TBA4565 Module GPS, Project 1
GPS Absolute (Point) positioning with code pseudorange

Relevant material (Blackboard):
Lecture notes (Slides)
Report: Individual report. Submit the report on the Blackboard.

Tasks:

1. Compute the satellite coordinates.
2. Compute the satellite coordinates disregarding the nine correction terms.
3. Compute the receiver position as well as the receiver clock error and PDOP.

In this exercise we numerically investigate how we can determine position of a GPS receiver located on any place on or above the Earth from observations to several GPS satellites. The GPS receiver in this exercise tracks 7 GPS satellites (SV= Satellite Vehicle). The broadcast ephemerides of these 7 GPS satellites SV08, SV10, SV21, SV24, SV17, SV03, and SV14 are given together with the code pseudorange observation P(L1), Satellite clock error. Ionospheric error and Tropospheric error to each satellite (see Table below):

	SV 08	SV 10	SV 21	SV 24
t_{oe}[s]	$5.544000000000E+05$	$5.544000000000E+05$	$5.544000000000E+05$	$5.54400000000E+05$
$\sqrt{a}$ [$\sqrt{\text{meter}}$]	$5.153727157593E+03$	$5.153683794022E+03$	$5.153599693298E+03$	$5.153645917892E+03$
e	$8.500933647156E-03$	$8.606069488451E-03$	$2.470061811619E-02$	$1.409416110255E-02$
M_0[rad]	$1.898270050224E+00$	$-2.367579231750E+00$	$1.883581467643E+00$	$4.100407232706E-02$
ω[rad]	$2.723083480843E-01$	$-2.439684130522E+00$	$-7.320073574218E-01$	$9.032047692583E-01$
i_0[rad]	$9.581793010882E-01$	$9.795369600004E-01$	$9.617064126694E-01$	$9.336960928300E-01$
λ_0[rad]	$2.9681666440550D-01$	$2.418130818020E+00$	$1.285046985834E+00$	$-1.859141459356E+00$
Δn[rad s^{-1}]	$4.719125090702E-09$	$4.184817381514E-09$	$4.415183774142E-09$	$5.135214031782E-09$
$\dot{i}$ [rad s^{-1}]	$-4.257320329604E-10$	$2.107230709369E-11$	$-2.553677824757E-10$	$4.471614745150E-10$
$\dot{\Omega}$[rad s^{-1}]	$-8.101051385268E-09$	$-7.985689443046E-09$	$-7.820682768056E-09$	$-8.495710801526E-09$
C_{uc}[rad]	$-4.470348358154E-07$	$-8.810311555862E-07$	$3.902241587639E-06$	$1.372769474983E-06$
C_{us}[rad]	$9.246170520782E-06$	$3.382563591003E-06$	$7.575377821922E-06$	$9.723007678986E-06$
C_{rc}[m]	$2.014375000000E+02$	$3.245625000000E+02$	$2.230625000000E+02$	$1.829687500000E+02$
C_{rs}[m]	$-1.250000000000E+01$	$-1.812500000000E+01$	$7.018750000000E+01$	$2.606250000000E+01$
C_{ic}[rad]	$-1.154839992523E-07$	$-3.911554813385E-08$	$2.961605787277E-07$	$-2.980232238770E-08$
C_{is}[rad]	$1.899898052216\,E-07$	$-8.009374141693E-08$	$1.490116119385E-07$	$2.738088369370E-07$
P(L1)[m]	22 550 792.660	22 612 136.900	20 754 631.240	23 974 471.500
dt^j[s]*	+0.000 133 456 32	0.000 046 155 711	−0.000 151 820 34	0.000 265 875 20
$\mathrm{d}_{\mathrm{ion}}$[m]	3.344	2.947	2.505	3.644
$\mathrm{d}_{\mathrm{trop}}$[m]	4.055	4.297	2.421	9.055

