(x)$
- it gives, as its output, the sum $f(x)+g(x)$

What is the domain of this new function $f+g$? In order for f to know how to act on x, we must have $x \in \mathcal{D}(f)$. In order for g to know how to act on x, we must also have $x \in \mathcal{D}(g)$. *Any* two real numbers $f(x)$ and $g(x)$ can be added. Thus:

$$\begin{aligned} \mathcal{D}(f+g) &= \{x \,|\, x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} \\ &= \mathcal{D}(f) \cap \mathcal{D}(g) \end{aligned}$$

EXERCISE 2

♣ 1. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the new function $f+g$. What is the domain of $f+g$? Be sure to write complete mathematical sentences.

♣ 2. Given arbitrary functions f and g, define a new function $f-g$ in the natural way. What is the domain of $f-g$? Be sure to write complete mathematical sentences.

♣ 3. Given arbitrary functions f and g, what should a function with the name fg do? Write down a precise definition. What is the domain of fg? Be sure to write complete mathematical sentences.

EXERCISE 3

Let f be any function, and let $k \in \mathbb{R}$. Define a new function kf by the rule:

$$(kf)(x) := k \cdot f(x)$$

♣ 1. What does the dot '$\cdot$' mean in the definition above? You should be able to figure this out from context.

♣ 2. In words, what does the function kf do?

♣ 3. What is the domain of the function kf? Write a complete mathematical sentence.

♣ 4. If $f(x) = x^3$ and $k = 4$, what is kf?

quotient function
$(\frac{f}{g})(x) := \frac{f(x)}{g(x)}$

As a second example, consider functions f and g. Define a new function, with name $\frac{f}{g}$, by the rule:

$$(\frac{f}{g})(x) := \frac{f(x)}{g(x)}$$

What is the domain of this new function $\frac{f}{g}$? There are three things to worry about. Firstly, f must know how to act on x in order to get $f(x)$, so we must have $x \in \mathcal{D}(f)$. Similarly, we must have $x \in \mathcal{D}(g)$. But there is an additional requirement; since division by zero is not allowed, we must also have $g(x) \neq 0$. Thus:

$$\mathcal{D}(\frac{f}{g}) = \{x \,|\, x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g) \text{ and } g(x) \neq 0\}$$

★★
associativity of the logical 'and'

An important point is being glossed over here. The logical 'and' is associative; that is,

$$(A \wedge B) \wedge C \iff A \wedge (B \wedge C)\ ,$$

as the truth table below illustrates. This allows us to say, without ambiguity, 'A and B and C' (without parentheses). Since intuition leads to this result anyway, students should believe the preceding argument without any digression.

A	B	C	$A \wedge B$	$(A \wedge B) \wedge C$	$B \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**F**	F	**F**
T	F	T	F	**F**	F	**F**
T	F	F	F	**F**	F	**F**
F	T	T	F	**F**	T	**F**
F	T	F	F	**F**	F	**F**
F	F	T	F	**F**	F	**F**
F	F	F	F	**F**	F	**F**
				↑ Same!		↑ Same!

EXERCISE 4

♣ 1. Define a function named $\sqrt{f}$ by the rule:

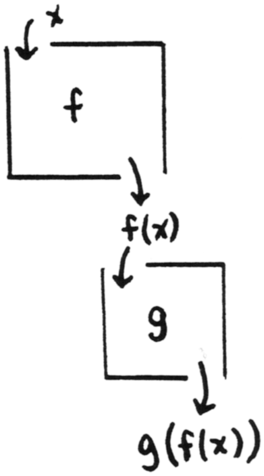
$$(\sqrt{f})(x) := \sqrt{f(x)}$$

What is the domain of $\sqrt{f}$? Be sure to write a complete mathematical sentence.

♣ 2. If $f(x) = x^3$, what is $\mathcal{D}(\sqrt{f})$?

♣ 3. If $g(x) = -x^2$, what is $\mathcal{D}(\sqrt{g})$?

composite functions



We next talk about a *very important* way of 'combining' functions: function composition.

Consider the 'combination' of functions illustrated in the diagram at left. Here, an input x is dropped into the f box, giving the output $f(x)$. Then, this output $f(x)$ is dropped into the g box, giving the output $g\big(f(x)\big)$. This type of combination of functions—characterized by one box acting in series with another box—is called *function composition.*

More precisely, consider functions f and g. A new function, named $g \circ f$ (read as 'g circle f' or 'g composed with f') is defined by the rule:

$$(g \circ f)(x) := g\big(f(x)\big)$$

This notation can be a little tricky: although the 'g' *appears* first in the name $g \circ f$, it doesn't ACT first! In the function $g \circ f$, the right-most function f acts first (it is 'closest' to x), then g acts. Analyze the definition again:

$$(g \circ f)(x) := \overbrace{g\big(\quad \underbrace{f(x)}_{f \text{ acts on } x \text{ first}} \quad\big)}^{g \text{ acts last}}$$

EXERCISE 5
practice with composition

♣ 1. Write down a precise definition of the function $f \circ g$. Draw a series of boxes that describes this function. Which function acts first?

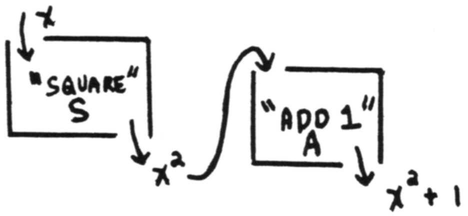
♣ 2. More than two functions can be composed. For example, one can define a new function $f \circ g \circ h$ by the rule:

$$(f \circ g \circ h)(x) := f(g(h(x)))$$

What function acts first? Second? Third? Draw a series of boxes that describes $f \circ g \circ h$.

EXAMPLE
viewing a function as a composition

Consider the function f defined by the rule $f(x) = x^2 + 1$. This function takes an input, squares it, and then adds 1. Thus, it can be viewed as a composition:



Define functions S (for 'square') and A (for 'add') by $S(x) = x^2$ and $A(x) = x + 1$. Then:

$$(A \circ S)(x) = A(S(x)) = A(x^2) = x^2 + 1$$

Thus, the function f has been viewed as a composition $A \circ S$.

EXERCISE 6

Consider the function f defined by the rule $f(x) = (x+1)^2$.

♣ 1. Describe, in words, what f does to a typical input x.

♣ 2. Draw a series of 'boxes' that describe what f does.

♣ 3. As in the previous example, view f as a composition of functions.

domain of the function $g \circ f$

Now, return to the function $g \circ f$ defined by the rule:

$$(g \circ f)(x) := g\big(f(x)\big)$$

What is the domain of this new function $g \circ f$? There are two things to worry about. Firstly, f must know how to act on x, so we must have $x \in \mathcal{D}(f)$. Secondly, g must know how to act on $f(x)$. That is, outputs from f are only acceptable IF they happen to be in the domain of g. To say this precisely:

$$\mathcal{D}(g \circ f) = \{x \,|\, x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\}$$

EXAMPLE

finding $\mathcal{D}(g \circ f)$

Let f and g be defined by $f(x) = -x^3$ and $g(x) = \sqrt{x}$. What is $\mathcal{D}(g \circ f)$? First,

$$\begin{aligned}(g \circ f)(x) &:= g\big(f(x)\big) \\ &= g(-x^3) \\ &= \sqrt{-x^3}\end{aligned}$$

By the domain convention, then:

$$\mathcal{D}(g \circ f) = \{x \,|\, -x^3 \geq 0\} = (-\infty, 0]$$

Although $\mathcal{D}(f) = \mathbb{R}$, not all these inputs are allowed for the new function $g \circ f$. It is also required that $f(x)$ (the input to g) be nonnegative; and this happens only when $x \leq 0$.

A slightly different technique is illustrated to find the domain of the 'reverse' composite, $\mathcal{D}(f \circ g)$:

$$\begin{aligned}\mathcal{D}(f \circ g) &= \{x \,|\, x \in \mathcal{D}(g) \text{ and } g(x) \in \mathcal{D}(f)\} \\ &= \{x \,|\, x \geq 0 \text{ and } \sqrt{x} \in \mathbb{R}\} \\ &= \{x \,|\, x \geq 0\} \\ &= [0, \infty)\end{aligned}$$

Be sure that you understand every step in this mathematical sentence!

EXERCISE 7

♣ In the example above, the domain of $g \circ f$ was found to be $(-\infty, 0]$. Find this result a different way, by completing the following mathematical sentence:

$$\begin{aligned}\mathcal{D}(g \circ f) &= \{x \,|\, x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\} \\ &= \{x \,|\, x \in \text{???} \text{ and } \text{?????????}\} \\ &= \text{?????}\end{aligned}$$

EXERCISE 8

Define functions f and g by $f(x) = |x|$ and $g(x) = \frac{1}{x}$.

♣ 1. Write a formula for $f \circ g$, and find its domain.

♣ 2. Write a formula for $g \circ f$, and find its domain.

♣ 3. Are $f \circ g$ and $g \circ f$ the same functions in this case? (That is, do they have the same domains, and use the same rule to obtain their outputs?)

♣ 4. Is it *always* true that $f \circ g = g \circ f$? If not, give an example where $f \circ g \neq g \circ f$.

When forming the composition $g \circ f$ from functions f and g, the nicest possible situation is when g knows how to act on ALL the outputs $f(x)$. We need a name for this set of outputs from f, and this is the next topic of discussion.

DEFINITION

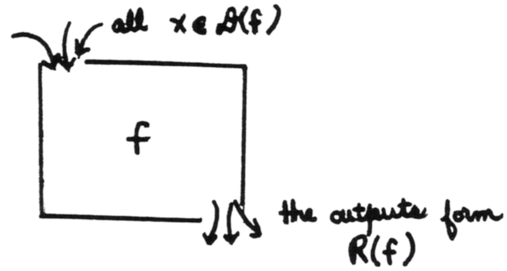
the range of a function f, $\mathcal{R}(f)$

Let f be a function with domain $\mathcal{D}(f)$. Then, the *range of f*, denoted by $\mathcal{R}(f)$, is the set of all outputs obtained from f as x takes on all possible input values. Precisely:

$$\mathcal{R}(f) := \{f(x) \,|\, x \in \mathcal{D}(f)\}$$

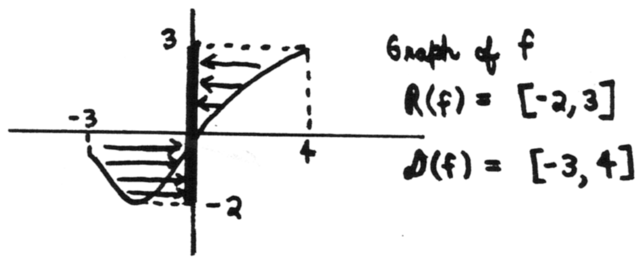
black box interpretation of the range

Here's the 'black box' interpretation of the range: drop *all possible* inputs into the top of the box. Gather together all the outputs that come out of the box. This set forms the range.



finding the range from a graph

If the graph of a function is available, then it is easy to determine the range: just imagine 'collapsing' the graph into the y-axis. The set of all y-values that are taken on forms the range of the function.



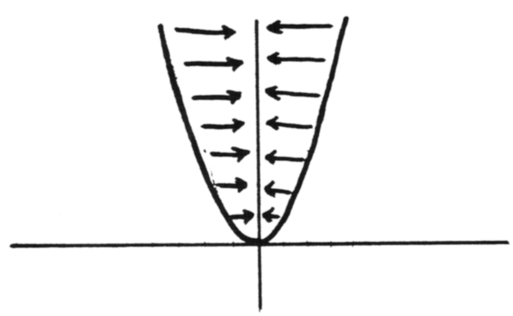
EXAMPLE

finding $\mathcal{R}(f)$

Let f be the function defined by the rule $f(x) = x^2$. By the domain convention, $\mathcal{D}(f) = \mathbb{R}$. What is the range?

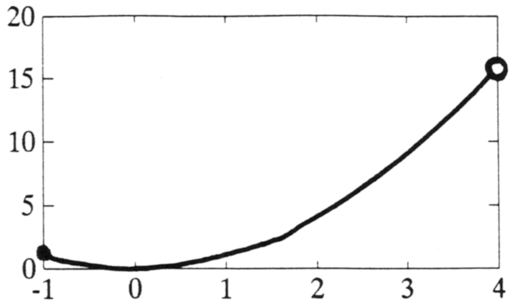
$$\begin{aligned} \mathcal{R}(f) &:= \{f(x) \,|\, x \in \mathcal{D}(f)\} \\ &= \{x^2 \,|\, x \in \mathbb{R}\} \\ &= [0, \infty) \end{aligned}$$

The question is also easily answered by studying the graph of f. If the graph is 'collapsed' into the y-axis, one obtains the interval $[0, \infty)$.



EXAMPLE
finding $\mathcal{R}(f)$

Now consider $f\colon [-1,4) \to \mathbb{R}$, $f(x) = x^2$. The graph of f is shown below:



As f runs through all the elements in its domain, the only outputs obtained are those in the set $[0,16)$. Thus:

$$\mathcal{R}(f) = [0,16)$$

EXAMPLE
finding the range of a constant function

Consider the constant function given by the rule $g(x) = 16$. By the domain convention, $\mathcal{D}(g) = \mathbb{R}$. But as g acts on all possible inputs, the only output obtained is 16. Thus, $\mathcal{R}(g) = \{16\}$. (♣ Why is it INCORRECT to say $\mathcal{R}(g) = 16$?)

EXERCISE 9
finding the range

Find the domains and ranges of the following functions. Be sure to write complete mathematical sentences.

♣ 1. $f_1(x) = \sqrt{x} + 1$

♣ 2. $f_2(x) = \sqrt{x+1}$

♣ 3. $g\colon [0,4] \to \mathbb{R}$ given by $g(x) = \sqrt{x}$

♣ 4. $h\colon \{0\} \cup (1,3] \to \mathbb{R}$ given by $h(x) = \sqrt{x}$

QUICK QUIZ
sample questions

1. Let A and B be sets. Write a precise definition of $A \cap B$. If $A = [1,3)$ and $B = \{1,2,3\}$, what is $A \cap B$?
2. TRUE or FALSE:
   - $[1,3] \subset \{1,3\}$
   - $\{1,3\} \subset [1,3]$
   - For all sets A and B, $A \cap B \subset A$.
3. Let f and g be functions. Give a precise definition of the new function $f+g$. What is the domain of $f+g$? Be sure to write a complete mathematical sentence.
4. Define functions a and b so that the function $f(x) = 2x - 1$ can be written as a composition, $f = a \circ b$.
5. What is the range of the function $f\colon \mathbb{Z} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 & \text{for } n > 1 \\ -1 & \text{for } n \le 1 \end{cases} \ ?$$

KEYWORDS *for this section*

Set union ($A \cup B$), set intersection ($A \cap B$), subset notation ($A \subset B$), assumption about functions in this text, combining functions to form new functions, determining the domain of the 'new' function, composite functions ($f \circ g$), domain of $f \circ g$, range of a function f, notation $\mathcal{R}(f)$.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN). The context will determine if a variable is a number, a function, or a set.

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $A \cup B$
2. $A \subset A \cup B$
3. $A \subset B$
4. $\mathcal{R}(f)$
5. $\mathcal{R}(f) = \mathbb{R}$
6. $\{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\}$
7. $x \in \mathcal{D}(f)$ and $f(x) \in \mathcal{D}(g)$
8. $(f+g)(x) := f(x) + g(x)$
9. $\{a\} \in \{a, b\}$
10. $\{a\} \subset \{a, b\}$

Find the range of each of the following functions. (You graphed these functions in §2.2, End-Of-Section Exercises, 11–14.)

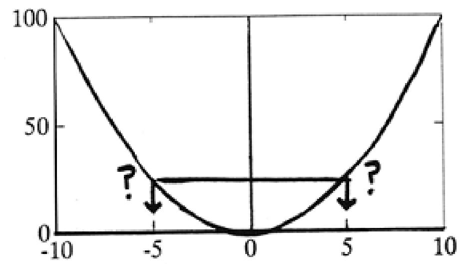
11. $f\colon [-1, 1] \to \mathbb{R},\ \ f(x) = (x+1)^3$
12. $g\colon (-\infty, 4] \to \mathbb{R},\ \ g(t) = 3|t-2| - 1$
13. $h\colon \{1, 4, 9, 16, 25\} \to \mathbb{R},\ \ h(t) = \sqrt{t}$
14. $f\colon \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \to \mathbb{R},\ \ f(x) = \frac{1}{x}$

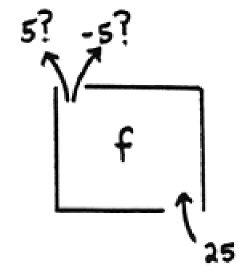
2.4 One-to-One Functions and Inverse Functions

Introduction

Recall that a *function* satisfies the property that for every input there exists a *unique* output. Thus, the graph of a function must pass a vertical line test.

Note what this definition does *not* say: it does not say that every *output* must be associated with a unique *input*. To illustrate this idea, consider the function f given by the rule $f(x) = x^2$:





When 5 is the input, $f(5) = 25$ is the output; and when -5 is the input, f(-5) = 25 is again the output. In terms of a 'black box', if we were to try to use the box 'backwards' and put the output 25 in the bottom, to see what input led to that output, the box would object: it doesn't know which input to choose. From a graphical point of view, functions need not pass a horizontal line test.

one-to-one functions; informal discussion

If the graph of a given function does indeed pass a horizontal line test, then this function has the additional property that for every *output*, there is a unique *input*. In this case, the function is given a special adjective: it is called a *one-to-one* function (abbreviated as '1-1'). The name is completely appropriate, for in this case there is a *one-to-one correspondence* between the outputs and the inputs: for every input, there exists a unique output (the 'function' condition), AND for every output, there exists a unique input (the '1-1' condition).

Observe that every *1-1 function* is firstly a *function*; we do not talk about the 1-1 property for non-functions.

In terms of a 'black box', the 1-1 property can be described as follows: we can stick an *output* in the bottom, and say, without ambiguity, what *input* must have been put in to produce this output. This is an extremely nice relationship between inputs and outputs: the inputs uniquely identify the outputs, *and* the outputs uniquely identify the inputs.

some symbols:
$\forall$, 'for all'
$\exists$, 'there exists'
!, 'a unique'

The following phrases occur so frequently in mathematics that there are special symbols for them:

The symbol ' $\forall$ ' is read as 'for all' or 'for every'.

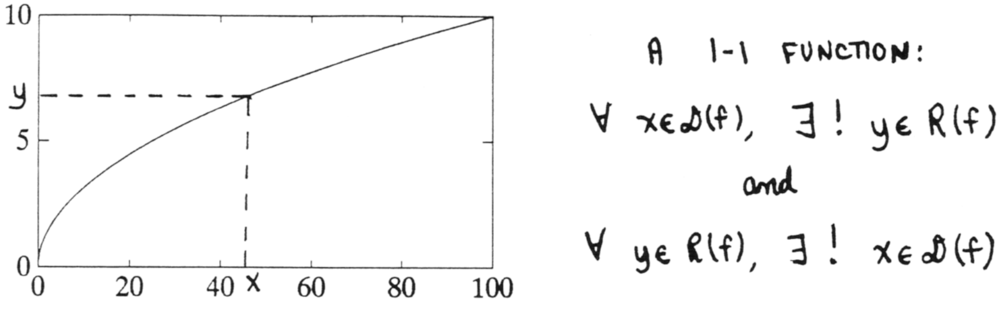
The symbol ' $\exists$ ' is read as 'there exists'.

The symbol ' ! ' is read as 'a unique'.

These symbols will be used freely throughout the text.

EXAMPLE
a 1-1 function

The function f given by $f(x) = \sqrt{x}$ is a one-to-one function. It is a *function* because for every $x \in \mathcal{D}(f)$, there exists a unique $y \in \mathcal{R}(f)$. Thus the graph of f passes the vertical line test. It is *one-to-one* because for every $y \in \mathcal{R}(f)$, there exists a unique $x \in \mathcal{D}(f)$. Thus, the graph of f passes the horizontal line test.



using the symbols ∀, ∃, !

The properties discussed in the previous paragraph can be stated concisely using the special symbols provided by the language of mathematics. For example, the 'function' condition

$$\forall\ x \in \mathcal{D}(f), \quad \exists\,!\ y \in \mathcal{R}(f)$$

is read as: *for all x in the domain of f, there exists a unique y in the range of f.* If this sentence were to appear in text, instead of being displayed (that is, set off and centered), then it would begin with the *words* 'For all' instead of the *symbol* ∀. It is not good style for a text sentence to begin with a symbol. However, in display mode, it is acceptable to begin a sentence with a symbol.

EXERCISE 1

♣ Write the following sentence using appropriate symbols: *For every y in the range of a function f, there exists a unique x in the domain of f.* What condition is being described here?

EXERCISE 2

♣ 1. Draw a graph of a non-function.

♣ 2. Draw a graph of a function that *is not* 1-1.

♣ 3. Draw a graph of a function that *is* 1-1.

♣ 4. Draw the graph of a relation between x and y such that y is a function of x and x is a function of y. Can you use a previous example?

♣ 5. Draw the graph of a relation between x and y such that y is a function of x but x is not a function of y. Can you use a previous example?

♣ 6. Is it possible to be a *one-to-one function* without being a *function*? Why or why not?

★★
precise definition of a 1-1 function

Here's the precise definition of a 1-1 function:

Let f be a function. Then:

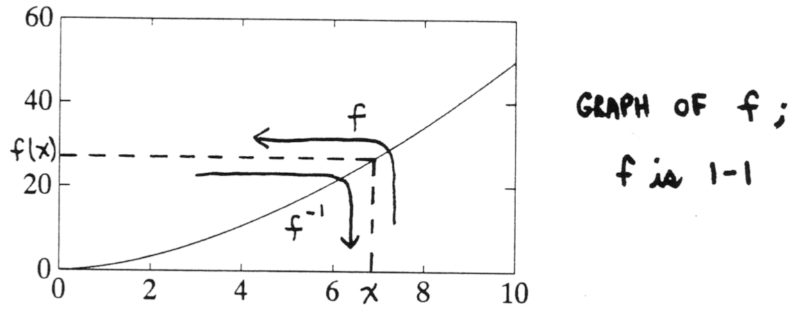
$$f \text{ is 1-1} \iff \forall\, x, y \in \mathcal{D}(f),\ f(x) = f(y) \implies x = y$$

This definition says: Whenever two outputs are the same, then the corresponding inputs must be the same. This is the form most often used if one is asked to *prove* that a function is 1-1.

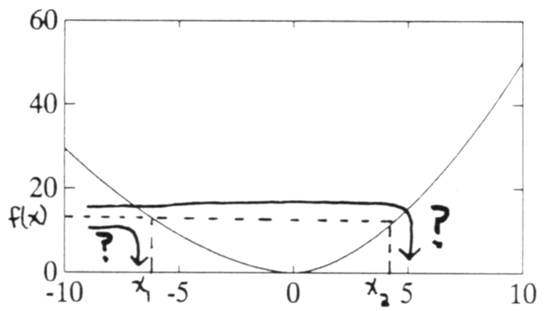
However, this definition requires an understanding of the mathematical sentence called an *implication*; that is, a sentence of the form $A \Rightarrow B$. Implications are discussed in a future section.

inverse function f^{-1}

Whenever a function f is 1-1, then another function, called f^{-1} (read as '*f inverse*') can be defined that 'undoes' what f does! The picture below illustrates this fact:

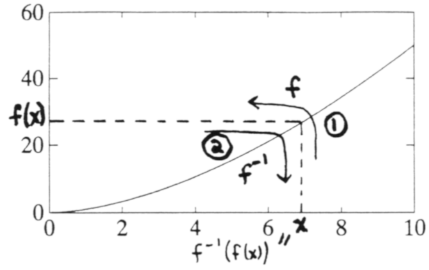


Given an input x, the function f sends it to the (unique) output $f(x)$.
Given this output $f(x)$, the function f^{-1} sends it back to the (unique) input x.
Note: if f is NOT 1-1, then we can't do this, as illustrated below.

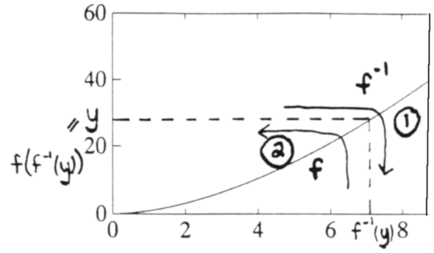


relationship between a function f and its inverse f^{-1}

For a 1-1 function f, the relationship between f and its inverse f^{-1} can be summarized via the sketches below. Look at the graph on the left. Begin with an x in the domain of f. The function f sends this input to $f(x)$. Then, the function f^{-1} takes $f(x)$ as its input, and sends us to the output $f^{-1}(f(x))$. But this output is just where we started: $f^{-1}(f(x)) = x$!



$\forall\ x \in \mathcal{D}(f),\ f^{-1}(f(x)) = x$



$\forall\ y \in \mathcal{R}(f),\ f(f^{-1}(y)) = y$

EXERCISE 3

♣ Look at the right-most preceding graph, labeled $\forall\ y \in \mathcal{R}(f),\ f\big(f^{-1}(y)\big) = y$. Explain, in words, what is happening here. Why do you suppose the dummy variable 'y' was used to represent a typical element of the range of f?

EXAMPLE

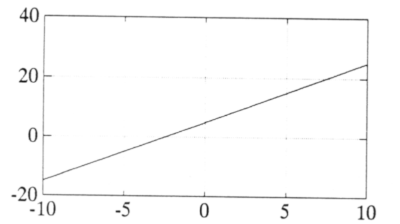
finding f^{-1}

Problem: Consider the function f given by the rule $f(x) = 2x + 5$. Show that f is 1-1. Then, find f^{-1}.

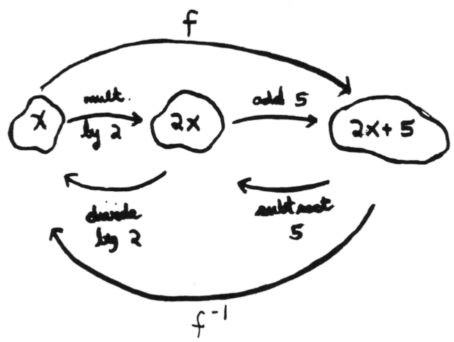
Solution: A quick graph of f shows that it passes both vertical and horizontal line tests, and hence is 1-1.

Now, let's find its inverse, in two different ways.

Method I (mapping approach): What does f DO? Well, it takes an input, multiplies it by 2, then adds 5. How could we UNDO this process? Just work backwards, 'reversing' the operations: first, subtract 5; then, divide by 2. Thus, $f^{-1}(x) = \frac{x-5}{2}$.



Graph of f,
f(x) = 2x + 5



Method II (algebraic approach): To find f^{-1}, use the fact that f and f^{-1} must satisfy the relationship $f(f^{-1}(x)) = x$. Treat the output $f^{-1}(x)$ as the 'unknown', and solve for it!

$$\begin{aligned}
f(f^{-1}(x)) &= x && (f \text{ and } f^{-1} \text{ must satisfy this}) \\
2\big(f^{-1}(x)\big) + 5 &= x && (\text{definition of } f) \\
f^{-1}(x) &= \frac{x-5}{2} && (\text{solve for } f^{-1}(x))
\end{aligned}$$

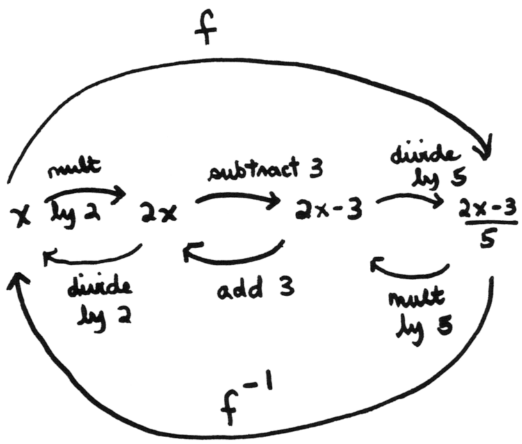
EXAMPLE

finding f^{-1}

Problem: Find the inverse function for $f(x) = \frac{2x-3}{5}$ by thinking about *undoing* what f does. Then check your answer by verifying that both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (for appropriate x).

Solution: Observe that f takes an input, multiplies it by 2, subtracts 3, then divides by 5. To undo this, f^{-1} must take its input, multiply by 5, add 3, and divide by 2, so that $f^{-1}(x) = \frac{5x+3}{2}$.

To check, observe that:



$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}\big(\frac{2x-3}{5}\big) \\
&= \frac{5(\frac{2x-3}{5}) + 3}{2} = \frac{2x-3+3}{2} \\
&= x
\end{aligned}$$

and

$$\begin{aligned}
f(f^{-1}(x)) &= f\left(\frac{5x+3}{2}\right) \\
&= \frac{2\big(\frac{5x+3}{2}\big) - 3}{5} \\
&= \frac{5x+3-3}{5} \\
&= x
\end{aligned}$$

EXERCISE 4

The following functions are $1-1$. (Why?) Find the inverse functions in two different ways: using a 'mapping approach', and an 'algebraic approach'. Verify the two conditions: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

♣ 1. $f(x) = 4x + 1$

♣ 2. $g(x) = \frac{3-5x}{7}$

relationship between the graphs of f and f^{-1}

The points on the graph of f are of the form:

$$\{(x, f(x)) \mid x \in \mathcal{D}(f)\}$$

The points on the graph of f^{-1} are of the form:

$$\{(f(x), x) \mid x \in \mathcal{D}(f)\}$$

That is, whenever a point (a, b) is on the graph of f, there is a point (b, a) on the graph of f^{-1}. This situation leads to a nice relationship between the graphs of f and f^{-1}, which is called *symmetry about the line $y = x$*, and is discussed next.

symmetry about the line $y = x$

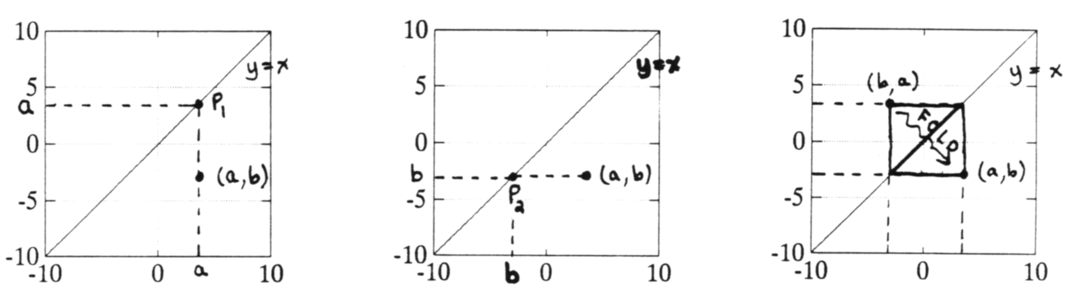
Graph the line $y = x$, and then plot a point (a, b) not on this line. Let's *use* the line $y = x$ to 'find' the point (b, a), as described below.

Refer to the left-most sketch below. First, go vertically from the point (a, b) to the line $y = x$, arriving at the point labeled P_1. The x-coordinate of P_1 is the same as that of (a, b) (why?), so P_1 must have coordinates $(a, ???)$. But, P_1 lies on the line $y = x$, so its x and y coordinates are the same. Thus, P_1 must be the point (a, a). Label the y-coordinate of P_1.

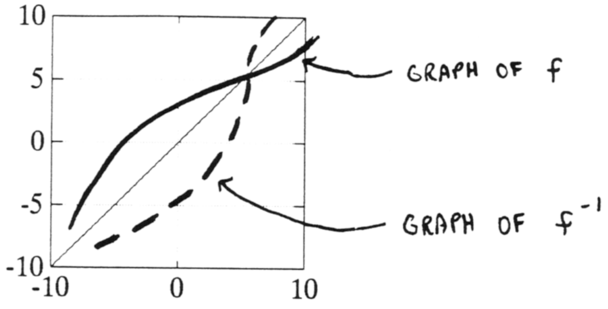
Now refer to the middle sketch below. Go horizontally from the point (a, b) to the line $y = x$, arriving at the point labeled P_2. This point P_2 has the same y-coordinate as (a, b) (why?), and hence P_2 is of the form $(???, b)$. But P_2 lies on the line $y = x$, so its x and y coordinates are the same. Thus, P_2 must be the point (b, b). Label the x-coordinate of P_2.

Look at the right-most sketch below. We used (a, b) and the line $y = x$ to find (b, a). A square is formed (with side of length $a - b$). The line $y = x$ forms a diagonal of this square. Thus, if we were to 'fold' the graph along the line $y = x$, the points would land on top of each other.

In other words, if you have a point (a, b) and WANT the point (b, a), just 'fold' along the line $y = x$! Curves that have the property that they lie atop each other, when folded along the line $y = x$, are said to be *symmetric about the line $y = x$*.

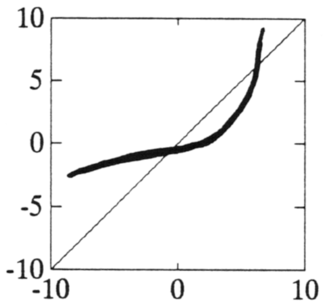


Now, given the graph of f, it is easy to find the graph of f^{-1}. Just 'fold' along the line $y = x$ (mentally), and sketch in the resulting curve, as illustrated below.



EXERCISE 5

♣ 1. On the graph below, sketch in a curve that is symmetric to the one drawn, about the line $y = x$.



♣ 2. Convince yourself that $f(x) = x^3$ is $1 - 1$. Then, find f^{-1}. Finally, graph both f and f^{-1} on the same graph.

a calculator exercise

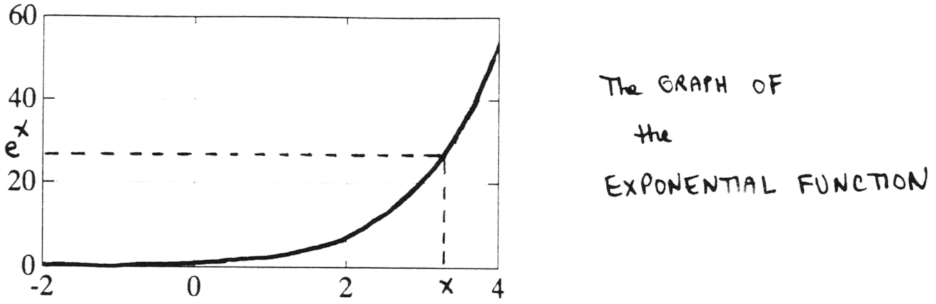
Pull out your hand-held calculator. Find keys that are labeled 'e^x' and 'ln'. (Ask you instructor for help, if necessary. Occasionally they are labeled slightly differently.) Now, try the following exercise:

`Input the number 2.`
`Press the` e^x `key. (Something like 7.39 will be displayed.)`
`Press the 'ln' key. (2 is displayed again.)`

Repeat this exercise, starting with 4.2 instead of 2.
Then repeat again, starting with -7.02.
What's happening here?

the exponential function; e^x

The key labeled e^x gives you access to the *exponential function*. The 'e' that appears here is the irrational number, $e \approx 2.72$. Thus, the exponential function takes an input x, and gives the output e^x. That is, it raises the number e to the input power. The graph of the exponential function is shown below.



Here are some important properties of the exponential function:

- The domain is $\mathbb{R}$.
- The range is $(0, \infty)$. In particular, e^x is *never* equal to zero.
- As x approaches infinity, so does e^x. (Which gets bigger *faster*, x or e^x?)
- As x approaches negative infinity, e^x approaches zero.

★★ In order to discuss the exponential function precisely, one must make sense of things like e^π; what is meant by e to an irrational power? There are a variety of approaches to resolving problems such as this. One approach is to show that the power series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for all x, and then use this series to define e^x. Since multiplication and addition is defined for all real numbers, things like e^π then make sense.

Why the name 'exponential function'?

The *exponential function* is named as such, because it satisfies the familiar laws of exponents. For all real numbers x and y:

$$\begin{aligned}
e^0 &= 1 \\
e^x e^y &= e^{x+y} && \text{(same base, multiplied, add the exponents)} \\
\frac{e^x}{e^y} &= e^{x-y} && \text{(same base, divided, subtract the exponents)} \\
(e^x)^y &= e^{xy} && \text{(power to a power, multiply the exponents)} \\
e^{-x} &= \frac{1}{e^x}
\end{aligned}$$

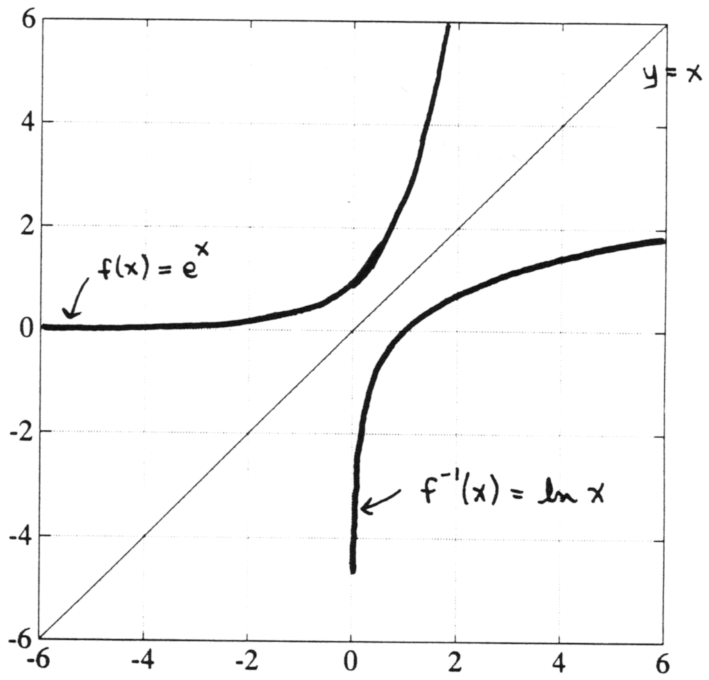
the natural logarithm function;

$\ln x$

Give the exponential function the name f; that is, define $f(x) := e^x$. Observe that f is 1-1. Therefore, it has an inverse f^{-1}. The name given to this inverse is *the natural logarithm function*, with output denoted by $\ln x$. That is, $f^{-1}(x) = \ln x$.

The *name* of the natural logarithm function is 'ln'; when the input is x, the output is often denoted by '$\ln x$' instead of '$\ln(x)$'. That is, the parentheses that are usually used in function notation are suppressed. However, in cases that might be ambiguous, parentheses *are* used: for example, '$\ln 2x + 5$' is ambiguous; does the author want '$(\ln 2x) + 5$', or '$\ln(2x + 5)$' ? In cases like this, parentheses are used to avoid confusion.

The key on your calculator labeled 'ln' (or, on some calculators, 'log') gives you access to the natural logarithm function. (Check that you've got the right key by finding $\ln 2.72$. You should get a number very close to 1, NOT close to 0.4!) The graph can be formed by reflecting the graph of $f(x) = e^x$ about the line $y = x$; this graph is shown below.



EXERCISE 6

Use your knowledge that the natural logarithm function is the inverse of the exponential function to answer the following questions:

♣ 1. What is the domain of the natural logarithm function?

♣ 2. What is the range of the natural logarithm function?

♣ 3. As x approaches infinity, what happens to $\ln x$?

♣ 4. As x gets closer and closer to zero (from the right), what happens to $\ln x$?

the classic 'inverse pair'

Since $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are inverse functions, they 'undo' each other. That is:

$$f^{-1}(f(x)) = x \ \ \forall \ x \in \mathcal{D}(f)$$

Rewriting in terms of e^x and $\ln x$, we have:

$$\ln(e^x) = x \ \ \forall \ x \in \mathbb{R}$$

Also:

$$f(f^{-1}(x)) = x \ \ \forall \ x \in \mathcal{R}(f)$$

That is:

$$e^{\ln x} = x \ \ \forall \ x \in (0, \infty)$$

EXERCISE 7

Let $f(x) = e^x$, so that $f^{-1}(x) = \ln x$.

♣ 1. Check that the correct translation of

$$f^{-1}(f(x)) = x \ \ \forall \ x \in \mathcal{D}(f)$$

is:

$$\ln(e^x) = x \ \ \forall \ x \in \mathbb{R}$$

♣ 2. Check that the correct translation of

$$f(f^{-1}(x)) = x \ \ \forall \ x \in \mathcal{R}(f)$$

is:

$$e^{\ln x} = x \ \ \forall \ x \in (0, \infty)$$

uses for the exponential and logarithmic functions

The exponential and natural logarithm functions arise in many practical applications in business and the life sciences. An application of the exponential function is addressed in the next example.

simple interest

Suppose that \$2,000 is put in a bank offering an annual interest rate of 10%. After one year, interest of (0.10)(\$2000) = \$200 is earned, so the total amount in the bank is:

$$\$2000 + \$200 = \$2200$$

Remember that simple interest is computed using the formula:

$$\text{INTEREST} = (\text{PRINCIPAL})(\text{RATE})(\text{TIME})$$

The units must agree; if the principal is in dollars, and the rate is annual (say, $\frac{10\%}{\text{year}}$), then time must have units of years. In this case, the time units will cancel, leaving units of:

$$(\$)(\frac{1}{\text{year}})(\text{year}) = \$$$

Simple annual interest is not very desirable: who wants to wait an entire year before having any interest added in?

compounding

Most banks offer 'compounding'; that is, they add in the accumulated interest at regular intervals.

For example, suppose a bank offers 10% annual interest, compounded semi-annually. Then, after 6 months (0.5 years), the accumulated interest of

$$(\text{principal})(\text{rate})(\text{time}) = (2000)(.10)(.5) = 100$$

will be added in, yielding a total of:

$$2000 + (2000)(.10)(.5) = 2100$$

It is conventional to suppress all units for intermediate calculations.

After six more months, interest of $(2100)(.10)(.5) = 105$ will be added in, yielding a total of:

$$2100 + 105 = 2205$$

This is \$5 more than the amount obtained with simple annual interest.

EXERCISE 8

♣ Figure out how much will be in the bank after one year, if compounded quarterly (every 3 months).

continuous compounding

The best situation that can occur is if, *at each instant*, the accumulated interest is added in. This is called *continuous compounding*. It can be shown that the exponential function describes this situation! That is, P dollars, compounded continuously at an annual interest rate r, for t years, will grow to Pe^{rt} dollars. For example, \$2000 at 10% annual interest rate, compounded continuously, will grow to $2000e^{(.10)(1)} = \$2,210.34$ after one year!

EXERCISE 9

Suppose \$5,000 is put in a bank. How much will there be after 2 years, assuming:

♣ 1. 8% simple annual interest?

♣ 2. 8% annual interest, compounded semi-annually?

♣ 3. 8% annual interest, compounded continuously?

♣ 4. How much money is gained by having continuous compounding, as opposed to simple annual interest?

we will see the exponential and natural logarithm functions again

There are further properties of the exponential and natural logarithm functions that make them useful tools in calculus. Thus, we will see these functions again as we proceed throughout this text.

On to calculus!

These first two chapters, now drawing to a close, have been preparatory chapters. Some basic algebra skills were reviewed. Elements of the language of mathematics that will be needed throughout the course were developed. Finally, we are ready to begin the study of calculus.

The central idea in calculus is that of a *limit*. Without this concept, it would be impossible to speak precisely about continuity, differentiation, or integration. Thus, the next chapter begins with the study of *limits*.

QUICK QUIZ
sample questions

1. Is $f(x) = x^2$ a one-to-one function? Why or why not?
2. Translate this mathematical sentence:

$$\forall\ y \in \mathcal{R}(f), \quad \exists\ !\ x \in \mathcal{D}(f)$$

   What condition is being described here?
3. Sketch the graph of a function f satisfying the following properties: f is one-to-one, $\mathcal{D}(f) = [0, \infty)$, $\mathcal{R}(f) = (-\infty, 0]$.
4. Write down the two equations that describe the relationship between a 1-1 function f and its inverse f^{-1}.
5. Show that $f(x) = \frac{1}{3}x - 1$ is 1-1 (say, by graphing). Then, find f^{-1}, using any appropriate method.

KEYWORDS
for this section

One-to-one function and defining condition, the symbols $\forall$, $\exists$, !, *inverse function* f^{-1}, *precise conditions describing the relationship between a function and its inverse, finding a formula for* f^{-1} *(in two ways), symmetry about the line* $y = x$, *graphing* f^{-1}, *the exponential function and its graph, the exponent laws, the natural logarithm function and its graph, conditions relating* e^x *and* $\ln x$.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN). The context will determine the appropriate variable type (number, set, function).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $f^{-1}(x)$
2. $\forall\ y \in \mathcal{R}(f), \quad \exists\ !\ x \in \mathcal{D}(f)$
3. If f and f^{-1} are inverse functions, then $f(f^{-1}(x)) = x \ \ \forall\ x \in \mathcal{R}(f)$.
4. If f and f^{-1} are inverse functions, then $f(f^{-1}(y)) = y \ \ \forall\ y \in \mathcal{R}(f)$.
5. $f(f^{-1}(x))$
6. f is 1-1
7. For all real numbers x, $\ln(e^x) = x$.
8. For all real numbers x, $e^{\ln x} = x$.
9. $\ln x$
10. $e^x = 3$

Sketch the graphs of the following functions, by 'building them up' from simpler pieces. Find the domain and range of each function.

11. $f(x) = 3x^2 - 2$
12. $g(x) = \ln(x + 2)$
13. $h(x) = e^{x+3} + 5$
14. $f(x) = -\ln(x - 4)$

____
NAME
SAMPLE TEST, worth 100 points, Chapters 1 and 2
Show all work that leads to your answers. Good luck!

1. (8 pts)

The following symbols are used for important sets of numbers. State the NAME of each set, and GIVE A PRECISE DESCRIPTION of the numbers in each set. The first one is done for you, as a sample.

(0 pt) $\mathbb{Q}$ is the set of *rational numbers.* These are numbers that can be written in the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

(2 pts) $\mathbb{R}$ ______________________________

(2 pts) $\mathbb{Z}$ ______________________________

(2 pts) $\mathbb{C}$ ______________________________

(1 pt) Give an example of an irrational real number. ________________

(1 pt) Give an example of a positive real number that is not an integer. __________

2. (12 pts)

Identify the following equations as TRUE, FALSE, or CONDITIONAL:

(1 pt) $\sqrt{(-3)^2} = -3$ ______________________________

(1 pt) $-4^2 = -16$ ______________________________

(1 pt) $x^3 = x \cdot x \cdot x$ ______________________________

(1 pt) $x = 3$ ______________________________

(2 pts) What does it mean to *solve an equation in* 1 *variable*? (Answer in English.)

(2 pts) How many variables are there in this equation? $3x + x^2 - 2 = \sqrt{x}$

(2 pts) Solve the equation $x^2 = 3$, taking the universal set to be the real numbers.

(1 pt) Solve the equation $x^2 = 3$, taking the universal set to be $\mathbb{Q}$.

(1 pt) What is the universal set assumed to be in this course, unless otherwise specified?

3. (14 pts) (2 pts each)

TRUE or FALSE. (Circle the correct response.)

T	F	For all real numbers a and b, $(a+b)^2 = a^2 + b^2$.
T	F	The number $\frac{3}{3 \cdot 2^{57} \cdot 5^{402}}$ has a finite decimal expansion.
T	F	$\{x \mid a < x < b\} = \{x \mid a < x \text{ and } x < b\}$
T	F	$0 \in (0,1]$
T	F	$\{0\} \subset [0,1]$
T	F	$x = 3 \iff x^2 = 9$
T	F	The symbol $\iff$ is used to compare numbers.

4.
(9 pts)

Consider the graph shown.

(3 pts) Is y a function of x? Why or why not?

(3 pts) Is x a function of y? Why or why not?

(3 pts) Sketch the graph of a function f satisfying the following properties: f is one-to-one, $\mathcal{D}(f) = [-1, 2)$, and $\mathcal{R}(f) = [0, 6]$.

5.
(12 pts)

In this question, you are asked to sketch several graphs. Put your graphs in the space provided below.

(3 pts) (a) Graph the equation $x = 5$, viewed as an equation in 1 variable.

(4 pts) (b) Graph the equation $x = 5$, viewed as an equation in 2 variables (x and y).

(5 pts) (c) Graph the equation $y - 3 = |x - 2|$.

6.
(15 pts)

The 'black box' shown corresponds to a function. Please answer the following questions:

(2 pt) What is the NAME of this function? ________________

(2 pt) What is the output of this function, when the input is y? ________________

(3 pts) Graph this function in the space provided below.

(2 pt) Is this a $1-1$ function? (YES or NO) ________________

(1 pt) What is $f(-2)$? (Write a complete sentence: $f(-2) = ???$) ________________

(1 pt) What is $f(x + h)$? ________________

(2 pt) What is $\mathcal{D}(f)$? ________________

(2 pts) What is $\mathcal{R}(f)$? ________________

7. (6 pts)

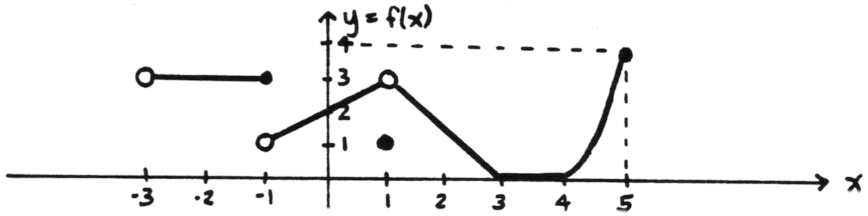
(1 pt) There are two equations that describe the relationship between a 1-1 function f and its inverse f^{-1}. One of these looks something like this: $f\big(f^{-1}(x)\big) = x \ \ \forall\, x \in$????. What must ???? be to get a true statement?

(2 pts) What is the second equation that describes the relationship between f and f^{-1}?

(3 pts) The function g given by $g(x) = 7x - 2$ is $1-1$. Find the inverse function g^{-1}.

8. (7 pts)

The following questions all refer to the function f whose graph is shown below.



Find the following, if they exist. Be sure to write complete mathematical sentences.

(2 pts) $f(1)$ ________________ $f(3.1)$ ________________

(1 pts) $\mathcal{D}(f)$

(1 pts) $\mathcal{R}(f)$

(2 pts) $\{x \mid f(x) = 0\}$

(1 pts) $\{x \mid f(x) = 4\}$

9. (3 pts)

Let f and g be functions.

(2 pts) Define, in the obvious way, a new function named $f + g$. Express the formula using the dummy variable t.

(1 pt) What is the domain of this new function $f + g$? Answer using a complete mathematical sentence.

10. (4 pts)

Solve the equation $x^3 - 3x + 2 = 0$. Be sure to write complete mathematical sentences. (HINT: Note that the number 1 is a root of $x^3 - 3x + 2$.)

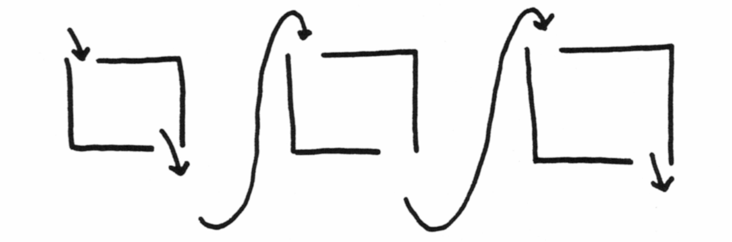
11. (3 pts)

(3 pts) Graph the function $f\colon (0,1)\cup(2,3)\to\mathbb{R}$, $f(x)=\sqrt{x}$.

12. (7 pts)

(3 pts) Let f and g be functions. What is the domain of the composite function $f\circ g$? (Write a complete mathematical sentence.)

(4 pts) Consider the function g defined by the rule $g(x)=(3x-1)^2$. This function can naturally be viewed as a composition of three functions. Tell me what each box below does.



This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 3

LIMITS AND CONTINUITY

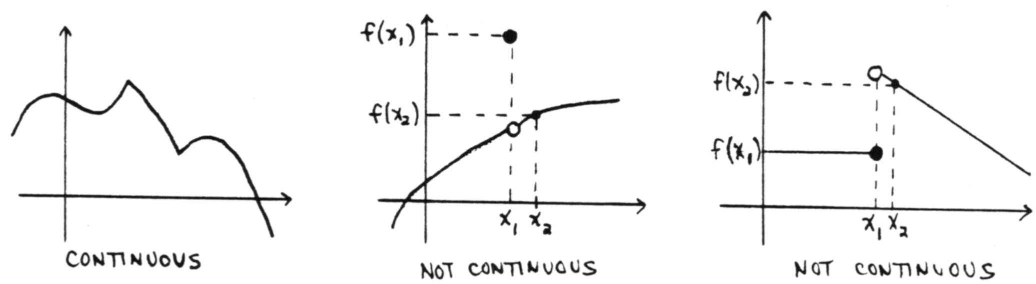
In this chapter, the study of calculus begins. The idea of a *limit*—getting arbitrarily close to something (without necessarily getting there!)—is fundamental to calculus. Without it, it would be impossible to speak precisely of continuity, differentiation, and integration.

This chapter, therefore, begins by studying limits. The definition of limit is then used to make precise the idea of a continuous function.

3.1 Limits—The Idea

continuity;
when inputs are 'close', corresponding outputs are 'close'

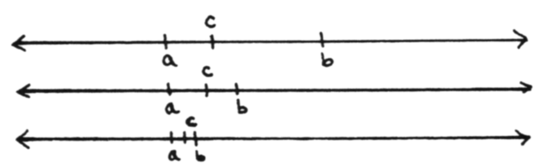
Certain functions have the property that when their inputs are close, so are their outputs. The mathematical idea that addresses this issue is *continuity.* Intuitively, a function is *continuous* if its graph can be traced without picking up your pencil; it can't have any 'breaks'. In other words, for a continuous function, when inputs are 'close together' the corresponding outputs should be 'close together'. This certainly doesn't happen in the second and third sketches below: in both cases, x_1 is 'close to' x_2, but $f(x_1)$ is not 'close to' $f(x_2)$.



What is meant by numbers being 'close'?

The idea of 'closeness' is not precise, at least in the English sense. Are the numbers 2 and 3 'close'? How about 2 and 2.01? How about 2 and 2.00001?

Just how 'close' can two different numbers be? The answer is really quite simple: *as close as you want.* The real numbers have a beautiful property: given any two real numbers a and b, if they're not equal, then there's another real number between them.



the mathematical tool that addresses the idea of 'numbers being close' is the **limit**

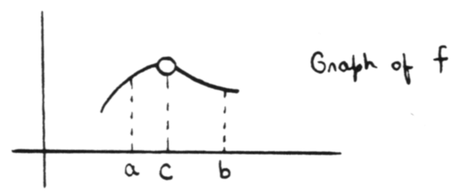
In order to discuss *continuity*, it is first necessary to have a mathematical tool that addresses, precisely, the notion of 'numbers being close'. The tool that accomplishes this is the mathematical *limit.* In our first—informal—discussion of limits, the word 'close' will be used in an intuitive sense. However, in the next section, you will see how this notion of 'closeness' is addressed precisely.

some initial assumptions about the functions we're working with

Suppose, for now, that any function f we're working with has the following property: it is defined near a number c, but not necessarily at c. We want to guarantee that there are inputs arbitrarily close to c (on both sides) that f knows how to act on. This requirement can be phrased more precisely—there must be numbers a and b such that:

$$(a,c) \cup (c,b) \subset \mathcal{D}(f)$$

This requirement will be weakened when things are made precise in the next section.



the mathematical sentence, $\lim\limits_{x\to c} f(x) = l$

The mathematical sentence

$$\lim_{x\to c} f(x) = l \qquad (*)$$

is read as:

The limit of $f(x)$, as x approaches c, is equal to l.

This sentence involves a function f, a constant c, and a constant l. The 'x' that appears twice (once in '$x \to c$', and once in '$f(x)$') is a dummy variable; it could equally well be called 't' or 'y' or 'ω'. As you'll see momentarily, x represents a number that is getting closer and closer to c.

The sentence can be true or false. We will be primarily interested in cases when it is true.

When is the sentence $\lim\limits_{x\to c} f(x) = l$ true?

In order for the mathematical sentence (*) to be true, the following two conditions must be satisfied:

- as x gets close to the number c coming in from the right-hand side, the corresponding function values $f(x)$ must get close to l; and
- as x gets close to c from the left-hand side, the corresponding function values $f(x)$ must also get close to l.

Thus, in order for the sentence $\lim_{x\to c} f(x) = l$ to be true, the following condition must be satisfied: when x is close to c, $f(x)$ must be close to l.

$\lim_{x\to c} f(x) = l$, text style

If the sentence $\lim\limits_{x\to c} f(x) = l$ is typed in text (instead of displayed), it requires extra space between the lines, to make room for the '$x \to c$'. Lots of people think that this extra space doesn't look very good. Therefore, the sentence is usually typeset differently in text, like this: $\lim_{x\to c} f(x) = l$. The phrase $x \to c$ is moved over, merely to prevent the excess space between lines.

Evaluate the limit $\lim_{x\to c} f(x)$

You will frequently be asked to:

$$\textit{Evaluate the limit } \lim_{x\to c} f(x)\,.$$

This means:

$$\textit{Find a number } l \textit{ so that the sentence } \lim_{x\to c} f(x) = l \textit{ is TRUE.}$$

(It will be shown that if there is such a number l, then it is unique.) If no such number l exists, then we say that:

$$\textit{The limit } \lim_{x\to c} f(x) \textit{ does not exist.}$$

EXAMPLE

finding the limit of a function

Let $f(x) = 2x$. Then:

$$\lim_{x\to 2} f(x) = 4$$

We could have instead said:

$$\textit{The mathematical sentence } \lim_{x\to 2} f(x) = 4 \textit{ is true.}$$

However, mathematicians usually have no need to say things that are false (except, perhaps, in a book on logic). Therefore, when a mathematical sentence is stated, it is assumed to be true. That is, when a mathematician states:

$$\lim_{x\to 2} f(x) = 4$$

this *means* that the sentence is true.

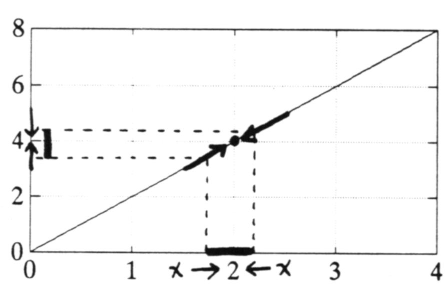
Now, why is it that this sentence is true? It is because as x approaches 2 from *either* side, the function values are getting close to 4.

For example, look at the table below. When x is 1.99, $f(x)$ is 3.98. That is, $f(1.99) = 3.98$. Also, when x is 2.001, $f(x)$ is 4.002. That is, $f(2.001) = 4.002$. These are examples of the fact that when x is close to 2, $f(x)$ is close to 4.

Now look at the graph of f, also given below. This graph clearly shows that when the inputs are close to 2, the corresponding function values are close to 4.

Note in this case that f is defined at 2, and $f(2) = 2 \cdot 2 = 4$. As x approaches 2, the corresponding function values $f(x)$ get close to $f(2)$. That is:

$$\lim_{x\to 2} f(x) = f(2)$$



GRAPH OF f, $f(x) = 2x$

x	$f(x)$	x	$f(x)$
1.9500	3.9000	1.9950	3.9900
1.9600	3.9200	1.9960	3.9920
1.9700	3.9400	1.9970	3.9940
1.9800	3.9600	1.9980	3.9960
1.9900	3.9800	1.9990	3.9980
2.0100	4.0200	2.0010	4.0020
2.0200	4.0400	2.0020	4.0040
2.0300	4.0600	2.0030	4.0060
2.0400	4.0800	2.0040	4.0080
2.0500	4.1000	2.0050	4.0100

As x approaches 2, $f(x)$ approaches 4

EXERCISE 1

♣ 1. Let $f(x) = 2x$. Evaluate each of the following limits. Be sure to write complete mathematical sentences. (That is, if asked to evaluate $\lim_{x\to 2} f(x)$, don't just say 4. Instead, write the complete sentence: $\lim_{x\to 2} f(x) = 4$.)

$$\lim_{x\to 3} f(x) \qquad \lim_{x\to 0} f(x) \qquad \lim_{x\to \pi} f(x) \qquad \lim_{x\to 2/3} f(x) \qquad \lim_{x\to -10.1} f(x)$$

♣ 2. Let c denote a particular real number, and let $f(x) = 2x$. What is

$$\lim_{x\to c} f(x)\ ?$$

EXAMPLE

finding a limit, f is not defined at c

Now, let $f(x) = 2x\frac{(x-2)}{(x-2)}$. In this case, f is *not defined* at $x = 2$; the graph in the previous example has been punctured.

Again:

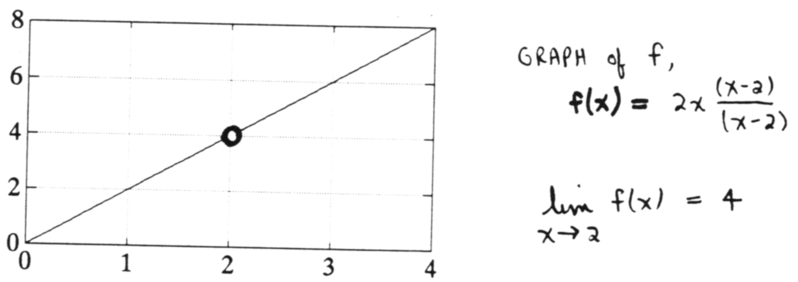
$$\lim_{x\to 2} f(x) = 4$$

This is because the two required conditions are satisfied: as x approaches 2 from the right AND the left, the corresponding function values are getting close to the number 4. That is, when x is close to 2 (but not equal to 2), the corresponding function values are close to 4.

In order to talk about the limit of a function f as x approaches c, the function f need NOT be defined at c. It need only be defined *near* c.

Here are some additional true limit statements about this function:

$$\lim_{x\to 3} f(x) = 2(3) = 6 \qquad \lim_{x\to \pi} f(x) = 2\pi \qquad \lim_{x\to 0} f(x) = 2(0) = 0$$



EXERCISE 2

♣ Evaluate the following limits. In each case, a quick sketch of the function may be helpful. Be sure to write complete mathematical sentences.

$$1.\ \lim_{x\to 3} 2x\frac{(x-3)}{(x-3)} \qquad 2.\ \lim_{x\to 2} 2x\frac{(x-3)}{(x-3)} \qquad 3.\ \lim_{x\to 1} x^2\frac{(x-1)}{(x-1)}$$

$$4.\ \lim_{x\to 0} x^2\frac{(x-1)}{(x-1)} \qquad 5.\ \lim_{x\to 3/2} \sqrt{x}\frac{(2x-3)}{(2x-3)}$$

EXAMPLE

finding a limit,
f is defined at c,
but in a strange way

Now let:

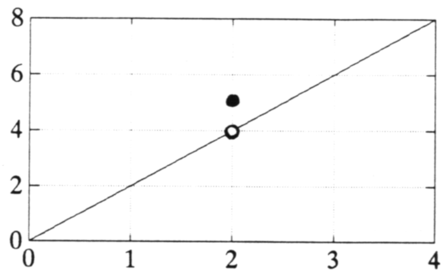
$$f(x) = \begin{cases} 2x & \text{for } x \neq 2 \\ 5 & \text{for } x = 2 \end{cases}$$

The graph of f is shown below. Again:

$$\lim_{x\to 2} f(x) = 4$$

This is because when x is close to 2 (but not equal to 2), the corresponding function values are close to 4.

When evaluating a limit as x approaches c, *x is not allowed to equal c*; the x values merely get *arbitrarily close to c*.



GRAPH OF f,

$$f(x) = \begin{cases} 2x & \text{for } x \neq 2 \\ 5 & \text{for } x = 2 \end{cases}$$

$$\lim_{x\to 2} f(x) = 4$$

EXERCISE 3

♣ 1. Sketch the graph of a function that satisfies the following conditions:
- $\lim_{x\to 3} f(x) = 4$
- $f(3) = 2$

♣ 2. Sketch the graph of a function that satisfies the following conditions:
- $\lim_{x\to 0} f(x) = 1$
- $0 \notin \mathcal{D}(f)$

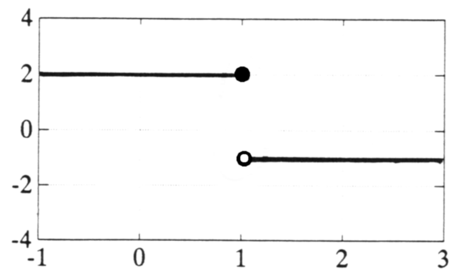
EXAMPLE

a limit that does not exist

Let:

$$f(x) = \begin{cases} 2 & x \le 1 \\ -1 & x > 1 \end{cases}$$

In this case, $\lim_{x\to 1} f(x)$ does not exist. The two required conditions cannot possibly be met for *any* real number l. As x gets close to 1 from the left-hand side, the function values are all equal to 2. As x gets close to 1 from the right-hand side, the function values are all -1. Thus, we are *not* getting close to *the same number* from both sides.



GRAPH OF f

$\lim_{x\to 1} f(x)$ DOES NOT EXIST

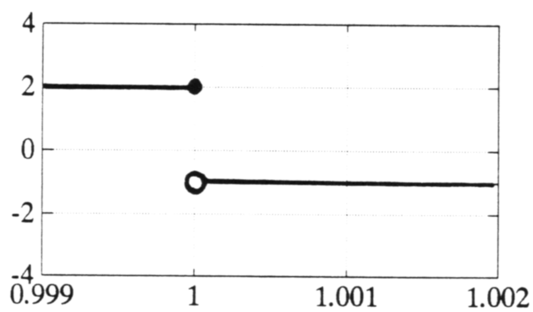
EXAMPLE

Let's work with the same function as in the previous example, but now consider some values of c different from 1.

What is $\lim_{x\to 2} f(x)$? As x approaches 2 from the right and left sides, $f(x)$ is -1. Thus, $\lim_{x\to 2} f(x) = -1$.

Similarly, $\lim_{x\to 0} f(x) = 2$.

What about $\lim_{x\to 1.001} f(x)$? When x is (sufficiently) close to 1.001, what (if anything) are the corresponding outputs close to? To answer this question, we need only 'magnify' what's happening for values of x near 1.001, as in the graph below. Now, it's clear that $\lim_{x\to 1.001} f(x) = -1$.



A MAGNIFICATION OF WHAT'S HAPPENING NEAR $x = 1.001$

$\lim_{x\to 1.001} f(x) = -1$

EXERCISE 4

Consider the function g given by:

$$g(x) = \begin{cases} -3 & \text{for } x < 2 \\ 5 & \text{for } x \geq 2 \end{cases}$$

♣ 1. Sketch the graph of g.

♣ 2. Find the following numbers, if they exist. Be sure to write complete mathematical sentences.

a) $g(2)$

b) $g(1.9782)$

c) $g(\pi)$

d) $\lim_{x\to 2} g(x)$

e) $\lim_{x\to 3} g(x)$

f) $\lim_{x\to 1.99999} g(x)$

g) $\lim_{z\to 0} g(z)$ (Hint: z is a dummy variable.)

h) $\lim_{y\to \pi} g(y)$

♣ 3. Let c be any number greater than 2. What is $\lim_{x\to c} g(x)$?

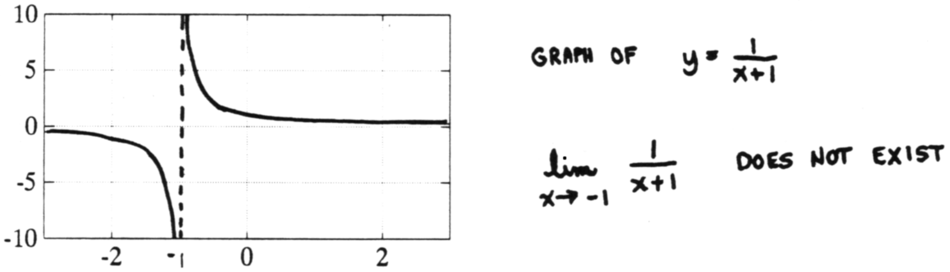
EXAMPLE

a limit that does not exist

The limit

$$\lim_{x\to -1} \frac{1}{x+1}$$

does not exist. As x approaches -1 from the left-hand side, $\frac{1}{x+1}$ approaches negative infinity. As x approaches -1 from the right-hand side, $\frac{1}{x+1}$ approaches positive infinity. The function values are not approaching any fixed real number.



The following sentences are all true:

$$\lim_{x\to 0} \frac{1}{x+1} = 1 \qquad \lim_{x\to 2} \frac{1}{x+1} = \frac{1}{3} \qquad \lim_{x\to -2} \frac{1}{x+1} = \frac{1}{-2+1} = -1$$

$$\lim_{y\to \pi} \frac{1}{y+1} = \frac{1}{\pi+1}$$

Also, for $c \neq -1$:

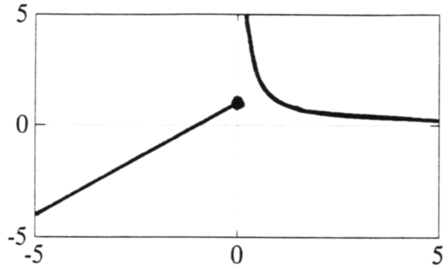
$$\lim_{x\to c} \frac{1}{x+1} = \frac{1}{c+1}$$

EXAMPLE

Let h be defined by:

$$h(x) = \begin{cases} \frac{1}{x} & \text{for } x > 0 \\ x+1 & \text{for } x \le 0 \end{cases}$$

The graph of h is shown below.



GRAPH OF h

$\lim_{x \to 0} h(x)$ DOES NOT EXIST

As x approaches 0 from the left-hand side, $h(x)$ approaches 1. However, as x approaches 0 from the right-hand side, $f(x)$ does not approach 1. Thus, $\lim_{x\to 0} h(x)$ does not exist.

The following sentences are all true:

$$\lim_{x\to 2} h(x) = 1/2 \qquad \lim_{t\to -1} h(t) = 0 \qquad \lim_{x\to 10^{-5}} h(x) = 10^5$$

$$\lim_{x\to -10^{-5}} h(x) = 1 - 0.00001 = 0.99999$$

EXERCISE 5

Let f be defined by:

$$f(x) = \begin{cases} \frac{1}{x-2} & \text{for } x > 2 \\ 1 - x^2 & \text{for } x \le 2 \end{cases}$$

♣ 1. Sketch the graph of f.

♣ 2. What is the domain of f?

♣ 3. Find the following numbers, if they exist. Be sure to write complete mathematical sentences.

a) $f(2)$

b) $\lim_{x\to 2} f(x)$

c) $f(c)$, for $c > 100$

d) $f(t)$, for negative t

e) $\lim_{t\to \pi+1} f(t)$

f) $\lim_{\omega\to 0} f(\omega)$

Be sure you understand the difference between the *expression*

$$\lim_{x \to c} f(x) \qquad (\dagger)$$

and the *sentence*:

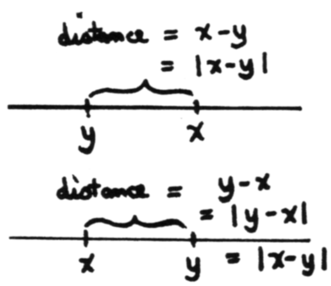
$$\lim_{x \to c} f(x) = l \qquad (\ddagger)$$

When (†) is defined, it is a NUMBER. What number? The number that $f(x)$ gets close to, as x gets close to c.

However, (‡) is a SENTENCE. Sentences have verbs; the verb in (‡) is the equals sign. This sentence (when it's true) is telling us that the number $\lim_{x \to c} f(x)$ is equal to l.

distance between real numbers

The use of the absolute value as a tool for measuring the distance between numbers is discussed next. This will help in understanding the precise definition of limit, which is the topic of the next section.



Let x and y be any two real numbers. Then:

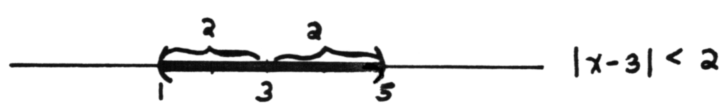
$$\text{the distance from } x \text{ to } y = |x - y|$$

Let's think about why this is true. If $x = y$, then the distance between them is 0, and the formula works.

If $x \neq y$, then one of the numbers lies further to the right on the number line. If x lies further to the right, the distance between the numbers is $x - y$. If y lies further to the right, the distance between the numbers is $y - x$. But in both cases, $|x - y|$ (which is equal to $|y - x|$) gives the distance between the two numbers.

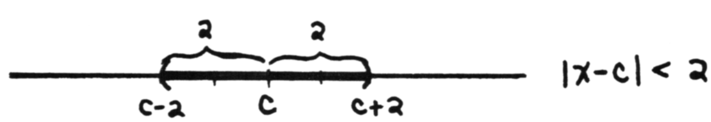
analyze the sentence $|x - 3| < 2$

Think about the sentence $|x - 3| < 2$. This sentence is an *inequality*; the verb is '$<$'. When is this sentence true? Using the interpretation of $|x - 3|$ as the distance between x and 3, the answer is easy: it is true for all numbers x whose distance from 3 is less than 2. Thus, it is true for $x \in (1, 5)$.



analyze the sentence $|x - c| < 2$

Now consider the sentence $|x - c| < 2$. By mathematical conventions, x is the variable, and c is a constant. When is this sentence true? Whenever x is a number whose distance from c is less than 2. Thus, the solution set of $|x - c| < 2$ is $(c - 2, c + 2)$.



analyze the sentence $0 < |x-c|$

For what values of x is $0 < |x-c|$ true? Reading from right to left, we must have the distance from x to c greater than 0. This happens as long as x is not equal to c; so the solution set of $0 < |x-c|$ is $(-\infty, c) \cup (c, \infty)$.

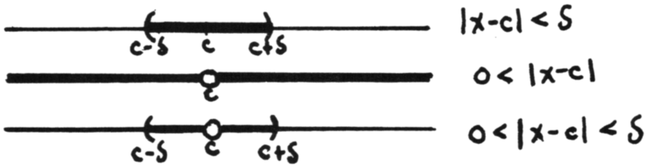
analyze the sentence $0 < |x-c| < \delta$

Now consider the sentence $0 < |x-c| < \delta$. By mathematical conventions, x is the variable, c and δ are constants. Usually, δ is thought of as a small positive number.

When is the sentence $0 < |x-c| < \delta$ true? Remember that this is short for *two* sentences, connected by the mathematical word 'and'. That is:

$$0 < |x-c| < \delta \iff 0 < |x-c| \text{ and } |x-c| < \delta$$

Thus, in order for the sentence to be true, we must have the distance from x to c less than δ, AND, x is not allowed to equal c. The solution set of $0 < |x-c| < \delta$ is shown below.



EXERCISE 6

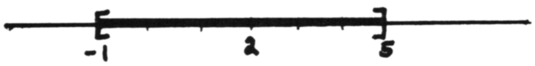
♣ 1. Write a mathematical sentence that is TRUE for all numbers whose distance from 4 is less than 2. What is the variable in your sentence?

♣ 2. Write a mathematical sentence that is TRUE for all numbers whose distance from -1 is greater than 5. What is the variable in your sentence?

♣ 3. Write a mathematical sentence that is TRUE for all numbers whose distance from π is greater than or equal to δ. (Here, δ is a constant.) What is the variable in your sentence?

EXERCISE 7

♣ 1. Write a mathematical sentence whose solution set is the set shown below.



♣ 2. Write a mathematical sentence whose solution set is the set shown below.

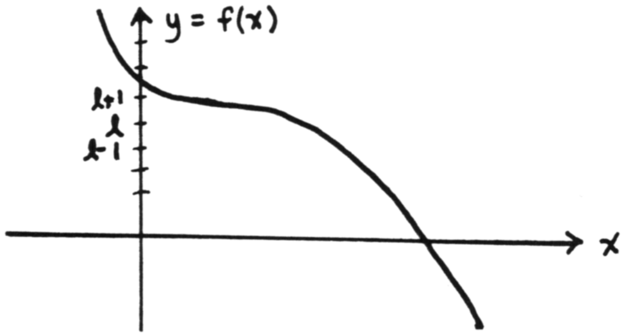
♣ 3. On a number line, show the solution set of $0 < |x-3| < 5$.

♣ 4. On a number line, show the solution set of $0 < |x+1| \le 2$. It may be helpful to rewrite $x+1$ as $x-(-1)$.

EXERCISE 8

♣ The graph of a function f is shown below. A specific number l is labeled on the y-axis.

On the x-axis, clearly show $\{x \ : \ |f(x) - l| < 2\}$. Here, the colon ':' is used instead of the vertical bar, '|', so there is less confusion with the adjacent absolute value symbol. The colon ':' is still read as 'such that' or 'with the property that'.



alternate notation for limits

Suppose that the sentence

$$\lim_{x \to c} f(x) = l$$

is true. Then, as x approaches c, the numbers $f(x)$ must approach l. One often writes this as:

$$\text{As } x \to c, \ \ f(x) \to l \ .$$

This is read as: *As x approaches c, $f(x)$ approaches l.* Thus, the arrow '$\to$' is read as 'approaches'.

QUICK QUIZ

sample questions

1 Evaluate the limit $\lim_{x \to -2} x^3$, if it exists.

2 Sketch the graph of $f(x) = x^2 \frac{x+1}{x+1}$. Then, evaluate the limit $\lim_{x \to -1} f(x)$, if it exists.

3 Sketch the graph of:

$$f(x) = \begin{cases} 3x & \text{for } \ x \neq 1 \\ 5 & \text{for } \ x = 1 \end{cases}$$

Then, evaluate the limit $\lim_{x \to 1} f(x)$, if it exists.

4 Sketch the graph of a function that satisfies the following conditions: $\lim_{x \to 2} f(x) = 5$, $f(2) = 1$.

5 Write a mathematical sentence that is TRUE for all numbers whose distance from -1 is less than or equal to 4. Use the variable t in your sentence.

KEYWORDS

for this section

Be familiar with the mathematical sentence:

$$\lim_{x \to c} f(x) = l$$

Roughly, when is this sentence true? Know that c and l are constants, and x is a dummy variable. Be able to evaluate simple limit statements. Know the difference between the expression $\lim_{x \to c} f(x)$ *and the sentence* $\lim_{x \to c} f(x) = l$. *Know that the distance between real numbers x and y is given by* $|x - y|$.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. $\lim_{x\to 1} 3x$
2. $\lim_{x\to 1} 3x = 3$
3. $\lim_{t\to 0} t^2 = 0$
4. $\lim_{t\to 0} t^2$
5. $\lim_{x\to c} f(x) = l$
6. As $x \to 1$, $2x \to 2$.
7. As $t \to 0$, $2t + 1 \to 1$.
8. $\lim_{x\to 2} f(x) = f(2)$
9. $\lim_{x\to 1} g(x) = g(1)$
10. $\lim_{x\to 0} \frac{x^2+x}{x} = 1$
11. $|x - y|$
12. $|x - 1| \le 2$
13. $|x - y| = |y - x|$
14. $|-2x - 2y| = 2|x + y|$
15. $|ab| = |a| \cdot |b|$
16. $|a + b| = |a| + |b|$
17. $|a - b| = |a| - |b|$
18. $|x| > 0 \iff x \in (-\infty, 0) \cup (0, \infty)$
19. For $\epsilon > 0$, $0 < |x| < \epsilon \iff x \in (-\epsilon, 0) \cup (0, \epsilon)$
20. For $0 < a < b$, $a < |x| < b \iff x \in (-b, -a)$ or $x \in (a, b)$

3.2 Limits—Making It Precise

a more precise way to view the sentence $\lim\limits_{x\to c} f(x) = l$

In the previous section, we said that when the sentence $\lim_{x\to c} f(x) = l$ is true, this means, roughly, that when x is close to c, then $f(x)$ is close to l.

Here's a more precise way to view this limit:

When the sentence $\lim_{x\to c} f(x) = l$ is true, then we can get the function values $f(x)$ *as close to l as desired*, merely by requiring that x be sufficiently close to c.

This idea is explored further in the next 'challenge'.

a friendly challenge

Suppose you are having a 'friendly' argument with a classmate. You have stated:

$$\lim_{x\to 2} 3x = 6$$

Your friend says: Prove it to me! Here's the resulting conversation:

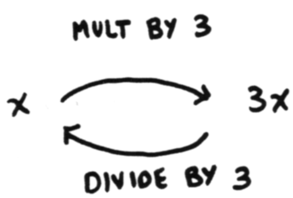
You: Okay, I will. I claim that I can get $3x$ *as close to* 6 as you want, just by requiring that x be close enough to 2. How close would you like to get $3x$ to 6?

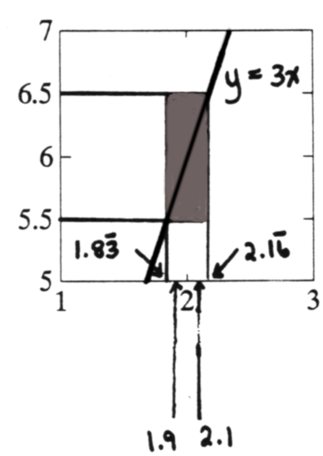
Friend: Within 0.5.

You: No problem. (Thinking out loud…) For the function $f(x) = 3x$, when x is the input, $3x$ is the output. So, to go from input to output, we multiply by 3. To go from output to input, divide by 3.

Now, you want me to get the output $3x$ within 0.5 of 6. That is, you want $3x$ to be in the interval $(6-0.5, 6+0.5) = (5.5, 6.5)$. The output 5.5 corresponds to the input $\frac{5.5}{3} = 1.8\overline{3}$; the output 6.5 corresponds to the input $\frac{6.5}{3} = 2.1\overline{6}$. As long as I keep x within the interval $(1.8\overline{3}, 2.1\overline{6})$, then $3x$ will be within the requested interval. We don't even need to 'cut things so close'. As long as x is within, say, 0.1 of 2, then $3x$ will be well within 0.5 of 6.

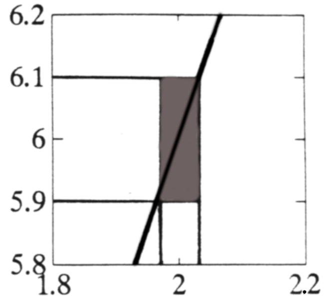
(Look at the sketch below.)





Friend: Well, now I want $3x$ no further than 0.1 from 6.

You: So, you want $3x$ to be in the interval $(5.9, 6.1)$. Well, $\frac{5.9}{3} \approx 1.97$, and $\frac{6.1}{3} \approx 2.03$. To be safe, let's just keep x within the interval $(1.98, 2.02)$. That is, as long as x is within 0.02 of 2, then $3x$ will be well within 0.1 of 6.

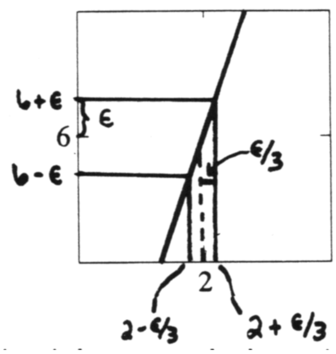


Friend: Well—now I want $3x$ within 0.0001 of 6!

You: (Calculates.) Just keep x within, say, 0.00003 of 2.

Friend: I'm thinking of a *really small number*; call it ϵ. I want $f(x)$ within ϵ of l.

You: (Draws the sketch below for Friend.) Just keep x within $\frac{\epsilon}{3}$ of 2.



Keep this 'challenge' in mind as you study the precise definition of the limit statement.

DEFINITION
the limit of a function
$\lim_{x\to c} f(x) = l$

The sentence '$\lim_{x\to c} f(x) = l$' is defined by:

$$\lim_{x\to c} f(x) = l \iff \text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - l| < \epsilon.$$

In this definition, two sentences are being compared with the 'is equivalent to' symbol, $\iff$. The sentence on the left is being given meaning by the sentence on the right. These two sentences always have precisely the same truth values, and hence can be used interchangeably. More precisely, the sentence

$$\lim_{x \to c} f(x) = l \tag{*}$$

on the left is being *defined* by the sentence

$$\begin{array}{l}\text{For every } \epsilon > 0\text{, there exists } \delta > 0\text{, such that if} \\ 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f)\text{, then } |f(x) - l| < \epsilon.\end{array} \tag{**}$$

on the right. This is how we determine the truth value of the sentence (*). If (**) is true, then so is (*). If (**) is false, then so is (*).

Next, we must carefully investigate (**), to see when it is true.

How close do you want $f(x)$ to be to l?

(within ϵ)

The sentence (**) begins with:

$$\text{For every } \epsilon > 0 \quad \cdots$$

Think of ϵ as being a small positive number, that says *how close you want the function values $f(x)$ to be to l.* In order for the sentence (**) to be true, the remainder of this sentence is going to have to be true for *every positive number* ϵ. In particular, it's going to have to be true when ϵ is arbitrarily close to zero; like $\epsilon = 10^{-2000}$.

The remainder of the sentence (**) addresses the question: How close do we need to get x to c in order to get $f(x)$ within ϵ of l?

How close must x be to c to accomplish this?

(within δ)

The sentence continues:

$$\cdots \quad \text{there exists } \delta > 0 \quad \cdots$$

Think of δ as a small positive number that says *how close we must get the x values to c, to ensure that the corresponding outputs fall within ϵ of l.* For most functions, the smaller ϵ is, the smaller δ is going to have to be.

check that δ really works

The sentence continues:

$$\cdots \quad \text{such that if } 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f) \quad \cdots$$

Up to this point, we have a 'challenge'; we want to get the function values within ϵ of l. We also have a 'proposed solution'; just keep the x values within δ of c. Now, we're going to show that this δ really works.

So we're saying: suppose x is a number that makes '$0 < |x-c| < \delta$ and $x \in \mathcal{D}(f)$' true. What x will make both of these sentences true? Well, $0 < |x - c| < \delta$ says that x must lie within δ of c, but not equal c. Remember that we don't ever let x equal c when evaluating a limit—x just gets arbitrarily close to c. Also, in order to talk about $f(x)$, we must certainly have x in the domain of f. Hence the sentence $x \in \mathcal{D}(f)$.

The sentence finishes:

$$\cdots \quad \text{then } |f(x) - l| < \epsilon.$$

As long as x is in the domain of f, and sufficiently close to c (but not equal to c), then the sentence $|f(x) - l| < \epsilon$ will be true. When is $|f(x) - l| < \epsilon$ true? Exactly when the distance between $f(x)$ and l is less than ϵ.

Rephrasing, in English

The mathematical sentence

$$\text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - l| < \epsilon. \qquad (**)$$

can be stated in English, as follows:

For every number epsilon greater than zero, there exists a number delta greater than zero, with the property that if x is within delta of c, but not equal to c, and if x is in the domain of f, then the distance between $f(x)$ and l is less than epsilon.

You must know the precise definition of the sentence:

$$\lim_{x \to c} f(x) = l$$

Also, you must be able to explain, in English, what this sentence means.

★

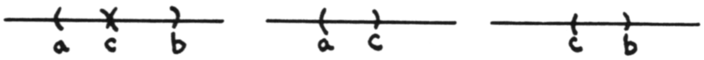
f must be defined on some interval near c

To avoid complications, we will only consider limits $\lim_{x\to c} f(x)$ in situations where f is defined on some interval near c; this interval may or may not include c.

More precisely, f must be defined at least on an interval of the form:

$$(a, c) \cup (c, b) \quad \text{or} \quad (a, c) \quad \text{or} \quad (c, b)$$

where $a < c$ and $c < b$.



The numbers a and b may, however, be arbitrarily close to c.

limits are fundamental to calculus

The concept of the limit of a function is fundamental to calculus. You will see limits again when we talk about continuous functions; when we talk about differentiating; when we talk about integrating. To truly *understand* calculus, you must *understand* limits.

The following examples and exercises should help in the learning process.

EXAMPLE

using the precise definition of the limit of a function

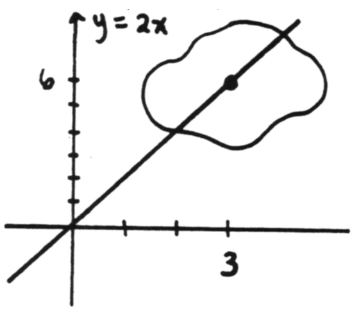
Problem: Use the precise definition of the limit of a function, to argue that:

$$\lim_{x \to 3} 2x = 6$$

Solution:

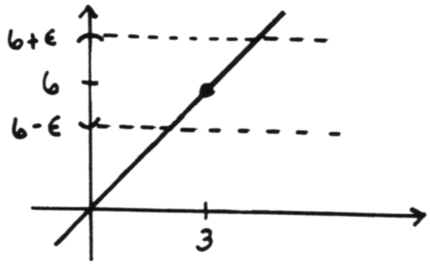
Step 1; state what you need to show, and sketch the function near c

Step 1. (State what you need to show, and sketch the function near c.) It must be shown that we can get $2x$ as close to 6 as desired (within ϵ), by requiring that x be sufficiently close to 3 (within δ).



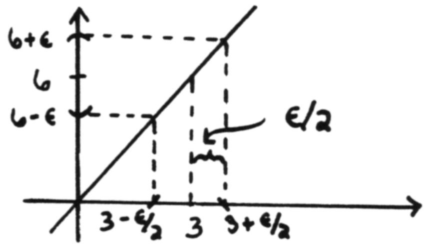
Step 2; set up an ϵ-interval about l

Step 2. (Set up an ϵ-interval about l.) Let $\epsilon > 0$. We want to get $2x$ within ϵ of 6; that is, within the interval $(6-\epsilon, 6+\epsilon)$. Show this interval on your sketch.



Step 3; 'pull back' to an appropriate interval about c

Step 3: ('Pull back' to an appropriate interval about c.) With the function $f(x) = 2x$, to go from an input x to an output $2x$, we multiply by 2; and to go from an output $2x$ to an input x, we divide by 2.

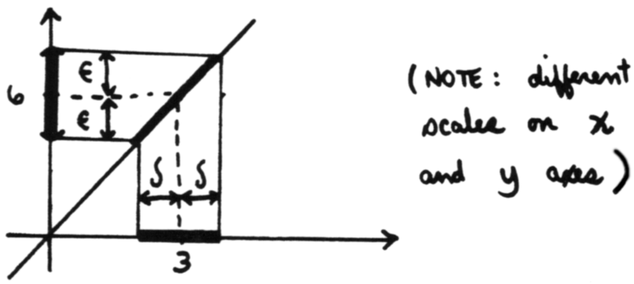


So, when the output is $6+\epsilon$, the corresponding input is $\frac{6+\epsilon}{2} = 3 + \frac{\epsilon}{2}$. And, when the output is $6-\epsilon$, the input is $\frac{6-\epsilon}{2} = 3 - \frac{\epsilon}{2}$.

Alternately, solving the equation $y = 2x$ for x yields $x = y/2$. When $y = 6+\epsilon$, we have $x = \frac{6+\epsilon}{2} = 3 + \frac{\epsilon}{2}$; and when $y = 6-\epsilon$, we have $x = \frac{6-\epsilon}{2} = 3 - \frac{\epsilon}{2}$.

Step 4; summarize results, by stating the 'δ that works'

Step 4. (Summarize your results, by stating the 'δ that works'.) Thus, if we take δ to be $\epsilon/2$, then whenever x is within δ of 3, we will have $2x$ within ϵ of 6.



the 4-step process for investigating limits

You must be able to investigate limits in the manner discussed in the previous example. Always follow the basic 4-step process that leads to a 'δ that works', and always make a sketch that summarizes what you are doing.

EXERCISE 1
elaborating on the previous example

♣ 1. In the preceding example, could δ have been taken to be a positive number *less than* $\frac{\epsilon}{2}$? Why or why not?

♣ 2. Is $\frac{\epsilon}{2} - .01$ necessarily a positive number less than $\frac{\epsilon}{2}$? Why or why not?

♣ 3. Is $\frac{\epsilon}{3}$ necessarily a positive number less than $\frac{\epsilon}{2}$? Why or why not?

♣ 4. Write down two positive numbers that are (always) less than $\frac{\epsilon}{2}$.

EXERCISE 2
using the 4-step process to investigate a limit

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 2} 4x = 8$$

DO NOT just 'copy' the preceding example. Close your book, and try to write down the argument yourself. If you get stuck, then re-read the previous example, and see where you went wrong. But then close your book again.

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 1} 2x + 3 = 5$$

EXERCISE 3
the limit of a constant function

Let f be the constant function defined by $f(x) = 5$.

♣ 1. Describe, in words, what the function f does.

♣ 2. Draw a 'black box' that describes f.

♣ 3. Does $\lim_{x \to 2} f(x)$ exist? Why or why not?

♣ 4. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 2} f(x) = 5$$

♣ 5. What did you choose for δ? Are there other 'natural' choices for δ?

EXAMPLE
evaluating a more general limit

Problem: Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 2} x^3 = 8$$

Step 1

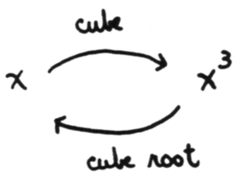
Step 1. It must be shown that we can get x^3 as close to 8 as desired (within ϵ), by requiring that x be sufficiently close to 2 (within δ).

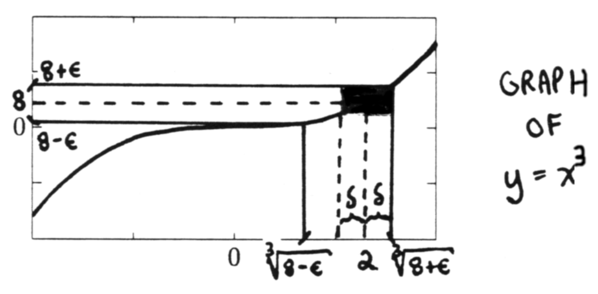
Step 2

Step 2. Let $\epsilon > 0$. We want to get x^3 within ϵ of 8; that is, in the interval $(8 - \epsilon, 8 + \epsilon)$.

Step 3

Step 3. Refer to the 'mapping diagram' below. When the output is $8 + \epsilon$, the corresponding input is $\sqrt[3]{8 + \epsilon}$; when the output is $8 - \epsilon$, the input is $\sqrt[3]{8 - \epsilon}$.





Step 4

Step 4. Now, which is the shorter distance: from 2 to $\sqrt[3]{8+\epsilon}$ or from 2 to $\sqrt[3]{8-\epsilon}$? Since the curve $y = x^3$ rises more steeply as x gets larger, the shorter distance is from 2 to $\sqrt[3]{8+\epsilon}$. (This fact will be proven later on in the course.) Thus, take the shorter distance $(\sqrt[3]{8+\epsilon}-2)$ to be δ.

Then, as long as x is within δ of 2, we will have x^3 within ϵ of 8.

EXERCISE 4

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 3} x^3 = 27$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to c} x^3 = c^3$$

where c is any positive real number.

EXERCISE 5

♣ 1. Let $c < 0$. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to c} x^3 = c^3$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} x^2 = 4$$

One of the beautiful things about the precise definition, is that now we don't have to worry about whether or not the function is defined on 'both sides' of c; the definition takes care of this for us, by requiring that x must be in the *domain* of the function! This is illustrated in the next example.

EXAMPLE
investigating a limit, when f is only defined on one side of c

Problem: Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 0} \sqrt{x} + 3 = 3$$

Step 1

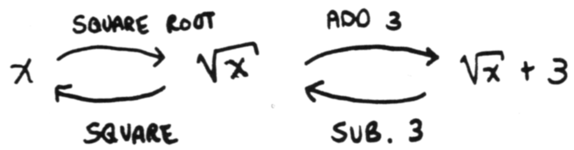
Step 1. Define $f(x) := \sqrt{x}+3$. It must be shown that we can get $\sqrt{x}+3$ as close to 3 as desired (within ϵ), merely by requiring that x be *in the domain of* f, and sufficiently close to 0 (within δ).

Step 2

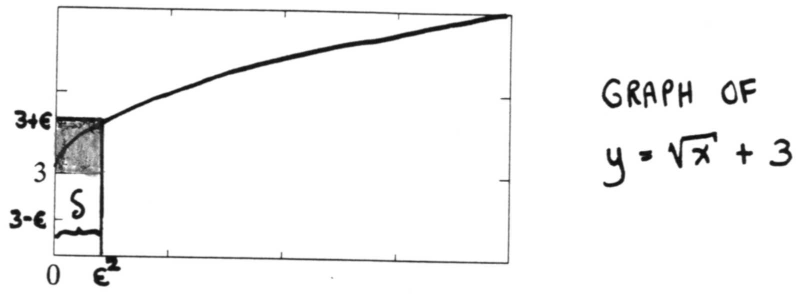
Step 2. Let $\epsilon > 0$. We must get $\sqrt{x}+3$ within ϵ of 3; that is, within the interval $(3-\epsilon, 3+\epsilon)$.

Step 3

Step 3. Refer to the 'mapping diagram' below:



When the output is $3+\epsilon$, the corresponding input is $((3+\epsilon)-3)^2 = \epsilon^2$. There is no input corresponding to the output $3-\epsilon$.



Step 4

Step 4. Referring to the sketch, we see that whenever x is within ϵ^2 of 0, and is within the domain of f, then $f(x)$ will be within ϵ of 3.

So, take $\delta = \epsilon^2$.

EXERCISE 6

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 0} \sqrt{x} + 2 = 2$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} f(x) = 4$$

where $f\colon [2,\infty) \to \mathbb{R}$ is defined by $f(x) = x^2$.

EXAMPLE

investigating a limit, when f is defined 'in a strange way' at c

The precise definition also 'covers' the situation when the function is defined in a 'strange way' at c. For example, consider the function f given by:

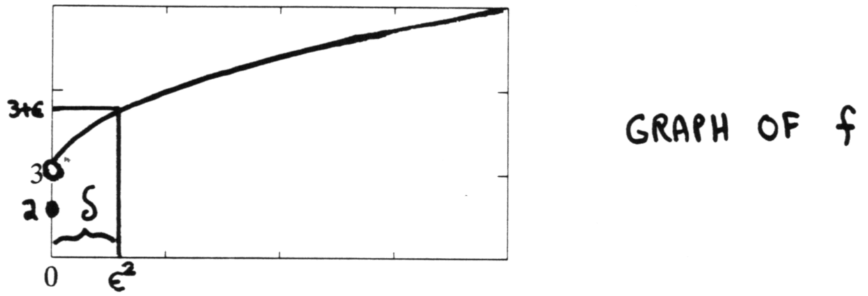
$$f(x) = \begin{cases} \sqrt{x} + 3 & x > 0 \\ 2 & x = 0 \end{cases}$$

Again, $\lim_{x\to 0} f(x) = 3$. We already found the 'δ that works' in the previous example. Then, as long as:

- x is within δ of 0
- x is in the domain of f
- x is *not equal to* 0

then $f(x)$ will be within ϵ of 3.

In mathematical language: if $0 < |x| < \delta$ and $x \in \mathcal{D}(f)$, then $|f(x) - 3| < \epsilon$.



EXERCISE 7

♣ 1. Which part of the sentence

$$0 < |x| < \delta$$

says that x must be within δ of 0?

♣ 2. Which part of the sentence says that x must not equal 0?

EXERCISE 8

Let $f\colon [1,\infty) \to \mathbb{R}$ be defined by the rule:

$$f(x) = \begin{cases} 3x-5 & \text{for } x > 1 \\ 1 & \text{for } x = 1 \end{cases}$$

♣ 1. Graph f.

♣ 2. What is the domain of f? In particular, is f defined at $x = 1$?

♣ 3. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 1} f(x) = -2$$

♣ 4. Discuss the phrase

$$0 < |x-c| < \delta \quad \text{and} \quad x \in \mathcal{D}(f)$$

relative to this example. What is c? What is δ? For what values of x is this phrase true, for the function f being considered here?

one-sided limits

Sometimes one is only interested in investigating the function values $f(x)$ as x approaches c from *only one side* (right or left), even though f *may* be defined on both sides of c. In other words, one can ask the question: as x approaches c from one side (right or left), do the corresponding function values $f(x)$ approach any particular real number? This leads to the notion of *one-sided limits.* Here's the precise definition of the *right-hand limit*:

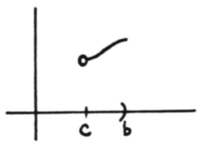
DEFINITION
right-hand limit

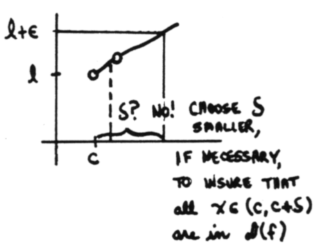
Let f be a function that is defined at least on an interval of the form (c,b), where $b > c$. Then:

$$\lim_{x\to c^+} f(x) = l \iff \text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } x \in (c, c+\delta), \text{ then } |f(x) - l| < \epsilon.$$

The phrase '$x \to c^+$' is read as '*x approaches c from the right-hand side*' or '*x approaches c from the positive side*'.

investigating this definition



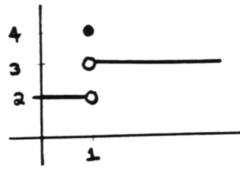


Let's investigate this definition. Here are the ways that it differs from the earlier (two-sided) limit:

- Since it is desired to let x approach c from the right-side, we require that f be defined at least on some small interval to the right of c.
- For the limit to exist (and equal l), we must be able to get $f(x)$ as close to l as desired (within ϵ), by requiring that x be close enough to c (within δ), *on the right-hand side.* Note that whenever $x \in (c, c+\delta)$, then x lies to the right of c. Also, note that x is not allowed to equal c, since c is not included in the interval $(c, c+\delta)$.
- The phrase '$x \in \mathcal{D}(f)$' was needed in the definition of the two-sided limit to 'cover the cases' when f was not defined on both sides of c. Now, however, we are *assuming* that f is defined to the right of c, so the phrase is not necessary. Delta (δ) can always be chosen small enough so that x will lie in the domain of f.

EXAMPLE

investigating a right-hand limit



Consider the function f given by:

$$f(x) = \begin{cases} 2 & x < 1 \\ 3 & x > 1 \\ 4 & x = 1 \end{cases}$$

The graph of f is shown. In this case, the (two-sided) limit $\lim_{x \to c} f(x)$ does not exist. (Why?) However:

$$\lim_{x \to 1^+} f(x) = 3$$

EXERCISE 9

left-hand limit

♣ 1. After studying the definition of right-hand limit, write down a precise definition of a *left-hand limit*. Denote the left-hand limit by

$$\lim_{x \to c^-} f(x)$$

and read the phrase '$x \to c^-$' as x *approaches* c *from the left-hand side* or x *approaches* c *from the negative side*. Be sure to write complete mathematical sentences.

Now, consider the function f given by:

$$f(x) = \begin{cases} x^2 & x < 3 \\ 5 & x > 3 \\ 7 & x = 3 \end{cases}$$

♣ 2. Graph this function f.

♣ 3. Evaluate the limits:

$$\lim_{x \to 3^+} f(x) \quad \text{and} \quad \lim_{x \to 3^-} f(x)$$

♣ 4. Why doesn't the (two-sided) limit $\lim_{x \to 3} f(x)$ exist?

♣ 5. Redefine the function for $x > 3$ so that the two-sided limit does exist for the redefined function.

In the next section, some tools will be developed to help us work with limit statements. Then, we will be in a position to discuss *continuity* precisely.

QUICK QUIZ

sample questions

1. Give a precise definition of the limit statement $\lim_{x\to c} f(x) = l$.
2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to -1} 3x = -3$$

3. Let f be defined by:

$$f(x) = \begin{cases} x^2 & \text{for } x > 2 \\ 6 & \text{for } x = 2 \\ x & \text{for } x < 2 \end{cases}$$

Sketch the graph of f, and evaluate the following limits, if they exist:

$$\lim_{x\to 2} f(x)\ , \qquad \lim_{x\to 2^+} f(x)\ , \qquad \lim_{x\to 2^-} f(x)$$

KEYWORDS

for this section

The precise definition of:

$$\lim_{x\to c} f(x) = l$$

You must be able to explain this definition in words, and with appropriate sketches. You must be able to use the 4-step process to show that certain limit statements are true. Also, you must understand one-sided limits.

END-OF-SECTION EXERCISES

Use the 4-step process to show that the following limit statements are true:

1. $\lim\limits_{x\to -2}(-x^2) = -4$

2. $\lim\limits_{x\to 2}(2 - x^3) = -6$

3. $\lim\limits_{t\to 16}\sqrt[4]{t} = 2$

4. $\lim\limits_{t\to 0}|t| = 0$

Graph the given function. Then, evaluate the specified limits. If a limit does not exist or is not defined, so state.

5. $f\colon (1,\infty) \to \mathbb{R},\ \ f(x) = x^2 + 2;\quad \lim\limits_{x\to 1} f(x),\ \ \lim\limits_{x\to 1^+} f(x),\ \ \lim\limits_{x\to 1^-} f(x)$

6. $f\colon (-\infty,1) \to \mathbb{R},\ \ f(x) = x^2 + 2;\quad \lim\limits_{x\to 1} f(x),\ \ \lim\limits_{x\to 1^+} f(x),\ \ \lim\limits_{x\to 1^-} f(x)$

7.
$$g(x) = \begin{cases} x & \text{for } \ x < -1 \\ 2 & \text{for } \ x = -1 \\ -x^2 & \text{for } \ x > -1 \ ; \end{cases}$$
$\lim\limits_{x\to -1} g(x),\ \ \lim\limits_{x\to -1^+} g(x),\ \ \lim\limits_{x\to -1^-} g(x)$

8.
$$g(x) = \begin{cases} 3 & \text{for } \ x \ge 2 \\ 1 & \text{for } \ x < 2 \ ; \end{cases}$$
$\lim\limits_{x\to 2} g(x);\ \ \lim\limits_{x\to 2^+} g(x);\ \ \lim\limits_{x\to 2^-} g(x)$

9. True or False: if $\lim\limits_{x\to c} f(x)$ exists and f is defined on both sides of c, then both $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ exist.

10. True or False: if both one-sided limits $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ exist, then $\lim\limits_{x\to c} f(x)$ exists.

3.3 Properties of Limits

This section establishes some useful properties of limits, the development of which provides additional practice with the concept of the limit of a function.

existence versus uniqueness

Mathematicians are extremely fond of *existence* and *uniqueness arguments*. An *existence argument* shows that a certain object *exists*, but does not address the issue: How many? A *uniqueness argument* answers the question 'How many?' with a definitive: Exactly one.

existence of $\lim_{x \to c} f(x)$

When does the limit $\lim_{x\to c} f(x)$ exist? The definition answers this question: it exists when there is a number l with the property that one can get $f(x)$ as close to l as desired, by requiring that $x \in \mathcal{D}(f)$ be sufficiently close to c, but not equal to c.

When $lim_{x\to c} f(x)$ exists, is it unique?

Is it possible that there are two different numbers l and k, *both* satisfying the definition of the limit of a function? Or, is the limit *unique*? If you stop to think about this for a moment, you'll probably conclude that $f(x)$ can't be close to *two different numbers* at the same time. But how can this be argued precisely?

the way mathematicians show uniqueness

The way mathematicians usually establish uniqueness is to:

- Suppose that there are *two*;
- Show that these two are the same.

a typical uniqueness argument

That is, suppose a mathematician is asked to prove the following theorem. (Remember, a *theorem* is a mathematical result that is both *important* and *true*.)

Theorem. An object with property P is unique.

Don't worry about what property P is; here we are discussing the *form* of a typical uniqueness argument, and are not concerned with specific *content*.

Here's how the proof would go:

Proof. Suppose that x and y both satisfy property P. (More stuff here.) Then, $x = y$. ∎

Early on in the proof, x could potentially be different from y; all that is known is that they both satisfy 'property P'. But then, information about 'property P' is used to show that x must equal y.

the symbol ∎ is used to mark the end of proofs

The symbol ∎ is an end-of-proof marker. It is really just a courtesy to the reader; a gentle reminder that the author has finished showing whatever was set out to be shown.

EXERCISE 1

♣ Prove that there is a unique solution to the linear equation

$$ax + b = c, \quad a \neq 0 \ ,$$

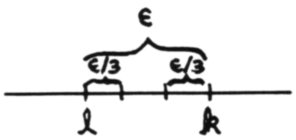
by supposing that both X and Y are solutions, and showing that $X = Y$. Be sure to write down complete mathematical sentences.

The next theorem states, in the language of mathematics, that limits are unique.

THEOREM *limits are unique*	Suppose that: $$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k$$ Then, $l = k$.

a motivation for the proof

Before jumping into the rigorous proof, just stop and think. How could it be shown that l must equal k?



If l is not equal to k, then there's some positive distance between them; call it ϵ. Since ϵ is positive, so is $\epsilon/3$. Looking back at the precise definition of the limit of a function, one observes that ϵ represents *any* positive number. The definition can certainly be applied, taking this positive number to be $\epsilon/3$. (If this seems awkward to you, rewrite the definition, using ω instead of ϵ. Then, take ω to be $\epsilon/3$.)

Since it is being assumed that *both*

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k \ ,$$

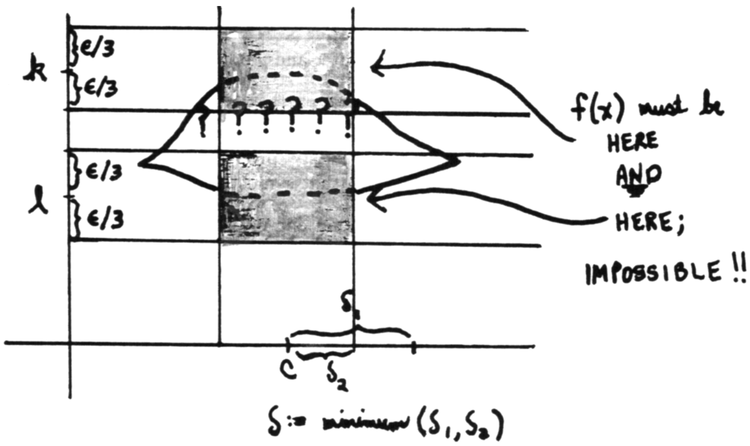
one must be able to get $f(x)$ within $\epsilon/3$ of both l and k, by requiring that x be sufficiently close to c.

So, get a number δ_1 such that whenever x is within δ_1 of c, $f(x)$ must be within $\epsilon/3$ of l.

And, get a number δ_2 so that whenever x is within δ_2 of c, then $f(x)$ must be within $\epsilon/3$ of k.

a contradiction

Take the minimum of δ_1 and δ_2, and call it δ. Then, whenever x is within δ of c, $f(x)$ must be within $\epsilon/3$ of *both* l and k. This is impossible; it is an example of what mathematicians call a *contradiction*. By assuming that k and l are different, one is led to a contradiction. Thus, it must be that k and l are NOT different; that is, they must be equal.



EXERCISE 2	♣ In the preceding argument, the author chose to get the function values $f(x)$ within $\epsilon/3$ of both l and k. Would $\epsilon/2$ have worked? How about $\epsilon/4$? Why do you suppose the author chose $\epsilon/3$?

The following proof 'formalizes' the ideas discussed above.

PROOF
limits are unique

Suppose that:

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k$$

If $l = k$, we're done. So suppose that $l \neq k$. Then, there is some positive distance between l and k; call it ϵ. Since ϵ is positive, so is $\epsilon/3$. Since $\lim_{x\to c} f(x) = l$, there exists δ_1 such that whenever $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_1$, it must be that $|f(x) - l| < \epsilon/3$.

Since $\lim_{x\to c} f(x) = k$, there exists δ_2 such that whenever $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_2$, it must be that $|f(x) - k| < \epsilon/3$.

Take δ to be the minimum of δ_1 and δ_2. Then, for any $x \in \mathcal{D}(f)$ with $0 < |x - c| < \delta$, we must have *both* $|f(x) - l| < \epsilon/3$ and $|f(x) - k| < \epsilon/3$, which is impossible.

Thus, it must be that $k = l$. ∎

EXERCISE 3

♣ Get another calculus book, and look up the *uniqueness of limits* theorem. Compare with what has been discussed here. Is the statement of the theorem the same? Read the proof (slowly and carefully). Is the proof exactly the same? Not every proof uses a contradiction argument. How does the other proof establish that $l = k$?

★★
the logical justification for proof by contradiction

The form of proof, called *proof by contradiction*, is justified by the following logical equivalence:

$$A \Rightarrow B \quad \iff \quad (\text{not}\, B \wedge A) \implies (S \wedge \text{not}\, S)\ ,$$

where S is any statement.

In the previous proof, the statement A is

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k\ ;$$

the statement B is:

$$l = k$$

The contradiction $(S \wedge \text{not}\, S)$ is the fact that $f(x)$ must be IN a certain interval (say, around l) and NOT IN this interval, at the same time.

Next, some rules are developed that tell us many situations in which limits are 'easy' to find.

in many cases, evaluating limits is easy;

direct substitution

For many 'nice' functions f, evaluating limits is as easy as *direct substitution*; that is:

$$\lim_{x \to c} f(x) = f(c)$$

This is called *direct substitution* because, to evaluate the limit, one need only substitute the number c into the expression for f.

For example:

$$\lim_{x \to 1} (x^2 - 4) = 1^2 - 4 = -3$$

and

$$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

(Functions that are 'nice' like this are given a special name—they are called *continuous!* This will be studied in more detail in the next section on continuity.)

The next two theorems tell us many 'nice' functions for which evaluating limits is this easy! The numbering scheme (e.g., P1, P2, P3) is merely for easy reference in the exercises and examples.

THEOREM

Properties of Limits

Let b and c denote real numbers; n is a positive integer.

P1) $\lim_{x \to c} b = b$ (The limit of a constant function is the constant.)

