TBA4565 Module GPS, Project 2
Accurate Relative Positioning with Carrier Phases

Relevant material (Blackboard):

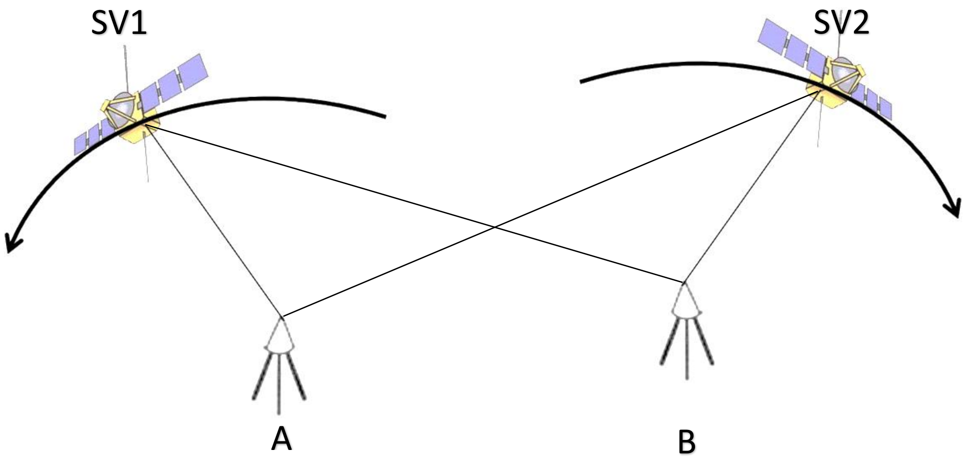
Lecture notes (Slides)

Report: Individual report. Submit the report on the Blackboard.

Tasks:

1. Relative positioning using carrier phase observations.
2. Decision on phase ambiguities.

In this exercise we numerically investigate how we can determine position of a GNSS receiver from relative positioning with carrier phases. We have two GNSS receivers A and B. The GNSS receivers track 5 GNSS satellites. The coordinates of these satellites in Earth Fixed Earth Centered system are already calculated from broadcast ephemerides and given together with the L1 carrier phase observations (see Tables below). L1 frequency is 1575.42 MHz and speed of the light is $c = 299\ 792\ 458\ {}^{m}/_{s}$. One double difference carrier phase between two receivers and two satellites are shown in figure below.



One double difference carrier phase consists of four carrier phase observations:

$$\Phi_{AB}^{12}(t) = \Phi_B^2(t) - \Phi_B^1(t) - \Phi_A^2(t) + \Phi_A^1(t)$$

Table 1. Satellite coordinates in Earth Fixed Earth Centered system in $t_1 = 172800$s seen from station A

Satellite no.	X (m)	Y (m)	Z (m)
154	-26916298.03	-2738678.66	-11996368.48
155	7525690.37	19506497.68	-20949608.30
159	-13393419.83	11968129.58	-23519577.49
174	-21266103.93	19852330.46	5502623.54
181	-25697148.88	8478510.63	-11983680.13

Table 2. L1 Carrier phase observations in $t_1 = 172800$s from station A

Satellite no.	L1 Carrier phase observations t_1 (Cycle)
154	143588831.82
155	130653024.41
159	126717695.82
174	135590276.69
181	132277486.86

Table 3. Satellite coordinates in Earth Fixed Earth Centered system in $t_1 = 172800$s seen from station B

Satellite no.	X (m)	Y (m)	Z (m)
154	-26916297.22	-2738678.33	-11996370.37
155	7525691.76	19506498.31	-20949607.21
159	-13393418.26	11968130.27	-23519578.04
174	-21266103.64	19852330.18	5502625.68
181	-25697149.62	8478511.10	-11983678.21

Table 4. L1 Carrier phase observations in $t_1 = 172800$s from station B

Satellite no.	L1 Carrier phase observations t_1 (Cycle)
154	144732675.77
155	131796866.21
159	127861541.27
174	136734190.29
181	133421353.12