	SV 17	SV 03	SN 14
t_{oe}[s]	$5.616000000000E+05$	$5.616000000000E+05$	$5.544000000000E+05$
$\sqrt{a}$ [$\sqrt{\text{meter}}$]	$5.153551399231E+03$	$5.153627956390E+03$	$5.153673786163E+03$
e	$1.359284052160E-02$	$5.056695663370E-03$	$3.410041797906E-03$
M_0[rad]	$2.896689411747E+00$	$-4.148605833742E-01$	$-1.706818405918E+00$
ω[rad]	$-1.407703052483E+00$	$1.070620602228E+00$	$-2.966001708162E+00$
i_0[rad]	$9.737887949892E-01$	$9.798129497365E-01$	$9.481087120078E-01$
λ_0[rad]	$3.866174181787E-01$	$2.420706131693D+00$	$-6.917513380141E-01$
Δn[rad s^{-1}]	$4.250534194668E-09$	$4.415183910132E-09$	$5.246647115340E-09$
$\dot{i}$ [rad s^{-1}]	$-3.493002640438E-10$	$-1.157191058795E-10$	$2.964409193828E-11$
$\dot{\Omega}$[rad s^{-1}]	$-7.656390347961E-09$	$-8.298202796312E-09$	$-8.448566202408E-09$
C_{uc}[rad]	$-2.551823854446E-07$	$-6.780028343201E-07$	$7.450580596924E-09$
C_{us}[rad]	$9.344890713692E-06$	$3.432855010033E-06$	$5.153939127922E-06$
C_{rc}[m]	$2.021875000000E+02$	$3.205000000000\ E+02$	$2.732187500000E+02$
C_{rs}[m]	$-6.96875000000E+00$	$-1.27500000000E+01$	$3.156250000000E+00$
C_{ic}[rad]	$2.719461917877E-07$	$-4.470348358154E-08$	$1.024454832077E-07$
C_{is}[rad]	$-2.793967723846E-08$	$4.470348358154E-08$	$-2.980232238770E-08$
P(L1)[m]	24 380 357.760	24 444 143.500	22 891 323.280
dt^j[s]*	−0.000 721 440 74	0.000 221 870 57	−0.000 130 207 19
$\mathrm{d_{ion}}$[m]	6.786	4.807	4.598
$\mathrm{d_{trop}}$[m]	9.756	10.863	4.997

* dt^j in this table includes other source of errors in addition to the satellite clock error.

Note: The satellite coordinates calculated in this project does not account for all necessary effects. The implementation of these effects is necessary to achieve several meters accuracy.

1) Compute the coordinates of the 7 GPS satellites. Observation epoch is T = 558 000 s. We must calculate the satellite coordinates in transmission time ($\boldsymbol{t^s}$) of signal. If we use T =558 000 s, it means that we calculate satellite coordinates in reception (observation) time. One easy way to correct this is to see back in time to find when satellite signal is emitted. Satellite signal takes around 0.07 s to reach the Earth. To be more accurate we divide the measured pseudorange of each satellite to the speed of the light. Of course, we must add the satellite clock error too. Therefore, to calculate the transmission time ($\boldsymbol{t^s}$), we have:

$$t^s = T - \frac{P}{c} + dt$$

Note that there are more accurate methods. See lecture notes.

The ephemerides are used to compute the satellite coordinates at the transmission time ($\boldsymbol{t^s}$) using the equations given in the Appendix. The coordinates of all 7 GPS satellites SV08, SV10, SV21, SV24, SV17, SV03, and SV14 should be computed.

2) Compute the coordinates of the 7 GPS satellites disregarding the 9 correction terms (put correction terms = 0). Show the differences with and without correction terms. Discuss the results.

The approximate geodetic receiver position is:

$\varphi_{r0} = 63.2°$,

$\lambda_{r0} = 10.2°$,

$h_{r0} = 100$ m.

3) Transform the receiver coordinates to Cartesian coordinates. The geodetic coordinates are transformed into the geocentric WGS84 Cartesian coordinates using equations given in the Appendix.

To estimate the receiver position, we need to establish observation equation $\boldsymbol{L} = \boldsymbol{AX}$ $(\Delta L = A\Delta X)$. How we design the observation equation is given in the Appendix.

4) Design the observation equation and estimate the receiver position in the Cartesian coordinates. The satellite coordinates together with the approximate receiver position are used to compute the design matrix $\boldsymbol{A}$. See the Appendix.