P2) $\lim_{x \to c} x = c$

P3) $\lim_{x \to c} x^n = c^n$

some remarks on proving theorems

The proofs of theorems that appear in mathematics books are usually precise, slick, clean, beautiful. Too often, students think that these proofs merely 'jump onto' the paper from the pencils of mathematicians. Not true. Mathematicians rarely 'jump into' a proof. Instead, they *play with* what they're trying to prove. They do things that help them *believe* that it is true. They may 'try out' the theorem in some simple cases, in an attempt to figure out what makes it work.

how you, as a reader, should approach theorems

When you read a theorem, you should do the following:

- Ask yourself: Do I understand what this is telling me that I can DO? Remember, theorems are usually statements of fact. But, *facts can tell you what to do*, if you understand the language.
- Ask yourself: Do I BELIEVE this result? Play with it. Try it in some simple cases. Draw some graphs. *Read and understand the proof.*

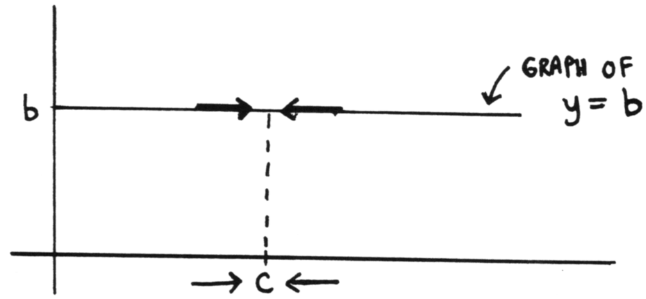
investigating the limit properties of the previous theorem

Let's investigate the properties in the previous theorem, the way a good reader should. Begin with property P1:

$$\lim_{x \to c} b = b$$

What does this say that you can DO? In words, this property states that the limit of a constant function is the constant. It tells you that evaluating the limit of a constant function is easy; just write down the constant.

Next, is this result BELIEVABLE? Recall that the graph of the constant function $f(x) = b$ is a horizontal line, that crosses the y-axis at the number b. No matter *what* the x-value happens to be, the function value is constant at b. Certainly the result is believable.



A precise proof of property P1 must appeal to the definition. It must be shown that one can get the function values as close to b as desired, by requiring that x be sufficiently close to c. Indeed, in this case, no matter what positive number one chooses for ϵ, *any* δ will work. Here's a precise proof:

PROOF of (P1) $\lim_{x \to c} b = b$	Let b and c be real numbers. Choose $\epsilon > 0$, and let $\delta = 1$. If $0 < \lvert x - c \rvert < 1$, then $\lvert b - b \rvert = 0 < \epsilon$. Thus, $\lim_{x \to c} b = b$. ∎

EXERCISE 4	♣ Are there any other values of δ that would work in the previous proof? Why do you suppose the author chose δ to be 1?

EXERCISE 5

Consider property P2:

$$\lim_{x \to c} x = c$$

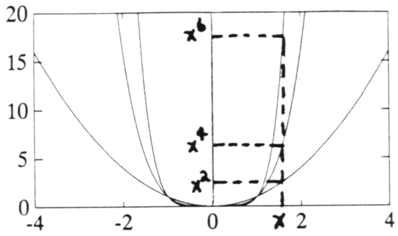
♣ 1. What is this telling you that you can DO?

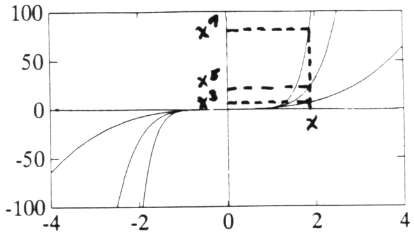
♣ 2. Do you believe it? Make a sketch that might help you believe this result.

♣ 3. Prove that $\lim_{x \to c} x = c$, by writing down a precise ϵ-δ argument. Use the 4-step process discussed in section 3.2 to find a 'δ that works'.

investigating $\lim_{x\to c} x^n = c^n$

Finding the 'δ that works' is more delicate when investigating $\lim_{x\to c} x^n$, in part due to the fact that different values of c and n will lead to different choices for δ. However, the sketches below certainly make plausible the idea that as x approaches c, x^n must approach c^n.





Next, some *Operations with Limits*.

THEOREM
Operations with Limits

Let b and c be real numbers; n is a positive integer. Suppose that *both* $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

O1) $\lim_{x\to c} bf(x) = b\big[\lim_{x\to c} f(x)\big]$
(You can 'pull constants out' of the limit.)

O2) $\lim_{x\to c} \big[f(x) + g(x)\big] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
(The limit of a sum is the sum of the limits.)

O3) $\lim_{x\to c} f(x)g(x) = \big[\lim_{x\to c} f(x)\big]\big[\lim_{x\to c} g(x)\big]$
(The limit of a product is the product of the limits.)

O4) If $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
(The limit of a quotient is the quotient of the limits.)

O5) $\lim_{x\to c} (f(x))^n = \big[\lim_{x\to c} f(x)\big]^n$
(Power rule)

EXERCISE 6

♣ 1. Does this theorem tell us that

$$\lim_{x\to c}\big[f(x) - g(x)\big] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) \ ,$$

whenever both individual limits exist? Why or why not?

♣ 2. Does this theorem tell us that

$$\lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \lim_{x\to c}\big[f(x) + g(x)\big] \ ,$$

whenever both individual limits exist? Why or why not?

EXERCISE 7

♣ 1. Evaluate the limit:

$$\lim_{x\to 0} x\cdot\frac{1}{x}$$

(Hint: Remember that x is not allowed to equal 0. What is the value of $x\cdot\frac{1}{x}$ for values of x *near* 0?)

♣ 2. Find the flaw in this student's argument.

Student's answer:

By O3:

$$\lim_{x\to 0} x\cdot\frac{1}{x} = \big(\lim_{x\to 0} x\big)\cdot\big(\lim_{x\to 0}\frac{1}{x}\big)$$

Since $\lim_{x\to 0}\frac{1}{x}$ does not exist, it must be that $\lim_{x\to 0} x\cdot\frac{1}{x}$ also does not exist.

investigating the operations with limits

Let's investigate property O1:

$$\lim_{x\to c} bf(x) = b\big[\lim_{x\to c} f(x)\big]$$

the hypotheses of a theorem;

singular: hypothesis

The *hypotheses* of a theorem are the things that are assumed to be true. (Singular: *hypothesis*.) One hypothesis of the previous theorem is that $\lim_{x\to c} f(x)$ exists. Thus, there is some number that $f(x)$ gets close to as x approaches c; in keeping with tradition, let's call this number l. How do the numbers $bf(x)$ differ from $f(x)$? They are each multiplied by b. Thus, as $f(x)$ gets close to l, $bf(x)$ must get close to $b\cdot l$. That is, if

$$\lim_{x\to c} f(x) = l$$

then:

$$\lim_{x\to c} bf(x) = b\cdot l = b\cdot\lim_{x\to c} f(x)$$

So the result does indeed seem plausible.

Similar reasoning should make the remaining operations plausible. We will look at one precise proof, which makes use of the *triangle inequality*, discussed next.

the triangle inequality,

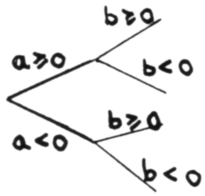
$|a+b|\le|a|+|b|$

Let a and b be real numbers. Then:

$$|a+b|\le|a|+|b|$$

PARTIAL PROOF
of the triangle inequality

Let a and b be real numbers. Since every real number is either nonnegative (≥ 0) or negative (< 0), there are several cases to be considered, as suggested by the 'tree diagram' below.



Recall the precise definition of the absolute value function:

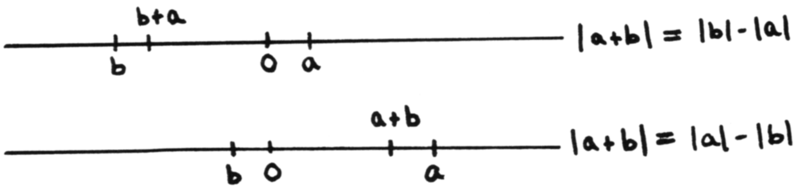
$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Also recall that the number $|x|$ is often called the *magnitude of x*.

Case 1 ($a \geq 0$ and $b \geq 0$). In this case, $|a| = a$ and $|b| = b$. (Why?) Also, since both a and b are nonnegative, so is $a + b$, so that $|a + b| = a + b$. In this case one actually obtains equality:

$$|a + b| = a + b = |a| + |b|$$

Case 2 ($a \geq 0$ and $b < 0$). In this case, writing down all the details often seems to obscure the simple idea, illustrated by the sketches below. The point is that when a and b have different signs, $|a + b|$ is either $|a| - |b|$ (if the magnitude of a is bigger) or $|b| - |a|$ (if the magnitude of b is bigger). But in either case, the difference is less than or equal to $|a| + |b|$.



EXERCISE 8

♣ 1. Write down the proof of

$$|a + b| \leq |a| + |b|$$

in the case when $a < 0$ and $b < 0$. Be sure to write complete mathematical sentences.

♣ 2. Is the case

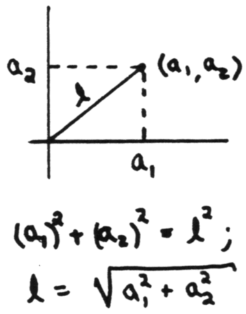
$$a < 0 \ \text{ and } \ b \geq 0$$

really any different from the case

$$a \geq 0 \ \text{ and } \ b < 0 \text{ ?}$$

Why or why not?

★
Why the name 'triangle inequality'?



★ The triangle inequality also holds when a and b are ordered pairs of real numbers. The 'length' of an ordered pair is found using Pythagorean's theorem:

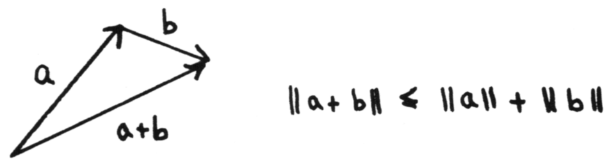
$$\|(a_1, a_2)\| := \sqrt{a_1^2 + a_2^2}$$

Although the absolute value symbol $|\cdot|$ is used to talk about the 'length' (magnitude) of a real number, the *norm* symbol $\|\cdot\|$ is traditionally used to talk about other lengths.

In this setting, the fact that

$$\|a + b\| \le \|a\| + \|b\|$$

has a nice geometric interpretation: in a triangle, the length of a side cannot exceed the sum of the lengths of the remaining two sides. This is the motivation for the name *triangle inequality*.



With the triangle inequality in hand, the precise proof of operation (O2) is now presented.

PROOF of (O2)
that the limit of a sum is the sum of the limits

Suppose that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, say:

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} g(x) = k$$

Choose $\epsilon > 0$. Then, $\epsilon/2$ is also positive, and there exists a corresponding δ_1 such that when $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_1$, it must be that $|f(x) - l| < \epsilon/2$. (♣ Why?)

Also, there exists δ_2 such that when $x \in \mathcal{D}(g)$ and $0 < |x - c| < \delta_2$, it must be that $|g(x) - k| < \epsilon/2$. (♣ Why?)

Let $\delta := \text{minimum}(\delta_1, \delta_2)$. Then, if $x \in \mathcal{D}(f) \cap \mathcal{D}(g)$ and $0 < |x - c| < \delta$, one obtains:

$$\begin{aligned} |f(x) + g(x) - (l + k)| &= |(f(x) - l) + (g(x) - k)| \\ &\le |f(x) - l| + |g(x) - k| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

This says that:

$$\begin{aligned} \lim_{x\to c}(f(x) + g(x)) &= l + k \\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \blacksquare \end{aligned}$$

EXERCISE 9

♣ In the proof above, supply reasons for each of these lines:

$$\begin{aligned} |f(x)+g(x)-(l+k)| &= |(f(x)-l)+(g(x)-k)| && \text{Reason:} \\ &\le |f(x)-l|+|g(x)-k| && \text{Reason:} \\ &< \epsilon/2+\epsilon/2 && \text{Reason:} \\ &= \epsilon && \text{Reason:} \end{aligned}$$

EXERCISE 10

♣ Let a and b be positive numbers. Convince yourself that if $m := \text{minimum}(a,b)$, then $m \le a$ and $m \le b$. (A number line sketch may be all you need to convince yourself of this fact.)

Where was this fact used in the proof of of (O2)?

extending the operations to more than two functions:

the 'treat it as a singleton' technique

Mathematicians realize that facts like

$$\lim_{x\to c}\bigl[f(x)+g(x)\bigr] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)\ ,$$

although seemingly holding only for *two* functions, actually hold for *any finite number of functions*. The proof uses a very common 'treat it as a singleton' technique. Assume in what follows that all the individual limits exist.

$$\begin{aligned} \lim_{x\to c} f(x)+g(x)+h(x) &= \lim_{x\to c}\bigl[f(x)+g(x)\bigr]+h(x) && \text{(associative law)} \\ &= \lim_{x\to c}\bigl[f(x)+g(x)\bigr]+\lim_{x\to c} h(x) && \text{(O2)} \\ &= \lim_{x\to c} f(x)+\lim_{x\to c} g(x)+\lim_{x\to c} h(x) && \text{(O2 again)} \end{aligned}$$

EXERCISE 11

♣ Assuming that all the individual limits exist, show that:

$$\lim_{x\to c} f(x)g(x)h(x) = \bigl[\lim_{x\to c} f(x)\bigr]\cdot\bigl[\lim_{x\to c} g(x)\bigr]\cdot\bigl[\lim_{x\to c} h(x)\bigr]$$

Be sure to write complete mathematical sentences, and give reasons supporting each step in your argument.

In closing, the tools developed in this section are used to show that evaluating limits of ANY polynomial is as easy as direct substitution:

THEOREM

Evaluating limits of polynomials

Let P be any polynomial:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Then:

$$\lim_{x\to c} P(x) = P(c)$$

PROOF

$$\begin{aligned}
\lim_{x\to c} P(x) &= \lim_{x\to c}\,(a_n x^n + \cdots + a_1 x + a_0) && \text{(definition of } P\text{)}\\
&= \lim_{x\to c} a_n x^n + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 && \text{(O2)}\\
&= a_n \lim_{x\to c} x^n + \cdots + a_1 \lim_{x\to c} x + a_0 && \text{(O1) and (P1)}\\
&= a_n c^n + \cdots + a_1 c + a_0 && \text{(P3)}\\
&= P(c) && \text{(polynomial } P\text{, evaluated at } c\text{)}
\end{aligned}$$

EXAMPLE

For example:

$$\lim_{x\to 1}(x^2 - 3x + \sqrt{2}) = 1^2 - 3(1) + \sqrt{2} = -2 + \sqrt{2}$$

QUICK QUIZ
sample questions

1. Explain, in a couple English sentences, how a mathematician often shows that an object is UNIQUE.
2. Under what condition(s) is the limit of a sum equal to the sum of the limits?
3. Give a precise statement of the 'triangle inequality' for real numbers.
4. Suppose you are told that, for a given function f and constant c, 'evaluating the limit $\lim_{x\to c} f(x)$ is as easy as direct substitution'. What does this mean?
5. Suppose that:

$$\lim_{x\to 1} f(x) = 3, \quad \lim_{t\to 1} g(t) = 5, \quad \text{and} \quad \lim_{y\to 1} h(y) = 2$$

Can you evaluate the following limit?

$$\lim_{z\to 1} \frac{-2f(z) + g(z)}{h(z)}$$

If so, do it.

KEYWORDS
for this section

Existence and uniqueness arguments, the end-of-proof symbol ▮, uniqueness of limits, direct substitution, properties of limits, how you should approach theorems, operations with limits, the triangle inequality, extending operations to more than two functions, limits of polynomials.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. If $\lim_{x\to c} f(x) = l$ and $\lim_{y\to c} f(y) = m$, then $l = m$.
2. If $\lim_{t\to c} f(t) = q$ and $\lim_{x\to c} f(x) = r$, then $q = r$.
3. $\lim_{x\to c} f(x) = \lim_{y\to c} f(y)$
4. $\lim_{x\to c} f(x) = \lim_{x\to d} f(x)$
5. If $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$.
6. If $\frac{\epsilon}{2} > 0$, then $\epsilon > 0$.
7. $\epsilon > 0 \iff \frac{\epsilon}{2} > 0$
8. $\epsilon > 0 \iff 2\epsilon > 0$
9. $\epsilon > 0 \iff (\epsilon - .1) > 0$
10. $\lim_{x\to c} d = d$ (Here, it is assumed that c and d are real numbers.)
11. $\lim_{x\to 2} x^{100} = 2^{100}$
12. $\lim_{y\to -1} y = -1$
13. $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
14. If the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

For the remaining problems, suppose that:

$$\lim_{x\to c} f(x) = -1, \quad \lim_{x\to c} g(x) = 2, \quad \text{and} \quad \lim_{x\to c} h(x) = 0$$

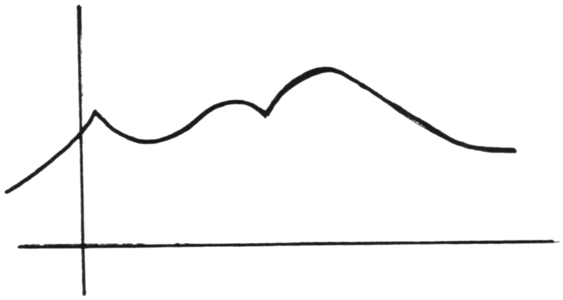
If possible, evaluate the following limits. If you don't have enough information to evaluate the limit, so state. Be sure to write complete mathematical sentences.

15. $\lim_{t\to c}[f(t) + g(t)]$
16. $\lim_{t\to c}(f - g)(t)$
17. $\lim_{y\to d}[f(y)g(y)]$
18. $\lim_{x\to c}\big([3g(x) - f(x)] \cdot h(x)\big)$

3.4 Continuity

Introduction

Intuitively, a function is *continuous* if its graph can be traced without lifting a pencil. This notion is made precise in this section.

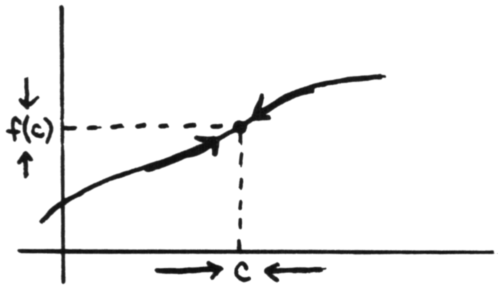


Mathematicians define what it means for a function to be *continuous at a point*: roughly, '*f is continuous at the point* $(c, f(c))$' means that:

- f is defined at c (so that $f(c)$ makes sense)
- as x approaches c, $f(x)$ approaches $f(c)$

The phrase '*f is continuous at the point* $(c, f(c))$' is usually shortened to '*f is continuous at c*'.

The precise definition follows:



DEFINITION
continuity at a point

A function f is *continuous at c* if f is defined at c, and:

$$\lim_{x \to c} f(x) = f(c)$$

this definition is saying three things

It is important to realize that the statement $\lim_{x\to c} f(x) = f(c)$ is saying **three things**:

1) $f(c)$ exists (i.e., c is in the domain of f)
2) $\lim\limits_{x \to c} f(x)$ exists
3) $\lim\limits_{x \to c} f(x) = f(c)$ (that is, the numbers above are equal!)

f is discontinuous at c

If *any one* of these three criteria fail, then f is *not* continuous at c. In this case, one says that f is *discontinuous at c.*

Finding limits at a point of continuity is easy! Use direct substitution.

Suppose f is continuous at c. Then, we know that f is defined at c; that is, $f(c)$ exists. Also, evaluating $\lim_{x\to c} f(x)$ is as easy as direct substitution, since *continuity at c* tells us that the limit is equal to $f(c)$!

Example

In the previous section, it was shown that for any polynomial P:

$$\lim_{x\to c} P(x) = P(c)$$

Here, c is any real number. This result says that *polynomials are continuous everywhere.*

Thus, for example:

$$\lim_{x\to 7.2}(4x^4 - \sqrt{2}x + \pi) = 4(7.2)^4 - \sqrt{2}(7.2) + \pi$$

EXERCISE 1

♣ 1. Evaluate the following limits:

a) $\lim_{x\to 2}(x^2 - x + 1)$

b) $\lim_{x\to \pi}(x^2 - x + 1)$

c) $\lim_{x\to b}(x^2 - x + 1)$ (here, b is a real number)

d) $\lim_{x\to n}(x^2 - x + 1)$ (here, n is an integer)

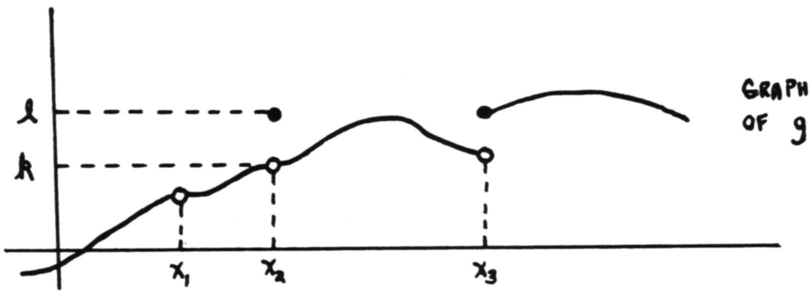
e) $\lim_{x\to d}(ax^2 + bx + c)$ (here, a, b, c and d are real numbers)

♣ 2. Find a function f and a number c such that f is continuous at c and $\lim_{x\to c} f(x) = 2$.

How can a function FAIL to be continuous at c?

Since there are really three requirements for a function to be continuous at c, there are also three ways that a function can *fail* to be continuous at c.

The function g with graph shown below illustrates the three ways that a function can be discontinuous at c. Refer to this graph for the discussions below.



f may not be defined at c

Whenever a function is not defined at c, (that is, $f(c)$ does not exist), then f is not continuous at c.

The function g is discontinuous at x_1, because g is not defined at x_1.

the limit as x approaches c may not exist

If $\lim_{x\to c} f(x)$ does not exist, then f is not continuous at c.

This function g is discontinuous at x_3, because $\lim_{x\to x_3} g(x)$ does not exist.

both $f(c)$ and $\lim\limits_{x\to c} f(x)$ may exist, but they aren't equal

It is possible for both $f(c)$ and $\lim_{x\to c} f(x)$ to exist, but not be equal.

The function g is discontinuous at x_2.

In this case, g is defined at x_2; $g(x_2) = l$.

Also, the limit of g as x approaches x_2 exists; $\lim_{x\to x_2} g(x) = k$.

However, these two numbers are *not equal*! That is:

$$\lim_{x\to x_2} g(x) \neq g(x_2)$$

two types of discontinuities

If a function is discontinuous at c, then the discontinuity can be classified, depending on how the definition of continuity fails.

DEFINITION
removable discontinuity

A function f has a *removable discontinuity at c* whenever $\lim_{x\to c} f(x)$ exists, but is not equal to $f(c)$.

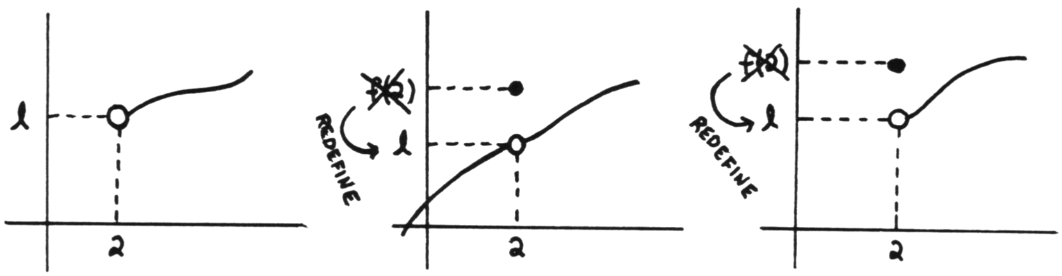
In this case, the discontinuity can be easily *removed* by merely defining (or redefining) the function f at c!

EXAMPLE
removable discontinuities

The functions whose graphs are shown below all have removable discontinuities at $x = 2$.

In the first case, $\lim_{x\to 2} f(x)$ exists, but f is not defined at 2. This discontinuity can be easily removed by defining $f(2) = l$.

In the second and third cases, $\lim_{x\to 2} f(x)$ exists, and $f(2)$ exists, but these numbers are not equal. These discontinuities can be easily removed by redefining f at 2 so that $f(2) = l$.



DEFINITION
nonremovable discontinuity

A function f has a *nonremovable discontinuity at c* whenever $\lim_{x\to c} f(x)$ does not exist.

In this case, the discontinuity can *not* be easily removed!

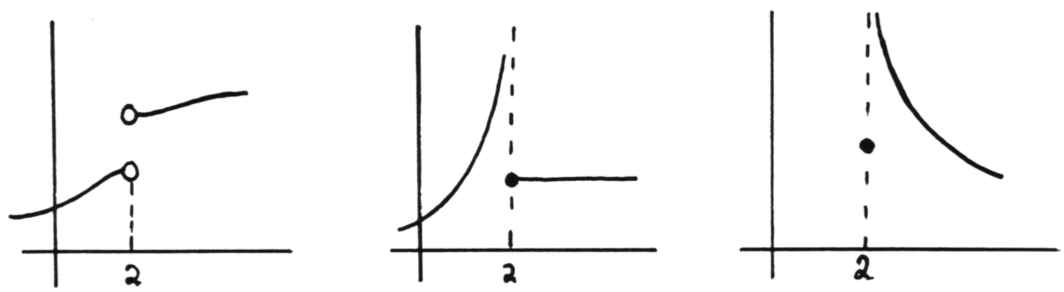
EXAMPLE

nonremovable discontinuities

The functions with graphs shown below all have nonremovable discontinuities at $x = 2$. In all cases, $\lim_{x\to 2} f(x)$ does not exist.

Observe that f may or may not be defined at a nonremovable discontinuity.

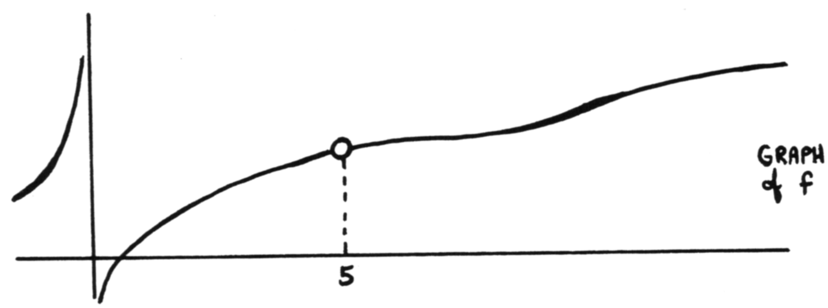
Any attempt to 'patch up' these discontinuities would require major reconstructive work. Essentially, one must grab both pieces of the graph and pull them together. No matter how this is done, it requires redefining the function on some entire *interval*, as opposed to just at a single point. Thus, this type of discontinuity is *not* easy to remove!



classifying discontinuities

To *classify a discontinuity* means to state if the discontinuity is removable or nonremovable.

For example, suppose you are asked to classify the discontinuities of the function shown below:



The correct response is:

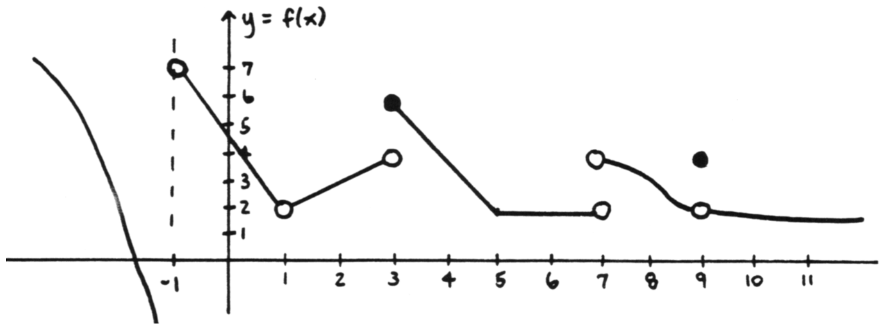
f has a removable discontinuity at 5.

f has a nonremovable discontinuity at 0.

Be sure to write complete mathematical sentences! Do *not* merely say: '*removable discontinuity at* 5'.

EXERCISE 2

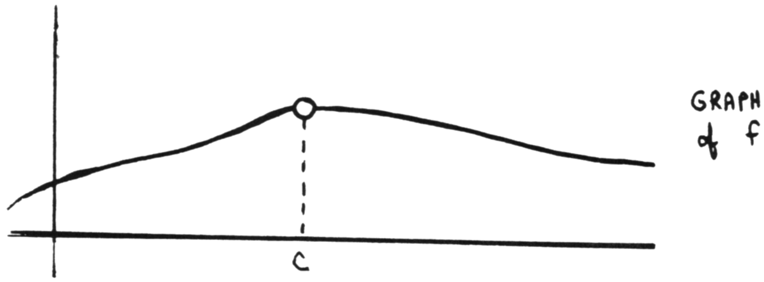
♣ 1. Classify the discontinuities of the function shown below.



♣ 2. If a discontinuity is removable, indicate how it can be 'removed'.

DON'T ASK THIS QUESTION!

Consider the function f shown below. Students like to ask the question: *Is this function continuous?*



Now, does this question really make sense? Continuity has only been defined *at a point*! That is, we have not defined what it means for a function f *to be continuous*; we have only defined what it means for a function f *to be continuous at a point* c.

The function shown is not continuous at c, because it's not defined at c. But, f IS continuous at every point where it is defined. Therefore, the absolutely correct answer to the not-so-correct question *Is this function continuous?* is:

The function f is continuous at every point in its domain.

or,

The function f is continuous at every point where it is defined.

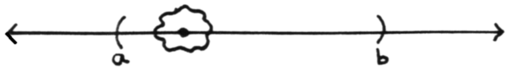
continuity on an interval

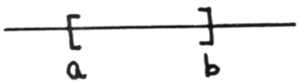
If a function happens to be continuous on an entire *interval* of real numbers, then it has some particularly nice properties. First, a brief discussion of *intervals*.

open intervals

closed intervals

A finite interval is said to be *open* if it does not include either endpoint. Thus, (a, b) is an open interval. The word 'open' refers to the fact that *every* point in this interval *has some space around it (on both sides) that remains entirely inside the interval.* In other words, each point in the interval has some room both to the left and to the right that is still in the interval. (Think of 'the wide open spaces'!)





A finite interval is said to be *closed* if it includes both endpoints. Thus, $[a, b]$ is a closed interval. Observe that the endpoints a and b do NOT have room both to the right and left that is still in the interval. There is no room to the left of a; and there is no room to the right of b.

A finite interval that includes only one endpoint is not open and not closed. Thus, the intervals $(a, b]$ and $[a, b)$ are not open, and not closed. *Thus, the words 'open' and 'closed' are used differently in mathematics than in English.* In English, if a door is not open, then it is closed. In mathematics, just because an interval is not open, does NOT mean that it is closed.

DEFINITION

continuity on an interval $[a, b]$

A function f is *continuous on the interval* $[a, b]$ if it satisfies the following conditions:

- f is defined on $[a, b]$
- f is continuous at each point in (a, b)
- As x approaches a from within the interval (from the right), $f(x)$ approaches $f(a)$. That is, f is well-behaved at the left endpoint.

  This can be stated in terms of a right-hand limit:

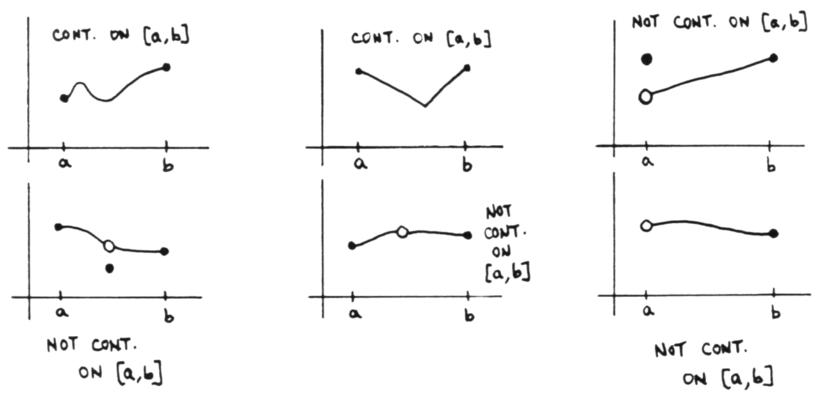
  $$\lim_{x \to a^+} f(x) = f(a)$$

- As x approaches b from within the interval (from the left), $f(x)$ approaches $f(b)$. That is, f is well-behaved at the right endpoint.

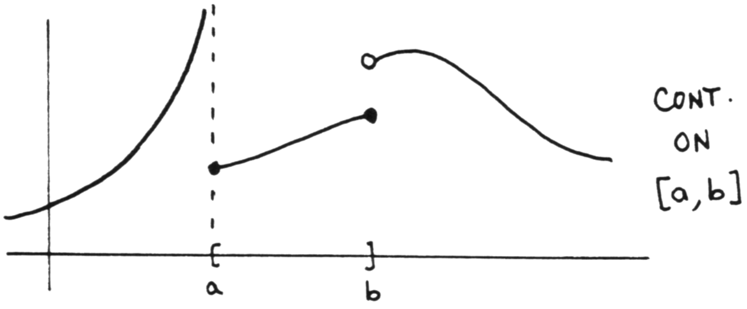
  This can be stated in terms of a left-hand limit:

  $$\lim_{x \to b^-} f(x) = f(b)$$

Some illustrative sketches appear below.



NOTE: A function f may be defined outside of $[a,b]$. However, when answering the question, '*Is f continuous on $[a,b]$?*', the function outside of $[a,b]$ is IGNORED. That is, to investigate the continuity of a function f on $[a,b]$, all one cares about is how f acts on $[a,b]$, and not outside of this interval.



A function that is continuous on a closed interval satisfies some particularly nice properties. These properties will be investigated in the last two sections of this chapter.

EXERCISE 3

Sketch graphs of functions satisfying the following requirements:

♣ 1. f is defined on $[a,b]$, but $\lim_{x\to a^+} f(x) \ne f(a)$

♣ 2. f is defined on $[a,b]$, but $\lim_{x\to b^-} f(x) \ne f(b)$

♣ 3. f is continuous on $[a,b]$, $f(a)=2$ and $f(b)=0$

♣ 4. f is defined on all of $\mathbb{R}$, f is continuous on $[a,b]$, but f is not continuous at a

continuity of sums, products, and scalar multiples

Suppose that functions f and g are both continuous at c. That is:

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c)$$

Then, by properties of limits:

$$\lim_{x\to c} f(x) + g(x) = f(c) + g(c)$$
$$\lim_{x\to c} f(x)\cdot g(x) = f(c)\cdot g(c)$$
$$\lim_{x\to c} k\cdot f(x) = k\cdot f(c)$$

Thus, if f and g are both continuous at c, then so are the sum $f+g$, the product $f\cdot g$, and the scalar multiple kf.

EXERCISE 4

quotients of continuous functions

♣ State a similar result regarding *quotients* of functions that are both continuous at c.

continuity of composite functions

Under what conditions should the composition $(f \circ g)$ be continuous at c? When x is close to c, we want $f(g(x))$ close to $f(g(c))$.

This can be guaranteed in two steps.

First, require that when x is close to c, then $g(x)$ is close to $g(c)$. That is, require that g be continuous at c.

Next, require that when the inputs to f are close to the number $g(c)$, then the outputs are close to $f(g(c))$. That is, require that f be continuous at $g(c)$.

Precisely, if g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

EXERCISE 5

Suppose that $g(3) = 9$ and $f(9) = 2$.

♣ 1. Under what conditions on f and g will the composition $f \circ g$ be continuous at 3?

♣ 2. Under these conditions, what is

$$\lim_{x \to 3} f(g(x)) \ ?$$

QUICK QUIZ
sample questions

1. Give a precise definition of what it means for a function f to be continuous at c.
2. Suppose that $\lim_{x \to c} f(x) = 2$ and $f(c) = 3$. Is f continuous at c? If not, classify the discontinuity.
3. Under what condition(s) does f have a nonremovable discontinuity at c?
4. For a given function f and constant $c \in \mathcal{D}(f)$, under what condition(s) is evaluating the limit $\lim_{x \to c} f(x)$ as easy as 'direct substitution'?
5. Sketch the graph of a function satisfying the following properties: $\mathcal{D}(f) = [1, 3]$, $f(1) = 2$, f is NOT continuous at $x = 1$, f IS continuous at $x = 3$.

KEYWORDS
for this section

Precise definition of continuity at a point: What three things is this definition saying? When can direct substitution be used to find a limit? Removable and nonremovable discontinuities, classifying discontinuities, open and closed intervals, difference between English and mathematical usage of the words 'open' and 'closed', continuity on an interval $[a, b]$, continuity of sums, products, scalar multiplies, quotients, and composite functions.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

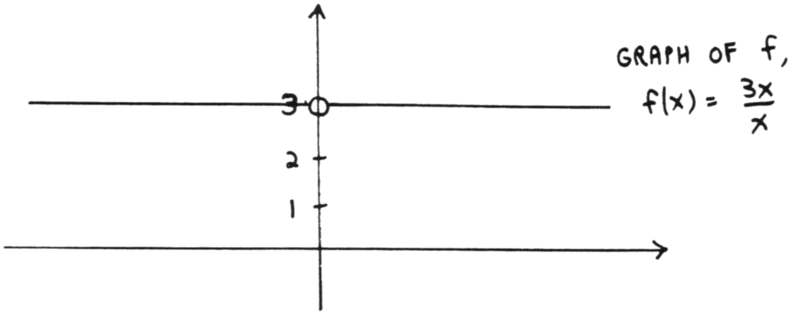
♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. f is continuous at c
2. $f(c) = 5$
3. $f(c)$
4. $\lim_{x\to c} f(x)$
5. $\lim_{x\to c} f(x) = f(c)$
6. If P is a polynomial, then $\lim_{x\to c} P(x) = P(c)$.
7. The function f has a removable discontinuity at $x = 2$.
8. If $\lim_{x\to c} f(x)$ does not exist, then f has a nonremovable discontinuity at c.
9. The function f is continuous on $[a, b]$.
10. The function $f(x) = x^2$ is continuous on $[1, 3]$.
11. If functions f and g are both continuous at c, then so is $f + g$.
12. If functions f and g are both continuous at $x = 2$, then so is the product function fg.
13. If a finite interval of real numbers is not open, then it is closed.
14. If a finite interval of real numbers is not closed, then it is open.
15. (a, b)
16. $(a, b]$ is an open interval

3.5 Indeterminate Forms

Introduction;
$\frac{0}{0}$ situations

Consider the function given by the rule $f(x) = \frac{3x}{x}$; its graph is shown below. Clearly, $\lim_{x\to 0} \frac{3x}{x} = 3$. Note, however, that *if one merely tried to plug in* 0 *for* x when investigating this limit, a '$\frac{0}{0}$' situation would have occurred.



Whenever direct substitution into $\lim_{x\to c} f(x)$ yields a $\frac{0}{0}$ situation, then the function f is *not* defined at c, since division by zero is not allowed. *But the limit MAY still exist.* Or, it may not exist. To see which of these two situations occurs, it is necessary to *rewrite* the function f to get it into a form where one can better analyze what's happening **near** c.

EXAMPLE
a $\frac{0}{0}$ situation;
the limit exists

Problem: Evaluate the limit $\lim_{x\to 0} \frac{3x}{x}$.

Solution: Remember that when investigating a limit as x approaches 0, x is not allowed to equal 0. And, for all values of x except 0, $\frac{3x}{x} = 3$. Therefore, the limit statement can be rewritten in an easier form:

$$\begin{aligned} \lim_{x\to 0} \frac{3x}{x} &= \lim_{x\to 0} 3 && (\frac{3x}{x} = 3 \text{ whenever } x \neq 0) \\ &= 3 && (\text{the limit of a constant function}) \end{aligned}$$

When evaluating the limit $\lim_{x\to c} f(x)$*, the function* f *may be replaced by ANY function that agrees with* f *NEAR* c *(but not necessarily AT* c*).*

EXERCISE 1

♣ Evaluate the limit:

$$\lim_{x\to 1} \frac{3 - 3x}{x - 1}$$

Be sure to write complete mathematical sentences.

EXAMPLE
a $\frac{0}{0}$ situation;
the limit does not exist

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{3x}{x^2}$$

Solution: Again, direct substitution yields a $\frac{0}{0}$ situation.

$$\lim_{x\to 0} \frac{3x}{x^2} = \lim_{x\to 0} \frac{3}{x}$$

Since $\lim_{x\to 0} \frac{3}{x}$ does not exist, neither does $\lim_{x\to 0} \frac{3x}{x^2}$.

EXAMPLE

Problem: Evaluate the limit:

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1}$$

Solution: Direct substitution of $x = 1$ into this limit statement yields a $\frac{0}{0}$ situation. It is necessary to rewrite $\frac{x^2+x-2}{x-1}$ in a way that better displays what is happening *near* $x = 1$. Since 1 is a zero of the numerator, $x - 1$ is a factor of the numerator. Indeed, factoring and canceling yields:

$$\begin{aligned} \frac{x^2+x-2}{x-1} &= \frac{(x-1)(x+2)}{x-1} && \text{(factor the numerator)} \\ &\overset{\text{for } x\neq 1}{=} x+2 && \text{(cancel factor of 1)} \qquad (*) \end{aligned}$$

Thus:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+x-2}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} \\ &= \lim_{x\to 1} (x+2) \\ &= 3 \end{aligned}$$

a 'restricted' equal sign

The second '=' sign that appears in (*) above is a sort of 'restricted' equal sign.

It is NOT completely correct to say that $\frac{(x-1)(x+2)}{x-1} = x + 2$, since these two expressions are NOT equal for all values of x. The left-most expression is not defined when x is 1; the right-most expression is 3 when x is 1.

To bring attention to this difference in the expressions, the necessary restriction is indicated over the equal sign. Thus, the reader becomes aware that the equality only holds when x is not equal to 1.

If the 'restricted' equal sign seem bothersome to you, there is an alternate (but more cumbersome) solution. One can instead say:

$$\frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1} \ ;$$

and for $x \neq 1$, this latter expression simplifies to $x + 2$.

an important distinction

The equal sign in the sentence

$$\lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1} (x+2)$$

is *not* a restricted equal sign. This equal sign is asserting that two *real numbers* are equal. Remember that every limit, if it exists, is a real number.

However, take away the limit instruction, and a restricted equal sign *is* needed:

$$\frac{(x-1)(x+2)}{x-1} \overset{\text{for } x\neq 1}{=} x+2$$

One way to view the '=' sign in this latter sentence, is that it is being used to compare two *functions*. Two *functions* are equal only if they have the same domains, and the outputs agree for all allowable inputs. A precise statement follows.

DEFINITION
equality of functions

Let f and g be functions of one variable. Then:

$$f = g \iff \begin{array}{l}\mathcal{D}(f) = \mathcal{D}(g) \text{ and} \\ f(x) = g(x) \text{ for all } x \text{ in} \\ \text{the common domain}\end{array}$$

EXERCISE 2

♣ 1. What mathematical sentence is being defined in the previous definition?

♣ 2. What does the symbol ‘ $\iff$ ’ mean in this definition?

♣ 3. Suppose you are told that g and h are both functions of one variable, and $g = h$. What can you conclude?

♣ 4. Suppose you are told that g and h are both functions of one variable, $\mathcal{D}(g) = \mathcal{D}(h) := \mathcal{D}$, and $g(x) = h(x) \ \ \forall\, x \in \mathcal{D}$. What can you conclude?

♣ 5. Let f and g be defined by the rules:

$$f(x) = x \cdot \frac{x-1}{x-1} \quad \text{and} \quad g(x) = x$$

Does $f = g$?

♣ 6. Let f and g be defined by the rules:

$$f(x) = \frac{3x}{x^2} \quad \text{and} \quad g(x) = \frac{3}{x}$$

Does $f = g$?

EXERCISE 3

Evaluate the following limits. Be sure to write complete mathematical sentences.

♣ 1. $\displaystyle\lim_{x\to 1} \frac{x^2 + x - 2}{3x^2 - x - 2}$

♣ 2. $\displaystyle\lim_{x\to -2} \frac{x^2 + x - 2}{x^3 - 7x - 6}$

EXERCISE 4

In the sentences below, replace the question marks with an equal sign ‘=’ or an appropriate ‘restricted’ equal sign.

♣ 1. $\displaystyle\lim_{x\to 2} \frac{e^x(x-2)}{2-x} \quad ? \quad -\lim_{x\to 2} e^x \quad ? \quad e^2$

♣ 2. $\frac{xe^x - 2e^x}{2-x} \quad ? \quad -e^x$

EXAMPLE
a $\frac{\infty}{\infty}$ situation

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{3/x}{1/x}$$

Solution: Direct substitution yields an $\frac{\infty}{\infty}$ situation. However, for $x \ne 0$ it is true that $\frac{3/x}{1/x} = 3$, and so:

$$\lim_{x\to 0} \frac{3/x}{1/x} = \lim_{x\to 0} 3 = 3$$

EXAMPLE

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{1/x^2}{1/x}$$

Solution:

$$\lim_{x\to 0} \frac{1/x^2}{1/x} = \lim_{x\to 0} \frac{1}{x}$$

Since $\lim_{x\to 0} \frac{1}{x}$ does not exist, neither does $\lim_{x\to 0} \frac{1/x^2}{1/x}$.
Hence, limits of the form $\frac{\infty}{\infty}$ may or may not exist.

EXAMPLE
a 1^∞ *situation*

Consider next the one-sided limit:

$$\lim_{x\to 0^+} (1+x)^{1/x}$$

The chart below suggests what happens as x approaches zero from the positive side:

x	$(1+x)^{1/x}$
0.10000000000000	2.59374246010000
0.01000000000000	2.70481382942153
0.00100000000000	2.71692393223559
0.00010000000000	2.71814592682493
0.00001000000000	2.71826823719230
0.00000100000000	2.71828046909575
0.00000010000000	2.71828169413208
0.00000001000000	2.71828179834736

Based on these results, it should not be surprising that:

$$\lim_{x\to 0^+} (1+x)^{1/x} = e$$

(This limit is sometimes taken as the *definition* of the irrational number e.)

indeterminate forms;
$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, 1^∞

Limits that result in a $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ or 1^∞ situation under direct substitution are called *indeterminate forms*. Such limits *may* or *may not* exist. They always require further analysis, to see what's really happening near the x-value of interest.

EXERCISE 5

Evaluate the following limits, if they exist:

♣ 1. $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x-1}$

♣ 2. $\displaystyle\lim_{x\to 2} \frac{x^3-4x^2-11x+30}{x-2}$

evaluating a limit by rationalizing the numerator

Recall that *rationalizing* means to rewrite in a form with *no radicals.* Thus, to *rationalize the numerator* means to rewrite the numerator in a form that contains no radicals.

Consider $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$. Observe that direct substitution would result in a '$\frac{0}{0}$' situation. To get better insight into this limit, you might want to plug numbers like $-.001$ and $.001$ into $\frac{\sqrt{x+1}-1}{x}$. Does it appear to be getting close to any particular number?

Here's how to "massage" the function into a form that is more suitable for seeing what happens for values *near* zero:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\ &= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{2}\end{aligned}$$

Do you believe this answer, based on your earlier analysis?

EXERCISE 6

Let's investigate the previous example a bit more closely.

♣ 1. Is the sentence

$$\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$$

true? Why or why not? Is a 'restricted' equal sign needed here?

♣ 2. Is the sentence

$$\lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$

true? Why or why not? Is a 'restricted' equal sign needed here?

EXERCISE 7

♣ Evaluate $\displaystyle\lim_{x\to 0} \frac{3x}{\sqrt{x+4}-2}$. If the limit does not exist, so state.

QUICK QUIZ

sample questions

1. What is an 'indeterminate form'? Answer in a complete sentence.
2. Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$. If the limit does not exist, so state. Be sure to write a complete mathematical sentence.
3. Is the sentence
$$\frac{x^2-1}{x-1} = x+1$$
true for ALL values of x? Why or why not?
4. Graph the equation $y = \frac{x^2-1}{x-1}$.
5. Graph the function $f(x) = \frac{x^2-1}{x-1}$.
6. Let f and g be functions of one variable. Give a precise definition of the sentence '$f = g$'.

KEYWORDS *for this section*	*What is meant by* $\frac{0}{0}$*,* $\frac{\pm\infty}{\pm\infty}$ *and* 1^∞ *situations? Restricted equal sign, equality of functions, indeterminate forms, rationalizing the numerator.*

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. For all real numbers x, $\frac{x^3-1}{x-1} = x^2 + x + 1$.
2. For all real numbers x except 1, $\frac{x^3-1}{x-1} = x^2 + x + 1$.
3. Let $f(x) = \frac{x^3-1}{x-1}$ and $g(x) = x^2 + x + 1$. Then, $f = g$.
4. $\lim_{x\to 1} \frac{x^3-1}{x-1}$
5. $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2 + x + 1)$
6. When evaluating a limit $\lim_{x\to c} f(x)$, the function f can be replaced by any function g that agrees with f, except possibly at c.
7. Suppose that whenever $x \ne c$, $f(x) = g(x)$. Then, $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.
8. $f = g$

♣ Evaluate the following limits. If a limit does not exist, so state. Be sure to write complete mathematical sentences.

9. $\displaystyle\lim_{x\to -1} \frac{x^3 + x^2 - 3x - 3}{x + 1}$
10. $\displaystyle\lim_{x\to 1} \frac{x^3 + x^2 - 3x - 3}{x + 1}$
11. $\displaystyle\lim_{x\to 2} \frac{x + 2}{x^2 + 4x + 4}$
12. $\displaystyle\lim_{x\to -2} \frac{x + 2}{x^2 + 4x + 4}$
13. $\displaystyle\lim_{t\to 0^+} (1 + t)^{1/t}$
14. $\displaystyle\lim_{y\to 0^+} (y + 1)^{1/y}$

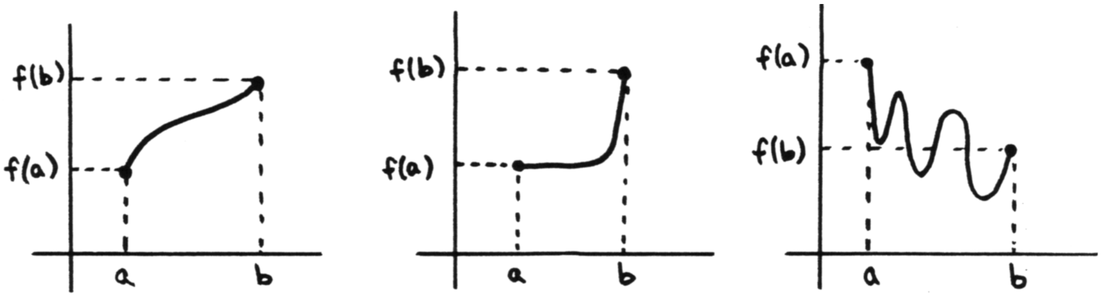
3.6 The Intermediate Value Theorem

Introduction

This section and the next present two fundamental properties of functions that are continuous on a closed interval.

the Intermediate Value Theorem

In this section, the *Intermediate Value Theorem* is discussed. Roughly, it says that a function continuous on $[a, b]$ must take on all values between $f(a)$ and $f(b)$; that is, all *intermediate* values. The idea is simple: since f is continuous on $[a, b]$, whenever the inputs are close, so must be the outputs. So if one begins at the point $(a, f(a))$ and traces the function, it is impossible to reach the point $(b, f(b))$ without passing through all y-values between $f(a)$ and $f(b)$. This idea is illustrated below.



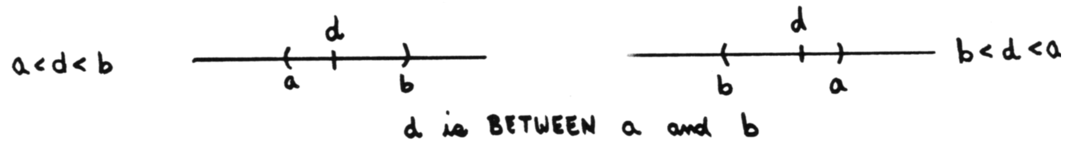
the word 'between';
d is between a and b

Precise statements of the Intermediate Value Theorem usually use the word 'between'. Here's the mathematical meaning of the word 'between':

Given real numbers a and b, one says that d is *between a and b* if:

- $a \ne b$, and
- d lies in the open interval bounded by a and b

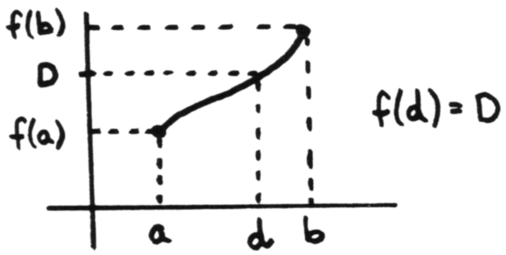
Perhaps a better phrase would be 'strictly between'; however, this is not in common usage.



THEOREM

the Intermediate Value Theorem (IVT)

Let f be continuous on $[a, b]$. If D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.



the IVT is an EXISTENCE theorem

The Intermediate Value Theorem is an *existence* theorem. That is, under appropriate hypotheses, it guarantees the *existence* of a number with a certain property.

Be a good reader!

check that all the hypotheses are really needed

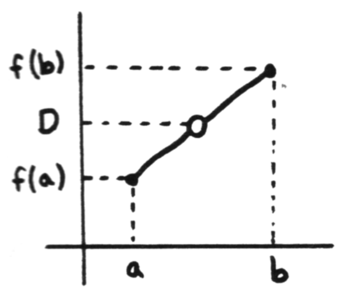
When presented with a theorem, a good reader will 'play with' the hypotheses, to see if they are all really needed to obtain the stated result.

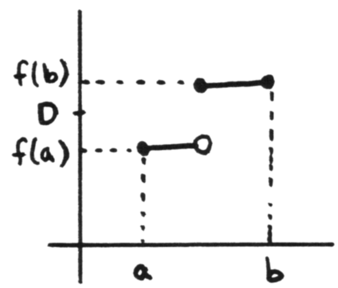
Remember that the *hypotheses* of a theorem are the things that are assumed to be true. The singular form of '*hypotheses*' is '*hypothesis*'. The *hypothesis* of the Intermediate Value Theorem is that f is continuous on $[a,b]$. Remember that this requires that f be defined on $[a,b]$, continuous on (a,b), and well-behaved at the endpoints. Are all these requirements really necessary?

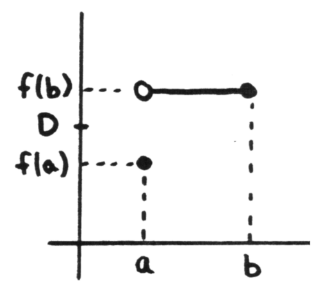
The sketches below illustrate that they are.

The first and second sketches illustrate situations where there is no d between a and b with $f(d)=D$. In both cases, f is not continuous at each point in the open interval (a,b).

The third sketch illustrates another situation where there is no d between a and b with $f(d)=D$. Here, f is not well-behaved at the left-hand endpoint, a.



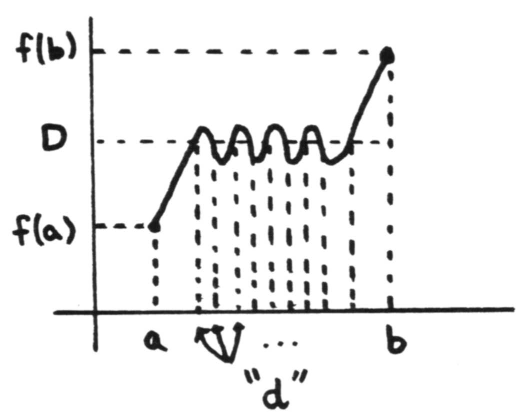




d may NOT be unique

Note that the Intermediate Value Theorem is NOT a uniqueness theorem. It only guarantees the *existence* of a certain number; it makes no claims about 'how many' such numbers there may be.

Indeed, the sketch below illustrates that an 'intermediate value' may be taken on ANY given number of times.



EXERCISE 1
practice with the IVT

♣ 1. Sketch the graph of a function f that satisfies the following requirements: f is continuous on $[a, b]$, $f(a) = 3$, $f(b) = 4$. Must there be a number $c \in (a, b)$ with $f(c) = \pi$? Why or why not? If so, label it on your graph.

♣ 2. Suppose f is continuous at each point in $(-5, 5)$. Must f be continuous on the interval $[-2, 2]$? Why or why not? Can you generalize this example?

♣ 3. Suppose f is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$. Must there exist a number c between a and b with $f(c) = 0$? Why or why not?

♣ 4. Suppose that $f(0) = 2$, $f(1) = 3$, but there is no number between 0 and 1 with function value 2.5. What conclusion, if any, can you make?

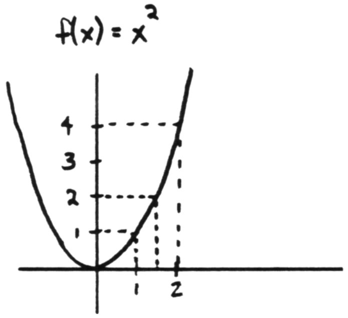
EXAMPLE
using the IVT to guarantee existence of a solution to an equation, and estimate its value

This example illustrates a very common use of the Intermediate Value Theorem: to guarantee existence of a solution to a given equation, and estimate the value of this solution.

Consider the equation $x^2 = 2$. We want a (real) number which, when squared, yields the number 2. How do we know that such a number exists? The Intermediate Value Theorem can be used to guarantee a solution, as follows:

Define $f(x) := x^2$. Then f is continuous on any interval $[a, b]$. We want to find a number x for which $f(x) = 2$. Since $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$, there must exist $d_1 \in (1, 2)$ with $f(d_1) = 2$ (Why?) Thus, $d_1^2 = 2$, so d_1 is a solution of the equation $x^2 = 2$. Now we have existence of a solution to this equation that lies between 1 and 2; and it has been approximated within 1 unit. That is, the solution lies in an interval of length 1.

Knowing that a solution d_1 lies between 1 and 2 is okay, but it would be nice to get a better estimate of the number d_1. So, let's refine our approach. Let's make a table of some functions values:



a solution lies between 1.4 and 1.5

x	x^2
1.0000	1.0000
1.1000	1.2100
1.2000	1.4400
1.3000	1.6900
1.4000	1.9600
1.5000	2.2500
1.6000	2.5600
1.7000	2.8900
1.8000	3.2400
1.9000	3.6100
2.0000	4.0000

Since $f(1.4) = (1.4)^2 = 1.96$ and $f(1.5) = (1.5)^2 = 2.25$, the intermediate value theorem guarantees the existence of $d_2 \in (1.4, 1.5)$ with $d_2^2 = 2$. The solution has been approximated within 0.1.

One more time. Here's another table of function values:

x	x^2
1.4000	1.9600
1.4100	1.9881
1.4200	2.0164
1.4300	2.0449
1.4400	2.0736
1.4500	2.1025
1.4600	2.1316
1.4700	2.1609
1.4800	2.1904
1.4900	2.2201
1.5000	2.2500

a solution to $x^2 = 2$ lies between 1.41 and 1.42

Since $f(1.41) = 1.9881$ and $f(1.42) = 2.0164$, there must exist $d_3 \in (1.41, 1.42)$ with $d_3^2 = 2$. The solution has been approximated within 0.01. It is easy to see how this process can continue.

The exact solution to $x^2 = 2$ in the interval being investigated is, of course, $\sqrt{2} \approx 1.4142$.

EXERCISE 2

♣ 1. Use the Intermediate Value theorem to find a solution to the equation $x^4 - 8x^2 = -15$ that lies in the interval $[0, 2]$. Approximate the solution to within 0.01; that is, get an interval of length .01 that contains a solution.

♣ 2. Find another solution to $x^4 - 8x^2 = -15$ that lies in the interval $[2, 3]$. Approximate it to within 0.1.

♣ 3. Find the exact solutions to the equation $x^4 - 8x^2 = -15$. Be sure to write a complete mathematical sentence. Which of these solutions were you finding in parts (1) and (2)?

EXAMPLE

Now consider the equation $x^3 = 2x + 3$. Suppose it is desired to locate and estimate a solution of this equation. Since the variable x appears on both sides of the equation, the approach taken in the previous example must be modified. The notion of *equivalence* comes to the rescue.

Since

$$x^3 = 2x + 3 \quad \iff \quad x^3 - 2x - 3 = 0 \ ,$$

these two equations have exactly the same truth values. They are interchangeable. We can work with whichever one is easier to work with. In this case, it is easier to work with $x^3 - 2x - 3 = 0$.

If we can find a value of x that makes $x^3 - 2x - 3 = 0$ true, then this same x will make $x^3 = 2x + 3$ true.

Define $f(x) := x^3 - 2x - 3$. A quick table shows that $f(1) = -4$ and $f(2) = 1$. Thus, the intermediate value theorem guarantees the existence of a number $d_1 \in (1, 2)$ with $f(d_1) = 0$. That is, $d_1^3 - 2d_1 - 3 = 0$.

x	f(x)
1	-4
2	1
3	18
4	53
5	112
6	201
7	326
8	493
9	708
10	977

1.0000	-4.0000
1.1000	-3.8690
1.2000	-3.6720
1.3000	-3.4030
1.4000	-3.0560
1.5000	-2.6250
1.6000	-2.1040
1.7000	-1.4870
1.8000	-0.7680
1.9000	0.0590
2.0000	1.0000

1.8000	-0.7680
1.8100	-0.6903
1.8200	-0.6114
1.8300	-0.5315
1.8400	-0.4505
1.8500	-0.3684
1.8600	-0.2851
1.8700	-0.2008
1.8800	-0.1153
1.8900	-0.0287
1.9000	0.0590

Another table of values shows that $f(1.8) = -0.7680$ and $f(1.9) = 0.0590$, so there must be a solution d_2 in $(1.8, 1.9)$.

Another table shows that $f(1.89) = -0.0287$ and $f(1.9) = 0.0590$. Thus, there must be a solution d_3 in $(1.89, 1.90)$.

Note that for this equation, it is not easy to find an exact solution.

EXERCISE 3

♣ Use the Intermediate Value Theorem to show the existence of a solution to the equation $x^3 - x^2 = 5x - 5$ that lies between 2 and 3. Then, approximate this solution to within 0.01.

EXERCISE 4

♣ 1. Suppose that f is continuous on $[a, b]$, and $f(a) = f(b) := D$. Must there be $d \in [a, b]$ with $f(d) = D$?

♣ 2. Suppose that f is continuous on $[a, b]$, and $f(a) = f(b) := D$. Must there be $d \in (a, b)$ with $f(d) = D$? Justify your answer. Be sure to write complete mathematical sentences.

EXERCISE 5

♣ 1. On a number line, show c, d and $\frac{c+d}{2}$ for various choices of c and d.

♣ 2. Let c and d be real numbers with $c < d$. Prove that $\frac{c+d}{2}$ is exactly half-way between c and d.

♣ 3. Suppose that f is continuous on $[a, b]$. Must there exist $d \in [a, b]$ with $f(d) = \frac{f(a)+f(b)}{2}$? Why or why not?

An 'implication' is a sentence of the form: If A, then B

The second sentence in the Intermediate Value Theorem is:

If D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.

This is a sentence of the form

If A, then B

IF $\boxed{D \text{ is any number between } f(a) \text{ and } f(b),}$

THEN $\boxed{\text{there exists a number } d \text{ between } a \text{ and } b \text{ with } f(d) = D.}$

A sentence of the form

If A, then B

is called an *implication.* Implications are the most common type of mathematical sentence. Therefore, to understand mathematics, you must understand implications. In this section, the study of implications begins. This study will continue throughout the text.

intuition for the sentence 'If A, then B'

Everyone is familiar with sentences of the form '*If A, then B*' because they are as common in English as they are in mathematics. Fortunately, there are a lot of similarities between the English and mathematical meanings. So let's review the English meaning, and then move on to the (more precise) mathematical meaning.

Suppose a person is trying to sell raffle tickets, and says to you,

If your ticket is chosen, then you'll get $1,000.

This is an (English) sentence of the form '*If A, then B*'. You know what it means: if the first part of the sentence is true—that is, if your ticket is chosen—then the second part of the sentence will also be true—you will get $1,000.

Suppose the big day arrives, and your ticket *isn't* selected. The first part of the sentence is *not true* in this case. And, you don't get $1,000. Does this make the person who sold you the raffle ticket into a liar? Of course not! The sentence

'If your ticket is chosen, then you'll get $1,000.'

is still true. It's just that this sentence only guarantees us that the second part will be true IF the first part is true!

another English example of an 'If A, then B' sentence

Here's another example. Suppose your parents say,

If you get an 'A' in calculus, then we'll take you out to dinner.

Now, if you get an 'A' and your parents *don't* take you out, then they've broken their promise. (In English, a sentence that is *false* is called a *broken promise* or a *lie*!) However, suppose you get a 'B'. Your parents know you worked really, really hard and decide to take you out anyway. Now, did they break their promise? Of course not. They promised that if you DO get an 'A', then they'll take you out. Their promise didn't give any information about what might happen if you *don't* get an 'A'.

In both of the preceding examples, it appears that in order for the sentence '*If A, then B*' to be TRUE, it must be that whenever A is true, B must also be true.

a true mathematical sentence of the form 'If A, then B'

Here's a very simple mathematical implication:

$$\text{If } x = 2, \text{ then } x^2 = 4$$

Use your intuition: would you want to say that this sentence is true or false? Based on English sentences of the same form, you'd probably want to say that it is true: because whenever the sentence $x = 2$ is true, then the sentence $x^2 = 4$ is also true. The only number that makes '$x = 2$' true is 2, and 2^2 is equal to 4.

Indeed, to a mathematician, the sentence

$$\text{If } x = 2, \text{ then } x^2 = 4$$

is true. This is because WHENEVER the sentence $x = 2$ is true, the sentence $x^2 = 4$ is also true.

★★ *implicit universal quantifier*

Really, the sentence

$$\text{If } x = 2, \text{ then } x^2 = 4$$

is a (true) implicit generalization,

$$\text{For all } x, \text{ if } x = 2 \text{ then } x^2 = 4.$$

However, it is common usage to leave the universal quantifier 'For all' implicit, rather than explicit.

a false mathematical sentence of the form 'If A, then B'

Now consider the implication:

$$\text{If } x^2 = 4, \text{ then } x = 2$$

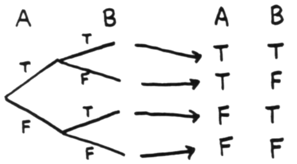
Where does your intuition lead you? Would you want to call this sentence *true* or *false*? In order to be true, you probably want to be assured that WHENEVER $x^2 = 4$ is true, then $x = 2$ must also be true. This is not the case. It is possible to choose x such that $x^2 = 4$ is true, but $x = 2$ is false. Just choose x to be -2. Then, $(-2)^2 = 4$ is true, but $-2 = 2$ is false. Thus, the mathematical sentence

$$\text{If } x^2 = 4, \text{ then } x = 2$$

is false.

the truth table for the sentence If A, then B

Now let's investigate the mathematical sentence '*If A then B*' even more precisely. The *sub-sentences* A and B can be true or false, and there are four possible combinations:



It is conventional to give the truth values of the sentence '*If A, then B*' by using a *truth table*:

the only way the sentence is FALSE, is if A is TRUE, and B is FALSE

A	B	If A, then B
T	T	T
T	F	F
F	T	T
F	F	T

the sentence is TRUE

Look at the truth table. Under what conditions is the sentence *If A, then B* TRUE? Well, it is true if A is true, and B is true. The first line of the truth table tells us this. This is not surprising.

The sentence '*If A, then B*' is also true if A is false. The third and fourth lines of the truth table tell us this. This is in perfect harmony with the English usage. If your parents say,

If you get an 'A' in calculus, then we'll buy you a dinner.

and then buy you a dinner when you get a 'B', they didn't lie. Their statement was still true.

'hypothesis' and 'conclusion' of an implication

In the sentence '*If A, then B*', A is called the *hypothesis* (of the implication) and B is called the *conclusion* (of the implication).

The third and fourth lines of the truth table tell us that if the hypothesis of an implication is false, then the sentence '*If A, then B*' is automatically true.

IMPORTANT!

To check that an implication is true, one need only check that whenever A is true, so is B

Here's an extremely important consequence of the definition of the sentence '*If A, then B*'. To see if a sentence of this form is TRUE, we need only verify that whenever A is true, so is B. We don't bother to check what happens if A is false: because if A is false, the sentence is automatically true.

EXERCISE 6

the word 'hypothesis' in mathematics

♣ We have run across the word *hypothesis* a few times now. We talked earlier about the *hypotheses* of a theorem; now we have the *hypothesis* of an implication. Does it make sense to use the same word in both situations? Comment.

An alternate form of the sentence 'If A, then B';

$A \implies B$

Since implications are extremely common in mathematics, it should not be surprising that there is more than one way to say the same thing. The sentence '*If A, then B*' can also be written in the form

$$A \implies B$$

and read as '*A implies B*'.

The next example gives some practice with implications.

EXAMPLE

practice with implications

Determine if the following implications are TRUE or FALSE. If an implication involves a variable, then in order to be true, it must be true for all possible choices of the variable.

- If $2 = 1$, then $2 = 5$

This sentence is true. Here, A is false, B is false, and '*If A, then B*' is true (line 4 of the truth table). Whenever the hypothesis of an implication is false, the implication is automatically true. Some students have trouble with this: they can't believe that a sentence can be *true* with so much *false* stuff floating around!

- $x > 2 \implies x > 1$

This sentence is true. Whenever x is a number greater than 2, then it is also greater than 1. That is, whenever $x > 2$ is true, then $x > 1$ must also be true. Note that this sentence is true for ALL real numbers x. In particular, if x is 1, then the sentence

$$1 > 2 \implies 1 > 1$$

is automatically true, because the hypothesis is false.

- If $x > 1$, then $x > 2$

This sentence is false. It is possible to make $x > 1$ true, but $x > 2$ false. Choose, say, $x = 1.5$.

showing that an implication is false

If an implication involves a variable, then to show that it is FALSE, you must produce a *specific* choice for the variable that makes the hypothesis TRUE, but the conclusion FALSE. Here's the format to use in such a situation:

Problem: TRUE or FALSE: $x > 1 \quad \implies \quad x > 2$

Solution: FALSE. Let $x = 1.5$. Then, the hypothesis

$$1.5 > 1$$

is true, but the conclusion

$$1.5 > 2$$

is false.

counterexample

A specific choice of variable(s) for which a sentence is false is called a *counterexample*.

Problem: Decide if the sentence 'If $x > 1$, then $x > 2$' is true or false. If false, give a counterexample.

Solution: The sentence is false. Let $x = 1.9$. Then, the hypothesis $1.9 > 1$ is true, but the conclusion $1.9 > 2$ is false.

EXAMPLE

Problem: Is the sentence

$$y^2 = 9 \implies y = 3$$

true or false? If false, give a counterexample.

Solution: The sentence is false. Let $y = -3$. Then, the hypothesis $(-3)^2 = 9$ is true, but the conclusion $-3 = 3$ is false.

EXERCISE 7

Decide if the following mathematical sentences are true or false. If false, give a counterexample, using the form illustrated above.

♣ 1. If $x = 3$, then $x^2 = 9$

♣ 2. If $x^2 = 9$, then $x = 3$

♣ 3. $x = 2 \implies |x| = 2$

♣ 4. If $|x| = 2$, then $x = 2$

♣ 5. $a < b \implies |a| < |b|$

♣ 6. If $0 < a < b$, then $|a| < |b|$

EXERCISE 8

For this entire exercise, assume that the sentence $A \implies B$ is TRUE.

♣ 1. What (if anything) can you conclude about the truth value of A?

♣ 2. What (if anything) can you conclude about the truth value of B?

♣ 3. Suppose you know that A is true. What (if anything) can you conclude about the truth value of B?

♣ 4. Suppose you know that B is true. What (if anything) can you conclude about the truth value of A?

♣ 5. Suppose you know that A is false. What (if anything) can you conclude about the truth value of B?

♣ 6. Suppose you know that B is false. What (if anything) can you conclude about the truth value of A?

QUICK QUIZ
sample questions

1 Give a precise statement of the Intermediate Value Theorem.

2 Suppose that f is continuous on the interval $[1, 3]$; $f(1)$ is negative, and $f(3)$ is positive. Must the function f take on the value 0 on $[1, 3]$? Why or why not?

3 TRUE or FALSE: $1 = 2 \implies 3 = 4$.

4 TRUE or FALSE: If $|x| = 1$, then $x = 1$. If the sentence is false, give a counterexample.

5 Give the truth table for the mathematical sentence $A \implies B$.

KEYWORDS
for this section

The Intermediate Value Theorem, use of the word 'between', using the IVT to guarantee and estimate solutions to equations. Implications: notation, truth table, hypothesis, conclusion, counterexample.

END-OF-SECTION EXERCISES

Determine if the following implications are TRUE or FALSE. If false, give a counterexample. The context will determine if the variable(s) used are numbers, functions, sentences, or sets.

Remember that if a sentence involves a variable, then to be TRUE, the sentence must be true for all possible choices of the variable.

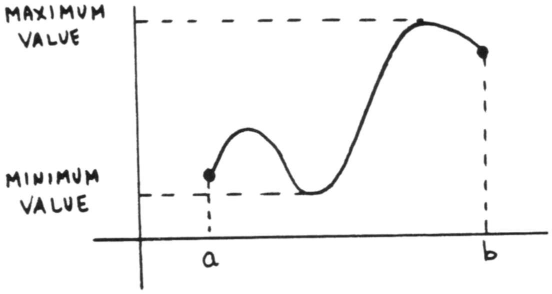
1. If f is continuous on $[a, b]$ and D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.
2. If f is continuous on $[0, 2]$, $f(0) = 1$, and $f(2) = 4$, then there exists $d \in (0, 2)$ with $f(d) = 3$.
3. If A is false, then the sentence $A \implies B$ is true.
4. If B is false, then the sentence $A \implies B$ is false.
5. If B is true, then the sentence 'If A, then B' is true.
6. If A is true, then the sentence 'If A, then B' is true.
7. If $|t| = 0$, then $t = 0$
8. If $|t| = 1$, then $t = 1$
9. If $t = 1$, then $|t| = 1$
10. If $t = -1$, then $|t| = 1$

3.7 The Max-Min Theorem

Introduction

This section presents a second fundamental property of functions that are continuous on a closed interval. Roughly, the *Max-Min Theorem* says that a function continuous on $[a, b]$ must attain both a maximum and minimum value on this interval.

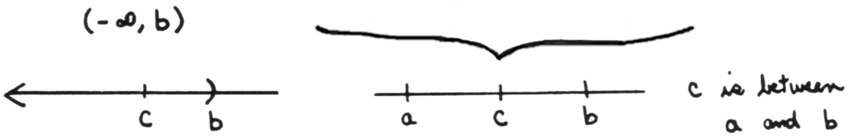
We begin with a discussion of maximum and minimum values on an interval.



interval I

In the next definition, I is an interval of real numbers containing c. For example, I may be of any of these forms:

(a, b) $[a, b]$ $(a, b]$ $[a, b)$ $(a, c]$ $[a, c]$ $[c, b)$ $[c, b]$



DEFINITION *minimum of f on I;* *maximum of f on I;* *extreme values of f on I*	Let f be defined on an interval I containing c. The number $f(c)$ is a *minimum (value) of f on I* $\iff$ $f(c) \le f(x) \ \ \forall\, x \in I$ The number $f(c)$ is a *maximum (value) of f on I* $\iff$ $f(c) \ge f(x) \ \ \forall\, x \in I$ When such maximum or minimum values do occur, they are called *extreme values of f on I*. Note that a 'value' is a *number*. One is usually interested not only in the number $f(c)$ but also the place or places where this number occurs. Such a point $(c, f(c))$ is called an *extreme (maximum or minimum) point* of f on I.

interpreting this definition

This definition assigns meaning to the phrase '$f(c)$ *is a minimum of* f *on* I'. The assigned meaning is this: $f(c) \le f(x) \ \ \forall\, x \in I$. That is, no matter what value of x is chosen from I, it must be that $f(c) \le f(x)$. Thus, $f(c)$ is the least number taken on by f over the interval I.

The definition can also be used 'from right to left'. That is, if it is known that $f(c) \le f(x) \ \ \forall\, x \in I$, then, by this definition, $f(c)$ is a minimum of f on I.

Definitions are *always* statements of *equivalence*. This definition states that the two sentences

$$f(c) \text{ is a minimum of } f \text{ on } I$$

and

$$f(c) \le f(x) \ \ \forall\, x \in I$$

are *equivalent*, and hence can be used interchangeably.

★ *every definition is (either implicitly or explicitly) a statement of equivalence*

Every definition is a statement of equivalence. Since mathematicians know this fact, they often get a bit sloppy about how they state definitions. It is common to see things like this:

DEFINITION. If object x has property P, then x is called a *glob*.

Or,

DEFINITION. The object x is called a *glob* if it has property P.

What the author really means here is:

DEFINITION. x has property $P \iff x$ is a glob

So: if x has property P, then it is a glob. And, if x is a glob, then x has property P. The two sentences are interchangeable.

That is, although definitions are commonly stated as sentences of the form '*If A, then B*', they are ALWAYS really statements of equivalence.

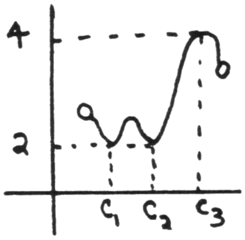
This is NOT true of theorems, however!

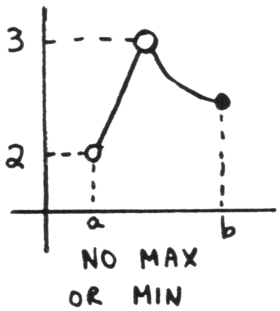
extreme values may or may not occur

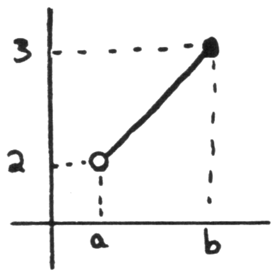
The following examples show that extreme values on an interval I may or may not exist.

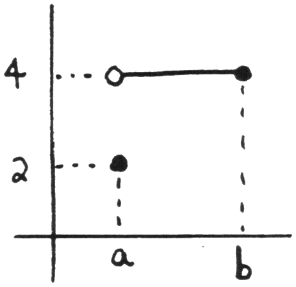
In the first sketch below, the minimum value of f on $I := (a, b)$ is 2, and is attained in two places; $f(c_1) = f(c_2) = 2$. Thus, $(c_1, 2)$ and $(c_2, 2)$ are both minimum points of f on I. Also, the maximum value of f on I is 4; $(c_3, 4)$ is a maximum point of f on I.

In the second sketch, take I to be the interval $(a, b]$. There is no minimum value. The number 2 is 'trying' to be the minimum value, but is never taken on. That is, there is no $c \in I$ with $f(c) = 2$. The only outputs taken on are those in the interval $(2, 3)$: does this set $(2, 3)$ have a least element? No! One can 'reach into' the output pile $(2, 3)$ and choose a number as close to 2 as desired; and then reach in again and choose a number even closer to 2. Since the number 2 is NOT in this pile, there is no least element. There is also no maximum value. ♣ Why?









In the third sketch, take $I := (a, b]$. The maximum value of f on I is 3; the point $(b, 3)$ is a maximum point. There is no minimum value.

In the last sketch, take $I := [a, b]$. The minimum value is 2; the point $(a, 2)$ is the only minimum point. The maximum value is 4, and is attained (taken on) by every $x \in (a, b]$. That is, the points $(x, 4)$ are all maximum points, for every $x \in (a, b]$.

Observe, in all these examples, that whenever a maximum or minimum value FAILS to exist, it is due either to a discontinuity of the function, or a missing endpoint.

EXERCISE 1

practice with extreme values

For each of the following, make a sketch illustrating a function f and an interval I satisfying the stated requirements:

♣ 1. I is an open interval, f is continuous at every point in I, 3 is the minimum value on I, there is no maximum value

♣ 2. I is neither open nor closed, f is not continuous at every point in I, -1 is the minimum value on I, 2 is the maximum value on I

♣ 3. f is defined on $[a, b]$, $\lim_{x\to a^+} f(x) = 2$, the minimum value of f on I is 0, the maximum value of f on I is 2

EXERCISE 2
minimum values versus minimum points

♣ 1. If a function f has a minimum value on I, must this minimum value be unique? That is, can there be two different numbers, both of which are minimum values on I?

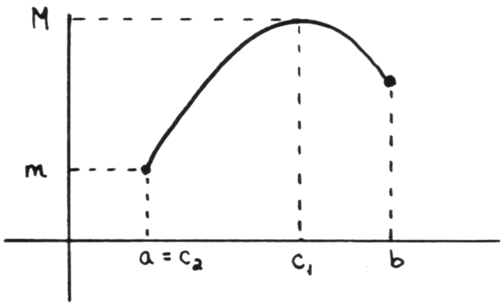
♣ 2. If a function f has a minimum point on I, must this point be unique? Or, can there be more than one point where the minimum value is attained?

conditions under which extreme values will always exist

The next theorem tells us that if a function is *continuous* on a *closed interval*, then it *must* take on both a maximum and minimum value on this interval.

THEOREM
the Max-Min Theorem

If a function f is continuous on a closed interval $[a, b]$, then f must take on both a maximum value M and a minimum value m on $[a, b]$. That is, there must exist $c_1 \in [a, b]$ for which $f(c_1) = M$. Also, there must exist $c_2 \in [a, b]$ for which $f(c_2) = m$.



★★
idea of proof of the Max-Min Theorem

To prove the Max-Min Theorem, one first shows that every continuous function on a closed interval is bounded on this interval. Let M be the least upper bound of the set $\{f(x) \mid x \in [a, b]\}$, and define:

$$g(x) := \frac{1}{M - f(x)}$$

Argue by contradiction. If f does NOT take on the value M, then g is continuous on $[a, b]$, and hence must be bounded on $[a, b]$. But, g is NOT bounded on $[a, b]$, since in this case $f(x)$ must take on values arbitrarily close to M. This provides the desired contradiction.

★★
a more general topological result

The Max-Min Theorem is a special case of an extremely important topological theorem: every continuous function on a compact set attains both a maximum and a minimum.

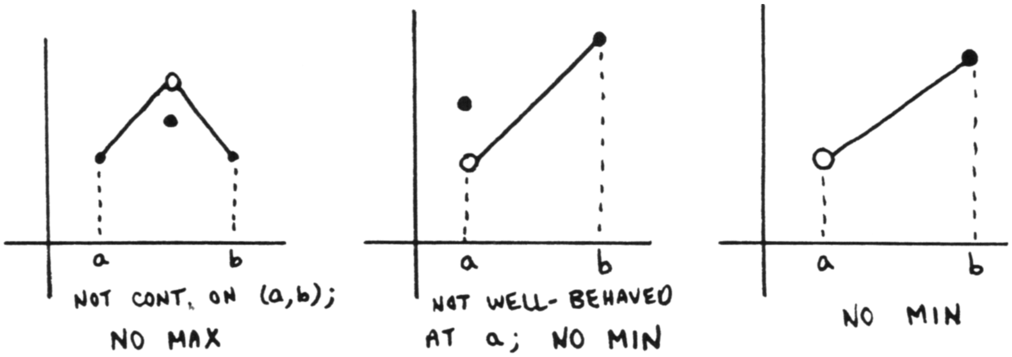
check that all the hypotheses are needed

To use the Max-Min Theorem, one must have a function f that is continuous on a closed interval $[a, b]$. That is, f must be defined on $[a, b]$, continuous on the open interval (a, b), and well-behaved at the endpoints. Take away any of these conditions, and extreme values are no longer guaranteed.

The first sketch below illustrates that continuity on (a, b) is needed.

The second sketch illustrates that the function must be well-behaved at the endpoints.

The third sketch illustrates that the function must be defined on a closed interval.



EXERCISE 3

♣ 1. Sketch the graph of a function that is NOT continuous on $[a, b]$, attains a minimum on $[a, b]$, does not attain a maximum on $[a, b]$.

♣ 2. Sketch the graph of a function that is NOT continuous on $[a, b]$, attains a maximum on $[a, b]$, but not a minimum.

♣ 3. Sketch the graph of a function that is NOT continuous on $[a, b]$, and attains both a maximum and minimum on $[a, b]$.

♣ 4. Sketch the graph of a function that is NOT continuous on $[a, b]$, and does not attain a maximum or minimum on $[a, b]$.

♣ 5. If f is NOT continuous on $[a, b]$, can the Max-Min Theorem be used to reach any conclusion about extreme values of f on $[a, b]$?

EXERCISE 4

♣ 1. Suppose you are given a function f and a closed interval I, and it is known that f does NOT attain a maximum value on I. Is f continuous on I?

♣ 2. Suppose f is defined on $[a, b]$ and continuous on (a, b). It is known that f does NOT attain a maximum value on $[a, b]$. Make some conclusion about the behavior of f on $[a, b]$.

In the next two chapters, calculus tools are developed to help locate maximum and minimum values, when they exist.

more on implications

This section is concluded with some additional study of *implications*. Note that the form of the Max-Min Theorem is an implication:

IF $\boxed{f \text{ is continuous on a closed interval } [a, b],}$

THEN $\boxed{f \text{ must take on both a maximum and minimum value on } I.}$

the 'contrapositive' of an implication

The *contrapositive* of the implication

$$\text{If } A\text{, then } B$$

is another implication:

$$\text{If (not } B\text{), then (not } A\text{)}$$

EXAMPLE

finding contrapositives

The contrapositive of the true implication

$$x = 1 \implies x^2 = 1$$

is:

$$x^2 \neq 1 \implies x \neq 1$$

EXAMPLE

finding contrapositives

The contrapositive of the true implication

$$\text{If } f \text{ is continuous on } [a,b]\text{, then } f \text{ attains a maximum value on } [a,b]$$

is:

If f does not attain a maximum value on $[a,b]$, then f is not continuous on $[a,b]$

relationship between an implication and its contrapositive

Is there any nice relationship between an implication and its contrapositive? Where does intuition lead you? Roughly, a true sentence '*If A, then B*' says that whenever A is true, B must also be true. So if B isn't true, then A can't be true; because if A WERE true, B would have to be true. This is the intuition behind the result:

An implication is equivalent to its contrapositive.

That is:

$$\text{If } A\text{, then } B \quad \iff \quad \text{If (not } B\text{), then (not } A\text{)}$$

In alternate notation:

$$A \Rightarrow B \quad \iff \quad \text{not } B \Rightarrow \text{not } A$$

The proof is easy: just show that both sentences have precisely the same truth values, regardless of the truth values of A and B!

not B	not A	A	B	A ⇒ B	NOT B ⇒ NOT A
F	F	T	T	T	T
T	F	T	F	F	F
F	T	F	T	T	T
T	T	F	F	T	T

IDENTICAL !!

EXERCISE 5

Determine if the following implications are true or false. Then, find their contrapositives.

♣ 1. If $x \in [1,2]$, then $x > 0$

♣ 2. If $x \in [0,1)$, then $x > 0$

♣ 3. $x \in [0,1) \implies x \ge 0$

♣ 4. If f is continuous on $[a,b]$, then f attains a minimum value on $[a,b]$.

♣ 5. Suppose that $a < b$, and D is a number between $f(a)$ and $f(b)$. Investigate this implication concerning f:

If f is continuous on $[a,b]$, then there exists a number $c \in [a,b]$ with $f(c) = D$.

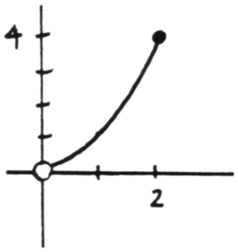
QUICK QUIZ
sample questions

1. Let f be defined on an interval I containing c. Give a precise definition of the sentence, 'the number $f(c)$ is a maximum of f on I'.
2. Sketch the graph of a function f that is defined on $I := [1,3]$, has a minimum value on I, but has no maximum value on I.
3. Sketch the graph of a function f that is continuous on (a,b) but attains NO maximum or minimum value on (a,b).
4. Give a precise statement of the Max-Min Theorem.
5. What is the contrapositive of $A \implies B$? What is the relationship between an implication and its contrapositive?

KEYWORDS
for this section

Extreme values for a function on an interval, extreme values may or may not exist, extreme values versus extreme points, the Max-Min Theorem, the contrapositive of an implication.

END-OF-SECTION EXERCISES



♣ Sketch the graph of each function f on the given interval I.

♣ Find the maximum and minimum value of f on I, if they exist.

♣ List all maximum points and minimum points (if any).

Be sure to answer using complete mathematical sentences. Here's a sample problem.

SAMPLE: $f(x) = x^2$, $I = (0, 2]$

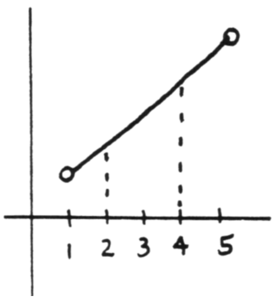
SOLUTION: The graph is shown at left. The maximum value of f on I is 4; there is no minimum value. The only maximum point is $(2, 4)$.

1. $f(x) = x^2$, $I = [0, 2)$
2. $f(x) = x^2$, $I = (0, 2)$
3. $f(x) = 4$, $I = \mathbb{R}$
4. $f(x) = -2$, $I = (0, \infty)$
5. $f(x) = (x - 2)^2 + 1$, $I = (1, 3)$
6. $f(x) = (x - 2)^2 + 1$, $I = [1, 3)$
7. $f(x) = |2x + 1|$, $I = (-1, 2]$
8. $f(x) = |2x + 1|$, $I = [-\frac{3}{4}, 0)$

♣ Determine if the following implications are true or false.

♣ If an implication is false, give a counterexample.

♣ Then, find the contrapositive of the implication.



Here's a sample problem:

SAMPLE: If f is continuous on $(1, 5)$, then f attains a maximum value on $(2, 4)$

SOLUTION: FALSE. Let f be the function graphed at left. Then the hypothesis 'f is continuous on $(1, 5)$' is TRUE, but the conclusion 'f attains a maximum value on $(2, 4)$' is FALSE.

The contrapositive is: If f does not attain a maximum value on $(2, 4)$, then f is not continuous on $(1, 5)$. (The contrapositive is of course also false.)

9. If f is continuous on $[a, b]$, then f attains a maximum value on $[a, b]$
10. If f does not attain a maximum value on $[a, b]$, then f is not continuous on $[a, b]$
11. If f is continuous on $(a, b]$, then f attains a maximum value on $(a, b]$
12. If f is continuous on $[a, b)$, then f attains a minimum value on $[a, b)$
13. If f is continuous on $(0, 5)$, then f attains both a maximum and minimum value on $[1, 2]$
14. If f is continuous on $(-5, -1)$, then f attains both a maximum and minimum value on $(-4, -2)$
15. If f is continuous on $\mathbb{R}$, then f attains a maximum value on $\mathbb{R}$; that is, there exists $c \in \mathbb{R}$ such that:
$$f(x) \le f(c) \quad \forall \quad x \in \mathbb{R}$$
16. If f is continuous on $\mathbb{R}$, then f attains a minimum value on $\mathbb{R}$; that is, there exists $c \in \mathbb{R}$ such that:
$$f(c) \le f(x) \quad \forall \quad x \in \mathbb{R}$$

NAME

SAMPLE TEST, worth 100 points, Chapter 3

Show all work that leads to your answers. Good luck!

5 pts	Give a precise (ϵ–δ) definition of the mathematical sentence: $\lim_{x \to c} f(x) = l$

15 pts

All the following questions have to do with the true limit statement $\lim_{x \to 2} x^2 = 4$.

(2 pts) Very roughly, $\lim_{x \to 2} x^2 = 4$ says (fill in the blanks):

Whenever ___________ is close to ___________, it must be that ___________ is close to ___________.

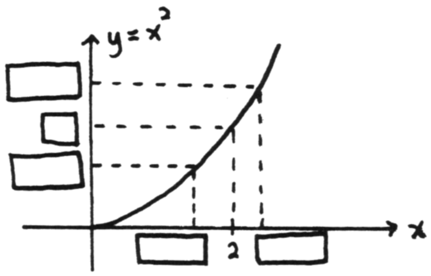
(2 pts) More precisely, the sentence says (fill in the blanks):

It is possible to get ___________ as close to ___________ as desired, merely by requiring that ___________ be sufficiently close to ___________.

The precise definition of $\lim_{x \to 2} x^2 = 4$ involves the sentence '$0 < |x - 2| < \delta$'.

(3 pts) For what value(s) of x is the sentence $0 < |x-2| < \delta$ true? Show these numbers on the number line below.

(8 pts) Fill in the boxes on the graph below with appropriate numbers/symbols that illustrate the '4-step process' showing that the limit statement $\lim_{x \to 2} x^2 = 4$ is true. Be sure to conclude with a 'δ that works'.



12 pts

TRUE or FALSE. (2 pts each) (Circle the correct response.)

T	F	For all real numbers a and b, $|a + b| \le |a| + |b|$.
T	F	If direct substitution into $\lim_{x \to c} f(x)$ yields a '$\frac{0}{0}$' situation, then the limit does not exist.
T	F	$(2 = 1)$ and $(1 + 1 = 2)$ $\implies$ $4 = 3$
T	F	If an interval of real numbers is not open, then it is closed.
T	F	If f is continuous on $[a, b]$, then f must attain a maximum value on $[a, b]$.
T	F	If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ for which $f(c) = \frac{f(a)+f(b)}{2}$.

8 pts

Evaluate the following limits, if they exist:

(3 pts) $\lim_{t \to -1} (t^3 - 2t^2 + 3)$

(5 pts) $\lim_{x \to 2} \dfrac{x^3 - x^2 - x - 2}{x^2 - 4}$

4 pts

Prove that an implication $A \implies B$ is equivalent to its contrapositive. (HINT: Make a truth table! I've got you started.)

A	B	A ⇒ B	
T	T		
T	F		
F	T		
F	F		

6 pts

(4 pts) The implication

$$\text{IF } x^2 = 4, \text{ THEN } x = 2$$

is false. Give a counterexample, by filling in the blanks:

Let $x =$ __________. Then __________________ is true, but __________________ is false.

(2 pts) TRUE or FALSE:

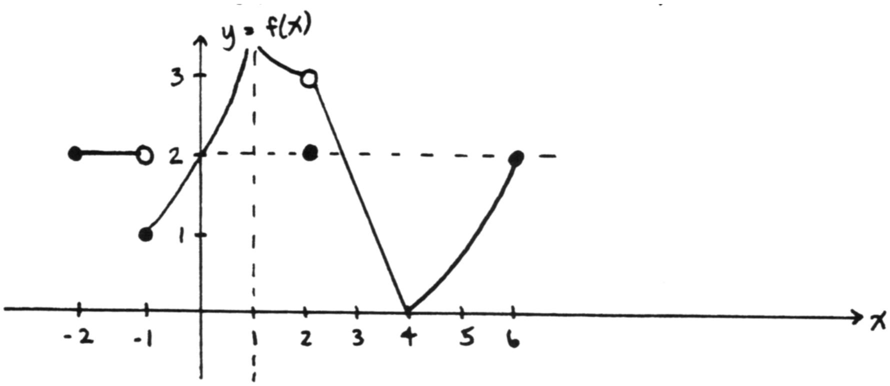
$$x = 2 \implies x^2 = 4$$

6 pts

Graph $f(x) = \frac{x^2-1}{x-1}$ in the space provided below.

24 pts

All the following questions refer to the graph of the function f shown below.



Find the following numbers/sets, if they exist. If they do not exist, so state. Be sure to write complete mathematical sentences. (2 pts each)

$f(-1)$ $\quad$ $\lim_{x\to -1} f(x)$ $\quad$ $\lim_{x\to -1.01} f(x)$

$\mathcal{D}(f)$ $\quad$ $\mathcal{R}(f)$ $\quad$ $\lim_{x\to -1^+} f(x)$

$\lim_{x\to 2} f(x)$ $\quad$ $\lim_{x\to -1^-} f(x)$ $\quad$ $f(-1.1)$

(2 pts) $\{x \mid f \text{ has a nonremovable discontinuity at } x\}$

(2 pts) $\{x \mid f(x) < 0\}$

(2 pts) Is f continuous on $[-1, 0]$? (YES or NO)

10 pts

Sketch the graph of a function f satisfying each set of requirements:

(5 pts) $\mathcal{D}(f) = (a, b)$, f is continuous on (a, b), $\lim_{x\to a^+} f(x) = 3$, $\lim_{t\to b^-} f(t) = -1$

(5 pts) f is continuous on $[0, 2]$, $f(0) = 1$, $f(2) = -1$. Must f attain a maximum value on $[0, 2]$? Why or why not?

10 pts

FILL IN THE BLANKS.

(5 pts) The Intermediate Value Theorem says: Let f be __________ on $[a, b]$. If D is any number between __________ and __________, then __________ a number d between __________ and __________ for which __________ .

(5 pts) A function f has a *removable discontinuity at c* whenever _______________ exists, but is not equal to _______________ .

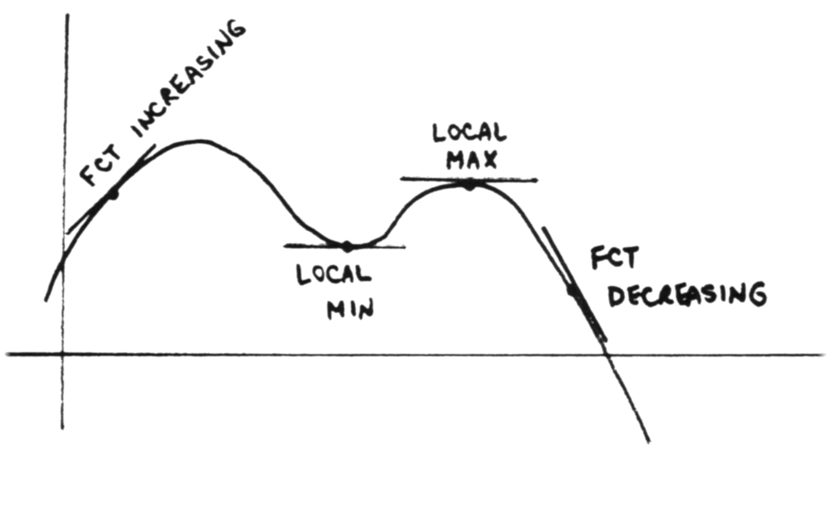
This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 4
THE DERIVATIVE

The *tangent lines* to the graph of a function can give us very important information about the function. Their slopes tell us where the function is increasing and decreasing. Where a tangent line has zero slope, the function *may* have a maximum or minimum point.

In this chapter, a new function is introduced, named f' and called the *derivative of* f, that tells us about the slopes of the tangent lines to the graph of f.



4.1 Tangent Lines

Introduction

Recall that the *slope of a line* tells us how fast the line rises or falls. Given distinct points (x_1, y_1) and (x_2, y_2), the slope of the line through these two points is

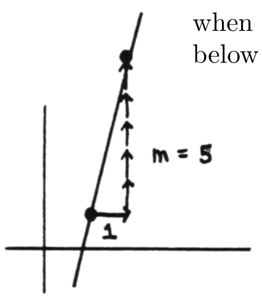
$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \ ,$$

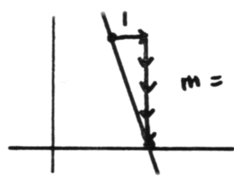
providing that $x_2 \neq x_1$. If $x_2 = x_1$, the line is vertical, and the slope *does not exist.*

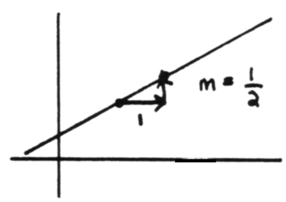
For given points (x_1, y_1) and (x_2, y_2) satisfying the additional requirement that $x_2 - x_1 = 1$, the slope of the line becomes:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{1}$$

This simple observation gives an important interpretation of the slope of a line: it is a number that tells the *vertical change per (positive) unit horizontal change* when traveling from point to point on the line. For example, the lines shown below have (from left to right) slopes 5, -4, and $\frac{1}{2}$.

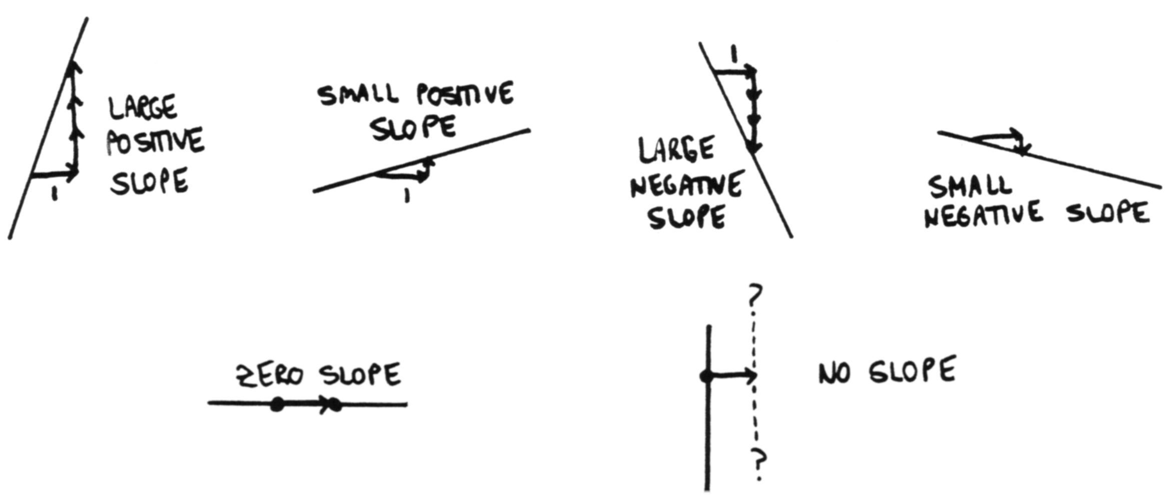






When traveling along a line from left to right:

- lines with large positive slopes are steep 'uphills';
- lines with small positive slopes are gradual 'uphills';
- lines with large negative slopes are steep 'downhills'; and
- lines with small negative slopes are gradual 'downhills'.



EXERCISE 1

♣ 1. Prove that:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Therefore, the order that the points are listed when calculating the slope of a line is unimportant.

♣ 2. A line has slope 3. If the x-values of two points on the line differ by 1, how much do their y-values differ by? If the x-values of two points differ by 2, how much do their y-values differ by?

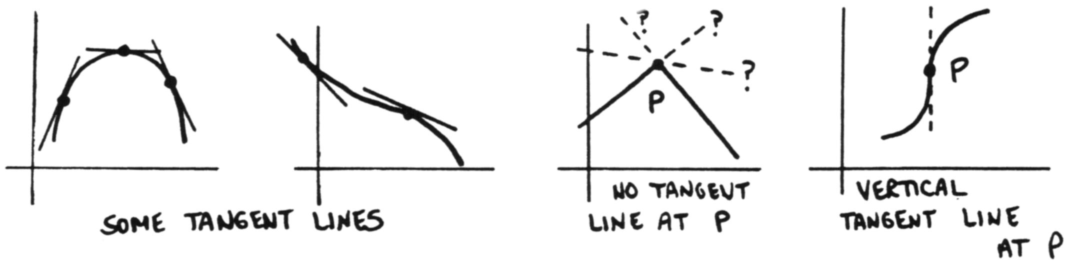
♣ 3. On the same graph, sketch lines that have slopes 1, 10, and $\frac{1}{10}$.

♣ 4. On the same graph, sketch lines that have slopes -1, -10, and $-\frac{1}{10}$.

tangent lines; informal discussion

The *tangent line* to a graph at a point P is the *line that best approximates the graph at that point.* In other words, it is the *best linear approximation at P.*

Tangent lines *may* or *may not* exist, as illustrated below. When they *do* exist, it is intuitively clear how they should be drawn.



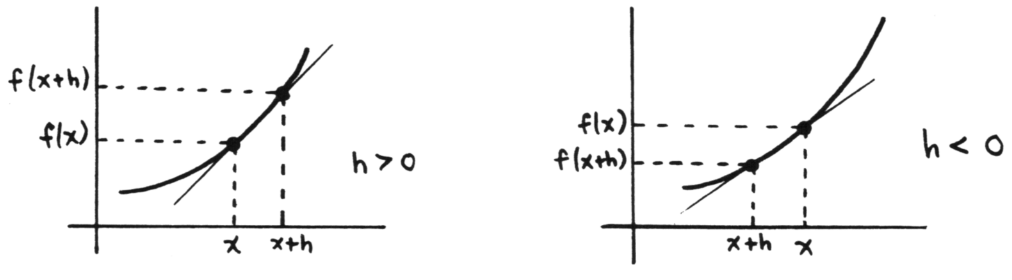
finding the slopes of tangent lines

GOAL: Find the slope of the tangent line to the graph of a function f at the point $(x, f(x))$.

PROBLEM: *Two* points are needed to find the slope of a line!

To remedy this problem, choose a second point that is close to $(x, f(x))$, and find the slope of the line through these two points. When the second point is very close to $(x, f(x))$, this line should be a good approximation to the tangent line.

Let h denote some small number, positive or negative. (Think of h as being, say, 0.1, 0.001 or -0.01.) Then, the point $(x+h, f(x+h))$ is close to $(x, f(x))$. If $h > 0$, the new point is to the right of $(x, f(x))$. If $h < 0$, the new point is to the left of $(x, f(x))$.



secant line

The line through these two points $(x, f(x))$ and $(x+h, f(x+h))$ is called a *secant line*. It serves as an approximation to the desired tangent line. In general, the *closer* the second point $(x+h, f(x+h))$ is to the initial point $(x, f(x))$, the better the approximation.

The slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$ is:

$$\begin{aligned} \text{slope of secant line} &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

difference quotient

The quantity

$$\frac{f(x+h) - f(x)}{h}$$

obtained above is called a *difference quotient*. It represents the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$.

let $h \to 0$

Since we expect the slope of the secant line to better approximate the slope of the tangent line as the second point moves closer to the first (which happens as h approaches 0), it is natural to investigate the limit:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

This limit *may* or *may not* exist. If it *does* exist, then there *is* a tangent line to the graph of f at the point $(x, f(x))$, and the limit value gives the *slope of the tangent line to the graph of f at the point $(x, f(x))$.* This result is summarized next.

DEFINITION

slope of the tangent line to the graph of f at the point $(x, f(x))$.

If the limit

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists, then there is a nonvertical tangent line to the graph of f at the point $(x, f(x))$, and the number m gives the slope of this tangent line.

investigating the limit; what are x and h?

The limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

uses two letters, x and h. The letter h is the *dummy variable* for the limit; it merely represents a number that is getting arbitrarily close to zero. The limit can equally well be written with a different dummy variable, say:

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{or} \qquad \lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

(The symbol Δx is read as '*delta x*', and denotes a *change in x*.)

The letter x that appears in the limit is the x-value of the point where the slope of the tangent line is desired. If, for example, the slope is desired at the point $(2, f(2))$, then the limit becomes:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

Note that the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ can only be investigated at a value of x where f is defined, so that $f(x)$ makes sense.

the limit is a $\frac{0}{0}$ situation

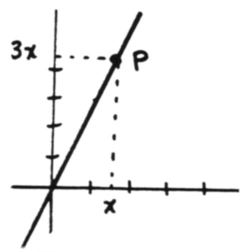
Observe that direct substitution of $h = 0$ into the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

yields a $\frac{0}{0}$ situation. Therefore, this limit can *never* be evaluated directly. It is necessary to get $\frac{f(x+h)-f(x)}{h}$ into a form that displays what is happening when h is *close to* zero, but not equal to zero. In many cases, one tries to simplify the difference quotient to a point where there is a factor of h in the numerator, that can be cancelled with the h in the denominator.

EXAMPLE

using the limit formula to find the slopes of tangent lines



It's always best to test a new result in a situation where you already know the answer. So, let's work first with the function $f(x) = 3x$. The graph of f is a line of slope 3. If P is *any point* on this line, then the tangent line at P is the line itself, and we should find that the slope of the tangent line is 3. Let's see if the above formula bears this out.

Let the 'first point' be $(x, f(x)) = (x, 3x)$, and let the 'second point' be $(x + h, f(x+h)) = (x+h, 3(x+h))$. The slope of the secant line between these two points is

$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)-3x}{h} = \frac{3h}{h}\ ,$$

and thus:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$$

Thus, for *any* point $(x, f(x))$ on the graph, the slope of the tangent line is 3, as expected.

EXERCISE 2

♣ 1. Consider the function $f(x) = -3x$. Using the limit formula, find the slope of the tangent line at the point $(1, -3)$.

♣ 2. Consider the function $f(x) = -3x$. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

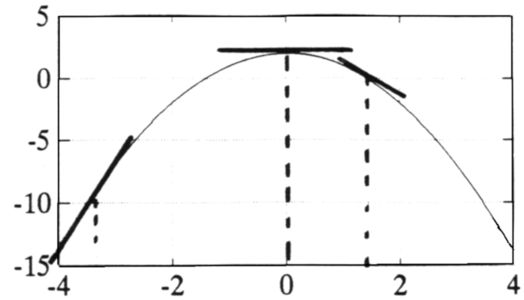
♣ 3. Consider the function $f(x) = kx$, where k is a nonzero constant. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

♣ 4. Consider the zero function $f(x) = 0$. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

EXAMPLE

using the limit formula to find the slopes of tangent lines

Next, consider the function $f(x) = -x^2 + 2$, with graph shown below.



We expect to find that:

- the slope of the tangent line at $x = 0$ is 0
- when x is small and positive, the slopes are small and negative
- when x is a large negative number, the slopes are large and positive

Let's see if this is borne out. Here, $f(x+h) = -(x+h)^2 + 2$, and we get:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\bigl(-(x+h)^2 + 2\bigr) - (-x^2 + 2)}{h} \\
&= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + 2 + x^2 - 2}{h} \\
&= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2x - h)}{h} \\
&= \lim_{h\to 0} (-2x - h) \\
&= -2x
\end{aligned}$$

Observe that this is a complete mathematical sentence. For a particular value of x, the '=' signs denote equality of real numbers. Do NOT drop the limit instruction until you actually let h go to 0. This sentence shows that the limit exists for every value of x, and is equal to $-2x$. That is, the slope of the tangent line at a point $(x, f(x))$ is $-2x$.

The expected results are obtained:

- When $x = 0$, the slope of the tangent to the point $(0, 2)$ is $-2(0) = 0$, as expected.
- When $x = 0.1$, the slope of the tangent line to the point $(0.1, 1.99)$ is $-2(0.1) = -0.2$, a small negative number, as expected.
- When $x = -4$, the slope of the tangent line to the point $(-4, -14)$ is $-2(-4) = 8$, a large positive number, as expected.

EXERCISE 3

♣ 3. Graph the function $f(x) = x^2$.

♣ 2. What do you expect for the slope of the tangent line when $x = 0$? When x is a small positive number? When x is a large negative number?

♣ 3. Using the limit formula, calculate the slope of the tangent line at a typical point $(x, f(x))$.

♣ 4. What is the slope of the tangent line at $(x, f(x))$? Does this agree with your expectations?

characterizing a two-sided limit by using one-sided limits

Suppose a function g is defined both to the left and to the right of c. In order for the two-sided limit

$$\lim_{x\to c} g(x)$$

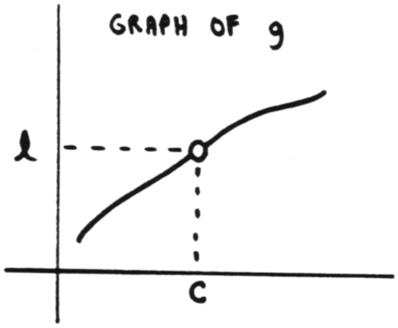
to exist, the function values $g(x)$ must approach *the same number* as x approaches c coming in from both sides.

That is, the two-sided limit $\lim_{x\to c} g(x)$ exists exactly when *both* one-sided limits

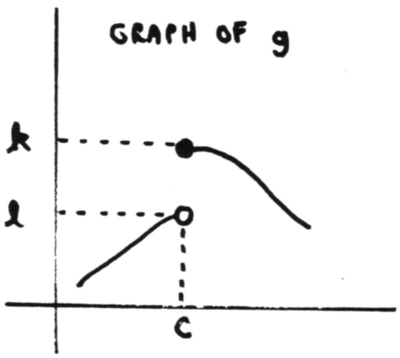
$$\lim_{x\to c^+} g(x) \quad \text{and} \quad \lim_{x\to c^-} g(x)$$

exist, *and have the same value.*

This observation is used in the next examples.



$$\left.\begin{array}{l} \lim_{x\to c^+} g(x) = \ell \\ \\ \lim_{x\to c^-} g(x) = \ell \end{array}\right\} \quad \lim_{x\to c} g(x) = \ell$$



$$\left.\begin{array}{l} \lim_{x\to c^+} g(x) = k \\ \\ \lim_{x\to c^-} g(x) = \ell \\ \\ \quad \ell \neq k \end{array}\right\} \quad \lim_{x\to c} g(x)$$

DOES NOT EXIST

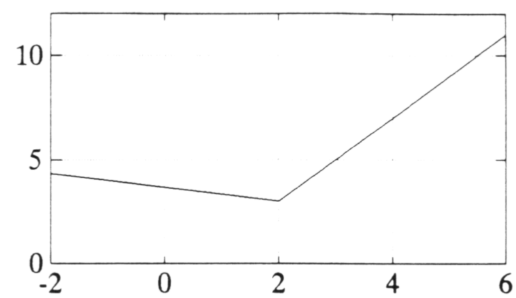
EXAMPLE

A function which does not have a tangent line at a point

Consider the function f defined piecewise as follows:

$$f(x) = \begin{cases} 2x - 1 & \text{when } x \ge 2 \\ -\frac{1}{3}x + \frac{11}{3} & \text{when } x < 2 \end{cases}$$

The graph of f is shown below.



First consider a point $(x, f(x))$ when $x > 2$. In this case, to the immediate left *and* right of the point $(x, f(x))$ the function f looks like:

$$f(x) = 2x - 1$$

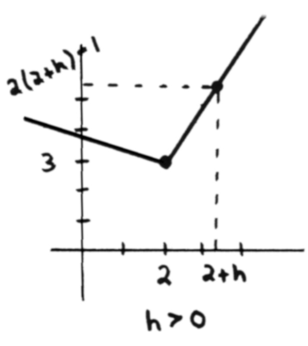
(♣ Why?) Thus, we find that:

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\big(2(x+h) - 1\big) - (2x - 1)}{h} \\ &= \text{(♣ You fill in the details.)} \\ &= 2 \end{aligned}$$

Similarly, if $x < 2$, the slopes of tangent lines are all $-\frac{1}{3}$. (♣ Be sure to check this yourself.)