Table 5. Satellite coordinates in Earth Fixed Earth Centered system in $t_2 = 175020\text{s}$ seen from station A

Satellite no.	X (m)	Y (m)	Z (m)
154	-28824855.18	-3331684.88	-5833025.24
155	2847363.30	17829892.74	-23451064.94
159	-18112952.18	10371378.82	-20977224.27
174	-21527125.49	20297088.12	-1169497.61
181	-23104316.90	6456482.66	-17326698.31

Table 6. L1 Carrier phase observations in $t_2 = 175020\text{s}$ from station A

Satellite no.	L1 Carrier phase observations t_2 (Cycle)
154	148445614.13
155	129436683.11
159	129024672.38
174	131540267.25
181	133044764.25

Table 7. Satellite coordinates in Earth Fixed Earth Centered system in $t_2 = 175020\text{s}$ seen from station B

Satellite no.	X (m)	Y (m)	Z (m)
154	-28824855.42	-3331684.93	-5833024.04
155	2847362.36	17829892.49	-23451065.25
159	-18112953.03	10371378.61	-20977223.64
174	-21527125.43	20297088.12	-1169498.87
181	-23104316.37	6456482.17	-17326699.19

Table 8. L1 Carrier phase observations in $t_2 = 175020\text{s}$ from station B

Satellite no.	L1 Carrier phase observations t_2 (Cycle)
154	147792428.87
155	128783480.33
159	128371483.32
174	130887130.25
181	132391579.66

GNSS receiver in A is a known station. The ellipsoidal coordinates of A are:

$\varphi_A = -32.003884648°$,

$\lambda_A = 115.894802001°$,

$h_A = 23.983$ m.

The second GNSS receiver in B is an unknown station that is going to be positioned in this exercise. The approximate ellipsoidal coordinates of B are:

$\varphi_{B0} = -31.9°$,

$\lambda_{B0} = 115.75°$,

$h_{B0} = 50$ m.

1) Transform the receiver coordinates to Cartesian coordinates. The geodetic coordinates are transformed into the Earth Fixed Earth Centered WGS84 Cartesian coordinates using equations given in the Appendix.

To estimate the position of receiver B, we need to establish observation equation $\boldsymbol{L} = \boldsymbol{AX}$. How we design the observation equation for double difference carrier phase measurements is given in the Appendix.

2) Design the observation equation and estimate the receiver position B in the Cartesian coordinates together with 4 unknown double difference phase ambiguities. The satellite coordinates together with the approximate receiver position are used to compute the design matrix $\boldsymbol{A}$. Estimate the variance covariance matrix of the unknown position and ambiguities. See the Appendix. This is a float solution. Also show the variance covariance matrix of the float solution in your report.

3) Float solution gives real phase ambiguity values. We must fix the real ambiguities to integer values. This process is called search technique. There are different search procedures. We use three scenarios below:

    a. Do nothing and use the real ambiguity in the next step,

    b. Fix the real ambiguity to nearest integer value,

    c. Use the standard deviation of ambiguities computed from LS in step 2 and fix the ambiguities considering the standard deviations.

Fix ambiguities according to the three scenarios above and with fixed ambiguities repeat the LS process in step 2 (now you have known ambiguities and only three unknowns for the position) and solve for final coordinates of the receiver B. This is a fixed solution. Report the fixed ambiguities by each of the three scenarios and compare them with each other in a Table. You will have three different values for the position of station B. Decide and choose the final coordinates for station B. Also report the variance covariance matrix of the final coordinates for station B.

4) Compute the position of station B in geodetic coordinates. An iteration might be necessary to do these computations. Equations are given in the Appendix.

Hints:

$h_B = 23.787$ m.