5) Compute the positional dilution of precision PDOP. The cofactor matrix $\boldsymbol{Q_X}$ is a [4x4] matrix in this exercise, where three components are contributed by the receiver position X, Y, Z and one component by the receiver clock. The equations are given in the Appendix.

6) Compute the receiver position in geodetic coordinates. An iteration might be necessary to do these computations. Equations are given in the Appendix.

7) Estimate the receiver clock error.

Note that you must account for the various effects and errors like satellite clock correction, atmospheric (troposphere and ionosphere) corrections. See Appendix.

Hints:

Satellite coordinates for SV 03: $\begin{bmatrix} X^3 \\ Y^3 \\ Z^3 \end{bmatrix} = \begin{bmatrix} 230\ 984\ 33.065 \\ -126\ 694\ 12.772 \\ 2\ 685\ 881.089 \end{bmatrix}$ m

$h_r = 115.032$ m.

Appendix

GPS satellite coordinate computations:

$GM = 3.986\,005 \times 10^{14} \dfrac{m^3}{s^2}$

$\omega_e = 7.292\,115\,1467 \times 10^{-5} \dfrac{rad}{s}$

$\pi = 3.141\,592\,653\,5898$

$t_k = t - t_{oe}$; The time from ephemerides reference epoch. t_k must account for beginning or end of GPS week crossovers. t and t_{oe} are expressed in seconds in the GPS week. Each week 604800 sec.
If $t_k > 302400$ sec, subtract 604800 sec from t_k.
If $t_k < -302400$ sec, add 604800 sec.

$$M_k = M_0 + \left(\sqrt{\frac{GM}{a^3}} + \Delta n\right) t_k$$

$E_k = M_k + e \sin E_k$; Should be solved by iteration. One method commonly used for this iteration is to do 3 iterations as below:

$E_0 = M_k$; Initial value (rad)

$E_j = E_{j-1} + \frac{M_k - E_{j-1} + e \sin E_{j-1}}{1 - e \cos E_{j-1}}$; Refined value, three iterations, ($j = 1, 2, 3$)

$E_k = E_3$; Final value

$f_k = 2\arctan\left(\sqrt{\frac{1+e}{1-e}} \tan\frac{E_k}{2}\right)$; (Unambiguous quadrant)

$$u_k = \omega + f_k + C_{uc} \cos 2\,(\omega + f_k) + C_{us} \sin 2\,(\omega + f_k)$$

$$r_k = a(1 - e \cos E_k) + C_{rc} \cos 2\,(\omega + f_k) + C_{rs} \sin 2\,(\omega + f_k)$$

$$i_k = i_0 + \dot{i} t_k + C_{ic} \cos 2\,(\omega + f_k) + C_{is} \sin 2\,(\omega + f_k)$$

$$\lambda_K = \lambda_0 + (\dot{\Omega} - \omega_e) t_k - \omega_e t_{oe}$$

Satellite coordinates in Earth fixed Earth centered WGS84:

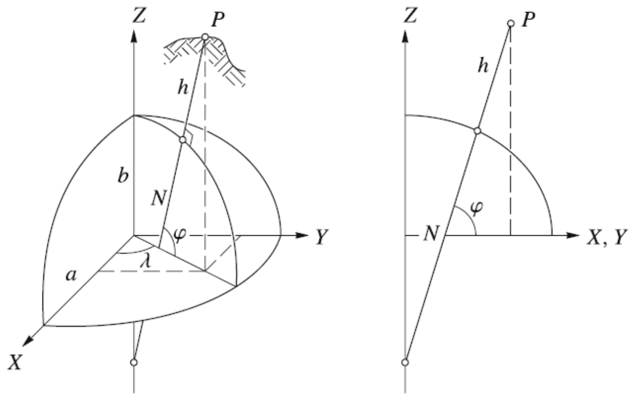
$$\begin{bmatrix} X^k \\ Y^k \\ Z^k \end{bmatrix} = R_3(-\lambda_k) R_1(-i_k) R_3(-u_k) \begin{bmatrix} r_k \\ 0 \\ 0 \end{bmatrix}$$

Transformation from ellipsoidal coordinates to Cartesian coordinates:

$X = (N + h) \cos\varphi \cos\lambda$

$Y = (N + h) \cos\varphi \sin\lambda$

$Z = \left(\frac{b^2}{a^2} N + h\right) \sin\varphi$



where *N* is the radius of curvature in the prime vertical and is obtained by:

$$N = \frac{a^2}{\sqrt{a^2 \cos^2\varphi + b^2 \sin^2\varphi}}$$

and *a*, *b* are the semi axes of the WGS84 ellipsoid.