The interesting situation occurs when $x = 2$; let us now investigate the limit:

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$



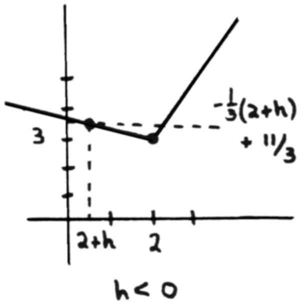
Remember that this limit is, in general, a *2-sided* object. Since the function f being investigated IS defined both to the right ($h > 0$) and left ($h < 0$) of 2, we must see what happens as h approaches 0 from the right-hand side *and* the left-hand side.

Whenever $h > 0$ (h approaches 0 from the right-hand side), we have $2 + h > 2$, so that

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{(2(2+h) - 1) - 3}{h} \\ &= 2 \end{aligned}$$

and so:

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = 2$$



Whenever $h < 0$ (so that h approaches 0 from the left-hand side), we have $2 + h < 2$, so that

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{(-\frac{1}{3}(2+h)+\frac{11}{3})-3}{h} \\ &= -\frac{1}{3}\end{aligned}$$

and so:

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h} = -\frac{1}{3}$$

Since the right and left hand limits do not agree, the two-sided limit

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

does not exist.

That is, there is *no tangent line to f at $x = 2$.* This result was, of course, expected!

EXERCISE 4

Consider the function f, with graph shown below.

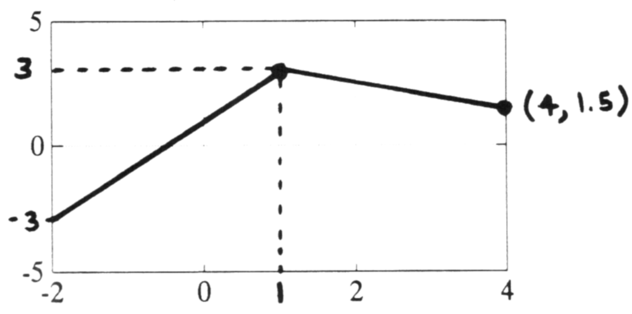
♣ 1. Give a piecewise description for this function f.

Now, attempt to find the tangent line at the point $(1, 3)$, as follows:

♣ 2. Find: $\displaystyle\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$

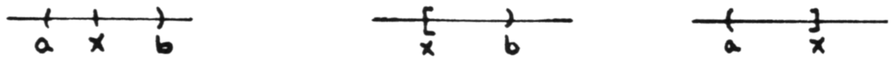
♣ 3. Find: $\displaystyle\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$

♣ 4. Does $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ exist? Why or why not?



EXERCISE 5

When h is a number near zero, $x+h$ is a number near x. So, in evaluating the limit
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\ ,$$
we require that f be defined on some interval containing x. This interval can be of any of these forms:



♣ 1. If f is defined on an interval (a,b) containing x, then is the limit
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
a genuine two-sided limit? Why or why not?

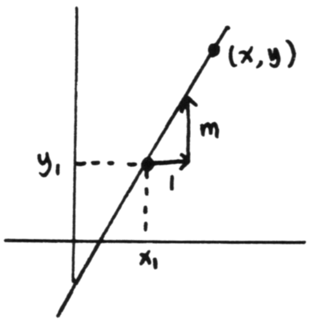
♣ 2. If f is only defined on an interval of the form $[x,b)$, then is the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ a genuine two-sided limit? If not, what type of limit is it?

♣ 3. If f is only defined on an interval of the form $(a,x]$, then is the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ a genuine two-sided limit? If not, what type of limit is it?

ALGEBRA REVIEW

point-slope form for lines

identifying lines



Two pieces of (non-contradictory, non-overlapping) information uniquely determine a line. The most common information given to identify a line is:

- two distinct (different) points on the line; or
- the slope of the line, and a point on the line.

Suppose that the slope of a line is known, call it m; and a point on the line is known, call it (x_1,y_1). Now, let (x,y) denote *any other* point on the uniquely identified line (so $x \neq x_1$). Using the points (x_1,y_1) and (x,y) to compute the (known) slope:
$$\frac{y-y_1}{x-x_1} = m \quad \iff \quad y-y_1 = m(x-x_1)$$
Thus, any point (x,y) lying on the line with slope m through (x_1,y_1) makes the equation $y-y_1=m(x-x_1)$ true; and any point that makes the equation true lies on the line.

point-slope form of a line

That is, the equation of a line that has slope m and passes through the point (x_1,y_1) is given by:
$$y-y_1 = m(x-x_1)$$
This is called the *point-slope form* of a line.

EXAMPLE

using point-slope form

Problem: Find the equation of the line that has slope 2, and passes through the point $(-1,3)$.

Solution: The information is ideally suited to point-slope form:
$$y-3 = 2(x-(-1)) \quad \iff \quad y = 3+2(x+1) \quad \iff \quad y = 2x+5$$

Problem: Find the equation of the line that passes through the points $(5, -2)$ and $(-1, 3)$.

Solution: First, find the slope of the line:

$$m = \frac{3 - (-2)}{(-1) - 5} = \frac{5}{-6} = -\frac{5}{6}$$

Then, use either point, the known slope, and point-slope form. Using the point $(5, -2)$, the equation is:

$$y - (-2) = -\frac{5}{6}(x - 5)$$

Using the point $(-1, 3)$, the equation is:

$$y - 3 = -\frac{5}{6}(x - (-1))$$

EXERCISE 6

♣ Verify that the two equations obtained above are equivalent; that is, they describe precisely the same line. That is, show that:

$$y - (-2) = -\frac{5}{6}(x - 5) \quad \iff \quad y - 3 = -\frac{5}{6}(x - (-1))$$

One way to do this is to put both equations into the same form; say, $y = mx + b$ form, or $ax + by + c = 0$ form. Once they're in the same form, they are easy to compare.

QUICK QUIZ
sample questions

1. Use a limit to compute the slope of the tangent line to the graph of $f(x) = x$ at $x = 2$. Be sure to write complete mathematical sentences.
2. In the expression $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, what is the dummy variable? Rewrite the limit using a different dummy variable (you choose).
3. In the expression $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, what does x represent?
4. In the limit $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, what does $\frac{f(x+h)-f(x)}{h}$ represent?
5. Let $f\colon [0, 3] \to \mathbb{R}$ be defined by $f(x) = x^2$. Graph f. Does

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

exist? If so, what is it?

KEYWORDS
for this section

Tangent lines, finding the slopes of tangent lines, secant lines, difference quotient, slope of the tangent line to the graph of a function f at the point $(x, f(x))$, characterizing a two-sided limit by using one-sided limits.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
2. $\displaystyle\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{h}$
3. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=m$
4. $\displaystyle\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{h}=m$
5. The slope of the tangent line to the graph of $f(x)=x^2$ at the point (x,x^2) equals $2x$.
6. The slope of the tangent line to the graph of $g(x)=5$ at the point $(x,5)$ equals 0.

♣ For the remaining problems, define a function g by

$$g(h):=\frac{f(x+h)-f(x)}{h}\ ,$$

where f is a function of one variable, with $x\in\mathcal{D}(f)$.

7. Find $g(0.1)$ and $g(\Delta x)$.
8. Rewrite the limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ in terms of the function g.
9. When is a number h in the domain of g? Answer using a complete mathematical sentence.
10. What does the number $g(h)$ tell us?
11. What does the number $\displaystyle\lim_{h\to 0}g(h)$ tell us, when it exists?
12. Write down the ϵ-δ definition of the sentence:

$$\lim_{h\to 0}g(h)=m$$

Be sure to write a complete mathematical sentence.

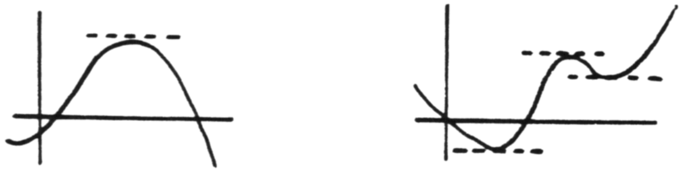
4.2 The Derivative

Introduction

In the previous section, it was shown that if a function f has a nonvertical tangent line at a point $(x, f(x))$, then its slope is given by the limit:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad (*)$$

This is potentially very powerful information about the function f. For example, places where a tangent line has slope 0 often correspond to maximum or minimum values of a function.



Also, the slope of the tangent line at $(x, f(x))$ tells how the function values $f(x)$ are changing at the instant one is 'passing through' the point $(x, f(x))$: whether the graph is rising or falling, and how quickly.

Because of the importance of this slope information, the limit $(*)$, when it exists, is given a special name: it is called the *derivative of f at x*, and denoted by $f'(x)$ (read '*f prime of x*'). This is summarized below.

DEFINITIONS

differentiable at x; the derivative of f at x; differentiation

For a given function f and $x \in \mathcal{D}(f)$, if the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

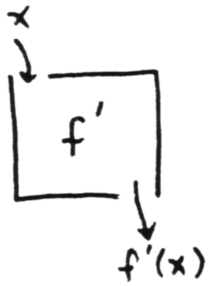
exists, then one says that *f is differentiable at x*, and writes:

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

The number $f'(x)$ is called the *derivative of f at x*.

The process of finding $f'(x)$ is called *differentiation*.

DEFINITION
f', the derivative function



Given a function f, we now have a way to construct a *new* function, named f'. This function f' is called the *derivative of* f. The domain of f' is the set of all $x \in \mathcal{D}(f)$ for which the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

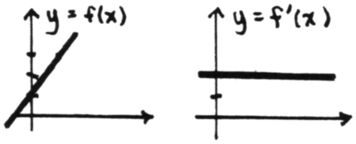
exists.

If f is differentiable at every point in its domain, then $\mathcal{D}(f') = \mathcal{D}(f)$. However, f may *not* be differentiable at every point in its domain. So, the domain of f' may be 'smaller' than the domain of f. In general, all that can be said is that $\mathcal{D}(f') \subset \mathcal{D}(f)$.

The derivative f' takes an input x, and gives as an output the slope of the tangent line to the graph of f at the point $(x, f(x))$.

EXAMPLE
differentiating $f(x) = 2x + 1$

Let f be defined by $f(x) = 2x + 1$. The graph of f is a line L with slope 2. At any point on this line, the tangent line is the line L itself. So, at every point, the slope of the tangent line is 2. Thus, the function f' has the same domain as f, and is defined by $f'(x) = 2$.



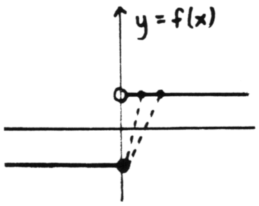
One usually abbreviates the problem as follows:

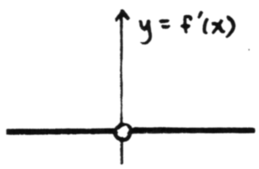
PROBLEM: Differentiate: $f(x) = 2x + 1$

SOLUTION: $f'(x) = 2$

In particular, $f'(0) = 2$, $f'(\pi) = 2$, and $f'(-1002.1) = 2$.

EXAMPLE
differentiating a function that is defined piecewise





PROBLEM: Differentiate:

$$f(x) = \begin{cases} -1 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$$

SOLUTION: For all positive and negative x, tangent lines exist and have slope zero. However, f is not differentiable at 0. Intuitively, this is clear; there is no obvious way to draw a tangent line at the point $(0, -1)$. This conclusion is confirmed by investigating the right-hand limit at $x = 0$:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-(-1)}{h} = \lim_{h\to 0^+} \frac{2}{h} \ ,$$

which does not exist. Thus, the two-sided limit does not exist.

Summarizing:

$$f'(x) = \begin{cases} 0 & \text{for } x \ne 0 \\ \text{not defined} & \text{for } x = 0 \end{cases}$$

In particular, $f'(0.1) = 0$, $f'(-0.001) = 0$, and $f'(0)$ does not exist.

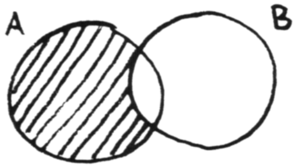
In this example, $\mathcal{D}(f) = \mathbb{R}$, and $\mathcal{D}(f') = \{x \mid x \ne 0\}$. Using interval notation, one can alternately write $\mathcal{D}(f') = (-\infty, 0) \cup (0, \infty)$. Unfortunately, both of these are pretty long expressions for the domain of f'. There is a simpler expression, that makes use of *set subtraction*, discussed next.

DEFINITION
set subtraction

Let A and B be sets. Define a new set, denoted by $A-B$, and read as 'A *minus* B', by:

$$A - B := \{x \,|\, x \in A \text{ and } x \notin B\}$$

Thus, the set $A-B$ consists of all the elements of A that are *not* elements of B. (In other words, take A and 'subtract off' any elements of B.)



EXAMPLE
set subtraction

For example, if

$$A = \mathbb{R} \text{ and } B = [1, 2)\ ,$$

then:

$$A - B = (-\infty, 1) \cup [2, \infty) \quad \text{and} \quad B - A = \emptyset$$

If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ then:

$$A - B = \{1, 2\} \quad \text{and} \quad B - A = \{4, 5\}$$

This notation gives an easier way to describe the domain of f' in the previous example: $\mathcal{D}(f') = \mathbb{R} - \{0\}$.

EXERCISE 1

♣ 1. Why is it incorrect to say $\mathcal{D}(f') = \mathbb{R} - 0$?

For each of the following sets A and B, find both $A-B$ and $B-A$. Be sure to answer using complete mathematical sentences.

♣ 2. $A = \mathbb{R}$, $B = (-\infty, 2]$

♣ 3. $A = (-3, 3]$, $B = [-1, 4)$

♣ 4. $A = \mathbb{R}$, B is the set of irrational numbers

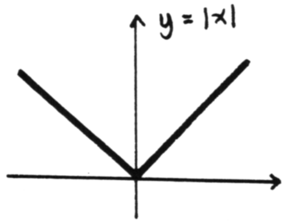
Find sets A and B so that $S = A - B$. (There is *not* a unique correct answer.)

♣ 5. $S = (-1, 0) \cup (0, 1]$

♣ 6. $S = \{1, 2, 3\}$

EXAMPLE

differentiating $f(x) = |x|$*; a function that is continuous at a point, but not differentiable there*



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Consider the function $f(x) = |x|$.

For $x > 0$, $f(x) = |x| = x$ is certainly differentiable with derivative $f'(x) = 1$.

For $x < 0$, $f(x) = |x| = -x$ is also differentiable with derivative $f'(x) = -1$.

When $x = 0$, there is no tangent line. In this case, both one-sided limits exist, but do not agree:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{|h| - |0|}{h} \\ &= \lim_{h \to 0^+} \frac{h}{h} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} &= \lim_{h \to 0^-} \frac{|h|}{h} \\ &= \lim_{h \to 0^-} \frac{-h}{h} \\ &= -1 \end{aligned}$$

Since the one-sided limits do not agree, the two-sided limit *does not exist*, and f is *not* differentiable at $x = 0$.

Summarizing:

$$f'(x) = \begin{cases} 1 & \text{for } x > 0 \\ \text{not defined} & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

Thus, for example, $f'(1.7) = 1$, $f'(-4/3) = -1$, and $f'(0)$ does not exist.

EXERCISE 2

Let $f(x) = 3x - 1$.

♣ 1. Graph f. What is $\mathcal{D}(f)$?

♣ 2. What is the function f'? In particular, what is $\mathcal{D}(f')$?

Now, let $f(x) = |x - 3|$.

♣ 3. Give a piecewise description for f.

♣ 4. Graph f. What is $\mathcal{D}(f)$?

♣ 5. For $x > 3$, what is $f'(x)$?

♣ 6. For $x < 3$, what is $f'(x)$?

♣ 7. Show that f' is not defined at $x = 3$, by investigating the limits:

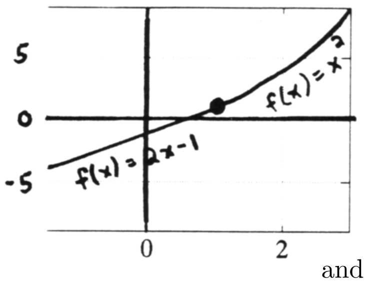
$$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h}$$

♣ 8. Write down a piecewise description of f'.

♣ 9. Graph f'.

EXAMPLE

a 'patched together' function that IS differentiable at the patching point



PROBLEM: Is the function

$$f(x) = \begin{cases} x^2 & x \geq 1 \\ 2x-1 & x < 1 \end{cases}$$

differentiable at $x = 1$?

SOLUTION: Note that $f(1) = 1^2 = 1$. Investigate both one-sided limits:

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+} \frac{(1+h)^2-1}{h} \\
&= \lim_{h\to 0^+} \frac{1+2h+h^2-1}{h} \\
&= \lim_{h\to 0^+} \frac{h(2+h)}{h} = 2
\end{aligned}$$

and

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(2(1+h)-1)-1}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$$

Since both one-sided limits agree,

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

exists and equals 2. Thus, f is differentiable at 1, and $f'(1) = 2$. That is, the tangent line to the graph of f at the point $(1,1)$ has slope 2.

EXERCISE 3

Let f be defined by:

$$f(x) = \begin{cases} x^2 & \text{for } x \leq -1 \\ -2x-1 & \text{for } x > -1 \end{cases}$$

♣ 1. Graph f. What is $\mathcal{D}(f)$?

♣ 2. Does $f'(x)$ exist for $x < -1$? If so, what is it?

♣ 3. Does the limit $\lim_{x\to -1^-} f'(x)$ exist? If so, what is it?

♣ 4. Does $f'(x)$ exist for $x > -1$? If so, what is it?

♣ 5. Investigate two appropriate one-sided limits to decide if $f'(-1)$ exists. If it does, what is it?

♣ 6. Is there a tangent line to the graph of f at the point with x-value -1? If so, what is its slope?

♣ 7. Graph f'. What is $\mathcal{D}(f')$?

EXAMPLE

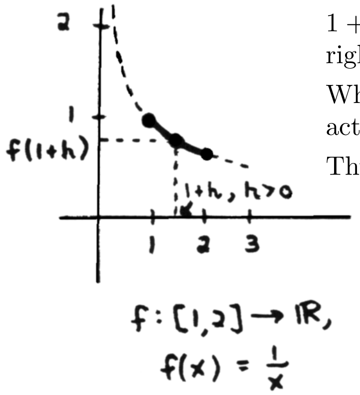
a function that is differentiable at an endpoint of its domain

PROBLEM: Consider the function $f\colon [1,2] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Is f differentiable at $x = 1$?

SOLUTION: Note that $f(1) = \frac{1}{1} = 1$. For f to be differentiable at $x = 1$, the limit

$$\lim_{h\to 0} \frac{f(1+h) - 1}{h}$$

must exist. Does it? Remember that to investigate this limit, one only considers values of h that are *close to* 0 and *in the domain of the function* $\frac{f(1+h)-1}{h}$. When is h in the domain of $\frac{f(1+h)-1}{h}$? Only when $1+h \in \mathcal{D}(f)$. And for this function, $1 + h \in \mathcal{D}(f)$ only when $h > 0$. Here, the 'two-sided limit' is identical to the right-hand limit.



When a function is only defined on one side of a point, the 'two-sided limit' is actually just a one-sided limit.

Thus, one has:

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h} && \text{(line 1)}\\
&= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} && \text{(line 2)}\\
&= \lim_{h\to 0^+} \frac{1 - 1 - h}{h(1+h)} && \text{(line 3)}\\
&= \lim_{h\to 0^+} \frac{-1}{1+h} = -1 && \text{(line 4)}
\end{aligned}
$$

Thus, f is differentiable at $x = 1$, and $f'(1) = -1$.

EXERCISE 4

♣ 1. Give a reason (or reasons) for each line of the preceding display. The lines are numbered for easy reference.

Consider the function $f\colon [0,4] \to \mathbb{R}$ given by $f(x) = (x-1)^2$.

♣ 2. Graph f.

♣ 3. Is f differentiable at $x = 0$? That is, does the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

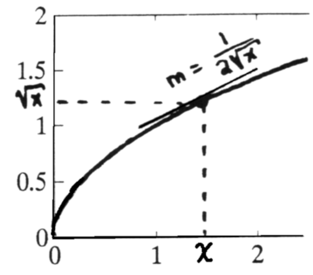
exist? Justify your answer. Be sure to write complete mathematical sentences.

EXAMPLE

a function that is not differentiable at an endpoint of its domain

Let $f(x) = \sqrt{x}$.

For all $x > 0$ one has:

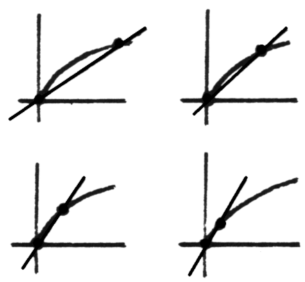


$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} && \text{(line 1)}\\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} && \text{(line 2)}\\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} && \text{(line 3)}\\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} && \text{(line 4)}\\
&= \frac{1}{2\sqrt{x}} && \text{(line 5)}
\end{aligned}
$$

Thus, at each point $(x, \sqrt{x})$ for $x > 0$, the tangent line exists and has slope $\frac{1}{2\sqrt{x}}$.

EXERCISE 5

♣ Give a reason (or reasons) for each line in the display above. The lines are numbered for easy reference.



Now think about what happens when $x = 0$. In this case, the limit is actually a right-hand limit, and reduces to:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h}$$

For $h > 0$, we can write $h = (\sqrt{h})^2$, so that:

$$\frac{\sqrt{h}}{h} = \frac{\sqrt{h}}{(\sqrt{h})^2} = \frac{1}{\sqrt{h}}$$

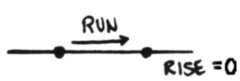
But as $h \to 0^+$, $\frac{1}{\sqrt{h}}$ does *not* approach a specific real number. It gets arbitrarily large. There is a vertical tangent line at the point $(0, 0)$, and a vertical line has no slope. So f is *not* differentiable at 0.

Caution!

'no slope' versus 'zero slope'

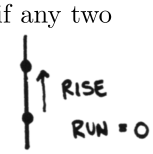
Every horizontal line has zero slope. Choosing any two points on the line, and traveling from one point to the other via the rule '*rise, then run*' yields:

$$\frac{\text{rise}}{\text{run}} = \frac{0}{\text{some nonzero number}} = 0$$



Every vertical line has no slope; that is, the slope is undefined. For if any two points are chosen on the line, computation of the slope yields

$$\frac{\text{rise}}{\text{run}} = \frac{\text{some nonzero number}}{0}\ ,$$

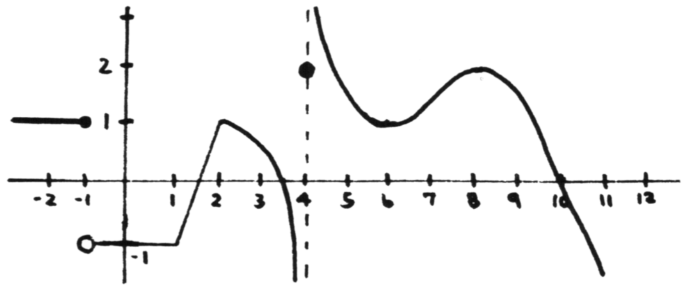


and division by zero is undefined.

Thus, *no slope* and *zero slope* have entirely different meanings. This can be confusing, because in English, the words 'no' and 'zero' are often used as synonyms.

reading info from a graph

Consider the function f whose graph is shown below:



Read the following information from the graph, if possible. If a quantity does not exist, so state.

$$f(-1), \quad f'(-1), \quad f'(-1.5), \quad f'(1.5), \quad f'(2), \quad f(4), \quad f'(6), \quad f'(7)$$

SOLUTION:

- $f(-1) = 1$
- $f'(-1)$ does not exist
- $f'(-1.5) = 0$
- $f'(1.5) = 2$ (Use the known points $(1, -1)$ and $(2, 1)$ to compute the slope.)
- $f'(2)$ does not exist
- $f(4) = 2$
- $f'(6) = 0$
- $f'(7) > 0$; one might estimate that $f'(7) \approx 1$

Now, answer the following questions about f:

- What is $\mathcal{D}(f)$?
- What is $\mathcal{R}(f)$?
- Where is f continuous?
- Where is f differentiable?
- What is $\{x \,|\, f(x) > 0\}$?
- What is $\{x \,|\, f(x) \in [-1, 1]\}$?
- What is $\{x \,|\, f(x) = 1\}$?

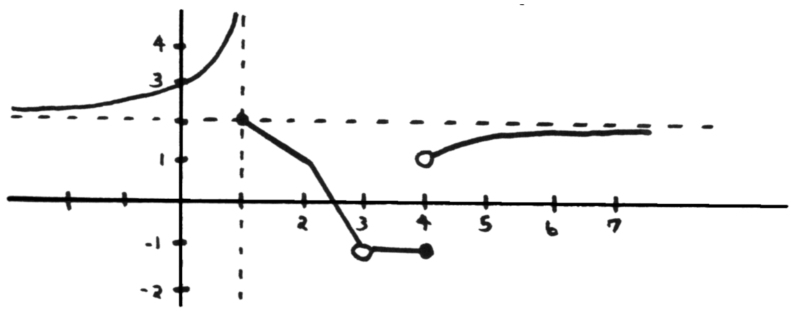
SOLUTION:

For all these answers, the assumption is made that the patterns indicated at the four borders of the graph continue.

- $\mathcal{D}(f) = \mathbb{R}$
- $\mathcal{R}(f) = \mathbb{R}$
- f is continuous at all x in the set $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$. A simpler notation for this set is $\mathbb{R} - \{-1\,,\,4\}$.
- f is differentiable at all x in the set $\mathbb{R} - \{-1\,,\,1\,,\,2\,,\,4\}$.
- Some approximation is necessary here.
  $\{x \mid f(x) > 0\} = (-\infty, -1] \cup (1.5, 3.5) \cup [4, 10)$
- Some approximation is necessary here.
  $\{x \mid f(x) \in [-1, 1]\} = (-\infty, 3.8) \cup \{6\} \cup (9.5, 10.5)$
- Some approximation is necessary here.
  $\{x \mid f(x) = 1\} = (-\infty, -1] \cup \{2, 6, 9.5\}$

EXERCISE 6

Consider the function f whose graph is shown below. Read the following information from the graph, if possible. Approximate, when necessary. If a quantity does not exist, so state. Be sure to write complete mathematical sentences.



♣ 1. $f(0)$, $f(1)$, $f'(1)$, $f'(2)$, $f'(1.34)$,
$f(3)$, $f(4)$, $f'(\pi)$, $f'(1000)$

♣ 2. What is $\mathcal{D}(f)$?

♣ 3. What is $\mathcal{D}(f')$?

♣ 4. What is $\mathcal{R}(f)$?

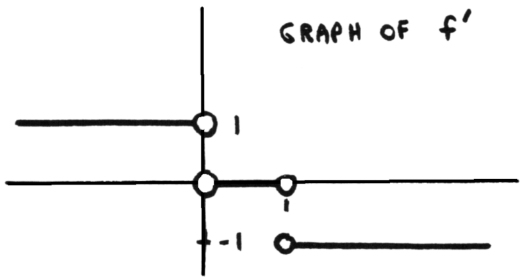
♣ 5. Where is f continuous? Classify any discontinuities.

♣ 6. What is $\{x \mid f(x) \le 0\}$?

♣ 7. What is $\{x \mid f'(x) < 0\}$?

'reconstructing' a function from its derivative

Suppose that a function f has *derivative* f' whose graph is shown below. What, if anything, can be said about the graph of f?



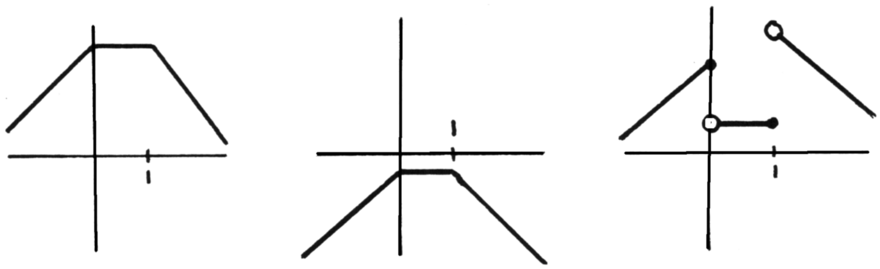
SOLUTION:

For $x < 0$, the tangent lines to the graph of f must all have slope 1.

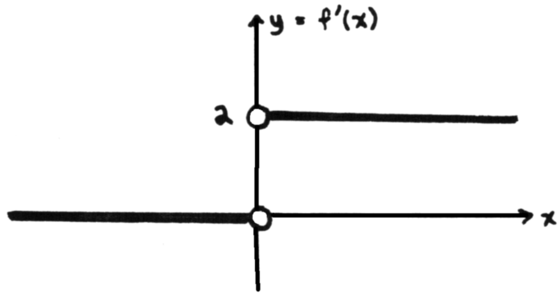
For $0 < x < 1$, the tangent lines to the graph of f must all have slope 0.

For $x > 1$, the tangent lines to the graph of f must all have slope -1.

There *is not* a unique function f that satisfies these requirements. For example, any of the following graphs would produce the specified derivative:



EXERCISE 7 Suppose that a function f has derivative f' whose graph is shown below:

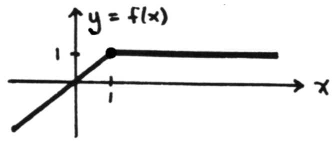


♣ 1. What, if anything, can be said about the graph of f?

♣ 2. Graph three different functions f that could have the specified derivative.

QUICK QUIZ
sample questions

1. Give a precise definition of $f'(x)$.
2. What is the difference between f' and $f'(x)$?
3. If $A = [0, 4)$ and $B = \{0, 2, 4\}$, find $A - B$ and $B - A$.
4. For the function given below, find and graph f'.



5 TRUE or FALSE: If f is differentiable at x, then f is defined at x.

KEYWORDS
for this section

Differentiable at x, $f'(x)$ is the derivative of f at x, differentiation, f' is the derivative function, finding derivatives using the definition, set subtraction.

END-OF-SECTION EXERCISES

For each function f listed below, do the following:

♣ Graph f. What is $\mathcal{D}(f)$?

♣ Find f'. When necessary, use the definition of derivative.

♣ Graph f'. What is $\mathcal{D}(f')$?

1. $f(x) = |x - 2|$
2. $f(x) = \begin{cases} 2 & \text{for } -3 < x \le 0 \\ \frac{1}{x} & \text{for } 0 < x < 4 \end{cases}$
3. $f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ 2x & \text{for } x > 1 \end{cases}$

♣ Use the definition of the derivative to find $f'(c)$ for each function f and number $c \in \mathcal{D}(f)$.

4. $f(x) = 3x^2 - 1,\ \ c = 2$
5. $f(x) = \frac{1}{x-1},\ \ c = 2$
6. $f(x) = \sqrt{x} + 1,\ \ c = 4$

♣ Find the equation of the tangent line to the graph of the function f at the specified point. Feel free to use any earlier results.
The point-slope form may be useful: remember that

$$y - y_1 = m(x - x_1)$$

is the equation of the line that has slope m and passes through the point (x_1, y_1).

7. $f(x) = x^2,\ \ c = 3$
8. $f(x) = \sqrt{x},\ \ c = 0$
9. $f(x) = (x + 2)^2 + 1,\ \ c = -2$

4.3 Some Very Basic Differentiation Formulas

Introduction

If a differentiable function f is quite simple, then it *is* possible to find f' by using the definition of derivative directly:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, this process is quite tedious. Also, as f gets more complicated, the limit gets increasingly more difficult to evaluate.

In this section, some differentiation formulas are developed to make life easier.

First, some notation:

NOTATION for the derivative

prime notation;

f has derivative f'

There are several notations used for the derivative.

So far, the *prime* notation has been used: if f is differentiable at x, then the slope of the tangent line at the point $(x, f(x))$ is the number $f'(x)$. The *name* of the derivative function is f'; $f'(x)$ is the function f', evaluated at x.

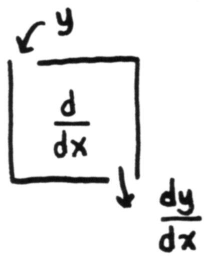
If y is a differentiable function of x, then its derivative can be denoted, using prime notation, by y'. For example, if $y = x^2$, then $y' = 2x$. If it is desired to emphasize that y' is being evaluated at a particular input c, one can write $y'(c)$.

NOTATION for the derivative

Leibniz notation;

y has derivative $\frac{dy}{dx}$;

$\frac{dy}{dx}$ evaluated at c is denoted by either $\frac{dy}{dx}(c)$ or $\frac{dy}{dx}|_{x=c}$



If y is a differentiable function of x, then its derivative can alternately be denoted by $\frac{dy}{dx}$. This is the *Leibniz* notation for the derivative. Read '$\frac{dy}{dx}$' as '*dee y, dee x*'.

For example, if $y = x^2$, then $\frac{dy}{dx} = 2x$. Again, if it is desired to emphasize that $\frac{dy}{dx}$ is being evaluated at a particular input c, one can write $\frac{dy}{dx}(c)$ or $\frac{dy}{dx}|_{x=c}$. These latter two expressions can both be read as: '*dee y, dee x, evaluated at c*'. In particular, the vertical bar '|' is read as '*evaluated at*'.

Similarly, if f is a differentiable function of x, its derivative in Leibniz notation is $\frac{df}{dx}$ (read as '*dee f, dee x*'). If one wants to emphasize that this derivative is being evaluated at a particular value, say c, then one writes $\frac{df}{dx}(c)$ or $\frac{df}{dx}|_{x=c}$.

One problem with Leibniz notation is that the *name of the function* and an *output of the function* are confused. When one says:

$$\textit{if } y = x^2, \textit{ then } \frac{dy}{dx} = 2x,$$

the symbol $\frac{dy}{dx}$ is really being used as *both* the function name and its output. Strictly speaking, one should write: if $y = x^2$, then $\frac{dy}{dx}(x) = 2x$. However, this is not common practice.

an important use of Leibniz notation: the operator $\frac{d}{dx}$

The notation $\frac{d}{dx}$ can be used to denote an instruction: $\frac{d}{dx}$ acts on a differentiable function of x to produce its derivative.

For example, one can write:

$$\frac{d}{dx}(3x - 1) = 3 \quad \text{and} \quad \frac{d}{dt}(t^2) = 2t \quad \text{and} \quad \frac{d}{dz}(2z + 1) = 2$$

This '$\frac{d}{dx}$' notation is often used in stating differentation formulas. Also, it is convenient if you are asked to differentiate a function that is not given a name.

EXERCISE 1

practice with notation

Let $f(x) = x^2$.

Rewrite the following sentences about f, using prime notation.

♣ 1. $\frac{df}{dx} = 2x$

♣ 2. $\frac{df}{dx}(3) = 6$

♣ 3. $\frac{df}{dx}|_{x=3} = 6$

♣ 4. $\frac{df}{dt} = 2t$

♣ 5. $\frac{df}{dt}(3) = 6$

♣ 6. $\frac{df}{dt}|_{t=3} = 6$

Rewrite the following sentences using Leibnitz notation.

♣ 7. $f'(x) = 2x$

♣ 8. $f'(3) = 6$

♣ 9. $f'(t) = 2t$

compiling some differentiation tools

We now begin to compile some tools that will help us differentiate functions more easily.

DIFFERENTIATION TOOL

the derivative of a constant is 0

Let $f(x) = k$, for $k \in \mathbb{R}$. Then $f'(x) = 0$.

alternate notation

This rule can be rewritten, using the '$\frac{d}{dx}$' operator, as follows:
For every real number k:

$$\frac{d}{dx}(k) = 0$$

PROOF

Proof. Let $f(x) = k$, for $k \in \mathbb{R}$. Then, for every x:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = 0$$

Thus, $f'(x) = 0$. ∎

EXAMPLE

Remember that the symbol '∎' merely marks the end of the proof.
If $f(x) = \sqrt{\pi^2 - 5}$, then $f'(x) = 0$.
If $y = e - 3$, then $\frac{dy}{dx} = 0$.
$\frac{d}{dx}\left(\frac{\sqrt{7}}{3\sqrt{2}}\right) = 0$
If $f(x) = a + b$, where a and b are constants, then $f'(x) = 0$.

EXERCISE 2

Rewrite each of these examples, using alternate notation.

DIFFERENTIATION TOOL

constants can be 'slid' out of the differentiation process

Suppose that f is differentiable at x, and let $k \in \mathbb{R}$. Recall that the function kf is defined by the rule:

$$(kf)(x) := k \cdot f(x)$$

Then:

$$(kf)'(x) = k \cdot f'(x)$$

In words, *the derivative of a constant times a differentiable function is the constant, times the derivative of the function.*

alternate notation

This rule can be rewritten, using a mixture of the '$\frac{d}{dx}$' operator and prime notation, as:

$$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$$

PROOF

Proof. Let f be differentiable at x, and let $k \in \mathbb{R}$. It is necessary to show that the function given by $(kf)(x) = k \cdot f(x)$ is differentiable at x.

$$\begin{aligned}
&\lim_{h\to 0} \frac{(kf)(x+h) - (kf)(x)}{h} \\
&\qquad = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} && \text{(defn of } kf\text{)} \\
&\qquad = \lim_{h\to 0} k \cdot \frac{f(x+h) - f(x)}{h} && \text{(factor out } k\text{)} \\
&\qquad = k \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{(prop. of limits, } f \text{ diff. at } x\text{)} \\
&\qquad = k \cdot f'(x) && (f \text{ is diff at } x)
\end{aligned}$$

Thus, the function kf is differentiable at x, and has derivative given by:

$$(kf)'(x) = k \cdot f'(x) \qquad \blacksquare$$

What made this proof work? Properties of limits!

Observe what made this proof work: since we knew, a priori, that f was differentiable at x (so that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists), we were able to use the property of *limits* to slide the constant out. The properties of limits will play a crucial role in the proofs of *all* the differentiation formulas.

EXAMPLE

If $f(x) = 2x$, then $f'(x) = 2 \cdot \frac{d}{dx}(x) = 2(1) = 2$.

If h is differentiable at x, and $f(x) = \sqrt{2}h(x)$, then $f'(x) = \sqrt{2}h'(x)$.

If $y = \frac{1}{2t} = \frac{1}{2} \cdot \frac{1}{t}$, then $\frac{dy}{dt} = \frac{1}{2} \cdot \frac{d}{dt}\big(\frac{1}{t}\big)$. (This last example can be completed after the statement of another differentiation tool, the *Simple Power Rule*.)

DIFFERENTIATION TOOL

differentiating sums and differences

Suppose that both f and g are differentiable at x. Then the functions $f + g$ and $f - g$ are also differentiable at x, and:

$$(f+g)'(x) = f'(x) + g'(x)$$
$$(f-g)'(x) = f'(x) - g'(x)$$

In words, *the derivative of a sum is the sum of the derivatives*, and *the derivative of a difference is the difference of the derivatives.*

alternate notation

This rule can be rewritten, using a mixture of the '$\frac{d}{dx}$' operator and prime notation, as:

$$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$$

PROOF

Proof. It is shown first that, under the stated hypotheses, $f+g$ is differentiable at x.

Recall that the function $f+g$ is defined by the rule $(f+g)(x) := f(x)+g(x)$.

Since, by hypothesis, both f and g are differentiable at x, it is known that $f'(x)$ and $g'(x)$ exist.

Then:

$$\begin{aligned}
&(f+g)'(x) \\
&\quad := \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} && \text{(defn. of derivative)} \\
&\quad = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} && \text{(defn of } f+g\text{)} \\
&\quad = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h} && \text{(regroup)} \\
&\quad = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(limit of a sum, hypotheses)} \\
&\quad = f'(x)+g'(x)
\end{aligned}$$

To see that $f-g$ is differentiable at x, we can now cite earlier results. Note that:

$$(f-g)(x) := f(x)-g(x) = f(x)+(-g(x)) = f(x)+(-g)(x)$$

So, the function $f-g$ can be written as a sum of two functions, with names f and $-g$. Then:

$$\begin{aligned}
(f-g)'(x) &= f'(x)+(-g)'(x) && \text{(Why?)} \\
&= f'(x)+(-g'(x)) && \text{(Why?)} \\
&= f'(x)-g'(x) \;\blacksquare
\end{aligned}$$

EXERCISE 3

♣ 1. Prove the previous result yourself, without looking at the book. You could be asked to write down a precise proof on an in-class exam.

♣ 2. Under what hypotheses is the limit of a sum equal to the sum of the limits? Was this result used in the previous proof? Where? Were the hypotheses met?

♣ 3. Re-prove the fact that $(f-g)'(x) = f'(x)-g'(x)$ (under suitable hypotheses), but this time DON'T cite earlier results. Just use the definition of derivative.

Does the rule apply when there are more than 2 terms?

Although the previous result is stated for only 2 terms, does it tell us that, say,

$$(f+g+h)'(x) = f'(x) + g'(x) + h'(x) \ ,$$

providing that f, g and h are all differentiable at x? Of course! Just pull out the old '*treat it as a singleton*' trick:

$$\begin{aligned} (f+g+h)'(x) &= \big((f+g)+h\big)'(x) && \text{(group)} \\ &= (f+g)'(x) + h'(x) && \text{(use result once)} \\ &= f'(x) + g'(x) + h'(x) && \text{(use result again)} \end{aligned}$$

EXERCISE 4

♣ Prove that, under suitable hypotheses:

$$(f+g+h+k)'(x) = f'(x) + g'(x) + h'(x) + k'(x)$$

SIMPLE POWER RULE

differentiating x^n

For all positive integers n:

$$\frac{d}{dx}x^n = nx^{n-1}$$

More generally, if n is a real number, and I is any interval on which *both* x^n and nx^{n-1} are defined, then x^n is differentiable on the interval I, and:

$$\frac{d}{dx}x^n = nx^{n-1}$$

EXAMPLE

Here are some very basic applications of the Simple Power Rule:

- If $f(x) = x^2$, then $f'(x) = 2x^{2-1} = 2x$. Here, the Simple Power Rule was applied with $n = 2$.
- $\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$
- If $y = x^{1007}$, then $\frac{dy}{dx} = 1007x^{1006}$. Here, the Simple Power Rule was applied with $n = 1007$.
- The slope of the tangent line to the graph of $f(x) = x^7$ at the point $(2, 2^7)$ is $f'(2) = 7(2^6)$.

EXAMPLE

rewriting the function, to make it 'fit' the Simple Power Rule

Here are some more advanced applications of the Simple Power Rule. The Simple Power Rule is used whenever the function being differentiated looks like (or *can be made to look like*) x^n. The laws of exponents, and fractional exponent notation, are used extensively to rewrite functions, to get them into a form where the Simple Power Rule can be applied. The Algebra Review in this section reviews the necessary tools.

Problem: Differentiate $f(x) = \frac{1}{x}$.

Solution: Rewrite the function as $f(x) = x^{-1}$. Taking $n = -1$ in the Simple Power Rule, one obtains:

$$f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

On what interval(s) is this formula valid? It is valid on any interval for which BOTH $\frac{1}{x}$ and $-\frac{1}{x^2}$ are defined. Both expressions are defined on $\mathbb{R} - \{0\}$. Thus, the formula is valid for all real numbers, except 0.

EXAMPLE

Problem: Differentiate $y = \sqrt{x}$.

Solution: Rewrite y, using fractional exponents, as $y = x^{1/2}$. Taking $n = \frac{1}{2}$ in the Simple Power Rule, one obtains:

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

On what interval(s) is this formula valid? The expression $\sqrt{x}$ is defined for $x \geq 0$. The expression $\frac{1}{2\sqrt{x}}$ is defined for $x > 0$. BOTH expressions are defined on $(0, \infty)$. Thus, the formula is valid for all positive real numbers.

put the derivative in a form that matches the original function form

It is always a good idea to put the derivative in a form that agrees, as closely as possible, with the form of the original function. Since the original function in this example was given in radical form, $y = \sqrt{x}$, the derivative was also rewritten in radical form, $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

$\frac{d}{dx}kx^n = nkx^{n-1}$

Using both the Simple Power Rule and the fact that constants can be 'slid out' of the differentiation process yields an extremely useful formula:

$$\frac{d}{dx}kx^n = k\frac{d}{dx}x^n = k(nx^{n-1}) = knx^{n-1}$$

Thus, for example:

- If $f(x) = 3x^2$, then $f'(x) = 6x$.
- $\frac{d}{dx}\pi x^{11} = 11\pi x^{10}$
- If $y = \sqrt{2}x$, then $\frac{dy}{dx} = (1)(\sqrt{2})x^{1-1} = \sqrt{2}x^0 = \sqrt{2}$.

  It is not necessary to write out all these steps. You should be able to recognize $y = kx$ as a *line* that has slope k. Thus, $\frac{dy}{dx} = k$.
- The slope of the tangent line to the graph of $y = 3x^5$ at the point $(1, 3)$ is $\frac{dy}{dx}|_{x=1}$. Here, $\frac{dy}{dx} = 15x^4$, so that $\frac{dy}{dx}|_{x=1} = 15(1)^4 = 15$.

EXERCISE 5

practice with the Simple Power Rule

For each of the functions listed below, do the following:

- Write the function in the form $f(x) = x^n$.
- Differentiate, using the Simple Power Rule.
- On what interval(s) is the formula obtained for the derivative valid?
- Find the equation of the tangent line to the graph of f when $x = 1$.

♣ 1. $f(x) = \sqrt[3]{x}$

♣ 2. $f(x) = \frac{1}{\sqrt{x}}$

♣ 3. $f(x) = \sqrt{x}\sqrt[3]{x^2}$

idea of proof of the Simple Power Rule

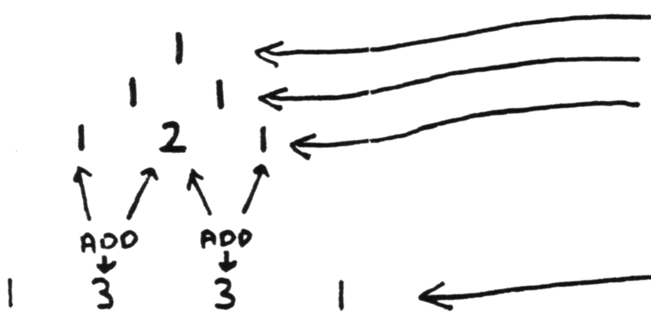
When the exponent is a positive integer, the idea of the proof of the Simple Power Rule is very simple. This idea is illustrated by considering a special case: Show that if $f(x) = x^3$, then $f'(x) = 3x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + \overbrace{3x^2h}^{\text{one factor of } h} + \overbrace{3xh^2 + h^3}^{\text{more than one } h}) - x^3}{h} \qquad (\text{expand } (x+h)^3) \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}$$

A brief review of Pascal's Triangle, a tool for easily expanding $(a+b)^n$ for positive integers n, will enable you to repeat this argument for higher values of n.

Pascal's Triangle

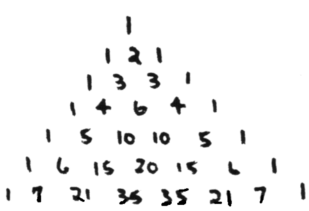
Let a and b be any real numbers. Observe the following pattern:



$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= (1)a + (1)b \\
(a+b)^2 &= (1)a^2 + 2ab + (1)b^2 \\
(a+b)^3 &= (a+b)(a+b)^2 \\
&= (a+b)(a^2 + 2ab + b^2) \\
&= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3 \\
&= (1)a^3 + 3a^2b + 3ab^2 + (1)b^3
\end{aligned}$$

A 'triangle' is formed. Each new row is easily obtained from the previous row by simple addition. It can be proven (★ say, by induction) that this pattern continues forever.

finding $(x+h)^7$



For example, suppose we want to expand $(x+h)^7$. Long multiplication would be extremely tedious. Instead, first write the appropriate *types* of terms in the expansion. Each term has variable part x^ih^j, where the exponents add up to 7. The first term has x^7 and h^0; the second x^6 and h^1, the third term has x^5 and h^2, and so on. So we get the term types:

$$x^7 \quad x^6h \quad x^5h^2 \quad x^4h^3 \quad x^3h^4 \quad x^2h^5 \quad xh^6 \quad h^7$$

Now, get the correct coefficients from Pascal's triangle (from the row beginning with the numbers '1 7 ...'):

$$(1)x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + (1)h^7$$

Thus:

$$(x+h)^7 = x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7$$

EXERCISE 6

♣ 1. Use Pascal's triangle to expand $(x+h)^9$.

♣ 2. Use Pascal's triangle to expand $(x-h)^4$.
Hint: Write $(x-h)^4 = (x+(-h))^4$, so the appropriate term 'types' are:

$$x^4 \quad x^3(-h) \quad x^2(-h)^2 \quad x(-h)^3 \quad (-h)^4$$

♣ 3. Prove that if $f(x) = x^4$, then $f'(x) = 4x^3$.

★★

The complete proof of the Simple Power Rule would take several pages, and we do not yet have at our disposal all the necessary tools. However, a sketch of the proof is as follows:

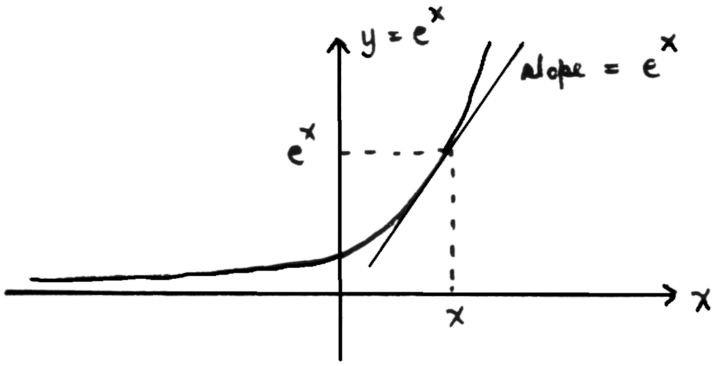
- First prove the result when x is a positive integer. (An easier proof than the one sketched above uses the product rule for differentiation.)
- Use the quotient rule for differentiation to extend the result to the negative integers.
- Use the formula for the derivative of an inverse function to extend the result to exponents of the form $\frac{1}{n}$.
- Write $x^{p/q} = (x^{1/q})^p$ to extend the result to all rational exponents.
- Use the exponential function to make sense of irrational exponents: $x^r = e^{r \ln x}$. (Here, we require x to be positive.) Differentiate to complete the proof.

DIFFERENTIATION TOOL

differentiating e^x

If $f(x) = e^x$, then $f'(x) = e^x$.

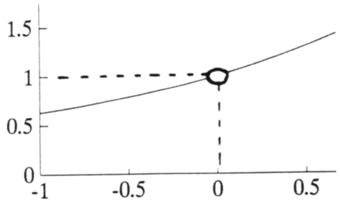
Thus, the *derivative of the exponential function is itself!* This is a property of the exponential function that is not shared by any other function. Make sure you understand what this fact is saying: if you look at any point on the graph of the function e^x, then the y-value of the point also tells you the slope of the tangent line to that point!



idea of proof

Let $f(x) = e^x$. Then:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} e^x \left(\frac{e^h-1}{h}\right)\end{aligned}$$



GRAPH OF $g(h) = \frac{e^h-1}{h}$

If it can be shown that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$, then we can complete the proof:

$$\begin{aligned}\lim_{h\to 0} e^x \left(\frac{e^h-1}{h}\right) &= e^x \cdot \lim_{h\to 0} \frac{e^h-1}{h} \\ &= e^x(1) = e^x\end{aligned}$$

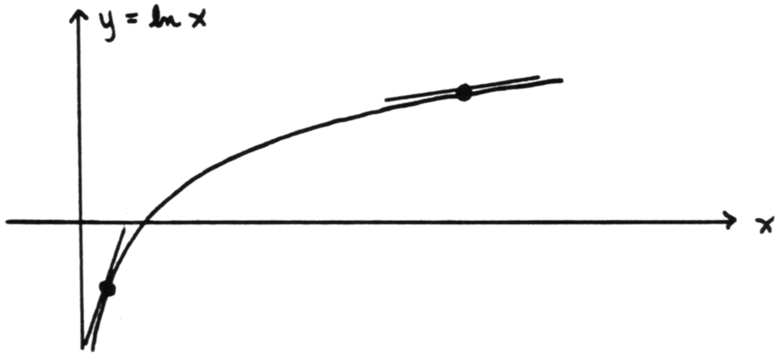
A graph of $g(h) := \frac{e^h-1}{h}$ for values of h close to 0 is shown, which illustrates the fact that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.

DIFFERENTIATION TOOL
differentiating $\ln x$

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.

the result is believable

Observe that this result *is* believable: when x is large, the slopes of tangent lines to the graph of $\ln x$ are small; and when x is close to 0, the slopes are large and positive.



EXAMPLE

To differentiate functions involving e^x and $\ln x$, it is often necessary to first rewrite the function, using properties of exponents and logs. These properties are reviewed in the Algebra Review at the end of this section.

Problem: Differentiate $f(x) = e^{2+x}$.

Solution: First write $f(x) = e^{2+x} = e^2 e^x$. Then,

$$f'(x) = e^2 \frac{d}{dx} e^x = e^2 e^x = e^{2+x} \ .$$

Another (easier) way to differentiate f will be possible after we study the Chain Rule for Differentiation.

EXAMPLE

Problem: Differentiate $g(x) = \ln 2x$.

Solution: First write $g(x) = \ln 2 + \ln x$. Then:

$$g'(x) = 0 + \frac{1}{x} = \frac{1}{x}$$

Another (easier) way to differentiate g will be possible after we study the Chain Rule for Differentiation.

EXERCISE 7

Differentiate each of the following functions. It will be necessary to first rewrite the functions, using properties of exponents and logarithms.

♣ 1. $f(x) = e^{x+5}$; interpret your result graphically.

♣ 2. $f(x) = \ln 7x$

♣ 3. Do you think that we have the necessary tools yet to differentiate $f(x) = e^{2x}$? Why or why not?

♣ 4. Do you think that we have the necessary tools yet to differentiate $g(x) = \ln(x + 3)$? Why or why not?

A chart summarizing the tools developed in this section is given below:

DIFFERENTIATION TOOLS

prime notation	$\frac{d}{dx}$ operator
if $f(x) = k$, then $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
$(kf)'(x) = k \cdot f'(x)$	$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$
$(f + g)'(x) = f'(x) + g'(x)$	$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$
$(f - g)'(x) = f'(x) - g'(x)$	$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$	$\frac{d}{dx}x^n = nx^{n-1}$
if $f(x) = e^x$, then $f'(x) = e^x$	$\frac{d}{dx}(e^x) = e^x$
if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$

ALGEBRA REVIEW

radicals and fractional exponents, properties of logarithms

radicals

A *radical* is an expression of the form

$$\sqrt[n]{x}\ , \qquad (*)$$

for $n = 2, 3, 4, \ldots$.

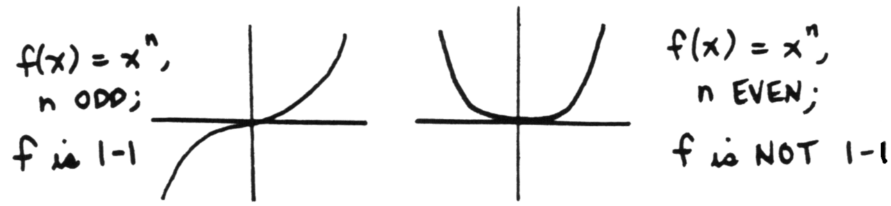
When $n = 2$, (*) is written more simply as $\sqrt{x}$, and is read as *the square root of* x .

When $n = 3$, $\sqrt[3]{x}$ is read as *the cube root of* x .

For $n \ge 4$, $\sqrt[n]{x}$ is read as *the* n^{th} *root of* x .

meaning of $\sqrt[n]{x}$

The purpose of radicals is to 'undo' exponents. That is, radicals provide a sort of inverse to the 'raise to a power' operation. Unfortunately, the 'raise to a power' functions $f(x) = x^n$ are only $1{-}1$ if n is odd. When n is even, special considerations need to be made.



ODD roots

First consider $f(x) = x^3$. Here, f is $1{-}1$, and its inverse is the cube root function, $f^{-1}(x) = \sqrt[3]{x}$. That is:

For all real numbers x and y:

$$y = \sqrt[3]{x} \iff y^3 = x$$

Rephrasing, y is the cube root of x if and only if y, when cubed, equals x.

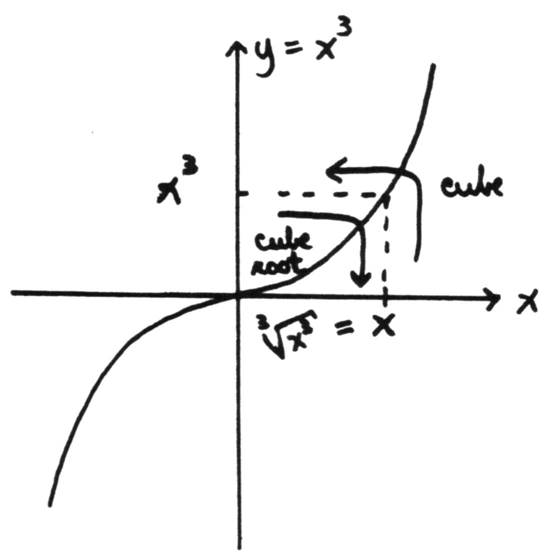
Thus, $\sqrt[3]{8} = 2$, since 2 is the unique number which, when cubed, equals 8.

Also, $\sqrt[3]{-8} = -2$, since -2 is the unique number which, when cubed, equals -8.

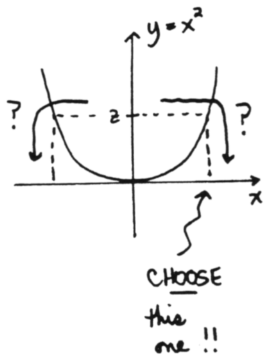
Indeed, for all real numbers x, and for $n = 3, 5, 7, 9, \ldots$,

$$\sqrt[n]{x^n} = x \ ,$$

since x is the unique real number which, when raised to an *odd* n^{th} power, equals x^n.



EVEN roots



When n is even, $f(x) = x^n$ is NOT 1−1. Consider, for example, $f(x) = x^2$. Here, (as for all even powers), $\mathcal{R}(f) = [0, \infty)$. Given $z \in \mathcal{R}(f)$, there are TWO inputs which, when squared, give z. *Mathematicians have agreed to choose the NONNEGATIVE number which works.* Precisely, we have:

For all $x \ge 0$ and for all real numbers y:

$$y = \sqrt{x} \qquad \iff \qquad y \ge 0 \ \text{ and } y^2 = x$$

That is, y is the square root of x if and only if y is nonnegative, and y, when squared, equals x.

Thus, $\sqrt{4} = 2$, since 2 is nonnegative, and $2^2 = 4$.

The expression $\sqrt{-4}$ is not defined, because there is NO real number, which when squared, equals -4.

What is $\sqrt{x^2}$? There are TWO real numbers which, when squared, give x^2: x and $-x$. We need to choose whichever is nonnegative. The absolute value comes to the rescue:

For all real numbers x:

$$\sqrt{x^2} = |x|$$

Indeed, for all nonnegative real numbers x, and for all $n = 2, 4, 6, 8, \ldots$, we have:

$$\sqrt[n]{x^n} = |x|$$

EXERCISE 8

practice with radicals

♣ 1. Consider this mathematical sentence:

For all real numbers x and y:

$$y = \sqrt[3]{x} \quad \iff \quad y^3 = x \qquad (*)$$

This sentence compares two 'component' sentences. What are they? What is (*) telling us that they have in common?

What is (*) telling us (if anything) when $y = 2$ and $x = 8$? How about when $y = -2$ and $x = 8$?

♣ 2. Consider this mathematical sentence:

For all $x \ge 0$ and for all real numbers y:

$$y = \sqrt{x} \qquad \iff \qquad y \ge 0 \ \text{ and } \ y^2 = x \qquad (**)$$

What two component sentences are being compared? What do they have in common?

What is (**) telling us (if anything) when $y = 2$ and $x = 4$? How about when $y = -2$ and $x = 4$?

Evaluate the following roots. Be sure to write complete mathematical sentences. State any necessary restrictions on x and y.

♣ 3. $\sqrt[5]{-32}$

♣ 4. $\sqrt[4]{(-2)^4}$

♣ 5. $\sqrt[6]{x^6}$

♣ 6. $\sqrt[9]{x^9}$

fractional exponent notation

When working with radicals in calculus, it is usually more convenient to use *fractional exponent notation* rather than *radical notation.*

Whenever $\sqrt[n]{x}$ *is defined, it can be alternately written as* $x^{\frac{1}{n}}$.

Thus:

- $\sqrt{5} = 5^{\frac{1}{2}}$
- $\sqrt[3]{x} = x^{1/3}$ for all real numbers x
- $\sqrt[4]{x} = x^{1/4}$ for all nonnegative real numbers x

Then, using properties of exponents (which are summarized below for your convenience), one can make sense of arbitrary rational exponents:

$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

or

$$x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p = (\sqrt[q]{x})^p \ ,$$

provided that both $\sqrt[q]{x^p}$ and $(\sqrt[q]{x})^p$ are defined. Use whichever representation is easiest for a given problem.

PROPERTIES OF EXPONENTS

Assume that a, b, n and m are restricted to values for which each expression is defined.

$a^m \cdot a^n = a^{m+n}$	(same base, multiplied, add exponents)
$\frac{a^m}{a^n} = a^{m-n}$	(same base, divided, subtract exponents)
$(a^m)^n = a^{mn}$	(power to a power, multiply exponents)
$(ab)^m = a^m b^m$	(product to a power, each factor gets raised to the power)
$(\frac{a}{b})^m = \frac{a^m}{b^m}$	(quotient to a power, both numerator and denominator get raised to the power)
$a^{-n} = \frac{1}{a^n}$	(definition of negative exponents)
$a^0 = 1$ for $a \neq 0$	(definition of zero exponent)

EXERCISE 9

♣ Convince yourself that each of these exponent laws 'makes sense'. Just look at special cases, where convenient.

For example, for positive integers m and n:

$$a^m \cdot a^n = \overbrace{(a \cdot \ldots \cdot a)}^{m \text{ factors of } a} \cdot \overbrace{(a \cdot \ldots \cdot a)}^{n \text{ factors of } a} = \overbrace{a^{m+n}}^{m+n \text{ factors of } a}$$

EXAMPLE

working with fractional exponents

Problem: Rewrite using fractional exponent notation. State any necessary restrictions on x and y. Where possible, write in two different ways.

1. $y = \sqrt[5]{x^3}$
2. $f(x) = \frac{\sqrt{x}\,\sqrt[3]{x^5}}{x}$

Solutions:

1. $y = \sqrt[5]{x^3} = (x^3)^{1/5} = x^{3\cdot\frac{1}{5}} = x^{3/5}$
2. Observe that $\mathcal{D}(f) = \{x \mid x > 0\}$. For such x:

$$\begin{aligned} f(x) &= \frac{\sqrt{x}\,\sqrt[3]{x^5}}{x} = \frac{x^{1/2}(x^5)^{1/3}}{x} \\ &= \frac{x^{1/2}x^{5/3}}{x} = \frac{x^{\frac{1}{2}+\frac{5}{3}}}{x} \\ &= \frac{x^{\frac{3}{6}+\frac{10}{6}}}{x} = \frac{x^{\frac{13}{6}}}{x^{\frac{6}{6}}} \\ &= x^{\frac{13}{6}-\frac{6}{6}} = x^{7/6} \\ &= (x^7)^{1/6} = \sqrt[6]{x^7} \end{aligned}$$

Alternately, if desired:

$$x^{7/6} = (x^{1/6})^7 = (\sqrt[6]{x})^7$$

All the steps were shown in the above display. You will probably be able to do a number of these steps in your head.

properties of $\ln x$

Next, some properties of logarithms are reviewed.

a precise view of functions

Whenever f is a function, then every input has a unique corresponding output. In other words, whenever two inputs are the same (and perhaps just have different names), then they must have the same output. Precisely, whenever f is a function with domain elements a and b:

$$a = b \quad \Longrightarrow \quad f(a) = f(b) \tag{1}$$

Thus, whenever the sentence '$a = b$' is true, so is the sentence '$f(a) = f(b)$'.

a precise view of a $1{-}1$ *function*

If f is, in addition, a $1{-}1$ function, then every output has a unique corresponding input. In other words, whenever two outputs are the same, then they must have come from the same input. Precisely, whenever f is a $1{-}1$ function with domain elements a and b:

$$f(a) = f(b) \quad \Longrightarrow \quad a = b \tag{2}$$

Thus, whenever the sentence '$f(a) = f(b)$' is true, so is the sentence '$a = b$'. Putting (1) and (2) together, whenever f is a $1{-}1$ function with domain elements a and b:

$$a = b \quad \Longleftrightarrow \quad f(a) = f(b)$$

Thus, if two inputs are the same, so are the corresponding outputs (the function condition); and whenever two outputs are the same, so are the corresponding inputs (the $1{-}1$ condition).

EXAMPLE

The function $f(x) = e^x$ is $1{-}1$ and has domain $\mathbb{R}$. Thus, for all real numbers x and y:

$$x = y \iff e^x = e^y$$

The function $g(x) = \ln x$ is $1{-}1$ and has as its domain the set of positive real numbers. Thus, for all positive real numbers x and y:

$$x = y \iff \ln x = \ln y$$

e^x and $\ln x$ are inverse functions

In addition, recall that e^x and $\ln x$ are inverse functions. Thus, a point (x, y) lies on the graph of $f(x) = e^x$ exactly when the point (y, x) lies on the graph of $g(x) = \ln x$. That is, for all $y > 0$ and for all real numbers x:

$$y = e^x \iff x = \ln y$$

We are now in a position to verify some important properties of logarithms, which are summarized below for convenience:

PROPERTIES OF LOGARITHMS

Assume that a and b are restricted to values for which each expression is defined

$$\begin{aligned}\ln(ab) &= \ln a + \ln b\\ \ln\frac{a}{b} &= \ln a - \ln b\\ \ln a^b &= b\ln a\end{aligned}$$

sample proof

The first equation says that *the log of a product is the sum of the logs.*

Here is its proof. The remaining proofs are left as exercises.

Let $a > 0$ and $b > 0$, so that all three expressions $\ln(ab)$, $\ln a$, and $\ln b$ are defined. Then:

$$\begin{aligned} y = \ln a + \ln b &\iff e^y = e^{\ln a + \ln b} && (e^x \text{ is a } 1{-}1 \text{ function})\\ &\iff e^y = e^{\ln a}e^{\ln b} && (\text{properties of exponents})\\ &\iff e^y = ab && (e^x \text{ and } \ln x \text{ ‘undo’ each other})\\ &\iff \ln e^y = \ln ab && (\ln x \text{ is a } 1{-}1 \text{ function})\\ &\iff y = \ln ab \end{aligned}$$

Thus, the sentences $y = \ln a + \ln b$ and $y = \ln ab$ always have the same truth values. That is, $\ln ab = \ln a + \ln b$.

EXERCISE 10

♣ 1. In words, what does

$$\ln\frac{a}{b} = \ln a - \ln b$$

say? Prove it. Be sure to justify every step of your proof.

♣ 2. Prove that:

$$\ln a^b = b\ln a$$

Be sure to write complete mathematical sentences, and justify every step of your proof.

QUICK QUIZ
sample questions

1. Differentiate $f(x) = \sqrt{x}$. Write the derivative using both prime notation, and Leibniz notation.
2. TRUE or FALSE: $\frac{d}{dx}(\frac{\pi\sqrt{2}}{7+\sqrt{3}}) = 0$
3. TRUE or FALSE: The slope of the tangent line to the graph of $y = x^3$ at the point $(2, 8)$ is 12. Show any work leading to your answer.
4. Expand $(a - b)^4$, using Pascal's Triangle.
5. Let $g(x) = e^x + \ln x$. Find $g'(x)$.

KEYWORDS
for this section

Prime notation for the derivative, Leibniz notation for the derivative, the operator $\frac{d}{dx}$*, the derivative of a constant, constants can be 'slid out' of the differentiation process, differentiating sums and differences, the Simple Power Rule for differentiation, Pascal's Triangle, differentiating* e^x *and* $\ln x$*, radicals and fractional exponent notation, properties of* $\ln x$*.*

END-OF-SECTION EXERCISES

♣ Differentiate the following functions. Feel free to use any tools developed in this section.

♣ 1. $f(x) = (2x + 1)^3$

♣ 2. $g(x) = \frac{\sqrt{x}+1}{\sqrt[7]{x}}$

♣ 3. $h(x) = \begin{cases} 3x^2 - 2x + 1 & x \ge 1 \\ 4x - 2 & x < 1 \end{cases}$

What is $\mathcal{D}(h)$?
What is $\mathcal{D}(h')$?

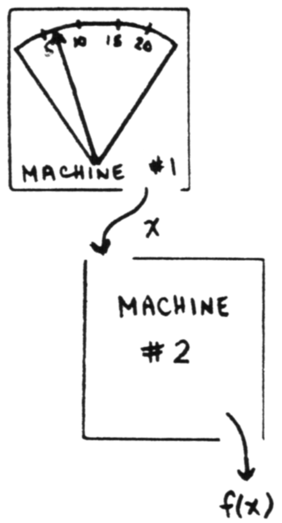
♣ 4. $h(x) = \begin{cases} 3x^2 - 2x + 1 & x \ge 1 \\ 3x - 1 & x < 1 \end{cases}$

What is $\mathcal{D}(h)$?
What is $\mathcal{D}(h')$?

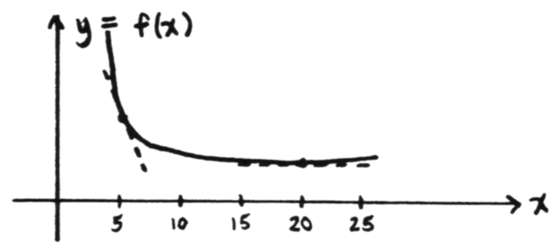
4.4 Instantaneous Rates of Change

Introduction

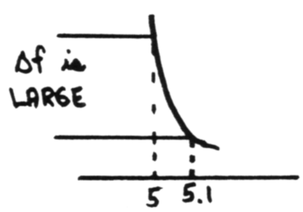
The number $f'(x)$ gives the *slope* of the tangent line to the graph of f at the point $(x, f(x))$ (when the tangent line exists and is not vertical).

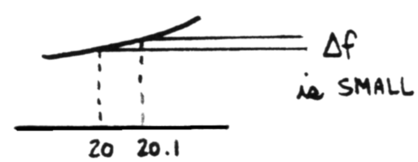


Let's think about this information, from a practical viewpoint. Suppose, in a certain laboratory, there are two machines; call them machine 1 and machine 2. Each day, you must take a reading x from machine 1. This reading is then input into machine 2, which produces an output $f(x)$. Suppose that the relationship between the input x and the output $f(x)$ is shown below.



When the input is 20, the slope of the tangent line to the graph of f is of small magnitude. That is, when x changes from 20 by some small amount, the function value will not change very much. So, if you have misread the information from machine 1 slightly, this will not dramatically affect the output from machine 2.



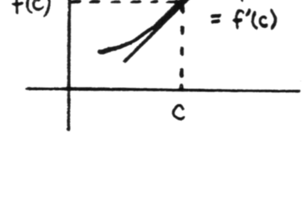


However, when the input is 5, the slope of the tangent line to the graph of f is of large magnitude. Thus, when x changes from 5 by some small amount, the function value will change dramatically. So, if you have misread the information from machine 1 slightly, this *will* dramatically affect the output from machine 2 (a bad situation).

Thus, the information about *how fast the function is changing at a point* can be vitally important.

instantaneous rates of change

There is an important interpretation of the information that $f'(x)$ gives us: *$f'(x)$ tells us how fast the function f is changing at the point $(x, f(x))$.*



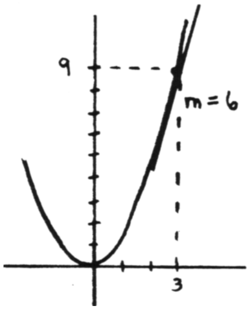
More precisely, for a fixed value of c, the number $f'(c)$ gives the *instantaneous rate of change* of the function values $f(x)$ with respect to x, at the point $(c, f(c))$.

That is, $f(x)$ changes $f'(c)$ times as fast as x at the point $(c, f(c))$.

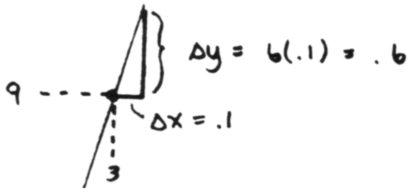
In many situations, we can *use this information to approximate nearby function values,* as illustrated in the next example.

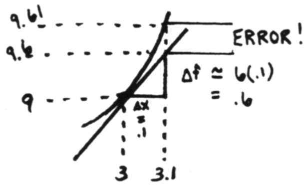
using $f'(x)$ to predict nearby function values

Consider the function $f(x) = x^2$, with derivative $f'(x) = 2x$. The point $(3, 9)$ lies on the graph of f, and the slope of the tangent line at this point is $f'(3) = 2(3) = 6$.



Suppose that knowledge of the function f is lost; *all you now know* is that the point $(3, 9)$ lies on *some graph*, and the slope of the tangent line at this point is 6.





You are asked to *approximate* the function value when $x = 3.1$. This is certainly possible. You know that when $x = 3$, the function values are changing 6 times as fast as the x values. So, if x changes by some small amount, it is reasonable to expect that $f(x)$ will change by approximately 6 times this amount.

The change in x from $x = 3$ to $x = 3.1$ is $\Delta x = 0.1$. So we expect $f(x)$ to change by approximately $6(\Delta x) = 6(0.1) = 0.6$. Thus, it is reasonable to approximate the *new* function value by the *old* function value, plus 0.6. Thus, $f(3.1) \approx 9 + 0.6 = 9.6$.

Now, you find the missing paper and remember that $f(x) = x^2$. Thus, it is now possible to compute the *actual value* of the function when $x = 3.1$: $f(3.1) = (3.1)^2 = 9.61$. How far off were you? You had *estimated* the value at 9.6; the actual value was 9.61. Not bad!

So *we can use the information about the value of the derivative <u>at a single point</u> to approximate values of the function that are nearby!*

the slopes of the tangent lines are changing as we move from point to point

Observe that the approximation we got in the previous example was just that—*an approximation*. That is because our answer was based on the fact that the slope of the tangent line at the point $(3, 9)$ is 6; but *as soon as we move away from that point, this is no longer true.* Indeed, the slopes of the tangent lines *increase* as we travel from $x = 3$ to $x = 3.1$; they increase from 6 to 6.2. So, actually, the rate of change of the function is *faster than* 6 over the interval from $x = 3$ to $x = 3.1$. This is why our approximation of 9.6 was a bit low. The actual function value is 9.61.

EXERCISE 1

Suppose that all you know about a function f is that the point $(3, 7)$ lies on the graph, and the slope of the tangent line at this point is 5.

♣ 1. Approximate, as best you can, $f(3.2)$ and $f(2.9)$.

♣ 2. Sketch two curves that satisfy $f(3) = 7$ and $f'(3) = 5$. On your sketches, show your *approximation* to $f(3.2)$, and the *actual value* $f(3.2)$.

♣ 3. Suppose you now learn that $f(x) = x^2 - x + 1$. Verify that the point $(3, 7)$ lies on the graph of f, and that the slope of the tangent line here is 5.

♣ 4. How far off were your estimates? That is, compare the actual values of $f(3.2)$ and $f(2.9)$ to your estimates from (1).

★★
f' must be continuous

An underlying assumption in this scheme is that f' is continuous in the interval about x under investigation. It is of course possible for a function f to be differentiable at x, and yet have f' NOT be continuous at x. Take, for example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function has as its derivative:

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So, f is differentiable at 0 and $f'(0) = 0$. However, f' is not continuous at 0.

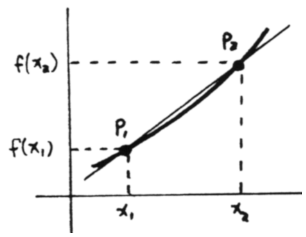
In a motivated class, this importance of the *continuity of f'* could be discussed. Perhaps note that, in analysis, the class of functions that are both *differentiable* on a set S AND have the property that *f' is continuous on S* are given a special name, $C^1(S)$, due to their importance!

DEFINITION
average rate of change

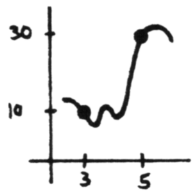
Given a function f and two points $P_1 = (x_1, f(x_1))$, $P_2 = (x_2, f(x_2))$ on the graph of f, we define:

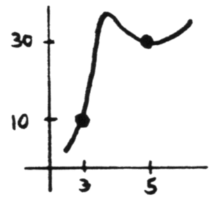
$$\text{the } \textit{average rate of change of } f \textit{ from } x_1 \textit{ to } x_2 := \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Thus, the *average rate of change of f from x_1 to x_2* represents the slope of the secant line through P_1 and P_2.



This seems entirely reasonable: if the points are $(3, 10)$ and $(5, 30)$, then the function has changed by 20 when x has changed by 2, and it seems reasonable to say that, on average, the function has changed by $\frac{20}{2}$ (per a unit change in x). Of course, as illustrated below, the function may behave *entirely differently* between these two points, and yet still exhibit the same average rate of change.





$\Delta f := f(x_2) - f(x_1)$
$\Delta x := x_2 - x_1$
average ROC $= \frac{\Delta f}{\Delta x}$

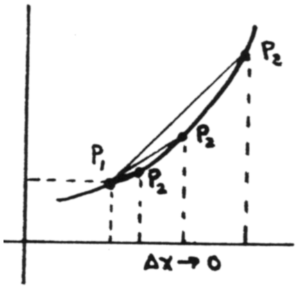
Letting Δf denote the change in function values $f(x_2) - f(x_1)$, and Δx denote the change in x-values $x_2 - x_1$, one can write:

$$\text{average rate of change of } f = \frac{\Delta f}{\Delta x}$$

as $\Delta x \to 0$, the average ROC approaches the instantaneous ROC

Suppose that, for a given function f, there IS a tangent line at the point P_1. If we fix this point P_1, and let the second point P_2 slide closer and closer to P_1 (thus letting $\Delta x \to 0$), then the secant line through P_1 and P_2 approaches the tangent line at P_1. In words, the *average rate of change approaches the instantaneous rate of change, as Δx approaches* 0 .

further appreciation for the Leibniz notation



Whereas the notation Δx is used to denote a *finite* change in x (say from $x = 3$ to $x = 3.1$), it is common in calculus to let (intuitively) dx denote an *infinitesimal* change in x. That is, somehow, dx is meant to represent an *arbitrarily small* change in x.

Similarly, df is used to denote an *arbitrarily small* change in function values.

Armed with this intuition, we can gain a further appreciation for the Leibniz *notation* for the derivative: As Δx approaches 0, $\frac{\Delta f}{\Delta x}$ approaches the slope of the tangent line at x. In general, the closer Δx is to 0, the closer $\frac{\Delta f}{\Delta x}$ will be to the slope of the tangent line at x. The Leibniz notation $\frac{df}{dx}$, therefore, is meant to connote the image of an *infinitesimal change in f* divided by an *infinitesimal change in x*.

More precisely, of course, the notation $\frac{df}{dx}$ should conjure the image of Δx going to 0: it should conjure up the *process* of the second point sliding ever closer to the first. If the notation $\frac{df}{dx}$ succeeds in reminding you of this process each time you see it, then the notation is a good notation.

EXERCISE 2

For the function $f(x) = x^3$, find the average rate of change of f from:

♣ 1. $x = 1$ to $x = 2$

♣ 2. $x = 1$ to $x = 1.5$

♣ 3. $x = 1$ to $x = 1.2$

♣ 4. Find the instantaneous rate of change at $x = 1$. Compare with the average rates of change you just found, and comment.

♣ 5. Why were all of the average rates of change *higher* than the instantaneous rate of change?

EXERCISE 3

For the function $f(x) = -x^2$, find the average rate of change of f from:

♣ 1. $x = -2$ to $x = -1$

♣ 2. $x = -2$ to $x = -1.5$

♣ 3. $x = -2$ to $x = -1.8$

♣ 4. Find the instantaneous rate of change at $x = -2$. Compare with the average rates of change you just found, and comment.

♣ 5. Why were all of the average rates of change *lower* than the instantaneous rate of change?

EXERCISE 4

♣ 1. Sketch the graph of a function f that satisfies the following properties:

- The average rate of change from $x = 0$ to $x = 1$ is 5.
- The instantaneous rate of change at $x = 0$ is -1 and the instantaneous rate of change at $x = 1$ is 2.
- $f(0.5) = 6$

♣ 2. Now, sketch a different curve that satisfies the same properties.

relationship between differentiability and continuity

This section is closed with a very important theorem, stating a relationship between differentiability and continuity.

THEOREM *differentiable at x implies continuous at x*

If a function is *differentiable* at x, then it is *continuous* at x.

differentiability is 'stronger' than continuity

One often refers to this fact by saying that *differentiability is a stronger condition than continuity.* That is, requiring a tangent line to exist at a point, forces the function to be continuous at that point.

proving an implication

This theorem is an implication; that is, it is of the form '*If A, then B*'. Remember that a sentence of this form is automatically true whenever A is false; in such cases, it is called *vacuously true*. To verify that the sentence is *always* true, then, we need only verify that whenever A is true, so is B.

direct proof of $A \implies B$

The proof of an implication '*If A, then B*' often takes the following form:

HYPOTHESIS:	Suppose A is true.
BODY OF PROOF:	Use the fact that A is true (and other necessary tools) to show that B is true.
CONCLUSION:	Conclude that B is true.

This form of proof, where we assume that A is true and then show that B must also be true, is called a *direct proof* of $A \implies B$.

In preparation for the proof of the preceding theorem, the next exercise addresses equivalent characterizations of continuity.

EXERCISE 5 *equivalent characterizations of continuity at x*

Recall that, by definition:

$$f \text{ is continuous at } c \iff \lim_{x\to c} f(x) = f(c)$$

This limit statement makes precise the following intuition: whenever the inputs to f are close to c, the corresponding outputs are close to the number $f(c)$.

♣ 1. What is the *dummy variable* in the limit statement $\lim_{x\to c} f(x) = f(c)$?

♣ 2. Rewrite $\lim_{x\to c} f(x) = f(c)$ with dummy variable y.

♣ 3. Now, using dummy variable y, write the limit statement corresponding to the sentence: f *is continuous at* x.

♣ 4. Convince yourself that the following sentences are all equivalent ways to say that 'f is continuous at x':

$$\begin{aligned} f \text{ is continuous at } x &\iff \lim_{y\to x} f(y) = f(x) \\ &\iff \lim_{h\to 0} f(x+h) = f(x) \\ &\iff \lim_{h\to 0}\big(f(x+h) - f(x)\big) = 0 \end{aligned}$$

For example, if the sentence $\lim_{h\to 0} f(x+h) = f(x)$ is true, then when h is close to 0, $f(x+h)$ must be close to $f(x)$. But when h is close to 0, $x+h$ is close to x. So this says that when the inputs are close to x, the corresponding outputs must be close to $f(x)$, as desired.

One of these equivalent characterizations is used in the next proof.

PROOF

that f differentiable at x implies f continuous at x

Proof. Suppose that f is differentiable at x. That is,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists, and is given the name $f'(x)$.

BODY OF PROOF

To show that f is *continuous* at x, it is shown equivalently that:

$$\lim_{h\to 0}\bigl(f(x+h)-f(x)\bigr)=0$$

To this end:

$$\begin{aligned}\lim_{h\to 0}\bigl(f(x+h)-f(x)\bigr) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot h && (\text{for } h\neq 0,\ \frac{h}{h}=1)\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0}h && (\text{property of limits})\\ &= f'(x)\cdot 0\\ &= 0\end{aligned}$$

CONCLUSION

Thus, f is continuous at x. ∎

EXERCISE 6

♣ 1. What is the *hypothesis* of the theorem just proved?

♣ 2. Where was this hypothesis used in the previous proof?

short form of the previous proof

As mathematicians get more and more proficient at writing proofs, typically the proofs become shorter and shorter. The previous result could be proven more briefly as follows:

Proof. Let f be differentiable at x. Then

$$\lim_{h\to 0}f(x+h)-f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot h=f'(x)\cdot 0=0. \quad ∎$$

Observe that *all the excess* has been cut out of this proof; only the hypothesis and the 'heart' of the body of the proof remain.

the contrapositive of the previous theorem

The previous result is an implication:

$$\text{IF } f \text{ is differentiable at } x, \text{ THEN } f \text{ is continuous at } x. \tag{1}$$

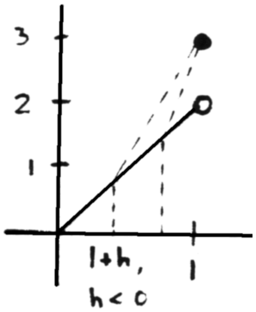
The *contrapositive* of this implication is:

$$\text{If } f \text{ is } \textit{not} \text{ continuous at } x, \text{ then } f \text{ is } \textit{not} \text{ differentiable at } x. \tag{2}$$

Since an implication is equivalent to its contrapositive, and since (1) is true (♣ Why?), sentence (2) is also true. Thus, whenever a function f is NOT continuous at x, we can conclude that f is NOT differentiable at x. This often gives an elegant way to prove that a function is not differentiable at a point, as illustrated next.

EXAMPLE

not continuous $\Longrightarrow$ *not differentiable*



Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 2x & x \in [0,1) \\ 3 & x = 1 \end{cases}$$

Since f is *not* continuous at $x = 1$, it is *not* differentiable at $x = 1$.

The fact that f is not differentiable at $x = 1$ could also be proven directly: the limit

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^-} \frac{2(1+h) - 3}{h} \\ &= \lim_{h\to 0^-} \frac{2h-1}{h} \\ &= \lim_{h\to 0^-} 2 - \frac{1}{h} \end{aligned}$$

does not exist.

However, citing the previous result is more elegant.

QUICK QUIZ

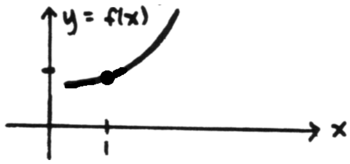
sample questions

1. Let $f(x) = x^3$. Find the average rate of change of f from $x = 1$ to $x = 2$. What is the graphical interpretation of this number?
2. Let $f(x) = x^3$. Find the instantaneous rate of change of f at $x = 1$. What is the graphical interpretation of this number?
3. Consider the function f graphed below. You are not given enough information to find average or instantaneous rates of change. However, you can answer the following question:

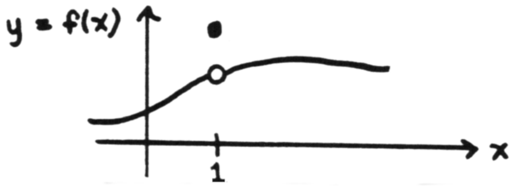
   the instantaneous rate of change of f at $x = 1$ is

   (circle one) (less than greater than equal to)

   the average rate of change of f from $x = 1$ to $x = 2$.

4. Sketch the graph of a function f that satisfies the following properties: $f(x) < 0$ for all $x \in [1,3]$; $f(1) = -5$; the average rate of change of f from $x = 1$ to $x = 3$ is 2; and $f'(2) = -1$.
5. Prove that the function f shown below is not differentiable at $x = 1$.

KEYWORDS

for this section

Instantaneous rate of change, using $f'(x)$ to predict nearby function values, average rates of change, relationship between the instantaneous and average rates of change, What process should the Leibniz notation $\frac{df}{dx}$ conjure up?, relationship between differentiability and continuity, direct proof of $A \implies B$, equivalent characterizations of continuity.

END-OF-SECTION EXERCISES

♣ In each question below, you are given a *point* on the graph of a function f, and the *instantaneous rate of change* of the function at this point.

♣ Use this limited information to predict the value of f at the given nearby point.

♣ Make a sketch that illustrates what you are doing.

1. point: $(1, 3)$
   instantaneous ROC at this point: 2
   nearby point: $(2, ?)$

2. point: $(2, 5)$
   instantaneous ROC at this point: -1
   nearby point: $(3, ?)$

3. point: $f(3) = -1$
   instantaneous ROC at this point: $f'(3) = 5$
   nearby point: $x = 4$

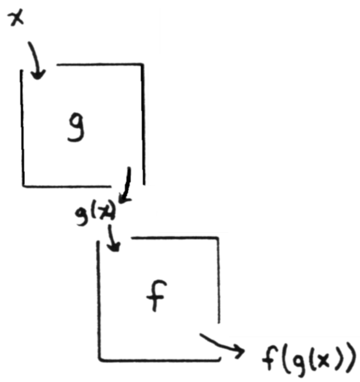
4. point: $f(-3) = 2$
   instantaneous ROC at this point: $f'(-3) = 1$
   nearby point: $x = -4$.

4.5 The Chain Rule
(Differentiating Composite Functions)

composite functions; review

Let f and g be functions of x. Recall that the composition $f \circ g$ is defined by

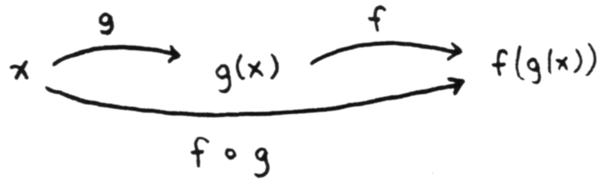
$$(f \circ g)(x) := f(g(x))$$



and has domain:

$$\mathcal{D}(f \circ g) = \{x \,|\, x \in \mathcal{D}(g) \text{ and } g(x) \in \mathcal{D}(f)\}$$

One reads $f \circ g$ as 'f *circle* g' or 'f *composed with* g'.

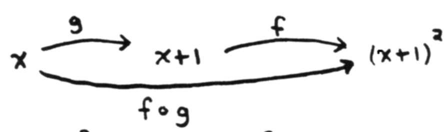


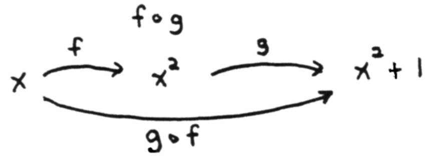
EXAMPLE

Let $f(x) = x^2$ and $g(x) = x + 1$. Find $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) := f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$
$$(g \circ f)(x) := g(f(x)) = g(x^2) = x^2 + 1$$

Here are the corresponding mapping diagrams:





composition of functions is NOT, in general, commutative!

Observe that for these functions f and g, $f \circ g \neq g \circ f$. Indeed, the only value of x for which $f(g(x)) = g(f(x))$ is $x = 0$ (♣ check this).

Thus, composition of functions is *not*, in general, commutative!

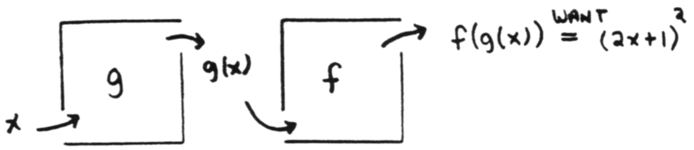
differentiating composite functions; introduction

If f and g are *differentiable*, then it would be reasonable to hope that $f \circ g$ is also differentiable. This is the idea investigated in this section.

EXERCISE 1

♣ 1. For $f(x) = x^2 - 1$ and $g(x) = -2x$, find both $f \circ g$ and $g \circ f$.

♣ 2. View the function $h(x) = (2x+1)^2$ as a composition of functions. That is, find functions f and g for which $h = f \circ g$. (There is not a unique answer.) To do this, appropriately 'name' the function boxes below.



a motivating example

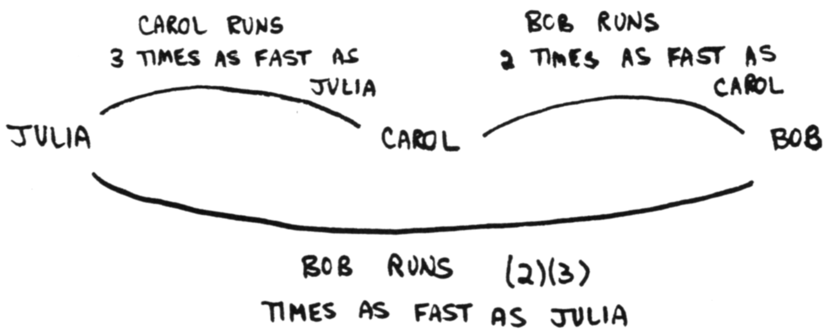
Consider the following scenario:

Suppose that Bob runs 2 times as fast as Carol, and Carol runs 3 times as fast as Julia.

If Julia runs 1 mile/hour, then Carol runs 3(1) miles/hour, and Bob runs 2(3) = 6 miles/hour. Thus, Bob runs 6 times as fast as Julia.

Observe that the rates *multiply*. This situation can be rephrased as follows (here, '*roc*' and '*wrt*' are abbreviations for '*rate of change*' and '*with respect to*', respectively):

$$\textit{roc of Bob wrt Julia} = (\textit{roc of Bob wrt Carol}) \cdot (\textit{roc of Carol wrt Julia})$$



rephrasing in terms of the derivative

Now reconsider the idea in the previous example, but this time in terms of functions and their derivatives. The critical idea is this:

If a function h is differentiable at c, then:

$$h(x) \text{ changes } h'(c) \text{ times as fast as } x \text{ at the point } (c, h(c)) \qquad (*)$$

Refer to the 'function box' sketch below as you read the following discussion.

We want to find $(f \circ g)'(c)$.

That is, we want to know how fast the function values $(f \circ g)(x)$ are changing with respect to x, at the point $\bigl(c, f(g(c))\bigr)$.

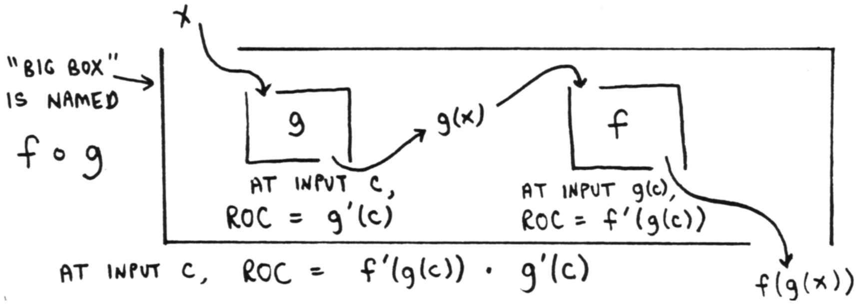
Well, $g(x)$ changes $g'(c)$ times as fast as x at $(c, g(c))$. (Rewrite (*), with 'h' replaced by 'g'.)

And, $f\bigl(g(x)\bigr)$ changes $f'(g(c))$ times as fast as $g(x)$ at $\bigl(g(c), f(g(c))\bigr)$. (Rewrite (*), with '$h$' replaced by '$f$', '$x$' replaced by '$g(x)$', and '$c$' replaced by '$g(c)$'.)

Thus, the rate of change of $(f \circ g)(x)$ wrt x at $(c, f(g(c)))$ should be:

$$f'(g(c)) \cdot g'(c)$$

And it is! The rule that tells us how to differentiate composite functions is called the *chain rule*; the name will be motivated shortly. A precise statement follows after a couple exercises.



EXERCISE 2

Consider the functions $f(x) = x^2$ and $g(x) = x + 1$ of an earlier example. It was found that $(f \circ g)(x) = x^2 + 2x + 1$. In this exercise, you will find the number $(f \circ g)'(2)$ in two different ways.

♣ 1. Differentiate $f \circ g$, and evaluate it at $x = 2$. What do you get?

Now, obtain the same result by doing the following:

♣ 2. Find out how fast the function values $g(x)$ are changing wrt x at $x = 2$. That is, find $g'(2)$.

♣ 3. Find out how fast the function values $f\bigl(g(x)\bigr)$ are changing wrt $g(x)$ at $g(2)$. That is, find $f'(g(2))$. (This is the function f', evaluated at the number $g(2)$.)

♣ 4. Multiply: $f'(g(2)) \cdot g'(2)$. Do your answers agree?

EXERCISE 3

♣ Repeat the previous exercise, except this time analyzing the function $g \circ f$ at $x = 2$. (You will need to make appropriate changes in the questions.)

THE CHAIN RULE Suppose that f and g are functions of x satisfying the following conditions:

- g is differentiable at x
- f is differentiable at $g(x)$

Then, the function $f \circ g$ is differentiable at x, and:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

What is $f'(g(x))$? Be sure that you understand what $f'(g(x))$ represents: *it is the function f', evaluated at $g(x)$.*

Very roughly, in words, *to find out how fast $f \circ g$ changes with respect to x, we find how fast f changes wrt $g(x)$, and multiply by how fast g changes wrt x.*

Remember, *the chain rule tells you how to differentiate composite functions.*

EXAMPLE Problem: For the functions $f(x) = 3x^2 - 2x$ and $g(x) = x^3$, find $(f \circ g)'$ in two different ways.

Solution:

Method I: First find the function $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f(x^3) = 3(x^3)^2 - 2x^3 = 3x^6 - 2x^3$$

Then, differentiation yields:

$$(f \circ g)'(x) = 18x^5 - 6x^2$$

Method II: By the chain rule, $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

$$\begin{aligned} f'(x) &= 6x - 2 \\ f'(g(x)) &= f'(x^3) = 6x^3 - 2 \\ g'(x) &= 3x^2 \\ f'(g(x)) \cdot g'(x) &= (6x^3 - 2)(3x^2) = 18x^5 - 6x^2 \end{aligned}$$

EXERCISE 4 ♣ Let $f(x) = x^3$ and $g(x) = 3x^2 - 2x$. Find $(f \circ g)'$ in two different ways.

motivation for the name 'Chain Rule'

The chain rule can be extended to compositions of more than 2 functions, as follows:

$$
\begin{aligned}
(a \circ b \circ c)'(x) &= \big(a \circ (b \circ c)\big)'(x) \\
&= a'\big((b \circ c)(x)\big) \cdot (b \circ c)'(x) \\
&= a'(b(c(x))) \cdot b'(c(x)) \cdot c'(x)
\end{aligned}
$$

(Make sure you understand *every step* here! The Chain Rule was applied twice; once in going from the first line to the second line; once in going from the second line to the third line.)

Similarly:

$$(a \circ b \circ c \circ d)'(x) = a'(b(c(d(x)))) \cdot b'(c(d(x))) \cdot c'(d(x)) \cdot d'(x)$$

Granted, the notation gets a bit unwieldy, but the important point is: see the *chains* that are forming? This is precisely the motivation for the name.

EXERCISE 5

♣ Write down the formula for the derivative of $a \circ b \circ c \circ d \circ e$ at x. Under what condition(s) do you think your formula holds?

★ *function composition is associative*

The previous argument used the fact that *composition of functions is associative.* This allows us to write things like $a \circ b \circ c$ without ambiguity. Indeed:

$$a \circ b \circ c = (a \circ b) \circ c = a \circ (b \circ c)$$

Leibniz notation for the chain rule

If y is a function of u, and u is a function of x, then the chain rule becomes, in Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Roughly, to find out how fast y changes with respect to x, we find how fast y changes with respect to u, and multiply by how fast u changes with respect to x.

EXAMPLE

Problem: Let $y = u^2$ and $u = 3x^2 - x$. Find $\frac{dy}{dx}$ in two ways.
Method I: Write y as a function of x, and differentiate.

$$\begin{aligned} y &= u^2 = (3x^2 - x)^2 = 9x^4 - 6x^3 + x^2 \\ \frac{dy}{dx} &= 36x^3 - 18x^2 + 2x \end{aligned}$$

Method II: Use the chain rule.

$$\begin{aligned} \frac{dy}{du} &= 2u \\ \frac{du}{dx} &= 6x - 1 \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (2u) \cdot (6x - 1) \\ &= 2(3x^2 - x)(6x - 1) \\ &= (6x^2 - 2x)(6x - 1) \\ &= 36x^3 - 18x^2 + 2x \end{aligned}$$

Note that since we want $\frac{dy}{dx}$ as a function of x, it was necessary to write $\frac{dy}{du}$ in terms of x.

EXERCISE 6

♣ 1. Let $y = 3u$ and $u = x^2 - 1$. Find $\frac{dy}{dx}$ in two ways.

♣ 2. Suppose y is a function of u, u is a function of v, and v is a function of x. Write down the formula for $\frac{dy}{dx}$, using Leibniz notation.

♣ 3. Let $y = u^2$, $u = 3v$ and $v = x^3$. Find $\frac{dy}{dx}$ in two ways.

some remarks on the proof of the chain rule

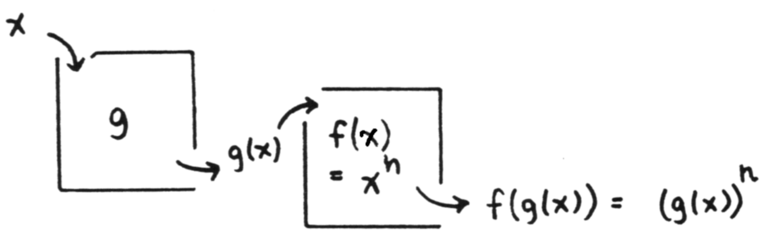
The proof of the chain rule is nontrivial; even in more advanced calculus books, it usually appears in an appendix, or as a supplement to the section on the chain rule.

appreciating the chain rule

It is often hard for the beginning calculus student to *appreciate the importance of the chain rule*. Perhaps this appreciation can begin by seeing all the new differentiation formulas that are an easy consequence of this rule…

how to differentiate $(g(x))^n$

Recall that the Simple Power Rule tells us how to differentiate x^n. However, we don't yet know a simple way to differentiate a *function* raised to a power, $(g(x))^n$. The chain rule will be used to tell us how to differentiate $(g(x))^n$! The trick comes in viewing $(g(x))^n$ as a *composition* of functions, as shown below:



Let $f(x) = x^n$, so that $(f \circ g)(x) = f(g(x)) = (g(x))^n$.

Thus, finding the derivative of $(g(x))^n$ reduces to finding the derivative of the composite function $f \circ g$.

This is easy, by the chain rule. First observe that $f'(x) = nx^{n-1}$ (by the Simple Power Rule), and then:

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= n(g(x))^{n-1} \cdot g'(x)\end{aligned}$$

This result is summarized below.

GENERAL POWER RULE

differentiating $(g(x))^n$

The *general power rule* tells us how to differentiate $(g(x))^n$:

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

Observe that the General Power Rule looks a lot like the Simple Power Rule. The new part is that you must remember to *multiply by the derivative of the function that is being raised to the power.*

EXERCISE 7

♣ Think about what restrictions are necessary (say, on the exponent n and the function g) in order for the formula

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

to make sense.

EXAMPLE

using the General Power Rule

Problem: Differentiate $f(x) = (3x-1)^7$.

Solution: Before the chain rule, we *could* differentiate this function f, but we would first need to multiply it out, and then differentiate term-by-term. The chain rule, however (under the guise of the General Power Rule) makes the problem easy:

$$f'(x) = 7(3x-1)^6 \cdot (3) = 21(3x-1)^6$$

The final form of the derivative obtained from using the chain rule is also *much more desirable* than the form obtained if we first multiplied f out, and then differentiated!

EXAMPLE

Problem: Differentiate $y = [x^2 - (x+1)^{-4}]^4$.

Solution: Be sure to write down complete mathematical sentences!

$$\begin{aligned}\frac{dy}{dx} &= 4[x^2 - (x+1)^{-4}]^3 \cdot \frac{d}{dx}[x^2 - (x+1)^{-4}] \\ &= 4[x^2 - (x+1)^{-4}]^3 \cdot [2x - (-4)(x+1)^{-5}(1)] \\ &= 4[x^2 - (x+1)^{-4}]^3 \cdot [2x + 4(x+1)^{-5}]\end{aligned}$$

Make sure you understand *every line* of this example. The General Power Rule was used twice—do you see where?

To find $\frac{dy}{dx}|_{x=0}$, just evaluate the formula at $x = 0$: $\frac{dy}{dx}|_{x=0} = 4[-1]^3 \cdot [4] = -16$.

EXAMPLE

Problem: Let f be a (differentiable) function of one variable.
Find $\frac{d}{dx}f(x^2+2x+1)$. (In other words, define h by $h(x) := f(x^2+2x+1)$, and find $\frac{d}{dx}h(x)$.)
Solution:

$$\frac{d}{dx}f(\overbrace{x^2+2x+1}^{g(x)}) = f'(\overbrace{x^2+2x+1}^{g(x)})\cdot(\overbrace{2x+2}^{g'(x)})$$

The Chain Rule was applied, taking $g(x) = x^2+2x+1$. The result is the function f', evaluated at x^2+2x+1, and then multiplied by $2x+2$.
Problem: Now, find $\frac{d}{dx}f(x^2+2x+1)|_{x=0}$.
Solution:

$$\begin{aligned}\frac{d}{dx}f(x^2+2x+1)|_{x=0} &= f'(x^2+2x+1)\cdot(2x+2)|_{x=0}\\ &= f'(0^2+2(0)+1)\cdot(2\cdot 0+2)\\ &= f'(1)\cdot 2\\ &= 2f'(1)\end{aligned}$$

This result cannot be simplified further, unless additional information is obtained about the function f.

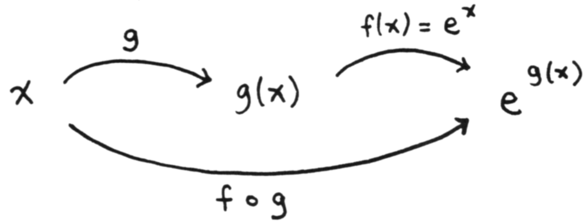
EXERCISE 8

Differentiate the following functions. Use any appropriate method. It may be necessary to rewrite the functions before differentiating.
Then, find $f'(0)$ and $f'(1)$ (if they exist).

♣ 1. $f(x) = (2x+1)^7$
♣ 2. $f(x) = -\frac{1}{\sqrt{x^2+3}}$
♣ 3. $f(x) = (g(h(x)))^3$, where g and h are differentiable functions of one variable
♣ 4. $f(x) = [x+(x^2-1)^{-2}]^{-3}$

differentiating $e^{g(x)}$

To differentiate $e^{g(x)}$, the technique is again to view it as a composition:



First, define $f(x) = e^x$, so that $(f\circ g)(x) = f(g(x)) = e^{g(x)}$.
We seek $(f\circ g)'(x)$.
Recall that $f'(x) = e^x$. Then:

$$\begin{aligned}(f\circ g)'(x) &= f'(g(x))\cdot g'(x)\\ &= e^{g(x)}\cdot g'(x)\end{aligned}$$

This result is summarized next.

DIFFERENTIATION TOOL

differentiating $e^{g(x)}$

Suppose that g is differentiable at x. Then:

$$\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x)$$

EXERCISE 9

$\frac{d}{dx} \ln g(x) = \frac{1}{g(x)} \cdot g'(x)$

♣ Use the chain rule to show that:

$$\frac{d}{dx} \ln g(x) = \frac{1}{g(x)} \cdot g'(x)$$

What restrictions must be placed on g in order that this formula make sense?

EXAMPLE

Problem: Differentiate the following functions. Use any appropriate techniques. Be sure to write complete mathematical sentences.

a) $y = e^{x^2-1}$

b) $f(x) = e^{\sqrt{2x+1}}$

c) $g(t) = \ln\sqrt{t}$

d) $y = \frac{3}{\ln(2x-1)}$

Solutions:

a) $\frac{dy}{dx} = e^{x^2-1} \cdot (2x) = 2xe^{x^2-1}$

b)

$$\begin{aligned} f'(x) &= e^{\sqrt{2x+1}} \cdot \frac{d}{dx}(\sqrt{2x+1}) = e^{\sqrt{2x+1}} \cdot \frac{d}{dx}\big((2x+1)^{1/2}\big) \\ &= e^{\sqrt{2x+1}} \cdot (\frac{1}{2})(2x+1)^{\frac{1}{2}-1}(2) = e^{\sqrt{2x+1}} \cdot (2x+1)^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{2x+1}}}{\sqrt{2x+1}} \end{aligned}$$

c) Whenever possible, simplify the function by using properties of logarithms, before differentiating: $g(t) = \ln\sqrt{t} = \ln(t^{1/2}) = \frac{1}{2}\ln t$

Then, $g'(t) = \frac{1}{2} \cdot \frac{1}{t} = \frac{1}{2t}$.

d) First, rewrite y in a form that 'fits' the general power rule:

$$y = 3[\ln(2x-1)]^{-1}$$

Then:

$$\begin{aligned} \frac{dy}{dx} &= 3(-1)[\ln(2x-1)]^{-2}\frac{d}{dx}(\ln(2x-1)) \\ &= -3[\ln(2x-1)]^{-2} \cdot \frac{1}{2x-1} \cdot 2 \\ &= \frac{-6}{[\ln(2x-1)]^2(2x-1)} \end{aligned}$$

We have now added several important results to the list of Differentiation Tools:

DIFFERENTIATION TOOLS

prime notation	$\frac{d}{dx}$ operator
if $f(x) = k$, then $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
$(kf)'(x) = k \cdot f'(x)$	$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$
$(f+g)'(x) = f'(x) + g'(x)$	$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x) + g'(x)$
$(f-g)'(x) = f'(x) - g'(x)$	$\frac{d}{dx}\big(f(x)-g(x)\big) = f'(x) - g'(x)$
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$
if $f(x) = (g(x))^n$, then $f'(x) = n(g(x))^{n-1} \cdot g'(x)$	$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$
if $f(x) = e^x$, then $f'(x) = e^x$	$\frac{d}{dx}(e^x) = e^x$
if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$	$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$
if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
if $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)} \cdot g'(x)$	$\frac{d}{dx}\big(\ln(g(x))\big) = \frac{1}{g(x)} \cdot g'(x)$

QUICK QUIZ
sample questions

1. Give a precise statement of the Chain Rule for differentiation. What type of function(s) does the Chain Rule tell you how to differentiate?
2. Let $f(x) = \sqrt{2}(1-x)^7$. Find $f'(x)$.
3. Suppose y is a function of w, w is a function of v, v is a function of u, and u is a function of t. Write a formula for $\frac{dy}{dt}$, using Leibniz notation.
4. Fill in the blanks: roughly, the formula

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

tells us that to find out how fast ________________ changes with respect to ________________, we find out how fast ________________ changes with respect to ________________, and multiply by how fast ________________ changes with respect to ________________.

5. Differentiate: $f(x) = \ln \sqrt[3]{2x+1}$

KEYWORDS
for this section

The chain rule (differentiating composite functions), motivation for the name 'chain rule', Leibniz notation for the chain rule, general power rule, differentiating $e^{g(x)}$ *and* $\ln(g(x))$.

END-OF-SECTION EXERCISES

The purpose of these exercises is to give you additional practice with all the differentiation formulas.

♣ Differentiate each of the following functions. Use any appropriate tools and notation. Be sure to write complete mathematical sentences. Write the derivative in a form that resembles, as closely as possible, the original function.

1. $f(x) = \frac{2}{\sqrt{e^x - 1}} + x$
2. $g(x) = \sqrt[3]{x^2 - 1}$
3. $y = (e^x)^3$
4. $y = e^{3x}$
5. $y = (3t - 4)^{11}$
6. $y = (2 - t)^8$
7. $g(t) = 3\sqrt[6]{t^2 + t + 1}$
8. $h(t) = -\sqrt[3]{\frac{1}{t^2 - 1}}$
9. $f(y) = 7e^{-y} + \ln(-y)$
10. $g(y) = \ln \sqrt[3]{-y}$
11. $y = (\ln x)^3$
12. $y = \ln(\sqrt{x}(x + 1))$
13. $y = \dfrac{-1}{t + \sqrt{t - 1}}$
14. $y = \dfrac{2}{(e^{3x} - 1)^4}$

4.6 Differentiating Products and Quotients

Introduction

The derivative of a sum of differentiable functions is always the sum of the derivatives. Is the derivative of a *product* of differentiable functions always the product of the derivatives? The next example shows that the answer is NO.

EXAMPLE

the derivative of a product is NOT the product of the derivatives

Let $f(x) = x$ and $g(x) = x^2$. Then, the product function fg is defined by

$$(fg)(x) := f(x) \cdot g(x) = x^3 \ ,$$

and has derivative $(fg)'(x) = 3x^2$. However, the product of the derivatives is $f'(x) \cdot g'(x) = (1) \cdot (2x) = 2x$. Note that:

$$(fg)'(x) \neq f'(x) \cdot g'(x)$$

EXERCISE 1

the derivative of a quotient is NOT the quotient of the derivatives

♣ Find a pair of functions f and g for which:

$$(\frac{f}{g})'(x) \neq \frac{f'(x)}{g'(x)}$$

The *correct* rule to be used for differentiating products is called the *Product Rule for Differentiation*, and is stated next. Since it is such a surprising result, it is absolutely necessary to study the proof carefully, to understand what gives rise to this formula!

PRODUCT RULE for differentiation

Suppose that f and g are both differentiable at x. Then, the product function fg is also differentiable at x, and:

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

In words, *the derivative of a product is the first times the derivative of the second, plus the derivative of the first times the second.*

The Product Rule is also commonly written as:

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

The trick in the proof of the Product Rule is to *add zero* in an appropriate form, in order to rewrite the difference quotient for fg in a way that brings the difference quotients for f and g into the picture. Study the following proof:

PROOF
of the Product Rule

Proof. Let f and g be differentiable at x, and let $(fg)(x) = f(x)g(x)$. Then $(fg)(x+h) = f(x+h)g(x+h)$. The lines below are numbered for easy reference:

$$
\begin{aligned}
&(1)\ (fg)'(x) := \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&(2) \qquad\qquad = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&(3) \qquad\qquad = \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{-f(x+h)g(x) + f(x+h)g(x)}^{\text{add } 0} - f(x)g(x)}{h} \\
&(4) \qquad\qquad = \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x) \\
&(5) \qquad\qquad = \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x) \\
&(6) \qquad\qquad = \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) g(x) \\
&(7) \qquad\qquad = f(x)g'(x) + f'(x)g(x) \ \blacksquare
\end{aligned}
$$

a discussion of each line of the previous proof; lines (1)–(5)

You must understand the previous proof. In particular, it is essential that you understand the justification for *each step* in this proof.