Appendix

Carrier phase observation equation in the Least Squares (LS) method:

The observation equations below consider two receivers, and 4 GNSS satellites at two epochs. You must develop the observation equations for two receivers and 5 GNSS satellites at two epochs.

Relative positioning aims at higher accuracies achievable only with carrier phases. We use double difference technique for relative positioning aiming very high accuracy. We start with considering two receivers in *A* and *B* and two satellites *i* and *j*. The double differences equation is (satellite and receiver clock errors are eliminated and the orbit, the ionospheric and the tropospheric errors are reduced significantly in double difference. In addition, the two stations A and B in this exercise are very close to each other and therefore the orbit, the ionospheric and the tropospheric errors can also be neglected):

$$\Delta\nabla\Phi = \Phi_{AB}^{ij}(t) = \rho_{AB}^{ij}(t) + \lambda N_{AB}^{ij}(t) \tag{1}$$

where

$$\rho_{AB}^{ij}(t) = \rho_B^j(t) - \rho_B^i(t) - \rho_A^j(t) + \rho_A^i(t)$$

and

$$\Phi_{AB}^{ij}(t) = \Phi_B^j(t) - \Phi_B^i(t) - \Phi_A^j(t) + \Phi_A^i(t)$$

are four geometry quantities and four carrier phase quantities, respectively, for a double difference. Please note that $\Phi_{AB}^{ij}(t)$ in Equation (1) is in meter, this means that the carrier phase observations in cycles should be multiplied by wavelength λ to be used in Equation (1) and all Φ's givens below.

Same as project 1, we have (we use general letters *i* for receiver and *j* for satellite):

$$\rho_i^j(t) = \sqrt{(X^j(t) - X_i)^2 + (Y^j(t) - Y_i)^2 + (Z^j(t) - Z_i)^2} = f(X_i, Y_i, Z_i)$$

where X_i, Y_i and Z_i are receiver position coordinates, and X^j, Y^j and Z^j are the satellite coordinates.

Linearizing $f(X_i, Y_i, Z_i)$ using a Taylor series expansion about approximate value $(X_{i0} + \Delta X_i, Y_{i0} + \Delta Y_i, Z_{i0} + \Delta Z_i)$ ("0" means approximate value):

$$f(X_i, Y_i, Z_i) = f(X_{i0}, Y_{i0}, Z_{i0}) + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial X_{i0}} \Delta X_i + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Y_{i0}} \Delta Y_i + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Z_{i0}} \Delta Z_i + \cdots$$

Then we have:

$$\rho_i^j\ (t) = \rho_{i0}^j\ (t) - \frac{X^j(t) - X_{io}}{\rho_{i0}^j\ (t)} \Delta X_i - \frac{Y^j(t) - Y_{io}}{\rho_{i0}^j\ (t)} \Delta Y_i - \frac{Z^j(t) - Z_{io}}{\rho_{i0}^j\ (t)} \Delta Z_i$$

where

$$\rho_{i0}^j(t) = \sqrt{(X^j(t) - X_{i0})^2 + (Y^j(t) - Y_{i0})^2 + (Z^j(t) - Z_{i0})^2}$$

We must linearize the four geometry terms in Equation (1), we then get (we come back to our definition at the beginning, two receivers in *A* and *B* and two satellites *i* and *j*:

$$\begin{aligned}
\rho_{AB}^{ij}(t) = \rho_{B0}^j\ (t) &- \frac{X^j(t) - X_{Bo}}{\rho_{B0}^j\ (t)} \Delta X_B - \frac{Y^j(t) - Y_{Bo}}{\rho_{B0}^j\ (t)} \Delta Y_B - \frac{Z^j(t) - Z_{Bo}}{\rho_{B0}^j\ (t)} \Delta Z_B \\
-\rho_{B0}^i\ (t) &+ \frac{X^i(t) - X_{Bo}}{\rho_{B0}^i\ (t)} \Delta X_B + \frac{Y^i(t) - Y_{Bo}}{\rho_{B0}^i\ (t)} \Delta Y_B + \frac{Z^i(t) - Z_{Bo}}{\rho_{B0}^i\ (t)} \Delta Z_B \\
-\rho_{A0}^j\ (t) &+ \frac{X^j(t) - X_{Ao}}{\rho_{A0}^j\ (t)} \Delta X_A + \frac{Y^j(t) - Y_{Ao}}{\rho_{A0}^j\ (t)} \Delta Y_A + \frac{Z^j(t) - Z_{Ao}}{\rho_{A0}^j\ (t)} \Delta Z_A \\
+\rho_{A0}^i\ (t) &- \frac{X^i(t) - X_{Ao}}{\rho_{A0}^i\ (t)} \Delta X_A - \frac{Y^i(t) - Y_{Ao}}{\rho_{A0}^i\ (t)} \Delta Y_A - \frac{Z^i(t) - Z_{Ao}}{\rho_{A0}^i\ (t)} \Delta Z_A
\end{aligned}$$

After substitution and rearranging, the linear observation equation becomes:

$$\begin{aligned}
\Delta\ell_{AB}^{ij}\ (t) &= a_{X_A}^{ij}\ (t)\Delta X_A + a_{Y_A}^{ij}\ (t)\Delta Y_A + a_{Z_A}^{ij}\ (t)\Delta Z_A \\
&+ a_{X_B}^{ij}\ (t)\Delta X_B + a_{Y_B}^{ij}\ (t)\Delta Y_B + a_{Z_B}^{ij}\ (t)\Delta Z_B + \lambda N_{AB}^{ij}(t)
\end{aligned}$$

where

$$\Delta\ell_{AB}^{ij}(t) = \Phi_{AB}^{ij}(t) - \rho_{B0}^j(t) + \rho_{B0}^i(t) + \rho_{A0}^j(t) - \rho_{A0}^i(t)$$

includes measurements and terms calculated from the approximate values.

And

$$a_{X_A}^{ij}\ (t) = \frac{X^j(t) - X_{Ao}}{\rho_{A0}^j\ (t)} - \frac{X^i(t) - X_{Ao}}{\rho_{A0}^i\ (t)}$$

$$a_{Y_A}^{ij}(t) = \frac{Y^j(t) - Y_{Ao}}{\rho_{A0}^j\ (t)} - \frac{Y^i(t) - Y_{Ao}}{\rho_{A0}^i\ (t)}$$

$$a_{Z_A}^{ij}(t) = \frac{Z^j(t) - Z_{Ao}}{\rho_{A0}^j\ (t)} - \frac{Z^i(t) - Z_{Ao}}{\rho_{A0}^i\ (t)}$$

$$a_{X_B}^{ij}(t) = -\frac{X^j(t) - X_{Bo}}{\rho_{B0}^j\ (t)} + \frac{X^i(t) - X_{Bo}}{\rho_{B0}^i\ (t)}$$

$$a_{Y_B}^{ij}(t) = -\frac{Y^j(t) - Y_{Bo}}{\rho_{B0}^j\ (t)} + \frac{Y^i(t) - Y_{Bo}}{\rho_{B0}^i\ (t)}$$

$$a_{Z_B}^{ij}\ (t) = -\frac{Z^j(t) - Z_{Bo}}{\rho_{B0}^j\ (t)} + \frac{Z^i(t) - Z_{Bo}}{\rho_{B0}^i\ (t)}$$

In relative positioning the coordinates of one point, for example *A* here, should be known. Therefore, the number of unknowns reduces, as

$$\varDelta X_A = \varDelta Y_A = \varDelta Z_A = 0$$

and

$$\varDelta\ell_{AB}^{ij}(t) = \varPhi_{AB}^{ij}(t) - \rho_{B0}^j(t) + \rho_{B0}^i(t) + \rho_A^j(t) - \rho_A^i(t)$$