The **inverse transformation** (to compute the ellipsoidal coordinates φ, λ and h from the Cartesian coordinates X, Y, Z) is solved iteratively:

From X and Y, the radius of a parallel can be computed easily from

$$p = \sqrt{X^2 + Y^2} = (N + h) \cos\varphi$$

Rearranging

$$h = \frac{p}{\cos\varphi} - N$$

We know $e^2 = \frac{a^2 - b^2}{a^2}$ or $1 - e^2 = \frac{b^2}{a^2}$ and substituting in formula for the "Z":

$$Z = (N + h - e^2 N) \sin\varphi = (N + h)\left(1 - e^2 \frac{N}{N + h}\right) \sin\varphi$$

Dividing by *p*

$$\frac{Z}{p} = \left(1 - e^2 \frac{N}{N + h}\right) \tan\varphi$$

and then

$$\tan\varphi = \frac{Z}{p}\left(1 - e^2\frac{N}{N+h}\right)^{-1}$$

Now we can find φ, λ and h as below:

Longitude λ can be calculated directly:

$$\tan\lambda = \frac{Y}{X}$$

We find φ and h iteratively:

1) We compute $p = \sqrt{X^2 + Y^2}$
2) We compute an approximate value φ_0 from $\tan\varphi_0 = \frac{Z}{p}(1 - e^2)^{-1}$
3) We compute an approximate value N_0 from $N_0 = \frac{\mathrm{a}^2}{\sqrt{\mathrm{a}^2\cos^2\varphi_0 + \mathrm{b}^2\sin^2\varphi_0}}$
4) We compute ellipsoidal height $h = \frac{p}{\cos\varphi_0} - N_0$
5) We Compute an improved value for the latitude by
$$\tan\varphi = \frac{Z}{p}\left(1 - e^2\frac{N_0}{N_0+h}\right)^{-1}$$
6) We check for another iteration step: if $\varphi = \varphi_0$, then the iteration is completed; otherwise set $\varphi = \varphi_0$ and continue with step 3.

There are many other computation methods.

Observation equation in the Least Squares (LS) method:

<u>The observation equations below consider 4 GPS satellites. You have to develop the observation equations for 7 GPS satellites.</u>

Code pseudorange observation for receiver "i" and satellite "j" reads:

$$\mathrm{P}_i^j(t) = \rho_i^j(t) + c\left(\mathrm{dt}^j(t) - \mathrm{dT}_i(t)\right) + \mathrm{d_{ion}}_i^j + \mathrm{d_{trop}}_i^j + \text{other terms}$$

where

$$\rho_i^j(t) = \sqrt{(X^j(t) - X_i)^2 + (Y^j(t) - Y_i)^2 + (Z^j(t) - Z_i)^2} = f(X_i, Y_i, Z_i)$$

where X_i, Y_i and Z_i are receiver position coordinates, and X^j, Y^j and Z^j are the satellite coordinates. c is the speed of light.

Linearizing $f(X_i, Y_i, Z_i)$ using a Taylor series expansion about approximate value $(X_{i0} + \Delta X_i, Y_{i0} + \Delta Y_i, Z_{i0} + \Delta Z_i)$ ("0" means approximate value):

$$f(X_i, Y_i, Z_i) = f(X_{i0}, Y_{i0}, Z_{i0}) + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial X_{i0}} \Delta X_i + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Y_{i0}} \Delta Y_i + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Z_{i0}} \Delta Z_i + \cdots$$

Then we have:

$$\rho_i^j(t) = \rho_{i0}^j(t) - \frac{X^j(t) - X_{i0}}{\rho_{i0}^j(t)} \Delta X_i - \frac{Y^j(t) - Y_{i0}}{\rho_{i0}^j(t)} \Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\rho_{i0}^j(t)} \Delta Z_i$$

where

$$\rho_{i0}^j(t) = \sqrt{(X^j(t) - X_{i0})^2 + (Y^j(t) - Y_{i0})^2 + (Z^j(t) - Z_{i0})^2}$$