In line (1), the definition of the derivative is used, applied to the function fg.

In line (2), the definition of the function fg is used.

In line (3), the number 0 is added in an appropriate form. The motivation for adding zero in this form is to bring the difference quotients for f and g into the picture!

In line (4), $f(x+h)$ is factored out of the first two terms; and $g(x)$ is factored out of the last two terms.

In line (5), the limit of a sum is written as the sum of the limits. Is this allowable? Only if the individual limits

$$\lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x)$$

exist! Do they? The next few steps show that, indeed, they do.

line (6)

In the first part of line (6), the limit of a product is written as the product of the limits. Again, this is allowable only if each individual limit

$$\lim_{h\to 0} f(x+h) \quad \text{and} \quad \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

exists. We see in the next step that, indeed, these limits DO exist!

In the second part of line (6), one notes that $g(x)$ is *constant* relative to the limit being investigated. That is, $g(x)$ has nothing to do with h; and constants can be 'slid out' of the differentiation process.

line (7)

In line (7), it is finally demonstrated that all the individual limits exist, thus justifying, (after the fact), the limit operations used in the previous few steps.

Both $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist and equal, respectively, $g'(x)$ and $f'(x)$, by the hypothesis that f and g are differentiable at x.

Why is $\lim_{h\to 0} f(x+h) = f(x)$*?*

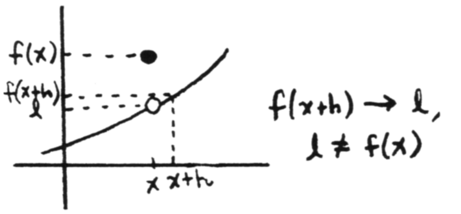
Why is
$$\lim_{h\to 0} f(x+h) = f(x)\ ?$$

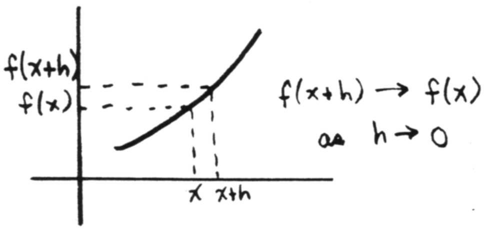
Note that this (true) sentence states that as h approaches 0, the numbers $f(x+h)$ approach $f(x)$. But when h is close to 0, $x+h$ is close to x; so, rephrasing, when the inputs to f are close to x, the corresponding outputs are close to $f(x)$. *This is precisely what it means for f to be continuous at x!* Indeed,

$$f \text{ is continuous at } x \quad \Longleftrightarrow \quad \lim_{h\to 0} f(x+h) = f(x)\ ,$$

so that the statement '$\lim_{h\to 0} f(x+h) = f(x)$' gives an equivalent characterization of continuity of f at x.

But how do we know that f is continuous at x? By hypothesis, f is differentiable at x. Since differentiability is a stronger condition than continuity, f must also be continuous at x.





EXERCISE 2 *prove the product rule*	♣ Prove the product rule for differentiation, *without looking at the book.* Be sure to justify each step in your proof. It would not be unreasonable for your instructor to ask you to prove the product rule on an in-class exam.

'testing' the Product Rule

When presented with a new result, it is always a good idea to 'test it out' in a situation where you already know the answer, to gain confidence. Therefore, we first differentiate a product whose derivative can be found by other means, so that we have a way to 'check' the answer derived from the product rule.

Problem: Let $f(x) = (x^2+1)(2x+5)$. Find f' by
(a) multiplying f out, and differentiating term-by-term; and
(b) using the product rule.

Solution:
(a) $f(x) = 2x^3 + 5x^2 + 2x + 5$, so:

$$f'(x) = 6x^2 + 10x + 2$$

(b) Using the product rule:

$$\begin{aligned} f'(x) &= (x^2+1)(2) + (2x)(2x+5) \\ &= 2x^2 + 2 + 4x^2 + 10x \\ &= 6x^2 + 10x + 2 \end{aligned}$$

Compare!

EXERCISE 3

♣ Let $f(x) = (x+4)(2x^3-1)$. Find f' in two ways: by multiplying out and differentiating term-by-term, and by using the product rule. Compare your results.

EXAMPLE

using the Product Rule

Problem: Differentiate $f(x) = x\sqrt{3x+2}$.

Solution: Note carefully how the $\frac{d}{dx}$ operator is used for intermediate steps in the solution that follows:

$$\begin{aligned} f'(x) &= x\frac{d}{dx}(3x+2)^{1/2} + (1)\sqrt{3x+2} && \text{(product rule)} \\ &= x\cdot\frac{1}{2}(3x+2)^{-1/2}(3) + \sqrt{3x+2} && \text{(general power rule)} \\ &= \frac{3x}{2\sqrt{3x+2}} + \sqrt{3x+2} && \text{(simplify)} \end{aligned}$$

It is sometimes desirable to write the formula for f' in a way that has no radicals in denominators. This is accomplished by rationalizing the denominator in the first term. The word 'rationalize' means to 'remove the radical'. Thus, to 'rationalize the denominator' means to 'remove the radical in the denominator'. Rewriting the first term yields

$$\begin{aligned} \frac{3x}{2\sqrt{3x+2}} &= \frac{3x}{2\sqrt{3x+2}}\cdot\frac{\sqrt{3x+2}}{\sqrt{3x+2}} \\ &= \frac{3x\sqrt{3x+2}}{2(3x+2)}\ , \end{aligned}$$

so that $f'(x)$ becomes:

$$f'(x) = \frac{3x\sqrt{3x+2}}{2(3x+2)} + \sqrt{3x+2}$$

The result can be expressed as a single term by getting a common denominator and combining fractions:

$$\begin{aligned} f'(x) &= \frac{3x\sqrt{3x+2}}{2(3x+2)} + \sqrt{3x+2}\cdot\frac{2(3x+2)}{2(3x+2)} \\ &= \frac{\sqrt{3x+2}(3x+2(3x+2))}{2(3x+2)} \\ &= \frac{\sqrt{3x+2}(9x+4)}{2(3x+2)} \end{aligned}$$

EXAMPLE

using the Product Rule

Problem: Differentiate $f(x) = 3x^2 \cdot h(2x-1)$, where h is a differentiable function of one variable.

Solution: Observe that $h(2x-1)$ denotes the function h, acting on the input $2x-1$, and NOT h times $2x-1$. Thus, $h(2x-1)$ is a composition of functions, which is differentiated using the chain rule. The *overall* form of the function f being differentiated is a product: it is the function $3x^2$, multiplied by $h(2x-1)$. Thus, the product rule is first applied; note how the $\frac{d}{dx}$ operator is conveniently used for intermediate steps:

$$\begin{aligned} f'(x) &= 3x^2 \cdot \frac{d}{dx}\big(h(2x-1)\big) + 6x \cdot h(2x-1) && \text{(product rule)} \\ &= 3x^2 \cdot h'(2x-1) \cdot 2 + 6x \cdot h(2x-1) && \text{(chain rule)} \\ &= 6x^2 \cdot h'(2x-1) + 6x \cdot h(2x-1) && \text{(simplify)} \end{aligned}$$

This expression for f' *cannot* be simplified further, unless we are given additional information about the function h.

EXAMPLE

generalizing the product rule

Problem: Find $\frac{d}{dx}a(x)b(x)c(x)$. Assume that a, b, and c are differentiable.

Solution: Use the 'treat it as a singleton' trick!

$$\begin{aligned} \frac{d}{dx}a(x)b(x)c(x) &= \frac{d}{dx}[\big(a(x)b(x)\big) \cdot c(x)] && \text{(group)} \\ &= a(x)b(x) \cdot c'(x) + \left(\frac{d}{dx}a(x)b(x)\right) \cdot c(x) && \text{(product rule)} \\ &= a(x)b(x)c'(x) + \big[a(x)b'(x) + a'(x)b(x)\big]c(x) && \text{(product rule)} \\ &= a(x)b(x)c'(x) + a(x)b'(x)c(x) + a'(x)b(x)c(x) && \text{(multiply out)} \\ &= a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x) && \text{(rearrange)} \end{aligned}$$

Observe the pattern? By defining the function abc via the rule

$$(abc)(x) := a(x)b(x)c(x) \ ,$$

this result can be written as:

$$(abc)'(x) = a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x)$$

It can also be shown that, (suppressing the '(x)', for convenience):

$$(abcd)' = a'bcd + ab'cd + abc'd + abcd'$$

The 'regular' product rule of course also follows this pattern:

$$(fg)' = f'g + fg'$$

These generalized results are extremely useful when differentiating products with more than two factors.

EXERCISE 4

♣ 1. Make a conjecture (educated guess) about the formula for $(abcde)'$. Assume that a, b, c, d and e are all differentiable. Feel free to suppress the '(x)' in your answer.

♣ 2. Use a 'generalized' product rule to differentiate $f(x) = (2x+1)(x^2-3)(4-x)$.

EXAMPLE

differentiating a quotient

Problem: Use any available differentiation tools to find $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$.

Solution:

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\big(f(x)\cdot\frac{1}{g(x)}\big) && \text{(rewrite as a product)}\\
&= f(x)\frac{d}{dx}(g(x))^{-1} + f'(x)\cdot\frac{1}{g(x)} && \text{(product rule)}\\
&= f(x)\cdot(-1)(g(x))^{-2}g'(x) + \frac{f'(x)}{g(x)} && \text{(general power rule)}\\
&= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}\cdot\frac{g(x)}{g(x)} && \text{(simplify, get common denom.)}\\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} && \text{(simplify)}
\end{aligned}
$$

The formula just derived tells us how to differentiate a quotient of functions, and is called the *Quotient Rule for Differentiation.* Note that the derivative of a quotient is *not* the quotient of the derivatives! That is:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

(This should not be a surprise; a quotient is a special kind of product—and we already learned that the derivative of a product is NOT the product of the derivatives.) A precise statement of the quotient rule follows:

QUOTIENT RULE for differentiation

Suppose that f and g are both differentiable at x, and $g(x) \neq 0$. Then, the function $\frac{f}{g}$ is also differentiable at x, and:

$$\big(\frac{f}{g}\big)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

In words, the derivative of a quotient is: *the bottom, times the derivative of the top, minus the top, times the derivative of the bottom, all over the bottom squared.*

memory device for the quotient rule

Some students find the following cute memory device helpful:

$$\frac{d}{dx}\big(\frac{HI}{HO}\big) = \frac{HO\,dHI - HI\,dHO}{HO\,HO}$$

(Note that 'HI' is 'high up' on the fraction.)

EXAMPLE
using the quotient rule

Problem: Differentiate $f(x) = \frac{2}{x}$ in two ways: using the simple power rule, and using the quotient rule.

Solution: Using the simple power rule:

$$f(x) = 2x^{-1}$$
$$f'(x) = -2x^{-2} = \frac{-2}{x^2}$$

Using the quotient rule:

$$f'(x) = \frac{x(0) - 2(1)}{x^2} = \frac{-2}{x^2}$$

EXERCISE 5

Differentiate each of the following functions in two ways: using the quotient rule, and NOT using the quotient rule. Compare your answers.

♣ 1. $f(x) = \frac{x}{2x-1}$

♣ 2. $g(x) = \frac{3}{(1-x)^4}$

return to the Simple Power Rule; proving that $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers n

Now that the product rule is in hand, it is possible to give an easy proof of the Simple Power Rule for Differentation in the case where n is a positive integer. An extremely important technique, called *proof by induction*, is used. This technique of *proof by induction* is discussed next.

Proof by Induction

A standard approach to proving that a formula is true *for all positive integers* is to use a *proof by induction*. The logic involved in this sort of proof is sometimes called *the domino principle*:

- STEP 1: First, show that the formula is true when $n = 1$.
- STEP 2: Next, show that whenever the formula is true for a positive integer K, then it must also be true for the next positive integer $K + 1$.

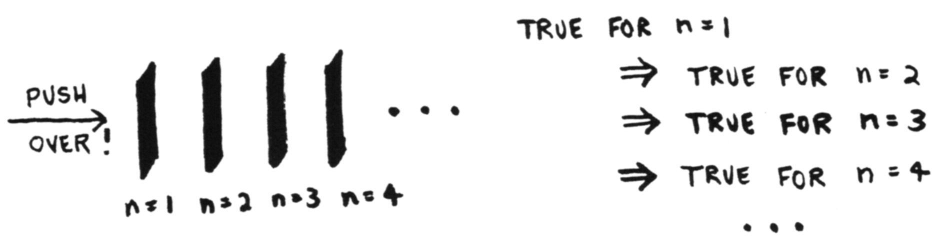
If both of these steps can be accomplished, then look what happens:

Since the formula is true for $n = 1$, it must also be true for $n = 2$.

Since the formula is true for $n = 2$, it must also be true for $n = 3$.

Since the formula is true for $n = 3$, it must also be true for $n = 4$.

- STEP 3: Continuing this scheme, conclude that the formula must be true for *all* positive integers!



Proof that $\frac{d}{dx}x^n = nx^{n-1}$ *for all positive integers* n

Problem: Prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all positive integers n.

Solution: Use a proof by induction.

- Step 1: Show that the formula $\frac{d}{dx}x^n = nx^{n-1}$ is true when $n = 1$.

  Solution to Step 1: When $n = 1$, $x^n = x^1 = x$ and $nx^{n-1} = (1)x^{1-1} = 1$. Since indeed $\frac{d}{dx}x = 1$, the formula is true when $n = 1$.

- Step 2: Let K be any positive integer, and *assume* that the formula is true when $n = K$. (This assumption is commonly referred to as the *inductive hypothesis*.) Show that it must also be true when $n = K + 1$.

  Solution to Step 2: Let K be a positive integer, and assume that $\frac{d}{dx}x^K = Kx^{K-1}$. It is now necessary to show that the formula holds when $n = K+1$. Each of the lines below is numbered for easy reference:

$$
\begin{aligned}
&(1) \qquad \frac{d}{dx}x^{K+1} &&= \frac{d}{dx}x \cdot x^K \\
&(2) &&= x(Kx^{K-1}) + (1)x^K \\
&(3) &&= Kx^K + x^K \\
&(4) &&= (K+1)x^K \\
&(5) &&= (K+1)x^{(K+1)-1}
\end{aligned}
$$

lines (1)–(3)

In line (1), x^{K+1} is viewed as the product $x \cdot x^K$ so that the product rule can be applied.

In line (2), the product rule is used to differentiate $x \cdot x^K$. Note that the *inductive hypothesis* $\frac{d}{dx}x^K = Kx^{K-1}$ is used to differentiate x^K.

In line (3), the expression is simplified, using the fact that $x \cdot x^{K-1} = x^1x^{K-1} = x^{1+K-1} = x^K$.

lines (4) and (5)

In line (4), x^K is factored out of each term.

In line (5), one notes that $K = (K+1) - 1$. Equating line (1) to line (5), we see that

$$\frac{d}{dx}x^{K+1} = (K+1)x^{(K+1)-1} \ ,$$

so that the formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds when n is replaced by $K + 1$. This completes Step (2).

- Step 3: Conclude that the result holds for *all* positive integers n.

  Solution to Step 3: Therefore, $\frac{d}{dx}x^n = nx^{n-1}$ for all positive integers n. ∎

EXERCISE 6

♣ 1. Use a *proof by induction* to prove that the formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

holds for all positive integers n. Be sure to clearly indicate where the inductive hypothesis is used.

Hint: To show that the formula holds for $n = K + 1$, you must show that:

$$1 + 2 + \ldots + K + (K+1) = \frac{(K+1)(K+2)}{2}$$

♣ 2. Use the previous formula to find:

$$1 + 2 + 3 + \ldots + 512$$

♣ 3. How could the formula be used to find

$$100 + 101 + \ldots + 512 \ ?$$

More important differentiation tools have been added in this section. The list is now complete, and is given below:

DIFFERENTIATION TOOLS

prime notation	$\frac{d}{dx}$ operator
if $f(x) = k$, then $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
$(kf)'(x) = k \cdot f'(x)$	$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$
$(f+g)'(x) = f'(x) + g'(x)$	$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$
$(f-g)'(x) = f'(x) - g'(x)$	$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$
if $f(x) = (g(x))^n$, then $f'(x) = n(g(x))^{n-1} \cdot g'(x)$	$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$
if $f(x) = e^x$, then $f'(x) = e^x$	$\frac{d}{dx}(e^x) = e^x$
if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$	$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$
if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
if $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)} \cdot g'(x)$	$\frac{d}{dx}\big(\ln g(x)\big) = \frac{1}{g(x)} \cdot g'(x)$
$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$	$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + f'(x)g(x)$
$(abc)' = a'bc + ab'c + abc'$	$\frac{d}{dx}(abcd) = a'bcd + ab'cd + abc'd + abcd'$
$(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

QUICK QUIZ
sample questions

1. Give a precise statement of the Product Rule for Differentiation.
2. Give a precise statement of the Quotient Rule for Differentiation.
3. Differentiate: $f(x) = x(x+1)^5$
4. Differentiate: $f(x) = \frac{2x+1}{e^{2x}}$
5. Differentiate: $y = x(x+1)(x^2+1)$ Use any correct method.

KEYWORDS
for this section

The product rule for differentiation, proof of the product rule, 'generalized' product rules, the quotient rule for differentiation, proof by induction.

END-OF-SECTION EXERCISES

The purpose of these exercises is to give you additional practice with all the differentiation formulas and notation.

♣ Differentiate the following functions. Use any appropriate formulas. Answer all the additional questions. If an object does not exist, so state. Be sure to write complete mathematical sentences.

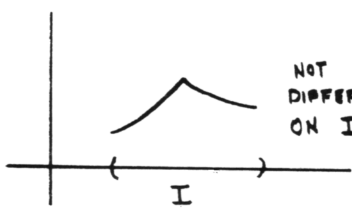
1. $y = x(2-x)^3$; find $y(0)$, $y(t^2)$, $y'(0)$, and $y'(t)$
2. $y = e^{-x}\sqrt{x^2-1}$; find $y(1)$, $y'(1)$, $\frac{dy}{dx}|_{x=\sqrt{2}}$, and $y(1)\cdot y'(\sqrt{2})$
3. $f(x) = e^x \ln x$; find $\mathcal{D}(f)$, $\mathcal{D}(f')$, $f'(e^x)$, and $f'(e^2)$
4. $f(x) = \ln(\ln x)$; find $\mathcal{D}(f)$, $\mathcal{D}(f')$, $f'(e^x)$, and $f'(f(e))$
5. $g(x) = e^{(e^x)}$; find $\lim_{x\to 0} g(x)$, $\lim_{x\to 0} g'(x)$, $\mathcal{D}(g)$, $g(g'(g(0)))$
6. $g(x) = (x-1)(2x+1)(1-x)^7$; find $\lim_{x\to 0} g(x)$, $\lim_{t\to 0} g(t)$, $\lim_{x\to 0} g'(x)$, and $g'(0)$. Is g' continuous at 0? Why or why not?
7. $h(x) = \ln\left(\frac{e^x}{x+1}\right)$; find the equation of the tangent line to the graph of h at $x = 0$
8. $h(x) = \sqrt{\ln x^3}$; find $\mathcal{D}(h)$ and the equation of the tangent line to the graph of h at $x = e$
9. $f(x) = e^{2x}(2x+1)^7$; find the equation of the tangent line to the graph of f at $x = 0$
10. $g(x) = (ax+b)^2(cx+d)^3(x+1)^4$. Assume that a, b, c and d are constants.
11. $h(t) = \frac{e}{(3t-1)^4}$; find the equation of the tangent line to the graph of h at $t = \frac{2}{3}$
12. $y = \frac{\ln t}{\sqrt{t+2}}$; what is the instantaneous rate of change of y with respect to t when $t = 1$?
13. Find all points on the graph of $y = [(x-3)(x+1)(2x-1)]^2$ where the tangent line is horizontal. (Hint: There are 5 such points.)

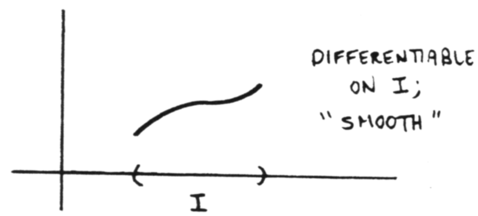
4.7 Higher Order Derivatives

Introduction; smooth functions

When a function f is differentiated, another function, f', is obtained. This *new* function f' may itself be differentiable. Thus, in many cases, one may continually repeat the differentiation process, obtaining the so-called *higher-order derivatives*. This section presents the notation for higher-order derivatives.

If the graph of a function f has a *kink* at x, then f is not differentiable at x. Thus, if f is differentiable at every point in some interval, it must not have any *kinks* in this interval. In this sense, a differentiable function is *smooth*. Mathematicians use the word '*smooth*' to describe the differentiability of a function, but the usage is not entirely consistent: to some, '*smooth*' means once-differentiable; to others, '*smooth*' means infinitely differentiable. In general, the more times a function is differentiable, the 'smoother' it is.





higher-order derivatives; notation

f', f'', f''', $f^{(4)}, \dots, f^{(n)}$

The following *prime notation* is used for the higher-order derivatives:

differentiate f to get $f\,'$;	f' is the (first) derivative of f
differentiate f' to get $f'\,'$;	f'' is the second derivative of f
differentiate f'' to get f''';	f''' is the third derivative of f
differentiate f''' to get $f^{(4)}$;	$f^{(4)}$ is the fourth derivative of f
differentiate $f^{(4)}$ to get $f^{(5)}$;	$f^{(5)}$ is the fifth derivative of f
	$\vdots$
differentiate $f^{(n-1)}$ to get $f^{(n)}$;	$f^{(n)}$ is the n^{th} derivative of f

The notation f'' can be read either as 'f double prime', or as 'the second derivative of f'.

It gets unwieldy to count the number of prime marks, so it is conventional to change to a *numerical* superscript, in parentheses, from about the fourth derivative on. The notation $f^{(4)}$ is usually read as 'the fourth derivative of f'. Observe that the *name* of the n^{th} derivative is $f^{(n)}$; this function, evaluated at x, is denoted by $f^{(n)}(x)$.

The functions f'', f''', $f^{(4)}, \dots$ are called the *higher-order derivatives of f*.

infinitely differentiable

If a function f has the property that $f^{(n)}$ exists (and has the same domain as f) for *all* positive integers n, then we say that f is *infinitely differentible.*

EXERCISE 1 What is the prime notation for each of the following?

- ♣ 1. the second derivative of g
- ♣ 2. the second derivative of g, evaluated at x
- ♣ 3. the derivative of f'''
- ♣ 4. the second derivative of $f^{(6)}$, evaluated at 3

EXAMPLE Let $P(x) = 2x^5 - x^4 + 2x - 1$. Then:

$$\begin{aligned}
P'(x) &= 10x^4 - 4x^3 + 2 \\
P''(x) &= 40x^3 - 12x^2 \\
P'''(x) &= 120x^2 - 24x \\
P^{(4)}(x) &= 240x - 24 \\
P^{(5)}(x) &= 240 \\
P^{(n)}(x) &= 0 \ , \quad \text{for } n \geq 6
\end{aligned}$$

EXERCISE 2 ♣ Find *all* derivatives of:

$$P(x) = 2x^7 - x^3 + 4$$

Be sure to write complete mathematical sentences.

It's a good exercise to differentiate an *arbitrary* polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \ ,$$

since this exercise offers an opportunity to introduce some important *summation* and *factorial* notation. So this is our next project. First, summation notation is introduced.

summation notation;
$\sum_{j=s}^{e} a_j$
the index of the sum is a dummy variable

Summation notation gives a convenient way to display a sum, when the terms share some common property.

For nonnegative integers s ('start') and e ('end') with $s < e$, one defines:

$$\sum_{j=s}^{e} a_j := a_s + a_{(s+1)} + \cdots + a_{(e-1)} + a_e$$

The symbol $\sum_{j=s}^{e} a_j$ is read as: *the sum, as j goes from s to e, of a_j*.

In particular, if $s = 1$ and $e = n$ one gets:

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

The variable j in the above notation is called the *index of the sum*; observe that *once the sum is expanded, this index j no longer appears*. In this sense, it is a *dummy variable*, and we need not be restricted to use of the letter j for this role. Traditionally, the letters i, j, k, m and n are used as indices for summation, precisely because of the strong convention dictating that these letters denote *integer* variables.

When summation notation appears in text (as opposed to in a display), it usually looks like this: $\sum_{j=1}^{n} a_j$. This way, it is not necessary to put extra space between the lines to make room for the '$j = 1$' and 'n'.

EXAMPLE
using summation notation

For example,

$$\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7$$

and:

$$\sum_{k=2}^{5} (k-3)^k = (2-3)^2 + (3-3)^3 + (4-3)^4 + (5-3)^5$$

Also:

$$\sum_{j=1}^{4} 5 = \overbrace{5}^{j=1} + \overbrace{5}^{j=2} + \overbrace{5}^{j=3} + \overbrace{5}^{j=4} = 4 \cdot 5 = 20$$

The sum

$$1 + 2 + \ldots + 207$$

could be written as:

$$\sum_{k=1}^{207} k \quad \text{or} \quad \sum_{n=1}^{207} n \quad \text{or} \quad \sum_{m=1}^{207} m$$

However, don't write something like $\sum_{i=1}^{207} k$, unless you *really want* the expression below!

$$\sum_{i=1}^{207} k = \overbrace{k + k + \cdots + k}^{\text{207 times!}} = 207k$$

EXERCISE 3

practice with summation notation

♣ 1. Expand the following sums. (You need not simplify the resulting sums.)

$$\sum_{j=1}^{6} b_j \ , \quad \sum_{k=1}^{5} (k+1)^k \ , \quad \sum_{m=0}^{4} (m+1) \ , \quad \sum_{i=1}^{n} 2i$$

♣ 2. Write the sum $\sum_{i=1}^{n} 2i$ using a different index.

♣ 3. Let k be a constant. Prove that:

$$\sum_{j=1}^{n} k a_j = k \sum_{j=1}^{n} a_j$$

(Thus, you can 'slide' constants out of a sum.) Be sure to write complete mathematical sentences.

♣ 4. Write the following sums using summation notation:

$$1 + 2 + 3 + \cdots + 100$$
$$34 + 35 + 36 + \cdots + 79$$
$$2 + 4 + 6 + \cdots + 78$$
$$5^2 + 6^3 + 7^4 + 8^5 + \cdots + 20^{17}$$

♣ 5. Prove the following statement:

$$\frac{d}{dx} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} f_i'(x)$$

You may assume that the functions f_i are all differentiable at x. Be sure to write complete mathematical sentences, and justify each step of your proof.

polynomials are infinitely differentiable

Now, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an arbitrary n^{th} order polynomial (so, $a_n \neq 0$). Using summation notation, one can write:

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

(Recall that $x^0 = 1$.) Differentiating once (and using the fact that the derivative of a sum is the sum of the derivatives) yields:

$$\begin{aligned} P'(x) &= \sum_{i=0}^{n} i \cdot a_i x^{i-1} \\ &= \sum_{i=1}^{n} i \cdot a_i x^{i-1} \end{aligned}$$

The index changed from a starting value of 0 to a starting value of 1 since when $i = 0$ the term $i \cdot a_i x^{i-1}$ vanishes, and hence contributes nothing to the sum. Continuing:

$$\begin{aligned} P''(x) &= \sum_{i=2}^{n} i(i-1)a_i x^{i-2} \\ P'''(x) &= \sum_{i=3}^{n} i(i-1)(i-2)a_i x^{i-3} \\ &\vdots \\ P^{(j)}(x) &= \sum_{i=j}^{n} i(i-1)(i-2)\cdots(i-(j-1))a_i x^{i-j} \quad \text{for } 1 \le j \le n \end{aligned}$$

factorial notation, $k!$

The previous formula for $P^{(j)}$ can be cleaned up a bit by using *factorial notation*, discussed next.

For a positive integer k, one defines:

$$k! := k(k-1)(k-2)\cdots(1)$$

The expression '$k!$' is read as 'k *factorial*'. By definition, $0! = 1$.

For example: $3! = 3 \cdot 2 \cdot 1 = 6$ and $200! = 200 \cdot 199 \cdot 198 \cdot \ \ldots \ \cdot 2 \cdot 1$

The product $20 \cdot 19 \cdot 18 \cdot \ldots \cdot 5$ can be written in factorial notation, if one first multiplies by 1 in an appropriate form:

$$\begin{aligned} 20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 5 &= 20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 5 \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{20!}{4!} \end{aligned}$$

This technique is used below, in order to 'clean up' the expression for $P^{(j)}$.

'cleaning up' the expression for $P^{(j)}$

Using the same 'multiply by 1 in an appropriate form' technique illustrated above, one gets:

$$\begin{aligned} &i(i-1)(i-2)\cdots(i-(j-1)) \\ &\qquad = i(i-1)(i-2)\cdots(i-(j-1))\cdot\frac{(i-j)(i-(j+1))\cdots(1)}{(i-j)(i-(j+1))\cdots(1)} \\ &\qquad = \frac{i!}{(i-j)!} \quad \text{for } i \ge j \end{aligned}$$

Thus, all the derivatives of an arbitrary n^{th} order polynomial P can be expressed as:

$$P^{(j)}(x) = \begin{cases} \displaystyle\sum_{i=j}^{n} \frac{i!}{(i-j)!} a_i x^{i-j} & \text{for } 1 \le j \le n \\ 0 & \text{for } j > n \end{cases}$$

Observe that although this notation is extremely compact, it can (especially for a beginner) make an easy idea seem difficult. For experts, however, the compactness of this notation can be extremely beneficial.

EXERCISE 4

Let $P(x) = \sum_{i=0}^{3} a_i x^i$.

♣ 1. Expand this sum. How many terms does P have?

♣ 2. Show that

$$P'(x) = \sum_{i=1}^{3} i \cdot a_i x^{i-1} \ ,$$

by expanding the sum, and verifying that it does indeed give a correct formula for P'.

♣ 3. Find formulas for P'' and P''', in summation notation.

♣ 4. What is $P^{(n)}$, for $n \ge 4$?

EXERCISE 5

practice with factorial notation

♣ 1. Express the following numbers as products. It is not necessary to multiply out these products.

$$5!\ ,\quad 0!\ ,\quad 100!$$

♣ 2. Write the following products using factorial notation:

$$10\cdot 9\cdot 8\cdot \ \ldots\ \cdot 2\cdot 1$$
$$207\cdot 206\cdot 205\cdot \ \ldots\ \cdot 1$$

♣ 3. Write the following product using factorial notation:

$$105\cdot 104\cdot 103\cdot \ \ldots\ \cdot 50$$

Leibniz notation for higher-order derivatives

Here is the Leibniz notation for higher-order derivatives. Let y be a function of x. Then:

$$\begin{aligned}
\frac{d}{dx}(y) &= \frac{dy}{dx} && \text{is the first derivative} \\
\frac{d}{dx}(\frac{dy}{dx}) &= \frac{d^2y}{dx^2} && \text{is the second derivative} \\
\frac{d}{dx}(\frac{d^2y}{dx^2}) &= \frac{d^3y}{dx^3} && \text{is the third derivative} \\
&\vdots \\
\frac{d}{dx}(\frac{d^{n-1}y}{dx^{n-1}}) &= \frac{d^ny}{dx^n} && \text{is the } n^{th} \text{ derivative}
\end{aligned}$$

If one wishes to emphasize that the derivative $\frac{d^ny}{dx^n}$ is being evaluated at a specific value of x, say $x = c$, then one can write either:

$$\frac{d^ny}{dx^n}(c) \quad \text{or} \quad \frac{d^ny}{dx^n}|_{x=c}$$

At first glance, the lack of symmetry in this notation is disturbing: for example, why should we write $\frac{d^2y}{dx^2}$, and not the more symmetric $\frac{d^2y}{d^2x}$?

However, it should be clear from the process illustrated above why this 'un-symmetry' arises. At the n^{th} step, one 'sees' n 'factors' of d upstairs, hence d^ny. Also, at the n^{th} step, one 'sees' n 'factors' of dx downstairs, hence $(dx)^n$, shortened to the simpler notation dx^n. (After all, it *is only notation*, so we want it to be as simple as possible, without sacrificing clarity.)

EXERCISE 6

What is the Leibniz notation for each of the following?

♣ 1. the second derivative of y (where y is a function of x)

♣ 2. the second derivative of y (where y is a function of t)

♣ 3. the second derivative of g (where g is a function of x)

♣ 4. the second derivative of g, evaluated at 2

♣ 5. the derivative of $\frac{d^3y}{dx^3}$

♣ 6. the second derivative of $\frac{d^3y}{dx^3}$, evaluated at 3

EXERCISE 7

In problems (1) and (2), find the second derivative of the given function. Use any appropriate notation.

♣ 1. $y = \frac{x}{e^x}$

♣ 2. $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$

♣ 3. Find the equation of the tangent line to the graph of the first derivative of $f(x) = \frac{x}{e^x}$ at $x = 0$.

QUICK QUIZ
sample questions

1. What is meant by the phrase, 'the higher derivatives of a function f'?
2. Write the second derivative of f, evaluated at x, using both prime notation and Leibniz notation.
3. Expand the sum: $\sum_{i=1}^{3} i^{i+1}$
4. Write $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ using factorial notation.
5. State that 'the derivative of a sum is the sum of the derivatives', using summation notation.

KEYWORDS
for this section

Smooth functions, higher-order derivatives, prime notation for higher-order derivatives, infinitely differentiable, summation notation, factorial notation, Leibniz notation for higher-order derivatives.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a SENTENCE (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. If f is differentiable at x, then the number $f'(x)$ gives the slope of the tangent line to the graph of f at the point $(x, f(x))$.
2. If f is differentiable at x, then the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, and gives the slope of the tangent line to the graph of f at the point $(x, f(x))$.
3. $f'(x)$
4. $f'(3)$
5. $f'(x) = 2x$
6. $y' = 3$
7. If f and g are differentiable at x, then $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$.
8. If f is differentiable at c, then $f'(c) = \frac{df}{dx}(c)$.
9. $\ln ab$
10. For $a > 0$ and $b > 0$, $\ln ab = \ln a + \ln b$.
11. $f'(g(x)) \cdot g'(x)$
12. $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$
13. $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1$
14. $10! = 10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1$
15. $\sum_{i=0}^{3} i = 6$
16. $\sum_{j=1}^{n} a_j$
17. If f is differentiable at c, then $f'(c) = 2$.
18. f is differentiable at c if and only if f is continuous at c

4.8 Implicit Differentiation
(Optional)

Introduction;

$y = f(x)$
explicit representation

You are used to seeing equations of the form:

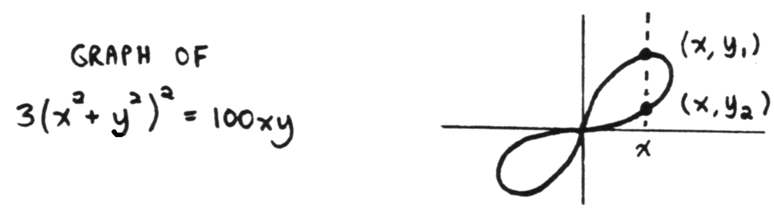
$$y = f(x)$$

Here, y is isolated on one side of the equation, and all the x's appear on the other side. In such a case, one says that y is given *explicitly* in terms of x. When such a representation is possible, y is truly a *function* of x; once a choice for x is made, substitution into the formula $f(x)$ yields the corresponding unique value of y.

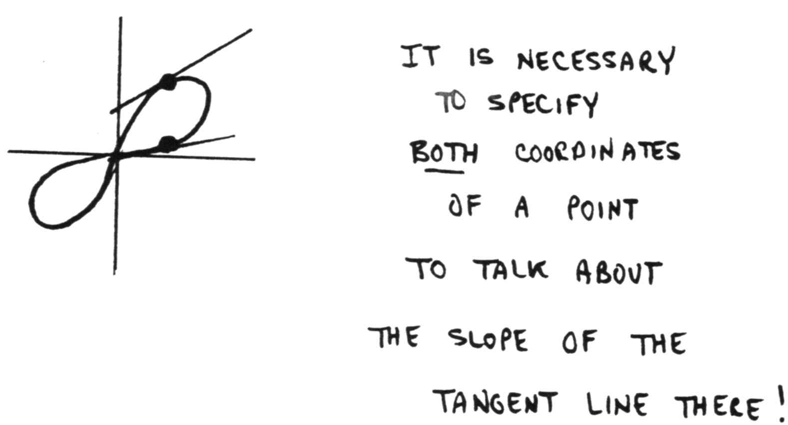
implicit representation

Often, it is *inconvenient* or *impossible* to solve for y in terms of x. In many such instances, the inability to solve uniquely for y in terms of x stems from the fact that y is *not* a function of x.

For example, the graph of $3(x^2 + y^2)^2 = 100xy$ is shown below. Although y is *not* a function of x, one can still talk about the slopes of tangent lines at various points on the graph. However, *since we are not dealing with a function*, to specify the location in the graph in which there is interest, it is necessary to specify *both* an x and y value.



If a relationship between x and y is such that y is not solved explicitly in terms of x, then one says that y is expressed *implicitly* in terms of x.

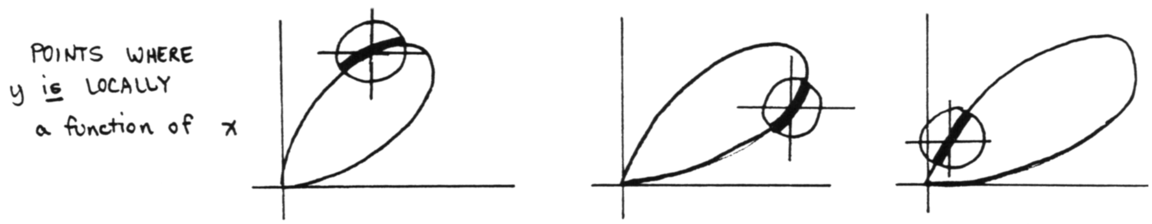


y is locally a function of x

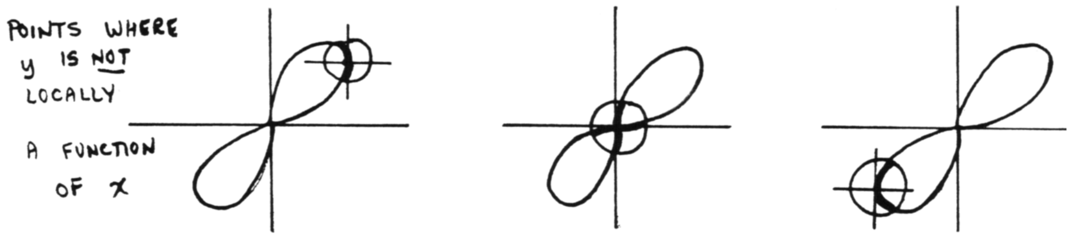
The technique of *implicit differentiation* is used to get information about slopes of tangent lines, in cases when y is given *implicitly* in terms of x. The key idea is this: although y is not (globally) a function of x, if attention is restricted to a *local* situation, then y *CAN* be viewed as a function of x (at most points).

Think about it this way: take a 'mini' coordinate system, and center the origin at a point on a curve. If it is possible to draw a circle (no matter how small!) around this coordinate system, within which one sees the graph of a *function*, then, locally, y is a function of x.

The sketches below show several points at which y IS locally a function of x.

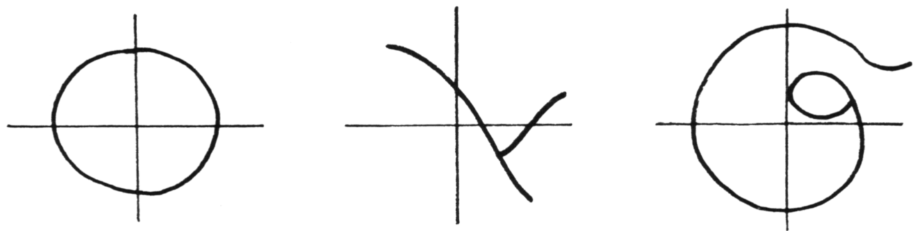


The sketches below show three points at which y is NOT locally a function of x. No matter how small a circle is drawn around the point, there is no way to enclose a piece of graph for which y is a function of x.



EXERCISE 1

♣ On the graphs below, identify any points where y is NOT locally a function of x.



the technique of implicit differentiation

Implicit differentiation works like this: given a relationship between x and y, differentiate both sides of the equation with respect to x, remembering that (locally, at least!) y *is* a function of x.

if y is a function of x, then it must be differentiated accordingly

Suppose that y is a function of x, say $y = y(x)$. Then, *y must be differentiated using the rules that are appropriate for differentiating functions of x.* For example:

$$\frac{d}{dx}y^3 = \frac{d}{dx}(y(x))^3 = 3(y(x))^2 \cdot y'(x)$$

This is usually written more simply as:

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$$

Similarly:

$$\frac{d}{dx}x\ln y = x(\frac{1}{y})\frac{dy}{dx} + \ln y$$

EXERCISE 2

Find the following derivatives, treating y as a function of x.

♣ 1. $\frac{d}{dx}(y^2)$

♣ 2. $\frac{d}{dx}(xy)$

♣ 3. $\frac{d}{dx}(x+y)^3$

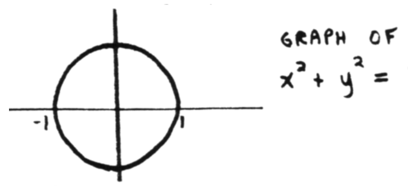
♣ 4. $\frac{d}{dx}(\ln y)$

when y is a function of x, the formula for $\frac{dy}{dx}$ is also a function of x

Whenever y is a (global) function of x, then each point on the curve is uniquely identified by its x-coordinate. In particular, if one wants to talk about the slope of a tangent line at a point, it is only necessary to specify the x-coordinate to locate the point. Therefore, whenever y is a function of x, $\frac{dy}{dx}$ is also a function of x.

However, if y is NOT a function of x, then to identify a point on the curve, BOTH its x and y coordinates are needed. So, to talk about the slope of a tangent line at a particular point, one also needs to specify both coordinates. In such cases, then, the formula for $\frac{dy}{dx}$ involves BOTH x AND y.

EXAMPLE Consider the equation $x^2 + y^2 = 1$. The set of all points (x, y) that make this equation true is the circle of radius 1, centered at the origin. (See the Algebra Review on circles at the end of this section.)



Observe that y is not (globally) a function of x. However, *at all points except* $(1, 0)$ *and* $(-1, 0)$, y is *locally* a function of x.

Differentiating both sides of $x^2 + y^2 = 1$ with respect to x, and remembering that (at least locally) y is a function of x, yields:

$$2x + 2y\frac{dy}{dx} = 0$$

In this case, it is possible to solve for $\frac{dy}{dx}$:

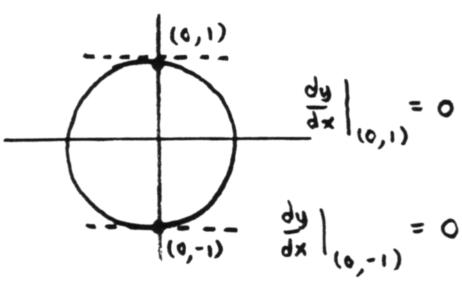
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Observe that this formula for $\frac{dy}{dx}$ depends on *both* x and y. This was expected, since *both* an x and y coordinate are needed to uniquely identify the point where the slope of the tangent line is desired.

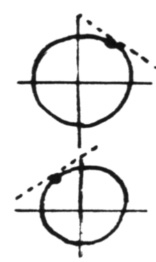
The formula seems to yield reasonable results. For example, $\frac{dy}{dx}|_{(0,1)} = -\frac{0}{1} = 0$. This information reflects the fact that the slope of the tangent line at the point $(0, 1)$ is horizontal.

Also, $\frac{dy}{dx}|_{(0,-1)} = -\frac{0}{-1} = 0$. Again, the tangent line at $(0, -1)$ is horizontal.

Some additional examples are given below. Note in particular that the formula for the derivative *fails* when $y = 0$; there are *vertical* tangent lines at these points.



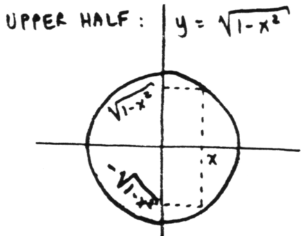
x	y	$\frac{dy}{dx}$
$1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$
$-1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$



same example, different viewpoint

In the previous example, the equation $x^2 + y^2 = 1$ *could have* been solved for y, to obtain:

$$y = \pm\sqrt{1 - x^2}$$



Here, the '+' sign yields the upper half of the circle, and the '−' sign the lower half of the circle. Differentiating $y = +\sqrt{1 - x^2}$ in the normal way yields the slopes of the tangent lines to the upper half of the circle:

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}$$

Thus, the formula is compatible with that obtained by implicit differentiation. However, differentiating implicitly was much easier than this latter approach.

EXERCISE 3

♣ Differentiate $y = -\sqrt{1 - x^2}$ to get a formula for $\frac{dy}{dx}$ that is valid for the lower half of the circle. Show that the result is compatible with the formula obtained by differentiating implicitly.

EXERCISE 4

♣ 1. Graph the equation $(y - 2)^2 + x^2 = 9$.

♣ 2. At what points on the graph is y NOT locally a function of x?

♣ 3. Find $\frac{dy}{dx}$ by differentiating implicitly. At what point(s) does the formula fail? Why?

further uses for implicit differentiation

There are two other common situations where implicit differentiation is extremely useful. These are discussed next.

differentiating complicated products & quotients

Recall that the log of a product is the sum of the logs; the log of a quotient is the difference of the logs. Since differentiating sums and differences is *much easier* than differentiating products and quotients, we can exploit the logarithm as illustrated in the next example.

EXAMPLE

logarithmic differentiation

Problem: Differentiate $y = \dfrac{x^2(x-2)}{\sqrt{2x-3}}$.

Solution: First, find the natural logarithm of y:

$$\begin{aligned}\ln y &= \ln(x^2(x-2)) - \ln\sqrt{2x-3}\\ &= \ln x^2 + \ln(x-2) - \ln(2x-3)^{1/2}\\ &= 2\ln x + \ln(x-2) - \frac{1}{2}\ln(2x-3)\end{aligned}$$

In the equation

$$\ln y = 2\ln x + \ln(x-2) - \frac{1}{2}\ln(2x-3)\ ,$$

y is given *implicitly* as a function of x. Implicit differentiation yields:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x-2} - \frac{1}{2}\cdot\frac{1}{2x-3}\cdot 2$$

Since y truly is a function of x in this example, we expect to be able to get a formula for the derivative as a function of x, and we certainly can:

$$\begin{aligned}\frac{dy}{dx} &= y\cdot\left[\frac{2}{x} + \frac{1}{x-2} - \frac{1}{2x-3}\right]\\ &= \frac{x^2(x-2)}{\sqrt{2x-3}}\left[\frac{2}{x} + \frac{1}{x-2} - \frac{1}{2x-3}\right]\end{aligned}$$

This process of differentiating a function y by first taking the logarithm and then using implicit differentiation is often referred to as *logarithmic differentiation.*

EXERCISE 5

Use logarithmic differentiation to differentiate:

♣ 1. $y = (\frac{1}{x})(\frac{1}{2x-1})(\frac{1}{3x-1})$

♣ 2. $y = \dfrac{x^4\sqrt[3]{x-1}}{\sqrt[5]{2x+1}}$

differentiating variable expressions to variable powers;

logarithmic differentiation

Another common use for implicit differentiation is in differentiating variable expressions raised to variable powers, illustrated next.

Suppose that $y = x^{2x}$. The extended power rule for differentiation *does not apply* here, since the exponent is *not* a constant. Instead, find the natural logarithm of y,

$$\ln y = \ln x^{2x} = 2x\ln x$$

and then differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = 2x\frac{1}{x} + (2)(\ln x) = 2(1+\ln x)$$

Since y *is* truly a function of x, we expect to be able to express the derivative as a function of x, and we can:

$$\frac{dy}{dx} = y\cdot 2(1+\ln x) = 2x^{2x}(1+\ln x)$$

EXERCISE 6

Use logarithmic differentiation to differentiate. In each case, write $\frac{dy}{dx}$ as a function of x.

- ♣ $y = x^x$
- ♣ $y = (2x)^x$
- ♣ $y = (2x)^{3x}$
- ♣ $y = (\sqrt{x+1})^{(x^2)}$

ALGEBRA REVIEW

circles

EXERCISE 7

the relationship between the sentences $a = b$ and $a^2 = b^2$

Consider the equations $a = b$ and $a^2 = b^2$.

♣ 1. Show that these equations are NOT equivalent. That is, find choices for a and b for which the sentences $a = b$ and $a^2 = b^2$ have different truth values.

♣ 2. Now consider the sentence:

$$\text{For } a \ge 0 \text{ and } b \ge 0, \quad a = b \iff a^2 = b^2 \,.$$

The phrase 'For ...' has been used to restrict the universal sets for a and b to the nonnegative real numbers. This sentence asserts that, as long as both a and b are nonnegative, then the equations $a = b$ and $a^2 = b^2$ WILL always have the same truth values. Convince yourself that this is true.

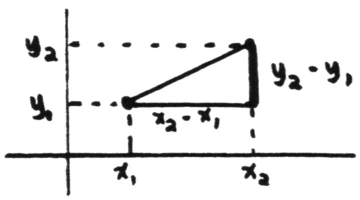
♣ 3. Conclude the following: if you are in a situation where it is known that both a and b are nonnegative, then the sentence $a = b$ can be replaced, if convenient, by the equation $a^2 = b^2$.

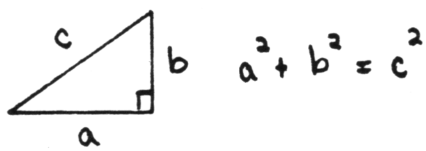
distance between two points

Recall first that the distance between points (x_1, y_1) and (x_2, y_2) is given by:

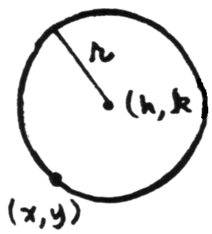
$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

This formula is an immediate consequence of Pythagorean's Theorem.





circles



Now, it is desired to find the equation of the circle with center (h,k) and radius r. That is, we seek an equation that is true for all points (x,y) that lie on the circle of radius r centered at the point (h,k).

This is easy to get: we want those points (x,y) whose distance from (h,k) is equal to r. That is, we want points (x,y) satisfying:

$$\sqrt{(y-k)^2 + (x-h)^2} = r$$

Since both sides of this equation are nonnegative (r is the radius of a circle, and square roots are nonnegative), an equivalent equation is obtained by squaring both sides (see Exercise #7):

$$(y-k)^2 + (x-h)^2 = r^2$$

This is the equation of the circle centered at (h,k), with radius r.

EXAMPLE

Problem: Graph $x^2 + y^2 = 1$.
Solution: Rewrite:

$$x^2 + y^2 = 1 \quad \iff \quad (x-0)^2 + (y-0)^2 = 1^2$$

This is the circle centered at $(0,0)$ with radius 1 .

Problem: Graph $(3-y)^2 + (x+1)^2 = 4$.
Solution: Rewrite:

$$(3-y)^2 + (x+1)^2 = 4 \quad \iff \quad (y-3)^2 + (x-(-1))^2 = 2^2$$

This is the circle centered at $(-1,3)$ with radius 2 .

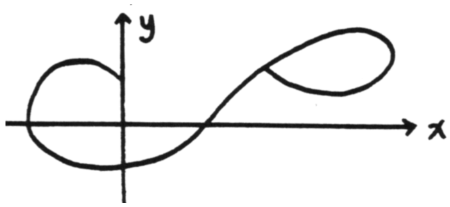
Problem: Graph $x^2 + y^2 + 3y = \frac{7}{4}$.
Solution: Rewrite, by completing the square:

$$\begin{aligned} x^2 + y^2 + 3y = \frac{7}{4} \quad &\iff \quad x^2 + (y^2 + 3y + (\frac{3}{2})^2) = \frac{7}{4} + (\frac{3}{2})^2 \\ &\iff \quad x^2 + (y + \frac{3}{2})^2 = \frac{7}{4} + \frac{9}{4} \\ &\iff \quad x^2 + (y - (-\frac{3}{2}))^2 = 2^2 \end{aligned}$$

This is the circle centered at $(0, -\frac{3}{2})$ with radius 2 .

QUICK QUIZ
sample questions

1. Let $xy^2 = 2$. Find $\frac{dy}{dx}$, by differentiating implicitly.
2. Let $y = x^{2x}$. Find y', by using logarithmic differentiation.
3. Graph $x^2 - 2x + y^2 = 8$.
4. Write an equation where y is given *explicitly* in terms of x; where y is given *implicitly* in terms of x.
5. On the sketch below, identify any point(s) where y is NOT locally a function of x.



KEYWORDS
for this section

Explicit versus implicit representations, y is locally a function of x, implicit differentiation, logarithmic differentiation, differentiating complicated products and quotients, differentiating variable expressions to variable powers, equations of circles.

END-OF-SECTION EXERCISES

♣ Graph the equation (each is a circle).

♣ Identify any point where y is NOT locally a function of x.

♣ Find y' by differentiating implicitly.

♣ Check that the given point(s) lie on the circle; write the equation of the tangent line at these points.

1. $x^2 + 4x + y^2 - 2y + 4 = 0$; $(-2, 2)$, $(-1, 1)$
2. $x^2 + 4x + y^2 - 2y = -4$; $(-2, 0)$, $(-3, 1)$
3. $4x - 2y = -x^2 - y^2 - 1$; $(-1, 1 + \sqrt{3})$
4. $4x - 2y = -x^2 - y^2 - 1$; $(-1, 1 - \sqrt{3})$

4.9 The Mean Value Theorem

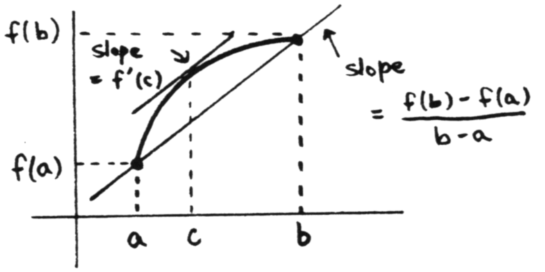
Introduction

The Mean Value Theorem is often referred to as *the Fundamental Theorem of Differential Calculus*. Its importance cannot be overemphasized! Several applications are given in the next section.

The Mean Value Theorem

Suppose that f is differentiable on an open interval (a, b), and continuous on the closed interval $[a, b]$. Then there is at least one number c in (a, b) for which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



motivation for the name

The word '*mean*' often has the same mathematical meaning as the word '*average*', and such is the case here. It has been seen that the quotient

$$\frac{f(b) - f(a)}{b - a}$$

represents the *average (mean) rate of change of the function f on* $[a, b]$. Recall that this quotient gives the slope of the line through the points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem states that there is at least one number $c \in (a, b)$ where the *instantaneous* rate of change $f'(c)$ is the same as the *average rate of change over the entire interval.*

EXERCISE 1

Consider the function $f\colon [a, b] \to \mathbb{R}$, $f(x) = x^2$. For each interval $[a, b]$ listed below, do the following:

- Sketch the graph of f on $[a, b]$.
- Find $c \in (a, b)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- On your graph, show both the tangent line at $(c, f(c))$ and the line through the endpoints of the interval.

♣ 1. $[a, b] = [1, 2]$

♣ 2. $[a, b] = [-1, 1]$

♣ 3. $[a, b] = [-1, 2]$

discussion of the hypotheses to the Mean Value Theorem

The phrase '*f is differentiable on the open interval* (a, b)' means that f is differentiable at x, for every $x \in (a, b)$.

Recall that if f is *differentiable* on (a, b), then it must also be *continuous* on (a, b). (Differentiability is '*stronger*' than continuity!) Thus, by requiring that f be differentiable on (a, b), one is also assured that f is continuous on (a, b). The additional requirement that f be *continuous on the closed interval* $[a, b]$ only adds the assurance that f 'behaves properly' at the endpoints. The following examples illustrate why this requirement is necessary.

EXAMPLE

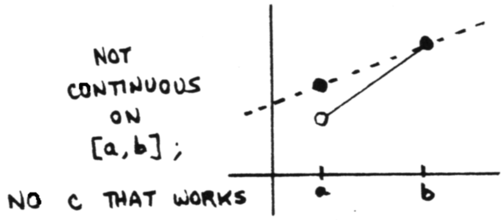
the conclusion of the MVT fails; there is no 'c that works'

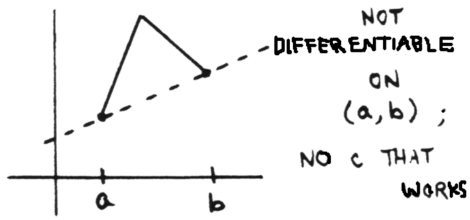
If the hypotheses of the Mean Value Theorem are not met, then its conclusion is not guaranteed.

In the first example below, f is differentiable on (a,b), but not continuous on $[a,b]$.

In the second example, f is not differentible on (a,b).

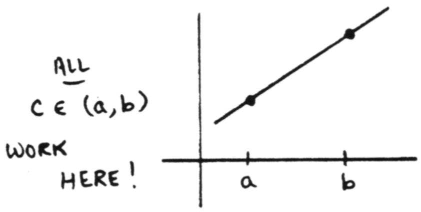
In both cases, there is no '*c that works*'. That is, there is NO POINT $(c, f(c))$ for $c \in (a,b)$ where the tangent line has the same slope as the line through the endpoints of the interval.

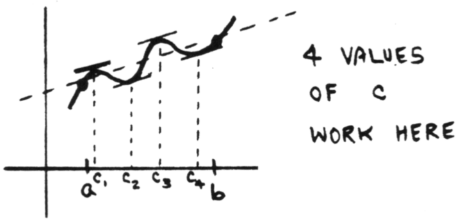




there's not necessarily a unique c that works

The Mean Value Theorem is an *existence* theorem, NOT a *uniqueness* theorem. Thus, it does *not* guarantee a *unique* value of c that works, as the sketches below illustrate.





EXERCISE 2

Sketch the graph of a function f that meets each of the following requirements:

♣ 1. f is differentiable on $(1,3)$, continuous on $[1,3]$, $f(3) = 10$, $f(1) = 0$, $f(2) \neq 5$, $f'(2) = 5$, and f is not linear on $[1,3]$

♣ 2. f is differentiable on $(1,3)$, continuous on $[1,3]$, $f(3) = 10$, $f(1) = 0$, $f(2) = 5$, $f'(2) = 5$, and f is not linear on $[1,3]$

♣ 3. f is differentiable on $(2,5)$, $f(2) = 1$, $f(5) = 3$, and there is NO $c \in (2,5)$ for which $f'(c) = \frac{2}{3}$

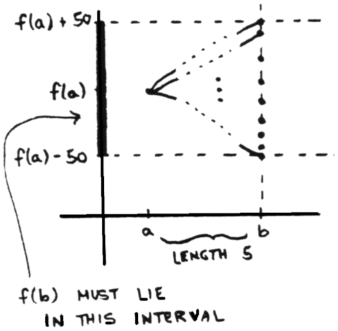
♣ 4. The average rate of change of f on $[0,2]$ is 4, and yet NOWHERE on $(0,2)$ does f have an instantaneous rate of change of 4 .

uses of the Mean Value Theorem

The Mean Value Theorem is the tool pulled out in most every situation where *derivative* information is to be used to gain information about the function itself. It is used extensively in analysis for approximate calculations and to obtain error estimates. Some typical uses are presented below. Also, the Mean Value Theorem is used in the next section to obtain some important results.

EXAMPLE

getting bounds on function values



Suppose that f is differentiable on $\mathbb{R}$. Also, suppose it is known that $|f'(x)| \le 10$ for all $x \in \mathbb{R}$. This means that, at any instant, the function values $f(x)$ *never* change at a rate of magnitude greater than 10 units per unit change in x. So if x changes by 1, what is the most that $f(x)$ could change by? Well, it could increase by $10 \cdot 1$. Or, it could decrease by $10 \cdot 1$.

If x changes by 2, what is the most that $f(x)$ could change by? It could increase by $10 \cdot 2$, or decrease by $10 \cdot 2$.

These ideas are made precise by using the Mean Value Theorem. That is, derivative information is used to get a bound on how much the function f can possibly change over any interval $[a, b]$, as follows.

Let $[a, b]$ be any interval. By the Mean Value Theorem, there exists $c \in (a, b)$ for which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But, by hypothesis, $|f'(c)| \le 10$. Thus:

$$\left| \frac{f(b) - f(a)}{b - a} \right| \le 10$$

That is:

$$|f(b) - f(a)| \le 10|b - a|$$

For example, suppose that $[a, b]$ is an interval of length 5, so that $|b - a| = 5$. Then, $|f(b) - f(a)| \le 10 \cdot 5$. That is, the distance from $f(b)$ to $f(a)$ must be less than or equal to 50. So, $f(b)$ must lie in the interval $\bigl(f(a) - 50\,,\ f(a) + 50\bigr)$.

Bounding techniques such as this are extremely important in analysis.

EXERCISE 3

Suppose f is differentiable on $\mathbb{R}$, and $|f'(x)| \le 2$ for all $x \in \mathbb{R}$. Answer the following questions.

♣ 1. Suppose $f(1) = 5$. In what interval must $f(2)$ lie?

♣ 2. Suppose $f(1) = 5$. In what interval must $f(3)$ lie?

♣ 3. How much can $f(x)$ change by, whenever x changes by an amount Δx?

consequences of the Mean Value Theorem

All the results listed below are consequences of the Mean Value Theorem. Some of these will be studied more thoroughly in future sections. Some use words that have not yet been defined. These results are listed simply to give you an appreciation for the kind of information that can be gleaned from the Mean Value Theorem (MVT)!

For all these results, assume that f is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then f and g differ by at most a constant on $[a, b]$.
- If $f'(x) \ge 0$ for all $x \in (a, b)$ and if $x_1 < x_2$ are in $[a, b]$, then $f(x_1) \le f(x_2)$.
- If $f'(x) > 0$ for all $x \in (a, b)$ and if $x_1 < x_2$ are in $[a, b]$, then $f(x_1) < f(x_2)$.
- If $f'(x) \ge 0$ for all $x \in (a, a + \delta)$, then $(a, f(a))$ is a relative minimum point of f.
- If $f'(x) \ge 0$ for all $x \in (b - \delta, b)$, then $(b, f(b))$ is a relative maximum point of f.

sample proof

Here, the Mean Value Theorem is used to prove the first result in the previous list:

Suppose f is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof. Let $K = f(a)$. It will be shown that $f(x) = K$ for all $x \in [a,b]$, so that f is constant on $[a,b]$.

Choose any point $x \in (a,b]$. Since f is differentiable on the subinterval (a,x) (why?) and continuous on $[a,x]$ (why?), there exists $c \in (a,x)$ for which:

$$f'(c) = \frac{f(x) - f(a)}{x - a} = \frac{f(x) - K}{x - a}$$

Since $f'(c) = 0$, it must be that

$$\frac{f(x) - K}{x - a} = 0 \ ,$$

so that $f(x) - K = 0$, and thus $f(x) = K$. Thus, f is constant on $[a,b]$. ∎

★★
the proof of the MVT

The proof of the Mean Value Theorem is nontrivial. It often goes like this:

- First, prove that if f is differentiable at x and $f'(x) > 0$, then

$$f(x - h) < f(x) < f(x + h)$$

  for all positive h sufficiently small. Prove the similar result for $f'(x) < 0$.
- Prove *Rolle's Theorem*: Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b) = 0$, then there exists at least one number $c \in (a,b)$ for which $f'(c) = 0$.

  Rolle's Theorem is a special case of the Mean Value Theorem.
- To prove the Mean Value Theorem, apply Rolle's Theorem to the function:

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right]$$

the mathematical phrase, 'for all'

The remainder of this section deals with the mathematical phrase, 'for all'. Fortunately, the conventional English usage of this phrase agrees very nicely with its mathematical meaning; for this reason, the author has been able to avoid a careful discussion up to this point. It is time, however, to make things precise.

$S(x)$ denotes a sentence involving the variable x

Let $S(x)$ denote a sentence involving the variable x. For example, $S(x)$ might represent the sentence '$x = 3$', or it might represent the sentence '$3x - 2 > 0$'. Let $\mathcal{U}$ denote the universal set for the variable x.

In order for the sentence

$$\text{` For all } x \in \mathcal{U}, \ \ S(x) \text{ '} \qquad (*)$$

to be true, $S(x)$ must be true, no matter what choice of x is made from the universal set. If there is at least one value of $x \in \mathcal{U}$ for which $S(x)$ is false, then sentence (*) is false.

EXAMPLES

The sentence

$$\text{‘For all } x \in \mathbb{R},\ \ x^2 \ge 0\text{ ’}$$

is true. No matter what real number x is chosen, the sentence ‘$x^2 \ge 0$’ is true.

The sentence

$$\text{‘ For all } x \in \mathbb{R},\ \ x^2 > 0\text{ ’}$$

is false. Choosing $x = 0$, the sentence ‘$0^2 > 0$’ is false.

The sentence

$$\text{‘ For all } x \in \mathbb{R},\ \ x^2 < 0\text{ ’}$$

is false. Here, no matter *what* value of x is chosen, the sentence ‘$x^2 < 0$’ is false.

The sentence

$$\text{‘ For all } x > 0,\ \ |x| = x\text{ ’}$$

is true. Whenever x is a positive number, the sentence ‘$|x| = x$’ is true.

The sentence

$$\text{‘ For all sets } A \text{ and } B,\ \ A \subset A \cup B\text{ ’}$$

is true. No matter what sets are chosen for A and B, A is always a subset of $A \cup B$.

What does it mean for (*) to be false?

If a sentence of the form

$$\text{‘ For all } x \in \mathcal{U},\ \ S(x)\text{ ’}$$

is false, then all that can be said (without additional information) is that *there is at least one $x \in \mathcal{U}$ for which $S(x)$ is false.*

EXERCISE 4

TRUE or FALSE:

♣ 1. For all $x \in \mathbb{R}$, $|x| \ge 0$

♣ 2. For all $x \in \mathbb{R}$, $|x| > 0$

♣ 3. For all $t < 0$, $|t| = -t$

♣ 4. For all $x \in (2, 3)$, $x \ge 0$

♣ 5. For all sets A and B, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

♣ 6. For all functions f and g that are differentiable at x, $(f + g)'(x) = f'(x) + g'(x)$

the universal set is sometimes omitted

Sometimes, mathematicians get a bit casual with their use of ‘for all’. For example, the universal set is frequently omitted, if it is understood from context.

For example, in a course such as this, where the universal set is understood to be $\mathbb{R}$ (unless otherwise stated), the (true) sentence

$$\text{‘ For all } x,\ \ |x| \ge 0\text{ ’}$$

is understood to be an abbreviation for the more correct sentence:

$$\text{‘ For all } x \in \mathbb{R},\ \ |x| \ge 0\text{ ’}$$

the words 'for all' are sometimes omitted

Even more annoying—the words 'for all' are often omitted, in certain types of situations, if they are understood from context! For example, the (true) sentence

$$\text{` } 2x + 1 = 0 \quad \iff \quad x = -\frac{1}{2} \text{ '}$$

is really an abbreviation for the more correct (true) sentence:

$$\text{` For all } x \in \mathbb{R},\ \ (2x + 1 = 0 \iff x = -\frac{1}{2}) \text{ '}$$

The connective ' $\iff$ ' is defined via the truth table given below:

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Note that the sentence '$A \iff B$' is true precisely when A and B have the same truth values (either they are both true, or both false). Thus, the sentence

$$\text{` For all } x \in \mathbb{R},\ \ (2x + 1 = 0 \quad \iff \quad x = -\frac{1}{2}) \text{ '}$$

is true because, no matter what real number is chosen, the sentences '$2x+1 = 0$' and '$x = -\frac{1}{2}$' always have the same truth values. Observe that this is in perfect agreement with earlier discussions of mathematical equivalence.

EXERCISE 5

How might a mathematician abbreviate the following (true) sentences, if appropriate information is understood from context?

♣ 1. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $x + y = y + x$

♣ 2. For all $x \in \mathbb{R}$, $x = 2 \iff 3x = 6$

♣ 3. For all sets A and B, $A \subset A \cup B$

♣ 4. For all statements P and Q, $\big((P \Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)\big)$

(A *statement* is merely a sentence that is either true, or false, but not both. For example, '$1 = 2$' is a false statement; '$2 = 1 + 1$' is a true statement; and '$x = 1$' is not a statement until a particular value of x is substituted into the equation.)

QUICK QUIZ
sample questions

1. Give a precise statement of the Mean Value Theorem.
2. What does the word 'mean' in the Mean Value Theorem refer to?
3. Let $f(x) = x^3$ and $[a, b] = [1, 3]$. Find the number c that is guaranteed by the Mean Value Theorem. Make a sketch that illustrates your work.
4. Suppose that f is differentiable on (a, b), but there is no $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. What can be said about the function f?
5. Suppose that f is continuous on $[a, b]$, but there is no $c \in (a, b)$ with $f'(c)$ equal to the average rate of change of f over $[a, b]$. What (if anything) can be said about the function f?

KEYWORDS
for this section

The Mean Value Theorem (MVT), motivation for the name, using the MVT to bound function values, some consequences of the MVT, the mathematical phrase 'for all', truth table for $A \iff B$.

END-OF-SECTION EXERCISES

These exercises review many of the ideas in Chapter 4.

1. What information does the limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ give (when it exists)? Answer in English.

2. What information does the limit $\displaystyle\lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ give (when it exists)? Answer in English.

3. Suppose that, for a given function f, it is known that $f'(2)=4$. What does this tell us about the function f? Sketch the graphs of two different functions satisfying this requirement.

4. Suppose that, for a given function f, it is known that $f(2)=1$ and $f'(2)=-1$. Sketch the graphs of two different functions satisfying this requirement.

5. Use the *definition of derivative* to find $f'(x)$ if $f(x)=-x^2$.

6. Use the *definition of derivative* to find $f''(x)$ if $f'(x)=3x$.

7. Sketch the graph of a function f that is continuous at 3, but not differentiable at 3.

8. Is it possible to sketch the graph of a function that is differentiable at 3, but not continuous at 3? Why or why not?

9. Differentiate $f(x)=xe^{2x}\ln(2-x)$. Use any appropriate tools. Then, find: $\mathcal{D}(f)$, $\mathcal{D}(f')$, the equation of the tangent line when $x=0$.

10. Let $f(x)=x^2$ and $g(x)=\frac{1}{x}$. Find $\frac{d}{dx}f(g(x))$ in two different ways (using the Chain Rule, and NOT using the Chain Rule).

NAME (1 pt)
SAMPLE TEST, worth 100 points, Chapter 4

Show all work that leads to your answers. Good luck!

1. (14 pts) TRUE or FALSE. (Circle the correct response.)

T	F	If f is continuous at x, then f is differentiable at x.
T	F	$\mathbb{R} - (1, 2] = (-\infty, 1] \cup (2, \infty)$
T	F	The Chain Rule tells us how to differentiate composite functions.
T	F	Let K and n denote positive integers, and let $P(n)$ denote some statement about n. Suppose that $P(1)$ is true. Also suppose that if $P(K)$ is true, then $P(K+1)$ must be true. Then $P(1007)$ is true.
T	F	$\sum_{i=1}^{3} i^{2i} = 1 + 2^4 + 3^6$
T	F	$72 \cdot 71 \cdot \ldots \cdot 49 = \frac{72!}{48!}$
T	F	For all functions f and g, if f and g are differentiable at x, and $g'(x) \neq 0$, then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g'(x)}$.

2. (8 pts) Use the DEFINITION of derivative to find $f'(x)$ if $f(x) = x^2 - 1$. Be sure to write down complete mathematical sentences. I'll get you started:

$$f'(x) = \lim_{h \to 0}$$

3. (5 pts) Use Pascal's triangle to expand $(a + b)^4$.

4. (28 pts)	Differentiate the following functions. Use any appropriate tools. Be sure to write complete and correct mathematical sentences.
	(7 pts) $f(x) = \frac{\sqrt{2}}{\sqrt{x}}$
	(7 pts) $y = xe^{2x-1}$
	(7 pts) $g(t) = \frac{\ln t}{\sqrt[3]{t^2-1}}$
	(7 pts) $y = (x+1)^{11}(e^x)(x^3)$ (A 'generalized product rule' may be helpful here.)

5. (10 pts)	Sketch the graph of a function f satisfying each set of requirements:
	(5 pts) f is continuous on $[0,2]$, $f(0) = 1$, $f(2) = -1$, f is not differentiable at $x = 1$
	(5 pts) $\mathcal{D}(f) = [1,2]$, $f(1) = -1$, the average rate of change of f on $[1,2]$ is 4, f is not linear on $[1,2]$

6. (10 pts)

(3 pts) Find the slope of the tangent line to the graph of $f(x) = x^3$ at $x = 1$.

(4 pts) Find the EQUATION of the tangent line to the graph of $f(x) = x^3$ at $x = 1$.

(3 pts) Find all points (x, y) on the graph of $f(x) = x^3$ where the tangent line has slope 12.

7. (4 pts)

(2 pts) Give the PRIME notation for each of the following:

- the second derivative of f ________________
- the second derivative of f, evaluated at 2 ________________

(2 pts) Give the LEIBNITZ notation for each of the following. Assume that y is a function of x.

- the first derivative of y ________________
- the second derivative of y, evaluated at 0 ________________

8. (5 pts)

Suppose that f, g and h are differentiable everywhere. Then:

$$\frac{d}{dx} f(g(h(x))) = \underline{\hspace{8cm}}$$

9. (8 pts)

Give a precise statement of the Mean Value Theorem, and make a sketch that illustrates what this theorem is saying.

10. (7 pts)

(Optional) Differentiate $f(x) = x^{2x}$.

This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 5

USING THE INFORMATION GIVEN BY THE DERIVATIVE

In this chapter, the information that can be gleaned from the first and second derivatives of a function is studied. Such information is extremely useful in locating extreme values (maxima or minima) and in graphing functions.

5.1 Increasing and Decreasing Functions

increasing and decreasing functions; roughly

Roughly, a function f is *increasing* if its graph moves UP, traveling from left to right; and is *decreasing* if its graph moves DOWN, traveling from left to right. The precise definitions follow.

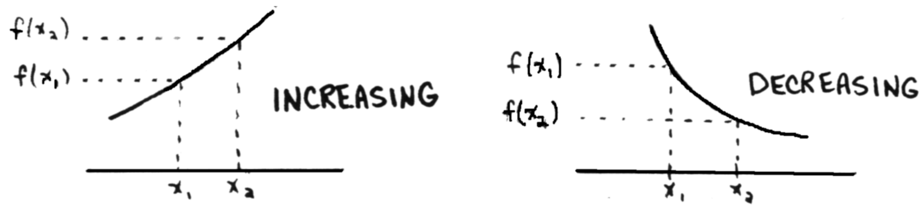
DEFINITION *increasing and decreasing functions*

A function f is *increasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) < f(x_2)$$

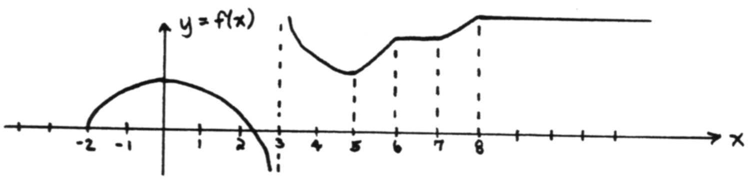
A function f is *decreasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) > f(x_2)$$



EXAMPLE

Problem: Identify the open intervals on which the function graphed below is increasing and decreasing.



Solution:

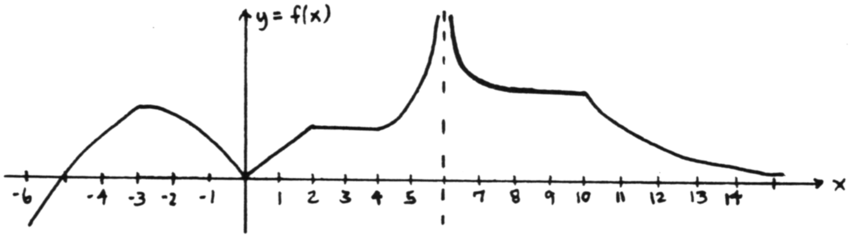
f increases on $(-2, 0) \cup (5, 6) \cup (7, 8)$

f decreases on $(0, 3) \cup (3, 5)$

f is neither increasing nor decreasing on $(6, 7) \cup (8, \infty)$

EXERCISE 1

♣ Identify the open intervals on which the function graphed below is increasing and decreasing. Be sure to write complete mathematical sentences.



EXERCISE 2

Sketch the graphs of functions satisfying the following properties:

♣ 1. f increases on $(1,3)$, and $f(x) < 0 \ \ \forall\ x \in (1,3)$

♣ 2. f increases on $(1,3)$, $f(2) = 0$, and $f(x) < -5 \ \ \forall\ x \in (1,1.5)$

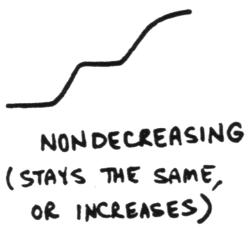
♣ 3. f increases on $(1,3)$, and is not differentiable at $x = 2$

♣ 4. f increases on $(1,3)$, decreases on $(3,5)$, and is differentiable at $x = 3$
What do you suspect that the number $f'(3)$ must be?

♣ 5. f increases on $(1,3)$, decreases on $(3,5)$, and is not differentiable at $x = 3$

EXERCISE 3

nonincreasing, nondecreasing functions



Sometimes it is important to know where a function *doesn't decrease.* A function *doesn't decrease* if it either increases or stays the same, and functions satisfying this property are called *nondecreasing* functions. Here's a precise definition:

A function f is *nondecreasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) \le f(x_2)$$

When you see the word 'nondecrease', think to yourself: 'does not decrease'.

♣ 1. How should you read aloud '$x_1 < x_2 \implies f(x_1) \le f(x_2)$'? What is the hypothesis of this implication? What is the conclusion?

♣ 2. Let $x_1 = 1$, $x_2 = 3$, $f(x_1) = -1$ and $f(x_2) = -0.5$. For these choices:

a Is the hypothesis of $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

b Is the conclusion of $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

c Is the sentence $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

♣ 3. Now let $x_1 = 1$, $x_2 = 3$, $f(x_1) = -0.5$ and $f(x_2) = -1$. Answer the same questions as in (2).

♣ 4. Let $I = (0,4)$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $f(1) = 1$, $f(2) = 2$, $f(3) = 1$. Based on this information alone, can the truth of the sentence

$$\text{for all } x_1, x_2 \in I, \ \ x_1 < x_2 \implies f(x_1) \le f(x_2)$$

be decided? If so, is it true or false?

♣ 5. Repeat (4) with $I = (0,4)$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $f(1) = 1$, $f(2) = 2$, $f(3) = 3$.

♣ 6. Sketch the graph of a function that is nondecreasing on the interval $(1,3)$, but *not* increasing on $(1,3)$.

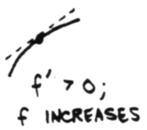
♣ 7. Sketch the graph of a function f that is increasing on $(0,1)$ and nondecreasing on $(1,2)$. Is f nondecreasing on $(0,2)$? Justify your answer.

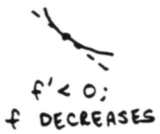
♣ 8. True or False: Every function that is increasing on I is nondecreasing on I.

True or False: Every function that is nondecreasing on I is increasing on I.

♣ 9. Based on your experience with nondecreasing functions, write down a precise definition of a *nonincreasing* function on an interval I.

getting increasing/ decreasing info from the derivative





If f is *differentiable* on (a,b), then for any $c \in (a,b)$, the number $f'(c)$ exists and tells us how fast the function values $f(x)$ are changing with respect to x at the point $(c, f(c))$. It seems plausible that this derivative information could be used to determine where f is increasing or decreasing: intuitively, where the slopes of the tangent lines are positive, the graph should be travelling UP (increasing), and where the slopes are negative, the graph should be travelling DOWN (decreasing).

It is indeed the case that increasing/decreasing information can be obtained from the sign of the derivative. The proof of this fact is a classic application of the Mean Value Theorem: using information about f' to glean information about f!

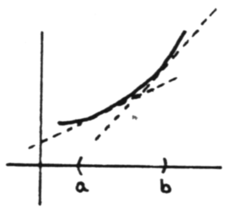
THEOREM

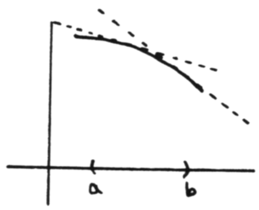
increasing, decreasing info from f'

Suppose that f is differentiable on (a,b).

If $f'(x) > 0$ for all $x \in (a,b)$, then f is *increasing* on (a,b).

If $f'(x) < 0$ for all $x \in (a,b)$, then f is *decreasing* on (a,b).





At first glance, one might be concerned that the Mean Value Theorem does not apply here, since the requirement about continuity at the endpoints of the interval has not been met. Make sure you understand how this 'problem' is circumvented in the following proof.

PARTIAL PROOF

Proof. Let f be differentiable on (a,b) and suppose that $f'(x) > 0 \ \forall\, x \in (a,b)$. Choose *any* x_1, x_2 in (a,b) with $x_1 < x_2$ (so that $x_2 - x_1 > 0$). Observe that x_1 cannot be a, since $a \notin (a,b)$. Similarly, x_2 cannot be b.

Since f is differentiable at x_1 and x_2 (by hypothesis), f must also be continuous at x_1 and x_2. (♣ Why?) Thus, f is not only differentiable on the open interval (x_1, x_2), but also continuous on the closed interval $[x_1, x_2]$. Thus, the Mean Value Theorem guarantees existence of a number c in (x_1, x_2) for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But since $c \in (a,b)$ and $f'(x) > 0 \ \forall\, x \in (a,b)$, we have $f'(c) > 0$. Thus:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Multiplying both sides of this inequality by the positive number $x_2 - x_1$ yields the equivalent inequality

$$f(x_2) - f(x_1) > 0 \ ,$$

that is, $f(x_2) > f(x_1)$. It has been shown that *whenever* $x_1, x_2 \in I$ satisfy $x_1 < x_2$, it is also true that $f(x_1) < f(x_2)$. So, f is increasing on I.

The remaining case is left as an exercise. ▮

EXERCISE 4

proof of the remaining case

♣ Prove the following result:

If f is differentiable on (a,b) and $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on (a,b).

Use the previous proof as a guide, and make appropriate changes. Be sure to write complete mathematical sentences.

shorter forms of the proof

More advanced students of mathematics would condense the proof a bit. Here's what a shorter proof might look like:

Proof. Let f be differentiable on (a,b) with $f'(x) > 0 \ \forall\, x \in (a,b)$. Choose any $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Since f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$, the MVT guarantees existence of a number $c \in (x_1, x_2)$ for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) > 0$ yields the desired conclusion that $x_1 < x_2 \implies f(x_1) < f(x_2)$. ▮

EXERCISE 5

write a shorter proof

♣ Prove that if f is differentiable on (a,b) and $f'(x) \ge 0$ for all $x \in (a,b)$, then f is nondecreasing on (a,b). Use the previous 'shorter proof' as a guide, and make appropriate changes. Be sure to write complete mathematical sentences.

shortest proof

A real expert would merely say the following:

Proof. The proof is a direct consequence of the Mean Value Theorem. ▮

EXAMPLE

Problem: Consider the function:

$$P(x) = 2x^3 + 3x^2 - 12x$$

Find the open intervals on which P increases; decreases.

First Solution: Differentiation yields:

$$\begin{aligned} P'(x) &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x-1)(x+2) \end{aligned}$$

Where is $P'(x)$ positive? Negative? Recall that a product ab of real numbers is positive in two situations:

$$ab > 0 \iff (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

EXERCISE 6

investigating $ab > 0$

♣ 1. What *are* the two situations for which ab is positive? Answer in English.

Remember that the symbol '$\iff$' in the sentence

$$ab > 0 \iff (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

tells us that the 'smaller' sentences being compared *always* have the same truth values. If one is true, so is the other; if one is false, so is the other.

♣ 2. If $a = 1$ and $b = 2$, is the sentence '$ab > 0$' true or false? How about the sentence '$(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$'?

♣ 3. Repeat (2), taking $a = 1$ and $b = -2$.

Returning to the example, we now investigate where $P'(x)$ is positive:

$$\begin{aligned} \{x \mid P'(x) > 0\} &= \{x \mid x - 1 > 0 \text{ and } x + 2 > 0\} \cup \{x \mid x - 1 < 0 \text{ and } x + 2 < 0\} \\ &= \{x \mid x > 1 \text{ and } x > -2\} \cup \{x \mid x < 1 \text{ and } x < -2\} \\ &= \{x \mid x > 1\} \cup \{x \mid x < -2\} \\ &= (1, \infty) \cup (-\infty, -2) \end{aligned}$$

Thus, $P'(x)$ is positive on $(-\infty, -2) \cup (1, \infty)$, so P increases on these intervals.

EXERCISE 7

♣ 1. What happened to the number 6 that appears in the formula for $P'(x)$?

♣ 2. One line in the previous display used the fact that:

$$x > 1 \text{ and } x > -2 \iff x > 1$$

Where was this fact used? Convince yourself that it is indeed true.

EXERCISE 8

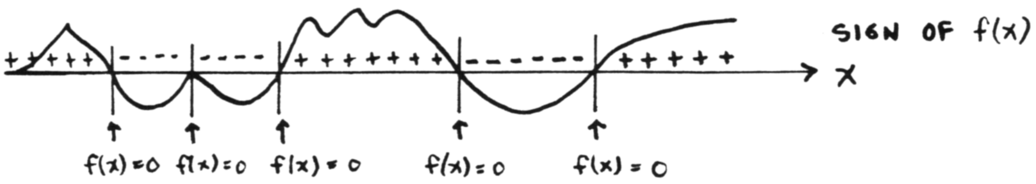
♣ Now use the fact that

$$ab < 0 \iff (a < 0 \text{ and } b > 0) \text{ or } (a > 0 \text{ and } b < 0)$$

to find the open intervals on which $P(x) = 2x^3 + 3x^2 - 12x$ decreases. Be sure to write complete mathematical sentences.

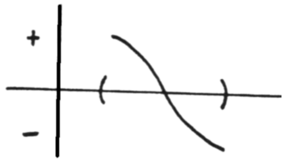
a better approach

There is a much easier approach to the previous problem, that exploits the *continuity* of P'. The places where a *continuous* function is positive and negative can be easily determined, merely by finding out where the function is *zero*, and then testing some points in between! This technique is discussed next.



a useful consequence of the Intermediate Value Theorem

If a function f is continuous on an interval and takes on *both* positive and negative values on this interval, then it must also take on the value 0. This fact is an immediate consequence of the Intermediate Value Theorem. (♣ Why?)



So suppose that f is continuous on I, and consider the implication:

IF f takes on both positive and negative values on I,
THEN f takes on the value 0 on I.

The contrapositive of this implication is:

IF f is nonzero on I,
THEN f is either positive or negative on I.

Since an implication is equivalent to its contrapositive, it has been shown that if a function is *continuous* and *nonzero* on an interval, then it must be *either positive or negative on this interval.*

EXERCISE 9

♣ 1. Convince yourself that:

$$\text{not}(f \text{ takes on the value } 0 \text{ on } I) \iff f \text{ is nonzero on } I$$

♣ 2. Convince yourself that:

$$\text{not}(f \text{ takes on both positive and negative values on } I) \iff f \text{ is either positive or negative on } I$$

finding out where an arbitrary function is positive and negative;

the 'number line' approach

Now, to find out where an *arbitrary* (not necessarily continuous) function is positive or negative, proceed as follows:

- Find all the discontinuities; mark these on a number line.
- Find all the places where the function is zero; mark these on the number line.
- The function must be continuous and nonzero on every subinterval. So, it must be either positive or negative on these subintervals!
- Choose a test point in each subinterval to determine the sign (+ or −) of the function there.

For lack of a better name, this will be called the 'number line' approach.

The 'number line' approach is used next to find where the function $P(x) = 20x^2 + 8x - 1$ is positive and negative.

EXAMPLE

using the 'number line' approach

Problem: Use the 'number line approach' to find where $P(x) = 20x^2 + 8x - 1$ is positive and negative.

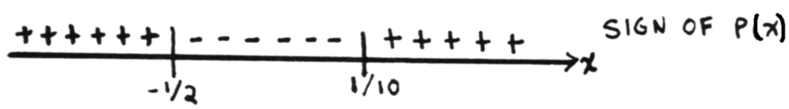
Solution: The function P is continuous everywhere. To determine where it is zero, either use the quadratic formula, or factor:

$$P(x) = (10x - 1)(2x + 1)$$

Then:

$$\begin{aligned} P(x) = 0 \quad &\iff \quad (10x-1)(2x+1) = 0 \\ &\iff \quad 10x - 1 = 0 \ \text{ or } \ 2x + 1 = 0 \\ &\iff \quad x = \frac{1}{10} \ \text{ or } \ x = -\frac{1}{2} \end{aligned}$$

Mark these zeroes on a number line. It is nice to label the number line as '*SIGN OF* $P(x)$'.



There are three subintervals formed; on each, P is continuous and nonzero. A test point must be chosen from each subinterval to determine the sign of $P(x)$ there. Choose the simplest test point to work with.

When $x = -1$: $P(-1) = (10(-1) - 1)(2(-1) + 1) = (-11)(-1) > 0$

When $x = 0$: $P(0) = (10(0) - 1)(2(0) + 1) = (-1)(1) < 0$

When $x = 1$: $P(1) = (10(1) - 1)(2(1) + 1) = (9)(3) > 0$

Note that, in all three cases, it was not necessary to determine the exact function values. Our only interest is the SIGN of the result.

In conclusion, $P(x)$ is positive on $(-\infty, -\frac{1}{2}) \cup (\frac{1}{10}, \infty)$ and negative on $(-\frac{1}{2}, \frac{1}{10})$.

using the 'number line' approach to determine where a function increases, decreases.

To simplify the problem of determining where a function is increasing and decreasing, just apply the 'number line' approach to the first derivative of the function, as in the next example.

EXAMPLE

Problem: Consider the function from an earlier example:

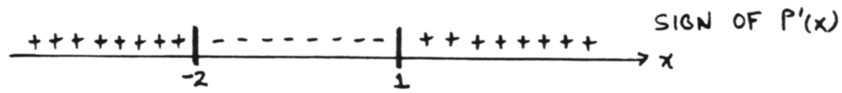
$$P(x) = 2x^3 + 3x^2 - 12x$$

Find the open intervals on which P increases; decreases.

Solution: The earlier example showed that P has derivative:

$$P'(x) = 6(x-1)(x+2)$$

This derivative P' is continuous everywhere; it is zero at $x = 1$ and $x = -2$. Plot these points on a number line, as shown below. It is nice to label the number line as '*SIGN OF* $P'(x)$'.



There are three subintervals formed, and a test point must be chosen from each.

When $x = -3$: $P'(-3) = 6(-3-1)(-3+2) = 6(-4)(-1) > 0$

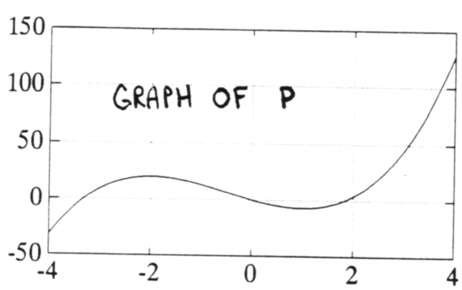
When $x = 0$: $P'(0) = 6(0-1)(0+2) = 6(-1)(2) < 0$

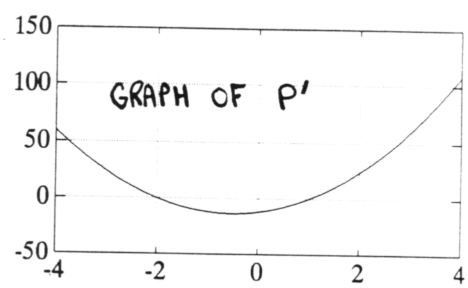
When $x = 2$: $P'(2) = 6(2-1)(2+2) = 6(1)(4) > 0$

Thus, $P'(x)$ is positive on $(-\infty, -2) \cup (1, \infty)$ and negative on $(-2, 1)$.

Thus, P increases on $(-\infty, -2) \cup (1, \infty)$ and decreases on $(-2, 1)$. This is (of course) in agreement with earlier results.

The graphs of P and P' are shown below.





The next problem is written in a very abbreviated form. Feel free to use this form when working the End-Of-Section exercises.

EXAMPLE

the 'number line' approach with a discontinuous function;

abbreviated form of solution

Problem: Find the open intervals on which the function $f(x) = \frac{x^2}{x-1}$ increases and decreases.

Solution:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

Discontinuity: at $x = 1$

Zeroes of f':

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad x(x-2) = 0 \\ &\iff \quad x = 0 \ \text{ or } \ x = 2 \end{aligned}$$

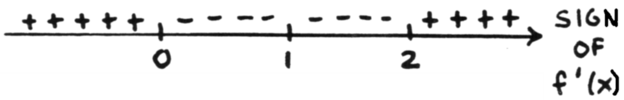
Test Points:

$-1 :\quad \frac{(-1)(-1-2)}{+} = \frac{(-)(-)}{+} > 0$

$\frac{1}{2} :\quad \frac{(\frac{1}{2})(\frac{1}{2}-2)}{+} = \frac{(+)(-)}{+} < 0$

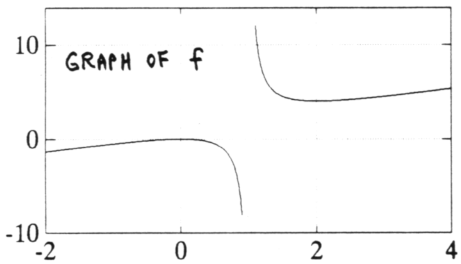
$\frac{3}{2} :\quad \frac{(\frac{3}{2})(\frac{3}{2}-2)}{+} = \frac{(+)(-)}{+} < 0$

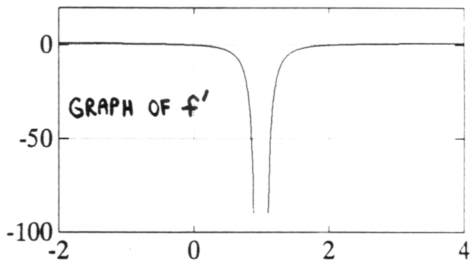
$3 :\quad \frac{3(3-2)}{+} = \frac{(+)(+)}{+} > 0$



Conclusion: f increases on $(-\infty, 0) \cup (2, \infty)$ and decreases on $(0, 1) \cup (1, 2)$.

The graphs of f and f' are shown below.





QUICK QUIZ

sample questions

♣ 1. Give a precise definition of what it means for a function f to increase on an interval I.

♣ 2. Find the open interval(s) on which $f(x) = x(x-1)$ is positive and negative. Use the 'number line' approach.

♣ 3. True or False: If f is nonzero and continuous on an interval I, then f is either positive or negative on I.

♣ 4. True or False: If $f'(x) > 0$ for all x in an interval I, then f increases on I.

KEYWORDS

for this section

Increasing and decreasing functions—precise definitions, nonincreasing and nondecreasing functions, getting inc/dec information from the derivative, using the Intermediate Value Theorem to decide where a function is positive and negative.

END-OF-SECTION EXERCISES

Find the open intervals on which the functions are positive and negative. Use any appropriate method. In many cases, the 'number line' approach may be easiest.

1. $P(x) = x^2 + x - 2$
2. $P(x) = x^2 - x - 2$
3. $P(x) = 2x^2 - 4x - 6$
4. $P(x) = 3x^2 + 6x - 9$
5. $P(x) = 12x^2 - 13x + 3$
6. $P(x) = 14x^2 + 3x - 2$
7. $P(x) = x^3 + 2x^2 + x$
8. $P(x) = x^3 - 2x^2 + x$

9 $P(x) = x^3 + 4x^2 - x - 4$ Hint: $x = 1$ is a zero of P

10 $P(x) = x^3 - 13x - 12$ Hint: $x = -1$ is a zero of P

11. $f(x) = x^2e^x - x^2$
12. $f(x) = e^x x^2 - e^x$

13 $f(x) = \ln(2x - 1)$ Hint: $\mathcal{D}(f) = (\frac{1}{2}, \infty)$, so the 'number line' approach is applied to a 'partial' number line. Also: $\ln(2x - 1) = 0 \iff 2x - 1 = 1$

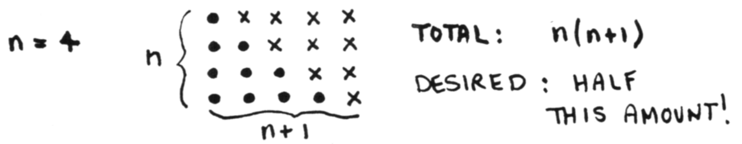
14 $f(x) = \ln(1 - 2x)$ Hint: $\mathcal{D}(f) = (-\infty, \frac{1}{2})$, and $\ln(1 - 2x) = 0 \iff 1 - 2x = 1$

Find the open intervals on which the functions increase and decrease. Use any appropriate method. In many cases, the 'number line' approach applied to the first derivative may be easiest.

15. $f(x) = 2x^3 + 3x^2 - 12x + 1$
16. $f(x) = x^3 - 3x^2 - 9x + 4$
17. $f(x) = xe^x$
18. $f(x) = x^2e^x$
19. $f(x) = x \ln x$
20. $f(x) = x^2 \ln x$

21 (formula for finding $1 + 2 + \cdots + n$) The picture below gives a geometric proof that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$



a. Explain how the formula 'comes from' the picture.

b. Find: $1 + 2 + \cdots + 67$

c. Find: $S := 64 + 65 + \cdots + 108$ Hint: $S = (1 + 2 + \cdots + 108) - (1 + 2 + \cdots + 63)$

22 (refer to (21)) It was seen in the previous problem that the function $S\colon \{1, 2, 3, \dots\} \rightarrow \mathbb{R}$ defined by $S(n) = \frac{n(n+1)}{2}$ gives as its outputs the sum of the first n positive integers.

As n increases, so does $S(n)$. (♣ Why?) To check this, first define a function $f\colon \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \frac{x(x+1)}{2}$, (the 'analogue' of S that is defined for all real numbers, not only the positive integers).

a. If it can be shown that f increases on an interval that contains the positive integers, then S must also be increasing. Convince yourself that this is true.

b. Show that f increases on $(-\frac{1}{2}, \infty)$. Conclude that S is an increasing function.

END-OF-SECTION EXERCISES

(continued)

23 For all real numbers x except 1, and for all positive integers n:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Here is how the formula is derived:

a. Define: $S := 1 + x + \cdots + x^n$

Find: $x \cdot S$

b. Show that: $xS - S = x^{n+1} - 1$

Conclude that the formula for S has the desired form.

c. Use the formula for S to find $1+2+2^2+2^3+2^4$. Verify that your result is correct, by performing the indicated additions and exponentiations.

d. Use the formula for S to find $2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$. Again, verify that your result is correct.

24 (refer to (23)) As in the previous example, define:

$$S(x) = 1 + x + x^2 + \cdots + x^n$$

a. For a fixed value of n and for positive x, one would suspect that as x increases, so does $S(x)$. Comment on why this is true.

b. Let $n = 2$ in the formula for $S(x)$. Determine where S increases and decreases. Think about your result.

c. Let $n = 3$ in the formula for $S(x)$. Determine where S increases and decreases. Think about your result.

25 (Probability). This problem shows a situation in probability that gives rise to sums of the form $1 + x + \cdots + x^n$.

On a fair dice (6 sides), the probability of getting at least one '2' in the first n throws is given by:

$$P(n) = \frac{1}{6}[1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{n-1}]$$

To get a feeling for where this formula comes from, let's check a couple cases:

a. Let $n = 1$. Discuss why the probability of getting a '2' on one throw is $\frac{1}{6}$. Verify that the formula is correct in this case.

b. Let $n = 2$. How can we get at least one '2' in two throws? We could get a 2 on the first throw; the probability of this is $\frac{1}{6}$. Or, we could get a number other than '2' on the first throw, and a '2' on the second throw; the probability of this is $(\frac{5}{6})(\frac{1}{6})$. The total probability is $\frac{1}{6} + (\frac{5}{6})(\frac{1}{6})$. Verify that the formula is correct in this case.

c. Conjecture that as n increases, so should $P(n)$. Why? Find $P(1)$, $P(2)$ and $P(3)$. Are these results in agreement with your conjecture?

d. Prove that, for all positive integers n, $P(n) < P(n + 1)$. Thus, P is indeed an increasing function. Hint: Compare the formulas for $P(n)$ and $P(n + 1)$. What do they differ by? Is the difference always positive?

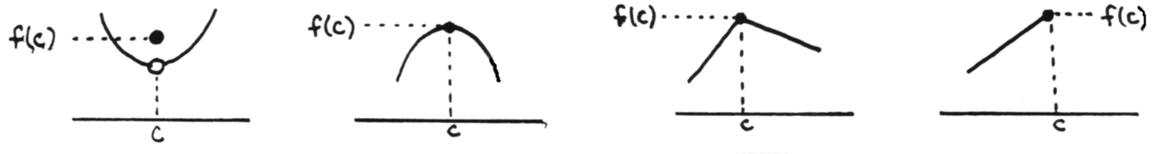
5.2 Local Maxima And Minima
Critical Points

Introduction

Often, one is interested in *maximizing* or *minimizing* functions; say, maximizing profits or minimizing costs. In the section on the Max-Min Theorem, maximum and minimum values of a function on an interval were discussed. Now, this idea is extended so that we can talk about *local maximum and minimum values for a function.*

local maximum;
local minimum;
informal discussion

A number $f(c)$ is called a *local maximum* for a function f, if, *locally*, it's the *highest value*. That is, for all x sufficiently close to c, it must be that $f(x) \le f(c)$. The sketches below illustrate some of the ways that this can happen.

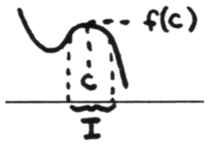


Analogously, a number $f(c)$ is called a *local minimum* for a function f, if, *locally*, it's the *lowest value.*

EXERCISE 1	♣ Make four sketches that illustrate different ways a function f may have a local minimum.

Here are the precise definitions of *local maximum* and *local minimum*:

DEFINITION
local maximum,
local minimum,
local extreme values



Let f be a function and let $c \in \mathcal{D}(f)$.

The number $f(c)$ is a *local maximum for* f if and only if there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in \mathcal{D}(f)$, then $f(x) \le f(c)$.

The number $f(c)$ is a *local minimum for* f if and only if there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in \mathcal{D}(f)$, then $f(x) \ge f(c)$.

When such maximum or minimum values occur, they are called *local extreme values of* f.

a slight
abuse of notation

Strictly speaking, the *number* $f(c)$ is the local extreme value for f. However, one often abuses notation and speaks of, say, the *local maximum (point)* $(c, f(c))$. This is because one is *not only interested in the local maximum value* $f(c)$*, but also the place* c *where it occurs.*

EXERCISE 2

Sketch the graph of a function satisfying:

♣ 1. f has a local maximum at $x = 2$, $f(2) = 4$, f is not differentiable at $x = 2$

♣ 2. $\mathcal{D}(f) = [1, 2]$, f has a local minimum at $x = 1$, $f(2) < f(1)$, the maximum value of f on $[1, 2]$ is 5

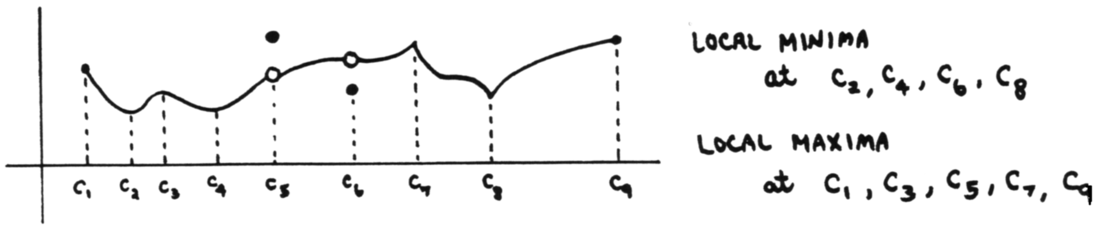
♣ 3. f has a local maximum at $x = 1$, $f(1) = 2$, $\{x \mid f(x) \in (2, 3)\} = \emptyset$, f has a local minimum at $x = 2$

EXERCISE 3	♣ Suppose a function f has the following property: $f(2) = 3$, and arbitrarily close to $x = 2$ there are inputs whose function values are greater than 3. Can f have a local maximum at $x = 2$? Why or why not? How about a local minimum at $x = 2$?

a function may have many local extrema

The plural of *maximum* is *maxima*. The plural of *minimum* is *minima*. The singular of *extrema* is *extremum*.

Observe that a function may have *many* local extrema, as illustrated below.

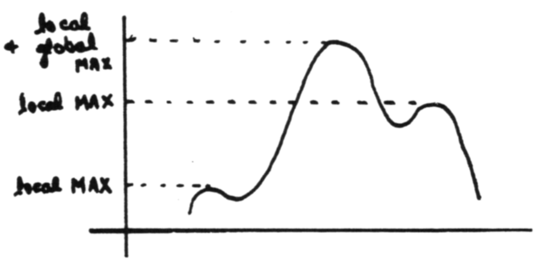


local versus global extrema

Occasionally, one is interested in the highest value that a function f attains on some specified set (like the highest value f attains on an interval; or the highest value f attains over its entire domain). Such a highest value (if it exists) is called a *global maximum*.

Similarly, one can speak of a *global minimum*.

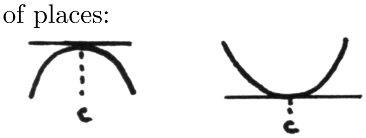
Right now, we won't concern ourselves with *global* extrema. However, later on we'll return to this idea. For the moment, just be aware that the adjective *local* is used to distinguish the discussion from the *global* case.



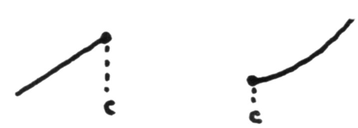
finding local maxima and minima

It appears, from the examples put forth thus far, that local maxima and minima seem to occur at certain types of places:

(a) places where $f'(c) = 0$



(b) endpoints of a domain



(c) places where the derivative does not exist

Indeed, *these are the ONLY types of places where local extreme values can occur.* This fact is proven in the next few paragraphs. Refer to the flow chart at left as you read through the following argument.

proof that local extreme values can only occur at certain types of places

Let c be in the domain of f.

Either c is an endpoint of the domain of f, or not. If it is, then there *may* be a local extreme value at c, as in situation (b) above.

Now suppose that c is not an endpoint. Either f is differentiable at c or not. If not, then there *may* be a local extreme value at c, as in case (c) above.

Now suppose f *is* differentiable at c. Then either $f'(c) \neq 0$, or $f'(c) = 0$. If $f'(c) = 0$, then there *may* be a local extreme value at c, as in case (a) above.

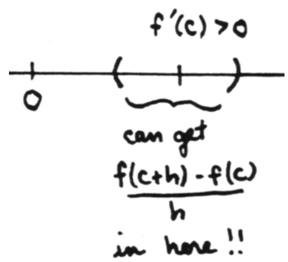
IF: $c \in \mathcal{D}(f)$, c is not an endpoint, f is differentiable at c, $f'(c) \neq 0$; then $f(c)$ is NOT a local extreme value for f

All that is needed to complete the proof is to show the following:

IF:

- $c \in \mathcal{D}(f)$,
- c is not an endpoint of the domain of f,
- f is differentiable at c, and
- $f'(c) \neq 0$,

THEN the number $f(c)$ *is not* a local extreme value for f.



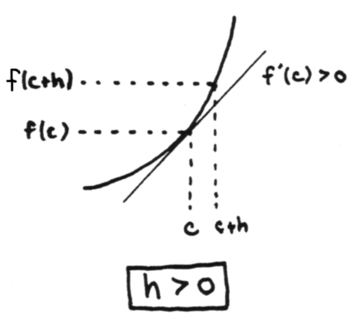
So assume all these things are true. Since $f'(c) \neq 0$, it must be that either $f'(c) < 0$ or $f'(c) > 0$. Suppose for the moment that $f'(c) > 0$. (You'll investigate the case $f'(c) < 0$ in the exercises.) That is:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0$$

This means that *we can get the values* $\frac{f(c+h)-f(c)}{h}$ *as close to the number* $f'(c)$ *as desired*, merely by restricting ourselves to values of h sufficiently close to 0.

So choose δ so small that whenever $|h - 0| < \delta$, one has:

$$\frac{f(c+h) - f(c)}{h} > 0 \qquad (*)$$

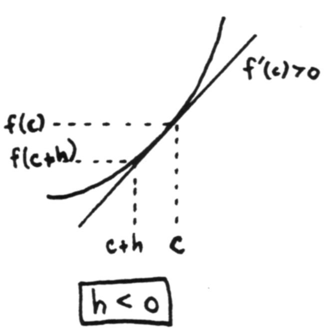


By hypothesis, c is *not* an endpoint of the domain of f, so the function is defined both to the right and left of c. Now, whenever $h > 0$ and $|h| < \delta$, multiplying both sides of (*) by the positive number h yields the equivalent inequality

$$f(c+h) - f(c) > 0 \ ,$$

so that $f(c+h) > f(c)$. That is, *arbitrarily close to* $(c, f(c))$*, on the right, is another point* $(c+h, f(c+h))$ *with a greater function value,* $f(c+h) > f(c)$. Thus, $f(c)$ is NOT a local maximum.

Similarly, whenever $h < 0$ and $|h| < \delta$, multiplying both sides of (*) by the negative number h yields the equivalent inequality



$$f(c+h) - f(c) < 0 \ ,$$

so that $f(c+h) < f(c)$. That is, *arbitrarily close to* $(c, f(c))$*, on the left, is another point* $(c+h, f(c+h))$ *with a lesser function value.* Thus, $f(c)$ is NOT a local minimum, either.

Thus, f does not have a local extreme value at $x = c$.

EXERCISE 4

♣ Prove the following:

If $c \in \mathcal{D}(f)$, c is not an endpoint of the domain of f, f is differentiable at c, and $f'(c) < 0$, then the number $f(c)$ is NOT a local extreme value for f.

What has been learned?

CRITICAL POINTS;

candidates for places where local extrema occur

Summarizing:

IF a function f has a local extremum at the point $(c, f(c))$, THEN

- c is an endpoint of the domain of f; or
- f is *not* differentiable at c; or
- $f'(c) = 0$

Any point that satisfies at least one of these three conditions is called a *CRITICAL POINT*. The critical points are the CANDIDATES for places where local extrema can occur.

EXERCISE 5

more practice with implications

The sentences $P \Rightarrow Q$ and $Q \Rightarrow P$ are *not equivalent*, as the truth table below shows.

P	Q	P ⇒ Q	Q ⇒ P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

♣ 1. What truth values of P and Q make $P \Rightarrow Q$ true, but $Q \Rightarrow P$ false?

♣ 2. What truth values of P and Q make $Q \Rightarrow P$ true, but $P \Rightarrow Q$ false?

♣ 3. What truth values of P and Q make both $P \Rightarrow Q$ and $Q \Rightarrow P$ true?

In particular, *even if it is known that $P \Rightarrow Q$ is true*, it is NOT possible to conclude that $Q \Rightarrow P$ must also be true.

♣ 4. Argue that the implication $x = 2 \implies x^2 = 4$ is true. That is, show that for *all* values of x, the sentence is true.

♣ 5. Argue that the implication $x^2 = 4 \implies x = 2$ is false. That is, show that there is *at least one value of x* for which the sentence is false.

CAUTION:

a critical point MAY or MAY NOT be a place where a local extreme value occurs

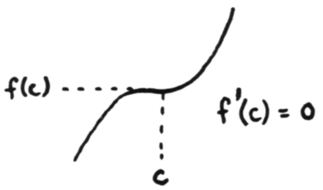
It has been proven that the following sentence is always true:

IF f has a local extreme value at the point $(c, f(c))$,
THEN the point $(c, f(c))$ is a critical point.

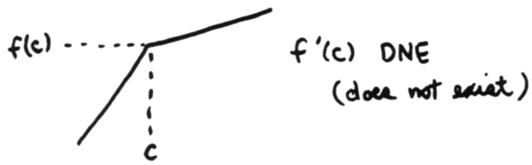
The previous exercise points out that this knowledge *DOES NOT* determine the truth of the sentence:

IF the point $(c, f(c))$ is a critical point,
THEN f has a local extreme value at $(c, f(c))$.

Indeed, the sketches below indicate that there *are* critical points that *do not* correspond to local extreme values.



(c,f(c)) is a CRITICAL PT., but not a local extremum



(c,f(c)) is a CRITICAL PT., but NOT a local extremum

EXAMPLE

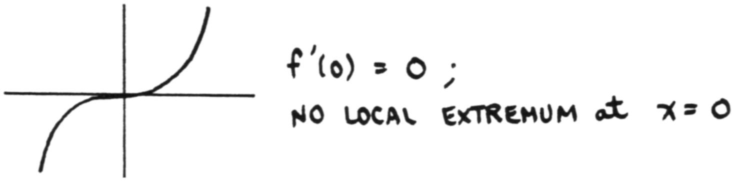
Problem: Determine if the following sentences are TRUE or FALSE. If false, make a sketch showing that it is false.

- *If f has a local minimum at $x = 2$, then the point $(2, f(2))$ is a critical point for f.*

Solution: True! Any local extrema MUST OCCUR at a critical point.

- *If $f'(c) = 0$, then f must have a local maximum or minimum at c.*

Solution: False! The function $f(x) = x^3$ satisfies $f'(0) = 0$. However, f does not have a local extremum at $x = 0$.