Assuming four satellites *i*, *j*, *k* and *l* and two epochs $t_1$and t_2, the observation equation in matrix form becomes:

$$\varDelta L = \begin{bmatrix} \varDelta\ell_{AB}^{ij}(t_1) \\ \varDelta\ell_{AB}^{ik}(t_1) \\ \varDelta\ell_{AB}^{il}(t_1) \\ \varDelta\ell_{AB}^{ij}(t_2) \\ \varDelta\ell_{AB}^{ik}(t_2) \\ \varDelta\ell_{AB}^{il}(t_2) \end{bmatrix} \qquad \varDelta X = \begin{bmatrix} \varDelta X_B \\ \varDelta Y_B \\ \varDelta Z_B \\ N_{AB}^{ij} \\ N_{AB}^{ik} \\ N_{AB}^{il} \end{bmatrix}$$

$$A = \begin{bmatrix} a^{ij}_{X_B}(t_1) & a^{ij}_{Y_B}(t_1) & a^{ij}_{Z_B}(t_1) & \lambda & 0 & 0 \\ a^{ik}_{X_B}(t_1) & a^{ik}_{Y_B}(t_1) & a^{ik}_{Z_B}(t_1) & 0 & \lambda & 0 \\ a^{il}_{X_B}(t_1) & a^{il}_{Y_B}(t_1) & a^{il}_{Z_B}(t_1) & 0 & 0 & \lambda \\ a^{ij}_{X_B}(t_2) & a^{ij}_{Y_B}(t_2) & a^{ij}_{Z_B}(t_2) & \lambda & 0 & 0 \\ a^{ik}_{X_B}(t_2) & a^{ik}_{Y_B}(t_2) & a^{ik}_{Z_B}(t_2) & 0 & \lambda & 0 \\ a^{il}_{X_B}(t_2) & a^{il}_{Y_B}(t_2) & a^{il}_{Z_B}(t_2) & 0 & 0 & \lambda \end{bmatrix}$$

This is a system of determined and solvable with 6 observations and 6 unknowns. Only one epoch observation gives an under-determined system with more unknowns than observations.

We then have:

$$\Delta L = A \Delta X$$

$$\Delta X = (A^T P A)^{-1} A^T P \Delta L$$

Variance-covariance matrix can be written:

$$\boldsymbol{C}_X = (\boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A})^{-1}$$

$$\boldsymbol{C}_X = \begin{bmatrix} q_{XX} & q_{XY} & q_{XZ} & q_{XN_{ij}} & q_{XN_{ik}} & q_{XN_{il}} \\ q_{XY} & q_{YY} & q_{YZ} & q_{YN_{ij}} & q_{YN_{ik}} & q_{YN_{il}} \\ q_{XZ} & q_{YZ} & q_{ZZ} & q_{ZN_{ij}} & q_{ZN_{ik}} & q_{ZN_{il}} \\ q_{XN_{ij}} & q_{YN_{ij}} & q_{ZN_{ij}} & q^{ij}_{NN} & q_{N_{ij}N_{ik}} & q_{N_{ij}N_{il}} \\ q_{XN_{ik}} & q_{YN_{ik}} & q_{ZN_{ik}} & q_{N_{ij}N_{ik}} & q^{ik}_{NN} & q_{N_{il}N_{ik}} \\ q_{XN_{il}} & q_{YN_{il}} & q_{ZN_{il}} & q_{N_{ij}N_{il}} & q_{N_{il}N_{ik}} & q^{il}_{NN} \end{bmatrix}$$

Weight matrix given in the next page.