The code pseudorange observation becomes:

$$\mathrm{P}_i^j(t) = \rho_{i0}^j(t) - \frac{X^j(t) - X_{i0}}{\rho_{i0}^j(t)} \Delta X_i - \frac{Y^j(t) - Y_{i0}}{\rho_{i0}^j(t)} \Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\rho_{i0}^j(t)} \Delta Z_i + c\left(\mathrm{dt}^j(t) - \mathrm{dT}_i(t)\right) + \mathrm{d_{ion}}_i^j + \mathrm{d_{trop}}_i^j + \text{other terms}$$

neglecting all other terms in the formula:

$$\mathrm{P}_i^j(t) - \rho_{i0}^j(t) - c\mathrm{dt}^j(t) - \mathrm{d_{ion}}_i^j - \mathrm{d_{trop}}_i^j$$
$$= -\frac{X^j(t) - X_{i0}}{\rho_{i0}^j(t)}\Delta X_i - \frac{Y^j(t) - Y_{i0}}{\rho_{i0}^j(t)}\Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\rho_{i0}^j(t)}\Delta Z_i - c\mathrm{dT}_i(t)$$

Let

$\Delta\ell^j$=$\mathrm{P}_i^j(t) - \rho_{i0}^j(t) - c\mathrm{dt}^j(\mathrm{t}) - \mathrm{d_{ion}}_i^j - \mathrm{d_{trop}}_i^j$, and

$$a_{X_i}^j = -\frac{X^j(t) - X_{i0}}{\rho_{i0}^j(t)},\ a_{Y_i}^j = -\frac{Y^j(t) - Y_{i0}}{\rho_{i0}^j(t)},\ a_{Z_i}^j = -\frac{Z^j(t) - Z_{i0}}{\rho_{i0}^j(t)}$$

Then the code pseudorange observation equation becomes:

$$\Delta\ell^j = a_{X_i}^j\Delta X_i + a_{Y_i}^j\Delta Y_i + a_{Z_i}^j\Delta Z_i - c\mathrm{dT}_i(t)$$

For 4 satellites then we have:

$$\Delta\ell^1 = a_{X_i}^1\Delta X_i + a_{Y_i}^1\Delta Y_i + a_{Z_i}^1\Delta Z_i - c\mathrm{dT}_i(t)$$
$$\Delta\ell^2 = a_{X_i}^2\Delta X_i + a_{Y_i}^2\Delta Y_i + a_{Z_i}^2\Delta Z_i - c\mathrm{dT}_i(t)$$
$$\Delta\ell^3 = a_{X_i}^3\Delta X_i + a_{Y_i}^3\Delta Y_i + a_{Z_i}^3\Delta Z_i - c\mathrm{dT}_i(t)$$
$$\Delta\ell^4 = a_{X_i}^4\Delta X_i + a_{Y_i}^4\Delta Y_i + a_{Z_i}^4\Delta Z_i - c\mathrm{dT}_i(t)$$

$\left(X^j(t), Y^j(t), Z^j(t)\right)$: Satellite positions (Computed from broadcast ephemeris)
$\mathrm{dt}^j(\mathrm{t}), \mathrm{d_{ion}}_i^j$ and $\mathrm{d_{trop}}_i^j$(are already computed and given, Formulas are given in lecture slides)
(X_{i0}, Y_{i0}, Z_{i0}): Approximate values of the receiver position
$\mathrm{P}_i^j(t)$: Pseudorange measurements to each satellite (measured)

$(\Delta X_i, \Delta Y_i, \Delta Z_i)$: Corrections to approximate values of the receiver position (estimated using LS)
$\mathrm{dT}_i(\mathrm{t})$: Receiver clock correction (estimated using LS)