EXERCISE 6

Determine if the following sentences are TRUE or FALSE. If false, make a sketch showing that it is false.

♣ 1. If f has a local maximum at $x = 4$, then the point $(4, f(4))$ must be a critical point for f.

♣ 2. If f has a local maximum at $x = 4$, then it must be that $f'(4) = 0$.

♣ 3. If $\mathcal{D}(f') = \mathbb{R}$ and f has a local maximum at $x = 4$, then it must be that $f'(4) = 0$.

♣ 4. If $f'(1) = 0$, then $(1, f(1))$ is a local extreme point for f.

strategy for finding local extrema

Given a function f, here is a strategy for finding all the local extrema of f:

First, find *all places* in $\mathcal{D}(f)$ where the derivative is zero, where the derivative doesn't exist, and all the endpoints of the domain of f. That is, find all the critical points for f.

Then, investigate each of these critical points to see if it *is* or *is not* a local extremum for f.

EXAMPLE

Problem: Find all local extreme values for the function $f\colon [0,5] \to \mathbb{R}$, $f(x) = (x-2)^2$.

Solution:

- Note that $\mathcal{D}(f) = [0,5]$. All critical points must come from the domain of f.
- The endpoints of $\mathcal{D}(f)$ are critical points. Calculate the function values at the endpoints:

$$f(0) = (0-2)^2 = 4$$
$$f(5) = (5-2)^2 = 9$$

  Thus, $(0,4)$ and $(5,9)$ are critical points.
- Find f'.

$$f'(x) = 2(x-2)(1) = 2(x-2)\ .$$

- Find all places where $f'(x) = 0$. Be sure to write a complete mathematical sentence!

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad 2(x-2) = 0 \\ &\iff \quad x = 2 \end{aligned}$$

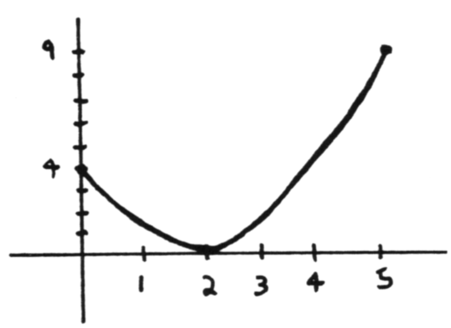
  Note that $f(2) = (2-2)^2 = 0$. So, $(2,0)$ is a critical point.
- There are no places where $f'(c)$ does not exist.

It is convenient to summarize all this information by constructing a table:

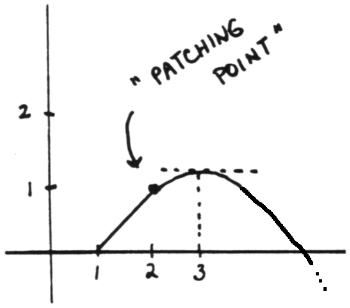
C	f(C)	WHY A CRIT. PT?	LOCAL EXTREMUM?
0	4	ENDPT OF DOMAIN	MAX
5	9	ENDPT OF DOMAIN	MAX
2	0	$f'(c) = 0$	MIN

Now, IF f has any local extrema, they must occur at these three points.

Indeed, a quick sketch of the graph of f is easy to get in this case: take the graph of $y = x^2$, and shift it two units to the right. Thus, we see that there are local maxima at $(0,4)$ and $(5,9)$, and a local minimum at $(2,0)$.



EXAMPLE



Problem: Find all local extreme values for the function $f\colon [1,\infty) \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ -\frac{1}{8}(x^2-6x) & \text{for } x > 2 \end{cases}$$

Observe that f is continuous at the point $(2,1)$. This is because

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x-1 = 2-1 = 1$$

and:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -\frac{1}{8}(x^2-6x) = -\frac{1}{8}(2^2 - 6\cdot 2) = 1$$

Clearly, f is differentiable everywhere, except possibly at the 'patching point' $(2,1)$, and:

$$f'(x) = \begin{cases} 1 & \text{for } x \in [1,2) \\ -\frac{1}{4}(x-3) & \text{for } x > 2 \end{cases}$$

Is f differentiable at $(2,1)$? It is possible, in this case, to answer without computing a difference quotient. We need only look at the formulas for f', and think.

What is the 'direction' at $(2,1)$, coming in from the left? Here, the graph is a line with slope 1, so the 'direction' coming in from the left is 1.

What is the 'direction' at $(2,1)$, coming in from the right? Arbitrarily close to $(2,1)$, on the right, the tangent lines are given by the function $-\frac{1}{4}(x-3)$. And as x approaches 2, this function approaches $-\frac{1}{4}(2-3) = \frac{1}{4}$. Thus, the 'direction' coming in from the right is $\frac{1}{4}$.

Since $1 \ne \frac{1}{4}$, f is not differentiable at the point $(2,1)$.

★★ The precise justification for the argument above is really quite subtle. It uses the continuity of f at and near c, the continuity of f' near c, and an interchange of limit operations. However, it should be intuitive to students that this is possible.

Finally:

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad -\frac{1}{4}(x-3) = 0 \\ &\iff \quad x = 3 \end{aligned}$$

Note that $f(3) = -\frac{1}{8}(3^2 - 6\cdot 3) = \frac{9}{8}$. Since the graph of any function of the form $f(x) = ax^2 + bx + c$ (for $a \ne 0$) is known to be a parabola, it is now possible to make a quick sketch of the function.

If f has any local extrema, they must occur at the places listed below:

C	F(C)	WHY A CRIT PT?	LOCAL EXTREMUM?
1	0	ENDPT of DOMAIN	MIN
2	1	f'(c) DNE	NO
3	9/8	f'(c) = 0	MAX

From the sketch, it is easy to see that $(1,0)$ is a local minimum, $(2,1)$ is not a local extremum, and $(3,\frac{9}{8})$ is a local maximum.

EXERCISE 7

Find all local extrema of the following functions, by finding and checking all the critical points. In each case, sketch the graph of the function.

♣ 1. $f\colon [1,\infty) \to \mathbb{R}$, $f(x) = (x-4)^3$

♣ 2. $f\colon [-1,4] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} -x^2+1 & \text{for } x \in [-1,1] \\ x-1 & \text{for } x \in (1,4] \end{cases}$$

♣ 3. $f\colon [-1,4] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} -x^2+1 & \text{for } x \in [-1,1] \\ -x+1 & \text{for } x \in (1,4] \end{cases}$$

using information from f' to help determine local extreme behavior

Given a critical point, it may be difficult to determine if it corresponds to a local maximum, a local minimum, or neither, particularly when the graph of the function is not easily available. In many cases, the information that f' gives can be used to help investigate local extreme behavior at critical points.

To see this, let $(c, f(c))$ be a critical point. Suppose that f is continuous at c and differentiable near c. If f increases to the left of c, and decreases to the right of c, then $f(c)$ must be a local maximum. That is, if $f'(x) > 0$ to the left of c, and $f'(x) < 0$ to the right of c, then $f(c)$ is a local maximum.

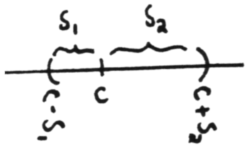
If f decreases to the left of c ($f'(x) < 0$), and increases to the right of c ($f'(x) > 0$), then $f(c)$ must be a local minimum.

If f increases both to the left and right of c, or if f decreases both to the left and right of c, then $f(c)$ is not a local extreme value.

These observations are the content of a test that is commonly called the *First-Derivative Test*. It is stated precisely below.

FIRST DERIVATIVE TEST

for examining behavior at critical points



Suppose that $(c, f(c))$ is a critical point for f, so that f *may* have a local maximum or minimum at c.

Suppose that f is continuous at c. Also suppose that f is differentiable on $(c - \delta_1, c) \cup (c, c + \delta_2)$ for some positive numbers δ_1 and δ_2; that is, f is differentiable on intervals immediately to the left and right of c.

- If $f'(x) > 0$ on $(c - \delta_1, c)$ and $f'(x) < 0$ on $(c, c + \delta_2)$, then $(c, f(c))$ is a local maximum for f.
- If $f'(x) < 0$ on $(c - \delta_1, c)$ and $f'(x) > 0$ on $(c, c + \delta_2)$, then $(c, f(c))$ is a local minimum for f.
- If $f'(x)$ has the same sign on both intervals $(c - \delta_1, c)$ and $(c, c + \delta_2)$, then $(c, f(c))$ is neither a local maximum or minimum for f.

f must be continuous at c!

Note that if one takes away the hypothesis that f is continuous at c, then one is no longer guaranteed the conclusion of the theorem. To see this, consider the functions f sketched below. In both cases, f decreases to the left of c, and f increases to the right of c. However, the point $(c, f(c))$ is NOT a local minimum for the function f in either case.

EXERCISE 8

♣ 1. Sketch a function f satisfying: f increases to the left of c, decreases to the right of c, but $(c, f(c))$ is NOT a local maximum.

♣ 2. Sketch a function f satisfying: f increases to the left of c, decreases to the right of c, f is not continuous at c, and $(c, f(c))$ IS a local maximum.

'First-Derivative Tests' at endpoints

Similar results can be stated for endpoints. For example, suppose c is an endpoint of the domain of f, f is continuous at c, and $f'(x) > 0$ on some interval to the right of c. Then, $(c, f(c))$ must be a local minimum for f.

EXERCISE 9

♣ Precisely state a '*First-Derivative Test*' that applies to each of the three sketches shown below.

EXAMPLE

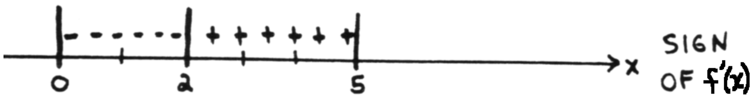
Problem: Reconsider an earlier example: Find all local extreme values for the function $f\colon [0,5] \to \mathbb{R}$, $f(x) = (x-2)^2$

There are three critical points, and first-derivative tests can be used to investigate each of these.

Since $f'(x) < 0$ to the right of $x = 0$, $(0,4)$ is a local maximum.

Since $f'(x) > 0$ to the left of $x = 5$, $(5,9)$ is a local maximum.

Since $f'(x) < 0$ to the left of $x = 2$, and $f'(x) > 0$ to the right of $x = 2$, $(2,0)$ is a local minimum.



EXERCISE 10

♣ Use first-derivative tests to investigate the critical points of an earlier example: the function $f\colon [1,\infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ -\frac{1}{8}(x^2 - 6x) & \text{for } x > 2 \end{cases}$$

EXAMPLE

Problem: Find all local extreme values for the function $f\colon [0,1) \cup (1,9] \to \mathbb{R}$ given by $f(x) = \frac{\sqrt{x}}{x-1}$.

Solution:

- Note that $\mathcal{D}(f) = [0,1) \cup (1,9]$. All critical points must come from the domain of f.
- The points $(0, f(0)) = (0,0)$ and $(9, f(9)) = (9, \frac{3}{8})$ are critical points, because they are endpoints of the domain of f.
- Find f' :

$$\begin{aligned} f'(x) &= \frac{(x-1)(\frac{1}{2}x^{-1/2}) - \sqrt{x}(1)}{(x-1)^2} \\ &= \ldots \quad \text{(simplify)} \\ &= \frac{-(x+1)}{2\sqrt{x}(x-1)^2} \end{aligned}$$

  The function f is defined at $x = 0$, but f' is not defined at 0, because this would produce division by 0. (Indeed, there is a vertical tangent line at $(0,0)$.)

- Find all places where $f'(x) = 0$:

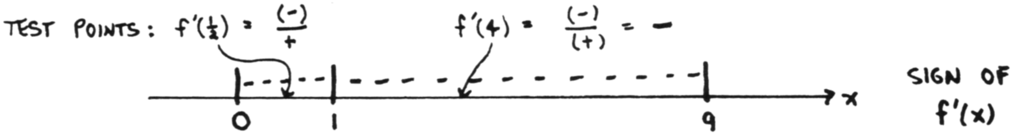
$$\begin{aligned} f'(x) = 0 \quad &\iff \quad -(x+1) = 0 \\ &\iff x = -1 \end{aligned}$$

  However, $x = -1$ is not in the domain of f.

This information is summarized below:

c	$f(c)$	WHY A CRIT. PT?	LOCAL EXTREMUM?
0	0	ENDPT of $\mathcal{D}(f)$, $f'(0)$ DNE	MAX
9	3/8	ENDPT OF $\mathcal{D}(f)$	MIN

To use first-derivative tests to investigate each critical point, the sign of $f'(x)$ must be determined. To do this, locate all places where $f'(x)$ is zero and discontinuous, and then use test points in each subinterval.



Conclude that $(0,0)$ is a local maximum, and $(9, \frac{3}{8})$ is a local minimum.

Observe that it was NOT necessary to have the graph of f to determine all local maxima and minima. Later on, we will see a graph of this function.

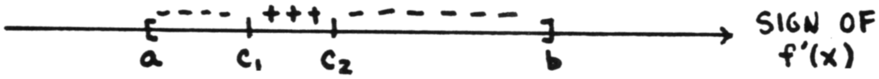
EXERCISE 11

♣ 1. Find all local extreme values for the function $f\colon [0,1) \cup (1,4) \to \mathbb{R}$ given by: $f(x) = \dfrac{e^x}{x-1}$

♣ 2. Find all local extreme values for the function $f\colon [0,8] \to \mathbb{R}$ given by: $f(x) = \sqrt[3]{x} + x^3$

QUICK QUIZ
sample questions

1. Suppose that a function f has a local extremum at the point $(c, f(c))$. What (if anything) can be said about the behavior of f at this point?
2. Suppose that $(c, f(c))$ is a critical point for f. Must this point be a local maximum or a local minimum?
3. What is a 'critical point'?
4. Suppose it is known that '$A \implies B$' is true. Does this information alone determine the truth of '$B \implies A$'?
5. Suppose that a function $f\colon [a,b] \to \mathbb{R}$ is differentiable. The sign of $f'(x)$ is summarized below. Find all local extrema for f on $[a,b]$.



KEYWORDS
for this section

Local maxima and minima, local extreme values, local versus global extrema, critical points, strategy for finding local extrema, First Derivative Tests.

END-OF-SECTION EXERCISES

1. Find at least three sentences in this section that are *implicit* 'for all' sentences. That is, the words 'for all' do not appear, but are assumed to be there.

The problems below review concepts from sections 5.1 and 5.2.

♣ Classify each entry as an expression or a sentence.

♣ For any sentence, state whether it is TRUE or FALSE.

♣ All the implications are (either implicitly or explicitly) 'for all' sentences. If an implication is FALSE, give a COUNTEREXAMPLE. That is, provide a specific example where the hypothesis is TRUE, but the conclusion is FALSE.

2. If f increases on an interval I, then for all $a, b \in I$ with $a < b$, $f(a) < f(b)$
3. If f decreases on an interval I, then for all $a, b \in I$ with $a > b$, $f(a) < f(b)$
4. If g increases on an interval I, then for all $x, y \in I$, $g(x) \le g(y)$
5. If g decreases on an interval I, then for all $x, y \in I$ with $x < y$, $g(y) \le g(x)$
6. If $f(x) \le f(y)$ for all numbers x and y in an interval I with $x < y$, then f is nondecreasing on I.
7. If $g(a) \ge g(b)$ for all numbers a and b in an interval I with $a < b$, then g is nonincreasing on I.
8. If f if differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
9. For all functions f, if f is increasing on an interval (a, b), then $f'(x) > 0$ on (a, b).
10. For all real numbers c and d:
$$cd > 0 \iff (c > 0 \text{ and } d > 0) \text{ or } (c < 0) \text{ and } d < 0)$$
11. For all real numbers c and d:
$$cd < 0 \iff (c > 0 \text{ and } d < 0) \text{ or } (c < 0 \text{ and } d > 0)$$
12. If f is continuous and nonzero on an interval I, then f is either positive or negative on I.
13. If f is continuous, and takes on both positive and negative values on an interval I, then f must also take on the value 0 on I.
14. If f has a local maximum at $(1, f(1))$, then the point $(1, f(1))$ is a critical point.
15. If f has a local minimum at $(2, f(2))$, then $f'(2) = 0$.
16. If $(c, f(c))$ is a critical point for f, then f has a local maximum or minimum at $(c, f(c))$.
17. If $\mathcal{D}(f) = \mathcal{D}(f') = \mathbb{R}$ and $(c, f(c))$ is a critical point for f, then $f'(c) = 0$.
18. If $P \Rightarrow Q$ is true, then $Q \Rightarrow P$ is true.
19. If f is continuous at c, $f'(x) > 0$ to the right of c, and $f'(x) < 0$ to the left of c, then the point $(c, f(c))$ is a local minimum for f.
20. If $f\colon [1, \infty) \to \mathbb{R}$ is continuous, and has the property that $f'(x) < 0$ for all $x \in (1, \frac{9}{8})$, then f has a local maximum at $x = 1$.

5.3 The Second Derivative; Inflection Points

the second derivative function, f''

If a function f is sufficiently smooth, then we can differentiate once to get f', and again to get f''. The function f'' is called the *second derivative of* f, and is by far the most important higher-order derivative.

Recall that at $x = c$:

$g'(c)$ *gives the instantaneous rate of change of* $g(x)$ *with respect to* x

Taking $g = f'$:

$(f')'(c) = f''(c)$ *gives the instantaneous rate of change of* $f'(x)$ *with respect to* x

Since the numbers $f'(x)$ give the slopes of the tangent lines to the graph of f, the function f'' tells *how fast the slopes of the tangent lines to the graph of* f *are changing.*

concave up

For example, if the slopes of the tangent lines are *increasing*, then the scenario is the following:

Thus, when $f''(x) > 0$, the shape illustrated above ('holding water') is generated. Such a graph is said to be *concave up.* Observe that the more positive $f''(x)$ is, the more quickly the slopes of the tangent lines increase, and hence the more rapidly the graph turns.

concave down

Similarly, if the slopes of the tangent lines are *decreasing*, then the scenario is the following:

Thus, when $f''(x) < 0$, the shape illustrated above ('shedding water') is generated. Such a graph is said to be *concave down.* Again note that the more negative $f''(x)$ is, the more quickly the slopes of the tangent lines decrease, and hence the more rapidly the graph turns.

EXERCISE 1

Sketch graphs satisfying the following properties:

♣ 1. $f(1) = f(3) = 2$, $f(2) = 3$; the slopes of the tangent lines increase on the interval $(1, 2)$ and decrease on $(2, 3)$; f is continuous at $x = 2$

♣ 2. $f(x) < 0$ on $[1, 3]$; the slopes of the tangent lines decrease on the interval $(1, 2)$ and increase on $(2, 3)$

The precise definitions of *concave up* and *concave down* on an interval follow.

DEFINITION
concave up on I
concave down on I

Let I be an interval of real numbers.

If $f''(x) > 0$ for all x in I, then f is *concave up* on I.

If $f''(x) < 0$ for all x in I, then f is *concave down* on I.

a common convention concerning DEFINITIONS

Every definition is, either implicitly or explicitly, a statement of *equivalence*. For this reason, there is a convention regarding definitions: they may be stated as 'If ... then ...' sentences, when in actuality they are 'if and only if' sentences.

Without this convention, the previous definition would have to be written something like this:

DEFINITION
concave up on I
concave down on I

Let I be an interval of real numbers.

f is *concave up* on I if and only if, for all $x \in I$, $f''(x) > 0$.

f is *concave down* on I if and only if, for all $x \in I$, $f''(x) < 0$.

EXAMPLE
concavity of a line

Consider the linear function $f(x) = ax + b$. As one moves from point to point, the slopes of the tangent lines *do not change at all*. This information is reflected in the second derivative:

$$\begin{aligned} f(x) = ax + b &\implies f'(x) = a \\ &\implies f''(x) = 0 \end{aligned}$$

Conversely, suppose a function g has the property that $g''(x) = 0$. Then it must be that $g'(x) = a$ for some real number a. (♣ Why?) But then, g must be a line with slope a, so that $g(x) = ax + b$ for some real number b.

The process of going from information about *derivatives* back to information about the *original function* is called *integration* or *antidifferentiation*, and is discussed in more detail in later sections.

NOTE about the word: 'Conversely'

Recall that the sentence $A \Rightarrow B$ is called an *implication*. The new sentence $B \Rightarrow A$ is called the *converse* of the sentence $A \Rightarrow B$.

It has been seen that the truth of $A \Rightarrow B$ in no way influences the truth of $B \Rightarrow A$. Each sentence must be investigated separately. If you have just investigated the sentence $A \Rightarrow B$, and now want to investigate its converse, $B \Rightarrow A$, it is common to say: *Conversely,* ... . This prepares the reader for the fact that you are about to investigate the converse.

EXERCISE 2

Sketch graphs satisfying the following properties:

♣ 1. $f(1) = 1$, $f'(3) = \frac{1}{2}$, and $f''(x) = 0 \ \ \forall\ x \in [1,5]$. What must $f(5)$ be?

♣ 2. $f(3) = 4$, $f'(3) = 2$, $f''(x) = 0$ on $(1,5)$, f is continuous at $x = 5$, f is defined but not continuous at $x = 1$

EXERCISE 3

Write down the *converse* of each of the following implications, using any correct notation. Is the original implication true? What about its converse?

♣ 1. If $x = 2$, then $x^2 = 4$

♣ 2. $1 = 2 \Rightarrow 1 + 1 = 2$

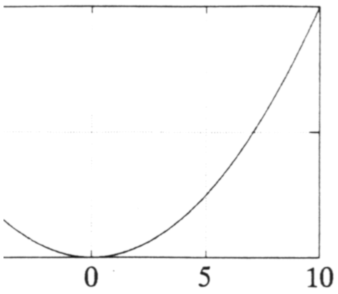
♣ 3. If $1 = 2$, then $2 = 3$

♣ 4. Now, let $A \Rightarrow B$ be an implication. What is the converse of the converse? What is the contrapositive of the converse? What is the converse of the contrapositive?

EXAMPLE

concavity of the squaring function

Consider the function $f(x) = x^2$. Here, $f'(x) = 2x$ and $f''(x) = 2$. The second derivative is constantly 2. The curve always 'turns' at exactly the same rate. Thus, at every point on the graph of f, the slopes of the tangent lines are changing twice as fast as the x value of the point is changing. Thus, when x changes by 1, one should find that the slope of the tangent line changes by 2 .



For example, consider the point $(0, f(0)) = (0, 0)$ on the graph of f. Here, the tangent line has slope $f'(0) = 2 \cdot 0 = 0$. Move one unit to the right, to the point $(1, f(1)) = (1, 1)$. Here, the tangent line has slope $f'(1) = 2 \cdot 1 = 2$. When x increased by 1, the slope of the tangent line increased by 2 !

Let's investigate this fact in more generality. Let $(x, f(x))$ be *any* point on the graph of f. Then, $(x+1, f(x+1))$ is the point with x value increased by 1 . The slope of the tangent line at $(x, f(x))$ is $f'(x) = 2x$. The slope of the tangent line at $(x+1, f(x+1))$ is:

$$\begin{aligned} f'(x+1) &= 2(x+1) \\ &= 2x + 2 \\ &= f'(x) + 2 \end{aligned}$$

$f(x) = x^2$

CONSTANT "TURNING RATE":

$f''(x) = 2$

Thus, when x increases by 1, the slope of the tangent line increases by 2. That is, when $\Delta x = 1$, we have:

$$\begin{aligned} \Delta f' &= f'(x+1) - f'(x) \\ &= (f'(x) + 2) - f'(x) \\ &= 2 \end{aligned}$$

EXERCISE 4

Consider the function $f(x) = 2x^2$.

♣ 1. When x changes by an amount Δx, how much do you expect the slopes of the tangent lines to change by?

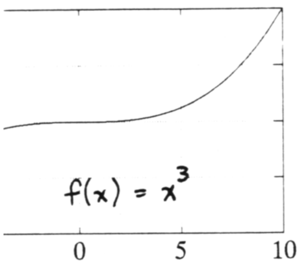
♣ 2. Find the slope of the tangent line at the point $(x + \Delta x, f(x + \Delta x)$ and compare it to the slope of the tangent line at $(x, f(x))$.

♣ 3. What is $\Delta f'$?

EXAMPLE

concavity of the cubing function

Now consider the function $f(x) = x^3$. Here, $f'(x) = 3x^2$ and $f''(x) = 6x$. In this case, the rate of change of the slopes of the tangent lines depends on *what point we are at*. The larger the magnitude of x, the more rapidly the curve 'turns'.



For example, when $x = 1$, we have $f''(1) = 6 \cdot 1 = 6$. Thus, when x changes by some small amount, we expect the slope of the tangent line to change by approximately six times this amount. Look at the chart below:

x	$f'(x) = 3x^2$
1.0	$3(1)^2 = 3$
1.1	$3(1.1)^2 = 3.63$

$\Delta x = 0.1$

$\Delta f' = .63$

$.63 \cong 6(.1)$, $\Delta f' \cong 6(\Delta x)$

Observe that the change in f' is *approximately* six times the change in x, but not exactly. Why? The answer is quite simple: $f''(1)$ gives us an *instantaneous* rate of change. However, as soon as we move away from the point $(1, 1)$, the rate of change is no longer exactly 6. Indeed, over the interval $[1, 1.1]$, f'' is actually *greater* than 6; f'' increases from 6 (at $x = 1$) to 6.6 (at $x = 1.1$). This is precisely why our calculation was a bit high.

EXERCISE 5

Consider the function $f(x) = x^3$.

♣ 1. At the point $(2, 8)$, how fast are the slopes of the tangent lines changing?

♣ 2. How much do you estimate the slopes will change by, in moving from the point $(2, 8)$ to the point $(2.1, (2.1)^3)$?

♣ 3. Find the slopes of the tangent lines at both $x = 2$ and $x = 2.1$. What is $\Delta f'$?

♣ 4. Was your estimate high or low? Why?

EXERCISE 6

Let $f(x) = -(x - 4)^4 + 20$.

♣ 1. Sketch the graph of f.

♣ 2. Plot the point when $x = 2$.

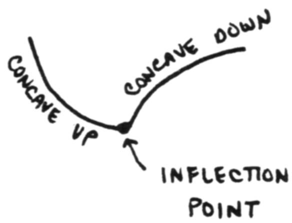
♣ 3. How fast are the slopes of the the tangent lines changing when $x = 2$?

♣ 4. If we move to the point $(2.1, f(2.1))$, how much do you estimate the slopes will change by?

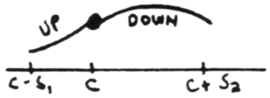
♣ 5. Find $f'(2)$ and $f'(2.1)$. How much did the slopes change by?

places where the concavity changes

Points on the graph of a function f where the concavity changes—from concave up to concave down, or from concave down to concave up—are particularly interesting. Thus, such points are given a special name—they are called *inflection points*. The precise definition appears next.



DEFINITION
inflection point



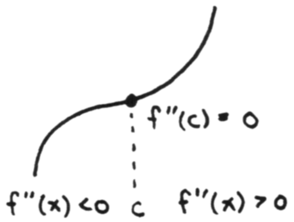
An *inflection point* is a point where the concavity of a function changes (from up to down, or down to up).

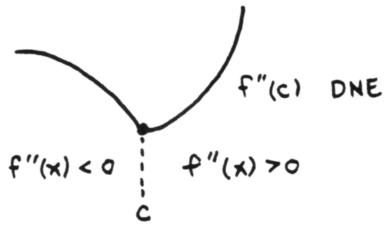
More precisely, the point $(c, f(c))$ is an *inflection point* for a function f if there is an interval $(c - \delta_1, c + \delta_2)$ about c, such that the concavity of f on $(c - \delta_1, c)$ differs from the concavity on $(c, c + \delta_2)$. Here, δ_1 and δ_2 are positive numbers.

Note that an inflection point cannot occur at an endpoint of the domain, because one has to be able to look on *both sides* to see if the concavity is different.

places where $f''(c) = 0$ or $f''(c)$ does not exist are the only CANDIDATES for inflection points

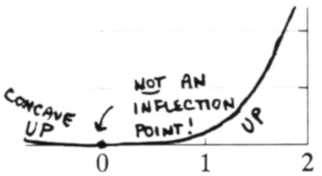
The sketches below illustrate two ways in which an inflection point can occur. It is possible to have an inflection point $(c, f(c))$ where $f''(c) = 0$. Also, it is possible to have an inflection point where $f''(c)$ does not exist. Indeed, a logical argument similar to that used in the previous section shows that *these are the only types of places where inflection points can occur.* Thus, the places where $f''(c) = 0$ and where $f''(c)$ does not exist give the *candidates* for places where inflection points occur.





Caution!

If $f''(c) = 0$, this *does not mean* that there must be an inflection point at $(c, f(c))$. Similarly, if $f''(c)$ does not exist, this *does not mean* that there must be an inflection point at $(c, f(c))$. The sketches below illustrate this fact.



$f(x) = x^4$, $f''(x) = 12x^2 \overset{\text{always}}{\geq} 0$

$f''(0) = 0$

Recall that the critical points of a function give the *candidates* for places where local maxima and minima occur. Similarly, the places where $f''(c) = 0$ or $f''(c)$ does not exist merely give the *candidates* for the places where there are inflection points. Each of these points must be checked to see if it is, or is not, an inflection point.

strategy for finding the inflection points of a function

Suppose it is desired to find all the inflection points of a function f. Proceed as follows.

- Find the domain of f.
- Find f', and then f''. Find all $c \in \mathcal{D}(f)$ where $f''(c) = 0$ or $f''(c)$ does not exist. Remember that an inflection point cannot occur at an endpoint of the domain. These are the *candidates* for inflection points.
- Find the sign of f'' everywhere. Use this information to check each candidate.

EXAMPLE
finding inflection points

Problem: Find all inflection points for the function:

$$P(x) = x^4 + 4x^3 - 18x^2 - 6x + 1$$

- The domain of f is $\mathbb{R}$.

- Find P'':

$$\begin{aligned} P'(x) &= 4x^3 + 12x^2 - 36x - 6 \\ P''(x) &= 12x^2 + 24x - 36 \\ &= 12(x^2 + 2x - 3) \\ &= 12(x+3)(x-1) \end{aligned}$$

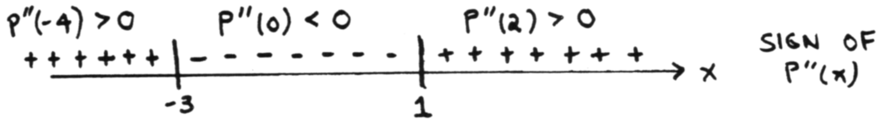
  Observe that $\mathcal{D}(P'') = \mathbb{R}$, and:

$$P''(x) = 0 \quad \iff \quad x = -3 \ \text{ or } \ x = 1$$

  When $x = 1$, $P(1) = (1)^4 + 4(1)^3 - 18(1)^2 - 6(1) + 1 = -18$. Thus, $(1, -18)$ is a candidate for an inflection point.

  Similarly, $(-3, -170)$ is a candidate for an inflection point.

- Determine the sign of P'' everywhere:



  To the left of $x = -3$, the graph is concave up; to the right, concave down. Thus, the concavity changes as one passes through the point $(-3, -170)$, so it is an inflection point. Similarly, $(1, -18)$ is an inflection point.

EXERCISE 7

Find all inflection points for each of the following functions:

♣ 1. $P(x) = x^4 - 4x^3 - 7x + 1$

♣ 2. $f(x) = \sqrt{x} + x^2$

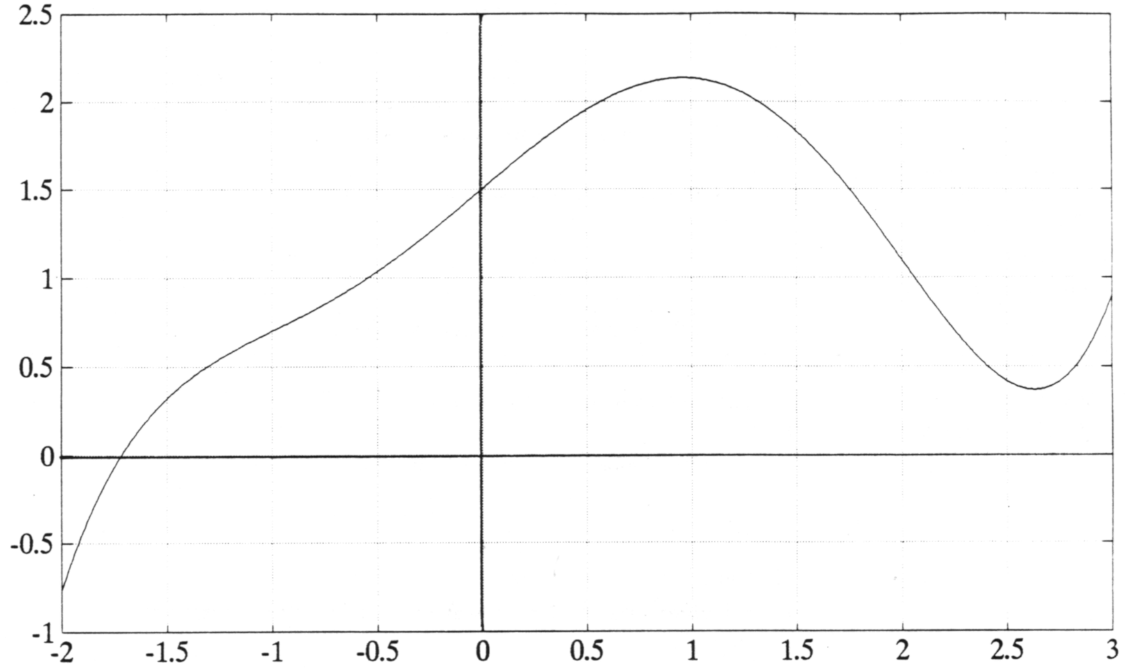
EXERCISE 8

Refer to the graph shown below to answer the following questions. Approximate where necessary.

♣ 1. On what open interval(s) is the function positive? Negative?

♣ 2. On what open interval(s) is the function increasing? Decreasing?

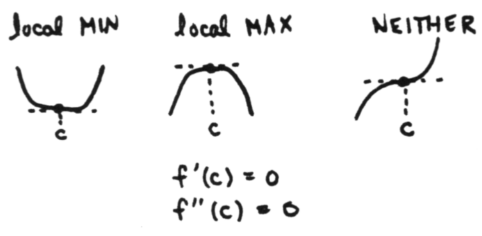
♣ 3. On what open interval(s) is the function concave up? Concave down?



a test for local extreme values that uses the second derivative

If $(c, f(c))$ is a critical point for f, and f is continuous at c, then one way to check if there is a local maximum or minimum at c is to investigate the sign of the first derivative near c. This is the content of the First Derivative Test.

In some cases, there is an easier test. Consider the sketches below. In each case, the point $(c, f(c))$ is a critical point because $f'(c) = 0$; that is, there is a horizontal tangent line at $(c, f(c))$.



In the first sketch, $f''(c) > 0$, so that the slopes of tangent lines are increasing as one passes through the point $(c, f(c))$. Since $f'(c) = 0$, it must be that the slopes are negative to the left of c, and positive to the right of c. That is, the function must decrease to the left of c, and increase to the right of c. Thus, $(c, f(c))$ must be a local minimum. That is, if $f''(c) > 0$, then the graph is concave up at c, and the point is a local minimum.

In the second sketch, $f''(c) < 0$. In this case, the graph is concave down at c, and the point is a local maximum.

If $f''(c) = 0$, anything is possible: no conclusion can be reached without further investigation. These observations lead to what is commonly known as the Second Derivative Test, stated and proved below.

The Second Derivative Test

for local maxima and minima

Suppose that $f'(c) = 0$, so that there is a horizontal tangent line at the point $(c, f(c))$.

If $f''(c) > 0$, then the point $(c, f(c))$ is a local minimum.

If $f''(c) < 0$, then the point $(c, f(c))$ is a local maximum.

If $f''(c) = 0$, no general conclusion is possible.

What does '$f''(c) > 0$' mean?

The proof is given for the case $f''(c) >$. The remaining case is left as an exercise.

One comment before we begin. When a mathematician says

$$f'(c) = 0 \ ,$$

this really means two things:

- f is differentiable at c, so that the number $f'(c)$ exists; and
- $f'(c) = 0$.

For the sake of brevity, the first sentence is usually omitted.

Similarly, when a mathematician says

$$f''(c) > 0 \ ,$$

this means that

- f is twice differentiable at c, so that the number $f''(c)$ exists; and
- $f''(c) > 0$.

PARTIAL PROOF

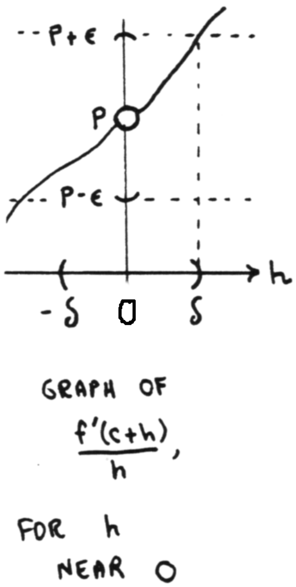
of the Second Derivative Test

Proof. Suppose that $f'(c) = 0$ and $f''(c) > 0$. Assume, for simplicity, that f is defined on both sides of c.

Recall that $f'' = (f')'$. Thus, $f''(c) > 0$ means that the limit

$$\lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h) - 0}{h}$$

exists, and is positive. Call the value of this limit P (for 'positive'). Thus, it is possible to get the values $\frac{f'(c+h)}{h}$ as close to P as desired, merely by requiring that h be sufficiently close to 0. Remember that when h is close to 0, $c + h$ is close to c. In particular, when $h < 0$, $c + h$ is to the left of c; and when $h > 0$, $c + h$ is to the right of c.



Refer to the sketch. Choose ϵ so that every number in the interval $I := (P - \epsilon, P + \epsilon)$ is positive. Then, find δ so that whenever h is within δ of 0, the numbers $\frac{f'(c+h)}{h}$ end up in I.

If $h < 0$, and within δ of 0, then multiplying both sides of the inequality

$$\frac{f'(c+h)}{h} > 0$$

by the negative number h yields

$$f'(c+h) < 0 \ ,$$

so the function is decreasing to the left of the point $(c, f(c))$.

Similarly, if $h > 0$ and within δ of 0, then we get

$$f'(c+h) > 0 \ ,$$

so the function is increasing to the right of the point $(c, f(c))$.

By the First Derivative Test, the point $(c, f(c))$ is a local minimum. ∎

EXERCISE 9

♣ 1. Prove the Second Derivative Test, in the case when $f''(c) < 0$.

♣ 2. Use the Second Derivative Test to find all local extreme values for $P(x) = 3x^4 + 4x^3 - 12x^2 + 1$. To do this, proceed as follows:

First, find all places where $P'(x) = 0$.

Next, check the sign of the second derivative at each value of c for which $P'(c) = 0$.

QUICK QUIZ

sample questions

1. What kind of information does the second derivative of a function give us?
2. Give a precise definition of what it means for a function f to be concave up on an interval I.
3. State the converse of this implication:

   If $x = 1$, then $x^2 = 1$

   Is the converse true or false?
4. Suppose that $f'(c) = 0$ and $f''(c) < 0$. What, if anything, can be said about the point $(c, f(c))$?
5. Let $f(x) = (x - 1)^3$. Find $f''(1)$.

KEYWORDS *for this section*	*Concave up and down, the word 'conversely', inflection points, candidates for inflection points, strategy for finding inflection points, the Second Derivative Test.*

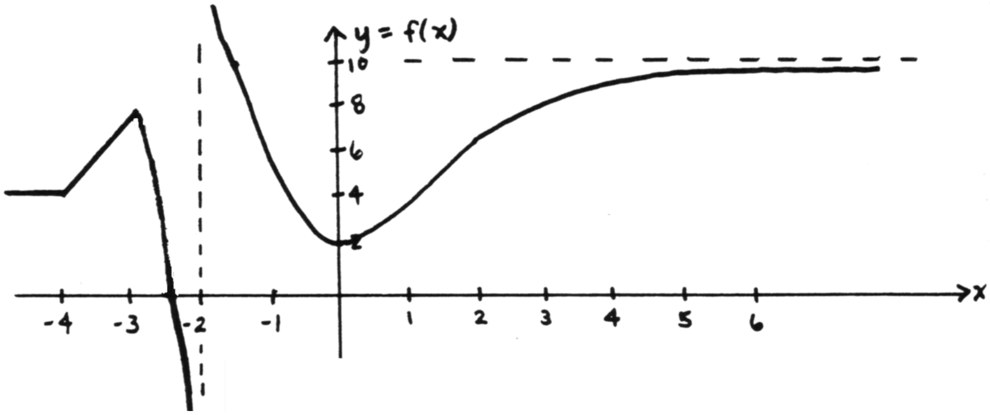
END-OF-SECTION EXERCISES

♣ Use BOTH the First and Second Derivative Tests to find all local extrema for the functions given below. Be sure that the results of both tests agree!

1. $P(x) = x^4 - 2x^3 + x^2 + 10$
2. $P(x) = 9x^4 + 16x^3 + 6x^2 + 1$

♣ All the remaining questions refer to the graph given below. Approximate where necessary. Assume that the patterns exhibited at the graph boundaries continue. If an object does not exist, so state.

♣ Read all the following information off the graph. Be sure to answer using complete mathematical sentences.



3. On what interval(s) is $f(x)$ positive? Negative?
4. On what interval(s) is f increasing? Decreasing?
5. On what interval(s) is f concave up? Concave down?
6. What is $\mathcal{D}(f)$?
7. What is $\mathcal{D}(f')$?
8. Find: $\{x \mid f'(x) = 0\}$
9. Find: $\{x \mid f(x) > 10\}$
10. Find: $\{x \mid f'(x) > 0\}$
11. Find: $\{x \mid f''(x) < 0\}$
12. Find: $\lim_{x\to 0} f(x)$
13. Find: $\lim_{t\to -2} f(t)$
14. Find: $\lim_{y\to -4} f(y)$
15. List all the critical points for this function.
16. Find: $f(0)$, $f'(0)$, $f(1000)$, $f'(1000)$
17. Find: $\{x \in \mathcal{D}(f) \mid f \text{ is not differentiable at } x\}$
18. Find all inflection points.
19. Find: $\displaystyle\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$
20. Find: $\displaystyle\lim_{x\to -3.5} f'(x)$

5.4 Graphing Functions
Some Basic Techniques

graphing a function of one variable

Given an arbitrary function of one variable, call it f, the *graph of* f is a 'picture' of the points $\{(x, f(x)) \,|\, x \in \mathcal{D}(f)\}$. Although the entire graph can rarely be shown (due to the fact that, say, $\mathcal{D}(f)$ is an infinite interval), one certainly wants to see everything interesting. These 'interesting' aspects usually include: local maxima and minima, global maxima and minima, inflection points, discontinuities, 'kinks', x and y-axis intercepts, asymptotes, and behavior at infinity.

Global maxima and minima are discussed in this section. Asymptotes and behavior at infinity are discussed in section 5.6.

Most of the tools necessary to take a systematic approach to graphing a function are now available. Some general guidelines are outlined below.

Graphing a function f

a systematic approach

Let f be a function of one variable. If the first two derivatives of f are reasonably easy to obtain, then the following strategy is suggested to obtain the graph of f:

- Find $\mathcal{D}(f)$, the domain of f. Sketch appropriate axes. Plot a few easy points. In particular, plot any endpoints of the domain of f.
- Note if the function is symmetric about the y axis or the origin. (See the Algebra Review in this section.) If so, the function only needs to be graphed for, say, nonnegative x, and the rest filled in from symmetry.
- Find $f'(x)$.

  Find all $c \in \mathcal{D}(f)$ where $f'(c) = 0$ or $f'(c)$ does not exist.

  Plot these points $(c, f(c))$ with the symbol ' $\times$ ' (if $f'(c)$ does not exist), or with the symbol '$-\!\times\!-$ ' (if $f'(c) = 0$).

  These points, together with the endpoints of $\mathcal{D}(f)$, are the critical points. They are the *candidates* for local maxima and minima.
- Find $f''(x)$.

  Find all $c \in \mathcal{D}(f)$ where $f''(c) = 0$ or $f''(c)$ does not exist.

  Plot these points $(c, f(c))$ with the symbol ' $\divideontimes$ '.

  These are the *candidates* for inflection points.
- Find the intervals where $f''(x) > 0$ (f is concave up) and $f''(x) < 0$ (f is concave down), using the now-familiar procedure:

  Draw a number line labeled *Sign of* $f''(x)$. On it, indicate all the places where f'' is *not continuous*, and all the places where $f''(x) = 0$.

  Choose a test point T in each interval, and see if $f''(T)$ is positive or negative.

  Use this information to sketch the graph.
- Fill in any necessary details, such as x-axis intercept(s), y-axis intercept, asymptote information, and behavior at $\pm\infty$.

global maximum; global minimum

Thus far in this text, we have discussed:

- maximum and minimum values *on an interval*
- *local* maximum and minimum values

Sometimes, it is of interest to know if a function attains a maximum or minimum value, as the inputs are allowed to vary over the *entire domain* of f. If such an extreme value exists, it is called a *global extreme value.*

The precise definition follows.

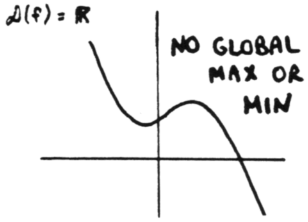
DEFINITION

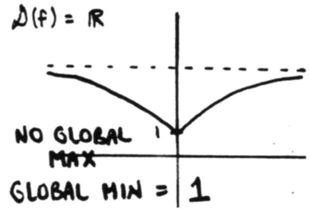
global maximum; global minimum

Let f be a function with domain $\mathcal{D}(f)$.

If there exists $c_1 \in \mathcal{D}(f)$ such that $f(c_1) \le f(x) \ \ \forall\, x \in \mathcal{D}(f)$, then the number $f(c_1)$ is the *global minimum* for f.

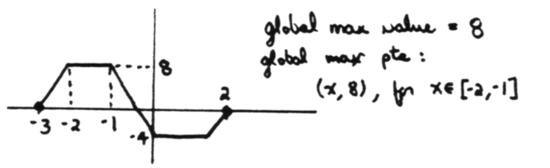
If there exists $c_2 \in \mathcal{D}(f)$ such that $f(c_2) \ge f(x) \ \ \forall\, x \in \mathcal{D}(f)$, then the number $f(c_2)$ is the *global maximum* for f.

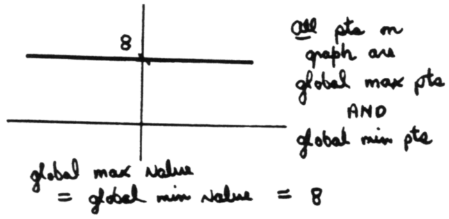




values (numbers) versus points

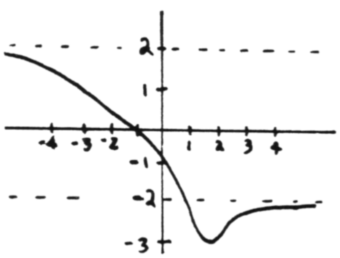
Note that if a global minimum or maximum *value* exists, then it must be unique. However, this value may be taken on by *more than one input*, as the examples below illustrate. As usual, one is often interested in knowing the input(s) that give rise to global extreme values. Thus, one frequently speaks of, say, a *global maximum point.*

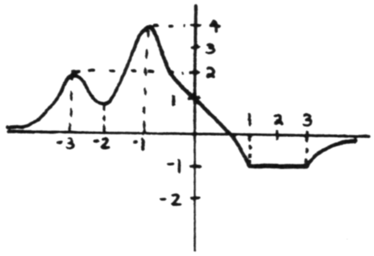




EXERCISE 1

♣ Decide if the graphs shown below have a global maximum value; global minimum value. If so, list all global maximum point(s); all global minimum point(s). Assume that the domain of each function is $\mathbb{R}$.





EXERCISE 2

If the following sentences are false, make a sketch which illustrates how they can fail.

♣ 1. True or False: If $(c, f(c))$ is a local maximum point for f, then it is a global maximum point for f.

♣ 2. True or False: If $(c, f(c))$ is a global maximum point for f, then it is a local maximum point for f.

♣ 3. True or False: If the number M is a global maximum value for f, then it is unique.

♣ 4. True or False: If the point (c, M) is a global maximum point for f, then it is unique.

Now, lots of graphing examples!!

EXAMPLE

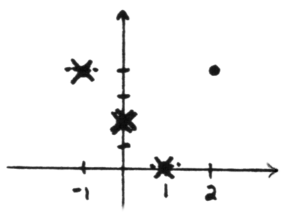
graphing a polynomial

Problem: Completely graph:

$$P(x) = x^3 - 3x + 2$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few simple points:

x	P(x)
0	2
1	0
-1	4
2	4



- Find the first derivative:

$$P'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

Observe that $\mathcal{D}(f') = \mathbb{R}$, so the only critical points come from places where $f'(x) = 0$. Be sure to write down complete mathematical sentences.

$$\begin{aligned} P'(x) = 0 \quad &\iff \quad 3(x-1)(x+1) = 0 \\ &\iff \quad x = 1 \ \text{ or } \ x = -1 \end{aligned}$$

So, $(1, f(1)) = (1, 0)$ and $(-1, f(-1)) = (-1, 4)$ are critical points. Plot these with an $\times$ to emphasize that they correspond to places where there is a horizontal tangent line.

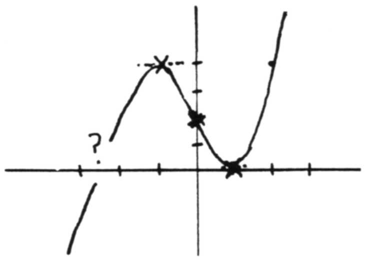
- Find the second derivative:

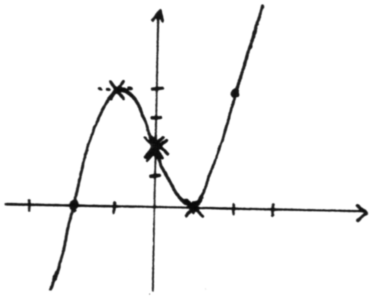
$$P''(x) = 6x$$

Again, $\mathcal{D}(f'') = \mathbb{R}$, so the only candidates for inflection points occur when $f''(x) = 0$.

$$\begin{aligned} P''(x) = 0 \quad &\iff \quad 6x = 0 \\ &\iff \quad x = 0 \end{aligned}$$

Thus, $(0, f(0)) = (0, 2)$ is the only candidate for an inflection point. Plot this point with an $\divideontimes$ to emphasize that there may be an inflection point here.

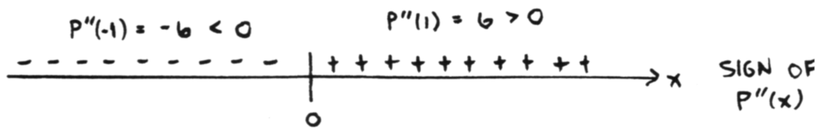




- Investigate the sign of the second derivative:

  P'' is continuous everywhere, and is zero only at $x = 0$. Make a number line, indicating the point $x = 0$. This yields two subintervals, $(-\infty, 0)$ and $(0, \infty)$.

  Choose a 'test point' from each of these intervals.



  Thus, P'' is positive on $(0, \infty)$ and negative on $(-\infty, 0)$, so P is concave up on $(0, \infty)$ and concave down on $(0, -\infty)$. Use this information to fill in the majority of the graph.

- Fill in any missing details. Here, it would be nice to know the second x-axis intercept. We can always 'zero in' on it, using the Intermediate Value Theorem. However, in this case, we can do even better. Since $x = 1$ is a root of P, $x - 1$ must be a factor of P. Do a long division:

$$\begin{array}{r} x^2 + x - 2 \\ x-1\,\overline{)\,x^3 - 3x + 2} \\ -(x^3 - x^2) \\ \hline x^2 - 3x + 2 \\ -(x^2 - x) \\ \hline -2x + 2 \\ -2x + 2 \\ \hline \end{array}$$

  Thus, $P(x) = (x-1)(x^2 + x - 2) = (x-1)^2(x+2)$. The remaining x-axis intercept occurs at $x = -2$.

- Once the graph of P is complete, read off all this important information:

  $(-1, 4)$ is a local maximum

  $(1, 0)$ is a local minimum

  no global maximum, no global minimum

  $(0, 2)$ is an inflection point

  concave down on $(-\infty, 0)$

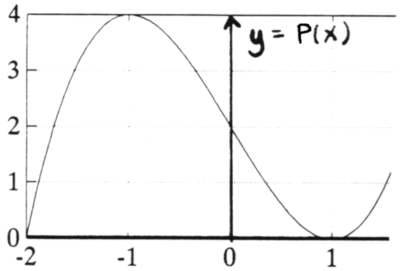
  concave up on $(0, \infty)$

  increasing on $(-\infty, -1) \cup (1, \infty)$

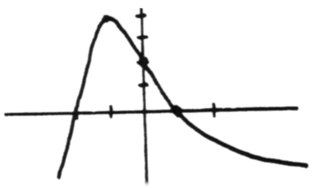
  decreasing on $(-1, 1)$

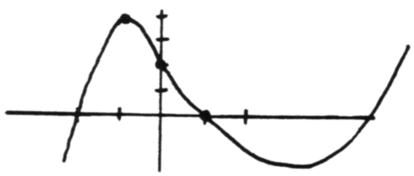
  Note that it was *not* necessary to investigate the sign of P' to find out where P increases and decreases.

  A graph of P is shown.



EXERCISE 3 Reconsider the previous example. It was found that P is concave up on $(0, \infty)$. Why couldn't the graph look like the two situations shown below? Comment.





checking behavior at infinity

In the last step of the previous example, one final check could have been made: the graph shows that as $x \to \infty$, $f(x) \to \infty$; and as $x \to -\infty$, $f(x) \to -\infty$. It would be prudent to verify that the function f really behaves this way.

approximating polynomials by their highest order term

To this end, an important property of polynomials is needed. Informally:

When x is large (positive or negative), then a polynomial P is well approximated by its highest order term.

Another way to state this is:

For large x, the highest order term of a polynomial dominates.

The *highest order term* of a polynomial in x is the term with the greatest exponent on x.

$x \gg 0$ means x is large and positive

$x \ll 0$ means x is large and negative

The phrase 'x is large' is sometimes used to mean that x is a number that is very, very far from zero on the number line. That is, either x is positive and $|x|$ is much greater than zero; or x is negative and $|x|$ is much greater than zero. Thus, one might say that both 10^7 and -2^{36} are 'large' numbers.

If x is large, then $P(x) = x^3 - 3x + 2$ is well approximated by the simpler polynomial $\tilde{P}(x) = x^3$. That is, for large x:

$$x^3 - 3x + 2 \approx x^3$$

(The symbol '$\approx$' is read as *is approximately equal to*.)

When x is large and positive, so is x^3. Thus, so must be $x^3 - 3x + 2$.

When x is large and negative, so is x^3. Thus, so must be $x^3 - 3x + 2$.

The sentence '$x \gg 0$' is read as 'x is much greater than zero'. So instead of saying 'x is large and positive', one can equivalently say '$x \gg 0$'.

The sentence '$x \ll 0$' is read as 'x is much less than zero'. So instead of saying 'x is large and negative', one can equivalently say '$x \ll 0$'.

more precisely: investigate a limit!

This idea of 'approximation by the highest order term' can be made precise. Consider an arbitrary polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \ , \qquad (*)$$

where $a_n \neq 0$. The highest order term is $a_n x^n$.

It is possible to get $P(x)$ *as close to $a_n x^n$ as desired*, by making x sufficiently large. To see that this is true, divide both sides of (*) by x^n, obtaining:

$$\frac{P(x)}{x^n} = a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}$$

For large (enough) values of x, all the terms on the right-hand side, except a_n, will be close to zero. That is, as x approaches $+\infty$ or $-\infty$, $\frac{P(x)}{x^n}$ approaches a_n. And when $\frac{P(x)}{x^n}$ is close to a_n, then $P(x)$ is close to $a_n x^n$. This idea will be made yet more precise in the final section of this chapter.

EXAMPLE

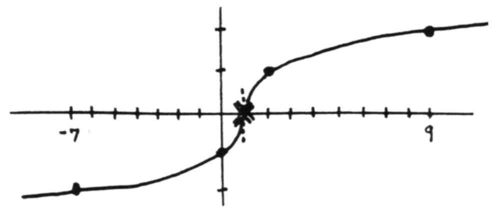
Problem: Completely graph:

$$f(t) = (t-1)^{1/3}$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few points:

t	$f(t)$
0	-1
1	0
2	1
9	2

t	$f(t)$
-7	-2



- $f'(t) = \frac{1}{3}(t-1)^{-2/3}$

  $\mathcal{D}(f') = \{x \,|\, x \neq 1\}$. Observe that f *is* defined at 1, but f' is *not* defined at 1. Thus, $(1, f(1)) = (1, 0)$ is a critical point. As x approaches 1 (from either side), $f'(x) \to \infty$, so there is a *vertical tangent line* at the point $(1, 0)$. Indicate this on the graph using the symbol $\times$.

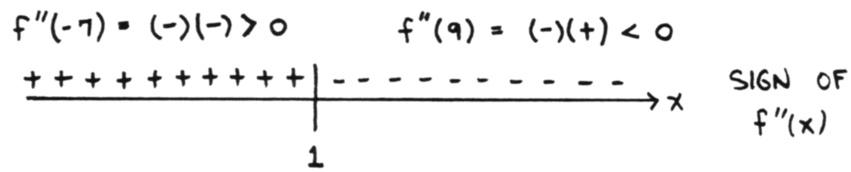
  f' is never equal to 0, so there are no other critical points.

- $f''(t) = -\frac{2}{9}(t-1)^{-5/3}$

  Again, $\mathcal{D}(f'') = \{x \,|\, x \neq 1\}$. So f'' is *not* defined at $x = 1$, but f is. Thus, $(1, 0)$ is also a candidate for an inflection point. Put a $\divideontimes$ over this point, to remind us of this fact.

  f'' is never equal to 0, so there are no other candidates for inflection points.

- f'' is continuous everywhere except at 1, and is never 0. Thus, one need only check the sign of f'' (plus or minus) on the intervals below.



- Details: check behavior at infinity.

  For large values of t:
  $$(t-1)^{1/3} \approx t^{1/3} = \sqrt[3]{t}$$
  So, as $t \to \infty$, $f(t) \to \infty$. And, as $t \to -\infty$, $f(t) \to -\infty$.

- Read off all important information:

  no local maxima or minima

  no global maximum or minimum

  $(1, 0)$ is an inflection point

  x-axis intercept: $(1, 0)$

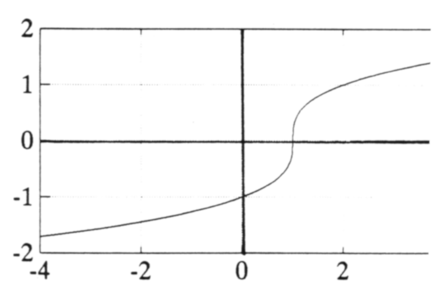
  y-axis intercept: $(0, -1)$

  concave up on $(-\infty, 1)$

  concave down on $(1, \infty)$

  increasing on $(-\infty, \infty)$

  A graph of f is shown at right.



EXAMPLE

Problem: Completely graph:

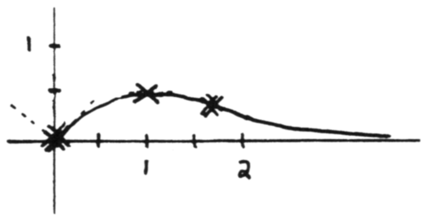
$$g(x) = \frac{|x|}{x^2+1}$$

- $\mathcal{D}(g) = \mathbb{R}$, since x^2+1 is never zero. Note that g is an even function, since:

$$g(-x) = \frac{|-x|}{(-x)^2+1} = \frac{|x|}{x^2+1} = g(x)$$

  Thus, g only needs to be graphed on $(0,\infty)$; the rest is filled in from symmetry.

- For $x > 0$, $|x| = x$, so that $g(x) = \frac{x}{x^2+1}$ and:



$$\begin{aligned} g'(x) &= \frac{(x^2+1)(1)-(x)(2x)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \\ &= \frac{(1-x)(1+x)}{(x^2+1)^2} \end{aligned}$$

Remember that this formula only holds for $x > 0$. When $x = 1$, there is a horizontal tangent line. So, $(1, g(1)) = (1, \frac{1}{2})$ is a critical point.

Is there a tangent line at $x = 0$? Note that:

$$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \frac{(1-x)(1+x)}{(x^2+1)^2} = \frac{(1)(1)}{(1)^2} = 1$$

So, as x approaches zero from the right, the tangent lines have slopes that approach 1. Sketch in a dashed line with slope 1 to the right of zero, as shown.

By symmetry, as zero is approached from the left, the tangent lines have slopes that approach -1. Thus, there is a 'kink' at zero. That is, $g'(0)$ does not exist. So, $(0, g(0)) = (0,0)$ is also a critical point.

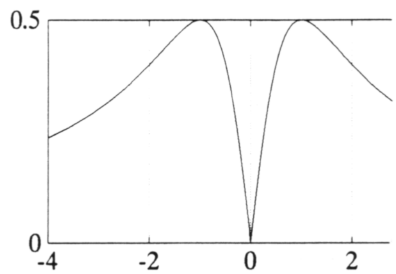
Observe that it has been shown that g' is not continuous at 0. Indeed, g' has a nonremovable discontinuity at $x = 0$.

- Since g' is not continuous at 0, g' is not differentiable at 0. (♣ Why?) That is, $g''(0)$ does not exist, and is a candidate for an inflection point.

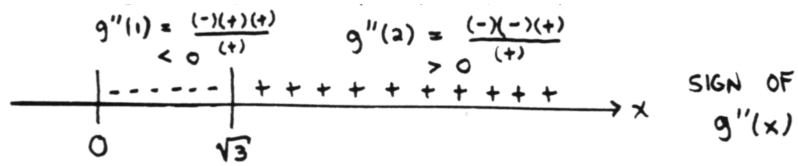
  For $x > 0$:

$$\begin{aligned} g''(x) &= \frac{(x^2+1)^2(-2x) - (1-x^2)2(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1)[(x^2+1)+2(1-x^2)]}{(x^2+1)^4} \\ &= \frac{-2x(3-x^2)}{(x^2+1)^3} \\ &= \frac{-2x(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+1)^3} \end{aligned}$$

  When $x = \sqrt{3} \approx 1.7$, $g''(x)$ is zero. Thus, $(1.7, g(1.7)) = (1.7, 0.4)$ is an (approximate) candidate for an inflection point.



- Investigate the sign of g'' on $(0,\sqrt{3})$ and $(\sqrt{3},\infty)$:



Thus, g is concave down on $(0,\sqrt{3})$ and concave up on $(\sqrt{3},\infty)$.

- Details: check behavior at infinity.
For $x \gg 0$:
$$\frac{x}{x^2+1} \approx \frac{x}{x^2} = \frac{1}{x}$$
Thus, as $x \to \infty$, $f(x) \to 0$.

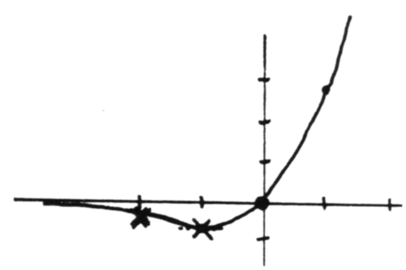
- Read off all important information:
$(0,0)$ is a local and global minimum
$(1,0.5)$ and $(-1,0.5)$ are local and global maxima
$(\sqrt{3},\frac{\sqrt{3}}{4})$ and $(-\sqrt{3},\frac{\sqrt{3}}{4})$ are inflection points
concave up on $(-\infty,-\sqrt{3}) \cup (\sqrt{3},\infty)$
concave down on $(-\sqrt{3},0) \cup (0,\sqrt{3})$
increasing on $(-\infty,-1) \cup (0,1)$
decreasing on $(-1,0) \cup (1,\infty)$
A graph of g is shown at left.

EXAMPLE

Problem: Completely graph:
$$f(x) = xe^x$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few points:

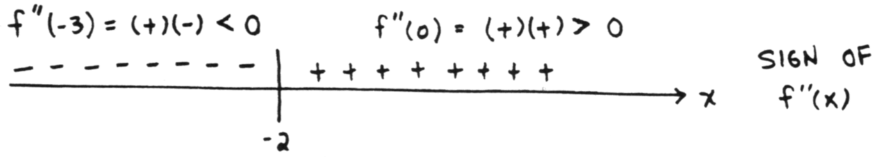
x	f(x)
0	0
1	e ≈ 2.7
-1	-1/e ≈ -.4



- $f'(x) = xe^x + (1)e^x = e^x(x+1)$
$\mathcal{D}(f') = \mathbb{R}$
$$\begin{aligned} f'(x) = 0 \quad &\iff \quad e^x(x+1) = 0 \\ &\iff \quad x+1 = 0 \\ &\iff x = -1 \end{aligned}$$
Thus, $(-1, f(-1)) = (-1, -e^{-1}) = (-1, -\frac{1}{e}) \approx (-1, -0.4)$ is the only critical point.

- $f''(x) = e^x(1) + e^x(x+1) = e^x(x+2)$
$\mathcal{D}(f'') = \mathbb{R}$
$$f''(x) = 0 \quad \iff \quad x = -2\ ,$$
so $(-2, -2e^{-2}) \approx (-2, -0.3)$ is the only candidate for an inflection point.

- Investigate the sign of the second derivative:



- Details: Note that

$$f(x) = 0 \quad \iff \quad xe^x = 0 \quad \iff \quad x = 0\ ,$$

so the only x-axis intercept is at 0.

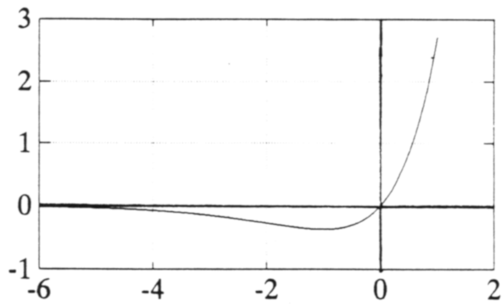
As $x \to \infty$, $f(x) \to \infty$.

As $x \to -\infty$, we run into a '$(-\infty)(0)$' situation, which requires further investigation. In this case, plotting some additional points, and using the fact that f cannot cross the x-axis again, we conclude that as $x \to -\infty$, $f(x) \to 0$.

- $(-1, -\frac{1}{e})$ is a local and global minimum

  $(-2, -\frac{2}{e^2})$ is an inflection point
  concave up on $(-2, \infty)$
  concave down on $(-\infty, -2)$
  increasing on $(-1, \infty)$
  decreasing on $(-\infty, -1)$

  A graph of f is shown below.



EXERCISE 4

Completely graph the following functions:

♣ 1. $P(x) = 3x^4 + 4x^3 - 12x^2 + 1$

♣ 2. $f(t) = (t + 2)^{1/5}$

♣ 3. $g(x) = \frac{|x|}{x^2 - 1}$

♣ 4. $f(x) = xe^{-x}$

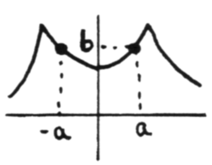
Read off all this information from your graphs:

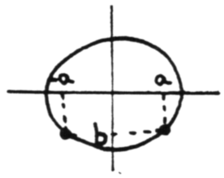
local maxima and minima
global maxima and minima
inflection points
x and y-axis intercepts (approximate, if necessary)
open intervals on which the graph is concave up and down
open intervals on which the graph is increasing and decreasing

ALGEBRA REVIEW
even and odd functions

DEFINITION
symmetry about the y-axis

If a graph has the property that whenever (a, b) is on the graph, so is $(-a, b)$, then the graph is *symmetric about the y-axis.*





Observe that if such a graph is folded the graph along the y-axis, the part of the graph to the right of the y-axis coincides with the part to the left. Why is this? Answer: By folding along the y-axis, one is *identifying* points that have the same magnitude x-values. For example, after folding, $x = 2$ ends up on top of $x = -2$. And, $x = 5$ ends up on top of $x = -5$. For a graph that is symmetric about the y-axis, such points have exactly the same y-values, so the points coincide.

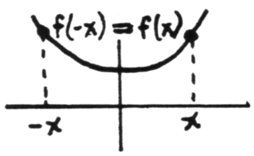
There is an equivalent characterization of symmetry about the y-axis, if one happens to be working with a function:

DEFINITION
even functions

If a function f satisfies the property that

$$f(-x) = f(x) \quad \forall\, x \in \mathcal{D}(f) \ ,$$

then f is an *even* function, and its graph is symmetric about the y-axis.



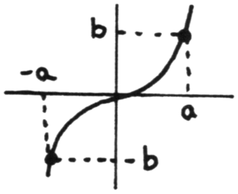
For example, $f(x) = x^4$ is an even function. To see this, one need only verify that:

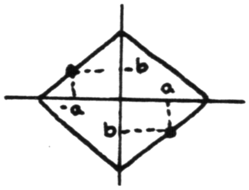
$$f(-x) = (-x)^4 = x^4 = f(x)$$

♣ Is $f(x) = x^6 + 2x^2$ an even function? How about $g(x) = \frac{1}{x^2+1}$?

DEFINITION
symmetry about the origin

If a graph satisfies the property that whenever (a, b) is on the graph, so is $(-a, -b)$, then the graph is *symmetric about the origin.*





Note that if such a graph is folded *twice*—once along the x-axis, and once along the y-axis—then the parts of the graph coincide.

♣ Think about why this 'coinciding' takes place.

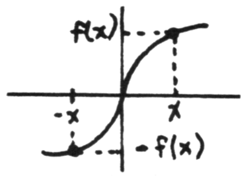
There is an equivalent characterization of symmetry about the origin, if one happens to be working with a function:

DEFINITION
odd functions

If a function f satisfies the property that

$$f(-x) = -f(x) \quad \forall\ x \in \mathcal{D}(f) \ ,$$

then f is an *odd* function, and its graph is symmetric about the origin.



So if f is an odd function, then whenever $(x, f(x))$ is on the graph, so is $(-x, -f(x))$.

♣ Show that $f(x) = x^3$ is an odd function; graph it.

♣ Is $f(x) = \dfrac{1}{x^3 - x}$ an odd function? How about $g(x) = \dfrac{x}{x^3 - x}$?

QUICK QUIZ
sample questions

1. Sketch the graph of a function that has a global maximum value of 10; there should be 3 global maximum points.
2. When $x \gg 0$, what does the graph of $P(x) = 127 - 3x + x^4 - 6x^7$ look like? How about when $x \ll 0$?
3. Is $f(x) = x^5 - x$ an even function? An odd function? Be sure to support your answers.
4. Completely graph $f(x) = 6x^2 - 7x - 3$, using the systematic approach discussed in this section.

KEYWORDS
for this section

A systematic approach to graphing a function, symmetry about the y-axis, even functions, symmetry about the origin, odd functions, global maximum and minimum values, global maximum and minimum points, checking behavior at infinity, approximating polynomials by their highest order term.

END-OF-SECTION EXERCISES

♣ Re-do each of the graphing examples from this section, *without looking at the text.* If you get stuck, then study the example, and try it on your own again.

5.5 More Graphing Techniques

graphing polynomials

Since polynomials are infinitely differentiable, the only critical points and candidates for inflection points arise from places where P' and P'' are equal to zero. *If* P' and P'' can be factored, then their zeroes are easy to find; if not, the zeroes can be *approximated* using the *Intermediate Value Theorem.*

In this section, some techniques concerned with factoring polynomials are reviewed. Most of these techniques should be familiar to you from algebra, and are merely gathered here for your convenience. We begin by studying quadratic polynomials.

factorable over the integers

Let $P(x) = ax^2 + bx + c,\ \ a \neq 0$, be a quadratic polynomial. The polynomial P is '*factorable over the integers*' if

$$P(x) = (K_1x + K_2)(K_3x + K_4)\ ,$$

where the K_i are all integers.

Thus, $P(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$ *is* factorable over the integers, but $P(x) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ *is not* factorable over the integers.

factoring $x^2 + bx + c$, b and c integers

If $P(x) = x^2 + bx + c$, where the coefficient of the x^2 term is 1, then one usually takes the approach illustrated below to try and factor P:

Problem: Factor $P(x) = x^2 + x - 6$.

Solution: A factorization of P must be of the form:

$$x^2 + x - 6 = (x + A)(x + B) = x^2 + (\overbrace{A + B}^{\text{must } = 1})x + \overbrace{AB}^{\text{must } = -6}$$

Thus, one seeks integers A and B that multiply together to give -6 (the constant term), and that add together to give 1 (the coefficient of the x term). In this case, taking $A = 3$ and $B = -2$ work, so that:

$$x^2 + x - 6 = (x + 3)(x - 2)$$

When $a \neq 1$, a similar approach can be taken, and is discussed next.

factoring $ax^2 + bx + c$, integer coefficients

Suppose that $P(x) = ax^2 + bx + c$, $a \neq 0$, has integer coefficients, and *is* factorable over the integers. That is, suppose there exist integers K_1, K_2, K_3 and K_4 for which:

$$\begin{aligned} ax^2 + bx + c &= (K_1x + K_2)(K_3x + K_4) \\ &= \overbrace{K_1K_3}^{a}\,x^2 + (\overbrace{\underbrace{K_2K_3}_{:=A} + \underbrace{K_1K_4}_{:=B}}^{b})x + \overbrace{K_2K_4}^{c} \quad \text{(multiplying out)} \end{aligned}$$

Defining $A := K_2K_3$ and $B := K_1K_4$, we see that

$$AB = (K_2K_3)(K_1K_4) = (K_1K_3)(K_2K_4) = ac$$

and:

$$A + B = K_2K_3 + K_1K_4 = b$$

What is all this saying? It says that:

Whenever a polynomial $ax^2 + bx + c$ is factorable over the integers, we can find integers A and B, where $AB = ac$ and $A + B = b$, that (we'll see) can be used to factor the polynomial for us!

The technique is illustrated in the next example.

EXAMPLE

factoring a quadratic, $a \neq 1$

Problem: Factor $8x^2 - 10x - 3$.

Solution: We seek integers A and B satisfying

$$AB = (\text{coefficient of } x^2 \text{ term}) \cdot (\text{constant term})$$

and:

$$A + B = \text{coefficient of } x \text{ term}$$

Thus, we want:

$$AB = (8)(-3) = -24 \quad \text{and} \quad A + B = -10$$

Choosing $A = -12$ and $B = 2$ works. Then:

$$\begin{aligned} 8x^2 - 10x - 3 &= 8x^2 + (2x - 12x) - 3 && \text{(rewrite middle term)} \\ &= (8x^2 + 2x) + (-12x - 3) && \text{(regroup)} \\ &= 2x(4x + 1) - 3(4x + 1) && \text{(factor each group)} \\ &= (2x - 3)(4x + 1) && \text{(factor out } (4x + 1)\text{)} \end{aligned}$$

Note that when the middle term is rewritten as a sum, the order *does not matter*:

$$\begin{aligned} 8x^2 - 10x - 3 &= 8x^2 + (-12x + 2x) - 3 && \text{(rewrite middle term)} \\ &= (8x^2 - 12x) + (2x - 3) && \text{(regroup)} \\ &= 4x(2x - 3) + (2x - 3) && \text{(factor each group)} \\ &= (4x + 1)(2x - 3) && \text{(factor out } (2x - 3)\text{)} \end{aligned}$$

EXERCISE 1

Use the technique described above to factor the following quadratics.

♣ 1. $3x^2 + 2x - 1$

♣ 2. $10x^2 - 13x - 3$

♣ 3. $14x^2 + 19x - 3$

★

When is $ax^2 + bx + c$, with integer coefficients, factorable over the integers?

Here's a precise statement of the factoring result discussed above:

THEOREM. Let $P(x) = ax^2 + bx + c$ have integer coefficients, $a \neq 0$. Then, P is factorable over the integers if and only if there exist integers A and B with $AB = ac$ and $A + B = b$.

Idea of Proof. It has already been shown that if P is factorable over the integers, then integers A and B with the desired property exist.

The other direction uses the fact that a polynomial with integer coefficients is factorable over $\mathbb{Z}$ iff it is factorable over $\mathbb{Q}$ (see, e.g., John B. Fraleigh, A First Course in Abstract Algebra, third edition, page 280). Suppose integers A and B exist with $AB = ac$ and $A+B = b$. If $c = 0$, then $ax^2+bx = x(ax+b)$ is factorable over $\mathbb{Z}$. Suppose $c \neq 0$. Then, since $a \neq 0$, and $AB = ac$, both A and B are nonzero. Further, $AB = ac \implies \frac{A}{a} = \frac{c}{B}$. Then:

$$\begin{aligned} ax^2 + bx + c &= ax^2 + (A + B)x + c \\ &= (ax^2 + Ax) + (Bx + c) \\ &= ax(x + \frac{A}{a}) + B(x + \frac{c}{B}) \\ &= (ax + B)(x + \frac{c}{B}) \end{aligned}$$

Thus, P is factorable over $\mathbb{Q}$, and hence over $\mathbb{Z}$. ∎

a technique that always works;

using the quadratic formula

The *quadratic formula* can always be used to factor *any quadratic polynomial*, whether or not it is factorable over the integers. Recall that the *quadratic formula* says that the equation $ax^2 + bx + c = 0$, $a \neq 0$, has solutions x_1 and x_2 given by:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The '+' sign gives one solution; the '−' sign gives the second solution.

These zeroes provide the factors of the polynomial:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Note that you must supply the constant factor a yourself.

EXAMPLE

factoring a quadratic by using the quadratic formula

Problem: Factor $8x^2 + 5x - 3$, using the quadratic formula.

Solution: First, find the roots of this quadratic. That is, solve:

$$8x^2 + 5x - 3 = 0$$

By the quadratic formula:

$$\begin{aligned} x_{1,2} &= \frac{-5 \pm \sqrt{5^2 - 4(8)(-3)}}{2(8)} \\ &= -1\ ,\ \frac{3}{8} \end{aligned}$$

Since -1 is a root, $x-(-1)=x+1$ is a factor.
Since $\frac{3}{8}$ is a root, $x-\frac{3}{8}$ is a factor.
Only the constant factor need be supplied:

$$\begin{aligned} 8x^2+5x-3 &= 8(x+1)(x-\frac{3}{8}) \\ &= (x+1)8(x-\frac{3}{8}) \\ &= (x+1)(8x-3) \end{aligned}$$

♣ Use the technique discussed earlier to factor $8x^2+5x-3$.

EXERCISE 2 ♣ Use the quadratic formula to factor each polynomial from Exercise 1.

EXAMPLE
graphing a more complicated polynomial

Problem: Completely graph $f(x)=(x-1)^2(2x+3)x$.

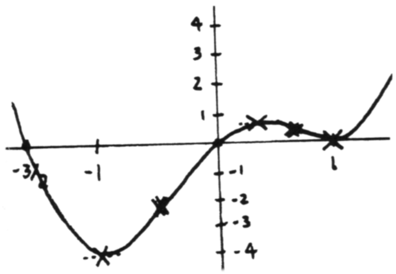
- Plot a few points:

x	P(x)
1	0
-3/2	0
0	0
2	14
-1	-4

x	P(x)
3/8	≃ .55
.72	≃ .25
-.47	≃ -2.09

- Find the first derivative. Use the 'generalized product rule': $\frac{d}{dx}(ABC)=A'BC+AB'C+ABC'$

$$\begin{aligned} f'(x) &= 2(x-1)(2x+3)x+(x-1)^2(2)x+(x-1)^2(2x+3)(1) \\ &= (x-1)\big[2x(2x+3)+2x(x-1)+(x-1)(2x+3)\big] \\ &= (x-1)(8x^2+5x-3) \\ &= (x-1)(x+1)(8x-3) \end{aligned}$$



Thus, $f'(x)=0$ when $x=1,-1,\frac{3}{8}$. Find the corresponding function values, and add these points to the table of points started above. Plot the points with a $\times$.

- Find the second derivative:

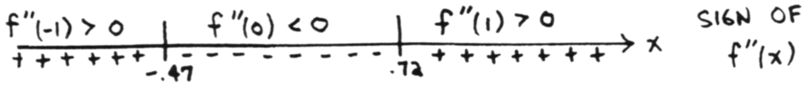
$$\begin{aligned} f''(x) &= (1)(x+1)(8x-3)+(x-1)(1)(8x-3)+(x-1)(x+1)(8) \\ &= 24x^2-6x-8 \\ &= 2(12x^2-3x-4) \end{aligned}$$

Using the quadratic formula, the solutions of $12x^2-3x-4=0$ are:

$$x_1=\frac{3+\sqrt{201}}{24}\approx 0.72 \quad \text{and} \quad x_2=\frac{3-\sqrt{201}}{24}\approx -0.47$$

Find the corresponding function values, and plot these points with a $\divideontimes$.

- Sign of f'':



  Use this concavity information to fill in the graph.
- Behavior at infinity: As $x \to \pm\infty$, $f(x) \approx 2x^4 \to \infty$, which agrees with the graph.

some final results

The remainder of this section is a collection of useful results and techniques concerning polynomials. These may be familiar to you from algebra. They are merely gathered here for your convenience.

RATIONAL ROOT THEOREM

Let $P(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with *integer* coefficients. Suppose that $a_n \neq 0$ and $a_0 \neq 0$.

If P has a rational zero $\frac{p}{q}$ (in lowest terms), then p is a factor of a_0 and q is a factor of a_n.

What if $a_0 = 0$?

Observe that if $a_0 = 0$ and $a_1 \neq 0$, then:

$$P(x) = x(\overbrace{a_n x^{n-1} + \cdots + a_2 x + a_1}^{\tilde{P}(x)})$$

Apply the Rational Root Theorem to $\tilde{P}(x)$.

★
PROOF
of the Rational Root Theorem

Proof. The notation $a|b$ (read as 'a divides b') means that a is a factor of b. Suppose $\frac{p}{q}$ is a rational root in lowest terms, so:

$$a_n(\frac{p}{q})^n + a_{n-1}(\frac{p}{q})^{n-1} + \cdots + a_1(\frac{p}{q}) + a_0 = 0$$

Multiplication by q^n yields:

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \qquad (*)$$

Observe that all terms except the last have a factor of p. Then:

$$p(a_n p^{n-1} + \cdots + a_1 q^{n-1}) = -a_0 q^n$$

Since p divides the left-hand side, it must divide the right-hand side. But $p \not| q$, so $p \not| q^n$, so it must be that $p | a_0$.

For the remaining result, observe that every term in (*) except the first has a factor of q. Repeat the argument, with obvious changes. ∎

negating 'A and B'

The Rational Root Theorem is an implication (with some additional hypotheses):

IF P has a rational zero $\frac{p}{q}$ (in lowest terms),
THEN (p is a factor of a_0) and (q is a factor of a_n).

The conclusion of this implication is a sentence of the form 'A and B'. Thus, to find the contrapositive of this implication, one must negate 'A and B'. How is this done?

Use your intuition: 'A and B' is true only when *both* A and B are true. So when is 'A and B' false? When A is false, or B is false. Precisely,

$$\text{not}(A \text{ and } B) \quad \iff \quad (\text{not}\,A) \text{ or } (\text{not}\,B)\ ,$$

as the truth table below confirms:

A	B	A and B	not (A and B)	not A	not B	(not A) or (not B)
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

logical symbols:
$\wedge$ *for 'and'*
$\vee$ *for 'or'*
$\neg$ *for 'not'*
DeMorgan's laws

The sentence 'A and B' can be written as $A \wedge B$. The symbol $\wedge$ is a synonym for the mathematical word 'and'.

The sentence 'A or B' can be written as $A \vee B$. The symbol $\vee$ is a synonym for the mathematical word 'or'.

The sentence 'not A' can be written as $\neg A$. The symbol $\neg$ is a synonym for the mathematical word 'not'.

With this notation, the previous logical equivalence can be more simply written as:

$$\neg(A \wedge B) \quad \iff \quad (\neg A) \vee (\neg B)$$

In the next exercise, you are asked to prove that:

$$\neg(A \vee B) \quad \iff \quad (\neg A) \wedge (\neg B)$$

These two logical equivalences are commonly known as *DeMorgan's Laws.*

EXERCISE 3

♣ Prove that:

$$\neg(A \vee B) \quad \iff \quad (\neg A) \wedge (\neg B)$$

That is, make a truth table which shows that $\neg(A \vee B)$ and $(\neg A) \wedge (\neg B)$ always have the same truth values.

Now, the contrapositive of the sentence:

IF P has a rational zero $\frac{p}{q}$ (in lowest terms),
THEN (p is a factor of a_0) and (q is a factor of a_n)

is:

IF (p is not a factor of a_0) or (q is not a factor of a_n),
THEN $\frac{p}{q}$ is not a zero of P

This latter sentence tells us that the only *candidates* for rational roots of P are numbers of the form $\frac{p}{q}$, where p is a factor of the constant term, and q is a factor of the leading coefficient. The next example illustrates how this information is used.

EXAMPLE
using the Rational Root Theorem

Problem: Find all rational roots of $P(x) = 14x^4 - x^3 - 17x^2 + x + 3$. Use these roots to factor P as completely as possible.

Solution: The leading coefficient is 14, with factors: ± 1, ± 2, ± 7, ± 14

The constant term is 3, with factors: ± 1 and ± 3

Thus, if $\frac{p}{q}$ is a root of P, it must be that:

$$p \in \{\pm 1, \pm 3\} \quad \text{and} \quad q \in \{\pm 1, \pm 2, \pm 7, \pm 14\}$$

That is:

$$\frac{p}{q} \in \{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{7}, \pm\frac{3}{14}\}$$

Each candidate is checked:

$$\begin{aligned}
P(1) &= 14(1)^4 - 1^3 - 17(1)^2 + 1 + 3 = 0 && \text{(root 1, factor } x-1) \\
P(-1) &= 14(-1)^4 - (-1)^3 - 17(-1)^2 + (-1) + 3 = 0 && \text{(root } -1, \text{ factor } x+1) \\
P(\frac{1}{2}) &= \cdots = 0 && \text{(root } \frac{1}{2}, \text{ factor } x - \frac{1}{2}) \\
P(-\frac{1}{2}) &= \cdots \ne 0 && (-\frac{1}{2} \text{ is not a root}) \\
&\vdots
\end{aligned}$$

Continuing, it is found that $P(1) = P(-1) = P(-\frac{3}{7}) = P(\frac{1}{2}) = 0$. This information is used to factor P:

$$\begin{aligned}
P(x) &= 14(x - \frac{1}{2})(x + \frac{3}{7})(x-1)(x+1) \\
&= 2(x - \frac{1}{2})7(x + \frac{3}{7})(x-1)(x+1) \\
&= (2x-1)(7x+3)(x-1)(x+1)
\end{aligned}$$

Note that we had to supply the constant factor of 14 ourselves.

EXAMPLE
using the Rational Root Theorem

Problem: Find all rational roots of $P(x) = x^4 - 2x^2 - 3x - 2$. Use these roots to factor P as completely as possible.

Solution: If $\frac{p}{q}$ is a rational root, then:

$$p \in \{\pm 1, \pm 2\} \quad \text{and} \quad q \in \{\pm 1\}$$

Thus:

$$\frac{p}{q} \in \{\pm 1, \pm 2\}$$

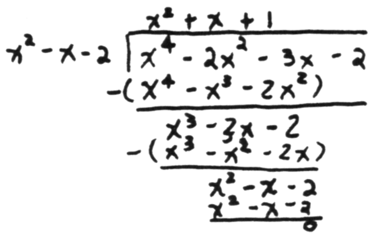
Indeed:

$$\begin{aligned}
P(1) &= 1 - 2 - 3 - 2 \ne 0 \\
P(-1) &= 1 - 2 + 3 - 2 = 0 \\
P(2) &= 16 - 8 - 6 - 2 = 0 \\
P(-2) &= 16 - 8 + 6 - 2 \ne 0
\end{aligned}$$

Thus, -1 and 2 are roots, so:

$$P(x) = (x+1)(x-2)(????) = (x^2-x-2)(????)$$

Use long division to find the remaining factor:



Thus:

$$P(x) = (x+1)(x-2)(x^2+x+1)$$

An application of the quadratic formula shows that the roots of x^2+x+1 are not real numbers. Thus, P cannot be factored any further, using only real numbers.

EXERCISE 4

♣ 1. Refer to the previous example. Find two more polynomials, different from P, that have precisely the same candidates for rational roots. (Hint: Only the leading coefficient and the constant term are used to find the candidates.)

♣ Use the rational root theorem to find all rational roots of the following polynomials. Use this information to factor the polynomial as completely as possible.

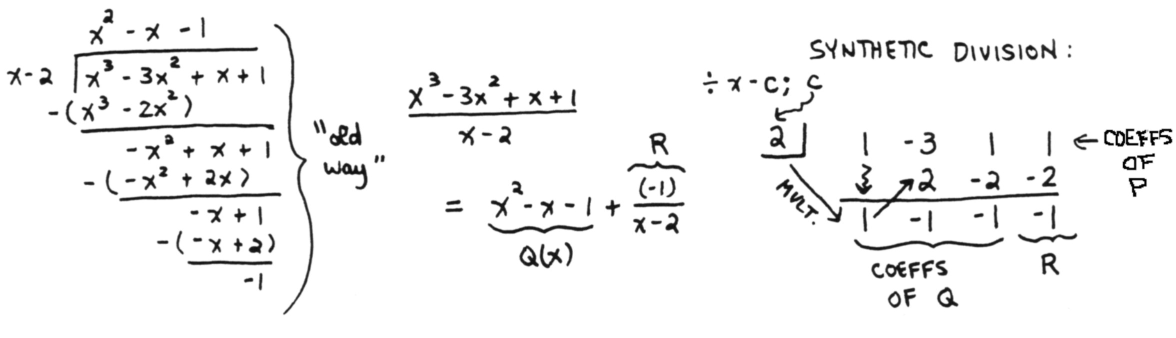
♣ 2. $5x^3 - 3x^2 - 12x - 4$

♣ 3. $4x^4 + 5x^3 - 2x^2 + 5x - 6$

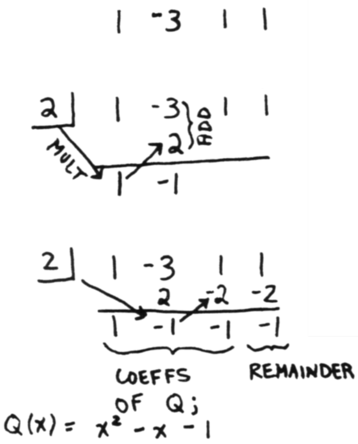
♣ 4. $3x^4 - x^3 + 12x^2 - 4x$ (Hint: First factor out an x. Then, apply the Rational Root Theorem to the remaining polynomial.)

SYNTHETIC DIVISION

Finding $\frac{P(x)}{x-c}$ via long division involves a lot of redundancy. Synthetic division suppresses all this redundancy and results in a useful tool for factoring. The process is illustrated below.



Here's how synthetic division is used to compute $\dfrac{P(x)}{x-c}$:



- Make sure P is written with decreasing powers of x.
- Write down the coefficients of P. Be sure to include 0 for any missing terms.
- To divide by $x-c$, put the number 'c' in a box to the left of the coefficients.

  For example, to divide by $x-2$, put a '2' in the box. To divide by $x+3 = x-(-3)$, put a '-3' in the box.
- Bring down the first coefficient.
- Multiply by c, and add to the next coefficient of P, as shown.
- Repeat as necessary. You have now computed

$$\frac{P(x)}{x-c} = Q(x) + \frac{R}{x-c} \quad\iff\quad P(x) = (x-c)Q(x) + R\ ;$$

  you need only read off the coefficients of Q and the remainder R.

  The last number computed is the remainder R. The preceding numbers are the coefficients of Q. Observe that the degree of Q is always one less than the degree of R.

REMAINDER THEOREM	If P is a polynomial and $P(x) = (x-r)Q(x) + R$, then $P(r) = R$.

The proof is trivial! $P(r) = (r-r)Q(r) + R = 0 \cdot Q(r) + R = R$. ∎

Usually, to evaluate a polynomial at a number r, we substitute r into the formula for P and crunch away. This theorem gives an alternate approach! It says that, to evaluate P at r, one can instead divide $P(x)$ by $x-r$; the remainder is precisely $P(r)$.

The Remainder Theorem, together with synthetic division, gives an efficient way to evaluate polynomials, as illustrated next.

EXAMPLE

using synthetic division and the Remainder Theorem to evaluate polynomials

Problem: Evaluate $P(x) = 14x^4 - x^3 - 17x^2 + x + 3$ at $x = 1$ and $x = -2$.

Solution: To find $P(1)$, use synthetic division to divide by $x-1$:

$$\begin{array}{r|rrrrr} 1 & 14 & -1 & -17 & 1 & 3 \\ & & 14 & 13 & -4 & -3 \\ \hline & \underbrace{14 \quad 13} & \underbrace{-4} & \underbrace{-3} & & \boxed{0} \end{array}$$

The remainder is 0, so $P(1) = 0$. Checking:

$$P(1) = 14 - 1 - 17 + 1 + 3 = 0$$

To find $P(-2)$, use synthetic division to divide by $x+2$:

$$\begin{array}{r|rrrrr} -2 & 14 & -1 & -17 & 1 & 3 \\ & & -28 & 58 & -82 & 162 \\ \hline & 14 & -29 & 41 & -81 & \boxed{165} \end{array}$$

The remainder is 165. Thus, $P(-2) = 165$. This was considerably easier than computing:

$$P(-2) = 14(-2)^4 - (-2)^3 - 17(-2)^2 + (-2) + 3$$

EXERCISE 5

Use synthetic division and the Remainder Theorem to evaluate the following polynomial at the specified values of x.

♣ $P(x) = x^4 - 2x^2 - 3x - 2$; $x = 1,\ -1,\ 2,\ -2$

Two additional tools for gaining information about the zeroes of polynomials are *Descartes' Rule of Signs* and the *Upper and Lower Bound Theorem.* Check your algebra book for more information.

QUICK QUIZ
sample questions

1. Factor $3x^2-2x-8$, by first finding numbers A and B that satisfy $AB = ???$ and $A + B = ???$
2. Factor $3x^2 - 2x - 8$, by using the Quadratic Formula.
3. What are the candidates for the rational roots of $P(x) = x^7 - 2x^5 + 2$?
4. Negate: A and B
5. Use the Remainder Theorem to find $P(1)$ if $P(x) = x^5 - 3x^2 + 2x - 1$.

KEYWORDS
for this section

Factorable over the integers, techniques for factoring $ax^2 + bx + c$, using the quadratic formula to factor ax^2+bx+c, the Rational Root Theorem, the symbols $\wedge$, $\vee$, $\neg$, negating $A \wedge B$ and $A \vee B$, DeMorgan's Laws, synthetic division, the Remainder Theorem.

END-OF-SECTION EXERCISES

♣ Use all available techniques to factor the following polynomials as completely as possible over $\mathbb{R}$.

1. $P(x) = 2x^3 - 3x^2 - 3x - 5$
2. $P(x) = 2x^6 - 4x^5 + 3x^4 - 2x^3 + x^2$
3. $P(x) = x^4 - 5x^2 + 6$
4. $P(x) = x^3 + x^2 - x$

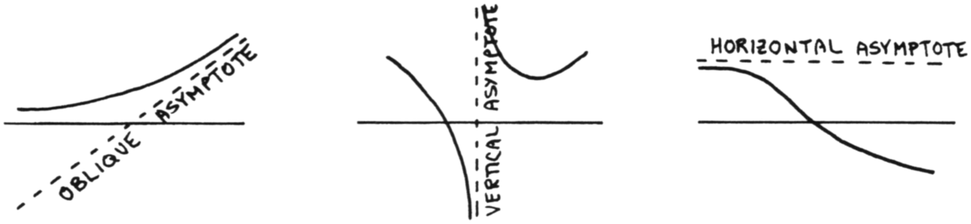
5.6 Asymptotes; Checking Behavior at Infinity

checking behavior at infinity

In this section, the notion of *checking behavior at infinity* is made precise, by discussing both *asymptotes* and *limits involving infinity.*

DEFINITION
asymptote

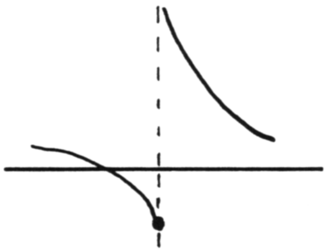
An *asymptote* is a curve (usually a line) that a graph gets arbitrarily close to as x approaches $\pm\infty$, or as x approaches some finite number.

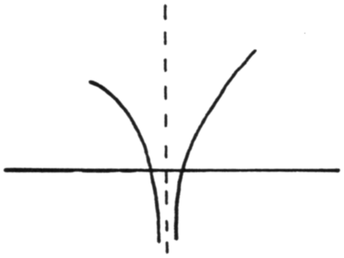


vertical asymptotes

An asymptote that is a vertical line is called a *vertical asymptote.*

That is, if the numbers $f(x)$ approach $\pm\infty$ as x approaches c from the right or left (or both), then the line $x = c$ is a vertical asymptote for f.





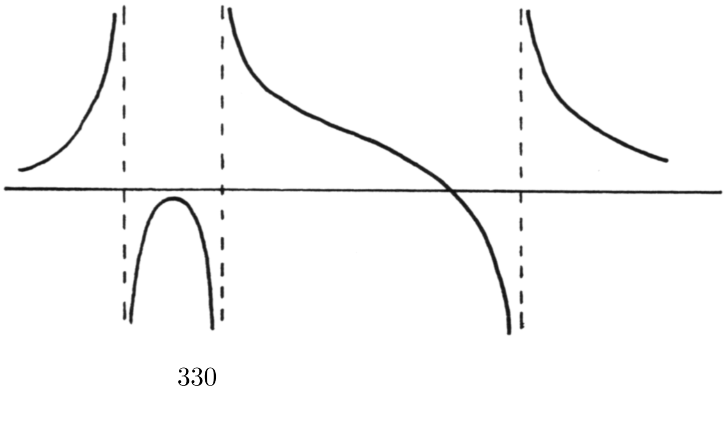
How do vertical asymptotes arise?

Vertical asymptotes arise most naturally when dealing with rational functions (ratios of polynomials):

$$f(x) = \frac{N(x)}{D(x)}$$

Any value of x for which the denominator is zero (and the numerator is non-zero) gives rise to a vertical asymptote.

A function can have an unlimited number of vertical asymptotes.



DEFINITION $\lim_{x\to c^+} f(x) = \infty$

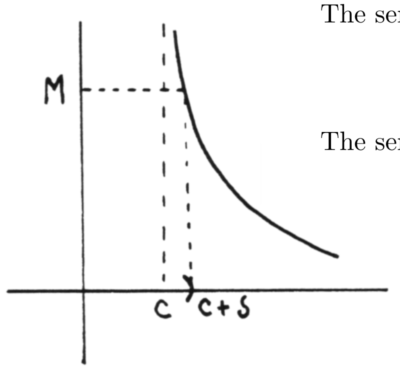
The limit statement

$$\lim_{x\to c^+} f(x) = \infty$$

means that $f(x)$ can be made as large and positive as desired, by requiring that x be sufficiently close to c (and greater than c).

Precisely:

$$\lim_{x\to c^+} f(x) = \infty \quad \iff \quad \begin{array}{l} \forall\ M > 0,\ \exists\ \delta > 0 \text{ such that if} \\ x \in (c, c+\delta), \text{ then } f(x) > M \end{array}$$



The sentence $\lim_{x\to c^+} f(x) = \infty$ can also be written:

$$\text{As } \ x \to c^+,\ \ f(x) \to \infty$$

The sentence '$\lim_{x\to c^+} f(x) = \infty$' is read as:

the limit of $f(x)$,
as x approaches c from the right-hand side,
is infinity

EXERCISE 1

♣ Give a precise definition of:

$$\lim_{x\to c^-} f(x) = \infty$$

Make a sketch that illustrates this limit statement. In English, what is this definition saying?

EXERCISE 2

♣ Give a precise definition of:

$$\lim_{x\to c^+} f(x) = -\infty$$

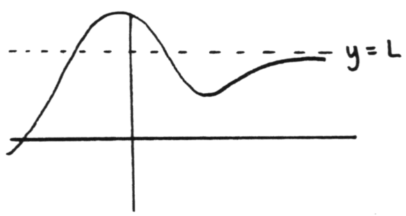
Make a sketch that illustrates this limit statement. In English, what is this definition saying?

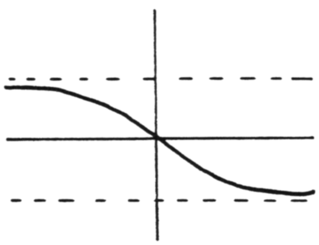
horizontal asymptotes

An asymptote that is a horizontal line is called a *horizontal asymptote.*

That is, if the line $y = L$ is a horizontal asymptote for f, then the function values $f(x)$ approach the finite number L as x approaches $+\infty$ or $-\infty$ (or both).

A *function* can have at most two horizontal asymptotes. (♣ Why?)





How do horizontal asymptotes arise?

Horizontal asymptotes also arise most naturally when dealing with rational functions,

$$f(x) = \frac{P(x)}{D(x)} \ ,$$

when the degrees of the numerator and denominator are the same.

For example, consider:

$$f(x) = \frac{3x^2 - 1}{x^2 - 2x + 2}$$

To investigate the behavior of f for large values of x, first multiply by 1 in an appropriate form (the highest power of x that appears, over itself):

$$\begin{aligned} f(x) &= \frac{3x^2 - 1}{x^2 - 2x + 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \frac{3 - \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{2}{x^2}} \end{aligned}$$

In this form, it is easy to see that when x is large (positive or negative), $f(x)$ is close to 3. Precisely, recall that the limit of a quotient is the quotient of the limits, *provided that each individual limit exists.* Since both 'numerator' and 'denominator' limits exist:

$$\lim_{x\to\pm\infty}(3 - \frac{1}{x^2}) = 3 \ \text{ and } \ \lim_{x\to\pm\infty}(1 - \frac{2}{x} + \frac{2}{x^2}) = 1 \ ,$$

it is correct to say that:

$$\lim_{x\to\pm\infty} \frac{3 - \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{2}{x^2}} = \frac{\lim_{x\to\pm\infty}(3 - \frac{1}{x^2})}{\lim_{x\to\pm\infty}(1 - \frac{2}{x} + \frac{2}{x^2})} = \frac{3}{1} = 3$$

Thus, the line $y = 3$ is a horizontal asymptote for the graph of f.

abbreviated form

Instead of writing out all the steps indicated above, the author usually summarizes things as follows:

Problem: Investigate the behavior of $f(x) = \frac{3x^2-1}{x^2-2x+2}$ for large values of x.

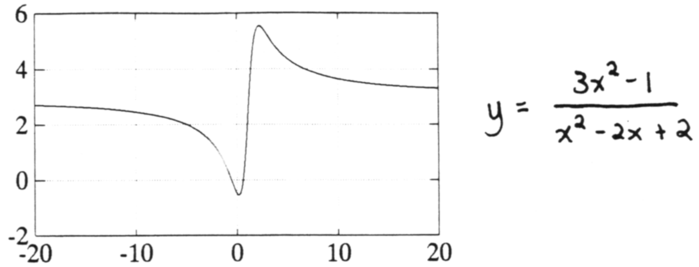
Solution: Think of approximating both the numerator and denominator polynomials by their highest order terms, and simply write:

$$\text{For large } x, \ \ f(x) \approx \frac{3x^2}{x^2} = 3$$

Thus, $y = 3$ is a horizontal asymptote for f.

This abbreviated analysis is fine, *provided that you understand why it is justified, and can fill in the details if pressed to do so.*

A MATLAB graph of f is shown below. Observe that there are no real numbers x for which the denominator $x^2 - 2x + 2$ equals zero, so f has no vertical asymptotes.



EXERCISE 3

♣ Investigate

$$f(x) = \frac{5x^3}{2x(x-1)(x+1)}$$

for horizontal asymptote behavior.

Write both a precise solution, and an abbreviated solution.

DEFINITION
$\lim_{x\to\infty} f(x) = L$

The limit statement

$$\lim_{x\to\infty} f(x) = L$$

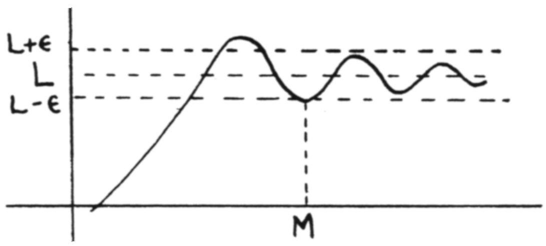
means that the numbers $f(x)$ can be made as close to L as desired, by requiring that x be sufficiently large and positive.

Precisely:

$$\lim_{x\to\infty} f(x) = L \quad \iff \quad \begin{array}{l} \forall\ \epsilon > 0,\ \exists\ M > 0 \text{ such that if } x > M, \\ \text{then } |f(x) - L| < \epsilon \end{array}$$

The sentence $\lim_{x\to\infty} f(x) = L$ can also be written:

$$\text{As } \ x \to \infty, \ \ f(x) \to L$$



EXERCISE 4

♣ Give a precise definition of:

$$\lim_{x\to-\infty} f(x) = L$$

Make a sketch that illustrates this limit statement. In English, what is this definition saying?

oblique asymptotes

An asymptote that is a line, but not a vertical or horizontal line, is called an *oblique asymptote*.

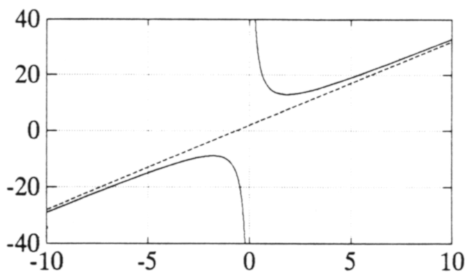
For example, consider the function:

$$f(x) = \frac{3x^2+2x+10}{x} = 3x + 2 + \frac{10}{x}$$

For large values of x (positive or negative), the number $\frac{10}{x}$ is close to zero. Thus, for large values of x,

$$f(x) \approx 3x + 2 \ ,$$

and the line $y = 3x+2$ is an oblique asymptote for f. The graph of f is shown below, along with the line $y = 3x + 2$.



Caution!

Do not 'abuse' the abbreviated solution technique! It is fine to say: for large values of x, $f(x) \approx \frac{3x^2}{x} = 3x$, and from this gain the information that when x is large, so is $f(x)$. However, it is *not* correct to infer that $y = 3x$ is an oblique asymptote! Observe that the 'multiply by 1 in an appropriate form' technique breaks down for this example:

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{3x^2+2x+10}{x} &= \lim_{x\to\pm\infty} \frac{3x^2+2x+10}{x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to\pm\infty} \frac{3+\frac{2}{x}+\frac{10}{x^2}}{\frac{1}{x}}\ ,\end{aligned}$$

but the limit of the quotient *cannot* be written as the quotient of the limits, since the denominator tends to 0.

When the degree of the numerator is greater than the degree of the denominator, the correct technique is to rewrite the rational function as a sum, by doing a long division. This is illustrated in the next example.

EXAMPLE

Problem: Find the oblique asymptote for $f(x) = \frac{2x^3+3x^2+x-2}{x^2-1}$.

Solution: As x gets large, so does $f(x)$. Do a long division:

$$
\begin{array}{r|l}
 & 2x \quad\ + 3 \\
\hline
x^2-1 & 2x^3 + 3x^2 + x - 2 \\
 & -(2x^3 \qquad\quad - 2x) \\
\hline
 & \qquad 3x^2 + 3x - 2 \\
 & \qquad -(3x^2 \qquad - 3) \\
\hline
 & \qquad\qquad 3x + 1
\end{array}
$$

Remember to stop when the degree of the remainder is strictly less than the degree of the divisor. Thus:

$$f(x) = 2x + 3 + \frac{3x+1}{x^2-1}$$

As $x \to \pm\infty$, $\frac{3x+1}{x^2-1} \to 0$. Thus, when x is large,

$$f(x) \approx 2x + 3 \ ,$$

and the line $y = 2x + 3$ is an oblique asymptote for f.

EXAMPLE

graphing a rational function

Problem: Completely graph $f(x) = \frac{x}{x-1}$.

Solution:

- $\mathcal{D}(f) = \{x \,|\, x \neq 1\}$

  Plot a few points:

x	$f(x)$
0	0
2	2
-2	2/3
-1	1/2

Check behavior near $x = 1$:

As $x \to 1^+$, $f(x) \to +\infty$. A convenient way to check this and write it down is:

$$f(1^+) \approx \frac{(+)}{(\text{small } +)} \to +\infty$$

The notation $f(1^+)$ connotes that f is being investigated on numbers that are a little bit greater than 1; say, 1.01 and 1.001 .

The notation $\frac{(+)}{(\text{small } +)}$ connotes a positive number divided by a small positive number, which yields a large positive number. For example: $\frac{1.01}{(1.01-1)} = \frac{1.01}{0.01} = 101$

Also:

$$f(1^-) \approx \frac{(+)}{(\text{small } -)} \to -\infty$$

That is, as $x \to 1^-$, $f(x) \to -\infty$.

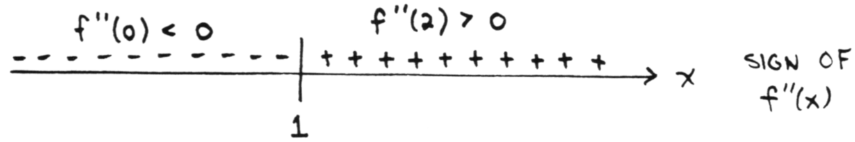
For example: $\frac{0.99}{0.99-1} = \frac{0.99}{-0.01} = -99$

Thus, $x = 1$ is a vertical asymptote.

- Compute the first derivative: $f'(x) = \frac{(x-1)(1)-x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

  $\mathcal{D}(f') = \mathcal{D}(f)$, and $f'(x)$ never equals 0. There are no critical points.
- Compute the second derivative: $f''(x) = \frac{2}{(x-1)^3}$

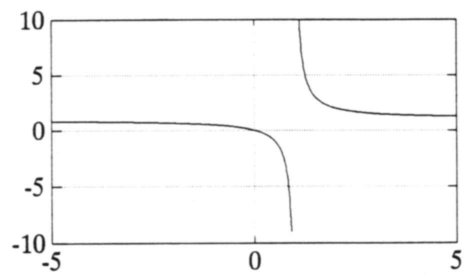
  $\mathcal{D}(f'') = \mathcal{D}(f)$, and $f''(x)$ never equals 0. There are no candidates for inflection points.
- Sign of the second derivative:



- Filling in some details:

  As $x \to \pm\infty$, $f(x) \approx \frac{x}{x} = 1$, so $y = 1$ is a horizontal asymptote.

A MATLAB graph of $f(x) = \frac{x}{x-1}$ is shown below.



EXAMPLE

graphing a rational function

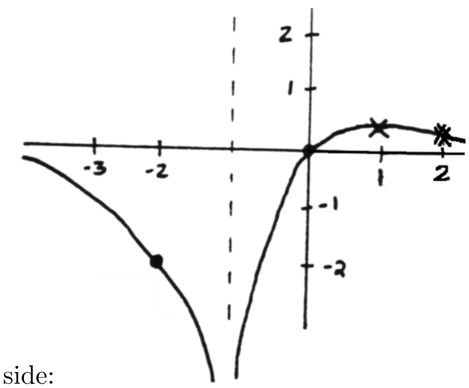
Problem: Completely graph $f(x) = \frac{x}{(x+1)^2}$.

Solution:

- $\mathcal{D}(f) = \{x \,|\, x \neq -1\}$

  Plot a few points:

x	f(x)
0	0
1	1/4
-2	-2
2	2/9



Check behavior near $x = -1$:

First, coming in to -1 from the right-hand side:

$$f(-1^+) \approx \frac{(-)}{(\text{small } +)} \to -\infty$$

Thus, as $x \to -1^+$, $f(x) \to -\infty$.

Next, coming in to -1 from the left-hand side:

$$f(-1^-) \approx \frac{(-)}{(\text{small } +)} \to -\infty$$

So, as $x \to -1^-$, $f(x) \to -\infty$.

The line $x = -1$ is a vertical asymptote.

- Compute the first derivative:

$$\begin{aligned} f'(x) &= \frac{(x+1)^2(1) - x \cdot 2(x+1)}{(x+1)^4} \\ &= \frac{(x+1)(x+1-2x)}{(x+1)^4} \\ &= \frac{1-x}{(x+1)^3} \end{aligned}$$

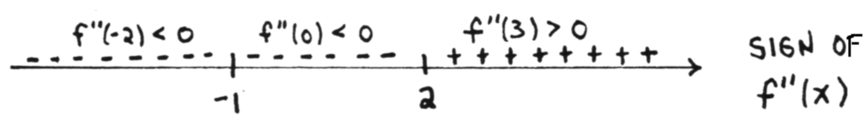
  $\mathcal{D}(f') = \mathcal{D}(f)$; $f'(x) = 0$ when $x = 1$, so $(1, f(1)) = (1, \frac{1}{4})$ is a critical point. Plot this point with a $\times$.

- Compute the second derivative:

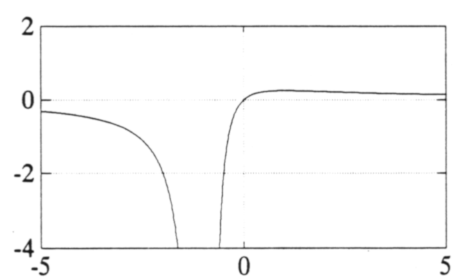
$$\begin{aligned} f''(x) &= \frac{(x+1)^3(-1) - (1-x)3(1+x)^2}{(x+1)^6} \\ &= \frac{(x+1)^2[-(x+1) - 3(1-x)]}{(1+x)^6} \\ &= \frac{2x-4}{(1+x)^4} \end{aligned}$$

  $\mathcal{D}(f'') = \mathcal{D}(f)$; $f''(x) = 0$ when $x = 2$. Thus, $(2, f(2)) = (2, \frac{2}{9})$ is a possible inflection point. Plot this point with a $⋇$.