Weight matrix *P* of double differences:

Double differencing introduces correlations. The double difference weight matrix can be calculated from the variance covariance matrix of carrier phase observations. The weight matrix P for 4 double differences for two receivers and 5 GNSS satellites at two epochs are given below. You can use it directly in your least squares formula. This is developed for five GNSS satellites at two epochs not four satellites as the formulas given above:

$$P = \frac{1}{2\sigma^2}\frac{1}{5}\begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

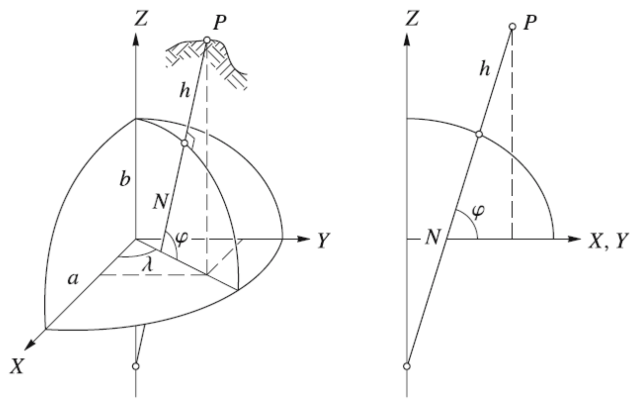
σ is the standard deviation of carrier phase observations and for this study (the phase observations have high quality) can be considered $\sigma = 5$ mm. How this weight matrix P is derived has been discussed and given in the slides in Blackboard (Weight matrix of phase double differences).

Transformation from ellipsoidal coordinates to Cartesian coordinates:

$X = (N + h) \cos\varphi \cos\lambda$

$Y = (N + h) \cos\varphi \sin\lambda$

$Z = \left(\frac{b^2}{a^2} N + h\right) \sin\varphi$



where N is the radius of curvature in the prime vertical and is obtained by:

$$N = \frac{a^2}{\sqrt{a^2 \cos^2\varphi + b^2 \sin^2\varphi}}$$

and a, b are the semi axes of the WGS84 ellipsoid.

The **<u>inverse transformation</u>** (to compute the ellipsoidal coordinates φ, λ and h from the Cartesian coordinates X, Y, Z) is solved iteratively:

From X and Y, the radius of a parallel can be computed easily from

$$p = \sqrt{X^2 + Y^2} = (N + h)\cos\varphi$$

Rearranging

$$h = \frac{p}{\cos\varphi} - N$$

We know $e^2 = \frac{a^2 - b^2}{a^2}$ or $1 - e^2 = \frac{b^2}{a^2}$ and substituting in formula for the “Z”:

$$Z = (N + h - e^2 N)\sin\varphi = (N + h)\left(1 - e^2 \frac{N}{N + h}\right)\sin\varphi$$

Dividing by p

$$\frac{Z}{p} = \left(1 - e^2 \frac{N}{N + h}\right)\tan\varphi$$

and then

$$\tan\varphi = \frac{Z}{p}\left(1 - e^2 \frac{N}{N + h}\right)^{-1}$$

Now we can find φ, λ and h as below:

Longitude λ can be calculated directly:

$$\tan\lambda = \frac{Y}{X}$$

We find φ and h iteratively:

1) We compute $p = \sqrt{X^2 + Y^2}$
2) We compute an approximate value φ_0 from $\tan\varphi_0 = \frac{Z}{p}(1 - e^2)^{-1}$
3) We compute an approximate value N_0 from $N_0 = \frac{\mathrm{a}^2}{\sqrt{\mathrm{a}^2\cos^2\varphi_0 + \mathrm{b}^2\sin^2\varphi_0}}$
4) We compute ellipsoidal height $h = \frac{p}{\cos\varphi_0} - N_0$
5) We Compute an improved value for the latitude by
$$\tan\varphi = \frac{Z}{p}\left(1 - e^2 \frac{N_0}{N_0 + h}\right)^{-1}$$
6) We check for another iteration step: if $\varphi = \varphi_0$, then the iteration is completed; otherwise set $\varphi = \varphi_0$ and continue with step 3.

There are many other computation methods.