For the speed of the light use $c = 299\ 792\ 458\ {}^{m}/_{s}$

Then, final estimated receiver position:

$$\hat{X}_i = X_{i0} + \Delta X_i$$
$$\hat{Y}_i = Y_{i0} + \Delta Y_i$$

$$\hat{Z}_i = Z_{i0} + \Delta Z_i$$

We use matrix form to show and solve the observation equation for one receiver, "r", and 4 satellites. We can write:

$$\boldsymbol{A} = \begin{bmatrix} a^1_{X_r} & a^1_{Y_r} & a^1_{Z_r} & -c \\ a^2_{X_r} & a^2_{Y_r} & a^2_{Z_r} & -c \\ a^3_{X_r} & a^3_{Y_r} & a^3_{Z_r} & -c \\ a^4_{X_r} & a^4_{Y_r} & a^4_{Z_r} & -c \end{bmatrix} = \begin{bmatrix} -\frac{X^1(t)-X_{r0}}{\varrho^1_{r0}(t)} & -\frac{Y^1(t)-Y_{r0}}{\varrho^1_{r0}(t)} & -\frac{Z^1(t)-Z_{r0}}{\varrho^1_{r0}(t)} & -c \\ -\frac{X^2(t)-X_{r0}}{\varrho^2_{r0}(t)} & -\frac{Y^2(t)-Y_{r0}}{\varrho^2_{r0}(t)} & -\frac{Z^2(t)-Z_{r0}}{\varrho^2_{r0}(t)} & -c \\ -\frac{X^3(t)-X_{r0}}{\varrho^3_{r0}(t)} & -\frac{Y^3(t)-Y_{r0}}{\varrho^3_{r0}(t)} & -\frac{Z^3(t)-Z_{r0}}{\varrho^3_{r0}(t)} & -c \\ -\frac{X^4(t)-X_{r0}}{\varrho^4_{r0}(t)} & -\frac{Y^4(t)-Y_{r0}}{\varrho^4_{r0}(t)} & -\frac{Z^4(t)-Z_{r0}}{\varrho^4_{r0}(t)} & -c \end{bmatrix}$$

$$\Delta X = \begin{bmatrix} \Delta X_r \\ \Delta Y_r \\ \Delta Z_r \\ \mathrm{dT}_r(t) \end{bmatrix}$$

$$\Delta L = \begin{bmatrix} \Delta\ell^1 \\ \Delta\ell^2 \\ \Delta\ell^3 \\ \Delta\ell^4 \end{bmatrix} = \begin{bmatrix} \mathrm{P}^1_r(t) - \rho^1_{r0}(t) - c\mathrm{dt}^1(t) - \mathrm{d_{ion}}{}^1_r - \mathrm{d_{trop}}{}^1_r \\ \mathrm{P}^2_r(t) - \rho^2_{r0}(t) - c\mathrm{dt}^2(t) - \mathrm{d_{ion}}{}^2_r - \mathrm{d_{trop}}{}^2_r \\ \mathrm{P}^3_r(t) - \rho^3_{r0}(t) - c\mathrm{dt}^3(t) - \mathrm{d_{ion}}{}^3_r - \mathrm{d_{trop}}{}^3_r \\ \mathrm{P}^4_r(t) - \rho^4_{r0}(t) - c\mathrm{dt}^4(t) - \mathrm{d_{ion}}{}^4_r - \mathrm{d_{trop}}{}^4_r \end{bmatrix}$$

We then have:

$$\Delta L = A\Delta X$$

In this exercise we use the same weight for all observations. For 4 satellites, degree of freedom=0:

$\Delta \mathrm{X} = A^{-1}\Delta \mathrm{L}$

If we track more than 4 satellites $\Rightarrow$ Least squares method

$\Delta X = (A^T A)^{-1} A^T \Delta L$

Variance-covariance matrix can be written:

$\boldsymbol{Q}_X = (\boldsymbol{A}^T\boldsymbol{A})^{-1}$

$$\boldsymbol{Q}_X = \begin{bmatrix} q_{XX} & q_{XY} & q_{XZ} & q_{XT} \\ q_{XY} & q_{YY} & q_{YZ} & q_{YT} \\ q_{XZ} & q_{YZ} & q_{ZZ} & q_{ZT} \\ q_{Xt} & q_{Yt} & q_{Zt} & q_{TT} \end{bmatrix}$$

The diagonal elements are used for the following PDOP definition:

$\mathrm{PDOP} = \sqrt{q_{XX} + q_{YY} + q_{ZZ}}$ ⋯ Positional dilution of precision.