- Sign of f'' :



A MATLAB graph of $f(x) = \frac{x}{(x+1)^2}$ appears below.



- Fill in some details:

  As $x \to \pm\infty$, $f(x) \approx \frac{x}{x^2} = \frac{1}{x} \to 0$, so the line $y = 0$ is a horizontal asymptote.

- Read off important information:
  $(1, \frac{1}{4})$ is a local maximum
  $(2, \frac{2}{9})$ is an inflection point
  The line $y = 0$ is a horizontal asymptote.
  The line $x = -1$ is a vertical asymptote.
  The graph is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.
  The graph is concave down on $(-\infty, -1) \cup (-1, 2)$ and concave up on $(2, \infty)$.

EXERCISE 5

Completely graph each of the following functions. Be sure to check for horizontal, vertical, and oblique asymptotes.

♣ 1. $f(x) = \dfrac{2x^3 - x^2 + 1}{x^2}$

♣ 2. $g(x) = \dfrac{x^2 - 3}{x^2 - 1}$

♣ 3. $y = \dfrac{1 - 4x^2}{x^2 + 1}$

EXERCISE 6

♣ Completely graph: $f(x) = \dfrac{x^3 + 2x^2 - x - 2}{x^2 - 1}$ (Be careful!)

QUICK QUIZ
sample questions

1. What is an *asymptote*?
2. Write down a *precise* definition for the limit statement: $\lim_{x \to c^-} f(x) = -\infty$
3. Find all asymptotes (horizontal, vertical, oblique) for: $f(x) = \dfrac{3x - 1}{x + 2}$
4. Under what condition(s) is the limit of a quotient equal to the quotient of the limit?

KEYWORDS
for this section

Asymptotes, vertical, horizontal and oblique asymptotes, precise definitions of limits involving infinity.

END-OF-SECTION EXERCISES

♣ Re-do each of the graphing examples in this section, *without looking at the text.* Be sure to write complete mathematical sentences. If you get stuck, then study the text example, close the book, and try it yourself again.

NAME (1 pt)
SAMPLE TEST, worth 100 points, Chapter 5

Show all work that leads to your answers. Good luck!

15 pts

Using the information that the first and second derivatives give, completely graph the function $P(x) = x^3 - 3x + 2$ in the space provided below. Clearly label any critical points, inflection points, x and y-axis intercept(s).

12 pts

TRUE or FALSE. Circle the correct response. (3 points each)

T	F	If $(c, f(c))$ is a critical point for f, then it is a local max or min.
T	F	If $f'(c) = 0$ and $f''(c) > 0$, then the point $(c, f(c))$ is a local min.
T	F	The second derivative of a function f tells us about the concavity of f.
T	F	Suppose that $(c, f(c))$ is a critical point for f. If $f'(x) > 0$ to the left of c, and $f'(x) < 0$ to the right of c, then $(c, f(c))$ is a local maximum for f.

18 pts

(4 pts) True or False: If a function f is continuous and nonzero on an interval I, then it must be either positive or negative on this interval. ____________________

(6 pts) Find where the function $f(x) = \frac{x(x-2)}{x+3}$ is positive and negative. (Hint: Draw a number line labeled 'Sign of $f(x)$'.)

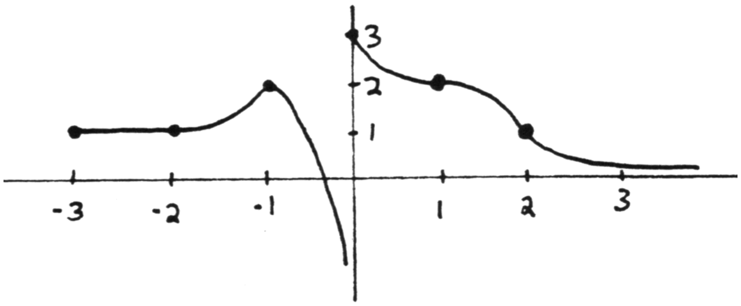
(8 pts) Find the open intervals on which f increases and decreases.

15 pts

Give a precise definition of $\lim_{x\to\infty} f(x) = L$. Make a sketch that illustrates the definition.

20 pts

The graph of a function f is shown below. Read the following information off the graph. Approximate where necessary. If a particular item does not exist, so state.



(1 ea) $f(-2.5)$ $f'(-2.5)$ $f'(1)$ $\lim_{x\to -1} f(x)$

(2 pts) open interval(s) where f increases: __________

(2 pts) open interval(s) where f' is negative: __________

(2 pts) open interval(s) where f is concave down: __________

(2 pts) open interval(s) where f'' is positive: __________

(2 pts) all local maximum point(s) for f: __________

(2 pts) all inflection point(s) for f: __________

(2 pts) all global maximum point(s) for f: __________

(2 pts) List all the critical points for f: __________

20 pts

Completely graph $f(x) = \dfrac{x+1}{x-1}$ in the space provided below. Clearly label all asymptotes.

This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

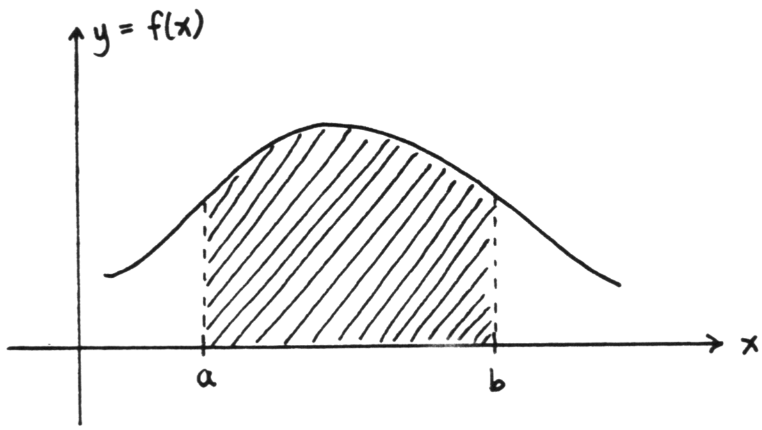
CHAPTER 6

ANTIDIFFERENTIATION

In the previous two chapters, the focus has been on differentiating a given function; i.e., given f, find f'.

Our attention now shifts to a new idea: given a function f, we seek another function F *whose derivative is* f. That is, given f, we seek F such that $F' = f$. So, in a sense, we are *undoing* differentiation, and hence the new function F is called an *antiderivative* of f.

The current chapter focuses on techniques for finding the antiderivatives of a function. It will be seen that if one has an antiderivative for a function f, then it can be used to find the *area bounded between the graph of* f *and the* x*-axis.*



6.1 Antiderivatives

undoing differentiation

In the previous sections, the focus has been on differentiating a given function: given f, find f'.

The question for the current chapter is this: given a function f, find another function F *whose derivative is* f. That is, given f, we seek F such that $F' = f$. So, in a sense, we are *undoing* differentiation.

a preliminary example

Let $f(x) = 3$. We want another function whose derivative is f. That is, we seek a function F satisfying $F'(x) = f(x) = 3$.

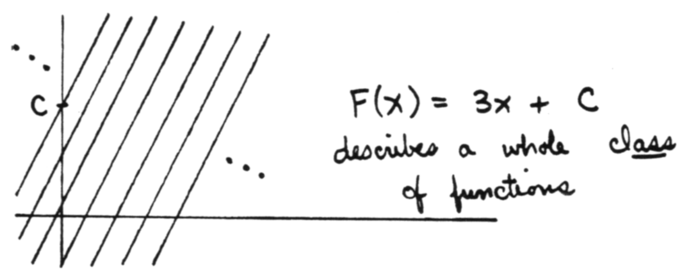
Clearly, $F(x) = 3x$ works, since in this case $F'(x) = 3$.

Also, $F(x) = 3x + 1$ works, since again $F'(x) = 3$.

Indeed, for any real number C, $F(x) = 3x + C$ is a function whose derivative is 3.

the equation $F(x) = 3x + C$ describes an ENTIRE CLASS of functions

Observe that the equation $F(x) = 3x + C$ describes an *entire class* of functions which have the *same shape*, but are translated up and down in the xy-plane. There is one function for each choice of the number C.



What does ANY function with derivative 3 look like?

Are there any functions other than those of the form $F(x) = 3x + C$ whose derivative is 3? We will see momentarily that the answer is 'No'.

Here's the way mathematicians address such a question: they *suppose* there is a function with derivative 3, and then proceed to show that it must actually be of the form $3x + C$.

an application of the Mean Value Theorem

Whenever derivative information is to be used to glean information about the function itself, you should not be surprised to see the Mean Value Theorem. Make sure you see how the Mean Value Theorem plays a crucial role in the next argument.

Suppose G is ANY function with derivative 3 …

Let $F(x) = 3x+C$, where C is any real number. Suppose that G is any function with derivative 3. Observe that we are *not* assuming that G must be of the form $G(x) = 3x + C$.

Then, we have both

$$F'(x) = 3 \text{ and } G'(x) = 3 \ ,$$

so that:

$$G'(x) - F'(x) = 0$$

Since the sum of the derivatives is the derivative of the sum, we can alternately write:

$$(G - F)'(x) = 0$$

Now recall a result from the end of Chapter 4. There, we learned that if the derivative of a function is zero, then the function must be constant: this was an application of the Mean Value Theorem. Thus, we must have

$$(G - F)(x) = K$$

for some constant K. That is,

$$G(x) - F(x) = K \ ,$$

or:

$$G(x) = F(x) + K = (3x + C) + K$$

... then, G must be of the form $3x + C$

Thus, we see that G *must actually be* of the form $3x +$ (some constant). It has therefore been established that *every* function with derivative 3 must look like $3x + C$ for some constant C.

the derivative of a function completely determines its shape

The preceding argument is now generalized slightly. Suppose that functions f and g are both differentiable (say on an open interval (a, b)), and suppose that:

$$f'(x) = g'(x) \ \ \forall \ x \in (a, b)$$

Then,

$$(f - g)'(x) = f'(x) - g'(x) = 0 \ ,$$

so that $(f - g)(x) = C$ for some constant C. That is,

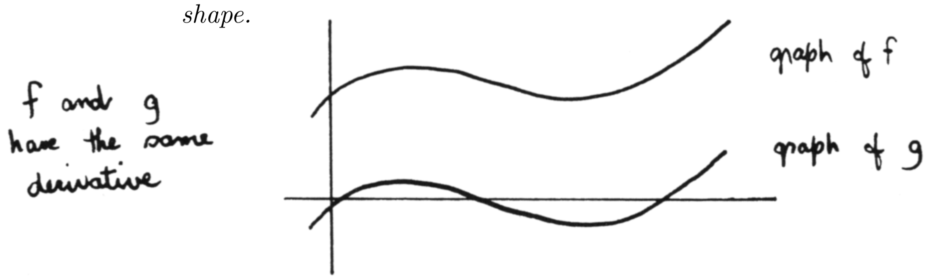
$$f(x) - g(x) = C \ ,$$

and hence:

$$f(x) = g(x) + C$$

Thus, *if two functions f and g have the same derivative, then they differ by at most a constant.* That is, *functions that have the same derivative must have the same shape.* The functions f and g might not be the same function, but the graph of one can be obtained from the graph of the other by a vertical translation.

In other words, *specifying the derivative of a function completely determines its shape.*



EXERCISE 1	♣ Let $f(x) = -1$. Find all functions F for which $F' = f$. How many are there? Sketch a few such functions F.

EXERCISE 2

Consider the function f with graph shown below. Note that $\mathcal{D}(f) = \mathbb{R} - \{c\}$.

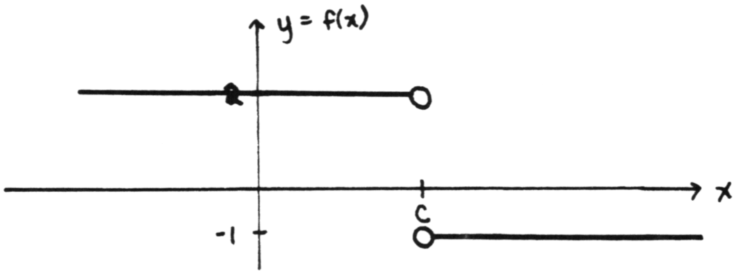
Sketch the graph of a function F satisfying each of the following properties:

♣ 1. F is continuous on $\mathbb{R}$, and $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$

♣ 2. Sketch another, different, function F satisfying the requirements above.

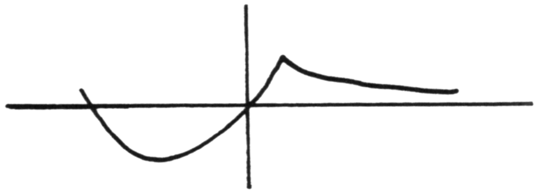
♣ 3. $\mathcal{D}(F) = \mathcal{D}(f)$, $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$, and F has a removable discontinuity at c

♣ 4. $\mathcal{D}(F) = \mathcal{D}(f)$, $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$, and F has a nonremovable discontinuity at c



EXERCISE 3

♣ Consider the function f shown below. On the same graph, sketch two different functions that have the same derivative as f.



DEFINITION
antiderivative;
arbitrary constant

A function F is called an *antiderivative* of a function f if

$$F'(x) = f(x)$$

for every x in the domain of f.

Thus, an antiderivative of f is a function *whose derivative is* f.

If you are able to find a *single* antiderivative of f, call it F, then there are an *infinite* number of antiderivatives, each of the form:

$$F(x) + C$$

Here, C represents any real number, and is called an *arbitrary constant.*

NOTATION
for antiderivatives:
indefinite integrals;
antidifferentiation;
integral sign;
integrand

The symbol

$$\int f(x)\,dx$$

is called the *indefinite integral of* f, and represents *all the antiderivatives of* f.

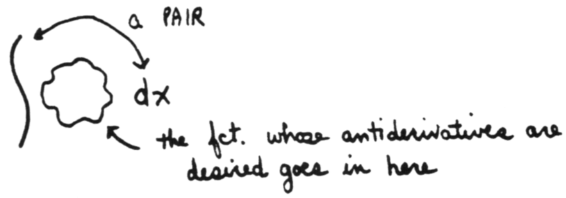
The process of finding $\int f(x)\,dx$ is called *antidifferentiation* ("undoing" differentiation!)

The symbol $\int$ is called the *integral sign.*

The function f that is being integrated is called the *integrand.*

$\int$ and dx are an instruction pair

It may be helpful to view the integral sign $\int$ and the symbol dx as an *inseparable instruction pair*. The function of x (call it f) whose antiderivatives are desired is placed between the symbols $\int$ and dx. The instruction is then to find all functions, whose derivatives *with respect to* x, equal f.



more notation: integrals; integration

Later on, we will study the *definite integral of f on* $[a, b]$, to be denoted by the symbol $\int_a^b f(x)\,dx$.

Both $\int f(x)\,dx$ (the indefinite integral) and $\int_a^b f(x)\,dx$ (the definite integral) are called *integrals*.

The process of finding either $\int f(x)\,dx$ or $\int_a^b f(x)\,dx$ is called *integration*.

EXAMPLE

the constant C is often called the 'constant of integration'

Problem: Evaluate $\int 3\,dx$.

Solution: We are asked to find all the antiderivatives of the function $f(x) = 3$. That is, we are asked to find all functions of x, whose derivative with respect to x is 3 . The solution is written concisely and correctly as:

$$\int 3\,dx = 3x + C$$

It is conventional that the letters C or K be used in this context to represent an arbitrary constant (i.e., any real number). This arbitrary constant is also referred to as the *constant of integration*.

It is important that you include the constant of integration. If you mistakenly write

$$\int 3\,dx = 3x \ ,$$

then you are claiming that the ONLY function whose derivative is 3 is the function $3x$. Not so! This is only *one* of an *infinite class* of functions that has derivative 3!

checking your answers by differentiating

If $\int f(x)\,dx = F(x) + C$, then $F(x) + C$ is an antiderivative of $f(x)$, so that:

$$\frac{d}{dx}\big(F(x) + C\big) = F'(x) + 0 = f(x)$$

Thus, answers can always be checked by differentiating.

Since the derivative of a constant is always zero, it is not necessary to include 'C' in the checking process. More simply, check that:

$$\frac{d}{dx}F(x) = F'(x) = f(x)$$

For example, to check that

$$\int 3\,dx = 3x + C \ ,$$

one verifies that:

CHECK!

$$\frac{d}{dx}(3x) = 3$$

practice with notation

Observe what happens when the element dx is changed:

$$\int 3\,dx = 3x + C$$

$$\int 3\,dy = 3y + C$$

$$\int 3\,d\omega = 3\omega + C$$

In the first case, $\frac{d}{dx}(3x) = 3$.
In the second case, $\frac{d}{dy}(3y) = 3$.
In the third case, $\frac{d}{d\omega}(3\omega) = 3$.

EXAMPLE

Problem: Evaluate $\int 2x\,dx$.

Solution: It is necessary to find *any* antiderivative of $2x$; that is, a function with derivative $2x$. Then, all other antiderivatives will differ by at most a constant. Observe that $F(x) = x^2$ works, since $F'(x) = 2x$.

Once we have a *single* antiderivative, we actually know them all. That is, any other function with the same derivative must have precisely the same shape. So:

$$\int 2x\,dx = x^2 + C$$

CAUTION!

Be careful not to write something like this:

$$\int 2t\,dx = t^2 + C$$

Taken literally, this says that

$$\frac{d}{dx}t^2 = 2t \ ,$$

which doesn't make any sense: if we're differentiating with respect to x, and are not told otherwise, then we would have to assume that t is constant with respect to x. Thus, $\frac{d}{dx}t^2 = 0$. The MORAL: make sure the letter x in the element 'dx' agrees with the variable of the function that you're integrating! (Unless, of course, you're doing something unusual.)

EXERCISE 4

Evaluate the following indefinite integrals. Be sure to write complete sentences. Don't forget to include the constant of integration.

♣ 1. $\int 3x^2\,dx$

♣ 2. $\int 2y\,dy$

♣ 3. $\int e^t\,dt$

♣ 4. $\int 2e^{2x}\,dx$

♣ 5. $\int \frac{1}{x}\,dx$ Here, just find an antiderivative of $\frac{1}{x}$ on the interval $(0,\infty)$.

EXERCISE 5

For this exercise, assume that $x>0$, so that $\ln x$ is defined.

Recall that $F(x)=\ln x$ has derivative $F'(x)=\frac{1}{x}$. Thus, $\ln x$ is an antiderivative of $\frac{1}{x}$, and hence:

$$\int \frac{1}{x}\,dx = \ln x + C \tag{*}$$

Also, $G(x)=\ln 2x$ has derivative $G'(x)=\frac{1}{2x}(2)=\frac{1}{x}$. Thus, $\ln 2x$ is an antiderivative of $\frac{1}{x}$, and hence:

$$\int \frac{1}{x}\,dx = \ln 2x + K \tag{**}$$

Equation (*) tells us that *every* antiderivative of $\frac{1}{x}$ must be of the form $\ln x + C$ for some constant C.

Equation (**) tells us that *every* antiderivative of $\frac{1}{x}$ must be of the form $\ln 2x + K$ for some constant K.

♣ Reconcile these two statements. That is, how can they both be true?

linearity of differentiation

If f and g are both differentiable functions of x, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

and:

$$\frac{d}{dx}k\cdot f(x) = k\cdot f'(x)$$

That is, the derivative of a sum is the sum of the derivatives, and constants can be 'slid out' of the differentiation process.

These two properties together are referred to as the *linearity of differentiation*. Alternately, one often says '*differentiation is a linear process*'.

We see next that the process of *antidifferentiation* obeys the same two properties: the integral of a sum is the sum of the integrals, and constants can be 'slid out' of the integral.

EXERCISE 6

♣ 1. What is meant by the phrase '*linearity of differentiation*'?

♣ 2. Identify all the places where the linearity of differentiation is used in the following sentence:

$$\begin{aligned}\frac{d}{dx}(x^2+3\sqrt{x}) &= \frac{d}{dx}(x^2+3x^{1/2})\\ &= \frac{d}{dx}x^2+\frac{d}{dx}3x^{1/2}\\ &= 2x+3\frac{d}{dx}x^{1/2}\\ &= 2x+3(\frac{1}{2}x^{-1/2})\\ &= 2x+\frac{3}{2\sqrt{x}}\end{aligned}$$

linearity of integration

Integration is a linear process, as is differentiation.
That is, the integral of a sum is the sum of the integrals:

$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Also, constants can be slid out of the integration process:

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Together, these two properties are referred to as the *linearity of the integral* or the *linearity of integration.*

partial proof of the linearity of integration

The fact that antidifferentiation is a linear process is a direct consequence of the linearity of differentiation, as the following discussion illustrates.

Problem: Show that:

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad (\dagger)$$

Solution: To begin, let F be an antiderivative of f (so that $F' = f$) and let G be an antiderivative of g (so that $G' = g$).
Then,

$$\int f(x)\,dx = F(x) + C_1$$

and

$$\int g(x)\,dx = G(x) + C_2\ ,$$

where C_1 and C_2 are arbitrary constants.

Then, the right-hand side of (†) becomes:

$$\begin{aligned}\int f(x)\,dx + \int g(x)\,dx &= (F(x)+C_1) + (G(x)+C_2)\\ &= F(x)+G(x)+(C_1+C_2)\\ &= F(x)+G(x)+C \qquad (1)\end{aligned}$$

Here, the two arbitrary constants have been lumped together into a single arbitrary constant.

Next, investigate the left-hand side of (†). What is $\int \big(f(x)+g(x)\big)\,dx$? We need a function with derivative $f(x)+g(x)$. But $F(x)+G(x)$ is such a function:

$$\frac{d}{dx}(F(x)+G(x)) = F'(x)+G'(x) = f(x)+g(x)$$

Thus, the left-hand side of (†) becomes:

$$\int \big(f(x)+g(x)\big)\,dx = F(x)+G(x)+C \qquad (2)$$

Compare (1) and (2)—they are identical. Thus, it has been shown that

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx\ ,$$

establishing that the integral of a sum is the sum of the integrals.

EXERCISE 7

♣ 1. Similar to the preceding argument, prove that:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

That is, constants can be 'slid out' of the integral.

♣ 2. Is $\int x^2\,dx = x\int x\,dx$? That is, can an '$x$' be slid out of the integral? Comment.

EXAMPLE

using the linearity of integration

The linearity of the integral can be used to solve a wide variety of integration problems. For example:

$$\begin{aligned}\int (2x-3)\,dx &= \int 2x\,dx + \int (-3)\,dx\\ &= \int 2x\,dx - \int 3\,dx\\ &= (x^2+C_1) - (3x+C_2)\\ &= x^2-3x+(C_1-C_2)\\ &= x^2-3x+C\end{aligned}$$

All arbitrary constants are always lumped into a single arbitrary constant. The previous problem is never written out in all the detail shown above. It is more simply written as:

$$\int (2x-3)\,dx = x^2 - 3x + C$$

That is: find an antiderivative of $2x$, subtract an antiderivative of 3, and add on an arbitrary constant.

EXERCISE 8 ♣ Supply a reason for *each line* in this mathematical sentence:

$$\begin{aligned}\int (2x-3)\,dx &= \int 2x\,dx + \int (-3)\,dx \\ &= \int 2x\,dx - \int 3\,dx \\ &= (x^2 + C_1) - (3x + C_2) \\ &= x^2 - 3x + (C_1 - C_2) \\ &= x^2 - 3x + C\end{aligned}$$

EXAMPLE Often, it is necessary to rewrite the integrand before integrating:

$$\begin{aligned}\int \frac{e^x - 1}{e^x}\,dx &= \int \frac{e^x}{e^x} - \frac{1}{e^x}\,\,dx \\ &= \int 1 - e^{-x}\,\,dx \\ &= x + e^{-x} + C\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dx}(x + e^{-x}) &= 1 - e^{-x} \\ &= \frac{e^x}{e^x}(1 - e^{-x}) \\ &= \frac{e^x - 1}{e^x}\end{aligned}$$

EXAMPLE As a second example:

$$\begin{aligned}\int \frac{2}{3x-7}\,dx &= 2\int \frac{1}{3(x - \frac{7}{3})}\,dx \\ &= \frac{2}{3}\int \frac{1}{x - \frac{7}{3}}\,dx \\ &= \frac{2}{3}\ln(x - \frac{7}{3}) + C\end{aligned}$$

♣ Do you see where two arbitrary constants were combined in this argument?

Check:

$$\begin{aligned}\frac{d}{dx}\frac{2}{3}(\ln(x-\frac{7}{3}) &= \frac{2}{3}(\frac{1}{x-\frac{7}{3}})\\ &= \frac{2}{3x-7}\end{aligned}$$

♣ Do you see where the linearity of differentiation was used in this check?

In the next few sections, additional tools are developed to help in the integration process.

EXERCISE 9

Evaluate the following integrals. Be sure to write complete mathematical sentences. Don't forget to include the constant of integration.

♣ 1. $\int(\frac{1}{x}+e^x-1)\,dx$

♣ 2. $\int\frac{3-t}{t}\,dt$

♣ 3. $\int\frac{1}{x-2}\,dx$

♣ 4. $\int\frac{1}{3x-5}\,dx$

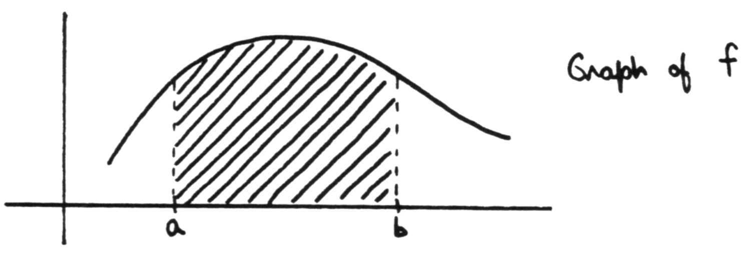
♣ 5. $\int(x+1)^2\,dx$

a preview of coming attractions

We will soon learn a very surprising fact: if f is a continuous nonnegative function, and if we can find an antiderivative F of f, then we can use this antiderivative to find the area trapped between the graph of f and the x-axis over an interval $[a,b]$!

All we have to do is this: evaluate the antiderivative F at b to get $F(b)$. Evaluate F at a, to get $F(a)$. Then:

$$\text{desired area} = F(b) - F(a)$$



This result is properly discussed in the next chapter. For now, just keep in mind that the antiderivatives of a function have a *very practical use*! To close this section, we look at a simple example of this surprising connection between antiderivatives and area.

EXAMPLE

finding area using an antiderivative

Problem: Find the area trapped beneath the graph of $f(x) = 2x$ on the interval $[a, b]$, where $0 < a < b$.

Solution: The desired area is a trapezoid, and calculus is certainly *not* needed, in this case, to find it:

$$
\begin{aligned}
\text{desired area} &= \frac{1}{2}(\text{altitude})(\text{sum of bases}) \\
&= \frac{1}{2}(b-a)(2a+2b) \\
&= \frac{1}{2}(b-a)2(a+b) \\
&= (b-a)(b+a)
\end{aligned}
$$

Now, let's use calculus to get the same answer. This time, we first find an antiderivative of f:

$$F(x) = x^2 \quad \text{has derivative} \quad F'(x) = 2x = f(x)$$

Then:

$$F(b) - F(a) = b^2 - a^2 = (b-a)(b+a)$$

Note that precisely the same result is obtained!

EXERCISE 10

♣ 1. Graph $f(x) = x$.

♣ 2. On your graph, show the area trapped beneath the graph of f and the x-axis on an interval $[a, b]$, where $0 < a < b$.

♣ 3. Compute this area, using the formula for the area of a trapezoid.

♣ 4. Next, observe that $F(x) = \frac{x^2}{2}$ is an antiderivative of f, since $F'(x) = \frac{1}{2}(2x) = x = f(x)$. Use calculus to find the area being investigated. Compare your answers.

QUICK QUIZ

sample questions

1. Suppose a function $f(x)$ has derivative 2 everywhere. What does the graph of f look like?
2. Fill in the Blank: specifying the derivative of a function completely determines its ________________.
3. Find $\int 2\,dt$.
4. Name one use for the antiderivatives of a function.
5. What is meant by the phrase, 'the linearity of differentiation'?

KEYWORDS

for this section

'Undoing' differentiation, the derivative of a function completely determines its shape, antiderivative, arbitrary constant, indefinite integrals, antidifferentiation, integral sign, integrand, integrals, integration, constant of integration, linearity of differentiation, linearity of integration, connection between antiderivatives and area.

END-OF-SECTION EXERCISES

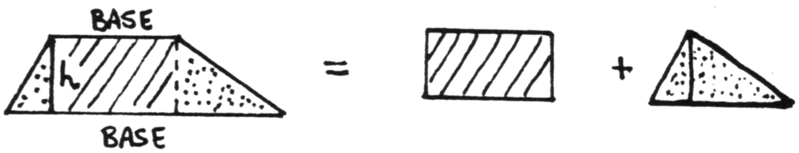
♣ Classify each entry below as an expression or a sentence.

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

(Feel free to assume that all functions appearing below are infinitely differentiable.)

1. $F'(x)$
2. $F'(x) = 2$
3. $\int f(x)\,dx$
4. $\int f(t)\,dt$
5. $\int f(x)\,dx = F(x) + C$
6. $\int 2\,dx = 2x + C$
7. $\int 2\,dt = 2t + C$
8. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
9. $\int kf(x)\,dx = k\int f(x)\,dx$
10. $\int f'(x)\,dx = f(x) + C$
11. (Deriving the formula for the area of a trapezoid) A trapezoid is any quadilateral with two parallel sides. The distance between the two parallel sides is called the *altitude* of the trapezoid. The two parallel sides are called the *bases* of the trapezoid.

    The area of any trapezoid can be found as the sum of a rectangle and a triangle, as illustrated below:

    ♣ Sum the areas of the rectangle and triangle, and conclude that:

$$\text{area of a trapezoid} = \frac{1}{2}(\text{altitude})(\text{sum of bases})$$

6.2 Some Basic Antidifferentiation Formulas

every differentiation formula has a 'counterpart' antidifferentiation formula

Every differentiation formula has a 'counterpart' antidifferentiation formula. For example:

$$\frac{d}{dx}(x) = 1 \quad \text{has 'counterpart'} \quad \int (1)\,dx = x + C$$

Why? The statement $\frac{d}{dx}(x) = 1$ tells us that x is an antiderivative of 1. That is, x is a function which, when differentiated, yields 1. Then, all other antiderivatives must have precisely the same shape; they can differ by at most a constant.

Similarly:

$$\frac{d}{dx}(x^2) = 2x \quad \text{has 'counterpart'} \quad \int 2x\,dx = x^2 + C$$

In this latter case, it would be more useful to have a formula for $\int x\,dx$, instead of $\int 2x\,dx$. Using the linearity of the integral, this is easy to get:

$$\begin{aligned}
\int 2x\,dx = x^2 + C \quad &\iff \quad 2\int x\,dx = x^2 + C && \text{(linearity)} \\
&\iff \quad \int x\,dx = \frac{(x^2 + C)}{2} && \text{(divide by 2)} \\
&\iff \quad \int x\,dx = \frac{x^2}{2} + K && \text{(rewrite constant)}
\end{aligned}$$

Since C is an arbitrary constant, so is $\frac{C}{2}$. There is no sense in giving an arbitrary constant a complicated name like $\frac{C}{2}$; so change the name to, say, K.

Thus, we have learned that:

$$\int x\,dx = \frac{x^2}{2} + C$$

using the formula $\int x\,dx = \frac{x^2}{2} + C$

With the formula

$$\int x\,dx = \frac{x^2}{2} + C$$

in hand, and linearity of the integral, a number of integration problems can be easily solved.

EXAMPLE

Problem: Evaluate $\int 3x\,dx$.

Solution: First, the solution is written in strictly correct, painstaking detail. Then, it is shown how the solution is commonly abbreviated.

$$\begin{aligned}
\int 3x\,dx &= 3\int x\,dx && \text{(linearity)}\\
&= 3(\frac{x^2}{2}+C) && \text{(use formula for integrating } x\text{)}\\
&= \frac{3x^2}{2}+3C && \text{(multiply)}\\
&= \frac{3x^2}{2}+K && \text{(rewrite constant)}
\end{aligned}$$

In practice, one recognizes that the final result will always have an added arbitrary constant. So: simply apply the formulas *without* the arbitrary constant, and in the final step, remember to include it. This yields the common solution appearance:

$$\int 3x\,dx = 3\int x\,dx = 3(\frac{x^2}{2}) + C = \frac{3x^2}{2} + C$$

Similarly, one writes

$$\int (\pi t - 4)\,dt = \pi\frac{t^2}{2} - 4t + C$$

and:

$$\int \frac{2-t}{7}\,dt = \frac{1}{7}(2t - \frac{t^2}{2}) + C = \frac{4t - t^2}{14} + C$$

EXERCISE 1

♣ 1. What is the antidifferentiation ‘counterpart’ to the differentiation formula

$$\frac{d}{dx}(x^3) = 3x^2\ ?$$

♣ 2. Use your ‘counterpart’ to obtain a formula for $\int x^2\,dx$.

♣ 3. Use your formula for integrating x^2 to evaluate $\int 5x^2\,dx$.

The next integration formula derives from the Simple Power Rule for Differentiation:

$$\frac{d}{dx}x^n = nx^{n-1}$$

It is thus appropriately named the ‘*Simple Power Rule for Integration*’.

Simple Power Rule for Integration

Let n be any number except -1. Then:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

This formula is referred to as the *Simple Power Rule for Integration.*

To verify this result, one need only check that:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}(n+1)x^{(n+1)-1} = x^n$$

Together with algebraic manipulation and linearity of the integral, this formula allows us to solve a wide variety of antidifferentiation problems, as the following examples illustrate.

EXAMPLE

Problem: Evaluate $\int x^{-3}\,dx$.

Solution:

$$\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{2x^2}\right) = \frac{d}{dx}(-\frac{1}{2}x^{-2}) = -\frac{1}{2}(-2)x^{-3} = x^{-3}$

EXAMPLE

Sometimes it is necessary to *rewrite* the integrand before integrating:

Problem: Evaluate $\int \frac{1}{x^2}\,dx$.

Solution:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

Check: $\frac{d}{dx}(-\frac{1}{x}) = \frac{d}{dx}(-x^{-1}) = -(-1)x^{-1-1} = x^{-2} = \frac{1}{x^2}$

Here, it was necessary to get the integrand into a form that could be handled by the Simple Power Rule for Integration.

EXAMPLE

Problem: Evaluate $\int t(t^2+1)\,dt$.

Solution:

$$\int t(t^2+1)\,dt = \int (t^3+t)\,dt = \frac{t^4}{4} + \frac{t^2}{2} + C$$

Check!

♣ Where was linearity of the integral used here?

EXAMPLE

Problem: Evaluate $\int (\sqrt[4]{x^3}+1)\,dx$.

Solution:

$$\begin{aligned}
\int (\sqrt[4]{x^3}+1)\,dx &= \int (x^{3/4}+1)\,dx && \text{(rewrite)} \\
&= \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + x + C && \text{(use formulas and linearity)} \\
&= \frac{4}{7}x^{7/4} + x + C && \left(\frac{3}{4}+1 = \frac{3}{4}+\frac{4}{4} = \frac{7}{4}\right) \\
&= \frac{4}{7}(x^7)^{\frac{1}{4}} + x + C && ((x^a)^b = x^{ab}) \\
&= \frac{4}{7}\sqrt[4]{x^7} + x + C
\end{aligned}$$

It's a good rule of thumb to get your final answer in a form that matches, as closely as possible, the original form of the problem. Since the original problem was given in radical form (not fractional exponent form), the final answer was also given in radical form.

EXAMPLE

Problem: Evaluate $\int \frac{x^2+1}{x^2}\,dx$.
Solution:

$$\begin{aligned}\int \frac{x^2+1}{x^2}\,dx &= \int 1 + \frac{1}{x^2}\,dx \\ &= \int 1 + x^{-2}\,dx \\ &= x + \frac{x^{-1}}{-1} + C \\ &= x - \frac{1}{x} + C \\ &= \frac{x^2-1}{x} + C\end{aligned}$$

EXAMPLE

Problem: Evaluate $\int (3y^2-1)^2\,dy$.
Solution:

$$\begin{aligned}\int (3y^2-1)^2\,dy &= \int (9y^4 - 6y^2 + 1)\,dy \\ &= 9(\frac{y^5}{5}) - 6(\frac{y^3}{3}) + y + C \\ &= \frac{9}{5}y^5 - 2y^3 + y + C\end{aligned}$$

EXERCISE 2

Evaluate the following integrals. Be sure to write complete mathematical sentences. Don't forget to include the arbitrary constant. Check your answers.

♣ 1. $\displaystyle\int (ax^2 + bx + c)\,dx$, where a, b and c are constants

♣ 2. $\displaystyle\int \frac{2\sqrt{t}-1}{t^2}\,dt$

♣ 3. $\displaystyle\int (1 + \sqrt[3]{x})^2\,dx$

♣ 4. $\displaystyle\int \sqrt{\frac{3\pi}{y^4}} - e^y\ dy$

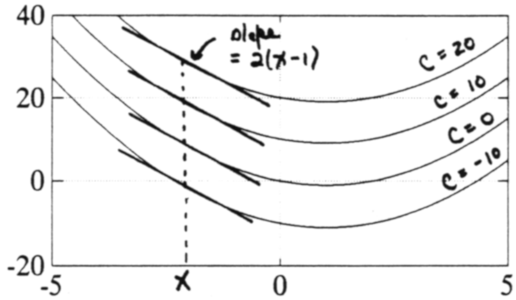
♣ 5. $\displaystyle\int \left(\frac{\sqrt{x}-1}{x}\right)^2 dx$

finding a particular solution

An integration problem like

$$\int 2(x-1)\,dx = 2(\frac{x^2}{2} - x) + C = x^2 - 2x + C$$

yields a whole *class* of functions, each of which has derivative $2(x-1)$. Some members of this class are shown below:

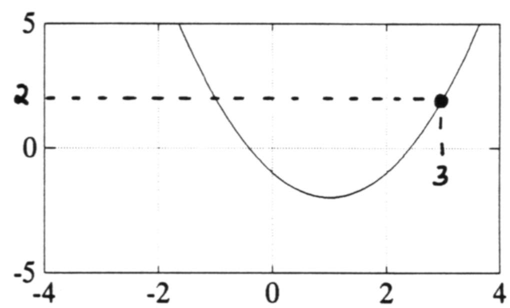


The class of functions
$y = x^2 - 2x + C$

Occasionally, it is desired to go into this class, and choose a *particular* member; one that passes through a specified point. For example, if we want a function f satisfying the two properties

- $f'(x) = 2(x-1)$, and
- $(3,2)$ lies on the graph of f

then we must find the constant C corresponding to the function shown below:



When will the function $f(x) = x^2 - 2x + C$ pass through the point $(3,2)$? Precisely when $f(3) = 2$:

$$\begin{aligned} f(3) = 2 \quad &\iff \quad 3^2 - 2(3) + C = 2 \\ &\iff \quad 3 + C = 2 \\ &\iff \quad C = -1 \end{aligned}$$

Thus, the desired function is $f(x) = x^2 - 2x - 1$. Problems such as this are called '*Finding a particular solution*'.

EXAMPLE

Problem: Find a function y satisfying:

- $\frac{dy}{dx} = x^2 + 2$, and
- the point $(1,5)$ lies on the graph of y

First, find *all* functions y with derivative $x^2 + 2$:

$$y = \int (x^2+2)\,dx = \frac{x^3}{3} + 2x + C$$

Since the desired curve is to contain the point $(1,5)$, C must be chosen to satisfy the property that $y = 5$ when $x = 1$:

$$\begin{aligned} (1,5) \text{ on curve} \quad &\iff \quad 5 = \frac{1^3}{3} + 2(1) + C \\ &\iff \quad C = 3 - \frac{1}{3} = \frac{8}{3} \end{aligned}$$

Thus, $y = \frac{x^3}{3} + 2x + \frac{8}{3}$ is the desired curve.

EXERCISE 3

finding particular solutions

♣ 1. Find a function y with derivative $2x - 3$, that passes through the point $(0,4)$.

♣ 2. Find a function f satisfying the following properties:

a) $f'(x) = \sqrt{x}$, and

b) $f(1) = -2$

EXERCISE 4

♣ 1. Find a function f satisfying all the following properties:

a) $f'(x) = 2$ for $x > 1$

b) $f'(x) = 3x^2$ for $x < 1$

c) $f(1) = 0$

d) f is continuous at $x = 1$

♣ 2. Find another function f satisfying all the properties above except the last: this time, f should have a nonremovable discontinuity at $x = 1$.

integrating e^{kx}

The antidifferentiation 'counterpart' of the differentiation formula $\frac{d}{dx}(e^{kx}) = ke^{kx}$ is:

$$\int ke^{kx}\,dx = e^{kx} + K \quad \iff \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

Summarizing:

integrating e^{kx}

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

EXAMPLE

Problem: Evaluate $\int e^{3x}\,dx$.

Solution:

$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

EXAMPLE

Problem: Evaluate $\int e^{2t-1}\,dt$.
Solution:

$$\begin{aligned}\int e^{2t-1}\,dt &= \int e^{2t}e^{-1}\,dt\\ &= e^{-1}\int e^{2t}\,dt\\ &= e^{-1}(\frac{1}{2}e^{2t}) + C\\ &= \frac{1}{2}e^{2t-1} + C\end{aligned}$$

Check: $\frac{d}{dt}(\frac{1}{2}e^{2t-1}) = \frac{1}{2}(e^{2t-1})(2) = e^{2t-1}$

integrating $x^{-1} = \frac{1}{x}$

Note that when $n = -1$, the Simple Power Rule for Integration does not apply, because the formula $\frac{x^{n+1}}{n+1}$ is not defined. Therefore, this rule cannot be used to tell us how to integrate $\int x^{-1}\,dx = \int \frac{1}{x}\,dx$.
However, we *do* know a function whose derivative is $\frac{1}{x}$:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Thus:

$$\int \frac{1}{x}\,dx = \ln x + C$$

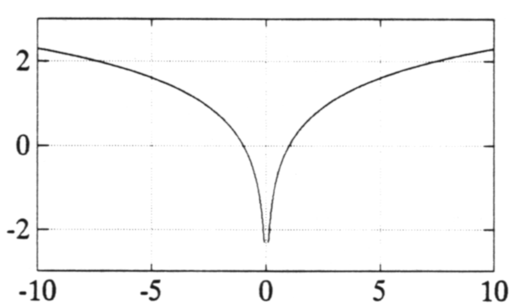
However, there's something undesirable about this formula. The function $\frac{1}{x}$ is defined for all x except 0; however the antiderivatives $\ln x + C$ are only defined for positive x. This problem can be remedied, and is the next topic of discussion.

investigating $\frac{d}{dx}\ln|x|$

The function $y = \ln|x|$ has the graph shown below. Note that:

$$\ln|x| = \begin{cases}\ln x & \text{for } x > 0\\ \ln(-x) & \text{for } x < 0\end{cases}$$

The domain of $\ln|x|$ is precisely the same as the domain of $\frac{1}{x}$: all nonzero x.
Now, is $\ln|x|$ an antiderivative of $\frac{1}{x}$? That is, does $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$? It is shown next that the answer is 'Yes'!



GRAPH OF
$y = \ln|x|$

Each 'piece' of the function is differentiated separately.
For $x > 0$: $\frac{d}{dx} \ln x = \frac{1}{x}$
For $x < 0$: $\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$
In either case the same formula is obtained, so that for all $x \neq 0$:

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

The antiderivative $\ln|x|$ should always be used when integrating $\frac{1}{x}$.

integrating $\frac{1}{x}$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

EXAMPLE

Problem: Find all the antiderivatives of $\frac{1}{3x}$.
Solution:

$$\begin{aligned} \int \frac{1}{3x}\, dx &= \frac{1}{3} \int \frac{1}{x}\, dx \\ &= \frac{1}{3} \ln|x| + C \end{aligned}$$

EXERCISE 5

Find all the antiderivatives of the following functions. Be sure to write your answers using complete mathematical sentences.

♣ 1. $f(x) = \frac{1-\sqrt{x}}{x}$

♣ 2. $y = \left(\frac{t+1}{t}\right)^2$

♣ 3. $g(x) = \frac{1}{7x} + e^{-x} + 1$

QUICK QUIZ
sample questions

1. What is the antidifferentiation 'counterpart' to the differentiation formula $\frac{d}{dx}e^{kx} = ke^{kx}$?
2. Find: $\int \sqrt{x}\, dx$
3. Find: $\int \frac{1}{2t}\, dt$
4. Find a function f satisfying: $f'(x) = x$ and $f(0) = 3$

KEYWORDS
for this section

Differentiation 'counterparts', Simple Power Rule for Integration, finding particular solutions, integrating e^{kx}, integrating $\frac{1}{x}$.

END-OF-SECTION EXERCISES

♣ Write three antidifferentiation problems, that can be solved with the tools available to you.

The first problem should involve a radical; the second a binomial squared, and the third a rational function.

Solve the three antidifferentiation problems, and then check, by differentiating.

6.3 Analyzing a Falling Object
(Optional)

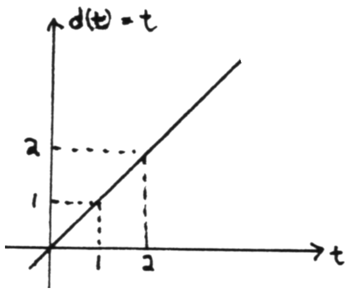
Introduction

In this section the motion of a falling object that is acted upon only by gravity is studied; this is a beautiful application of antidifferentiation to a real-life problem. Such an object travels in a (vertical) line, and it is thus first necessary to understand motion along a line. This is the next topic of discussion.

a particle traveling along a line

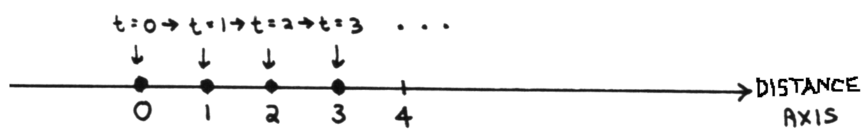
Suppose the function d tells the position of a particle along a line at time t. For convenience, distance along the line is measured in units of feet; time is measured in seconds.

For example, the function $d(t) = t$ describes a particle that is:

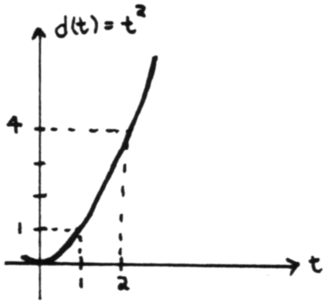


$$\begin{array}{llll} \text{at position} & d(0) = 0 & \text{at} & t = 0 \\ \text{at position} & d(1) = 1 & \text{at} & t = 1 \\ \text{at position} & d(2) = 2 & \text{at} & t = 2 \\ \text{at position} & d(T) = T & \text{at} & t = T \end{array}$$

The particle travels to the right at a constant speed of 1 foot per second.

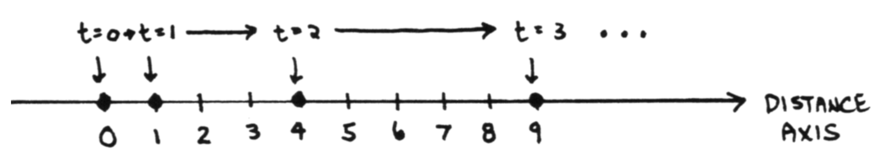


The function $d(t) = t^2$ describes a particle that is:

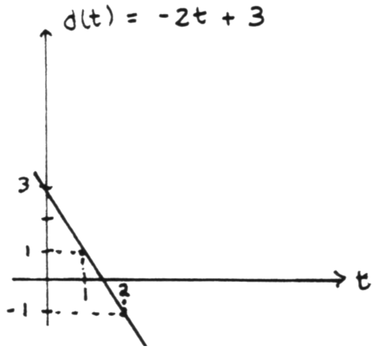


$$\begin{array}{llll} \text{at position} & d(0) = 0 & \text{at} & t = 0 \\ \text{at position} & d(1) = 1 & \text{at} & t = 1 \\ \text{at position} & d(2) = 4 & \text{at} & t = 2 \\ \text{at position} & d(T) = T^2 & \text{at} & t = T \end{array}$$

The particle travels to the right, and continually picks up speed as it travels.

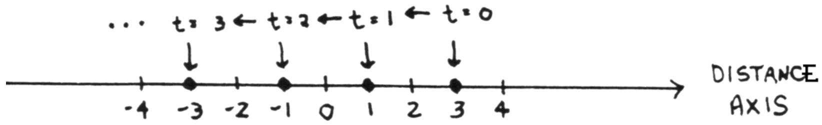


The function $d(t) = -2t + 3$ describes a particle that is:



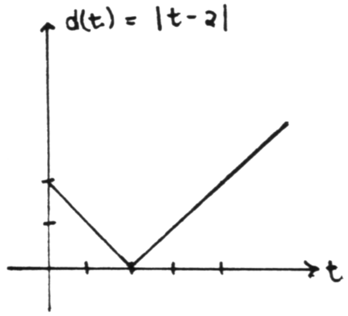
$$
\begin{aligned}
&\text{at position } \ d(0) = 3 \ \text{ at } \ t = 0\\
&\text{at position } \ d(1) = 1 \ \text{ at } \ t = 1\\
&\text{at position } \ d(2) = -1 \ \text{ at } \ t = 2\\
&\text{at position } \ d(3) = -3 \ \text{ at } \ t = 3\\
&\text{at position } \ d(T) = -2T + 3 \ \text{ at } \ t = T
\end{aligned}
$$

This particle starts at position 3, and travels to the left at a uniform speed of 2 feet per second.

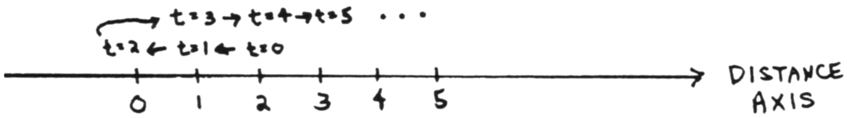


The function $d(t) = |t - 2|$ describes a particle that is:

$$
\begin{aligned}
&\text{at } \ d(0) = 2 \ \text{ at } \ t = 0\\
&\text{at } \ d(1) = 1 \ \text{ at } \ t = 1\\
&\text{at } \ d(2) = 0 \ \text{ at } \ t = 2\\
&\text{at } \ d(3) = 1 \ \text{ at } \ t = 3\\
&\text{at } \ d(4) = 2 \ \text{ at } \ t = 4\\
&\text{at } \ d(T) = T - 2 \ \text{ at } \ T > 2
\end{aligned}
$$



This particle starts at 2, moves backward to zero, then turns around and travels to the right. Except when it turns, the particle moves at a constant speed of one unit per second.



EXERCISE 1	A particle traveling along a line is at position $d(t)$ feet at t seconds. Describe the resulting motion, if: ♣ 1. $d(t) = 3t$ ♣ 2. $d(t) = -3t$ ♣ 3. $d(t) = -t^2$ ♣ 4. $d(t) = 2\|t - 1\|$

instantaneous velocity,
$v(t) := d'(t)$

Recall the instantaneous rate of change interpretation of the derivative: $f'(c)$ gives the instantaneous rate of change of the numbers $f(x)$ with respect to x, at the point $(c, f(c))$.

Specializing to the current setting, let $d(t)$ represent the position of a particle at time t. Then, $d'(T)$ gives the instantaneous rate of change of the numbers $d(t)$ with respect to time t, at the point $(T, d(T))$. That is, $d'(T)$ gives a *change in distance* per *change in time*. This type of information is commonly called *velocity*. Thus, $d'(t)$ gives the (instantaneous) velocity at time t, and is commonly denoted by $v(t)$. That is:

$$v(t) := d'(t) = \text{instantaneous velocity at time } t$$

Remember that ':=' means '*equals, by definition*'. Here, the name $v(t)$ ('v', for velocity) is being *assigned* to the derivative $d'(t)$. If distance along the line is measured in units of feet, and time is measured in seconds, then:

$$\text{units of } v(t) = \frac{\text{units of position}}{\text{units of time}} = \frac{\text{feet}}{\text{second}}$$

EXAMPLE
finding $v(t)$

Consider the earlier examples.

When $d(t) = t$, then $v(t) := d'(t) = 1$. At *every* time t, the instantaneous velocity is 1 foot per second. No matter where the particle is currently sitting on the line, it travels to the right at one foot per second.

EXAMPLE

When $d(t) = t^2$, then $v(t) := d'(t) = 2t$. In this case, the velocity of the particle *depends on* the time at which we are investigating the particle.

At $t = 0$, the particle is at position $d(0) = 0^2 = 0$ ft, and has instantaneous velocity $d'(0) = 2 \cdot 0 = 0$ ft/sec.

At $t = 1$, the particle is at position $d(1) = 1^2 = 1$ ft, and has instantaneous velocity $d'(1) = 2 \cdot 1 = 2$ ft/sec.

At $t = 2$, the particle is at position $d(2) = 2^2 = 4$ ft, and has instantaneous velocity $d'(2) = 2 \cdot 2 = 4$ ft/sec.

At $t = 3$, the particle is at position $d(3) = 3^2 = 9$ ft, and has instantaneous velocity $d'(3) = 2 \cdot 3 = 6$ ft/sec.

The particle moves faster and faster as it travels along the line.

EXERCISE 2

♣ Find $v(t)$ for each of the distance functions from Exercise 1. Does this velocity information agree with the description of the motion you gave in Exercise 1?

EXAMPLE

'velocity' versus 'speed'

When $d(t) = -2t + 3$, then $v(t) := d'(t) = -2$. At *every* time t, the particle has velocity -2 ft/second. That is, when t increases by 1, $d(t)$ *decreases* by 2. Thus, the negative sign indicates that the particle is moving to the left.

The word *speed* is commonly used to describe how fast something moves, regardless of the *direction* in which it moves. For example, if a particle travels to the right, covering 2 feet per second, it has speed 2 ft/second. If a particle travels to the left, covering 2 feet per second, it still has speed 2 ft/second.

Precisely, the *speed* of a particle at time t is given by the *magnitude* of velocity. That is:

$$\text{speed at time } t = |v(t)|$$

Observe that *velocity has both magnitude (size) and direction*, but *speed has only magnitude*.

EXAMPLE

Problem: Suppose the position of a particle traveling along a line is given by $d(t) = t^2 - 5t + 3$. Find the position, velocity, and speed of the particle at $t = 1$. Suppose distance along the line is measured in meters; time is measured in minutes.

Solution: The position of the particle at $t = 1$ is $d(1) = 1^2 - 5 \cdot 1 + 3 = -1$ meters.

$v(t) := d'(t) = 2t - 5$; so the velocity at $t = 1$ is $v(1) = 2 \cdot 1 - 5 = -3$ meters/minute.

The speed at $t = 1$ is $|v(1)| = |-3| = 3$ meters/minute.

At $t = 1$, the particle is traveling to the left, at the rate of 3 meters per minute.

EXERCISE 3

♣ Suppose the position of a particle traveling along a line is given by $d(t) = t^3 - 2t^2 + 3$. Suppose distance is measured in meters, and time is measured in seconds. Find the position, velocity, and speed of the particle at: $t = 1$, $t = -1$, $t = 0$, $t = T$

instantaneous acceleration, $a(t) := v'(t) = d''(t)$

A change in velocity per change in time is commonly called *acceleration*. For example, when a car 'accelerates', this means that its speed is increasing.

The function v' gives the change in velocity per change in time. Thus, this function v' is renamed a, and called the 'acceleration function'. Observe that $v'(t) = \frac{d}{dt}v(t) = \frac{d}{dt}d'(t) = d''(t)$. Precisely:

$$a(t) := v'(t) = d''(t) = \text{instantaneous acceleration at time } t$$

What are the units of acceleration? Since acceleration is a change in velocity per change in time, it has units of $\frac{\text{velocity}}{\text{time}}$. For example, if distance is measured in feet and time in seconds, then:

$$\text{units of acceleration} = \frac{\text{ft/sec}}{\text{sec}} = \frac{\text{ft}}{\text{sec}^2}$$

Going 'backwards': when you see units of (say) ft/sec^2, it may be valuable to remind yourself that this is 'feet per second, per second'.

For example, consider the distance function $d(t) = t$. Here, differentiating once yields $v(t) = 1$, and differentiating once more yields $a(t) = 0$. The particle always travels to the right with speed 1. Its velocity is not changing. Thus, its acceleration is 0.

Next consider the distance function $d(t) = t^2$. Here, $v(t) = 2t$ and $a(t) = 2$. When time increases by 1, the velocity of the particle increases by 2. The particle is speeding up. And no matter what time we look at the particle, it is always speeding up at the same rate. It has a constant acceleration of 2 ft/sec^2.

EXERCISE 4

♣ 1. Find the acceleration functions for each of the distance functions from Exercise 1. Think about your results.

♣ 2. Find the acceleration function for $d(t) = 2t^3 + t^2 - 3t + 1$.

vectors

A *vector* is a mathematical object that is completely characterized by two pieces of information: a magnitude (size, absolute value) and a direction. Vectors are conveniently represented using arrows: the length of the arrow represents the magnitude of the vector; the direction that the arrow is pointing represents the direction of the vector. The directions that vectors are allowed to take on is determined by the 'space' in which the vectors live, as illustrated by the examples below.

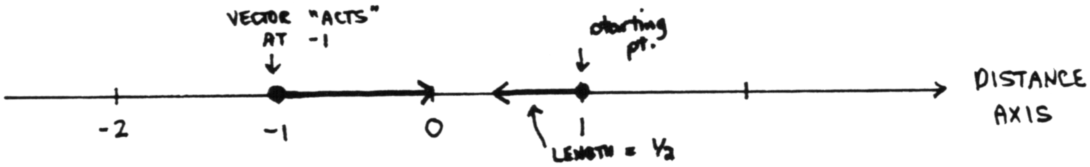
vectors in a line

Suppose the 'space' in which the vectors 'live' is a line. In a line, there are only two possible directions to move. If the line is positioned so that it is horizontal, these two directions are conveniently referred to as 'left' and 'right'. If the line is positioned so that it is vertical, these two directions are conveniently referred to as 'up' and 'down'. For other orientations of the line, names for the two directions are not so clear.

the starting point of a vector shows where it is 'acting'

Some vectors in a line are shown below. Note that each vector has a starting point (the non-arrow end). This starting point indicates where the vector is 'acting'.

For example, if vectors are being used to display velocity information of a particle traveling along a line (distance measured in feet, time in seconds) then the right-most vector below shows that when the particle is at position 1, it is moving left at a speed of $\frac{1}{2}$ foot/sec. The left-most vector below shows that when the particle is at position -1, it is moving right at a speed of 1 foot/sec.



EXERCISE 5

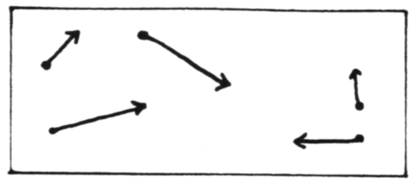
♣ 1. Suppose that when a particle is at position 5 on a line, it is moving left at 2 feet/sec. Illustrate this information using a vector.

♣ 2. Suppose that when a particle is at position -2 on a line, its velocity is 1 ft/sec. Illustrate this information using a vector.

♣ 3. The distance function $d(t) = t^2 - 1$ describes a particle's motion along a line (distance in feet, time in seconds). Illustrate the velocity information on a distance axis, at $t = 2$.

vectors in the plane

If vectors 'live' in a plane, then there are a lot more directions to move. Some vectors in a plane are illustrated below.



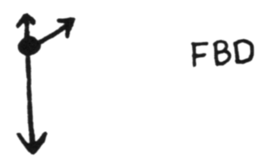
free-body diagram

We are now in a position to begin study of the motion of a falling object. A famous law from physics, known as *Newton's Second Law of Motion*, says that the sum of the forces acting on an object completely determines the acceleration of the object. Precisely:

$$\sum(\text{forces acting on an object}) = (\text{mass of object}) \cdot (\text{acceleration of object})$$

In physics, *vectors* are commonly used to illustrate the forces acting on an object; the resulting picture is called a *free-body diagram* (FBD).

For example, the object shown below has three forces acting on it. If this object is viewed as a *falling* object, then these forces can be interpreted: the force acting down is the force due to gravity; the small force acting upwards is air resistance; and the remaining force could be due to a wind current.



acceleration due to gravity;
$g \approx 32.2$ ft/sec^2

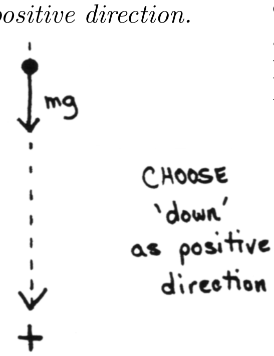
If air resistance and other minor forces are neglected, then the only force acting on a falling body is the force due to gravity. For a particle falling relatively close to the earth's surface, the force due to gravity is given by

$$\text{force due to gravity} = (\text{mass of object})(g)\ ,$$

where g denotes the acceleration due to gravity: $g \approx 32.2$ ft/sec^2

What shall we call the 'positive' vertical direction?

Initially, we'll agree that 'down' is the positive direction.



Newton's Second Law is used to analyze the motion of a falling object; an object traveling along the vertical line shown. First, however, an agreement must be reached about what is the 'positive' direction of this vertical line. Very often, 'up' is considered the positive direction. However, when working with a falling object (which will be traveling *down*), it is often more convenient to decide that 'down' will be the positive direction. Either way will work, as long as one is consistent. Here, we will choose 'down' to be the positive direction. In the exercises, you will get a chance to re-do this example, with 'up' being the positive direction.

using Newton's Second Law

Letting m denote the mass of the falling object, and letting $a(t)$ denote its acceleration at time t, an application of Newton's Second Law (with 'down' the positive direction) says that:

$$\sum \text{forces acting on object} = (\text{mass of object}) \cdot (\text{acceleration of object})$$

That is:

$$mg = m \cdot a(t)$$

Observe that the simplifying assumptions have resulted in only one force acting on the object. This is assumed to be the only force acting on the object throughout its entire fall (until it hits the earth). Note that the force mg appears in this equation as a *positive* constant; this is because the force mg points *down*, and it has been agreed upon that 'down' is the positive direction.

correct sign for $a(t)$ is DETERMINED by the equation

In this equation, the resulting acceleration $a(t)$ of the object is the 'unknown'. The correct sign for $a(t)$ is *determined* by the equation, based on the forces present. That is, the unknown acceleration always appears simply as '$a(t)$'; it would never enter the equation as, say, '$-a(t)$'.

using antidifferentiation to find $v(t)$ and $d(t)$

Once $a(t)$ is found, this information (together with some additional information, to be discussed momentarily) can be used to determine the velocity and distance functions for the particle, by antidifferentiating. How? Well, the falling object has some distance function d that describes its motion along the vertical line; and it must be that $d''(t) = a(t)$. Roughly, we will 'undo' the known derivative $d''(t) = a(t)$ to get information about d' and d.

mass cancels out of the equation of motion

Observe that in the equation $mg = m \cdot a(t)$, the mass cancels out. Thus, under the simplifying assumptions, the resulting acceleration of the object is NOT dependent on the mass of the object. This has an extremely important physical interpretation: if you simultaneously drop a penny and a concrete block from the top of a tall building, they will both hit the ground at the same time!

After cancellation of m, the resulting equation is $g = a(t)$. Since the unknown is $a(t)$, and it is common to put the unknown on the left, the equation is rewritten as $a(t) = g$. Remember that g is a *constant*, $g \approx 32\ \text{ft/sec}^2$.

finding $v(t)$

Since $a(t) = v'(t)$, the equation $a(t) = g$ can be rewritten as:

$$v'(t) = g$$

The unknown velocity function v has derivative g. Do we know ANY function of t that has derivative g? Of course: $y = gt$ has derivative g. Thus, ANY OTHER function with derivative g must have exactly the same shape, but may be translated vertically. That is, any function with derivative g must be of the form $gt + C$ for some constant C.

These thoughts are commonly written down as a list of implications:

$$\begin{aligned} v'(t) = g \quad &\implies \quad \int v'(t)\,dt = \int g\,dt \\ &\implies \quad v(t) = gt + C \end{aligned}$$

There are two important things to note about this mathematical sentence:

two constants of integration have been combined

- When the integration was performed, *two* constants of integration were really obtained: one from the integral on the left, and one from the integral on the right. However, these two constants were combined into a single constant, called C.

$A \implies B \implies C$ means $A \Rightarrow B$ and $B \Rightarrow C$

- This mathematical sentence is of the form '$A \Rightarrow B \Rightarrow C$', which is shorthand for '$A \Rightarrow B$ and $B \Rightarrow C$'. So whenever A is true, then B must be true. And whenever B is true, then C must be true. It follows that whenever A is true, C must be true.

  Letting:

$$\begin{aligned} &A \text{ be the sentence } v'(t) = g \\ &B \text{ be the sentence } \int v'(t)\,dt = \int g\,dt \\ &C \text{ be the sentence } v(t) = gt + C\ , \end{aligned}$$

  we conclude that whenever $v'(t) = g$, then $v(t) = gt + C$ for some constant C.

interpreting the constant of integration; initial velocity, v_0

Read 'v_0' as 'v naught'

Let's investigate the resulting equation $v(t) = gt + C$. This equation gives ALL functions that have derivative g. At time zero, $v(0) = g \cdot 0 + C = C$. Thus, the constant C represents the *initial velocity of the object*, and is commonly denoted by v_0. Read 'v_0' as 'v naught'. Thus, if the *initial velocity* of the falling object is known, then the velocity of the object at ALL times t is known (until some other force enters the picture, like the ground). If the object starts from rest, then the initial velocity is zero.

integrate once more to find $d(t)$

Now, use the fact that $v(t) = d'(t)$, and integrate again:

$$\begin{aligned} v(t) = gt + v_0 \quad &\implies \quad d'(t) = gt + v_0 \\ &\implies \quad \int d'(t)\,dt = \int (gt + v_0)\,dt \\ &\implies \quad d(t) = g \cdot \frac{t^2}{2} + v_0 t + K \end{aligned}$$

initial position, d_0

Read 'd_0' as 'd naught'

At time zero, $d(0) = g \cdot 0 + v_0 \cdot 0 + K = K$, so the constant K represents the *initial position* of the falling object. This initial distance is commonly denoted by d_0. Read 'd_0' as 'd naught'.

choosing the zero reference point on the vertical line

To measure distance along a vertical line, one MUST know where the number '0' lies. There are two common choices: the reference point '0' can coincide with the initial position of the falling object; or, '0' can coincide with the ground. Either choice is fine, providing one remains consistent when interpreting the results. This should become clear in the examples below.

summary

In summary, it has been found that if an object is acted on only by gravity, then its distance function d is given by

$$d(t) = \frac{gt^2}{2} + v_0 t + d_0\ ,$$

where v_0 represents the initial velocity of the object, and d_0 represents the initial position of the object.

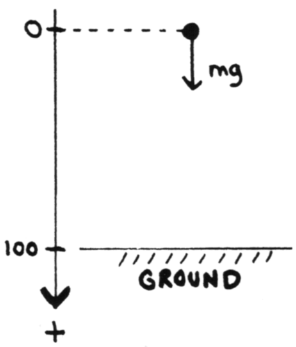
This equation was derived under the assumption that the positive direction of the vertical line is 'down'. The equation is valid until forces other than gravity (like the ground) act on the object.

EXAMPLE

Problem: Suppose that an object is dropped from a height of 100 feet. Answer the following questions:

- What is its distance function?
- How long does it take the object to hit the ground?
- What is the speed of the object when it hits the ground?

Solution #1



Solution #1. It is usually safest to re-derive the equations yourself. It doesn't take very long, and this way you are CERTAIN of the conventions about what is the positive direction, and what is the initial position.

Make a sketch, clearly showing the initial position of the object and the ground. Show the initial force acting on the object. On a vertical line, clearly label your choice for the positive direction, and your choice for '0'. Here, 'down' has been chosen as the positive direction, and '0' coincides with the initial position of the object.

Observe that with this choice of measuring scale, $d(0) = 0$.

$$
\begin{aligned}
\overbrace{mg = m \cdot a(t)}^{\text{Newton's second law}} \quad &\Longrightarrow \quad v'(t) = g && \text{(cancel } m,\ a(t) = v'(t) \text{, switch sides)} \\
&\Longrightarrow \quad v(t) = gt + v_0 && \text{(integrate, } v(0) = v_0) \\
&\Longrightarrow \quad v(t) = gt && (v_0 = 0) \\
&\Longrightarrow \quad d'(t) = gt && (v(t) = d'(t)) \\
&\Longrightarrow \quad d(t) = \frac{gt^2}{2} + d_0 && \text{(integrate, } d(0) = d_0) \\
&\Longrightarrow \quad d(t) = \frac{gt^2}{2} && (d_0 = 0)
\end{aligned}
$$

Thus, the distance function is:

$$d(t) = \frac{gt^2}{2}$$

For the chosen measuring scale, the ground is at position +100. So to answer the question: '*How long does it take the object to hit the ground?*', the distance function is set to 100, and solved for t:

$$
\begin{aligned}
\frac{gt^2}{2} = 100 \quad &\iff \quad t^2 = \frac{200}{g} \\
&\iff \quad t = \pm\sqrt{\frac{200}{g}}
\end{aligned}
$$

The nonnegative number t that makes this true is:

$$t = \sqrt{\frac{200}{g}} \approx \sqrt{\frac{200 \text{ ft}}{32 \text{ ft/sec}^2}} = 2.5 \text{ seconds}$$

The object will hit the ground in approximately 2.5 seconds.

What does $A = B \approx C = D$ *mean to us?*

Let's be sure we agree upon what the sentence

$$t = \sqrt{\frac{200}{g}} \approx \sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}} = 2.5\text{ seconds}$$

really means. Earlier in the text, it was decided that when a 'chain' like

$$A = B \approx C = D$$

appears, the symbols (in this case, '$\approx$' and '$=$') always compare the objects to their immediate left and right.
Thus,

$$t = \sqrt{\frac{200}{g}}$$

is a true equality, because $\sqrt{\frac{200}{g}}$ is the exact desired time. However,

$$\sqrt{\frac{200}{g}} \approx \sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}}$$

is an approximation, because the value of g is being approximated. And,

$$\sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}} = 2.5\text{ seconds}$$

is a true equality, because $\sqrt{\frac{200}{32}}$ is precisely 2.5.
Note that *if there is at least one '$\approx$' in a chain*, then the first thing in the chain is only approximately equal to the last in the chain. That is, in a chain like

$$A = B \approx C = D\ ,$$

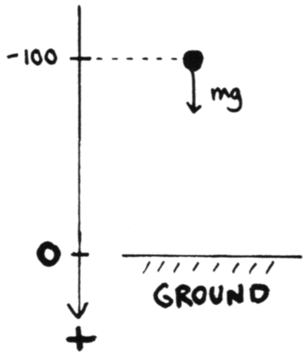
it follows that $A \approx D$. The 'strength' of a chain is determined by its weakest link!

The velocity function was found above to be $v(t) = gt$. Thus, the velocity at time $t = 2.5$ is:

$$v(2.5) = g \cdot (2.5) \approx (32\frac{\text{ft}}{\text{sec}^2})(2.5\text{ sec}) = 80\text{ ft/sec}$$

Observe that the parentheses in $v(2.5)$ are being used for *function evaluation*, NOT multiplication. That is, $v(2.5)$ means the function v, evaluated at 2.5.

Solution #2



Solution #2. This time, a different choice for '0' is made; '0' coincides with the ground. Since 'down' is still the positive direction, the choices lead to $d(0) = -100$. Now we get:

$$\begin{aligned} mg = m \cdot a(t) &\implies v'(t) = g \\ &\implies v(t) = gt + v_0 \\ &\implies v(t) = gt \\ &\implies d'(t) = gt \\ &\implies d(t) = \frac{gt^2}{2} + d_0 \\ &\implies d(t) = \frac{gt^2}{2} - 100 \end{aligned}$$

♣ Fill in a reason for each step in the preceding derivation.

This time, the distance function looks slightly different; it is given by:

$$d(t) = \frac{gt^2}{2} - 100$$

However, *we will obtain precisely the same information* as we did previously. (We must!)

The object hits the ground at time t for which $d(t) = 0$. That is:

$$\frac{gt^2}{2} - 100 = 0$$

This happens when $t = \sqrt{\frac{200}{g}} \approx 2.5$ seconds.

The velocity function is still $v(t) = gt$, so still $v(2.5) \approx 80$ ft/sec.

EXERCISE 6

♣ Re-do the previous example, with the conventions:

- 'up' is the positive direction
- '0' coincides with the initial position of the object

Be sure that you obtain the same answers!

EXERCISE 7

♣ Re-do the previous example, with the conventions:

- 'up' is the positive direction
- '0' coincides with the ground

Be sure that you obtain the same answers!

EXERCISE 8

Suppose an object is dropped from rest at a height of 200 feet. Answer the following questions, being careful to distinguish '=' from '≈' in your solutions:

♣ 1. What is the distance function for the falling object? What conventions have you used in your derivation?

♣ 2. How long will it take the object to hit the ground?

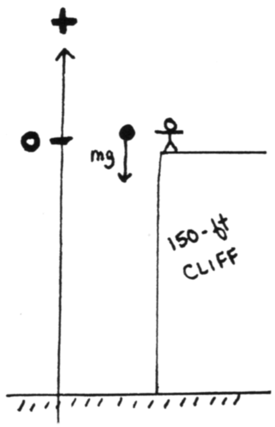
♣ 3. Where is the object after 1 second? 2 seconds?

♣ 4. The object falls past a 100 foot building. How long does it take to reach the top of this building?

♣ 5. What is the velocity of the object at 1 second? 2 seconds? When it hits the ground?

♣ 6. For how many seconds is the equation of motion that you derived valid?

EXAMPLE



Problem: Suppose a person standing at the top of a 150 foot cliff reaches out and throws an object upwards with an initial speed of 10 ft/sec. Answer the following questions:

- What is the distance function for the object? (Derive it.)
- What is the velocity function for the object?
- How long will it go up, before it starts to come down again?
- What is the maximum height that the object will reach?
- How long will it be before the object passes the person who threw it?
- When will the object hit the ground?

Solution. Choose 'up' to be the positive direction, and '0' to coincide with the initial position of the object. Observe that the force acting on the object points DOWN, which is now the negative direction. Then:

$$\begin{aligned}
-mg = m \cdot a(t) \quad &\Longrightarrow \quad a(t) = -g \\
&\Longrightarrow \quad v'(t) = -g \\
&\Longrightarrow \quad v(t) = -gt + v_0 \\
&\Longrightarrow \quad v(t) = -gt + 10 \\
&\Longrightarrow \quad d'(t) = -gt + 10 \\
&\Longrightarrow \quad d(t) = -\frac{gt^2}{2} + 10t + d_0 \\
&\Longrightarrow \quad d(t) = -\frac{gt^2}{2} + 10t
\end{aligned}$$

Thus, the distance and velocity functions are given by:

$$d(t) = -\frac{gt^2}{2} + 10t \quad \text{and} \quad v(t) = -gt + 10$$

♣ Fill in reasons justifying each step in the preceding derivation.

When the object reaches its maximum height, its velocity is 0 :

$$0 = -gt + 10 \quad \Longleftrightarrow \quad t = \frac{10}{g} \approx 0.31 \text{ seconds}$$

Thus, the object rises for about 0.31 seconds, before it turns around to come down again.

At $t = 0.31$:

$$d(0.31) = -\frac{g(0.31)^2}{2} + 100(.31) \approx 1.56 \text{ feet}$$

Remember that this position is relative to the '0' mark; the top of the cliff. Thus, the maximum height the object reaches is $150 + 1.56 = 151.56$ feet. (Observe that the height of the person who threw the object is being neglected.)

The person on the cliff is at position 0 relative to the chosen scale. Thus, we must set $d(t)$ equal to 0 and find the nonnegative value of t that makes this true:

$$\begin{aligned} -\frac{gt^2}{2} + 10t = 0 \quad &\iff \quad t(-\frac{gt}{2} + 10) = 0 \\ &\iff \quad t = 0 \text{ or } -\frac{gt}{2} + 10 = 0 \\ &\iff \quad t = 0 \text{ or } t = \frac{20}{g} \approx 0.63 \text{ seconds} \end{aligned}$$

It takes the object about 0.63 seconds to pass the person who threw it.

The object hits the ground when $d(t) = -150$, relative to the chosen scale:

$$\begin{aligned} -\frac{gt^2}{2} + 10t = -150 \quad &\iff \quad -\frac{gt^2}{2} + 10t + 150 = 0 \\ &\iff \quad t = \frac{-10 \pm \sqrt{(10)^2 - 4(-\frac{g}{2})(150)}}{2(-g/2)} \\ &\iff \quad t \approx -2.77 \text{ secs} \text{ or } t \approx 3.39 \text{ secs} \end{aligned}$$

Choosing the nonnegative answer, the object hits the ground after approximately 3.39 seconds.

QUICK QUIZ
sample questions

1. What is the difference between 'speed' and 'velocity'?
2. Suppose that the distance function for an object is given by $d(t) = t^2 + 2t$. Let distance be measured in feet, time in seconds. Find the position, velocity, speed, and acceleration of the object at $t = 1$.
3. What is a 'vector'?
4. What is a 'free body diagram'?
5. Suppose that $v(t) = gt$. In the sentence '$v(2) = g(2)$', what does '$v(2)$' mean? What does '$g(2)$' mean?

KEYWORDS
for this section

Motion along a line, instantaneous velocity and acceleration, velocity versus speed, vectors, vectors in a line, vectors in space, free-body diagrams, acceleration due to gravity, Newton's second law of motion, using antidifferentiation to find $v(t)$ and $d(t)$, interpreting the constants of integration, distinguishing between '$=$' and '$\approx$'.

END-OF-SECTION EXERCISES

♣ Suppose a person standing at the top of a 75 foot cliff reaches out and throws an object upwards with an initial speed of 20 ft/sec. You may ignore the height of the person throwing the object. Answer the following questions:

1. What is the distance function for the object? (Derive it. Use any appropriate conventions.)
2. What is the velocity function for the object?
3. How long will it go up, before it starts to come down again?
4. What is the maximum height that the object will reach?
5. How long will it be before the object passes the person who threw it?
6. When will the object hit the ground?

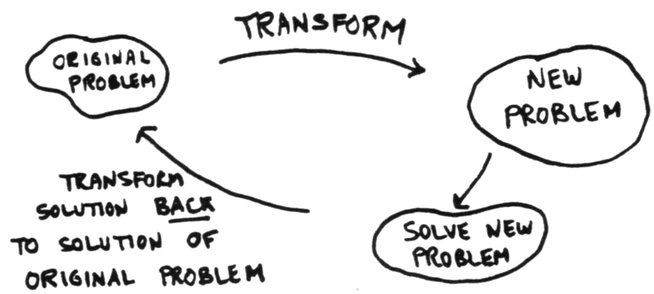
6.4 The Substitution Technique for Integration

a recurrent theme in mathematics; transforming a difficult problem into an easier one

A recurrent theme in mathematics is that of *transforming* a problem that is difficult to solve into one that is easier to solve.

This idea has already been used extensively: in the process of solving an equation, one *transforms* the original equation into an equivalent one (that is, one with the same solution set) that is easier to work with.

In this section, a method is studied by which it is often possible to transform a difficult integration problem into one that is much easier. The *transformed* problem is then solved, and the solution used to obtain the solution of the *original* problem. The technique is referred to as *substitution.*



EXAMPLE

the substitution technique for integration

Here's an example that illustrates the technique. Suppose one wants to find:

$$\int (3-4x^2)^{100}(-8x)\,dx$$

Theoretically at least, this problem *is* solvable with the tools currently available: one need 'only' multiply out $(3-4x^2)^{100}$, multiply this by $-8x$, and then integrate the resulting polynomial term-by-term. Practically speaking,

there must be a better way,

and there is.

do some renaming

Let's do some 'renaming'. Define a new variable u by $u := 3 - 4x^2$, and differentiate to see that $\frac{du}{dx} = -8x$. There just happens to be a $-8x$ in the integrand. So, the integral can be rewritten in terms of u:

$$\int (\overbrace{3-4x^2}^{u})^{100}\,\overbrace{(-8x)}^{\frac{du}{dx}}\,dx \;=\; \int u^{100}\,\frac{du}{dx}\,dx$$

Motivated by 'cancelling the dx's', one might conjecture that an equivalent problem is

$$\int u^{100}\,du\ ,$$

which is a problem that *can* be solved easily: $\int u^{100}\,du = \frac{u^{101}}{101} + C$

♣ What is a 'conjecture'?

Indeed, $\frac{u^{101}}{101} + C$ *is* the solution of $\int u^{100}\frac{du}{dx}\,dx$, since by the extended power rule for differentiation:

$$\frac{d}{dx}\frac{u^{101}}{101} = \frac{1}{101}(101u^{101-1})\frac{du}{dx} = u^{100}\frac{du}{dx}$$

(Remember that u is a function of x, and differentiate accordingly.) Next, transform the solution $\frac{u^{101}}{101} + C$ back to the variable x. Since $u = 3 - 4x^2$, the solution to the original problem is:

$$\int (3-4x^2)^{100}(-8x)\,dx = \frac{(3-4x^2)^{101}}{101} + C$$

EXERCISE 1 ♣ Check, by differentiating, that:

$$\int (3-4x^2)^{100}(-8x)\,dx = \frac{(3-4x^2)^{101}}{101} + C$$

simplified notation for the previous problem

Henceforward, here's how the previous problem will be written down:

$$\int (\overbrace{3-4x^2}^{u})^{100}\,\overbrace{(-8x)\,dx}^{du} = \int u^{100}\,du$$

$$= \frac{u^{101}}{101} + C$$

$$= \frac{(3-4x^2)^{101}}{101} + C$$

$u = 3 - 4x^2$

USUALLY LEAVE THIS STEP OUT → $\cancel{\frac{du}{dx} = -8x}$

$du = -8x\,dx$

Observe several important features of this solution:

- Write the substitution ($u = 3 - 4x^2$, in this case) *directly under* the integration problem.
- When u is a function of x, du is found by first differentiating u with respect to x
$$\frac{du}{dx} = -8x$$
and then 'multiplying' both sides by dx to obtain du. The justification for this procedure was motivated by the first example.
Usually, one doesn't bother to write down the intermediate step $\frac{du}{dx} = -8x$.
- Line up the equal signs as you are solving the problem. This form makes it easy to see the *original integration problem* and the *solution* at a glance.
- Once the solution in terms of the new variable u is obtained, rewrite this solution in terms of the original variable, x.

EXERCISE 2

♣ Supply a reason for each step:

$$\begin{aligned} \int (\overbrace{3 - 4x^2}^{u})^{100}\, \overbrace{(-8x)\, dx}^{du} &= \int u^{100}\, du \\ &= \frac{u^{101}}{101} + C \\ &= \frac{(3 - 4x^2)^{101}}{101} + C \end{aligned}$$

$$\boxed{\begin{aligned} u &= 3 - 4x^2 \\ du &= -8x\, dx \end{aligned}}$$

Don't mix variables!

Don't ever 'mix' variables when writing down your solution, like in:

$$\int (3 - 4x^2)^{100} x\, dx = \int \overbrace{\underbrace{u^{100} x}_{u \text{ and } x \text{ mixed}}}^{\text{BAD!}}\, dx = \cdots$$

Get everything ready to change to the new variable, and then do it—all at once.

choosing a 'u that works'

Strategy: choose something for u such that $\frac{du}{dx}$ also appears in the integrand

Not all problems are solvable by substitution, but many are. If you are faced with a difficult integration problem, the technique of substitution should always be tried. The challenge is, of course, to find a choice for u that 'works'. Here's the general strategy:

- Choose something for u so that its derivative $\frac{du}{dx}$ appears as a factor in the integrand (possibly off by a constant).

Often, as examples will illustrate, u is something that is raised to a power, or under a radical.

In the previous example, u was chosen to be $3-4x^2$ because it was noted that the derivative, $-8x$, was also a factor in the integrand. Actually, it is only critical that the *variable part* of the derivative appear in the integrand; linearity of the integral can be used to take care of *constants*, as the next example illustrates.

EXAMPLE

introducing a constant; multiply by 1 *in an appropriate form*

Problem: Evaluate $\int (3-4x^2)^{100}\, x\, dx$.

Solution: Note the similarity to the previous example. The only difference is that this time the '-8' is missing.

The substitution $u = 3-4x^2$ is still a good choice, since $\frac{d}{dx}(3-4x^2) = -8x$, and the *variable* part of this derivative, x, appears as a factor in the integrand.

To transform the problem into an integral in u, it is necessary to bring a -8 into the picture, *without changing the problem.* This can be accomplished by the usual technique of *multiplying by* 1 *in an appropriate form*:

$$\int (3-4x^2)^{100} x\, dx = \int (3-4x^2)^{100} \left(\frac{-8}{-8}\right) x\, dx \qquad \text{(multiply by 1 in form } \frac{-8}{-8}\text{)}$$

$$\boxed{\begin{aligned} u &= 3-4x^2 \\ du &= -8x\, dx \end{aligned}}$$

$$\begin{aligned}
&= \frac{1}{-8}\int (\overbrace{3-4x^2}^{u})^{100}\, \overbrace{(-8x)\, dx}^{du} && \text{(linearity of integral)} \\
&= -\frac{1}{8}\int u^{100}\, du && \text{(transform to } u\text{)} \\
&= -\frac{1}{8}\cdot \frac{u^{101}}{101} + C && \text{(solve problem in } u\text{)} \\
&= -\frac{1}{8}\cdot \frac{(3-4x^2)^{101}}{101} + C && \text{(rewrite in } x\text{)}
\end{aligned}$$

Since *constants* can be 'slid out' of the integral, we were able to 'get rid of' the undesired '$\frac{1}{-8}$' in the integrand. Only the -8 was left in the integrand, since this was needed as part of *du*.

EXERCISE 3

♣ 1. Check, by differentiating, that:

$$\int (3-4x^2)^{100} x\, dx = -\frac{1}{8}\cdot \frac{(3-4x^2)^{101}}{101} + C$$

♣ 2. Where and how was the linearity of the integral used in arriving at this solution?

The technique of substitution is further illustrated with a number of examples. Pay particular attention to the *complete mathematical sentences* in each of these examples.

EXAMPLE

evaluate an integral

Problem: Evaluate $\int (t+10)^7\, dt$.

Solution:

$$\begin{aligned}
\int (\overbrace{t+10}^{u})^7\ \overbrace{dt}^{du} &= \int u^7\, du \\
&= \frac{u^8}{8} + C \\
&= \frac{(t+10)^8}{8} + C
\end{aligned}$$

$$\boxed{\begin{aligned} u &= t+10 \\ du &= dt \end{aligned}}$$

Check: $\dfrac{d}{dt}\dfrac{(t+10)^8}{8} = \dfrac{1}{8}\cdot 8(t+10)^7(1) = (t+10)^7$

EXAMPLE

find all the antiderivatives of a function

Problem: Find all the antiderivatives of $\dfrac{x^2}{\sqrt{x^3-1}}$.

Solution:

$$
\begin{aligned}
\int \frac{x^2}{\sqrt{x^3-1}}\,dx &= \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3-1}}\,dx \\
&= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{3}\int u^{-1/2}\,du \\
&= \frac{1}{3}\cdot\frac{u^{1/2}}{1/2}+C \\
&= \frac{2}{3}\sqrt{x^3-1}+C
\end{aligned}
$$

$$\boxed{\begin{aligned} u &= x^3-1 \\ du &= 3x^2\,dx \end{aligned}}$$

EXERCISE 4

♣ 1. Why was u chosen to be x^3-1 in the previous example?

♣ 2. Supply reasons for each step in the previous example. In particular, make sure you identify where the linearity of the integral was used.

♣ 3. Check the previous solution, by differentiating.

EXAMPLE

integrate

Problem: Integrate: $\displaystyle\int \frac{y+1}{(y^2+2y+1)^3}\,dy$

Solution:

$$
\begin{aligned}
\int \frac{y+1}{(y^2+2y+1)^3}\,dy &= \int \frac{(\frac{1}{2})(2)(y+1)}{(y^2+2y+1)^3}\,dy \\
&= \frac{1}{2}\int \frac{2y+2}{(y^2+2y+1)^3}\,dy \\
&= \frac{1}{2}\int \frac{1}{u^3}\,du \\
&= \frac{1}{2}\int u^{-3}\,du \\
&= \frac{1}{2}\cdot\frac{u^{-2}}{-2}+C \\
&= -\frac{1}{4u^2}+C \\
&= -\frac{1}{4(y^2+2y+1)^2}+C
\end{aligned}
$$

$$\boxed{\begin{aligned} u &= y^2+2y+1 \\ du &= (2y+2)\,dy \end{aligned}}$$

EXERCISE 5

♣ 1. Why was u chosen to be y^2+2y+1 in the previous example?

♣ 2. Rewrite the previous example, using the dummy variable x instead of the dummy variable y. Do not look at the text while you are solving the problem.

♣ 3. Check the solution to the previous example, by differentiating.

EXAMPLE
two different approaches to the same problem

Problem: Find $\int e^{4+x}\,dx$ in two different ways.

Old way:

$$\begin{aligned}\int e^{4+x}\,dx &= \int e^4 e^x\,dx\\ &= e^4\int e^x\,dx\\ &= e^4\cdot e^x + C\\ &= e^{4+x}+C\end{aligned}$$

New way:

$$\begin{aligned}\int e^{4+x}\,dx &= \int e^u\,du\\ &= e^u + C\\ &= e^{4+x}+C\end{aligned}$$

$$\boxed{\begin{array}{l} u = 4+x \\ du = dx \end{array}}$$

Which was easier?

EXAMPLE
$\int e^{kx}\,dx = \frac{1}{k}e^{kx}+C$

Problem: Find a formula for integrating e^{kx}, for any nonzero constant k.
Solution:

$$\begin{aligned}\int e^{kx}\,dx &= \frac{1}{k}\int k\cdot e^{kx}\,dx\\ &= \frac{1}{k}\int e^u\,du\\ &= \frac{1}{k}e^u + C\\ &= \frac{1}{k}e^{kx}+C\end{aligned}$$

$$\boxed{\begin{array}{l} u = kx \\ du = k\,dx \end{array}}$$

This is a nice formula to remember. Thus, for example:

$$\int 7e^{3x}\,dx = 7(\frac{1}{3})e^{3x} + C = \frac{7}{3}e^{3x}+C$$

EXAMPLE

Some people take a slightly different approach when solving problems like $\int e^{4+x}\,dx$ and $\int e^{3x}\,dx$, as illustrated below:

$$\begin{aligned}\int e^{4+x}\,dx &= \int du\\ &= u + C\\ &= e^{4+x}+C\end{aligned}$$

$$\boxed{\begin{array}{l} u = e^{4+x} \\ du = e^{4+x}\,dx \end{array}}$$

$$\begin{aligned}\int e^{3x}\,dx &= \frac{1}{3}\int 3e^{3x}\,dx\\ &= \frac{1}{3}\int du\\ &= \frac{1}{3}u + C\\ &= \frac{1}{3}e^{3x}+C\end{aligned}$$

$$\boxed{\begin{array}{l} u = e^{3x} \\ du = 3e^{3x}\,dx \end{array}}$$

Variety is the spice of life. Which way do you prefer?

EXAMPLE
finding a particular solution

Problem: Find a function f satisfying the following two conditions:

- the graph of f passes through the point $(0, 1)$
- $f'(x) = \dfrac{1}{3x+5}$

Solution: First, find ALL functions f that have derivative $\frac{1}{3x+5}$. That is, find all the antiderivatives of f':

$$\boxed{\begin{array}{l} u = 3x + 5 \\ du = 3\,dx \end{array}}$$

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \frac{1}{3x+5}\,dx \\ &= \frac{1}{3}\int \frac{3}{3x+5}\,dx \\ &= \frac{1}{3}\int \frac{1}{u}\,du \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|3x+5| + C \end{aligned}$$

A problem like this was integrated earlier in the chapter, via a different technique. (See, for example, page 350.) Which technique do you prefer?

Check: Remember:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

An application of the Chain Rule gives:

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}\cdot f'(x)$$

Then: $\frac{d}{dx}(\frac{1}{3}\ln|3x+5|) = \frac{1}{3}\cdot\frac{1}{3x+5}\cdot 3 = \frac{1}{3x+5}$

Second, choose the antiderivative that passes through the desired point:

$$\begin{aligned} (0,1) \text{ lies on graph of } f(x) = \frac{1}{3}\ln|3x+5| + C \quad &\iff \quad f(0) = 1 \\ &\iff \quad \frac{1}{3}\ln 5 + C = 1 \\ &\iff \quad C = 1 - \frac{\ln 5}{3} \\ &\iff \quad C = \frac{3-\ln 5}{3} \end{aligned}$$

Note how this was written down using a *complete mathematical sentence*. The desired function is therefore:

$$\begin{aligned} f(x) &= \frac{1}{3}\ln|3x+5| + \frac{3-\ln 5}{3} \\ &= \frac{\ln|3x+5| + 3 - \ln 5}{3} \end{aligned}$$

EXERCISE 6

♣ 1. Use the Chain Rule to prove that:

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}\cdot f'(x)$$

♣ 2. Verify that the function

$$f(x) = \frac{\ln|3x+5| + 3 - \ln 5}{3}$$

has a graph that passes through the point $(0,1)$, and has derivative $f'(x) = \frac{1}{3x+5}$.

EXAMPLE
antidifferentiate

Problem: Antidifferentiate $\dfrac{\ln x}{x}$.

Solution:

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int u\,du\\ &= \frac{u^2}{2} + C\\ &= \frac{1}{2}(\ln x)^2 + C\end{aligned}$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

Check: $\dfrac{d}{dx}(\frac{1}{2}(\ln x)^2) = \frac{1}{2}\cdot 2(\ln x)(\frac{1}{x}) = \dfrac{\ln x}{x}$

EXAMPLE
using a letter different than 'u' for the substitution variable

Problem: Evaluate $\int (2-u)^4\,du$.

Solution: Just use a letter different than 'u' for the substitution variable! Here, the letter 'w' is used.

$$\begin{aligned}\int (2-u)^4\,du &= -\int (2-u)^4(-du)\\ &= -\int w^4\,dw\\ &= -\frac{w^5}{5} + C\\ &= -\frac{1}{5}(2-u)^5 + C\end{aligned}$$

$w = 2 - u$
$dw = -du$

QUICK QUIZ
sample questions

1. What is the idea behind the substitution technique for integration?
2. Solve $\int \frac{1}{2x-1}\,dx$ two ways; without using substitution, and using substitution. Do your answers agree?
3. Where is linearity of the integral used in the substitution technique?
4. Solve: $\int e^{3x}\,dx$
5. Is $\int (3x+\pi)^5\,dx = \frac{(3x+\pi)^6}{18} + C$? Justify your answer.

KEYWORDS
for this section

Transforming a difficult problem into an easier one, the substitution technique for integration, choosing a 'u that works', multiplying by 1 in an appropriate form.

END-OF-SECTION EXERCISES

♣ Evaluate the following indefinite integrals. Be sure to write complete mathematical sentences. Check your answers by differentiating.

1. $\displaystyle\int (2x-1)^{17}\,dx$
2. $\displaystyle\int 5t\sqrt{t^2+3}\,dt$
3. $\displaystyle\int \frac{3\ln 4x}{x}\,dx$
4. $\displaystyle\int (4e^{2t}+e^{1+t})\,dt$
5. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
6. $\displaystyle\int \frac{-1}{2u+5}\,du$
7. $\displaystyle\int \frac{4t+2}{\sqrt{(t^2+t+1)^3}}\,dt$
8. $\displaystyle\int (e^x+1)^5\cdot 3e^x\,dx$
9. Find a function f whose graph passes through the point $(0,4)$, and that has derivative $f'(x)=e^x(e^x+1)^3$.
10. A particle traveling along a line has velocity function given by:

$$v(t)=(t-2)^3$$

It is known that at $t=1$, the particle is at position $\frac{1}{2}$. Find the distance function for this particle.

11. A student passed in the following solution to an integration problem:

$$\begin{aligned}
\int (x^2+1)^5\,dx &= \int \frac{2x}{2x}(x^2+1)^5\,dx\\
&= \frac{1}{2x}\int (x^2+1)^5(2x\,dx)\\
&= \frac{1}{2x}\int u^5\,du\\
&= \frac{1}{2x}\frac{u^6}{6}+C\\
&= \frac{1}{2x}\frac{(x^2+1)^6}{6}+C\\
&= \frac{(x^2+1)^6}{12x}+C
\end{aligned}$$

♣ a) Do you believe that this is a correct solution? If not, where has the student made a mistake?

♣ b) Check the student's solution by finding $\frac{d}{dx}\frac{(x^2+1)^6}{12x}$. (Use the quotient rule.) Is the student's solution correct?

6.5 More on Substitution

integration is more difficult than differentiation

Integration is more difficult than differentiation. To *differentiate* a function, one only needs to recognize the *form* of the function—a product, a quotient, a composite function—and then use the appropriate differentiation tool. There is a much bigger 'bag of tricks' associated with integration. For the most part, people who are good at integrating are people who have had *lots of experience* integrating. Over time, with lots of practice, you will learn to recognize different types of integration problems, and apply appropriate tools.

more advanced substitution techniques

The substitution technique discussed in the previous section is the 'basic model' of substitution. In this section, more advanced substitution techniques are investigated.

a slight twist on the 'basic model' of substitution

Consider the integration problem:

$$\int x(x+1)^{10}\,dx$$

From a theoretical viewpoint, since $x(x+1)^{10}$ is just a polynomial, the problem is easy. From a computational viewpoint, one certainly doesn't want to multiply out $(x+1)^{10}$. And, the 'basic model' of substitution doesn't seem to work at first glance: one could try letting $u = x+1$, but there's an extra 'x' in the integrand, that cannot be pulled out of the integral.

Study the next example, to see how this 'problem' is overcome.

EXAMPLE

Problem: Solve $\int x(x+1)^{10}\,dx$.

Solution: Define $u := x+1$. Then, $du = dx$, and (writing x in terms of u), $x = u-1$. Transforming the integral in x to an integral in u yields:

$$\begin{aligned}\int x(x+1)^{10}\,dx &= \int (u-1)u^{10}\,du\\ &= \int u^{11} - u^{10}\,du\\ &= \frac{u^{12}}{12} - \frac{u^{11}}{11} + C\\ &= \frac{(x+1)^{12}}{12} - \frac{(x+1)^{11}}{11} + C\end{aligned}$$

$u = x+1$; $x = u-1$
$du = dx$

What made this work? Firstly, it was possible to rewrite the entire integrand in terms of u. Secondly, the resulting function of u was easier to integrate than the original function of x.

'role reversal'

Note that the substitution $u = x+1$ in the previous example transformed

$$\int x(x+1)^{10}\,dx \quad \text{to} \quad \int (u-1)u^{10}\,du\ ;$$

in the first integral, the *sum* is raised to the tenth power, and in the second integral, the *singleton* is raised to the tenth power. Hence, the substitution provided a sort of 'role reversal'. The next few examples illustrate the use of substitution for this type of 'role reversal'.

EXAMPLE
'role reversal'

Problem: Find $\displaystyle\int \frac{x}{x+1}\,dx\,.$

Solution: The problem is of the form $\int \frac{\text{singleton}}{\text{sum}}\,dx$. If the problem were instead of the form $\int \frac{\text{sum}}{\text{singleton}}\,dx$, then it would be easy, since, for example, $\frac{x+1}{x} = 1 + \frac{1}{x}$. Thus, the denominator is 'transformed to a singleton' by defining $u := x + 1$:

$u = x+1\ ;\ x = u-1$
$du = dx$

$$\begin{aligned}\int \frac{x}{x+1}\,dx &= \int \frac{u-1}{u}\,du \\ &= \int (1 - \frac{1}{u})\,du \\ &= u - \ln|u| + C \\ &= (1+x) - \ln|1+x| + C \\ &= x - \ln|1+x| + K\end{aligned}$$

In the last step, the constant 1 was absorbed into the constant of integration, to obtain a simpler answer.

Check: $\displaystyle\frac{d}{dx}(x - \ln|1+x|) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x}$

alternate solution;
long division

Here's an alternate solution to the integration problem $\displaystyle\int \frac{x}{1+x}\,dx\,.$

Alternate Solution: First, do a long division. Remember that when you divide by a polynomial, you want to write the divisor so that the powers of x decrease as you go from left to right:

$$\begin{array}{r|l} & \quad 1 \\ \hline x+1 & x \\ & -(x+1) \\ \hline & \quad -1 \end{array}$$

Thus, $\displaystyle\frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then:

$$\begin{aligned}\int \frac{x}{x+1}\,dx &= \int (1 - \frac{1}{x+1})\,dx \\ &= x - \ln|x+1| + C\end{aligned}$$

EXERCISE 1

♣ Find $\displaystyle\int \frac{3t}{t-1}\,dt$ in two ways. First, use the 'role reversal' substitution technique. Second, use long division.

EXAMPLE

Problem: Find all the antiderivatives of $\dfrac{3t}{2t+1}$.

Solution: In problems such as this, it is often easier to keep track of things if 'dt' is solved for in terms of 'du':

$$\begin{aligned}
\int \frac{3t}{2t+1}\,dt &= 3\int \frac{t}{2t+1}\,dt \\
&= 3\int \frac{\frac{u-1}{2}}{u}\,\frac{du}{2} \\
&= \frac{3}{4}\int \frac{u-1}{u}\,du \\
&= \frac{3}{4}\int 1-\frac{1}{u}\,du \\
&= \frac{3}{4}(u-\ln|u|)+C \\
&= \frac{3}{4}(2t+1-\ln|2t+1|)+C \\
&= \frac{3}{4}(2t-\ln|2t+1|)+K
\end{aligned}$$

$u = 2t+1;$

$t = \dfrac{u-1}{2}$

$du = 2dt;$

$dt = \dfrac{du}{2}$

The technique worked, because it was possible to rewrite the integrand entirely in terms of u, AND the resulting function of u was easier to integrate than the initial function of x.

♣ What was done in the last step of the previous integration?

EXERCISE 2

♣ Evaluate $\int \frac{2t}{3t-1}\,dt$ in two ways. First, use the 'role reversal' substitution technique. Second, use long division.

EXAMPLE

Problem: Find $\displaystyle\int \frac{3x}{(2x-1)^5}\,dx$.

Solution:

$$\begin{aligned}
\int \frac{3x}{(2x-1)^5}\,dx &= 3\int \frac{(u+1)/2}{u^5}\,\frac{du}{2} \\
&= \frac{3}{4}\int \frac{u+1}{u^5}\,du \\
&= \frac{3}{4}\int (u^{-4}+u^{-5})\,du \\
&= \frac{3}{4}\left(\frac{u^{-3}}{-3}+\frac{u^{-4}}{-4}\right)+C \\
&= \frac{3}{4}\left(-\frac{1}{3(2x-1)^3}-\frac{1}{4(2x-1)^4}\right)+C
\end{aligned}$$

$u = 2x-1;$

$x = \dfrac{u+1}{2}$

$du = 2dx;$

$dx = \dfrac{du}{2}$

EXAMPLE

Problem: Find $\int \frac{x}{2\sqrt{3x-1}}\,dx$.

Solution:

$$\begin{aligned}\int \frac{x}{2\sqrt{3x-1}}\,dx &= \frac{1}{2}\int \frac{u+1}{3}\cdot\frac{1}{\sqrt{u}}\,\frac{du}{3}\\ &= \frac{1}{18}\int \frac{u+1}{u^{1/2}}\,du\\ &= \frac{1}{18}\int (u^{1/2}+u^{-1/2})\,du\\ &= \frac{1}{18}(\frac{2}{3}u^{3/2}+2u^{1/2})+C\\ &= \frac{1}{18}\big(\frac{2}{3}(3x-1)^{3/2}+2(3x-1)^{1/2}\big)+C\\ &= \frac{1}{27}\sqrt{(3x-1)^3}+\frac{1}{9}\sqrt{3x-1}+C\end{aligned}$$

$u = 3x-1$;
$x = \frac{u+1}{3}$

$du = 3dx$;
$dx = \frac{du}{3}$

EXERCISE 3 Solve the following integration problems. Use any appropriate techniques.

♣ 1. $\int t(t+1)^7\,dt$

♣ 2. $\int \frac{5x}{\sqrt{(3-2x)^3}}\,dx$

♣ 3. $\int u\sqrt{u^2+1}\,du$

rationalizing substitutions

Remember that to 'rationalize' means to 'get rid of the radical'. Sometimes, an appropriate substitution can be used to get rid of a radical, and transform a difficult problem into a more manageable one. The technique is illustrated in the next example.

EXAMPLE

a rationalizing substitution

Problem: Find $\int \frac{1}{1+\sqrt{x}}\,dx$.

Solution: To *rationalize the integrand*, let $u=\sqrt{x}$, so that $u^2=x$. Remember that u is a function of x, and differentiate both sides of $u^2=x$ with respect to x, getting:

$$2u\frac{du}{dx}=1$$

Thus,

$$2u\,du = dx\ .$$

Now, transforming to an integral in u yields:

$$\begin{aligned}\int \frac{1}{1+\sqrt{x}}\,dx &= \int \frac{1}{1+u}\,(2u\,du)\\ &= 2\int \frac{u}{1+u}\,du\end{aligned}$$

$u = \sqrt{x}$;
$u^2 = x$

$2u\frac{du}{dx} = 1$;

$2u\,du = dx$

At this point, the previous *reversal of roles* procedure can be used:

$w = 1 + u;$
$u = w - 1$
$dw = du$

$$
\begin{aligned}
2\int \frac{u}{1+u}\,du &= 2\int \frac{w-1}{w}\,dw \\
&= 2\int 1 - \frac{1}{w}\,dw \\
&= 2(w - \ln|w|) + C \\
&= 2\big((1+u) - \ln|1+u|\big) + C \\
&= 2u - 2\ln|1+u| + K \\
&= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + K
\end{aligned}
$$

Remember that since we started with an integration problem involving x, it was necessary to end up with the antiderivatives in terms of x.

EXERCISE 4

♣ 1. Re-do the previous problem, without looking at the text.

♣ 2. Check that: $\dfrac{d}{dx}(2\sqrt{x} - 2\ln|1+\sqrt{x}|) = \dfrac{1}{1+\sqrt{x}}$

EXERCISE 5

♣ Solve the integral $\int \frac{x}{\sqrt{x-1}}\,dx$ in two ways. First, let $u = x-1$ and make a 'role reversal'. Second, let $u = \sqrt{x-1}$, so that $u^2 = x-1$, and make a rationalizing substitution. Compare your answers. Which way do you think was easier?

tables of integrals

In closing, it must be remarked that there are extensive tables of integrals available. One such compilation is:

Tables of Integrals and other Mathematical Data
Herbert Bristol Dwight, third edition
The MacMillan Company, New York, 1957

(This was my Dad's, so it is very special to me! There are obviously newer books available.)

To use such tables, one identifies the *form* of the integrand, finds a corresponding form in the table, and applies the formula.

For example, suppose one must integrate:

$$\int \frac{1}{x(1+3x^7)}\,dx$$

One finds the following entry in a table of integrals:

$$\int \frac{dx}{x(a+bx^m)} = \frac{1}{am}\log\left|\frac{x^m}{a+bx^m}\right|$$

Letting $a = 1$, $b = 3$, and $m = 7$, one applies the formula, getting:

$$\int \frac{1}{x(1+3x^7)}\,dx = \frac{1}{7}\log\left|\frac{x^7}{1+3x^7}\right|$$

♣ Check!

QUICK QUIZ
sample questions

1. Which is harder, in general, differentiation or integration?
2. Find all the antiderivatives of $\frac{x}{2+x}$. Use any appropriate technique.
3. What tools are available to help with integration?

KEYWORDS
for this section

Reversal of roles substitution technique, a rationalizing substitution, tables of integrals.

END-OF-SECTION EXERCISES

♣ The purpose of these exercises is to provide you with additional practice using *all* the antidifferentiation techniques discussed thus far in this chapter. Be sure to write complete mathematical sentences.

1. $\displaystyle\int \frac{e^{2x}+1}{5}\,dx$
2. $\displaystyle\int xe^{(3x^2-1)}\,dx$
3. $\displaystyle\int \frac{t}{\sqrt[3]{4t^2-1}}\,dt$
4. $\displaystyle\int \frac{x}{2x-1}\,dx$
5. $\displaystyle\int x(x+1)^3(x-1)^3\,dx$
6. $\displaystyle\int \frac{2t-1}{t}\,dt$
7. $\displaystyle\int \frac{(\ln x)^3}{3x}\,dx$

6.6 Integration by Parts Formula

Introduction

An attentive reader may have noticed that we have not yet learned how to integrate $\ln x$. Indeed, the integral $\int \ln x\,dx$ is a *classic* example of an integral that requires the *integration by parts* formula, which is the topic of this section. First, a derivation.

derivation of the Integration By Parts formula

The integration by parts formula is an easy consequence of the product rule for differentiation. Suppose that u and v are differentiable functions of x. Then, the product uv is also differentiable, and:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x (and using the linearity of the integral) yields:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

the remaining integrals absorb the constant of integration

Look at the left-hand side of this equation, and answer the following (trick) question: do we know a function whose derivative with respect to x is $\frac{d}{dx}(uv)$? Of course! The function uv has derivative $\frac{d}{dx}(uv)$! So we can replace the left-hand side by $uv + C$ to obtain:

$$uv + C = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

This equation can be simplified considerably. First, observe that the indefinite integrals remaining on the right-hand side will generate their own constant of integration, so it is not necessary to include the constant C on the left-hand side.

Furthermore, the integrals $\int u\frac{dv}{dx}\,dx$ and $\int v\frac{du}{dx}\,dx$ can be replaced by the simpler notation $\int u\,dv$ and $\int v\,du$. Thus, we have:

$$uv = \int u\,dv + \int v\,du$$

The final result is rearranged slightly, by solving for $\int u\,dv$:

Integration by Parts formula

$$\int u\,dv = uv - \int v\,du$$

This formula is commonly referred to more simply as the 'parts formula'.

EXERCISE 1 ♣ Derive the integration by parts formula, without looking at the text.

using the Integration by Parts formula;

hopefully, the new integral is easier

The idea in using the integration by parts formula is a familiar one: take a difficult integration problem, and try to *transform* it into an easier problem. When using the integration by parts formula, one takes an integral of the form $\int u\,dv$ and rewrites it in the form $uv - \int v\,du$. The hope is that the 'new' integral $\int v\,du$ is easier than the original integral $\int u\,dv$.

The general scheme is outlined below, and then illustrated in the example that follows.

a general procedure for using the parts formula

- Suppose that $\int f(x)\,dx$ cannot be solved by either elementary formulas, or substitution. It is decided to try integration by parts.
- You must *choose* u and dv to *rewrite* the integral in the form $\int u\,dv$. There will often be several possible choices for u and dv; this is the part of the problem that requires some expertise.

  A general strategy for choosing a u and dv that 'work' is presented after the example.
- From u, obtain du by differentiation.
- From dv, obtain v by integration. *Any* antiderivative can be used—usually (but not always), the constant of integration C is chosen to be zero, to obtain the simplest antiderivative.
- At this point, all the ingredients are at hand to rewrite the integral using the parts formula:

  $$\int u\,dv = uv - \int v\,du$$

  Look at the new integral $\int v\,du$. The hope is that this *new* integral $\int v\,du$ is easier to handle than the original integral $\int u\,dv$.

EXAMPLE

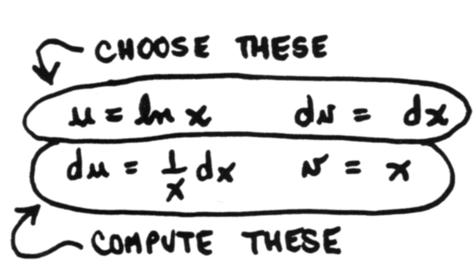
a classic; integrating $\int \ln x\,dx$

Problem: Find $\int \ln x\,dx$.

Solution: No previous technique seems to work here, so we are motivated to try the integration by parts formula. First, u and dv must be chosen to rewrite $\int \ln x\,dx$ in the form $\int u\,dv$.

The choices $u = \ln x$ and $dv = dx$ are made; following the example, the motivation for these choices is discussed.

Then:



$$\begin{aligned}\int \overbrace{\ln x}^{u}\,\overbrace{dx}^{dv} &= \overbrace{(\ln x)}^{u}(\overbrace{x}^{v}) - \int \overbrace{x}^{v}\cdot\overbrace{\frac{1}{x}\,dx}^{du}\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + C\end{aligned}$$

Check: $\dfrac{d}{dx}(x\ln x - x) = \big[x(\frac{1}{x}) + (\ln x)(1)\big] - 1 = 1 + \ln x - 1 = \ln x$

So, the result is correct.

a strategy for choosing u and dv

Here is a general strategy for choosing u and dv :

- The choice for dv must include dx. Also, since dv must be integrated to obtain v, *you must choose something for dv that you know how to integrate.* Sometimes, this consideration will completely determine the choice. (Observe that once dv is chosen, u must be everything that is left.)
- If there are several possible choices for dv, then *choose something for u that gets EASIER when you differentiate it.* This is motivated by the fact that du appears in the new integral: the simpler du is, the better.

In many problems, these two considerations will lead to a correct choice for u and dv. If not—experience, trial and error, and luck can all be factors in obtaining a correct choice for u and dv (if one exists).

return to the previous example; choosing u and dv

Reconsider the problem of finding $\int \ln x\, dx$. Here's how we arrived at the choices for u and dv :

- Choose something for dv that includes dx, and *that you know how to integrate.* We *can't* choose dv to be $\ln x\, dx$, since we don't know how to integrate this (that's the problem). So we are forced to choose $dv = dx$.
- Now, the choice for u is completely determined: u must equal everything else. Thus, $u = \ln x$.

EXAMPLE

choosing u and dv

Problem: Evaluate $\int xe^x\, dx$.

- There are several possible choices for dv here, since there are several 'pieces' that we know how to integrate. We could choose:

$$\begin{aligned} & dv = dx \\ \text{or}\quad & dv = x\, dx \\ \text{or}\quad & dv = e^x\, dx \end{aligned}$$

  Since this first consideration has not solved the 'choice' problem, we move on to the next consideration.
- Choose something for u that gets *simpler* when you differentiate it. If we choose $u = e^x$, then $\frac{du}{dx} = e^x$, which is no simpler. But if we choose $u = x$, then $\frac{du}{dx} = 1$, which is certainly simpler.
- Thus, choose $u = x$. Then dv must be everything else: $dv = e^x\, dx$. Here's how the problem is written down:

$$\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}}$$

$$\begin{aligned} \int x\, e^x\, dx &= (x)(e^x) - \int e^x\, dx \\ &= x\, e^x - e^x + C \end{aligned}$$

EXERCISE 2 ♣ Check that: $\frac{d}{dx}(xe^x - e^x) = xe^x$

In the following examples, use the strategy to see how we arrived at the choices for u and dv.

EXAMPLE

Problem: Evaluate $\int \frac{x}{e^x}\,dx$.

Solution:

$$
\begin{aligned}
\int \frac{x}{e^x}\,dx &= \int xe^{-x}\,dx \\
&= -xe^{-x} - \int (-e^{-x})\,dx \\
&= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C \\
&= -e^{-x}(x+1) + C
\end{aligned}
$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

EXERCISE 3

♣ 1. In applying the parts formula to $\int xe^{-x}\,dx$, list three possible choices for dv.

♣ 2. Corresponding to each choice for dv, what would u have to be? In which case is $\frac{du}{dx}$ simpler than u?

EXAMPLE

Problem: Evaluate $\int x^2 \ln x\,dx$.

Solution: One could choose either $dv = dx$ or $dv = x^2\,dx$, since both of these pieces can be integrated with prior techniques. If $dv = dx$ is chosen, then u must be $x^2 \ln x$, which gets much more complicated when differentiated. If $dv = x^2\,dx$ is chosen, then u must be $\ln x$, with the relatively simply derivative $\frac{1}{x}$. Thus, it is decided to initially try $dv = x^2\,dx$:

$$
\begin{aligned}
\int x^2 \ln x\,dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}(\frac{1}{x})\,dx \\
&= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \\
&= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C \\
&= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C
\end{aligned}
$$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^3}{3}$

EXERCISE 4

Use the parts formula to evaluate the following integrals. Use the 'strategy' to decide on your choices for u and dv.

♣ 1. $\int x \ln x\,dx$

♣ 2. $\int xe^{3x}\,dx$

♣ 3. $\int x^3 \ln x\,dx$

♣ 4. $\int \ln 3x\,dx$

a problem that's easier if a nonzero constant of integration is chosen when finding v

Problem: Evaluate $\int \ln(x+3)\,dx$.

Solution: We must choose $dv = dx$ and hence $u = \ln(x+3)$. If the 'traditional' approach is taken, where the constant of integration is chosen to be 0 when going from dv to v, then here's what happens:

$$\int \ln(x+3)\,dx = x\ln(x+3) - \int x \cdot \frac{1}{x+3}\,dx$$

$$\boxed{\begin{array}{ll} u = \ln(x+3) & dv = dx \\ du = \frac{1}{x+3}\,dx & v = x \end{array}}$$

This is fine, except that to solve the resulting integral $\int \frac{x}{x+3}\,dx$, either a 'role-reversing' substitution or long division is required. However, if we're a bit clever, this can be avoided:

$$\begin{aligned} \int \ln(x+3)\,dx &= (x+3)\ln(x+3) - \int (x+3)\frac{1}{x+3}\,dx \\ &= (x+3)\ln(x+3) - \int (1)\,dx \\ &= (x+3)\ln(x+3) - x + C \end{aligned}$$

$$\boxed{\begin{array}{ll} u = \ln(x+3) & dv = dx \\ du = \frac{1}{x+3}\,dx & v = x+3 \end{array}}$$

In obtaining v, we merely need a function whose derivative with respect to x is 1 ($dv = dx \iff \frac{dv}{dx} = 1$). Usually, we use $v = x$, because it's simplest. Here, however, it was certainly to our advantage to choose a nonzero constant of integration.

EXERCISE 5

♣ 1. Check that: $\frac{d}{dx}[(x+3)\ln(x+3) - x] = \ln(x+3)$

♣ 2. Find all the antiderivatives of $3\ln(x+1)$.

♣ 3. Evaluate $\int \ln(t - \frac{1}{2})\,dt$.

EXAMPLE
repeated parts

Problem: Evaluate $\int x^2 e^x \, dx$.

Solution:

$u = x^2 \qquad dv = e^x dx$
$du = 2x dx \qquad v = e^x$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx \end{aligned}$$

$u = x \qquad dv = e^x dx$
$du = dx \qquad v = e^x$

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x + C \end{aligned}$$

Combining results:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= e^x (x^2 - 2x + 2) + C \end{aligned}$$

Check:

$$\begin{aligned} \frac{d}{dx}\bigl(e^x (x^2 - 2x + 2)\bigr) &= e^x (2x - 2) + e^x (x^2 - 2x + 2) \\ &= e^x (2x - 2 + x^2 - 2x + 2) \\ &= x^2 e^x \end{aligned}$$

After the first application of parts, it was noted that the resulting 'new' integral $\int x e^x \, dx$ was easier than the one started with: the power of x was knocked down by one. Thus, we were motivated to repeat the process.

It's very important to write things down neatly and carefully!

EXERCISE 6

♣ 1. Re-do the previous example without looking at the text.

♣ 2. Evaluate $\int x^2 e^{3x} \, dx$. Be sure to write a complete mathematical sentence.

EXERCISE 7

Evaluate the integral $\int \frac{x}{(1+x)^6} \, dx$ in two ways:

♣ 1. First, use an appropriate 'role-reversal' substitution. Differentiate to verify that you have a correct solution.

♣ 2. Second, use parts with $u = x$ and a corresponding dv. Differentiate to verify that you have a correct solution.

♣ 3. The answers obtained from the two different approaches probably look a bit different. However, they must differ by at most a constant. Express each answer as a fraction with the same denominator, so that you can better compare them.

The antidifferentiation tools studied in this chapter are summarized next for your convenience:

ANTIDIFFERENTIATION TOOLS

$F'(x) = f(x)$	F is an *antiderivative* of f
$\int f(x)\,dx$	*all* antiderivatives of f
$\int f'(x)\,dx = f(x) + C$	all antiderivatives differ by a constant
$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$	the integral of a sum is the sum of the integrals
$\int kf(x)\,dx = k\int f(x)\,dx$	constants can be 'slid out' of the integral
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	Simple Power Rule for integration, $n \ne -1$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	integrating $\frac{1}{x}$
$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$	integrating e^{kx}
$\int f'(u)\frac{du}{dx}\,dx = f(u) + C,\ \ u$ a function of x	substitution technique
$\int u\,dv = uv - \int v\,du$	integration by parts formula

QUICK QUIZ
sample questions

1. What is the *Integration By Parts* formula? Where does it come from?
2. Evaluate $\int \ln 2t\,dt$.
3. Evaluate $\int \ln(x-1)\,dx$.
4. What must you think of when choosing dv for use in the Parts formula?

KEYWORDS
for this section

Integration by Parts formula, derivation of the parts formula, a strategy for choosing u and dv, choosing a nonzero constant when obtaining v, repeated parts.

END-OF-SECTION EXERCISES

The purpose of these exercises is to provide you with additional practice using *all* the antidifferentiation techniques discussed thus far in this chapter. Be sure to write complete mathematical sentences. Properties of exponents and logarithms may be needed to rewrite the integrand before integrating.

1. $\displaystyle\int (e^x - 1)^2\,dx$
2. $\displaystyle\int \frac{\ln(x^2 + 2x + 1)}{x+1}\,dx$
3. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$
4. $\displaystyle\int \ln\frac{1+x}{x}\,dx$
5. $\displaystyle\int \sqrt{\frac{e^t}{2}}\,dt$
6. $\displaystyle\int \frac{x}{\sqrt{x^4 \ln x}}\,dx$

NAME

SAMPLE TEST, worth 100 points, Chapter 6

Show all work that leads to your answers. Good luck!

8 pts

TRUE or FALSE. Circle the correct response. (2 points each)

T F $F(x) = x \ln x - x + 2$ is an antiderivative of $f(x) = \ln x$.

T F If $f'(x) = g'(x)$ for all $x \in \mathbb{R}$, then f and g differ by at most a constant.

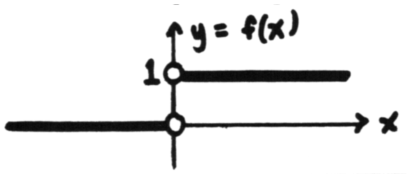
T F $\int t^2 \, dt = t^3 + C$

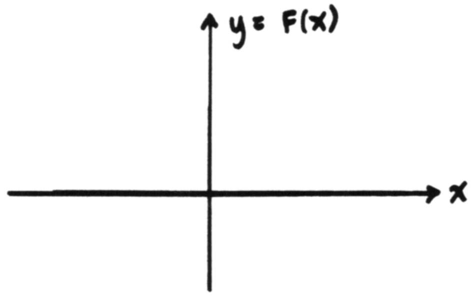
T F An antiderivative of f is a function with derivative f.

6 pts

The graph of a function f is given below. In the space provided, graph a function F satisfying:

- $F'(x) = f(x) \ \ \forall \ x \in \mathcal{D}(f)$;
- F is continuous at 0 ; and
- $F(-1) = 2$.





13 pts

(6 pts) These questions have to do with the indefinite integral $\int f(x)\,dx$.

Fill in the blanks:

The symbol $\int$ is called the ______________________.

The function f being integrated is called the ______________________.

The process of finding $\int f(x)\,dx$ is called ______________________.

(There are two possible correct answers here.)

(2 pts) Rewrite the integral $\int x^2 \, dx$ using a different dummy variable.

(5 pts) What is meant by the phrase ‘the linearity of the integral’?

6 pts

Classify each entry as an EXPRESSION or a SENTENCE.

If a *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

(2 pts) a) $\int f(t)\,dt + \int g(t)\,dt$

(2 pts) b) $\int x\,dx = \frac{1}{2} \int 2x\,dx$

(2 pts) c) $f(1) = 2$

32 pts

Evaluate the following indefinite integrals. Be sure to write complete mathematical sentences. Use any appropriate methods.

(8 pts) a) $\int (\frac{2}{x} + e^{3x} - 1)\,dx$

(8 pts) b) $\int \ln(x-1)\,dx$

(8 pts) c) $\int \frac{t}{\sqrt[3]{t^2-1}}\,dt$

(8 pts) d) $\int \frac{3x}{2x+1}\,dx$

6 pts

Find a function g satisfying $g'(x) = \frac{1}{\sqrt{2x-1}}$ and $g(1) = 2$.

4 pts

Give an antidifferentiation 'counterpart' to the differentiation formula:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

9 pts

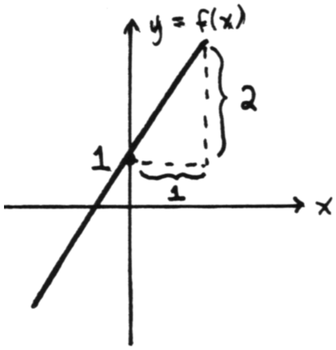
(5 pts) Fill in a reason for each step below:

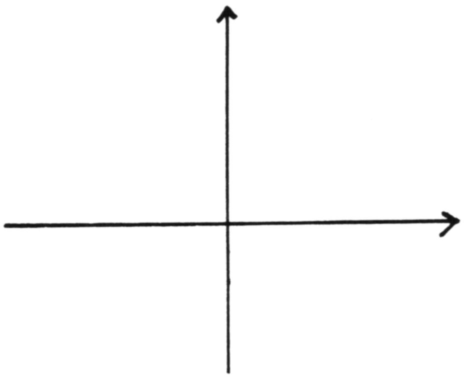
$$\begin{aligned}
\int (2x+1)^3\,dx &= \int (2x+1)^3 \cdot \frac{2}{2}\,dx && (\qquad\qquad\qquad\qquad) \\
&= \frac{1}{2}\int (2x+1)^3 2\,dx && (\qquad\qquad\qquad\qquad) \\
&= \frac{1}{2}\int u^3\,du && (\qquad\qquad\qquad\qquad) \\
&= \frac{1}{2}\cdot\frac{u^4}{4} + C && (\qquad\qquad\qquad\qquad) \\
&= \frac{1}{8}(2x+1)^4 + C && (\qquad\qquad\qquad\qquad)
\end{aligned}$$

(4 pts) Now, CHECK this antidifferentiation problem.

8 pts

A function f is graphed below. Find all the antiderivatives of f. Graph two of these antiderivatives in the space provided.





8 pts

A function f is graphed below. On the same graph, graph another function that has the same derivative as f, and has a nonremovable discontinuity at $x = 1$.

This page is intentionally left blank,
to keep odd-numbered pages
on the right.

Use this space to write
some notes to yourself!

CHAPTER 7

THE DEFINITE INTEGRAL

In the previous chapter, the *indefinite integral* $\int f(x)\,dx$ was studied. This integral gives all the antiderivatives of the function f.

In this chapter, another type of integral is studied, called the *definite integral of f on* $[a,b]$, and denoted by $\int_a^b f(x)\,dx$. Under suitable conditions, $\int_a^b f(x)\,dx$ gives information about the area trapped between the graph of f and the x-axis over the interval $[a,b]$.

The integrals $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are, in one sense, very different: $\int f(x)\,dx$ is a class of *functions*, (all the antiderivatives of f), but $\int_a^b f(x)\,dx$ is a *number*. However, in another sense, the integrals are very much related: the Fundamental Theorem of Integral Calculus tells us that if we know just one antiderivative of f, then we can compute the number $\int_a^b f(x)\,dx$.

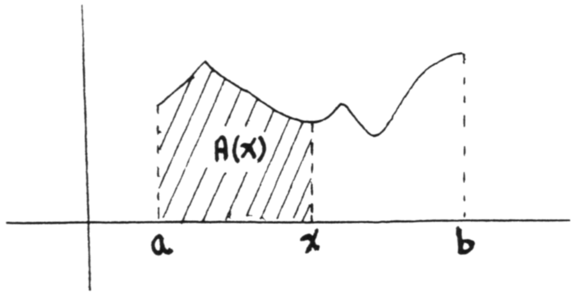
7.1 Using Antiderivatives to find Area

Introduction

In this section a formula is obtained for finding the area of the region bounded between the graph of a *continuous, nonnegative* function f and the x-axis. As mentioned in the previous chapter, it is seen that the antiderivatives of f play a *crucial* role in this process.

finding the area under the graph of a nonnegative, continuous function f

Let f be a function that is continuous on $[a, b]$. Also suppose that f is nonnegative, so that its graph lies on or above the x-axis. In this case, it makes sense to talk about the area under the graph of f; we seek the area between $x = a$ and $x = b$.



the area function; $A(x)$

First, define:

$$A(x) := \text{the area under the graph of } f\text{, from } a \text{ to } x$$

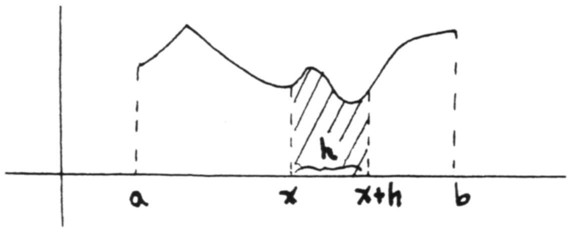
Observe that $A(a) = 0$, and $A(b)$ is the area being sought.

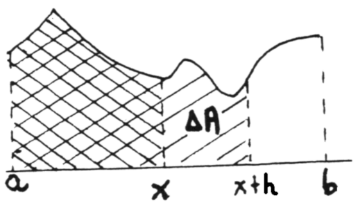
in the pictures, h is positive

Now, let x be a number between a and b, and let h be a small *positive* number. In the exercises accompanying this section, you will consider the case where h is a small *negative* number.

ΔA; a little piece of area

Focus attention on the little piece of area between x and $x+h$, as shown below.





This area can be obtained as follows: take the area under the graph from a to $x + h$, and subtract off the area from a to x. What's left is the area under the graph between x and $x + h$, as shown.

Thus, this little piece of area can be written in terms of the area function A as:

$$\Delta A := A(x + h) - A(x)$$

The symbol ΔA is read as '*delta A*' and denotes a *change in A*.

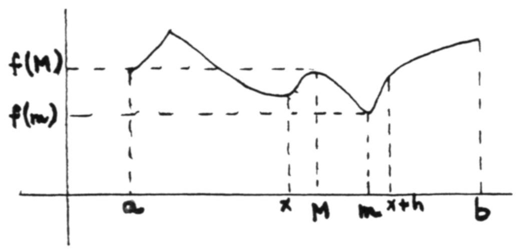
EXERCISE 1

♣ 1. If h is a small negative number, where is $x + h$ in relation to x?

♣ 2. Make a sketch showing x and $x + h$. What is the correct formula for ΔA in this case?

using the Max-Min Theorem

By hypothesis, f is continuous on the entire interval $[a, b]$, so it is also continuous on the subinterval $[x, x+h]$. Therefore, the Max-Min Theorem guarantees that f attains a minimum value $f(m)$ and a maximum value $f(M)$ on $[x, x+h]$, as illustrated below. Observe that both m and M come from the interval $[x, x+h]$.



under- and over-approximating the area with rectangles

The actual area ΔA of the little piece under inspection is under-approximated by the rectangle of height $f(m)$ and width h. Also, ΔA is over-approximated by the rectangle of height $f(M)$ and width h. That is:

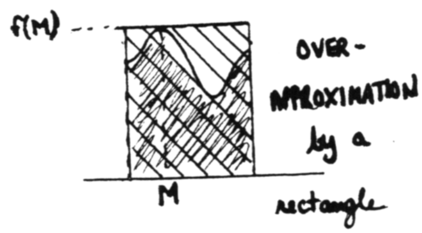
$$f(m) \cdot h \le \Delta A \le f(M) \cdot h$$

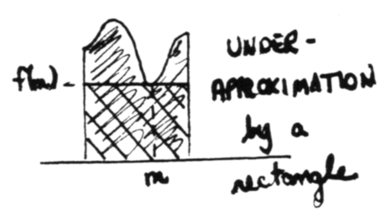
Division by the positive number h yields

$$f(m) \le \frac{\Delta A}{h} \le f(M) \ ,$$

and substituting in the definition of ΔA yields:

$$f(m) \le \frac{A(x+h) - A(x)}{h} \le f(M)$$





Be aware!

The numbers m and M depend on: the function f the number x the number h

What is about to be said applies to *both* m and M. For simplicity, it is stated only for m.

It's important that you understand that the number m depends on:

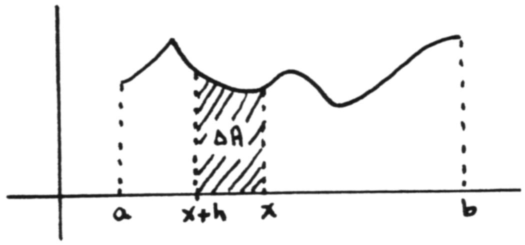
- the function (f) that you're working with
- the small interval $[x, x+h]$ that is currently under investigation; this interval depends on both x and h

Change any of these (f , x , or h) and the number m could change!

For this reason, a name like '$m_{f,x,h}$' (with three subscripts) might be better than just m. But then the notation would be so cumbersome that it could make things appear harder than they really are! So, we'll stick with just 'm'.

EXERCISE 2

You should have discovered in the previous exercise that if $h < 0$, then $\Delta A = A(x) - A(x+h)$, and the picture becomes the one shown below:



♣ 1. Why is the area of the under-approximating rectangle given by the formula $f(m) \cdot (-h)$ in this case?

♣ 2. What is the formula for the area of the over-approximating rectangle?

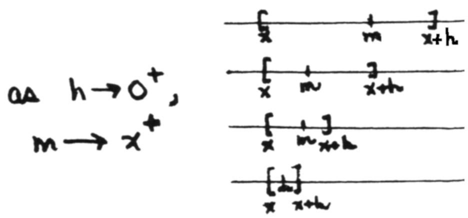
♣ 3. Provide a justification for each step in the mathematical sentence below. Remember that $h < 0$, and $\Delta A = A(x) - A(x+h)$.

$$\begin{aligned} f(m)(-h) \le \Delta A \le f(M)(-h) \quad &\iff \quad f(m) \le \frac{\Delta A}{-h} \le f(M) \\ &\iff \quad f(m) \le \frac{A(x) - A(x+h)}{-h} \le f(M) \\ &\iff \quad f(m) \le \frac{A(x+h) - A(x)}{h} \le f(M) \end{aligned}$$

Thus, precisely the same inequality is obtained as when h is positive.

let h approach 0; then m must approach x

Now let h approach 0 (from the right-hand side, since h is positive). Remember that m is trapped in the interval $[x, x+h]$, so as h approaches zero, m is forced to get close to x. That is, as $h \to 0^+$, it must be that $m \to x^+$.



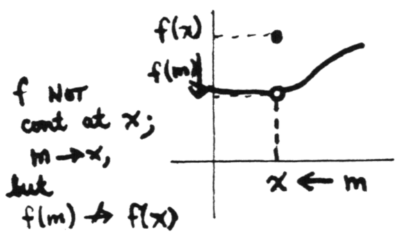
Note: Here, we're holding x fixed and letting h change. Since h is changing, m can change! The same label, 'h', is used in all four sketches above, even though h is getting smaller. The same label, 'm', is used, even though it can change. This can be confusing—same labels, different numbers—so be aware!

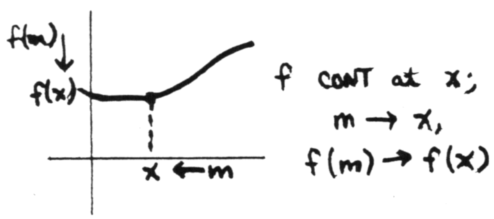
EXERCISE 3

♣ Rewrite the previous paragraph so that it applies when $h < 0$.

using the continuity of f

By hypothesis, f is continuous at x. Therefore, when the inputs are close to x, the corresponding outputs must be close to $f(x)$. In particular, when m is close to x, $f(m)$ must be close to $f(x)$. More precisely, as $m \to x^+$, we must have $f(m) \to f(x)$.





as h approaches 0, both m and M must get close to x

Similarly, since M is trapped between x and $x+h$, as h approaches 0, M must approach x. And as M gets close to x, the continuity of f at x tells us that $f(M)$ approaches $f(x)$.

the quotient $\frac{A(x+h)-A(x)}{h}$ is pinched between numbers that are both going to $f(x)$

Reconsider the previous inequality in light of our new information:

$$\cancelto{f(x)}{f(m)} \le \frac{A(x+h)-A(x)}{h} \le \cancelto{f(x)}{f(M)}$$

As h approaches 0 (from the right-hand side), both $f(m)$ and $f(M)$ are approaching $f(x)$. So the quotient

$$\frac{A(x+h)-A(x)}{h}$$

is pinched between numbers which are *both* going to the *same number*, $f(x)$! Therefore, $\frac{A(x+h)-A(x)}{h}$ must also be getting close to $f(x)$! (This observation is sometimes formalized in a result called the *Pinching Theorem for Limits.*) That is, it must be that:

$$\lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} = f(x)$$

EXERCISE 4

♣ Rewrite the necessary paragraphs, and conclude that:

$$\lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} = f(x)$$

the limit $\displaystyle\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$ exists

Now it is known that

$$\lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} = f(x)\ ;$$

and, if you've been doing the exercises, it is also known that:

$$\lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} = f(x)$$

Putting these two pieces of information together, we conclude that the two-sided limit exists and equals $f(x)$:

$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)$$

But when the limit

$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$

exists, it is given a special name: $A'(x)$! So it is now known that:

$$A'(x) = f(x)$$

That is, the area function A is a function which, when differentiated, yields f. That is, A is an antiderivative of f.

$A(x)$ is an antiderivative of $f(x)$

The fact just discovered is so important that it is worth repeating. *The area function A is an antiderivative of f.* In particular, it has been shown that whenever f is continuous and nonnegative on $[a,b]$, *an antiderivative of f always exists*! This is an extremely beautiful and important result.

now we know what **all** *the antiderivatives look like*

Getting our hands on *one* antiderivative is always the hard part; now we know what *all* the antiderivatives of f must look like—they must differ from A by at most a constant. That is, if F denotes *any* antiderivative of A, then:

$$A(x) = F(x) + C \qquad (*)$$

solving for the constant C

Remember that we want to find $A(b)$, since this represents the area under the graph of f between a and b. Using the fact that $A(a) = 0$, equation (*) yields

$$0 = A(a) = F(a) + C$$

so that $C = -F(a)$. Then (*) can be rewritten as:

$$A(x) = F(x) - F(a)$$

Now, letting x equal b, we obtain:

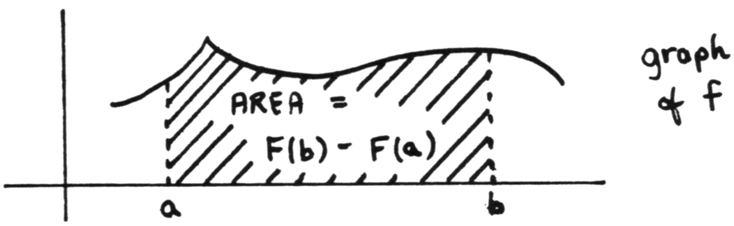
$$\text{desired area} = A(b) = F(b) - F(a)$$

This is the formula for the desired area, given in terms of *any* antiderivative of f. The result is summarized below.

formula for the area beneath the graph of a nonnegative, continuous function f on $[a,b]$

Let f be nonnegative and continuous on the interval $[a,b]$. Let F be *any* antiderivative of f on $[a,b]$. Then:

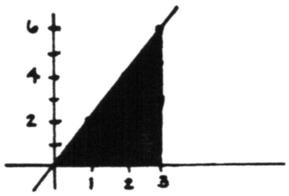
$$\text{the area under the graph of } f \text{ on } [a,b] \; = \; F(b) - F(a)$$



EXAMPLE

testing the formula in a case where the answer is already known

It's always a good idea to test a new result in a situation where you can find the answer by alternate means. So let's find the area under the graph of $f(x) = 2x$ between $x = 0$ and $x = 3$.



Calculus is certainly not needed, since the area is just a triangle:

$$\frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}(3)(6) = 9$$

Now, use the formula. An antiderivative of $f(x) = 2x$ is needed; the easiest one is $F(x) = x^2$. Then,

$$F(b) - F(a) = F(3) - F(0) = 3^2 - 0 = 9\ ,$$

which agrees with the first result.

EXERCISE 5

♣ 1. Show that $F(x) = x^2 + 7$ is an antiderivative of $f(x) = 2x$.

♣ 2. Find the area discussed in the previous example, using the antiderivative $F(x) = x^2 + 7$. What happens to the '7'?

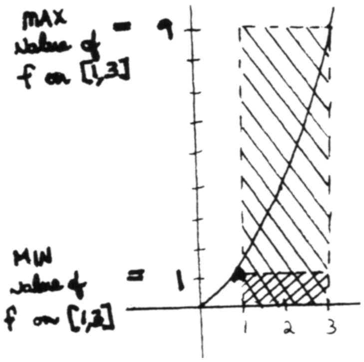
EXERCISE 6

Find the area under the graph of $f(x) = 2x$ between $x = 1$ and $x = 4$ in two ways:

♣ 1. Show that the desired area is a trapezoid; find the area of this trapezoid.

♣ 2. Use an antiderivative of f to find the area.

EXAMPLE



Problem: Find the area beneath the graph of $f(x) = x^2$ on $[1, 3]$.

Solution: Here, the area of the region is *not* easily obtainable from geometry. However, we can get some rough bounds on the desired area, as follows.

The minimum value of f on $[1, 3]$ is $1^2 = 1$. Thus, the desired area is under-approximated by a rectangle of width $3 - 1 = 2$ and height 1.

The maximum value of f on $[1, 3]$ is $3^2 = 9$. Thus, the desired area is over-approximated by a rectangle of width 2 and height 9. Together:

$$(1)(2) \le \text{actual area} \le (9)(2)$$

The actual area must lie between 2 and 18. Also, from the sketch, we expect the actual area to be near the middle of this range of numbers.

applying the formula

Now apply the formula. We need any antiderivative of $f(x) = x^2$; take $F(x) = \frac{x^3}{3}$, since it's the simplest one. Then:

$$F(b) - F(a) = F(3) - F(1) = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = 8\frac{2}{3}$$

The answer is certainly believable, based on the earlier estimates.

EXERCISE 7

♣ 1. Consider the function $f(x) = x^2$ on the interval $[2, 5]$. As in the previous example, get an under-approximation and an over-approximation of the area under f on $[2, 5]$.

♣ 2. Find the area, using an antiderivative of f.

♣ 3. Find the area, using a different antiderivative of f.

EXERCISE 8

♣ Use calculus to find the area under the graph of $f(x) = x^2$ on $[-2, -1]$. Here, $[a, b] = [-2, -1]$, so $a = -2$ and $b = -1$. Make a sketch of the graph of f, and the area that you are finding.

EXERCISE 9

♣ 1. Graph $f(x) = -x^2$. Show the area trapped between the graph of f and the x-axis on $[1, 3]$.

♣ 2. Using any antiderivative F of f, compute $F(3) - F(1)$. How does your answer compare to the area under the graph of $f(x) = x^2$ on $[1, 3]$?

♣ 3. Make a conjecture, based on this example.

QUICK QUIZ
sample questions

1. Suppose $h > 0$, and f is continuous on the interval $[x, x + h]$. What does the Max-Min Theorem guarantee?
2. Under what condition(s) does a function f have the property that as $x \to a$, $f(x) \to f(a)$?
3. Make a sketch that illustrates a function f, and $a \in \mathcal{D}(f)$, for which $f(x) \not\to f(a)$ as $x \to a$.
4. Find the area under the graph of $y = 3x^2$ on the interval $[0, 2]$.
5. Suppose f is continuous and nonnegative on $[c, d]$, and F is an antiderivative of f. Give a formula for the area under the graph of f on $[c, d]$.

KEYWORDS
for this section

Finding the area under the graph of a continuous, nonnegative function f on the interval $[a, b]$; a formula for this area in terms of any antiderivative F of f.

END-OF-SECTION EXERCISES

In each problem below, an area is described.

♣ Sketch the area that is described.

♣ Approximate the area in any reasonable way.

♣ Use calculus to find the area.

1. area bounded between the graph of $y = \ln x$ and the x-axis on the interval $[1, e]$
2. area under the graph of $y = \frac{1}{t}$ on $[1, 2]$
3. area bounded by the graph of $y = \sqrt{x}$, the x-axis, the line $x = 1$, and the line $x = 4$
4. area bounded by the graph of $y = x^2 + 1$, the line $y = 1$, the y-axis, and the line $x = 1$

7.2 The Definite Integral

the definite integral

In the previous section, it was found that if a function f is continuous and nonnegative, then the area under the graph of f on $[a, b]$ is given by $F(b) - F(a)$, where F is any antiderivative of f.

This result is usually expressed in terms of an integral, called *the definite integral of f on $[a, b]$*, and denoted by:

$$\int_a^b f(x)\,dx$$

(Read $\int_a^b f(x)\,dx$ as '*the definite integral of f, from a to b*'.)

In this section, study of the definite integral begins.

the actual definition of $\int_a^b f(x)\,dx$ is a bit complicated

The actual *definition* of the definite integral $\int_a^b f(x)\,dx$ is a bit complicated, due mainly to the fact that (not surprisingly!) it is defined in terms of a *limit.* The precise definition of $\int_a^b f(x)\,dx$ is presented in the next section. This definition reveals the following facts (which you can take on faith for the moment, and start understanding *now*):

For a continuous function f,

- $\int_a^b f(x)\,dx$ is a NUMBER; and
- if f happens to be *nonnegative* on $[a, b]$, then this number $\int_a^b f(x)\,dx$ has a very nice interpretation; it gives the area under the graph of f on $[a, b]$. Since $F(b) - F(a)$ gives this same area, where F is any antiderivative of f, we can in this case write:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

lower limit of integration;

upper limit of integration

The definite integral

$$\int_a^b f(x)\,dx$$

and the indefinite integral

$$\int f(x)\,dx$$

have similar appearances. The only difference is that the definite integral has numbers a and b adorning the integral sign. These two new components have names:

- a is called the *lower limit of integration*
- b is called the *upper limit of integration*

comparing the definite and indefinite integrals

Since the appearance of the two integrals is so similar, you should be asking yourself the following questions: Why the similar appearance? How are these integrals the same? How are they different? Here are some answers to these questions.

How are the integrals different?

The definite integral $\int_a^b f(x)\,dx$ is a NUMBER. If f is nonnegative on $[a,b]$, then this number has a nice interpretation as the area under the graph of f on $[a,b]$. However, the indefinite integral $\int f(x)\,dx$ is an INFINITE CLASS OF FUNCTIONS; all the antiderivatives of f. So, in one sense, the two integrals are very, very different.

How are the integrals the same?

In another sense, however, they are very much the same. It will be seen that if JUST ONE antiderivative of f is known, then the definite integral can be computed. This fact has already been established in a special case—when f is continuous and nonnegative on $[a,b]$—and we will see that it actually holds for *any* continuous function f. This is precisely the content of the Fundamental Theorem of Integral Calculus (to be presented momentarily); and is the justification for the similarity in the appearance of the two integrals.

EXERCISE 1

TRUE or FALSE:

♣ 1. The definite integral $\int_a^b f(x)\,dx$ is a function.

♣ 2. The number $\int_1^5 x^2\,dx$ gives the area under the graph of x^2 on the interval $[1,5]$.

♣ 3. For a continuous function f, $\int_a^b f(x)\,dx = \int f(x)\,dx$.

♣ 4. This text has not yet presented the actual definition of $\int_a^b f(x)\,dx$.

We've run across a situation before where a (precise) definition was hard to work with, and—fortunately—we could often get away with NOT working with the definition. Remember the *definition* of the derivative of a function?

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Evaluating this limit was a nuisance, even for fairly simple functions f. Fortunately, this definition rarely needs to be used any more, because the definition was USED to develop tools that allow us to thereafter BYPASS the definition; tools such as the simple power rule, chain rule, product rule, and quotient rule. Similarly, the Fundamental Theorem of Integral Calculus gives a convenient tool for computing the definite integral of f, whenever we can get our hands on an antiderivative of f. Here's a precise statement of the fundamental theorem:

Fundamental Theorem of Integral Calculus

Let f be continuous on $[a,b]$. If F is *any* antiderivative for f on $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

★★ *antiderivative of f on $[a,b]$*

For F to be an antiderivative of f on $[a,b]$, not only must $F'(x) = f(x)$ for all $x \in (a,b)$, but F must also be continuous on the closed interval $[a,b]$. In particular, F must 'behave properly' at the endpoints a and b.

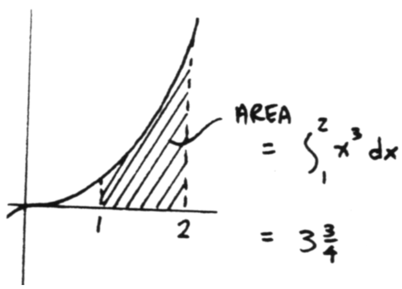
some notation used in connection with the definite integral; $F(x)\big|_a^b$

Let F be an antiderivative of f. The following notation is used in connection with evaluating the definite integral:

$$\begin{aligned}\int_a^b f(x)\,dx &= F(x)\big|_a^b \\ &= F(b) - F(a)\end{aligned}$$

That is, the notation $F(x)\big|_a^b$ is used to represent the operation of evaluating the antiderivative F at b, evaluating it at a, and then subtracting these two numbers, as illustrated in the next example.

EXAMPLE



Problem: Compute the definite integral:

$$\int_1^2 x^3\,dx$$

Solution: Find an antiderivative of x^3, and use the Fundamental Theorem.

The simplest antiderivative of $f(x) = x^3$ is $F(x) = \frac{x^4}{4}$. Using this antiderivative to evaluate the definite integral yields:

$$\begin{aligned}\int_1^2 x^3\,dx &= \frac{x^4}{4}\Big|_1^2 \\ &= \frac{(2)^4}{\mathbf{4}} - \frac{(1)^4}{\mathbf{4}} \\ &= 4 - \frac{1}{4} = 3\frac{3}{4}\end{aligned}$$

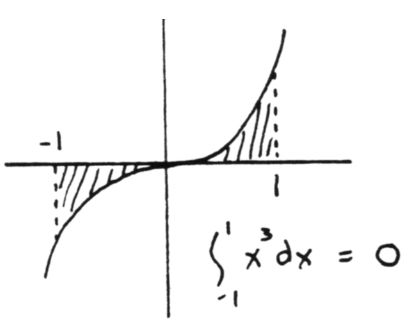
Since x^3 is positive on $[1, 2]$, the number $3\frac{3}{4}$ gives the area under the graph of f on $[1, 2]$.

factor constants out first

Note that the constant $\frac{1}{4}$ appears in **both terms** (shown bold above) in the evaluation process. It is usually easiest to factor this constant out first, and more simply write:

$$\begin{aligned}\int_1^2 x^3\,dx &= \frac{x^4}{4}\Big|_1^2 = \frac{1}{4}(2^4 - 1^4) \\ &= \frac{1}{4}(16 - 1) = \frac{15}{4} = 3\frac{3}{4}\end{aligned}$$

EXAMPLE



Problem: Compute the definite integral:

$$\int_{-1}^1 x^3\,dx$$

Solution: Find an antiderivative of x^3, and use the Fundamental Theorem.

Observe that, this time, x^3 is NOT positive over the entire interval of integration. Applying the fundamental theorem:

$$\begin{aligned}\int_{-1}^1 x^3\,dx &= \frac{x^4}{4}\Big|_{-1}^1 = \frac{1}{4}(1^4 - (-1)^4) \\ &= \frac{1}{4}(1 - 1) = \frac{1}{4}(0) = 0\end{aligned}$$

Momentarily, it will be made clear why the answer is 0. ♣ Any speculation?

EXERCISE 2

Evaluate the following definite integrals. Be sure to write complete mathematical sentences. When possible, interpret your answer in terms of area.

♣ 1. $\int_0^1 x^5\,dx$

♣ 2. $\int_0^4 e^x\,dx$

♣ 3. $\int_{-4}^0 e^x\,dx$

♣ 4. $\int_1^2 \frac{1}{x}\,dx$

♣ 5. $\int_{1/2}^2 \frac{1}{x}\,dx$

dummy variable of integration

Since the definite integral is a *number*, the variable of integration is irrelevant. That is, once $\int_a^b f(x)\,dx$ is evaluated, the letter 'x' is gone. Any letter may be used; for example, one can write

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(\omega)\,d\omega\ ;$$

they are all equal. Just be sure to carry this same letter through your computations; for example:

$$\int_1^2 t^3\,dt = \frac{t^4}{4}\Big|_1^2 = \frac{1}{4}(2^4 - 1^4) = 3\frac{3}{4}$$

The letter used in $\int_a^b f(x)\,dx$ is called the *dummy variable of integration.*

EXERCISE 3

♣ Suppose you KNOW $\int_a^b f(x)\,dx$. Then, do you know $\int_a^b f(t)\,dt$? How about $\int_a^b f(s)\,ds$? How about $\int_c^d f(x)\,dx$? How about $\int_a^b g(x)\,dx$?

To begin to better understand the definite integral, some properties that it satisfies are stated next.

Properties of the Definite Integral

linearity

Suppose that f is continuous on $[a,b]$.

For all constants k :

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

That is, constants can be pulled out of the definite integral.

Also:

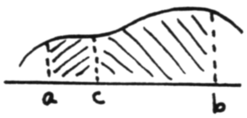
$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

That is, the integral of a sum is the sum of the integrals.

Together, these two properties are referred to as the *linearity of the (definite) integral.*

Properties of the Definite Integral

additivity



There are also properties that relate to the limits of integration. Again suppose that f is continuous on $[a, b]$.

For any $c \in (a, b)$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

That is, to integrate from a to b, one can choose any number c between a and b, and integrate instead in two pieces: from a to c, and then from c to b.

This property is referred to as the *additivity of the integral*.

Finally, for all real numbers a and b:

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

The last property says that if you integrate 'backwards', you must introduce a minus sign.

If F is an antiderivative of f, then the fundamental theorem can be used to find the definite integral, and we see that:

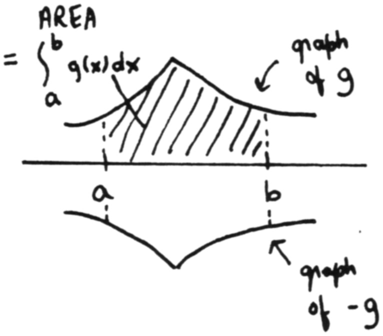
$$\begin{aligned}\int_b^a f(x)\,dx &= F(a) - F(b)\\ &= -(F(b) - F(a))\\ &= -\int_a^b f(x)dx\end{aligned}$$

This explains the last property above.

the definite integral treats area under the x-axis as negative

Let's investigate the property:

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



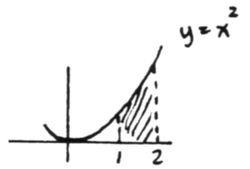
This property shows that if a *negative* function is integrated, then a *negative* number will be obtained. The *magnitude* of this negative number corresponds to the magnitude of the area beneath the x-axis.

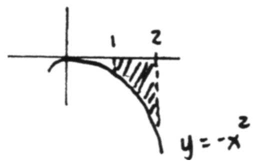
To see this, suppose that g is positive on $[a, b]$. Then, $-g$ is negative on $[a, b]$, and:

$$\int_a^b (-g(x))\,dx = -\int_a^b g(x)\,dx$$

The graphs of g and $-g$ are symmetric about the x-axis; $\int_a^b g(x)\,dx$ gives the area under the graph of g. Thus, *the definite integral treats area under the x-axis as negative.*

EXAMPLE





$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3}$$

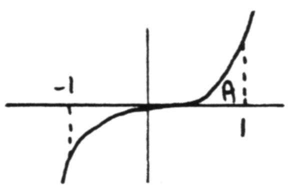
and

$$\int_1^2 -x^2\,dx = -\int_1^2 x^2\,dx = -\frac{7}{3}$$

Here, since the function $-x^2$ is negative over the entire interval $[1,2]$, the result $-\frac{7}{3}$ is interpreted as follows:

- The magnitude of the result, $|-\frac{7}{3}| = \frac{7}{3}$, indicates that there is $\frac{7}{3}$ units of area trapped between the graph of $-x^2$ and the x-axis.
- The fact that the answer $-\frac{7}{3}$ is *negative* indicates that this area lies *beneath* the x-axis.

Caution!!
$\int_a^b f(x)\,dx = 0$
does **not** *imply that*
$f(x) = 0$



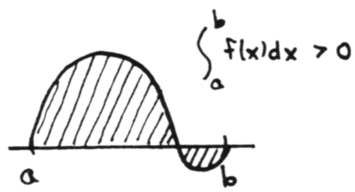
If one integrates over an interval $[a,b]$ on which the area trapped between the graph of f and lying *above* the x-axis is the same as that area *below* the x-axis, then the additivity of the integral shows that the definite integral will have value 0.

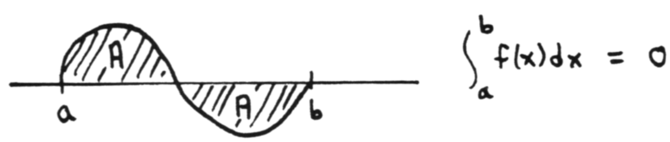
For example, it was seen earlier that $\int_{-1}^1 x^3\,dx = 0$. This is because, by additivity:

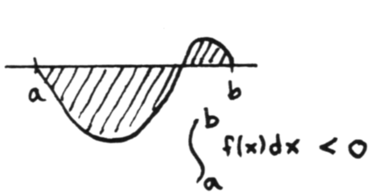
$$\begin{aligned}\int_{-1}^1 x^3\,dx &= \int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx \\ &= (-A) + (A) = 0\ ,\end{aligned}$$

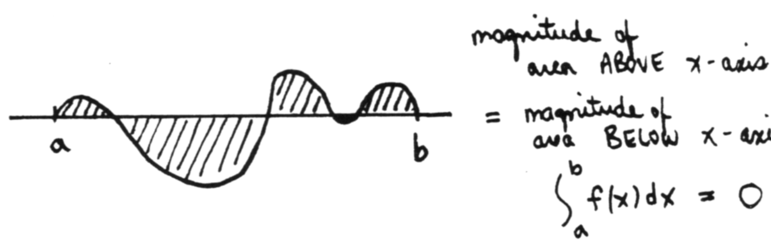
where A represents the magnitude of the area trapped between the graph of x^3 and the x-axis on $[0,1]$.

So, *just because* $\displaystyle\int_a^b f(x)\,dx = 0$ *does not necessarily mean that* $f(x) = 0$ *on* $[a,b]$. Instead, it means that the area trapped between the graph of f and lying *above* the x-axis, is the same as the area trapped between the graph of f and lying *below* the x-axis, on the interval $[a,b]$.









EXERCISE 4

The graph of a function f is shown below, and certain areas are labeled. Based on this information, evaluate the following integrals, if possible. If this is not possible based on the given information, so state.

♣ 1. $\int_{-3}^{-2} f(x)\,dx$

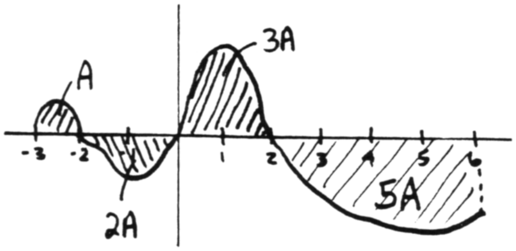
♣ 2. $\int_{-3}^{0} f(t)\,dt$

♣ 3. $\int_{-3}^{2} f(s)\,ds$

♣ 4. $\int_{0}^{5} f(x)\,dx$

♣ 5. $\int_{-2}^{2} f(t)\,dt$

♣ 6. $\int_{-3}^{-1} f(y)\,dy$



This section is concluded with some examples that illustrate how the properties of the definite integral can be used to help in its evaluation.

EXAMPLE

Problem: Evaluate $\int_0^1 (x^2 - 2x + 3)\,dx$.

Solution: By linearity:

$$\begin{aligned}
\int_0^1 (x^2 - 2x + 3)\,dx &= \int_0^1 x^2\,dx - 2\int_0^1 x\,dx + 3\int_0^1 (1)\,dx \\
&= \frac{x^3}{3}\Big|_0^1 - 2\cdot\frac{x^2}{2}\Big|_0^1 + 3\cdot x\Big|_0^1 \\
&= \frac{1}{3}(1-0) - \frac{2}{2}(1-0) + 3(1-0) \\
&= \frac{1}{3} - 1 + 3 = 2\frac{1}{3}
\end{aligned}$$

The solution is usually written down in a much more abbreviated form:

$$\begin{aligned}
\int_0^1 (x^2 - 2x + 3)\,dx &= \left(\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 3x\right)\Big|_0^1 \\
&= \left(\frac{1}{3} - 1 + 3\right) - (0) = 2\frac{1}{3}
\end{aligned}$$

EXAMPLE

find the indefinite integral first; then use any antiderivative to find the definite integral

When the integrand in a definite integral problem is complicated, some people prefer to *first* solve the companion indefinite integral, and then *use* any antiderivative to find the definite integral. This prevents having to 'carry around' the limits of integration.

Problem: Find the area under the graph of $\frac{1}{3x+1}$ on $[0, 2]$.

Solution: It is not necessary to graph $\frac{1}{3x+1}$; it is only necessary to recognize that whenever $x \in [0, 2]$, $\frac{1}{3x+1} > 0$. Thus, the graph lies entirely above the x-axis on this interval, and the desired area is given by the definite integral:

$$\int_0^2 \frac{1}{3x+1}\, dx$$

In a future section, we will discuss how to use the technique of substitution *directly* with definite integrals. For now, find an antiderivative by first solving the companion indefinite integral problem:

$u = 3x + 1$
$du = 3dx$

$$\begin{aligned}\int \frac{1}{3x+1}\, dx &= \frac{1}{3}\int \frac{1}{3x+1} 3\, dx = \frac{1}{3}\int \frac{1}{u}\, du \\ &= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+1| + C\end{aligned}$$

Use the simplest antiderivative to evaluate the desired definite integral:

$$\int_0^2 \frac{1}{3x+1}\, dx = \frac{1}{3}\ln|3x+1|\ \Big|_0^2 = \frac{1}{3}(\ln 7 - \ln 1) = \frac{1}{3}\ln 7 \approx 0.65$$

One more time!

Since variety is the spice of life, the previous problem is solved in a different way:

$$\begin{aligned}\int_0^2 \frac{1}{3x+1}\, dx &= \int_0^2 \frac{1}{3(x+\frac{1}{3})}\, dx = \frac{1}{3}\int_0^2 \frac{1}{x+\frac{1}{3}}\, dx \\ &= \frac{1}{3}\ln|x+\frac{1}{3}|\ \Big|_0^2 = \frac{1}{3}(\ln\frac{7}{3} - \ln\frac{1}{3}) \\ &= \frac{1}{3}(\ln\frac{7/3}{1/3}) = \frac{1}{3}(\ln 7) \approx 0.65\end{aligned}$$

EXERCISE 5	♣ Evaluate $\int_0^1 \frac{1}{5x+1}\, dx$ in two ways.

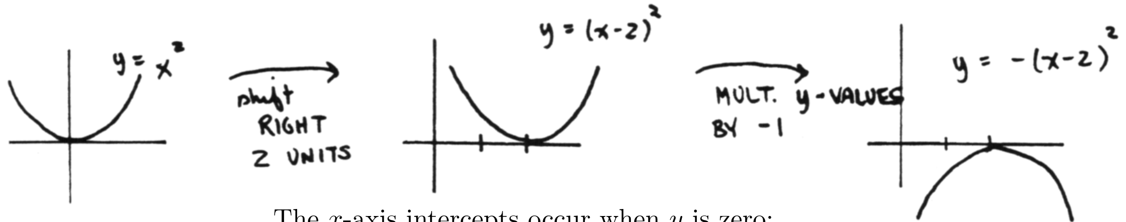
EXAMPLE

Problem: Determine the area of the region bounded by the graph of

$$y = -(x-2)^2 + 1$$

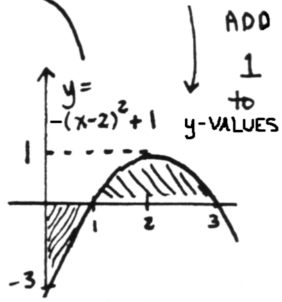
and the x-axis on the interval $[0, 3]$.

Solution: A quick sketch is easy to get and helpful. View y as being 'built up' as follows:



The x-axis intercepts occur when y is zero:

$$\begin{aligned} y = 0 \quad &\iff \quad -(x-2)^2 + 1 = 0 \\ &\iff \quad (x-2)^2 = 1 \\ &\iff \quad |x-2| = 1 \\ &\iff \quad x = 3 \ \text{ or } \ x = 1 \end{aligned}$$



Here we used the facts that:

- For all real numbers x, $\sqrt{x^2} = |x|$; and
- $|x-2|$ tells us how far x is from 2. Thus, $|x-2| = 1$ is true exactly when x is a number whose distance from 2 equals 1.

Now, to find the desired area, we MUST integrate in two pieces:

$$\begin{aligned} \int_0^1 \big(-(x-2)^2 + 1\big)\,dx &= -\frac{(x-2)^3}{3} + x\ \Big|_0^1 \\ &= \Big(-\frac{(1-2)^3}{3} + 1\Big) - \Big(-\frac{(0-2)^3}{3} + 0\Big) \\ &= \frac{4}{3} - \frac{8}{3} = -\frac{4}{3} \end{aligned}$$

The answer is negative, because the area is beneath the x-axis.

Also:

$$\int_1^3 \big(-(x-2)^2 + 1\big)\,dx = \ldots = \frac{4}{3}$$

The desired area is therefore:

$$\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

EXERCISE 6

♣ What would have happened if, in the previous problem, you had tried to compute the desired area by finding $\int_0^3 \big(-(x-2)^2 + 1\big)\,dx$? Evaluate this integral to confirm your answer.

EXERCISE 7

♣ Determine the area of the region bounded by the graph of $y = (x+2)^2 - 1$ on the interval $[-3, 0]$. Make a sketch showing the area that you are finding.

QUICK QUIZ
sample questions

1. In a few words, explain why there is such a similar appearance between the indefinite integral $\int f(x)\,dx$ and the definite integral $\int_a^b f(x)\,dx$.
2. Give a precise statement of the Fundamental Theorem of Integral Calculus.
3. What does the notation $F(x)\big|_a^b$ mean, when used in the context of evaluating definite integrals?
4. Compute: $\int_{-1}^2 x^2\,dx$
5. Show that: $\int_{-1}^1 x^3\,dx = 0$
   Interpret your answer in terms of area.

KEYWORDS
for this section

Notation for the definite integral, upper and lower limits of integration, comparing the definite and indefinite integrals, the Fundamental Theorem of Integral Calculus, the notation $F(x)\big|_a^b$, dummy variable, properties of the definite integral, linearity, additivity, integrating backwards introduces a minus sign, the definite integral treats area under the x-axis as negative.

END-OF-SECTION EXERCISES

Evaluate the following integrals. Use any appropriate methods. Be sure to write complete mathematical sentences.

1. $\displaystyle\int_0^2 \frac{3}{2}x^4\,dx$
2. $\displaystyle\int_1^8 t^{1/3}\,dt$
3. $\displaystyle\int_{-1}^1 (2x-3)\,dx$
4. $\displaystyle\int_0^1 (ax+b)\,dx$
5. $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$
6. $\displaystyle\int_{\ln 2}^{\ln 3} e^{2t}\,dt$

Find the area bounded by the graph of the given function and the x-axis on the stated interval. Make a sketch showing the area that you are finding. You may have to evaluate more than one integral to obtain your final answer.

7. $f(x) = 1 + e^x;\ [0,2]$
8. $f(x) = (x-1)(x+3);\ [-2,2]$
9. $f(x) = 2x^2 + 5x - 3;\ [0,2]$

7.3 The Definite Integral as the Limit of Riemann Sums

Introduction

This section presents the actual *definition* of the definite integral. As previously noted, one is often able to bypass this definition, due to the Fundamental Theorem of Integral Calculus. However, *it is still extremely important that you see this definition*, for three reasons:

- The definition provides the motivation for the notation
$$\int_a^b f(x)\,dx$$
that is used in connection with the definite integral.
- The definition provides the *intuition* that mathematicians use to help them develop many useful formulas involving the definite integral; e.g., finding the area between two curves and finding volumes of revolution. These formulas are presented later on in this chapter.
- The definition provides the justification for numerical methods used to approximate $\int_a^b f(x)\,dx$, when one is unable to obtain an antiderivative of f.

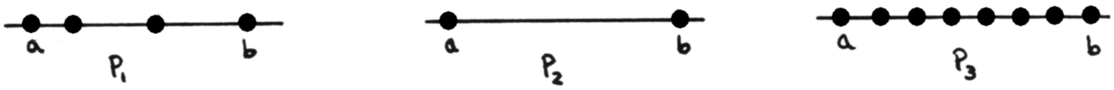
EXERCISE 1	♣ What are the three reasons for which it is important that you see the *definition* of the definite integral?

partition of an interval $[a,b]$

We begin with some definitions.

A *partition* of the interval $[a,b]$ is a finite collection (set) of points from $[a,b]$ that includes the endpoints a and b.

Some partitions of $[a,b]$ are shown below:



By convention, when one writes a partition

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$$

of $[a,b]$, it is assumed that:

- $x_0 = a$; that is, the first point in the partition is the left-hand endpoint a
- $x_n = b$; that is, the last point in the partition is the right-hand endpoint b
- The points are listed in *increasing* order, so that:

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$$

Observe that a partition of $[a,b]$ naturally breaks the interval $[a,b]$ into *non-overlapping subintervals* whose union is the entire interval $[a,b]$:

$$[\overbrace{x_0}^{=a}, x_1) \cup [x_1, x_2) \cup \cdots \cup [x_{n-2}, x_{n-1}) \cup [x_{n-1}, \overbrace{x_n}^{=b}]$$

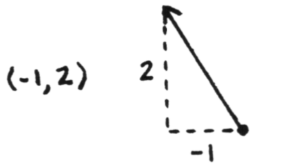
EXERCISE 2

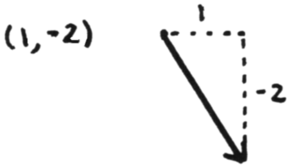
♣ 1. How many points are in the partition $P = \{1\,,\,2\,,\,2.5\,,\,3\}$ of $[1,3]$? Show these points on a number line. Into how many subintervals is $[1,3]$ divided by this partition?

♣ 2. How many points are in the partition $P = \{x_0, x_1, \dots, x_n\}$ of an interval $[a,b]$? Into how many subintervals is $[a,b]$ divided by this partition?

norm

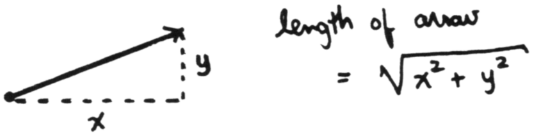
A *norm* is a tool used in mathematics to measure the *size* of objects.





For example, the absolute value $|\cdot|$ measures the size of real numbers; the function that maps a real number x to its 'size' $|x|$ is a *norm* on $\mathbb{R}$.

As a second example, a natural way to 'measure the size' of a pair of real numbers (x,y) is to first look at the arrow (vector) representing (x,y), and then measure its length;



the function that maps a pair (x,y) of real numbers to its 'size' $\sqrt{x^2+y^2}$ is a *norm* on the set of all ordered pairs.

measuring the 'size' of a partition

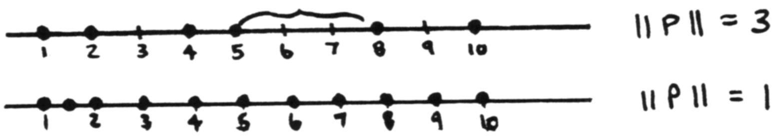
We need a way of measuring the *size* of a partition of $[a,b]$. We want to say that the partition is 'small' if the lengths of *all* the subintervals are small. Observe that if the length of the *longest* subinterval is small, then the lengths of *all* the subintervals must be small. This motivates the next definition.

norm of a partition; $\|P\|$

Define $\|P\|$ (read as the '*norm of the partition* P') to be the length of the *longest* subinterval in the partition P.

For example, if P is the partition $\{1,2,4,5,8,10\}$ of $[1,10]$, then $\|P\| = 3$, since the length of the longest subinterval is 3.

Also, if $P = \{1\,,\,1.5\,,\,2\,,\,3\,,\,4\,,\,5\,,\,6\,,\,7\,,\,8\,,\,9\,,\,10\}$, then $\|P\| = 1$, since the length of the longest subinterval is 1.



The closer $\|P\|$ is to zero, the smaller the subintervals, and hence the more points there are in P.

EXERCISE 3

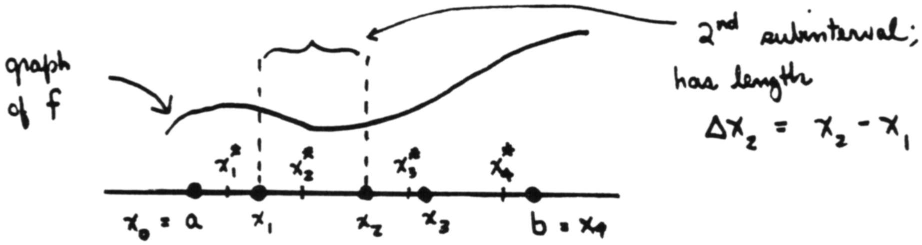
♣ 1. Give a partition of $[0,1]$ that has norm $\frac{1}{2}$. How many points are in this partition?

♣ 2. Give a different partition of $[0,1]$ that has norm $\frac{1}{2}$. How many points are in this partition?

♣ 3. What are the *fewest* number of points that you must have in a partition of $[0,1]$, in order for it to have norm $\frac{1}{2}$?

Riemann Sum for f; x_i^* is our choice from the i^{th} subinterval, which has length Δx_i

Let f be continuous on $[a, b]$, and let $P = \{x_0, \dots, x_n\}$ be any partition of the interval $[a, b]$, as illustrated below.



In each of the n subintervals, choose *any point*; let x_i^* denote the choice from the i^{th} subinterval.

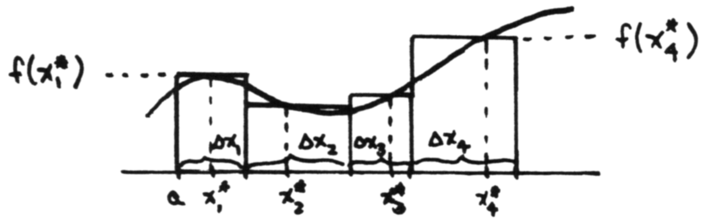
Also, let $\Delta x_i := x_i - x_{i-1}$ denote the length of the i^{th} subinterval.

Then, the sum

$$R(P) := f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a *Riemann sum for f*, corresponding to the partition P. ('Riemann' is pronounced REE-mon.)

Observe that if f is nonnegative, then the sum $R(P)$ represents the sum of the areas of the rectangles shown below, which approximates the area under the graph of f on $[a, b]$.



EXERCISE 4

Consider the partition $P = \{0, 1, 2, 3, 4\}$ of $[0, 4]$. Let $f(x) = x^2$.

♣ 1. Choose the midpoint from each subinterval of P. That is, choose:

$$x_1^* = 0.5\,,\ x_2^* = 1.5\,,\ x_3^* = 2.5\,,\ \ x_4^* = 3.5$$

Make a sketch that shows the graph of f, the partition P, and the choices x_i^*.

♣ 2. On each subinterval, draw a rectangle with height $f(x_i^*)$.

♣ 3. Sum the areas of these rectangles. That is, find the Riemann sum for f corresponding to the choices x_i^*.

♣ 4. What is the *actual* area under the graph under f on $[0, 4]$?

EXERCISE 5 ♣ Repeat the previous exercise, except this time with the partition

$$\{0\,,\,0.5\,,\,1\,,\,1.5\,,\,2\,,\,2.5\,,\,3\,,\,3.5\,,\,4\}$$

of $[0,4]$. Again choose the x_i^* to be the midpoints of each subinterval.

This time, what is the Riemann sum for f corresponding to the partition P and choices x_i^*?

obtain the definite integral by letting $\|P\| \to 0$

Under the hypothesis that f is continuous on $[a,b]$, it can be proven that as one chooses partitions with smaller and smaller norms, the corresponding Riemann sums approach a unique number.

We *define* this unique number to be the *definite integral of f on* $[a,b]$, denoted by $\int_a^b f(x)\,dx$.

more precisely

More precisely, as $\|P\| \to 0$, $R(P) \to \int_a^b f(x)\,dx$.

That is, we can get the numbers $R(P)$ as close to $\int_a^b f(x)\,dx$ as desired, merely by choosing a partition P of $[a,b]$ with norm sufficiently close to 0.

In other words, for every $\epsilon > 0$, there exists $\delta > 0$, such that if a partition P is chosen with $\|P\| < \delta$, then:

$$\left| R(P) - \int_a^b f(x)\,dx \right| < \epsilon$$

Rephrasing yet one more time, we can get the Riemann sum $R(P)$ as close to the number $\int_a^b f(x)\,dx$ as desired, by choosing a partition P of $[a,b]$ that has sufficiently small subintervals.

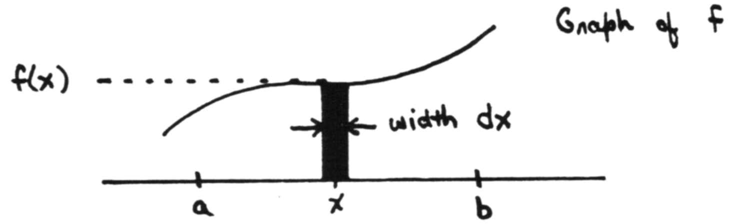
It is clear from the definition of $\int_a^b f(x)\,dx$ that this integral gives information about the *area* trapped between the graph of f and the x-axis.

If f is positive on $[a,b]$, then any Riemann sum $R(P)$ is also positive, and approximates the area under the graph of f on $[a,b]$.

If f is negative on $[a,b]$, then any Riemann sum $R(P)$ is also negative. (♣ Why?) The magnitude of the negative number $R(P)$ approximates the area trapped between the graph of f and the x-axis on $[a,b]$.

motivation for the notation $\int_a^b f(x)\,dx$; $f(x)\,dx$ is the (signed) area of a rectangle, with width dx, and height $f(x)$

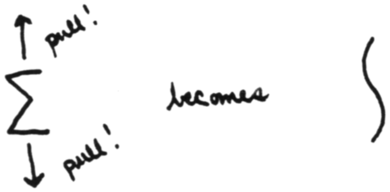
The definition of the definite integral of f on $[a,b]$ provides the motivation for the notation $\int_a^b f(x)\,dx$ used, as follows:



Think of dx as an *infinitesimally small piece of the x-axis*. At a point x between a and b, construct a rectangle of width dx and height $f(x)$. Then (using calculus!) ‘sum’ these rectangles as x varies from a to b .

$\sum$ becomes $\int$

The integral sign $\int$ is, therefore, a kind of *super sum*; indeed, one can think of obtaining it from the summation sign $\sum$ used for finite sums by stretching it out!



integration is an (infinite) summation process

That is, *integration is really an (infinite) summation process.*

If seeing the notation $\int_a^b f(x)\,dx$ conjures an image of a limit of Riemann sums, then it is a successful notation.

QUICK QUIZ
sample questions

1. What is a *partition* of an interval $[a,b]$?
2. Give two different partitions of $[1,3]$ that have norm $1/2$.
3. Let $f(x) = x^2$, and take the partition $\{0,1,2,3\}$ of the interval $[0,3]$. Is there a unique Riemann sum for f corresponding to this partition? Comment.
4. What picture might you think of when you see the notation $\int_a^b f(x)\,dx$?

KEYWORDS
for this section

Three reasons for seeing the definition of the definite integral, partition of an interval, norm, norm of a partition, Riemann sum for f, obtain the definite integral by letting $\|P\| \to 0$, motivation for the notation $\int_a^b f(x)\,dx$, integration is an (infinite) summation process.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\int x^2\,dx$
2. $\int_0^1 x^2\,dx$
3. $\int_0^1 x^2\,dx = \frac{1}{3}$
4. The integral $\int_a^b f(x)\,dx$ gives the magnitude of the area bounded between the graph of f and the x-axis on $[a,b]$.
5. If $a < b$, then the integral $\int_a^b e^x$ gives the magnitude of the area bounded between the graph of $y = e^x$ and the x-axis on $[a,b]$.
6. If P is a partition of $[a,b]$, then a Riemann sum $R(P)$ corresponding to f is an approximation to $\int_a^b f(x)\,dx$.
7. If g is twice differentiable on the interval $[a,b]$, then $\int_a^b g'(x)\,dx = g(b) - g(a)$.
8. If $a < b$ and f is continuous on $[a,b]$, then $\int_a^b |f(x)|\,dx \ge 0$.
9. If $a < b$ and f is continuous on $[a,b]$, then $\int_a^b (-|f(x)|)\,dx \le 0$.
10. For all real numbers a and b, $\int_a^b x^2\,dx = \int_a^b t^2\,dt$.

7.4 The Substitution Technique applied to Definite Integrals

Introduction

Consider the definite integral:

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

To find an antiderivative of $x\sqrt{1-x^2}$ requires a substitution; when this substitution is performed in the context of the definite integral, one must be careful how things are written down.

There are two basic approaches for using substitution in definite integral problems. Both are discussed in this section.

Approach #1

first find an antiderivative;

use it to solve the definite integral

The first approach, which has already been illustrated in an earlier section, is to recognize that *once we have an antiderivative, solving the definite integral problem is easy.* So we can *first solve the corresponding indefinite integral problem,* and then use the simplest antiderivative to compute the desired definite integral.

EXAMPLE

approach #1

Problem: Find $\int_0^1 x\sqrt{1-x^2}\,dx$.

Solution #1: First solve the corresponding indefinite integral problem:

$$\begin{aligned}\int x\sqrt{1-x^2}\,dx &= \frac{1}{-2}\int -2x\sqrt{1-x^2}\,dx \\ &= -\frac{1}{2}\int u^{1/2}\,du \\ &= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C \\ &= -\frac{1}{3}(1-x^2)^{3/2}+C \\ &= -\frac{1}{3}\left(\sqrt{1-x^2}\right)^3+C\end{aligned}$$

$$\boxed{\begin{aligned}u &= 1-x^2 \\ du &= -2x\,dx\end{aligned}}$$

The simplest antiderivative is when $C=0$. Then:

$$\begin{aligned}\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{3}\left(\sqrt{1-x^2}\right)^3\Big|_0^1 \\ &= 0-\left(-\frac{1}{3}\cdot 1\right)=\frac{1}{3}\end{aligned}$$

EXAMPLE

approach #2;

transform the original definite integral into a NEW definite integral;

changing the limits of integration

Another approach, *that allows the solution to be written down more compactly*, is to transform the original definite integral into a NEW definite integral, as illustrated in this alternate solution:

Solution #2:

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_1^0 u^{1/2}\,du \\
&= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^0 \\
&= -\frac{1}{3}u^{3/2}\Big|_1^0 \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}$$

NEW limits of integration

$u = 1 - x^2$

$du = -2x\,dx$

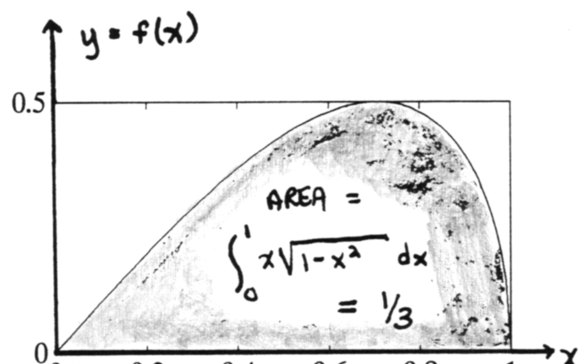
$x = 0 \Rightarrow u = 1 - 0^2 = 1$

$x = 1 \Rightarrow u = 1 - 1^2 = 0$

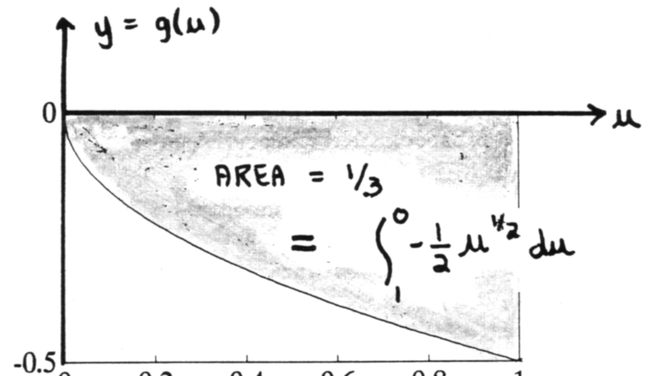
KEY OBSERVATIONS for using Approach #2

- Decide upon an appropriate substitution, just as you do with indefinite integral problems.
- Write the substitution directly under the definite integral, as usual.
- Directly under the substitution, calculate the limits of integration for the *new* definite integral (in the variable u). Remember: *don't change the limits of integration UNTIL you've rewritten the integral in terms of the new variable!*
- With this method, you never need to transform the antiderivative back to a function in the original variable.

Below is a sketch illustrating what is happening, from a graphical point of view, in this process.



Graph of $f(x) = x\sqrt{1-x^2}$ on $[0,1]$



Graph of $g(u) = -\frac{1}{2}u^{1/2}$ on $[0,1]$;

area beneath u-axis introduces a "$-$" sign;

integrating 'backwards' introduces another "$-$" sign!

variation on approach #2;

don't actually calculate the new limits, just note that they are different

There is a variation on the second approach that is often useful. Instead of *actually calculating* the new limits of integration, just make the reader aware that the limits have changed in the transformed problem. That is, when an 'old' limit of integration is 'a', the 'new' limit of integration is denoted by '$u(a)$' (the function u, evaluated at a). The technique is illustrated below:

Solution #3:

$$\boxed{\begin{aligned} u &= 1 - x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_{u(0)}^{u(1)} u^{1/2}\,du \\
&= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{u(0)}^{u(1)} \\
&= -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}$$

NOTE that the new limits are different but don't actually calculate them

CHANGE back to an antiderivative in x and use original limits

This technique is useful if the limits of integration for the transformed problem would be particularly messy, or difficult to compute.

EXERCISE 1

♣ Find $\int_0^1 x(3x^2-1)^5\,dx$. Write down your solution in three different ways. Be sure to write complete and correct mathematical sentences.

lurking in the background

The theoretical justification for this section lies in the following *change of variables formula*:

Change of Variables Formula

Let f and g' be continuous. Then:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Observe that this formula states exactly what we've been doing in this section: letting $u = g(x)$, one obtains $du = g'(x)\,dx$; when $x = a$, $u = g(a)$ and when $x = b$, $u = g(b)$.

$$\int_a^b f(\overbrace{g(x)}^{u})\,\overbrace{g'(x)\,dx}^{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

★★

existence of antiderivatives

If a function f is continuous on $[a,b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$.

Thus, every continuous function has an antiderivative. This fact is needed in (the first line of) the following proof.

PROOF
of the Change of Variables Formula

Proof. Let F be any antiderivative for f, so $F' = f$. Then, by the Chain Rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

so that $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$. Thus:

$$\begin{aligned}\int_a^b f(g(x))\, g'(x)\, dx &= F(g(x))\big|_a^b \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Also:

$$\begin{aligned}\int_{g(a)}^{g(b)} f(u)\, du &= F(u)\big|_{g(a)}^{g(b)} \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Compare! ∎

using integration by parts with definite integrals

When using the integration by parts formula with definite integrals, one again has to be careful how things are written down.

As usual, one option is to first solve the corresponding indefinite integral problem, and use any antiderivative to evaluate the definite integral. However, it is more compact to evaluate the definite integral directly, as illustrated in the next example.

EXAMPLE
using parts with a definite integral

Problem: Find $\int_1^2 \ln x\, dx$.

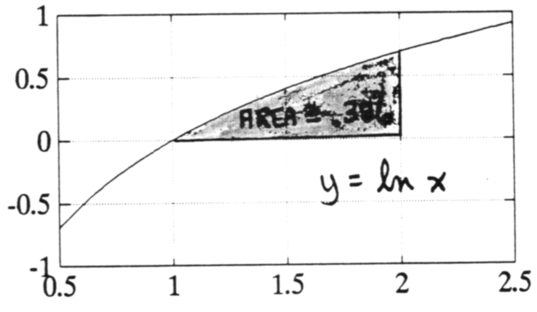
Solution:

$$\begin{aligned}\int_1^2 \ln x\, dx &= x \ln x\big|_1^2 - \int_1^2 x \cdot \frac{1}{x}\, dx \\ &= (2\ln 2 - 1\ln 1) - \left[x\big|_1^2\right] \\ &= 2\ln 2 - [2-1] \\ &= 2\ln 2 - 1 \approx 0.386\end{aligned}$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}}$$

Thus, the area under the graph of $y = \ln x$ on $[1, 2]$ is approximately 0.386 .

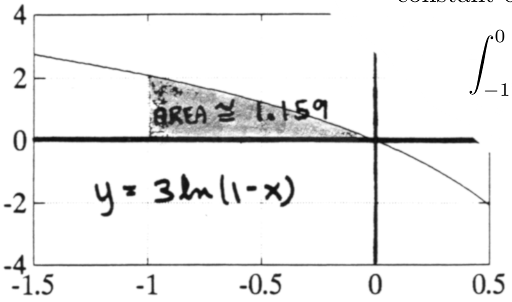
Note that it was necessary to evaluate *each part* of the antiderivative from 1 to 2 . In both cases, the symbol '$\big|_1^2$' is read as '*evaluated from* 1 *to* 2'.



EXAMPLE

Problem: find $\int_{-1}^{0} 3\ln(1-x)\,dx$.

Solution: It is usually easiest to use the linearity of the integral to factor the constant out first:



$$\begin{aligned}
\int_{-1}^{0} 3\ln(1-x)\,dx &= 3\left[\int_{-1}^{0} \ln(1-x)\,dx\right] \\
&= 3\left[(x-1)\ln(1-x)\Big|_{-1}^{0} - \int_{-1}^{0}(x-1)\frac{1}{x-1}\,dx\right] \\
&= 3\left[(0+2\ln 2) - \int_{-1}^{0}(1)\,dx\right] \\
&= 3\left[2\ln 2 - x\Big|_{-1}^{0}\right] \\
&= 3[2\ln 2 - (0+1)] \\
&= 6\ln 2 - 3 \approx 1.159
\end{aligned}$$

$u = \ln(1-x)$ $\quad dv = dx$

$du = \frac{1}{1-x}(-1)\,dx$ $\quad v = x-1$

$= \frac{1}{x-1}\,dx$

Observe how v was chosen to be $x-1$, instead of simply x, to simplify the integral $\int v\,du$.

EXERCISE 2

♣ Find $\int_0^1 xe^x\,dx$, by using parts. Do not solve the corresponding indefinite integral problem first; work directly with the definite integral.

QUICK QUIZ
sample questions

1. Find $\int_0^{\frac{1}{2}} (2x-1)^3\,dx$ by first solving the companion indefinite integral problem.
2. Find $\int_0^{\frac{1}{2}} (2x-1)^3\,dx$ by transforming it into a definite integral in the variable u, with correct limits of integration.
3. Solve $\int_1^e \ln x\,dx$ directly. That is, do NOT first solve the companion indefinite integral problem.

KEYWORDS
for this section

Various approaches to using the substitution technique in the context of definite integrals, the Change of Variables formula, using parts with definite integrals.

END-OF-SECTION EXERCISES

Evaluate the following definite integrals. Use any correct solution technique. Be sure to write complete mathematical sentences. Approximate answers to three decimal places.

1. $\displaystyle\int_{-1}^{1} x\sqrt{1+x^2}\,dx$
2. $\displaystyle\int_0^3 \frac{2}{3x+4}\,dx$
3. $\displaystyle\int_1^2 \frac{1}{(5-t)^3}\,dt$
4. $\displaystyle\int_1^3 \ln 3x\,dx$
5. $\displaystyle\int_2^3 5\ln(x-1)\,dx$

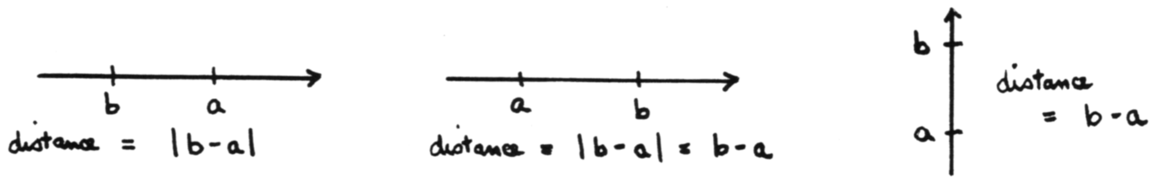
7.5 The Area Between Two Curves

Introduction

The definition of the definite integral provides mathematicians with *intuition* that helps to develop formulas involving the definite integral. First, a formula is developed for finding the area of the region between two curves.

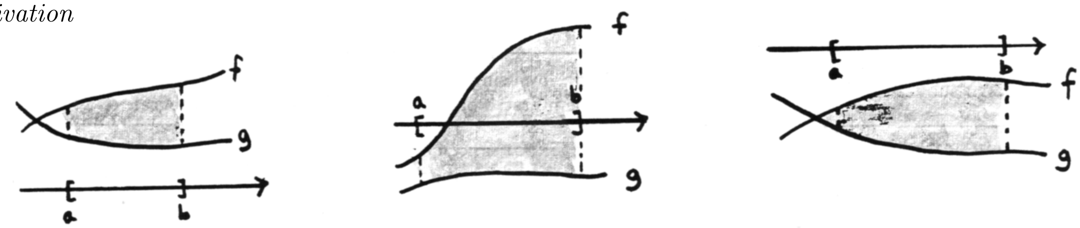
distance between two real numbers

Recall that if a and b are *any* two real numbers, then the distance between them is $|b-a|$. If, in addition, $b \ge a$, then the distance between them is $|b-a| = b-a$.



finding the area between two curves; a motivation

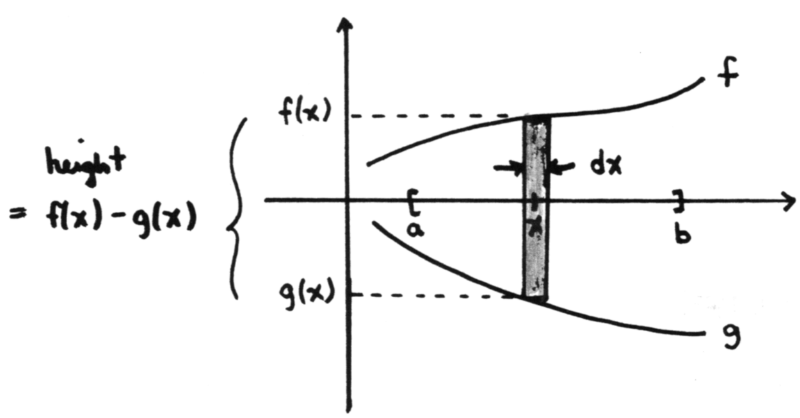
Let f and g both be continuous on the interval $[a,b]$, and suppose that $f(x) \ge g(x)$ for all $x \in [a,b]$, so that the graph of f lies above the graph of g on $[a,b]$.



We want to find the area between the graphs of f and g on $[a,b]$. To motivate the formula, proceed as follows:

Investigate a typical 'infinitesimal slice' of the desired area. First, choose a value of x between a and b, and look at a slice of the desired area at this value x. Denote the width of this typical slice by dx (think of dx as denoting an infinitesimal piece of the x-axis).

Since $f(x) \ge g(x)$, the height of the slice is $f(x) - g(x)$. Observe that this is the height of the slice, *regardless* of the signs (plus or minus) of f and g.



Therefore, the area of this typical slice is:

$$\overbrace{\big(f(x) - g(x)\big)}^{\text{height}} \overbrace{dx}^{\text{width}}$$

Now, use calculus to 'sum' these slices:

$$\text{desired area} = \int_a^b \big(f(x) - g(x)\big)\, dx$$

Although this is certainly not a rigorous development of the formula (which would require partitioning $[a, b]$ and investigating Riemann sums), the result is correct. The process illustrates how intuition about the definite integral can be used to gain some useful results.

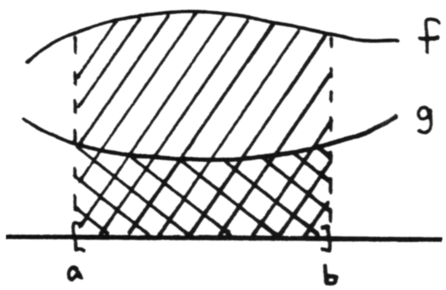
EXERCISE 1

♣ 1. Show that whenever $f(x) \geq g(x)$, then $f(x) - g(x) \geq 0$. Be sure to write a complete mathematical sentence. (This is a one-liner.)

♣ 2. Suppose that $f(1) = -2$ and $g(1) = -4$. Plot the two points described here. Is $f(1) \geq g(1)$? What is $f(1) - g(1)$ in this case?

another viewpoint

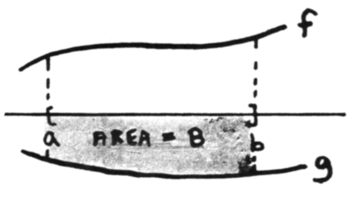
Now, view the previous problem from a different perspective. Suppose for the moment that *both* f and g are positive, and the graph of f lies above the graph of g. Then, the area between f and g can be found by finding the area under f, and subtracting off the area beneath g:



$$\begin{aligned}\text{area between } f \text{ and } g &= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx \\ &= \int_a^b \big(f(x) - g(x)\big)\, dx \quad \text{(by linearity)}\end{aligned}$$

Similarly, if f is positive and g is negative, then the graph of f necessarily lies above the graph of g, and the desired area can be found as follows:

Keep in mind that the word 'area' always refers to a *nonnegative* quantity.



In this situation (illustrated in the sketch), $\int_a^b g(x)\, dx$ is a *negative* number, since the definite integral treats area beneath the x-axis as negative. Thus, $B = -\int_a^b g(x)\, dx$. The desired area between the two curves on $[a, b]$ is then:

$$\begin{aligned}\text{desired area} &= \int_a^b f(x)\, dx + \left(-\int_a^b g(x)\, dx\right) \\ &= \int_a^b \big(f(x) - g(x)\big)\, dx \quad \text{(by linearity)}\end{aligned}$$

The same formula is again obtained.

EXERCISE 2

♣ Suppose that f and g are *both negative* on the interval $[a,b]$, and that $f(x) \geq g(x)$ on $[a,b]$. Make a sketch that illustrates this situation. Then, proceeding as in the previous example, find the formula for the area between f and g on $[a,b]$.

The result concerning the area between two curves is summarized below:

AREA BETWEEN TWO CURVES

Let f and g be continuous on $[a,b]$, and suppose that $f(x) \geq g(x)$ on $[a,b]$, so that the graph of f lies above the graph of g. Then, the area between the graphs of f and g on $[a,b]$ is given by:

$$\int_a^b \big(f(x) - g(x)\big)\,dx$$

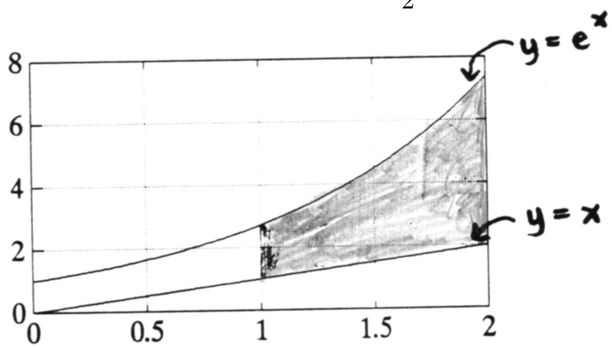
EXAMPLE

finding the area between two curves

Problem: Find the area between $y = e^x$ and $y = x$ on $[1,2]$.

Solution: A quick sketch shows that the graph of $y = e^x$ lies above the graph of $y = x$ on the interval $[1,2]$. Thus, the desired area is given by:

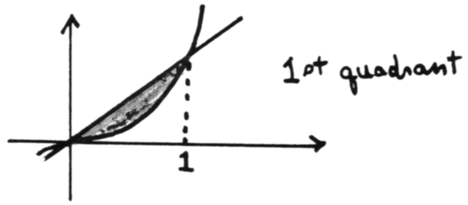
$$\begin{aligned}\int_1^2 (e^x - x)\,dx &= (e^x - \frac{x^2}{2})\Big|_1^2 \\ &= (e^2 - 2) - (e - \frac{1}{2}) \\ &= e^2 - e - \frac{3}{2} \approx 3.171\end{aligned}$$



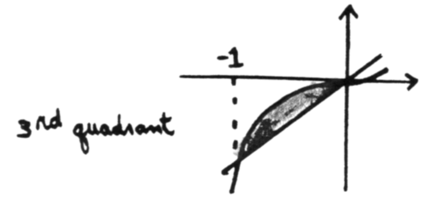
the phrase ‘bounded by’

The phrase ‘*bounded by …*’ can be roughly interpreted as ‘*having edges (boundary) given by the graphs of …*’. The idea is illustrated in the following examples:

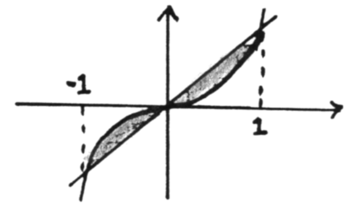
- The area in the first quadrant bounded by $y = x$ and $y = x^3$ is shown below. This is the area in the first quadrant that has as its boundary *only* the graphs of $y = x$ and $y = x^3$.

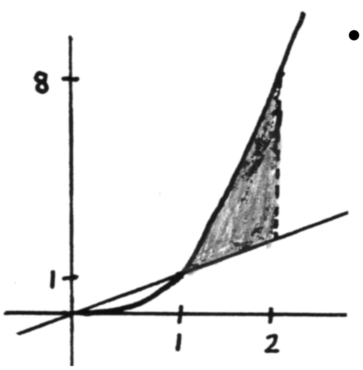


- The area in the third quadrant bounded by $y = x$ and $y = x^3$ is shown below. This is the area in the third quadrant that has as its boundary *only* the graphs of $y = x$ and $y = x^3$.

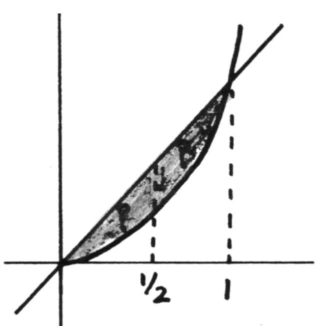


- The area bounded by $y = x$ and $y = x^3$ is shown below. This is the area that has as its boundary the graphs of $y = x$ and $y = x^3$. Observe that this area is naturally composed of two pieces.



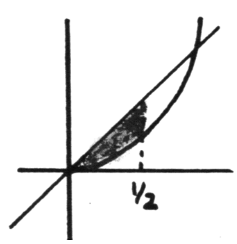


- The area bounded by $y = x$, $y = x^3$, and $x = 2$ is shown at left. This is the area having as its boundary the graphs of the given equations. Remember that the graph of $x = 2$ (viewed as an equation in two variables) is the set of all points (x, y) with $x = 2$. That is, the graph of $x = 2$ consists of all points with x-value equal to 2; thus, it is the vertical line that crosses the x-axis at 2.



- The phrase 'the area bounded by $y = x$, $y = x^3$ and $x = \frac{1}{2}$' is *ambiguous*; there are two adjacent pieces of area with the given edges. Do we want just one of these? Both of these? If both are desired, why wasn't the simpler description 'the area in the first quadrant bounded by $y = x$ and $y = x^3$' given?

  Because of this ambiguity, the desired area is clarified by using, say, the description 'the area bounded by $y = x$, $y = x^3$, $x = 0$, and $x = \frac{1}{2}$'. Alternately, the description 'the area between $y = x$ and $y = x^3$ on $[0, \frac{1}{2}]$' can be used. In either case, there is no doubt that the area being described is the one shown below.



EXERCISE 3 Sketch each of the areas described below:

♣ 1. The area bounded by $y = x^2$ and $y = x$.

♣ 2. The area in the first quadrant bounded by $y = x^2$ and $y = x^4$.

♣ 3. The area in the second quadrant bounded by $y = x^2$ and $y = x^4$.

♣ 4. The area bounded by $y = x^2$ and $y = x^4$.

♣ 5. The area bounded by $y = x^2$, the x-axis, $x = 1$ and $x = 3$.

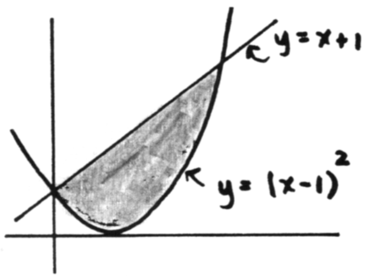
♣ 6. The area bounded by $y = x^2$, $y = -x^2$, $x = 1$ and $x = 3$.

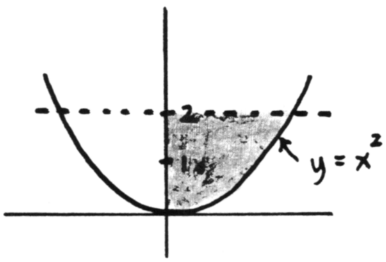
♣ 7. The area bounded by $y = x^2$, $y = 1$, and $y = 2$.

♣ 8. The area bounded by $y = x^2$ and $y = 4x - 3$.

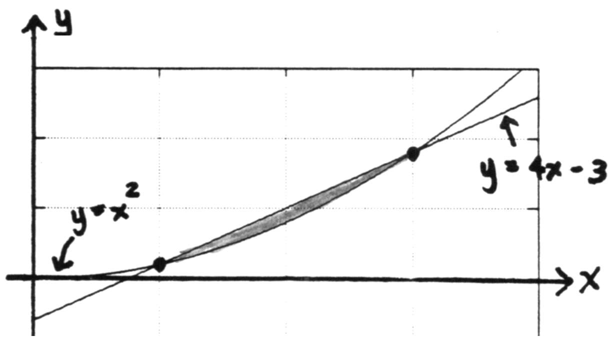
EXERCISE 4 ♣ Describe each of the areas shown below, using an appropriate variation of the phrase:

'*the area bounded by* ...'





EXAMPLE Problem: Find the area bounded by $y = x^2$ and $y = 4x - 3$.



finding intersection points

Solution: It is first necessary to find the intersection points. To do this, we seek points (x, y) that make *both* equations true (so that the point (x, y) lies on *both* curves).

If (x, y) is an intersection point, then when the number x is substituted into *either* equation, the *same* value of y results. Therefore, values of x are sought for which the y values on *both curves* are the same. To find such values, set the y values of both curves equal to each other, and solve for the corresponding value(s) of x:

$$\begin{aligned} x^2 = 4x - 3 \quad &\iff \quad x^2 - 4x + 3 = 0 \\ &\iff \quad (x-1)(x-3) = 0 \\ &\iff \quad x = 1 \ \text{ or } \ x = 3 \end{aligned}$$

EXERCISE 5

review of equivalence and the mathematical words 'or' and 'and'

♣ 1. In English, what does the sentence '$x^2 = 4x - 3 \iff x = 1$ or $x = 3$' mean?

♣ 2. For what value(s) of x is the sentence '$x = 1$ or $x = 3$' true? (If necessary, review the mathematical meaning of the word 'or'.)

♣ 3. Suppose that, for a given value of x, the sentence '$x^2 = 4x - 3$' is false. Can anything be said about the truth value of the sentence '$x = 1$ or $x = 3$'?

♣ 4. Is it correct to say

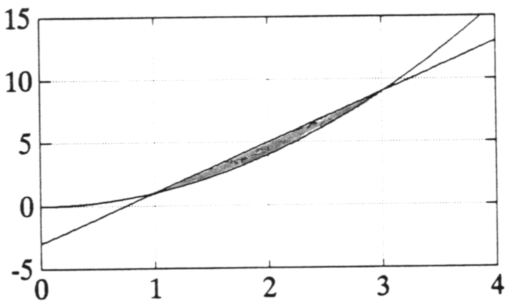
$$(x-1)(x-3) = 0 \iff x = 1 \ \text{ and } \ x = 3 \ ?$$

Why or why not? (If necessary, review the mathematical meaning of the word 'and'.)

So the curves intersect when $x = 1$ and when $x = 3$. Since the points with these x values lie on *both* curves, *either* curve, $y = x^2$ or $y = 4x - 3$, can be used to find the corresponding y-values:

$$\begin{aligned} & x = 1 \implies y = 1^2 = 1 && \text{(substituting into } y = x^2\text{)} \\ \text{or} \quad & x = 1 \implies y = 4(1) - 3 = 1 && \text{(substituting into } y = 4x - 3\text{)} \end{aligned}$$

$$\begin{aligned} & x = 3 \implies y = 3^2 = 9 && \text{(substituting into } y = x^2\text{)} \\ \text{or} \quad & x = 3 \implies y = 4(3) - 3 = 9 && \text{(substituting into } y = 4x - 3\text{)} \end{aligned}$$



use the simplest curve to find the corresponding y-values

Since *either* curve can be used to find the corresponding y-values, one usually chooses the simplest one. (In this example, it would be a toss-up as to which curve is simpler.)

From the sketch, the graph of $y = 4x - 3$ lies above the graph of $y = x^2$ on $[1, 3]$. (Momentarily, it will be observed that it is not really necessary to know which curve is on top.) Thus, the desired area is given by:

$$\begin{aligned}\int_1^3 \left((4x-3) - x^2\right) dx &= 2x^2 - 3x - \frac{1}{3}x^3\Big|_1^3 \\ &= (18 - 9 - 9) - (2 - 3 - \frac{1}{3}) \\ &= \frac{4}{3}\end{aligned}$$

EXAMPLE

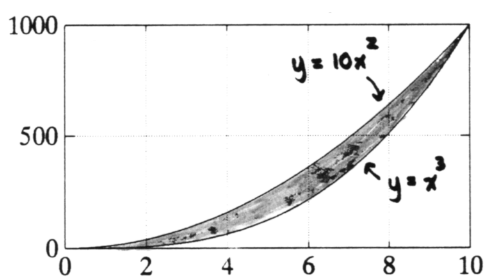
it's not really necessary to know which curve is on top

Problem: Find the area in the first quadrant bounded by $y = 10x^2$ and $y = x^3$.

Solution: It's not necessary to graph the functions; just find the intersection points:

$$\begin{aligned}10x^2 = x^3 \quad &\iff \quad x^3 - 10x^2 = 0 \\ &\iff \quad x^2(x - 10) = 0 \\ &\iff \quad x = 0 \ \text{ or } \ x = 10\end{aligned}$$

The curves intersect at $x = 0$ and $x = 10$. So, on the interval $[0, 10]$, either the graph of x^3 is on top, or the graph of $10x^2$ is on top. (If one were on top for only *part* of the time, then there would have to be another intersection point.) Just *guess* that x^3 is on top, and calculate:



$$\begin{aligned}\int_0^{10} \left(x^3 - 10x^2\right) dx &= \frac{1}{4}x^4 - \frac{10}{3}x^3\Big|_0^{10} \\ &= (\frac{10000}{4} - \frac{10000}{3}) \\ &= 10000(\frac{3}{12} - \frac{4}{12}) \\ &= -\frac{2500}{3}\end{aligned}$$

Since the answer is negative, the guess was incorrect: actually, $y = 10x^2$ is on top. But the desired area is still known:

$$\text{desired area} = \left|-\frac{2500}{3}\right| = \frac{2500}{3}$$

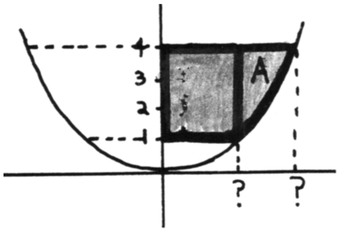
EXAMPLE

jigsaw puzzles

Problem: Find the area bounded by $y = x^2$, $y = 1$ and $y = 4$.

Solution: If you like jigsaw puzzles, you'll like this sort of problem. Just figure out the easiest 'pieces' that make up the desired area.

Symmetry can be used to simplify the problem. Find the area in the first quadrant, and double it.



One way to view the area in the first quadrant is as being composed of a rectangle, and an extra piece.

The height of the rectangle is $4 - 1 = 3$. What is the width? To answer this, we need to know where $y = 1$ and $y = x^2$ intersect:

$$x^2 = 1 \quad \Longleftrightarrow \quad x = \pm 1$$

(Remember that '$x = \pm 1$' is shorthand for '$x = 1$ or $x = -1$'.)

Thus, the width of the rectangle is $1 - 0 = 1$. The rectangle has area $(3)(1) = 3$.

What is the area of the remaining piece? We need to know where $y = 4$ and $y = x^2$ intersect:

$$x^2 = 4 \quad \Longleftrightarrow \quad x = \pm 2$$

Then, calling the area of this piece A:

$$\begin{aligned} A = \int_1^2 (4 - x^2)\,dx &= 4x - \frac{x^3}{3}\Big|_1^2 \\ &= (8 - \frac{8}{3}) - (4 - \frac{1}{3}) \\ &= 4 - \frac{7}{3} = \frac{12}{3} - \frac{7}{3} = \frac{5}{3} = 1\frac{2}{3} \end{aligned}$$

Therefore, the area bounded by $y = x^2$, $y = 1$ and $y = 4$ is:

$$2 \cdot (3 + 1\frac{2}{3}) = 9\frac{1}{3}$$

QUICK QUIZ
sample questions

1. Suppose that g and f are continuous functions, and that $g(x) \geq f(x)$ on the interval $[c, d]$. Give a formula for the area between f and g on $[c, d]$.
2. Find the area bounded by $y = -x^2 + 1$ and the x-axis.
3. Is the phrase 'the area bounded by $y = (x - 2)^2$, $x = 1$ and $y = 4$' ambiguous? Why or why not?
4. Find the area between $y = e^x$ and $f(x) = -x$ on $[0, 1]$. Make a sketch showing the area that you are finding.

KEYWORDS
for this section

The distance between two real numbers, finding the area between two curves, the phrase 'bounded by', finding intersection points, it is not necessary to know which curve is on top.

END-OF-SECTION EXERCISES

♣ Find the area of each region described below. Make a sketch, and shade the area that you are finding. Be sure to write complete mathematical sentences.

1. In the first quadrant, bounded by: $y = x^2$ and $y = x^4$
2. In the third quadrant, bounded by: $y = x$ and $y = x^3$
3. Bounded by $y = -(x - 2)^2 + 3$ and $y = -1$
4. Bounded by $y = x$ and $y = x^3$
5. Bounded by $y = x^2$, $y = 1$ and $y = 2$
6. Bounded by $y = x^2$ and $y = -1$, $x = 0$ and $x = 2$
7. Bounded by $y = x^3$, $y = 8$ and $x = -1$

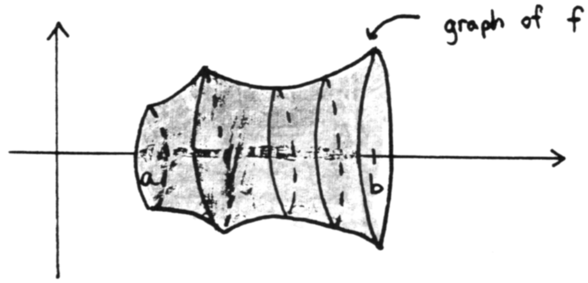
7.6 Finding the Volume of a Solid of Revolution—Disks

Introduction

Again, in this section, intuition gained from the definition of the definite integral helps to motivate some useful formulas for finding the volume of solids of revolution. Keep in mind that strict derivations of these formulas would require partitioning and investigating Riemann sums.

generating a solid of revolution; revolving about the x-axis

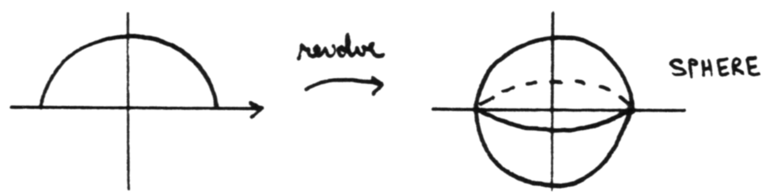
Let f be continuous on an interval $[a, b]$. If the area between the graph of f and the x-axis on $[a, b]$ is rotated about the x-axis, then a *solid of revolution* is generated. Our goal is to use calculus to find the volume of this solid of revolution.



EXAMPLE

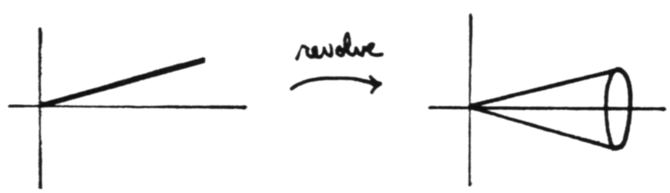
For example, consider the upper-half-circle shown below. When this graph is rotated about the x-axis, a sphere results.

The phrase 'this graph is rotated about the x-axis' is shorthand for the more correct phrase, 'the area between the graph and the x-axis is rotated about the x-axis'.



EXAMPLE

If the line shown below is revolved about the x-axis, a right circular cone is obtained.



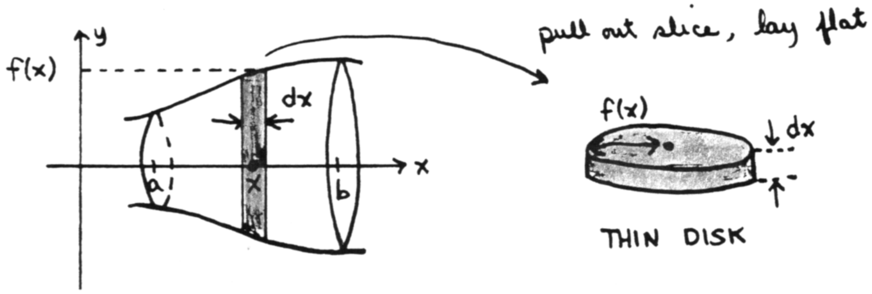
These two examples will be used to 'test' our formula, after its derivation.

motivational derivation of the formula;

Let f be continuous on $[a,b]$. For the moment, suppose that f is nonnegative, so that its graph lies above the x-axis. (This restriction will be removed momentarily.)

a typical slice of the solid

Revolve the graph about the x-axis. Let's investigate a *typical infinitesimal slice* of the resulting solid of revolution.



the slice is a disk with volume $\pi(f(x))^2\,dx$

Choose a value x between a and b. Imagine holding a saw, perpendicular to the xy-plane, and cutting a thin slice from the desired solid at this value x. Call the thickness of this slice dx, and think of dx as representing an *infinitesimally small* piece of the x-axis. Pull this slice out and lay it down. It looks like a disk! (Consequently, this technique is often referred to as *the disk method.*) The radius of the disk is $f(x)$, and hence its volume is:

$$\text{(area of circle)(thickness)} = \pi(f(x))^2 \cdot dx$$

Now, use integration to 'sum' these slices, as x travels from a to b:

$$\text{desired volume} = \int_a^b \pi\big(f(x)\big)^2\,dx$$

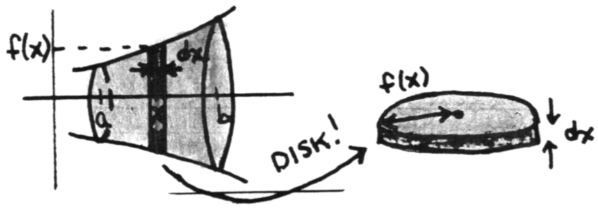
f can be negative

Observe that if f is negative, the radius of the resulting slice is $|f(x)|$, but, (since this radius is squared in finding the area of the circle), the volume of the typical slice is still $\pi(f(x))^2\,dx$. Thus, the formula holds for all continuous functions f.

The result is summarized below:

DISK METHOD

Let f be continuous on $[a,b]$. If the area between the graph of f and the x-axis on the interval $[a,b]$ is revolved about the x-axis, then the volume of the resulting solid of revolution is:



$$\int_a^b \pi(f(x))^2\,dx$$

EXAMPLE

testing the formula;

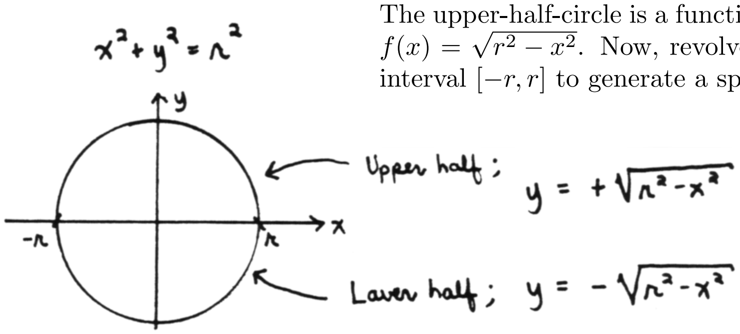
finding the volume of a sphere

Problem: Recall that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Derive this formula by investigating an appropriate integral.

Solution: The circle of radius r with center at the origin is the set of all points (x, y) satisfying the equation $x^2 + y^2 = r^2$; solving for y yields:

$$y^2 = r^2 - x^2 \iff |y| = \sqrt{r^2 - x^2} \iff y = \pm\sqrt{r^2 - x^2}$$

The upper-half-circle is a function; its equation is obtained by using the + sign: $f(x) = \sqrt{r^2 - x^2}$. Now, revolve this upper-half-circle about the x-axis on the interval $[-r, r]$ to generate a sphere of radius r.



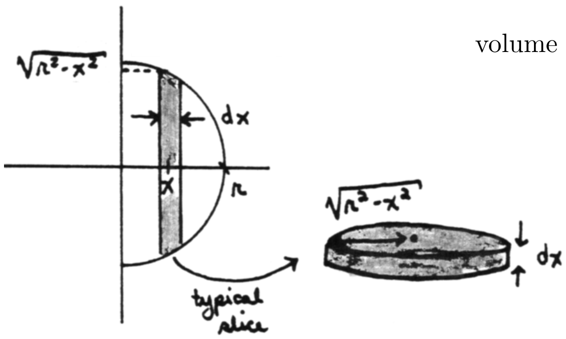
take advantage of symmetry

To cut down on the algebra, we can take advantage of symmetry and find the volume of the half sphere over the interval $[0, r]$; doubling this yields the desired result.

A typical infinitesimal slice of the desired solid at $x \in [0, r]$ has volume:

$$\pi(\sqrt{r^2 - x^2})^2\,dx$$

Then, using calculus to 'sum' these slices yields:



$$\begin{aligned}
\text{volume of sphere} &= 2 \cdot \int_0^r \pi\big(\sqrt{r^2 - x^2}\big)^2\,dx \\
&= 2\pi \int_0^r (r^2 - x^2)\,dx \\
&= 2\pi(r^2 x - \frac{x^3}{3})\Big|_0^r \\
&= 2\pi\big[(r^3 - \frac{r^3}{3}) - 0\big] = 2\pi\frac{2r^3}{3} \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

The expected result is indeed obtained.

EXERCISE 1 ♣ Repeat the previous exercise, without looking at the text.

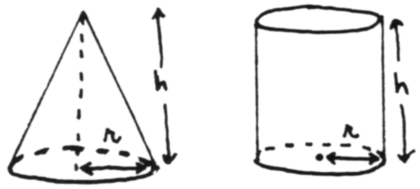
EXAMPLE

testing the formula; finding the volume of a right circular cone

Problem: Recall that the volume of a right circular cone of height h and base radius r is:

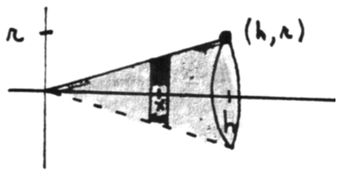
$$\frac{1}{3}\pi r^2 h$$

Thus, three such cones would completely fill the cylinder of height h and base radius r. Derive the formula for the volume of a right circular cone by investigating an appropriate integral.



Solution #1

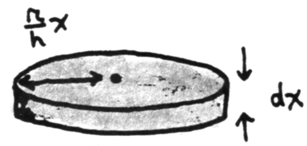
Solution #1: First, find the equation of the line passing through the point $(0,0)$ and (h,r); it has slope $\frac{r}{h}$ and passes through the origin, so has equation $y = \frac{r}{h}x$.



Revolving this graph about the x-axis on $[0,h]$ yields the desired solid.

An infinitesimal slice at $x \in [0,h]$ has volume

$$\pi(\frac{r}{h}x)^2\,dx$$



and integration over $[0,h]$ yields

$$\begin{aligned}
\text{desired volume} &= \int_0^h \pi(\frac{r}{h}x)^2\,dx \\
&= \pi\frac{r^2}{h^2}\int_0^h x^2\,dx \\
&= \pi\frac{r^2}{h^2}\cdot\frac{x^3}{3}\Big|_0^h \\
&= \pi\frac{r^2}{h^2}\big[(\frac{h^3}{3}) - 0\big] \\
&= \frac{1}{3}\pi r^2 h\ ,
\end{aligned}$$

which is of course the anticipated result.

EXERCISE 2

♣ 1. Repeat the previous example, without looking at the text.

♣ 2. Use calculus to find the volume of a cylinder of height h and radius r, by investigating an appropriate integral.

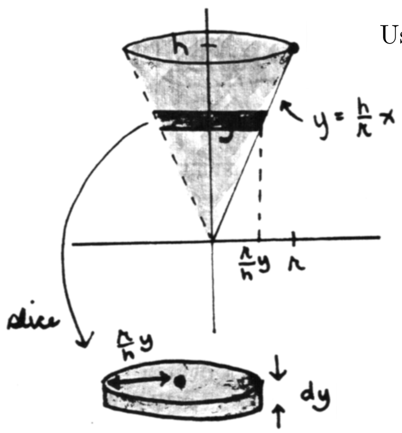
Solution #2

Solution #2. The 'disk formula' can also be applied to find the volume of a solid that results from revolution about the y-axis. It is only necessary that a typical 'slice' be a disk. This is illustrated in the next, alternate, derivation of the volume of a right circular cone.

This time, generate the cone by revolving the line $y = \frac{h}{r}x$ about the y-axis.

Observe that $y = \frac{h}{r}x \iff x = \frac{r}{h}y$. Make a thin (thickness dy) horizontal slice at a typical distance y, where $y \in [0, h]$. The volume of this slice is:

$$\pi(\frac{r}{h}y)^2 \cdot dy$$



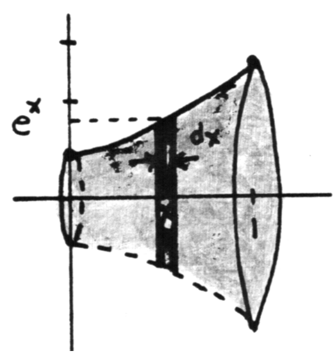
Using calculus to sum these slices as y varies from 0 to h yields:

$$\begin{aligned} \text{desired volume} &= \int_0^h \pi(\frac{r}{h}y)^2\,dy \\ &= \cdots = \frac{\pi r^2 h}{3} \end{aligned}$$

EXAMPLE

Problem: Revolve the graph of e^x about the x-axis on $[0, 1]$. Find the volume of the resulting solid of revolution.

Solution:



$$\begin{aligned} \text{desired volume} &= \int_0^1 \pi(e^x)^2\,dx \\ &= \pi\int_0^1 e^{2x}\,dx \\ &= \pi \cdot \frac{1}{2}e^{2x}\Big|_0^1 \\ &= \frac{\pi}{2}(e^2 - 1) \approx 10.036 \end{aligned}$$

EXAMPLE

Problem: Take the graph of x^2 on $[0, 2]$ and revolve it about the y-axis. Find the volume of the resulting solid of revolution.

Solution: Observe that the top of the desired solid is at $y = 4$, and the bottom is at $y = 0$.

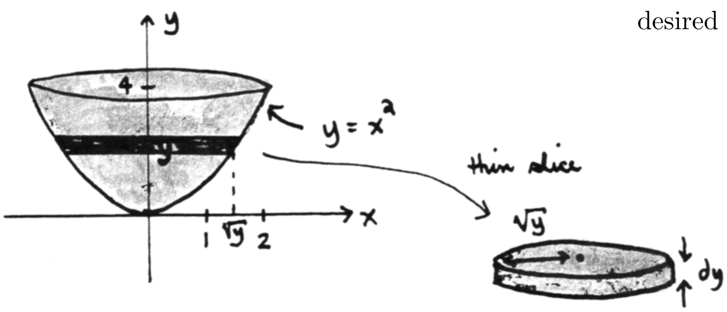
Make a thin (thickness dy) horizontal slice through the solid at distance $y \in [0, 4]$. For $y \geq 0$,

$$y = x^2 \quad \Longleftrightarrow \quad x = \pm\sqrt{y}\ ,$$

so the radius of the thin slice is $\sqrt{y}$, and has volume:

$$\pi(\sqrt{y})^2\, dy$$

Using calculus to 'sum' the disks as y goes from 0 to 4 yields:



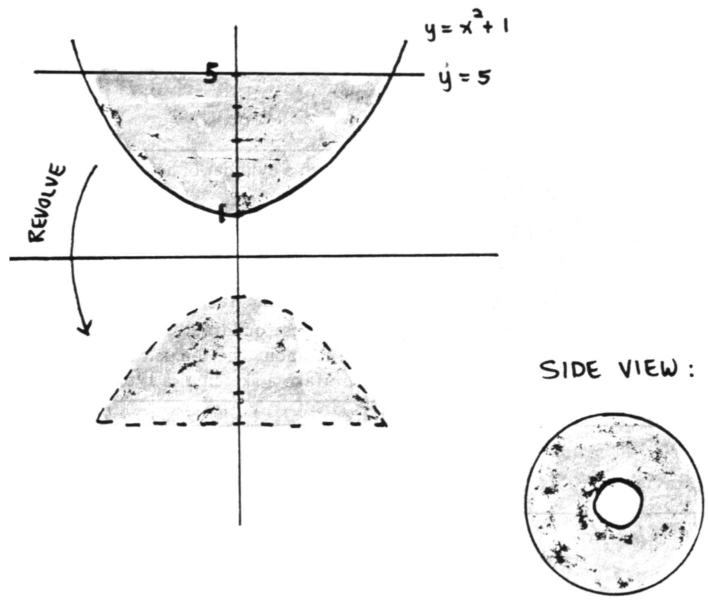
$$\begin{aligned}\text{desired volume} &= \int_0^4 \pi(\sqrt{y})^2\, dy \\ &= \pi \int_0^4 y\, dy \\ &= \pi \cdot \frac{y^2}{2}\Big|_0^4 \\ &= \frac{\pi}{2}(16 - 0) = 8\pi\end{aligned}$$

EXAMPLE

a solid with a hole

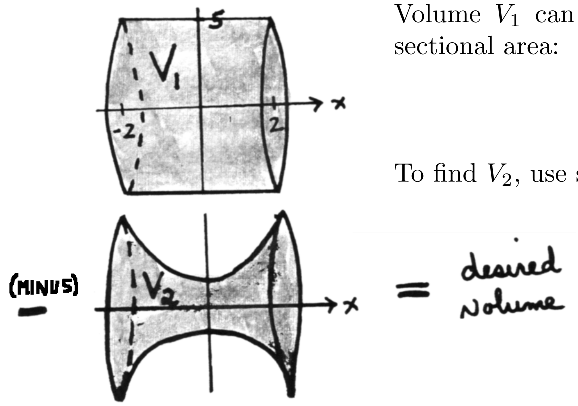
Problem: Find the volume of the solid generated by taking the region bounded by $y = 5$ and $y = x^2 + 1$, and revolving it about the x-axis.

Solution: The resulting solid of revolution has a hole in it. Note that the graphs $y = 5$ and $y = x^2 + 1$ intersect at values of x for which $5 = x^2 + 1$; solving this equation gives $x = \pm 2$. This problem will be solved in two different ways.



Approach #1;
view the desired volume as a difference of volumes

Approach #1. The desired volume can be viewed as a difference of volumes:
Revolve $y = 5$ about the x-axis on $[-2, 2]$; call this volume V_1 .
Revolve $y = x^2 + 1$ about the x-axis on $[-2, 2]$; call this volume V_2 .
The desired volume is $V_1 - V_2$.



Volume V_1 can be found *without* calculus, since it is has a constant cross-sectional area:

$$V_1 = (\text{area of circle})(\text{height}) = \pi(5)^2 \cdot 4 = 100\pi$$

To find V_2, use symmetry and calculus:

$$\begin{aligned} V_2 &= 2\int_0^2 \pi(x^2+1)^2\,dx \\ &= 2\pi\int_0^2 (x^4+2x^2+1)\,dx \\ &= \cdots = \frac{412\pi}{15} \end{aligned}$$

Thus, the desired volume is $V_1 - V_2 = 100\pi - \frac{412\pi}{15} = 72.5\bar{3}\pi$.

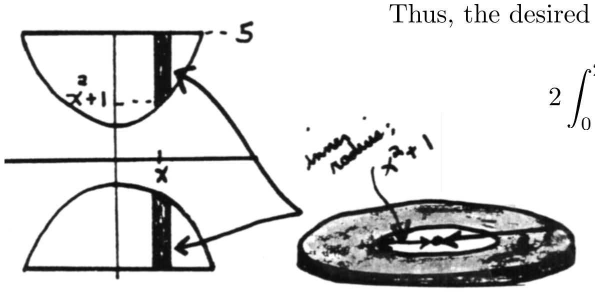
Approach #2;
look at an infinitesimal slice

Approach #2. This time, let's investigate a typical thin slice of the desired solid, at a distance $x \in [0, 2]$. It is shaped like a donut, and has volume:

$$\pi(5)^2\,dx - \pi(x^2+1)^2\,dx = \pi[5^2 - (x^2+1)^2]\,dx$$

Thus, the desired volume is (again using symmetry):

$$\begin{aligned} 2\int_0^2 \pi[5^2-(x^2+1)^2]\,dx &= 2\pi\int_0^2 24 - x^4 - 2x^2\,dx \\ &= \cdots = 72.5\bar{3}\pi \end{aligned}$$



QUICK QUIZ *sample questions*	1. Show two ways in which a cylinder of height h and radius r can be generated as a solid of revolution. 2. Show two ways in which a right circular cone of height h and base radius r can be generated as a solid of revolution. 3. Revolve the graph of x^2 about the x-axis on $[0, 1]$. Find the volume of the resulting solid of revolution. Make a sketch of a typical 'slice'. 4. Take the area in the first quadrant bounded by $y = x^2$, the y-axis, and $y = 1$, and revolve it about the y-axis. Find the volume of the resulting solid of revolution. Make a sketch of a typical 'slice'.

KEYWORDS *for this section*	*Generating a solid of revolution by revolving about the x-axis; what is the volume of a typical thin slice? The disk method, revolving about the y-axis, a solid with a hole.*

END-OF-SECTION EXERCISES

♣ Revolve each region described below about the x-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

1. Bounded by: $y = 2x$, $x = 0$, $x = 1$, and the x-axis
2. Bounded by: $y = x^3$, $x = 1$, $x = 2$, and the x-axis
3. Bounded by: $y = \frac{1}{x}$, $x = 1$, $x = 2$, and the x-axis
4. Bounded by: $y = |x|$, $x = -1$, $x = 1$, and the x-axis
5. Bounded by: $y = \sqrt{x}$, $x = 0$, $x = 4$, and the x-axis
6. Bounded by: $y = e^x + 1$, $x = 0$, $x = 1$, and the x-axis
7. In the first quadrant, bounded by: $y = x^2$, $y = 0$, $y = 4$, and the y-axis
   (A typical slice will have a hole—be careful.)
8. Bounded by: $y = x^3$, $y = 0$, $y = 8$, and the y-axis
   (A typical slice will have a hole—be careful.)

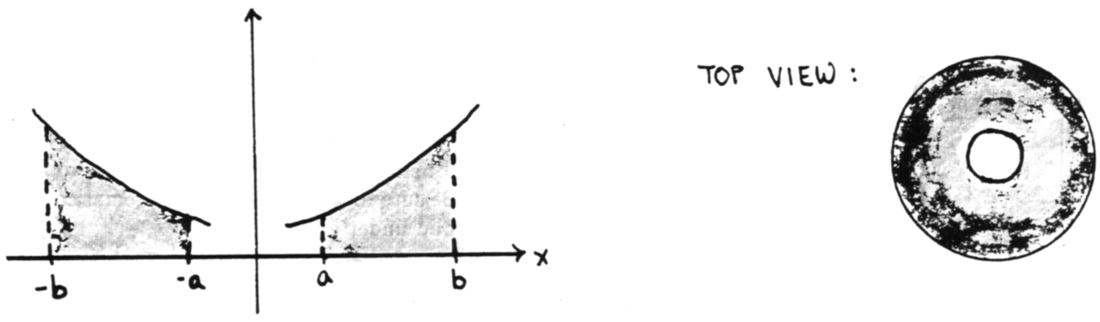
Revolve each region described below about the y-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

9. Bounded by: $y = x$, $y = 0$, $y = 2$, and the y-axis
10. Bounded by: $y = 2x$, $y = 1$, $y = 3$, and the y-axis
11. Bounded by: $y = \frac{1}{x}$, $y = 1$, $y = 2$, and $x = \frac{1}{2}$
    (The resulting solid will have a hole—be careful.)

7.7 Finding the Volume of a Solid Of Revolution—Shells

generating a volume of revolution; revolving about the y-axis

Let f be continuous and nonnegative on $[a,b]$. Take the area bounded by the graph of f and the x-axis on the interval $[a,b]$, and revolve it about the y-axis. In some instances, the volume of the resulting solid of revolution can be found by looking at disks (or disks with holes) that are sliced *horizontally*, that is, perpendicular to the y-axis. However, it is shown in this section that there is a more natural way to view the resulting solid in this case; as being built up from *thin shells.*



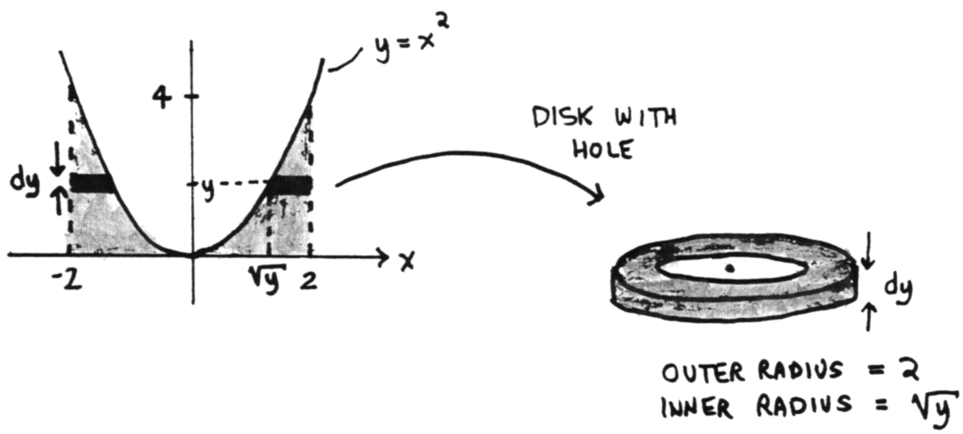
EXAMPLE

horizontal disks with holes

Problem: Revolve the area bounded by $f(x) = x^2$ and the x-axis on $[0,2]$ about the y-axis. Find the volume of the resulting solid of revolution, using horizontal disks.

Solution: The method discussed here was introduced in the previous section. This is a review problem.

When $x = 2$, $y = 2^2 = 4$. Let y denote a typical value in $[0,4]$, and cut a thin horizontal slice (thickness dy) from the desired volume at this value of y. As usual, view dy as an *infinitesimally small* piece of the y-axis. The slice is a disk with a hole (a donut); what is its volume?



get an expression for x in terms of y

Given y, it is necessary to know the corresponding value of x (since the x-value of the point determines the inner radius of the donut). That is, a formula for x in terms of y is needed. Solving $y = x^2$ for x yields:

$$y = x^2 \quad \iff \quad |x| = \sqrt{y} \quad \iff \quad x = \pm\sqrt{y}$$

Two answers are obtained, since, viewed from the y-axis, the curve is *not* a function of y. The positive number $+\sqrt{y}$ is chosen to give the inner radius of the donut.

The volume of this slice is found by first getting the volume of the slice when it doesn't have a hole, and then subtracting off the volume of the hole:

$$\pi(2)^2\,dy - \pi(\sqrt{y})^2\,dy \;=\; \pi(4-y)\,dy$$

Then, 'sum' these slices, as y travels from 0 to 4:

$$\begin{aligned}\text{desired volume} &= \int_0^4 \pi(4-y)\,dy \\ &= \pi(4y - \frac{y^2}{2})\Big|_0^4 \\ &= \pi(16 - \frac{16}{2}) = 8\pi\end{aligned}$$

EXERCISE 1

♣ Problem: Revolve the area bounded by $f(x) = x^3$ and the x-axis on $[0, 2]$ about the y-axis. Find the volume of the resulting solid of revolution, by using horizontal disks. Make a sketch of the volume that you are finding. Also make a sketch of a typical 'slice'.

disadvantages of the disk approach in this setting

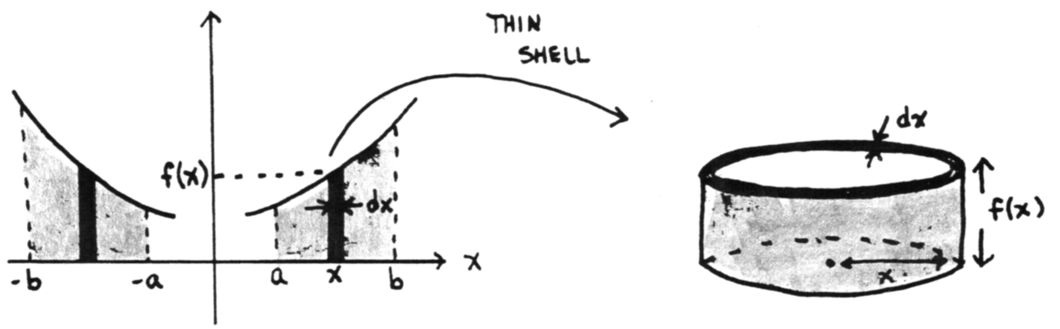
The previous approach was 'hard' in two ways:

- It was necessary to solve for x in terms of y. This is unnatural, since although y *is* a function of x, x may *not* be a function of y.
- The typical slice was not a simple disk, but a disk with a hole, which is more difficult to work with.

These disadvantages are overcome by viewing the volume in a different way, as discussed below.

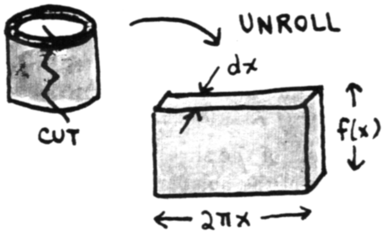
the shell method

Let f be continuous and nonnegative on $[a, b]$. Take the area bounded by the graph of f and the x-axis on $[a, b]$, and revolve it about the y-axis. Take a 'donut cutter' of radius x (where x is a number between a and b), and, coming down from the top, punch a thin shell (thickness dx) from the solid of revolution.



the shell has volume $2\pi x f(x)\,dx$

To calculate the volume of this thin shell, observe first that its circumference is $2\pi(\text{radius}) = 2\pi x$, and its height is $f(x)$. Cut the shell and unroll it. The volume is now easy to calculate:



$$(\text{width})(\text{height})(\text{thickness}) = (2\pi x)f(x)(dx)$$

Summing the volumes of these shells as x travels from a to b yields the desired volume of revolution:

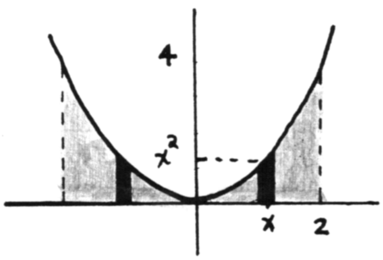
$$(\text{desired volume}) = \int_a^b 2\pi x\, f(x)\, dx$$

Remember that a rigorous derivation of this formula would require partitioning, and looking at Riemann sums.

EXAMPLE

Problem: Revolve the area bounded by $f(x) = x^2$ and the x-axis on $[0,2]$ about the y-axis. Using shells, find the volume of the resulting solid of revolution.

Solution: The solution is now much easier than when the volume was viewed as being 'built up' from horizontal disks:



$$\begin{aligned}(\text{desired volume}) &= \int_0^2 2\pi x\,(x^2)\,dx \\ &= 2\pi \int_0^2 x^3\,dx \\ &= 2\pi\ \frac{x^4}{4}\Big|_0^2 \\ &= \frac{\pi}{2}(16-0) \\ &= 8\pi\end{aligned}$$

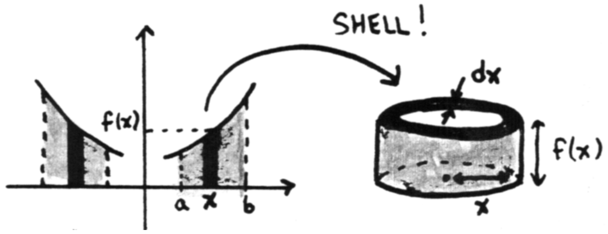
Note that you only integrate from 0 to 2; yet the volume being found extends from -2 to 2. (♣ Why is this?)

The result is summarized below.

SHELL METHOD

Let f be continuous and nonnegative on $[a,b]$. If the area between the graph of f and the x-axis on $[a,b]$ is revolved about the y-axis, then the volume of the resulting solid of revolution is:

$$\int_a^b 2\pi x\, f(x)\, dx$$

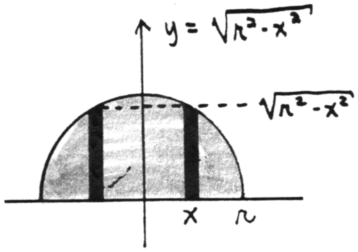


EXAMPLE

finding the volume of a sphere, using shells

Problem: Derive the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere of radius r, using shells.

Solution: As shown in the previous section, the upper-half circle of radius r has equation $y = \sqrt{r^2 - x^2}$. Take the area bounded by this curve and the x-axis on $[0, r]$ and revolve it about the y-axis. Double this volume to obtain the desired result.



$$u = r^2 - x^2$$
$$du = -2x\,dx$$
$$x = 0 \Rightarrow u = r^2$$
$$x = r \Rightarrow u = 0$$

$$\begin{aligned}
\text{desired volume} &= 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\,dx \\
&= 4\pi\int_0^r x\sqrt{r^2 - x^2}\,dx \\
&= \frac{4\pi}{(-2)}\int_0^r (-2)x\sqrt{r^2 - x^2}\,dx \\
&= -2\pi\int_{r^2}^0 u^{1/2}\,du \\
&= -2\pi \cdot \frac{2}{3}u^{3/2}\big|_{r^2}^0 \\
&= -\frac{4\pi}{3}[0 - (r^2)^{3/2}] \\
&= -\frac{4\pi}{3}(-r^3) \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

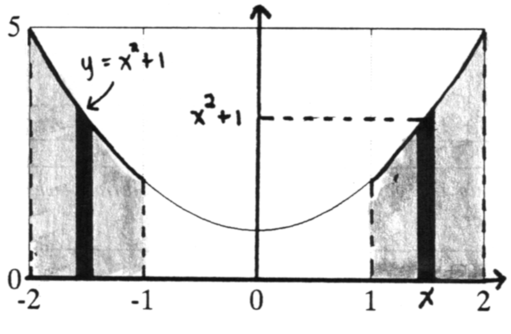
EXERCISE 2	♣ Cite a reason for every step in the previous example.

EXAMPLE

Problem: Revolve the region bounded by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$ about the y-axis. Find the volume of the resulting solid of revolution in *two ways*: using shells, and using disks. In each case, sketch the typical 'slice' (or 'slices'). Be sure to write complete mathematical sentences.

Solution using shells:

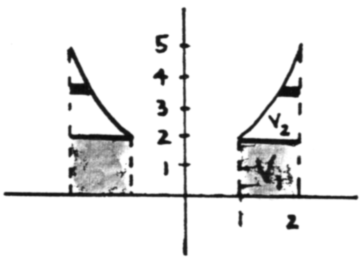
$$\int_1^2 2\pi x(x^2 + 1)\,dx = 2\pi\int_1^2 x^3 + x\,dx$$



$$= 2\pi\left(\frac{x^4}{4} + \frac{x^2}{2}\right)\Big|_1^2$$

$$= 2\pi\left[(4 + 2) - \left(\frac{1}{4} + \frac{1}{2}\right)\right]$$

$$= 2\pi\left(6 - \frac{3}{4}\right) = \frac{21\pi}{2}$$



Solution using disks: This time, the solid must be separated into two pieces, because the 'slices' look different, depending upon the value chosen for y.

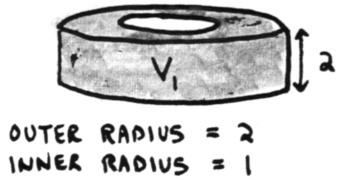
The volume V_1 of the bottom piece can be found without calculus; it is a cylinder, with a hole, of height 2. The outer radius is 2 and the inner radius is 1:

$$V_1 = \pi 2^2 \cdot 2 - \pi 1^2 \cdot 2 = 2\pi(2^2 - 1^2) = 2\pi(3) = 6\pi$$

The second volume V_2 requires using disks with holes:

Let $y > 2$, and find the corresponding x-value:

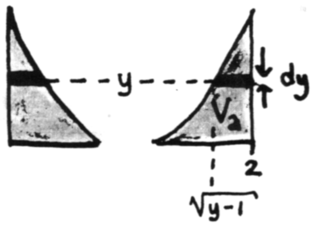
$$y = x^2 + 1 \iff x^2 = y - 1 \iff x = \pm\sqrt{y-1}$$



Therefore, a typical slice for the upper section has inner radius $\sqrt{y-1}$, and outer radius 2, and thus has volume:

$$\pi 2^2 \cdot dy - \pi(\sqrt{y-1})^2 \cdot dy = \pi(4 - (y-1))\, dy = \pi(5-y)\, dy$$

'Summing' these disks as y travels from 2 to 5 yields:



$$\begin{aligned}\int_2^5 \pi(5-y)\, dy &= \pi(5y - \frac{y^2}{2})\Big|_2^5 \\ &= \pi\big[(25 - \frac{25}{2}) - (10-2)\big] \\ &= \pi(\frac{9}{2})\end{aligned}$$

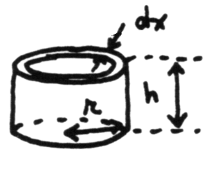
The total volume is

$$V_1 + V_2 = 6\pi + \frac{9}{2}\pi = \frac{21\pi}{2}\ ,$$

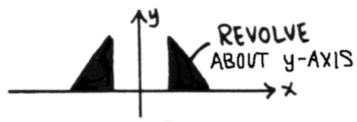
which agrees with the earlier result. You should be convinced that shells were *much easier* in this situation!

QUICK QUIZ
sample questions

1. What is the volume of the thin shell sketched here?



2. Revolve the area bounded by $y = x$ and the x-axis on $[0,2]$ around the y-axis. Use shells to find the volume of the resulting solid of revolution.
3. In the sketch below, would it be easier to use horizontal disks or shells to find the volume? Justify your answer.



KEYWORDS
for this section

The shell method for finding the volume of a solid of revolution; what is the volume of a typical thin slice?

END-OF-SECTION EXERCISES

Revolve each region described below about the y-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

1. Bounded by: $y = 2x$, $x = 0$, $x = 1$, and the x-axis
   (Find the volume in *two ways*; using shells, and using disks.)
2. Bounded by: $y = 2x$, $x = 1$, $x = 2$, and the x-axis
   (Find the volume in *two ways*; using shells, and using disks.)
3. Bounded by: $y = e^x$, $x = 0$, $x = 1$, and the x-axis
4. Bounded by: $y = e^x$, $x = 1$, $x = 2$, and the x-axis

Now, do these additional problems:

5. Derive the formula for the volume of a right circular cone of base radius r and height h, using shells.
6. Derive the formula for the volume of a cylinder of height h and base radius r, using shells.

NAME (1 pt)
SAMPLE TEST, worth 100 points, Chapter 7

Show all work that leads to your answers. Good luck!

1. (15 pts)

TRUE or FALSE. Circle the correct response. (3 points each)

T F Suppose f is continuous on $\mathbb{R}$ and $F' = f$. Then, $\int_a^b f(x)\,dx = F(b) - F(a)$.

T F If f is continuous on $\mathbb{R}$, then $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

T F If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ on $[a, b]$.

T F If f is continuous, then $\int_a^b f(x)\,dx$ is a function of x.

T F For all functions f that are defined at a, if $x \to a$, then $f(x) \to f(a)$.

2. (8 pts)

Find the area bounded by the graph of $\ln x$ and the x-axis on the interval $[e, e^2]$. Make a sketch that shows the area you are finding.

3. (12 pts)

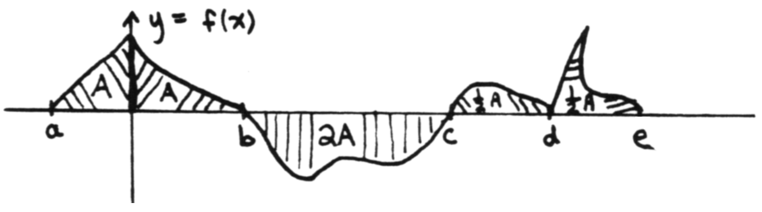
Refer to the sketch below, where certain areas are labeled. Evaluate the following integrals, if possible. If not possible with the given information, so state.

(3 pts) $\int_a^b f(t)\,dt$

(3 pts) $\int_b^0 f(x)\,dx$

(3 pts) $\int_a^c f(u)\,du$

(3 pts) Let $z \in (b, c)$, and find $\int_b^z f(x)\,dx + \int_z^c f(x)\,dx$



4.
(5 pts)

In a few words, discuss why the notation $\int_a^b f(x)\,dx$ is used for definite integrals.

5.
(20 pts)

Evaluate the following integrals. Use any appropriate techniques. Be sure to write complete sentences.

(5 pts) $\displaystyle\int_{-1}^{0} e^{3x}\,dx$

(5 pts) $\displaystyle\int_{0}^{1} (2x-1)^7\,dx$

(5 pts) $\displaystyle\int \frac{2t}{t-1}\,dt$

(5 pts) $\displaystyle\int \frac{1}{t\ln t}\,dt$

6.
(8 pts)

Find the area in the first quadrant, bounded by $y = x^2$ and $y = x^4$. Sketch the area that you are finding. Show all work that leads to your answer.

7.
(13 pts)

(5 pts) Give two different partitions of the interval $[0,1]$, each with norm $\frac{1}{3}$.

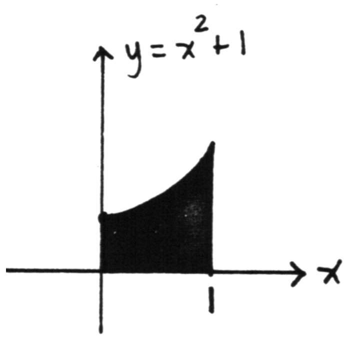
(8 pts) Find a Riemann sum for $f(x) = x^2$ corresponding to the partition $\{0,1,2\}$ of the interval $[0,2]$. (There are many correct answers possible.) What is your Riemann sum an approximation to?

8.
(18 pts)

Revolve the area shaded below around the y axis.

(8 pts) Find the volume of the resulting solid of revolution by using SHELLS.

(10 pts) Find the volume of the resulting solid of revolution by using horizontal DISKS.



ABBREVIATED SOLUTIONS TO
QUICK QUIZ QUESTIONS and ODD-NUMBERED END-OF-SECTION EXERCISES

CHAPTER 1. ESSENTIAL PRELIMINARIES

Section 1.1 The Language of Mathematics—Expressions versus Sentences

Quick Quiz:

1. a mathematical expression
2. numbers, functions, sets
3. $x = \frac{x}{2} + \frac{x}{2}$ (many others are possible)
4. $\sqrt{x} > 2$ and $4 - 3 = 7$ are sentences

End-of-Section Exercises:

1. EXP
3. SEN, T
5. SEN, F
7. SEN, T
9. SEN, ST/SF
11. SEN, T
13. EXP
15. SEN, T
17. SEN, ST/SF
19. SEN, T
21. SEN, F
23. SEN, T
25. SEN, ST/SF
27. EXP
29. SEN, T
31. SEN, T
33. SEN, ST/SF
35. SEN, F
37. Commutative Property of Addition
39. Distributive Property
41. If $x = 1$ and $y = 3$: $1 - 3 = 1 + (-3) = -2$
    If $x = 1$ and $y = -3$: $1 - (-3) = 1 + (-(-3)) = 1 + 3 = 4$
43. The expression xyz is not ambiguous; if one person computes this as $(xy)z$ and another as $x(yz)$, the same results are obtained.

Section 1.2 The Role of Variables

Quick Quiz:

1. The variables are x and y; the constants are A, B, and C.
2. With universal set $\mathbb{R}$, $x^2 = 3$ has solution set $\{\sqrt{3}, -\sqrt{3}\}$. With universal set $\mathbb{Z}$, the solution set is empty.
3. To 'solve' an equation means to find all choices (from some universal set) that make the equation true.
   Three solutions of $x + y = 4$: $(0, 4)$, $(4, 0)$, and $(2, 2)$. There are an infinite number of solution pairs!
4. The equation $x^2 \geq 0$ is (always) true. The equation $x > 0$ is conditional; it is true for $x \in (0, \infty)$, and false otherwise.
5. Choose two from the following list:
   - variables are used in mathematical expressions to denote quantities that are allowed to vary (like in the formula $A = \pi r^2$);
   - variables are used to denote a quantity that is initially unknown, but that one would like to know (for example, 'solve $2x + 3 = 5$');
   - variables are used to state a general principle (like the commutative law of addition).

End-of-Section Exercises:

1. EXP
3. SEN, F
5. SEN, T
7. SEN, F

9. SEN, F

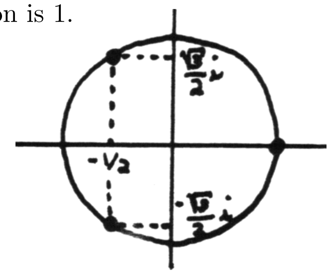
11. SEN, T

13. EXP

15. $\mathbb{R}$: the only solution is 1;
    the rational numbers: the only solution is 1;
    the integers: the only solution is 1.

17. $\mathbb{R}$: setting each factor to 0, the real number solutions are 1, $-\pi$, and $\frac{3}{2}$;
    the rational numbers: the only rational solutions are 1 and $\frac{3}{2}$;
    the integers: the only integer solution is 1.

19. a) The points are plotted at right:



b) Since $(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = \frac{1}{4} + \frac{3}{4} = 1$, the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ lies on the unit circle. Same for the remaining point.

c) Clearly, the number 1 satisfies the equation. To see that $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ satisfies $x^3 = 1$, observe that:

$$\begin{aligned}
(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3 &= (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= (\frac{1}{4} - \frac{\sqrt{3}}{2}i - \frac{3}{4})(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\
&= \frac{1}{4} + \frac{3}{4} \\
&= 1
\end{aligned}$$

Similarly for the remaining number.

d) The equation $x^3 - 1 = 0$ has the same solutions as the equation $x^3 = 1$, so the problem has already been solved.

Section 1.3 Sets and Set Notation

Quick Quiz:

1. F; the set has only 3 members
2. F
3. T
4. F
5. $105 = 5 \cdot 3 \cdot 7$; F

End-of-Section Exercises:

1. EXP; this is a set
3. SEN, T
5. SEN, F
7. SEN, C. The truth of this sentence depends upon the set S and the element x.
9. SEN, C. The truth depends on x. If x is 1, 2, or 3, then the sentence is true. Otherwise, it is false.
11. EXP; this is a set

13. SEN, T
15. SEN, C. The only number that makes this true is 1.
17. SEN, T. No matter what real number is chosen for x, both component sentences '$|x| \geq 0$' and '$x^2 \geq 0$' are true.
19. SEN, T. The two elements are both sets: $\{1\}$ and $\{1, \{2\}\}$
21. SEN, F. The number $\frac{3}{7}$ is in reduced form; the denominator has factors other than 2's and 5's.

Section 1.4 Mathematical Equivalence

Quick Quiz:

1. F; when x is -2, the first sentence is false, but the second is true.
2. THEOREM: For all real numbers a, b and c:

$$a = b \iff a + c = b + c$$

3. THEOREM: For all real numbers a, b and c:

$$a > b \iff a + c > b + c$$

4. $\{(x, y) \mid x \neq 3 \text{ and } y \neq 0\}$
5. equivalent
6. expressions; sentences

End-of-Section Exercises:

3. SEN, T. Both sentences have the same implied domain, and the same solution set, $\{4\}$.
5. SEN, T. Both sentences have the same implied domain, and the same solution set, $\{0\}$.
7. EXP
9. SEN, T. Both sentences have the same implied domain, and the same solution set, $\{-2\}$.
11.

$$\begin{aligned} 5x - 7 = 3 &\iff 5x = 10 \quad \text{(add 7)} \\ &\iff x = 2 \quad \text{(divide by 5)} \end{aligned}$$

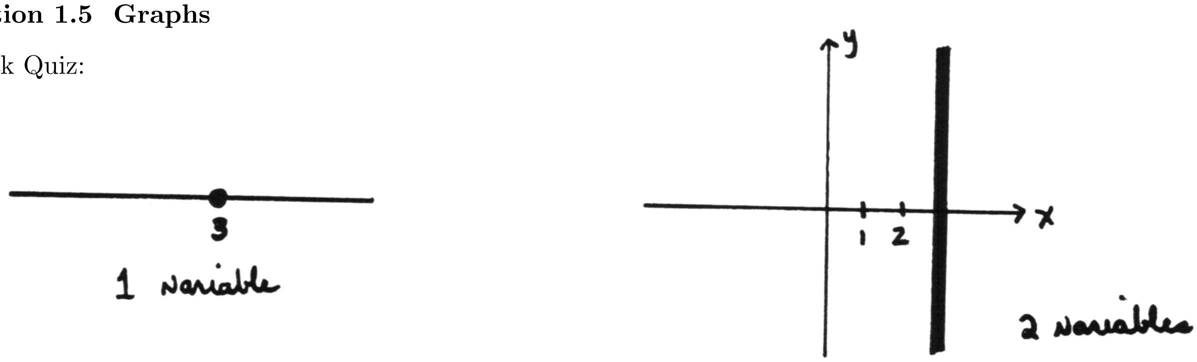
13.

$$\begin{aligned} 3x < x - 11 &\iff 2x < -11 \quad \text{(subtract } x\text{)} \\ &\iff x < -\frac{11}{2} \quad \text{(divide by the positive number 2)} \end{aligned}$$

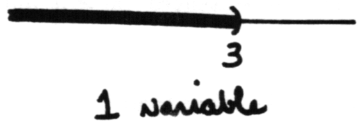
Section 1.5 Graphs

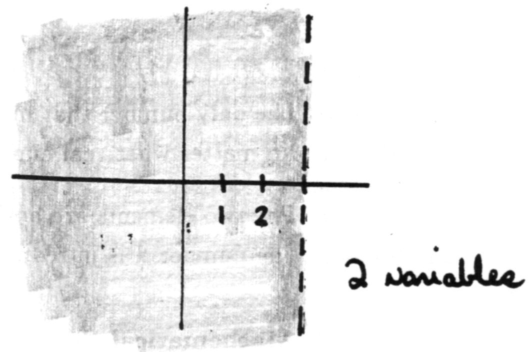
Quick Quiz:

1.

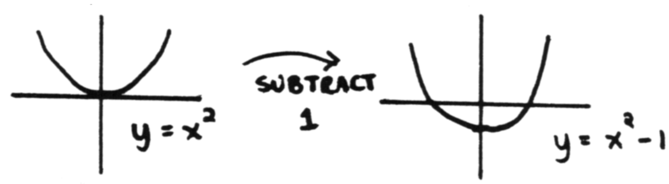


2.

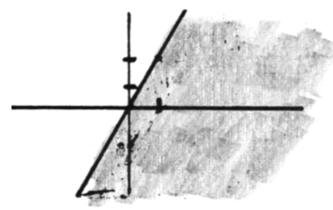




3. $y - x^2 + 1 = 0 \iff y = x^2 - 1$;

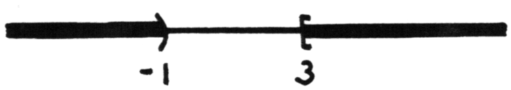


4. First graph the boundary, $y = 2x$. We want all points on or below this line.



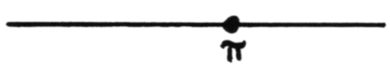
5. TRUE

6.



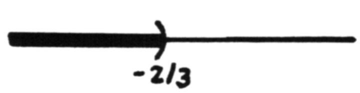
End-of-Section Exercises:

1. $x = \pi$



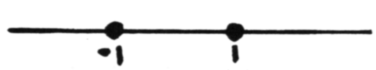
3. $|x| = 2 \iff x = 2$ or $x = -2$

5. $3x < -2 \iff x < -\frac{2}{3}$



7. $x = 0$ or $|x| = 1$

9. $x = 1$ or $|x| = 1$

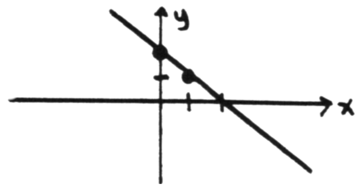


11. The critical observation here is that there are TWO numbers whose absolute value is 7: 7, and -7. Thus:

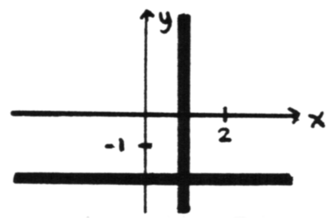
$$\begin{aligned} |3x+1| = 7 &\iff 3x+1 = 7 \ \text{ or } \ 3x+1 = -7 \\ &\iff 3x = 6 \ \text{ or } \ 3x = -8 \\ &\iff x = 2 \ \text{ or } \ x = -\frac{8}{3} \end{aligned}$$

The solution set is $\{2, -\frac{8}{3}\}$.

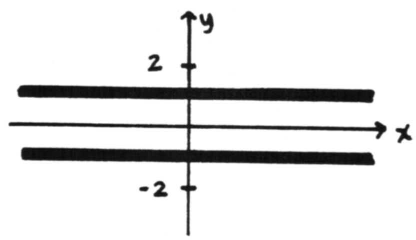
13. $x + y = 2 \iff y = -x + 2$. The graph is the line that crosses the y-axis at 2, and has slope -1.



15. The graph of '$x = 1$ or $y = -2$' is the set of all points with x-coordinate 1, together with all points with y-coordinate -2. The graph is shown below.



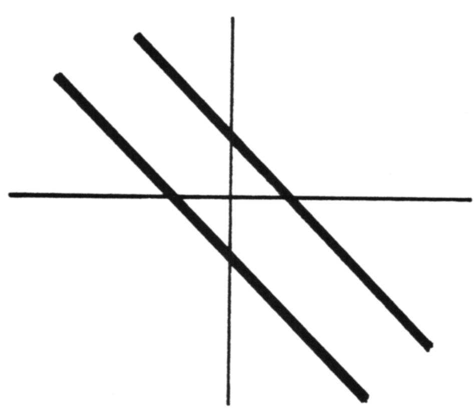
17. The solution set of $|y| = 1$, viewed as an equation in two variables, is $\{(x, y) \mid x \in \mathbb{R}, |y| = 1\}$. Thus, we seek all points with y-coordinates 1 or -1. See the graph below.



19.

$$
\begin{aligned}
|x + y| = 1 \quad &\iff \quad x + y = 1 \ \text{ or } \ x + y = -1 \\
&\iff \quad y = -x + 1 \ \text{ or } \ y = -x - 1
\end{aligned}
$$

The graph is the two lines shown below.



CHAPTER 2. FUNCTIONS

Section 2.1 Functions and Function Notation

Quick Quiz:

1. In the graph shown, y is a function of x, because for every x, there is a unique y. That is, the graph passes a vertical line test.

   However, x is not a function of y. It is NOT true that for every y, there is a unique x. That is, the graph does NOT pass a horizontal line test.

2. $$x^2 - y + 1 = 0 \iff y = x^2 + 1$$

   For every value of x, there is a unique value of y. Thus, y is a function of x.

3. $$\begin{aligned} x^2 - y + 1 = 0 &\iff x^2 = y - 1 \\ &\iff x = \pm\sqrt{y-1} \end{aligned}$$

   For each allowable y-value, there are *two* x-values. Therefore, x is NOT a function of y.

4. Calling the function f: $f(x) = (\frac{x}{2} - 3)^2$

5. $g(-1) = 2(-1)^2 - 1 = 2 - 1 = 1$
   $g(x^2) = 2(x^2)^2 - 1 = 2x^4 - 1$

End-of-Section Exercises:

1. $f(0) = 0^3 - 1 = -1$
   $f(1) = 1^3 - 1 = 0$
   $f(-1) = (-1)^3 - 1 = -1 - 1 = -2$
   $f(t) = t^3 - 1$
   $f(f(2))$; first find $f(2)$: $f(2) = 2^3 - 1 = 7$; then, $f(f(2)) = f(7) = 7^3 - 1 = 342$

3. $f(-2) = |-2| = 2$

   $$f(t) = |t| = \begin{cases} t & \text{if } t \ge 0 \\ -t & \text{if } t < 0 \end{cases}$$

   $f(-t) = |-t| = |t|$
   $f(x^2) = |x^2| = |x|^2$

5. $h(-x) = \frac{1}{-x} = -\frac{1}{x}$
   $h(h(x)) = h(\frac{1}{x}) = \frac{1}{1/x} = x$
   $h(\frac{1}{x}) = \frac{1}{1/x} = x$
   $h(x + \Delta x) = \frac{1}{x+\Delta x}$
   $h(|x|) = \frac{1}{|x|} = \left|\frac{1}{x}\right|$

7. $h(1,1) = 1^2 + 1^2 - 1 = 1$
   $h(x,x) = x^2 + x^2 - 1 = 2x^2 - 1$
   $h(y,x) = y^2 + x^2 - 1 = h(x,y)$
   $h(x + \Delta x, y + \Delta y) = (x + \Delta x)^2 + (y + \Delta y)^2 - 1$

Section 2.2 Graphs of Functions

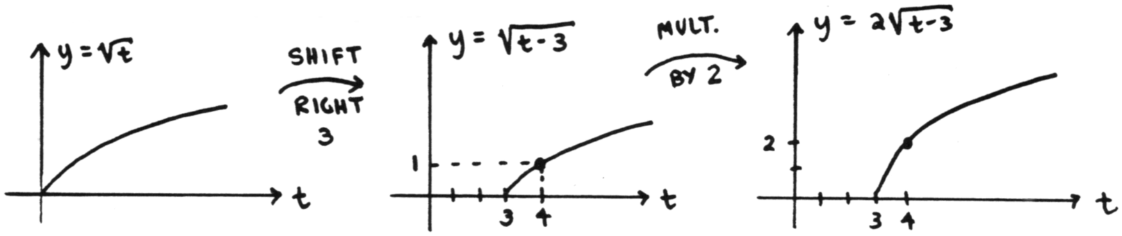
Quick Quiz:

1.

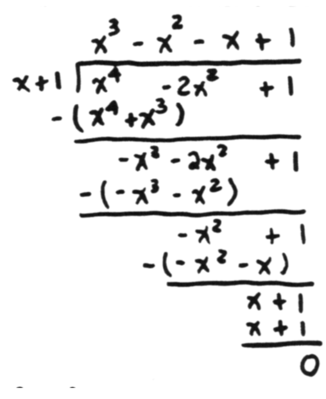
$$\begin{aligned}\mathcal{D}(f) &= \{x \mid 2x - 1 \geq 0 \text{ and } x^2 - 9 \neq 0\}\\ &= \{x \mid 2x \geq 1 \text{ and } x^2 \neq 9\}\\ &= \{x \mid x \geq \frac{1}{2} \text{ and } |x| \neq 3\}\\ &= \{x \mid x \geq \frac{1}{2} \text{ and } x \neq 3 \text{ and } x \neq -3\}\\ &= \{x \mid x \geq \frac{1}{2} \text{ and } x \neq 3\}\end{aligned}$$

Note that if $x \geq \frac{1}{2}$, then automatically, x is not equal to -3.

2. TRUE. The order that elements are listed in a set is unimportant. In this sentence, the '=' sign is being used for equality of SETS.
3. By definition, the *graph of f* is the set of points $\{(x, f(x)) \mid x \in \mathcal{D}(f)\}$. More precisely, the graph usually refers to a (partial) *picture* of this set of points, in the xy-plane.
4. $\mathcal{D}(f) = [3, \infty)$; the graph is 'built up' below:



5. $P(-1) = (-1)^4 - 2(-1)^2 + 1 = 1 - 2 + 1 = 0$; therefore -1 is a root of P. Long division by $x - (-1) = x + 1$ yields:

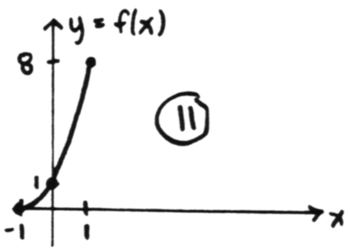


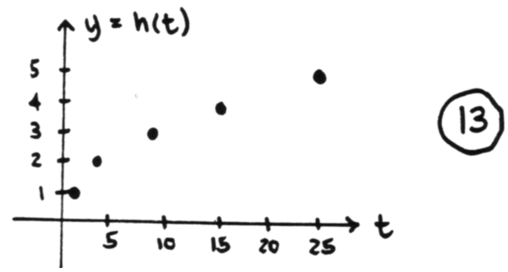
Therefore: $P(x) = (x + 1)(x^3 - x^2 - x + 1)$

End-Of-Section Exercises:

1. EXP; this is a set.
3. SEN; TRUE. The component sentences '$x \geq 2$ and $x \neq 1$' and '$x \geq 2$' always have the same truth values. Both are true on $[2, \infty)$ and false elsewhere.
5. SEN; TRUE
7. SEN; TRUE. Both sets are equal to $\{3\}$.
9. SEN; this is TRUE (by definition), providing g is a function of one variable.

The graphs requested in problems 11 and 13 are given below:





Section 2.3 Composite Functions

Quick Quiz:

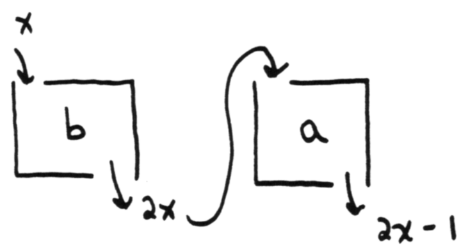
1. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. If $A = [1, 3)$ and $B = \{1, 2, 3\}$, then $A \cap B = \{1, 2\}$ since the only elements that are in *both* A and B are 1 and 2 .
2. The sentence '$[1, 3] \subset \{1, 3\}$' is FALSE. For example, $2 \in [1, 3]$, but $2 \notin \{1, 3\}$.

   The sentence '$\{1, 3\} \subset [1, 3]$' is TRUE.

   The sentence 'For all sets A and B, $A \cap B \subset A$' is TRUE. Everything that is in BOTH A and B, is also in A .
3. $(f + g)(x) := f(x) + g(x)$

   $\mathcal{D}(f + g) = \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} = \mathcal{D}(f) \cap \mathcal{D}(g)$
4. The function f takes an input x, multiplies by 2, then subtracts 1 . Define $b(x) = 2x$ and $a(x) = x - 1$; then:

$$\begin{aligned}(a \circ b)(x) &:= a(b(x)) \\ &= a(2x) \\ &= 2x - 1 \\ &:= f(x)\end{aligned}$$



5. $\mathcal{R}(f) = \{1, -1\}$. The only two output values taken on by f are 1 and -1 .

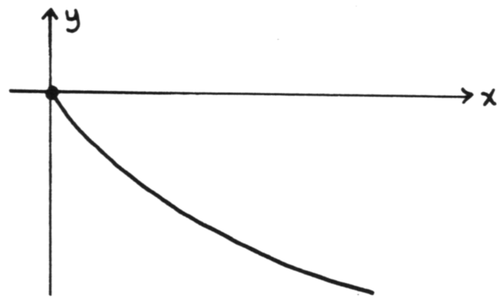
End-Of-Section Exercises:

1. EXP; $A \cup B$ is a set.
3. SEN; CONDITIONAL. The truth of the sentence '$A \subset B$' depends upon the sets A and B.
5. SEN; CONDITIONAL. The sentence '$\mathcal{R}(f) = \mathbb{R}$' states that the range of a function is the set of real numbers; the truth of this sentence depends upon the function f being referred to.
7. SEN; CONDITIONAL. The truth of this sentence depends upon the choice of functions f and g, and the choice of x.
9. SEN; FALSE. The set $\{a\}$ is NOT an element of the set $\{a, b\}$.
11. $\mathcal{R}(f) = [0, 8]$
13. $\mathcal{R}(h) = \{1, 2, 3, 4, 5\}$

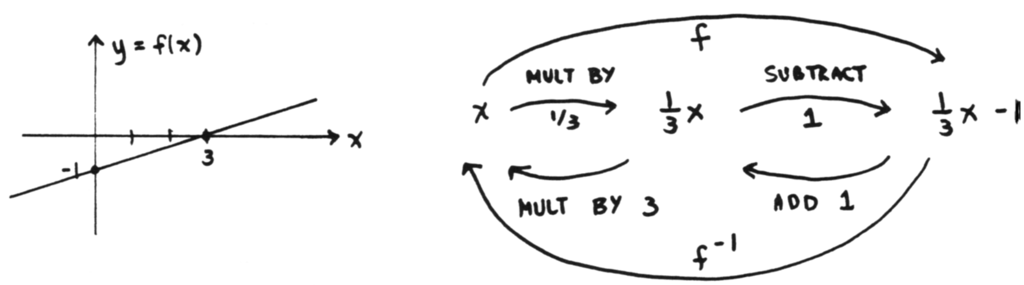
Section 2.4 One-to-One Functions and Inverse Functions

Quick Quiz:

1. The function $f(x) = x^2$ is NOT a one-to-one function. It does NOT have the property for every y, there is a unique x. That is, it does NOT pass the horizontal line test.
2. Translation: 'For every y in the range of f, there exists a unique x in the domain of f.' This is the 'one-to-one' condition for a function f.
3. One correct graph is shown below:

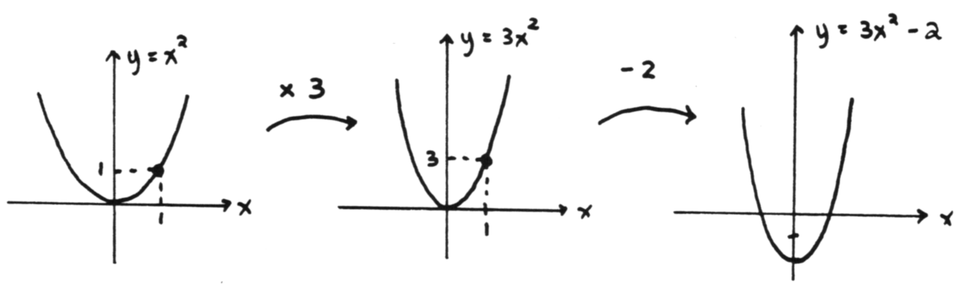


4. $f(f^{-1}(x)) = x \ \ \forall\, x \in \mathcal{R}(f)$
   $f^{-1}(f(x)) = x \ \ \forall\, x \in \mathcal{D}(f)$
5. The graph of f is the line shown below; it is clearly $1-1$. The function f takes an input x, multiplies by $\frac{1}{3}$, then subtracts 1; to 'undo' this, f^{-1} must take an input x, add 1, then divide by $\frac{1}{3}$ (that is, multiply by 3). Thus: $f^{-1}(x) = 3(x+1)$

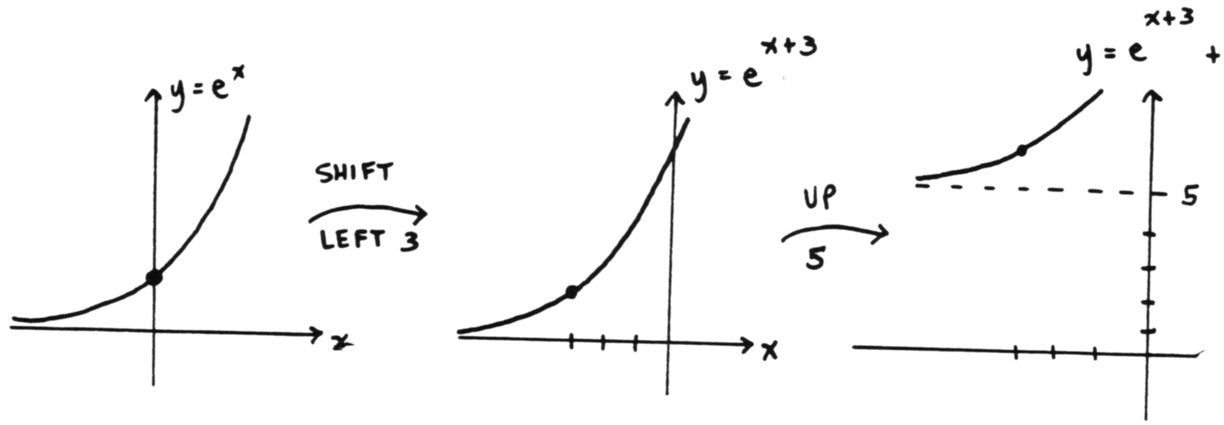


End-Of-Section Exercises:

1. EXP; this is a function f^{-1}, evaluated at x
3. SEN; T
5. EXP
7. SEN; T
9. EXP
11. $\mathcal{D}(f) = \mathbb{R}$, $\mathcal{R}(f) = (-2, \infty)$



13. $\mathcal{D}(h) = \mathbb{R}$, $\mathcal{R}(h) = (5, \infty)$

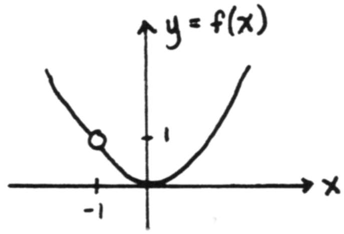


CHAPTER 3. LIMITS AND CONTINUITY

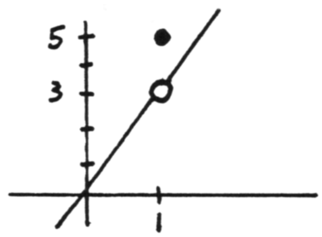
Section 3.1 Limits—The Idea

Quick Quiz:

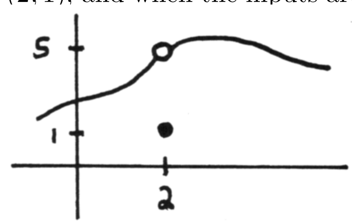
1. $\lim_{x \to -2} x^3 = (-2)^3 = -8$
2. $\lim_{x \to -1} f(x) = (-1)^2 = 1$



3. $\lim_{x \to 1} f(x) = 3(1) = 3$



4. There are many correct graphs. The graph must contain the point $(2, 1)$; and when the inputs are close to 2 (but not equal to 2), the outputs must be close to 5.



5. $|t - (-1)| \le 4$

End-of-Section Exercises:

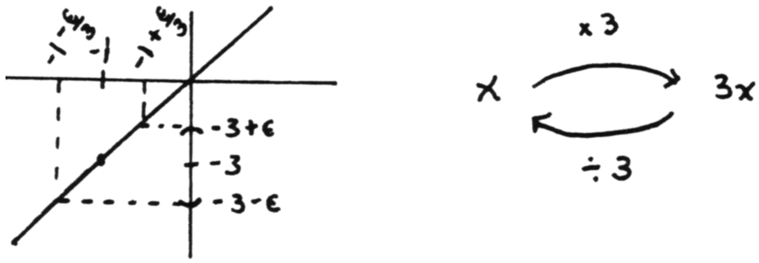
1. EXP
3. SEN; T
5. SEN; C
7. SEN; T
9. SEN; C
11. EXP
13. SEN; (always) T
15. SEN; (always) T
17. SEN; C
19. SEN; T

Section 3.2 Limits—Making It Precise

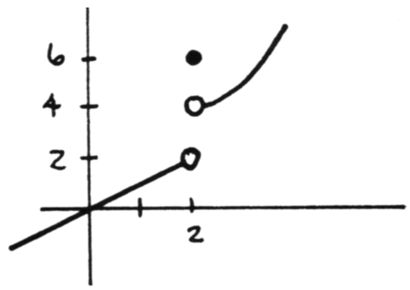
Quick Quiz:

1. $$\lim_{x \to c} f(x) = l \iff \forall\ \epsilon > 0,\ \exists\ \delta > 0,\ \text{such that if } 0 < |x-c| < \delta \text{ and } x \in \mathcal{D}(f),\ \text{then } |f(x)-l| < \epsilon$$

2. The ‘four step process’ is summarized using the mapping diagram and and sketch given below. Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{3}$.

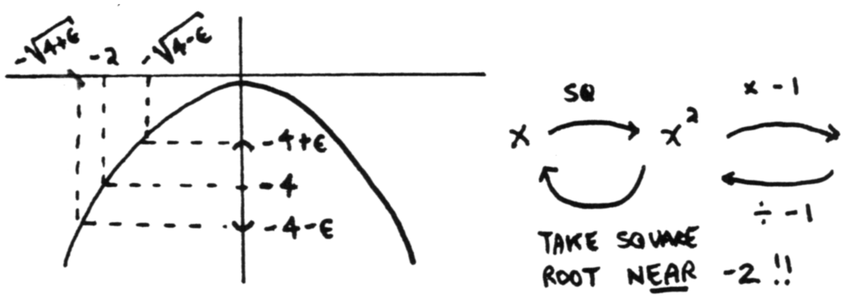


3. $\lim_{x\to 2} f(x)$ does not exist
$\lim_{x\to 2^+} f(x) = 4$
$\lim_{x\to 2^-} f(x) = 2$

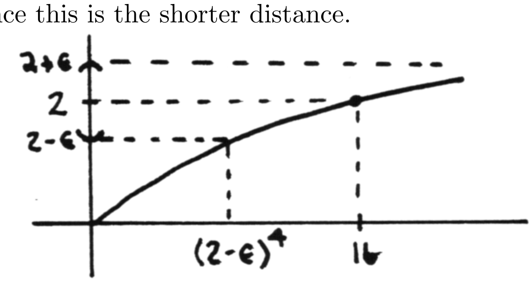


End-of-Section Exercises:

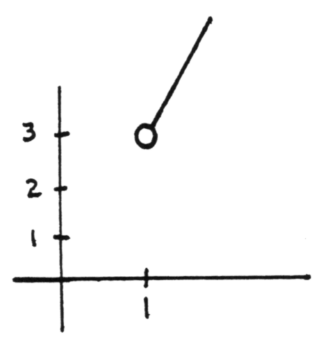
1. When ‘undoing’ the output $-4-\epsilon$, it is important to take the input that lies near -2! Take: $\delta := -2 + \sqrt{4+\epsilon}$



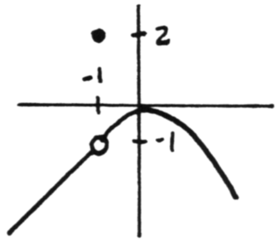
3. Take $\delta := 16 - (2-\epsilon)^4$, since this is the shorter distance.



5. $\lim\limits_{x\to 1} f(x) = 3$
$\lim\limits_{x\to 1^+} f(x) = 3$
$\lim\limits_{x\to 1^-} f(x)$ is not defined, since f is not defined to the left of 1



7. $\lim\limits_{x\to -1} g(x) = -1$
$\lim\limits_{x\to -1^+} g(x) = -1$
$\lim\limits_{x\to -1^-} g(x) = -1$



9. TRUE! Indeed, if $\lim\limits_{x\to c} f(x) = l$ and f is defined on both sides of c, then both one-sided limits must also exist and equal l.

Section 3.3 Properties of Limits

Quick Quiz:

1. To show that an object is unique, a mathematician often supposes that there are TWO, and then shows that they must be equal.
2. As long as both 'component' limits exist, the limit of a sum is the sum of the limits.
3. For all real numbers a and b:
$$|a+b| \le |a| + |b|$$
4. To evaluate the limit, just evaluate the function f at c; that is, substitute the value c into the formula for f.
5. All the component limits exist, so:
$$\lim_{z\to 1} \frac{-2f(z)+g(z)}{h(z)} = \frac{(-2)(3)+5}{2} = -\frac{1}{2}$$

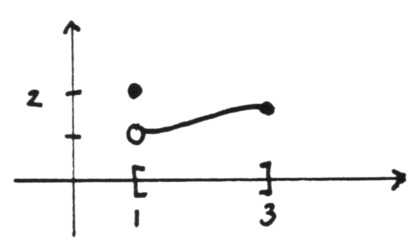
End-of-Section Exercises:

1. SEN; TRUE
3. SEN; TRUE
5. SEN; TRUE
7. SEN; TRUE
9. SEN; FALSE
11. SEN; TRUE
13. SEN; CONDITIONAL
15.
$$\begin{aligned}\lim_{t\to c}[f(t)+g(t)] &= \lim_{t\to c} f(t) + \lim_{t\to c} g(t)\\ &= (-1)+2\\ &= 1\end{aligned}$$
17. There is not enough information to evaluate this limit. We don't know anything about the behavior of f and g, as the inputs approach the number d.

Section 3.4 Continuity

Quick Quiz:

1. A function f is continuous at c if f is defined at c, and $\lim_{x\to c} f(x) = f(c)$.
2. NO! If f were continuous at c, the value of the limit would have to be 3. The discontinuity is removable.
3. f has a nonremovable discontinuity at c if $\lim_{x\to c} f(x)$ does not exist.
4. When f is continuous at c.
5. There are many correct graphs. Here is one:

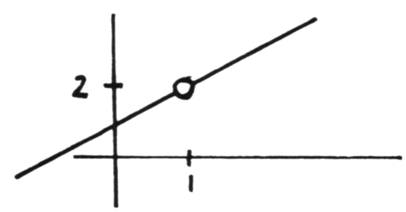


End-of-Section Exercises:

1. SEN; CONDITIONAL
3. EXP
5. SEN; CONDITIONAL
7. SEN; CONDITIONAL
9. SEN; CONDITIONAL
11. SEN; TRUE
13. SEN; FALSE
15. EXP. Out of context, it is not known if this is a POINT (a, b), or an open interval of real numbers. In either case, however, it is an EXPRESSION.

Section 3.5 Indeterminate Forms

Quick Quiz:

1. An 'indeterminate form' is a limit that, upon direct substitution, results in one of the forms: $\frac{0}{0}$, 1^∞, or $\frac{\pm\infty}{\pm\infty}$.
2. $$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1}(x+1) = 1+1 = 2\end{aligned}$$
3. NO! It is true for all values of x except 1. When x is 1, the left-hand side is not defined; the right-hand side equals 2 .
4. $$y = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} \overset{\text{for } x\neq 1}{=} x+1$$
   The graph is:
   
5. The graph of f is the same as the graph of $y = \frac{x^2-1}{x-1}$. See (4).
6. $f = g$ if and only if $\mathcal{D}(f) = \mathcal{D}(g)$, and $f(x) = g(x)$ for all x in the common domain.

End-of-Section Exercises:

1. SEN; FALSE
3. SEN; FALSE
5. SEN; TRUE
7. SEN; TRUE. (Either both limits do not exist; or they both exist, and are equal.)
9. $$\begin{aligned}\lim_{x\to -1}\frac{x^3+x^2-3x-3}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-3)}{x+1}\\ &= \lim_{x\to -1} x^2-3\\ &= (-1)^2-3 = -2\end{aligned}$$
11. $$\lim_{x\to 2}\frac{x+2}{x^2+4x+4} = \frac{2+2}{2^2+4(2)+4} = \frac{4}{16} = \frac{1}{4}$$

13. $\lim_{t \to 0^+} (1+t)^{1/t} = e$

Section 3.6 The Intermediate Value Theorem

Quick Quiz:

1. If f is continuous on $[a,b]$, and D is any number between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ with $f(c) = D$.
2. Since f is continuous on $[1,3]$ and 0 is a number between $f(1)$ and $f(3)$, the Intermediate Value Theorem guarantees the existence of a number c with $f(c) = 0$.
3. TRUE. When the hypothesis of an implication is false, the implication is (vacuously) true.
4. FALSE. Let $x = -1$. Then the hypothesis $|-1| = 1$ is true, but the conclusion $-1 = 1$ is false.
5.

A	B	A ⇒ B
T	T	T
T	F	F
F	T	T
F	F	T

END-OF-SECTION EXERCISES:

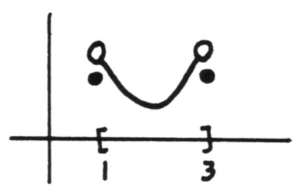
1. TRUE
3. TRUE
5. TRUE
7. TRUE
9. TRUE

Section 3.7 The Max-Min Theorem

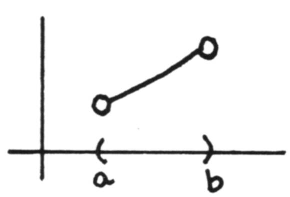
Quick Quiz:

1. The symbol ‘ $\iff$ ’ can also be read as ‘if and only if’.

   The number $f(c)$ is a maximum of f on I if and only if $f(x) \le f(c)$ for all $x \in I$.
2. There are many possible correct graphs.



3. There are many possible correct graphs.

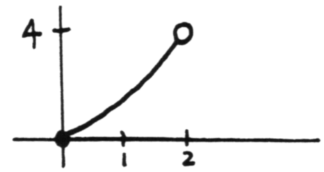


4. If f is continuous on $[a,b]$, then f attains both a maximum and minimum value on $[a,b]$.
5. The contrapositive of the sentence ‘$A \implies B$’ is the sentence ‘not $B \implies$ not A’.

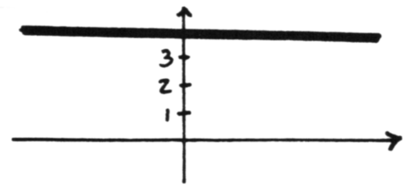
   An implication is equivalent to its contrapositive. That is, the sentences ‘$A \implies B$’ and ‘not $B \implies$ not A’ always have the same truth values, regardless of the truth values of A and B.

END-OF-SECTION EXERCISES:

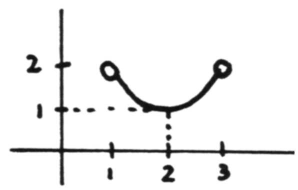
1. The minimum value of f on I is 0; there is no maximum value. The only minimum point is $(0,0)$.



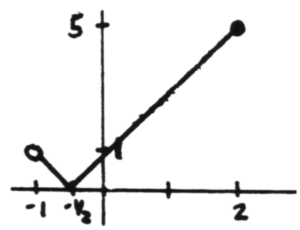
3. The maximum value of f on I is 4; the minimum value of f on I is 4. The points $(x,4)$ for $x \in \mathbb{R}$ are all both maximum and minimum points.



5. The minimum value of f on I is 1; there is no maximum value. The point $(2,1)$ is the only minimum point.



7. The minimum value of f on I is 0; the maximum value is 5. The only minimum point is $(-\frac{1}{2},0)$; the only maximum point is $(2,5)$.

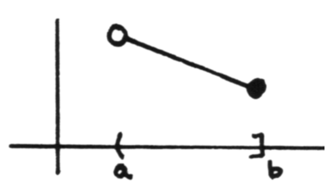


9. TRUE

   Contrapositive: If f does not attain a maximum value on $[a,b]$, then f is not continuous on $[a,b]$.

11. FALSE

   Counterexample: Let f be the function graphed below. Then, the hypothesis 'f is continuous on $(a,b]$' is TRUE, but the conclusion 'f attains a maximum value on $(a,b]$' is FALSE.



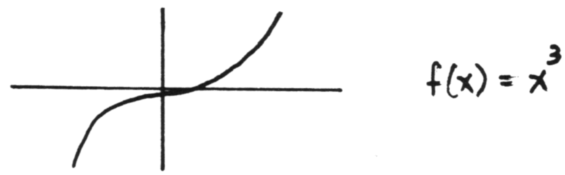
   Contrapositive: If f does not attain a maximum value on $(a,b]$, then f is not continuous on $(a,b]$.

13. TRUE

   Contrapositive: If f does NOT attain both a maximum and minimum value on $[1,2]$, then f is not continuous on $(0,5)$.

15. FALSE.

Counterexample: Let f be the function graphed below. Then the hypothesis 'f is continuous on $\mathbb{R}$' is TRUE, but the conclusion 'f attains a maximum value on $\mathbb{R}$' is FALSE.



Contrapositive: If f does not attain a maximum value on $\mathbb{R}$, then f is not continuous on $\mathbb{R}$.

CHAPTER 4. THE DERIVATIVE

Section 4.1 Tangent Lines

Quick Quiz:

1. Let $f(x) = x$. Then:

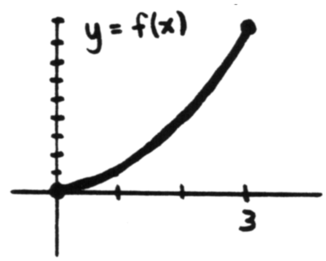
$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{(2+h)-2}{h}\\ &= \lim_{h\to 0}\frac{h}{h} = 1\end{aligned}$$

Thus, as expected, the slope of hte tangent line to f at the point $(2,2)$ is 1.

2. The dummy variable is h. Using the dummy variable t, the limit can be rewritten as:

$$\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

3. In the limit, x represents the x-value of a point where the slope of the tangent line is desired.
4. In the limit, the difference quotient $\frac{f(x+h)-f(x)}{h}$ represents the slope of a secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$. This secant line is being used as an approximation to the tangent line at the point $(x, f(x))$.
5. The function f is graphed below. Since f is only defined to the right of 0, the limit is actually a right-hand limit:



$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{h^2-0}{h}\\ &= \lim_{h\to 0^+} h = 0\end{aligned}$$

The slope of the tangent line at the point $(0,0)$ is 0.

End-of-Section Exercises:

1. EXP
3. SEN; CONDITIONAL
5. SEN; TRUE
7. $g(0.1) = \frac{f(x+0.1)-f(x)}{0.1}$; $g(\Delta x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$
9. $h \in \mathcal{D}(g) \iff \big(h \ne 0 \text{ and } x+h \in \mathcal{D}(f)\big)$
11. When $\lim\limits_{h\to 0} g(h)$ exists, it tells the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Section 4.2 The Derivative

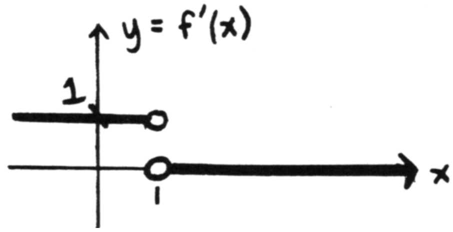
Quick Quiz:

1. When the limit exists:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

2. f' is the derivative *function*; $f'(x)$ is a particular output of this function, when the input is x.
3. $A - B = (0,2)\cup(2,4)$; $B - A = \{4\}$

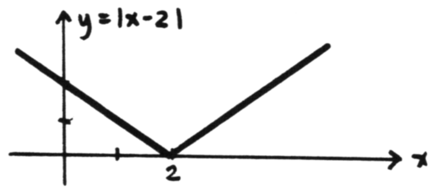
4. $\mathcal{D}(f') = \mathbb{R} - \{1\}$; its graph is:



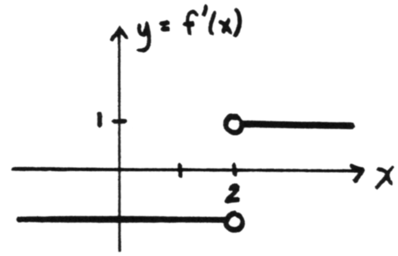
5. TRUE. If the limit $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists, then, in particular, f must be defined at x (so that $f(x)$ makes sense).

End-of-Section Exercises:

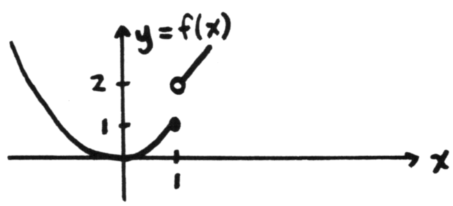
1. The graph of f is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.



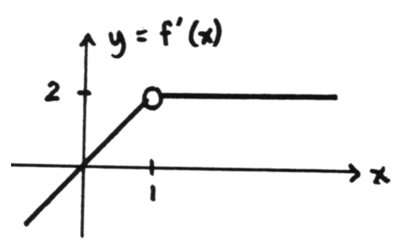
When $x > 2$, the slopes of the tangent lines equal 1.
When $x < 2$, the slopes of the tangent lines equal -1.
There is no tangent line at $x = 2$.
The graph of f' is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{2\}$.



3. The graph of f is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.



When $x > 1$, the slopes of the tangent lines equal 2.
When $x < 1$, the slopes of the tangent lines equal $2x$ (as per an example in the text).
There is no tangent line at $x = 1$.
The graph of f' is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{1\}$.



5. Note that $f(2) = \frac{1}{2-1} = 1$. Then:

$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{\frac{1}{(2+h)-1}-1}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{1+h}-\frac{1+h}{1+h}}{h}\\ &= \lim_{h\to 0}\frac{1-(1+h)}{h(1+h)}\\ &= \lim_{h\to 0}\frac{-1}{1+h} = -1\end{aligned}$$

Thus, $f'(2) = -1$. That is, the slope of the tangent line to the graph of f at the point $(2,1)$ is -1.

7. $y - 9 = 6(x-3)$
9. $y = 1$

Section 4.3 Some Very Basic Differentiation Formulas

Quick Quiz:

1. $f(x) = x^{1/2}$; $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. In Leibniz notation: $\frac{df}{dx} = \frac{1}{2\sqrt{x}}$
2. TRUE. The derivative of a constant equals zero.
3. $y' = 3x^2$; the slope of the tangent line at $x = 2$ is $y'(2) = 3(2^2) = 12$. TRUE.
4.

$$\begin{aligned}(a-b)^4 = (a+(-b))^4 &= (1)a^4 + (4)a^3(-b) + (6)a^2(-b)^2 + (4)a(-b)^3 + (1)(-b)^4\\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

1
1 2 1
1 3 3 1
1 4 6 4 1

5. $g'(x) = e^x + \frac{1}{x}$

End-of-Section Exercises:

1. Multiply out, differentiate term-by-term, and simplify: $f'(x) = 6(2x+1)^2$
3.

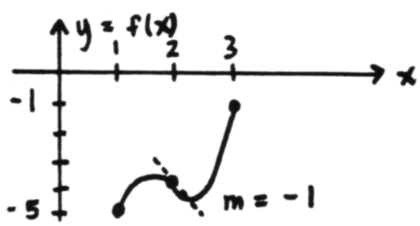
$$h'(x) = \begin{cases} 6x-2 & \text{for } x \ge 1\\ 4 & \text{for } x < 1\end{cases}$$

$\mathcal{D}(h) = \mathcal{D}(h') = \mathbb{R}$

Section 4.4 Instantaneous Rates of Change

Quick Quiz:

1. $\frac{f(2)-f(1)}{2-1} = \frac{2^3-1^3}{1} = 8 - 1 = 7$; this number represents the slope of the secant line through the points $(1,1^3)$ and $(2,2^3)$
2. $f'(x) = 3x^2$; $f'(1) = 3(1) = 3$. This number represents the slope of the tangent line at the point $(1,1^3)$.
3. less than; once we move to the right of $x = 1$, the rates of change increase
4. One correct sketch is given:

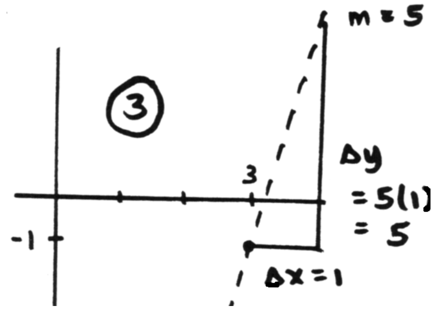


5. Since f is not continuous at $x = 1$, f is not differentiable at $x = 1$.

End-of-Section Exercises:

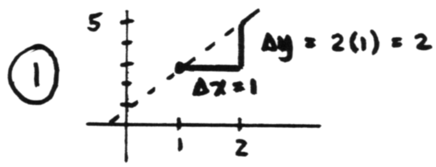
In all cases, the ‘predicted value’ for $f(x_2)$ from known information at x_1 is given by

$$f(x_2) \approx f(x_1) + (\Delta x)(f'(x_1)) \ ,$$



where $\Delta x_2 = x_2 - x_1$.

1. Here, $\Delta x = 2 - 1 = 1$; $f(2) \approx 3 + (1)(2) = 5$
3. Here, $\Delta x = 4 - 3 = 1$; $f(4) \approx -1 + (1)(5) = 4$



Section 4.5 The Chain Rule (Differentiating Composite Functions

Quick Quiz:

1. See page 231. The Chain Rule tells us how to differentiate composite functions.
2. $f'(x) = 7\sqrt{2}(1-x)^6(-1) = -7\sqrt{2}(1-x)^6$
3. $\frac{dy}{dt} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dt}$
4. ... tells us that to find out how fast $f \circ g$ changes with respect to x, we find out how fast f changes with respect to $g(x)$, and multiply by how fast g changes with respect to x
5. $f(x) = \frac{1}{3}\ln(2x+1)$, $f'(x) = \frac{1}{3} \cdot \frac{1}{2x+1} \cdot 2 = \frac{2}{3(2x+1)}$

End-of-Section Exercises:

1. $f'(x) = \dfrac{-e^x}{\sqrt{(e^x-1)^3}} + 1$
3. $\dfrac{dy}{dx} = 3e^{3x}$
5. $y' = 33(3t-4)^{10}$
7. $g'(t) = \dfrac{2t+1}{2\sqrt[6]{(t^2+2t+1)^5}}$
9. $f'(y) = -7e^{-y} + \dfrac{1}{y}$
11. $\dfrac{dy}{dx} = \dfrac{3}{x}(\ln x)^2$
13. $\dfrac{dy}{dt} = \dfrac{2\sqrt{t-1}+1}{2\sqrt{t-1}(t+\sqrt{t-1})^2}$

Section 4.6 Differentiating Products and Quotients

Quick Quiz:

1. See page 239.
2. See page 244.
3. $f'(x) = x \cdot 5(x+1)^4(1) + (1)(x+1)^5$
4. Using the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{e^{2x}(2) - (2x+1)\cdot 2e^{2x}}{(e^{2x})^2} \\ &= \frac{2e^{2x}(1-(2x+1))}{e^{4x}} \\ &= \frac{-4xe^{2x}}{e^{4x}} \end{aligned}$$

5. Using a 'generalized' product rule:

$$y' = (1)(x+1)(x^2+1) + x(1)(x^2+1) + x(x+1)(2x)$$

End-of-Section Exercises:

1.
$$\begin{gathered} y' = 2(2-x)^2(1-2x) \\ y(0) = 0\,, \quad y(t^2) = t^2(2-t^2)^3 \\ y'(0) = 8\,, \quad y'(t) = 2(2-t)^2(1-2t) \end{gathered}$$

3.
$$\begin{gathered} f'(x) = \mathrm{e}^x(\frac{1}{x} + \ln x) \\ \mathcal{D}(f) = (0,\infty)\,, \quad \mathcal{D}(f') = (0,\infty) \\ f'(\mathrm{e}^x) = \mathrm{e}^{(\mathrm{e}^x)}(\frac{1}{\mathrm{e}^x} + x)\,, \quad f'(\mathrm{e}^2) = \mathrm{e}^{(\mathrm{e}^2)}(\frac{1}{\mathrm{e}^2} + 2) \end{gathered}$$

5.
$$\begin{gathered} g'(x) = \mathrm{e}^{x+\mathrm{e}^x} \\ \lim_{x\to 0} g(x) = \mathrm{e}\,, \quad \lim_{x\to 0} g'(x) = \mathrm{e} \\ \mathcal{D}(g) = \mathbb{R}\,, \quad g(g'(g(0))) = \mathrm{e}^{\mathrm{e}^{(\mathrm{e}^{\mathrm{e}+\mathrm{e}^{\mathrm{e}}})}} \end{gathered}$$

7. $h'(x) = \frac{x}{x+1}$; the tangent line is horizontal, and has equation $y = 0$
9. $f'(x) = 4\mathrm{e}^{2x}(2x+1)^6(x+4)$; the tangent line has equation $y = 16x + 1$
11. $h(t) = \frac{-12\mathrm{e}}{(3t-1)^5}$; the tangent line has equation $y - \mathrm{e} = -12\mathrm{e}(t - \frac{2}{3})$
13.
$$y' = 0 \iff \big(x = 3 \text{ or } x = -1 \text{ or } x = \frac{1}{2} \text{ or } x = \frac{3 \pm \sqrt{17}}{2}\big)$$

Section 4.7 Higher Order Derivatives

Quick Quiz:

1. The 'higher derivatives' of a function f are the derivatives of the form $f^{(n)}$ for $n \geq 2$. That is, the second derivative, third derivative, fourth derivative, etc., are the 'higher derivatives' of f.
2. prime notation: $f''(x)$

   Leibniz notation: $\dfrac{d^2 f}{dx^2}(x)$
3. $\sum_{i=1}^{3} i^{i+1} = 1^{1+1} + 2^{2+1} + 3^{3+1} = 1 + 8 + 81 = 90$
4.
$$\begin{aligned} 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \frac{5!}{5!} \\ &= \frac{10!}{5!} \end{aligned}$$
5. $\displaystyle \frac{d}{dx} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} {f_i}'(x)$

End-of-Section Exercises:

1. SEN; TRUE
3. EXP

5. SEN; CONDITIONAL
7. SEN; TRUE
9. EXP
11. EXP
13. EXP
15. SEN; TRUE
17. SEN; CONDITIONAL

Section 4.8 Implicit Differentiation (Optional)

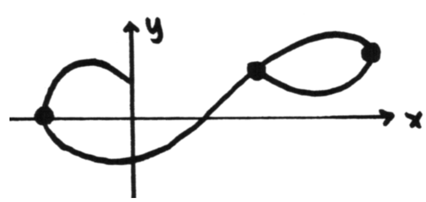
Quick Quiz:

1.
$$\begin{aligned}\frac{d}{dx}(xy^2) &= \frac{d}{dx}(2)\\ x(2y^1)\frac{dy}{dx} + (1)y^2 &= 0\\ \frac{dy}{dx} &= \frac{-y^2}{2xy}\end{aligned}$$

2. For $x > 0$:
$$\begin{aligned}\ln y &= \ln(x^{2x}) = 2x\ln x\\ \frac{1}{y}\frac{dy}{dx} &= (2x)\frac{1}{x} + (2)\ln x = 2 + 2\ln x = 2(1+\ln x)\\ \frac{dy}{dx} &= y\cdot 2(1+\ln x) = 2x^{2x}(1+\ln x)\end{aligned}$$

3. Put the equation in standard form, by completing the square:
$$\begin{aligned}x^2 - 2x + y^2 = 8 &\iff (x^2 - 2x + (\tfrac{-2}{2})^2) + y^2 = 8 + 1\\ &\iff (x-1)^2 + (y-0)^2 = 3^2\end{aligned}$$
The equation graphs as the circle centered at $(1,0)$ with radius 3.

4. There are *many* possible correct answers. Here are two:
y given explicitly in terms of x: $y = x^2 + 2x + 1$
y given implicitly in terms of x: $xy^2 = x + y$

5.


End-of-Section Exercises:

1. The graph is the circle centered at $(-2,1)$ with radius 1.
y is NOT locally a function of x at the points $(-1,1)$ and $(-3,1)$. (There are vertical tangent lines here.)
The equation of the tangent line at the point $(-2,2)$ is $y = 2$.
The equation of the tangent line at the point $(-1,1)$ is $x = -1$.

3. The graph is the circle centered at $(-2,1)$ with radius 2.
y is NOT locally a function of x at the points $(0,1)$ and $(-4,1)$; there are vertical tangent lines here.
The equation of the tangent line at the point $(-1, 1+\sqrt{3})$ is:
$$y - (1+\sqrt{3}) = -\frac{1}{\sqrt{3}}(x - (-1))$$

Section 4.9 The Mean Value Theorem

Quick Quiz:

1. See page 266.
2. The word 'mean' refers to 'average'; the Mean Value Theorem guarantees (under certain hypotheses) a place in an interval (a, b) where the *instantaneous* rate of change is the same as the *average* rate of change over the entire interval.
3. The average rate of change of f on the interval $[1, 3]$ is:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13$$

The instantaneous rates of change are given by $f'(x) = 3x^2$. We esek $c \in (1, 3)$ for which $f'(c) = 13$:

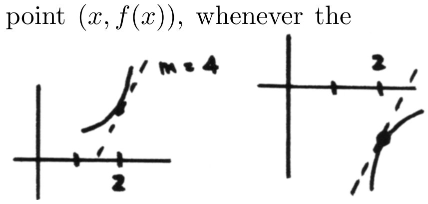
$$\begin{aligned} f'(c) = 13 &\iff 3c^2 = 13 \\ &\iff c^2 = \frac{13}{3} \\ &\iff c = \pm\sqrt{\frac{13}{3}} \end{aligned}$$

Choosing the value of c in the desired interval, we get $c = \sqrt{\frac{13}{3}}$.

4. If f WERE continuous on $[a, b]$, then there would have to be (by the MVT) a number $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. Thus, it must be that f is NOT continuous on $[a, b]$; that is, f 'goes bad' at (at least one) endpoint.
5. If f WERE differentiable on (a, b), then the MVT would guarantee that there must be $c \in (a, b)$ with $f'(c)$ equal to the average rate of change of f over $[a, b]$. Therefore, we can conclude that f is NOT differentiable on (a, b). That is, there is at least one value of x in the interval (a, b) where $f'(x)$ does not exist.

End-of-Section Exercises:

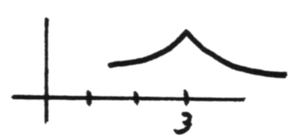
1. The limit gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, whenever the tangent line exists and is non-vertical.
3. There is a tangent line to the graph of f when $x = 2$, and its slope is 4.



5. Let $f(x) = -x^2$. Then:

$$\begin{aligned} f'(x) &:= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-(x+h)^2 - (-x^2)}{h} \\ &= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + x^2}{h} = \lim_{h\to 0} \frac{h(-2x - h)}{h} \\ &= \lim_{h\to 0} (-2x - h) = -2x \end{aligned}$$

7. Put a 'kink' in the graph when $x - 3$.



9.

$$f'(x) = e^{2x}\ln(2 - x) + 2xe^{2x}\ln(2 - x) - \frac{xe^{2x}}{2 - x}$$

$$\mathcal{D}(f) = (-\infty, 2)\,, \quad \mathcal{D}(f') = (-\infty, 2)$$

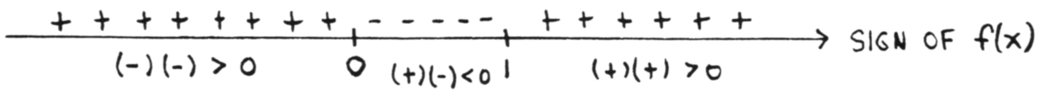
The tangent line when $x = 0$ has equation $y = (\ln 2)x$.

CHAPTER 5. USING THE INFORMATION GIVEN BY THE DERIVATIVE

Section 5.1 Increasing and Decreasing Functions

Quick Quiz:

1. See page 276.
2. Zeroes of f: $f(x) = 0 \iff (x = 0 \text{ or } x = 1)$. Choose the test points -1, $\frac{1}{2}$, and 2. The information is summarized below.



3. TRUE
4. TRUE

END-OF-SECTION EXERCISES:

1. Positive: $(-\infty, -2) \cup (1, \infty)$
   Negative: $(-2, 1)$
3. Positive: $(-\infty, -1) \cup (3, \infty)$
   Negative: $(-1, 3)$
5. Positive: $(-\infty, \frac{1}{3}) \cup (\frac{3}{4}, \infty)$
   Negative: $(\frac{1}{3}, \frac{3}{4})$
7. Positive: $(0, \infty)$
   Negative: $(-\infty, -1) \cup (-1, 0)$
9. Positive: $(-4, -1) \cup (1, \infty)$
   Negative: $(-\infty, -4) \cup (-1, 1)$
11. Positive: $(0, \infty)$
    Negative: $(-\infty, 0)$
13. Positive: $(1, \infty)$
    Negative: $(\frac{1}{2}, 1)$
15. The function f increases on $(-\infty, -2) \cup (1, \infty)$ and decreases on $(-2, 1)$.
17. The function f decreases on $(-\infty, -1)$ and increases on $(-1, \infty)$.
19. The function f decreases on $(0, \frac{1}{e})$, and increases on $(\frac{1}{e}, \infty)$.
21. b) 2278
    c) 3870
23. c) $1 + 2 + 2^2 + 2^3 + 2^4 = 31$
    d) $2^6 + \cdots + 2^{10} = 1984$

Section 5.2 Local Maxima and Minima—Critical Points

Quick Quiz:

1. The point $(c, f(c))$ must be a critical point. Thus, either it is an endpoint of the domain of f, or $f'(c) = 0$, or $f'(c)$ does not exist.
2. NO! There are critical points that are not local extreme points.
3. The 'critical points' for a function f are the CANDIDATES for the local extreme points of f.
4. NO! When $A \Rightarrow B$ is true, $B \Rightarrow A$ may be either true or false.
5. Since f is differentiable, it is also continuous. By the First Derivative Test, there is a maximum at $x = a$; a minimum at $x = c_1$; a maximum at $x = c_2$; and a minimum at $x = b$.

END-OF-SECTION EXERCISES:

3. TRUE
5. TRUE

7. TRUE 9. FALSE 11. TRUE 13. TRUE
15. FALSE 17. TRUE 19. TRUE

Section 5.3 The Second Derivative—Inflection Points

Quick Quiz:

1. The second derivative of a function tells us the rate of change of the slopes of the tangent lines. This information is referred to as the *concavity* of the function.
2. f is concave up on I if and only if $f''(x) > 0$ for every $x \in I$
3. The converse is: If $x^2 = 1$, then $x = 1$.

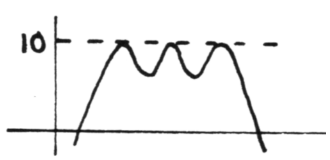
   The sentence is false. Choose x to be -1. Then the hypothesis '$(-1)^2 = 1$' is true, but the conclusion '$-1 = 1$' is false.
4. By the Second Derivative Test, the point $(c, f(c))$ is a local maximum point for f.
5. $f'(x) = 3(x-1)^2$, $f''(x) = 6(x-1)$, so $f''(1) = 6(1-1) = 0$

END-OF-SECTION EXERCISES:

1. local minima at $x = 0$ and $x = 1$; local maximum at $x = \frac{1}{2}$
3. $f(x)$ is positive on $(-\infty, -2.5) \cup (-2, \infty)$
   $f(x)$ is negative on $(-2.5, -2)$
5. f is concave up on $(-2, 2)$
   f is concave down on $(-3, -2) \cup (2, \infty)$
7. $\mathcal{D}(f') = \mathbb{R} - \{-4, -3, -2\}$
9. $\{x \mid f(x) > 10\} = (-2, -1.5)$
11. $\{x \mid f''(x) < 0\} = (-3, -2) \cup (2, \infty)$
13. $\lim_{t \to -2} f(t)$ does not exist
15. The critical points are: $\{(x, 4) \mid x \in (-\infty, -4)\}$, $(0, 2)$, $(-4, 4)$ and $(-3, 8)$
17. $\{x \in \mathcal{D}(f) \mid f \text{ is not differentiable at } x\} = \{-4, -3\}$
19. $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0) = 0$

Section 5.4 Graphing Functions—Some Basic Techniques

Quick Quiz:

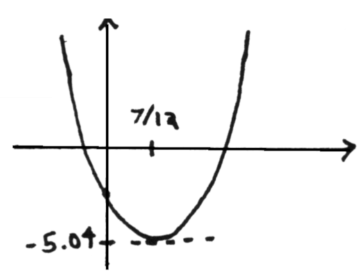
1. 

2. For $x \gg 0$ and $x \ll 0$, $P(x) \approx -6x^7$. So as $x \to \infty$, $P(x) \to -\infty$.
   As $x \to -\infty$, $P(x) \to \infty$.
3. $f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$. Thus, f is ODD, but not EVEN.
4. $f'(x) = 12x - 7$; $f'(x) = 0 \iff x = \frac{7}{12}$

   There is a horizontal tangent line at $(\frac{7}{12}, f(\frac{7}{12}))$; $f(\frac{7}{12}) = 6(\frac{7}{12})^2 - 7(\frac{7}{12}) - 3 \approx -5.04$

   $f''(x) = 12$, so $f''(x) > 0$ for all x

   Also, $f(x) = 0$
   $\iff (2x-3)(3x+1) = 0$
   $\iff x = \frac{3}{2}$ OR $x = -\frac{1}{3}$



Section 5.5 More Graphing Techniques

Quick Quiz:

1. Find A and B for which $AB = (3)(-8) = -24$ and $A + B = -2$; take $A = -6$ and $B = 4$. Then:

$$\begin{aligned} 3x^2 - 2x - 8 &= 3x^2 - 6x + 4x - 8 \\ &= 3x(x-2) + 4(x-2) \\ &= (3x+4)(x-2) \end{aligned}$$

2. First, solve $3x^2 - 2x - 8 = 0$ using the Quadratic Formula:

$$x = \frac{2 \pm \sqrt{4 - 4(3)(-8)}}{6} = \frac{2 \pm 10}{6} = 2,\ -\frac{4}{3}$$

Then:

$$\begin{aligned} 3x^2 - 2x - 8 &= 3(x-2)(x + \frac{4}{3}) \\ &= (x-2)(3x+4) \end{aligned}$$

3. CANDIDATES: $\frac{\pm 1, \pm 2}{\pm 1} = \pm 1,\ \pm 2$
4. not(A and B) $\iff$ (not A) or (not B)
5. $P(1) = -1$; the remainder upon division by $x - 1$ equals -1

END-OF-SECTION EXERCISES:

1. $P(x) = 2x^3 - 3x^2 - 3x - 5 = (x^2 + x + 1)(2x - 5)$
3. $P(x) = x^4 - 5x^2 + 6 = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3})$

Section 5.6 Asymptotes—Checking Behavior at Infinity

Quick Quiz:

1. An *asymptote* is a curve (often a line) that a graph gets close to as x approaches $\pm\infty$, or some finite number.
2. $\lim_{x \to c^-} f(x) = -\infty \quad \iff \quad \forall\ M < 0\ \ \exists\ \delta > 0$ such that if $x \in (c - \delta, c)$, then $f(x) < M$
3. VERTICAL: $x = -2$

   HORIZONTAL: $y = 3$
4. Both individual limits (the 'numerator' limit and the 'denominator' limit) must exist. Also, the 'denominator' limit cannot equal zero.

CHAPTER 6. ANTIDIFFERENTIATION

Section 6.1 Antiderivatives

Quick Quiz:

1. The graph of f is a line with slope 2. Thus, $f(x) = 2x + C$, for some constant C.
2. Specifying the derivative of a function completely determines its SHAPE.
3. $\int 2\,dt = 2t + C$
4. The antiderivatives of a function can be used to find the area trapped between the graph of the function and the x-axis.
5. The phrase refers to the facts that the derivative of a sum is the sum of the derivatives; and constants can be 'slid out' of the differentiation process.

END-OF-SECTION EXERCISES:

1. EXP
3. EXP
5. SEN; CONDITIONAL
7. SEN; TRUE
9. SEN; TRUE

Section 6.2

Quick Quiz:

1. The 'counterpart' is:

$$\int ke^{kx}\,dx = e^{kx} + C$$

A more useful version of this formula is found as follows:

$$\int ke^{kx}\,dx = e^{kx} + C \quad \iff k\int e^{kx}\,dx = e^{kx} + C \iff \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + K$$

2. Rewrite the integrand, and use the Simple Power Rule:

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{x^3} + C$$

3. $$\int \frac{1}{2t}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln|t| + C$$

4. If $f'(x) = x$, then antidifferentiating yields $f(x) = \frac{x^2}{2} + C$. Then:

$$f(0) = 3 \quad \iff \quad 0 + C = 3 \quad \iff \quad C = 3$$

Take: $f(x) = \frac{x^2}{2} + 3$

Section 6.3

Quick Quiz:

1. 'Speed' has only magnitude (size); 'velocity' has both magnitude and direction.
2. Position at $t = 1$: $d(1) = 1^2 + 2(1) = 3$ feet

   $v(t) = d'(t) = 2t + 2$; Velocity at $t = 1$: $v(1) = 2(1) + 2 = 4$ feet/second

   Speed at time $t = 1$: $|v(1)| = |4| = 4$ feet/second

   $a(t) = v'(t) = 2$; Acceleration at $t = 1$: $a(1) = 2$ feet/second^2

3. A 'vector' is a mathematical object that is completely described by two pieces of information: a magnitude (size), and a direction. Vectors are conveniently represented by arrows.
4. A *free-body diagram* is a picture that illustrates the forces acting on an object.
5. '$v(2)$' means the velocity function, *acting on* the input 2; this is function notation. However, '$g(2)$' means the constant g, *times* the number 2. Context is important!

END-OF-SECTION EXERCISES:

1. If 'down' is chosen as the positive direction, and '0' coincides with the ground, then: $d(t) = g\frac{t^2}{2} - 20t - 75$
3. approximately 0.63 seconds
5. approximately 1.25 seconds

Section 6.4

Quick Quiz:

1. With appropriate renaming, transform a difficult integration problem into one that is easier to handle. Solve the 'new' integral, then transform the solution back into a solution of the original problem.
2. Substitution:

$$\boxed{\begin{aligned} u &= 2x-1 \\ du &= 2\,dx \end{aligned}}$$

$$\begin{aligned} \int \frac{1}{2x-1}\,dx &= \frac{1}{2}\int \frac{1}{2x-1} 2\,dx \\ &= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|2x-1| + C \end{aligned}$$

Without substitution:

$$\int \frac{1}{2x-1}\,dx = \int \frac{1}{2(x-\frac{1}{2})}\,dx = \frac{1}{2}\int \frac{1}{x-\frac{1}{2}}\,dx = \frac{1}{2}\ln|x-\frac{1}{2}| + C$$

To see that the answers differ by only a constant, write:

$$\frac{1}{2}\ln|2x-1| = \frac{1}{2}\ln|2(x-\frac{1}{2})| = \frac{1}{2}[\ln 2 + \ln|x-\frac{1}{2}|] = \frac{1}{2}\ln 2 + \frac{1}{2}\ln|x-\frac{1}{2}|$$

Thus, the two answers differ only by the constant $\frac{1}{2}\ln 2$.

3. After multiplying by '1' in an appropriate form, the linearity of the integral is used to 'pull' the unwanted constant part out of the integral.
4. $$\int e^{3x}\,dx = \frac{1}{3}\int e^{3x} 3\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$$

$$\boxed{\begin{aligned} u &= 3x \\ du &= 3\,dx \end{aligned}}$$

5. We need only check if $\frac{(3x+\pi)^6}{18}$ is an antiderivative of $(3x+\pi)^5$:

$$\frac{d}{dx}\left(\frac{(3x+\pi)^6}{18}\right) = \frac{1}{18}(6)(3x+\pi)^5 \cdot (3) = (3x+\pi)^5$$

Thus, it IS true that: $\int (3x+\pi)^5\,dx = \frac{(3x+\pi)^6}{18} + C$

END-OF-SECTION EXERCISES:

1. $\frac{1}{36}(2x-1)^{18} + C$
3. $\frac{3}{2}(\ln 4x)^2 + C$
5. $2e^{\sqrt{x}} + C$
7. $\frac{-4}{\sqrt{t^2+t+1}} + C$

9. $f(x) = \dfrac{(e^x+1)^4}{4}$

Section 6.5

Quick Quiz:

1. In general, integration is harder than differentiation.
2.

$u = 2 + x$; $x = u - 2$
$du = dx$

$$\int \frac{x}{2+x}\,dx = \int \frac{u-2}{u}\,du = \int 1 - \frac{2}{u}\,du = u - 2\ln|u| + C$$
$$= (2+x) - 2\ln|2+x| + C = x - 2\ln|2+x| + K$$

3. There are extensive tables of integrals, and computer programs that can do symbolic integration.

END-OF-SECTION EXERCISES:

1. $\frac{1}{5}(\frac{1}{2}e^{2x} + x) + C$
3. $\frac{3}{16}\sqrt[3]{(4t^2-1)^2} + C$
5. $\dfrac{(x^2-1)^4}{8} + C$
7. $\dfrac{(\ln x)^4}{12} + C$

Section 6.6

Quick Quiz:

1. The Integration By Parts formula is:

$$\int u\,dv = uv - \int v\,du$$

It is a sort of 'integration counterpart' to the product rule for differentiation.

2.

$u = \ln 2t$ $\quad dv = dt$
$du = \frac{1}{2t}\cdot 2\,dt$ $\quad v = t$
$= \frac{1}{t}\,dt$

$$\int \ln 2t\,dt = (\ln 2t)(t) - \int t\cdot\frac{1}{t}\,dt$$
$$= t\ln 2t - \int (1)\,dt$$
$$= t\ln 2t - t + C$$

3.

$$\int \ln(x-1)\,dx = (x-1)\ln(x-1) - \int \frac{1}{x-1}(x-1)\,dx$$
$$= (x-1)\ln(x-1) - x + C$$

$u = \ln(x-1)$ $\quad dv = dx$
$du = \frac{1}{x-1}\,dx$ $\quad v = x-1$

4. The choice for 'dv' must be something that you *know how to integrate*!

END-OF-SECTION EXERCISES:

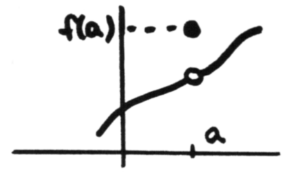
1. $\frac{1}{2}e^{2x} - 2e^x + x + C$
3. $\ln|1+e^x| + C$
5. $\sqrt{2e^t} + C$

CHAPTER 7. THE DEFINITE INTEGRAL

Section 7.1 Using Antiderivatives to find Area

Quick Quiz:

1. The Max-Min Theorem guarantees numbers $m \in [x, x+h]$ and $M \in [x, x+h]$ for which $f(m)$ is the minimum value of f on $[x, x+h]$, and $f(M)$ is the maximum value of f on $[x, x+h]$.
2. If f is continuous at a, then as $x \to a$, it must be that $f(x) \to f(a)$.
3. Any sketch where f IS defined at a, but f is NOT continuous at a, will work!



4. $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$. Then, the desired area is given by: $F(2) - F(0) = 2^3 - 0^3 = 8$
5. The desired area is given by: $F(d) - F(c)$

END-OF-SECTION EXERCISES:

1. 

approximation by a triangle: $\frac{1}{2}(1)(e-1) \approx 0.86$

actual area: Using integration by parts, an antiderivative of $f(x) = \ln x$ is $F(x) = x\ln x - x$. Then:

$$F(e) - F(1) = (e\ln e - e) - (1\ln 1 - 1) = (e-e) - (0-1) = 1$$

3. 

approximation by a trapezoid: $\frac{1}{2}(4-1)(1+2) = \frac{1}{2}(9) = \frac{9}{2} = 4.5$

actual area: An antiderivative of $f(x) = \sqrt{x} = x^{1/2}$ is $F(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$. Then:

$$F(4) - F(1) = \frac{2}{3}\sqrt{4^3} - \frac{2}{3}\sqrt{1^3} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{2}{3}(7) = \frac{14}{3} \approx 4.67$$

Section 7.2 The Definite Integral

Quick Quiz:

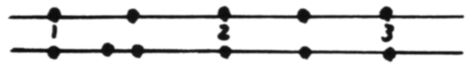
1. The indefinite integral $\int f(x)\,dx$ gives all the antiderivatives of the function f; by the Fundamental Theorem of Integral Calculus, if just *one* of these antiderivatives is known, then the definite integral $\int_a^b f(x)\,dx$ can be computed!
2. See page 409.
3. The notation $F(x)\big|_a^b$ means $F(b) - F(a)$.
4. $\int_{-1}^{2} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{2} = \frac{1}{3}(2^3 - (-1)^3) = \frac{1}{3}(8-(-1)) = \frac{1}{3}(9) = 3$
5. $\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4}(1^4 - (-1)^4) = 0$. On the interval $[-1,1]$, there is the same amount of area *above* the graph of $y = x^3$, as there is *below*.

END-OF-SECTION EXERCISES:

1. $\frac{48}{5}$
3. -6
5. $\frac{1}{3}\ln 2$
7. $1 + e^2$
9. The desired area is: $\frac{19}{24} + \frac{81}{8} = \frac{131}{12}$

Section 7.3 The Definite Integral as the Limit of Riemann Sums

Quick Quiz:

1. A *partition* of an interval $[a,b]$ is a finite set of points from $[a,b]$ that includes both a and b.
2. The length of the *longest* subinterval must be $\frac{1}{2}$:
   $P_1 = \{1, 1.5, 2, 2.5, 3\}$
   $P_2 = \{1, 1.3, 1.5, 2, 2.5, 3\}$
   

3. There is NOT a unique Riemann sum for f corresponding to this partition; any number x_1^* may be chosen from the subinterval $[0,1)$; any number x_2^* may be chosen from the second subinterval $[1,2)$, etc.
4. Think of a rectangle with 'width' dx and 'height' $f(x)$, where x is a number between a and b.

END-OF-SECTION EXERCISES:

1. EXP
3. SENTENCE; TRUE
5. SENTENCE; TRUE
7. SENTENCE; TRUE
9. SENTENCE; TRUE

Section 7.4 The Substitution Technique applied to Definite Integrals

Quick Quiz:

1. $\int (2x-1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{1}{2}\frac{u^4}{4} + C = \frac{1}{8}(2x-1)^4 + C$;
   $\int_0^{1/2} (2x-1)^3\,dx = \frac{1}{8}(2x-1)^4\,\big|_0^{1/2} = \frac{1}{8}(0-1) = -\frac{1}{8}$
   $u = 2x-1$
   $du = 2dx$
2. $\int_0^{1/2} (2x-1)^3\,dx = \frac{1}{2}\int_0^{1/2}(2x-1)^3\,2\,dx = \frac{1}{2}\int_{-1}^0 u^3\,du = \frac{1}{2}\frac{u^4}{4}\,\big|_{-1}^0 = \frac{1}{8}(0-1) = -\frac{1}{8}$
   $x = 0 \Rightarrow u = -1$
   $x = \frac{1}{2} \Rightarrow u = 0$
3. $u = \ln x \quad dv = dx$
   $du = \frac{1}{x}dx \quad v = x$

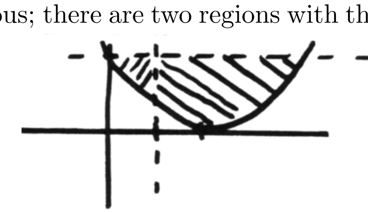
$$\begin{aligned}\int_1^e \ln x\,dx &= x\ln x\,\big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx\\ &= (e\ln e - 1\ln 1) - x\,\big|_1^e\\ &= e - (e-1) = 1\end{aligned}$$

END-OF-SECTION EXERCISES:

1. 0
3. ≈ 0.024
5. ≈ 1.931

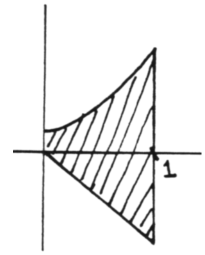
Section 7.5 The Area Between Two Curves

Quick Quiz:

1. $\int_c^d \big(g(x) - f(x)\big)\,dx$
2. The x-axis is described by $y = 0$. The intersection points are found by:
   
   $$-x^2 + 1 = 0 \quad\iff\quad x^2 = 1 \quad\iff\quad x = \pm 1$$
   Using symmetry, the desired area is:
   $$2\int_0^1 (-x^2+1)\,dx = (x - \frac{x^3}{3})\,\big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$
3. A quick sketch shows that the phrase IS ambiguous; there are two regions with the indicated boundaries. Which is desired? Or, are both desired?
   

4.

$$\int_0^1 (e^x - (-x))\,dx = \int_0^1 (e^x + x)\,dx$$
$$= (e^x + \frac{x^2}{2})\,\Big|_0^1$$
$$= (e + \frac{1}{2}) - e^0 = e + \frac{1}{2} - 1 = e - \frac{1}{2}$$



END-OF-SECTION EXERCISES:

1. $\dfrac{2}{15}$

3. $\dfrac{32}{3}$

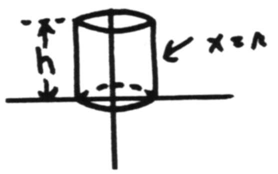
5. ≈ 2.438

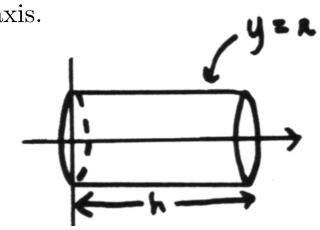
7. $20\frac{1}{4}$

Section 7.6 Finding the Volume of a Solid of Revolution—Disks

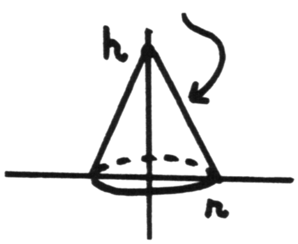
Quick Quiz:

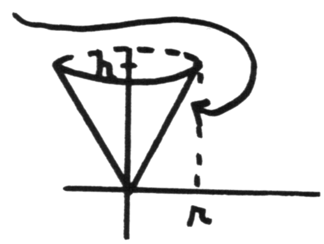
1. Revolve $x = r$ about the y-axis; or revolve $y = r$ about the x-axis.



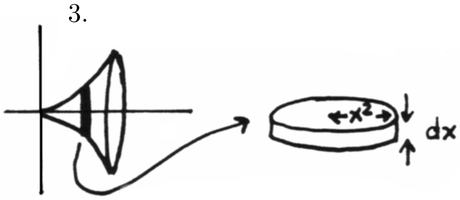


2. Revolve $y = -\frac{h}{r}x + h$ about the y-axis; or revolve $y = \frac{h}{r}x$ about the y-axis. (There are other correct answers.)





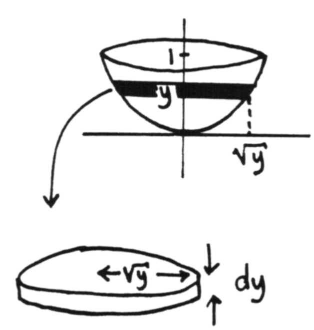
3.



$$\int_0^1 \pi(x^2)^2\,dx = \pi\frac{x^5}{5}\,\Big|_0^1 = \frac{\pi}{5}(1-0) = \frac{\pi}{5}$$

4. intersection points of $y = x^2$ and $y = 1$: $x^2 = 1 \iff x = \pm 1$
   Also: $y = x^2 \iff x = \pm\sqrt{y}$
   A typical 'slice' at a distance y has volume $\pi(\sqrt{y})^2\,dy$. The desired volume is:

$$\int_0^1 \pi(\sqrt{y})^2\,dy = \int_0^1 \pi y\,dy = \pi\frac{y^2}{2}\,\Big|_0^1 = \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$



END-OF-SECTION EXERCISES:

1. $\frac{4\pi}{3}$

3. $\frac{\pi}{2}$

5. 8π

7. $\frac{128\pi}{5}$

9. $\frac{8\pi}{3}$

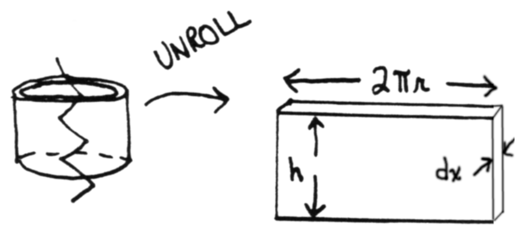
11. $\frac{\pi}{4}$

Section 7.7 Finding the Volume of a Solid of Revolution—Shells

Quick Quiz:

1. 'Cut' the shell and unroll it; the volume is:

$$2\pi r \cdot h \cdot dx$$



2.
$$\int_0^2 2\pi x(x)\,dx = 2\pi \frac{x^3}{3}\Big|_0^2 = \frac{2\pi}{3}(8-0) = \frac{16\pi}{3}$$

3. To use horizontal disks would require disks 'with holes'. Thus, in this case, shells are easier to use.

END-OF-SECTION EXERCISES:

1. $\frac{4\pi}{3}$

3. 2π

INDEX

Cover image by Olga
https://studio.envato.com/users/CrArt

The cover cats are very special
in my extended family:

Mr. Nels took care of the litter
after his mother died.
I've always pictured him as the fierce protector!

Kitsa is so regal!
Nothing fazes her.
I think of her as the queen of the four cats.

Don Paquito is totally lovable,
and totally driven by food.

Amelia was rescued as a kitten
on a road named after Amelia Earhart.