Summary of
TTK4150 Nonlinear Control Systems

Morten Fyhn Amundsen & Erik Liland

August 31, 2016

Contents

Todo

- Estimate of R_A
- K/KL-based stability (p28 slide6)
- L/L-infinity stability
- Def. 5.1
- Theorem 10.4
- Nonvanishing perturbation.
- Theorem 6.3

1 Math

1.1 Metric norms

General p-norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{1}$$

Taxicab norm ($p = 1$)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \tag{2}$$

Euclidean norm ($p = 2$)

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} \tag{3}$$

1.2 Boundedness and $\mathcal{L}_p$-norms

$\mathcal{L}_p$-norm

$$\left\|f\right\|_p = \left(\int_a^b \left|f(\tau)\right|^p \,\mathrm{d}\tau\right)^{1/p} \tag{4}$$

$\mathcal{L}_\infty$-norm

$$\left\|f\right\|_\infty = \sup_{a \le t \le b} \left|f(t)\right| \tag{5}$$

Boundedness

$$f \in \mathcal{L}_p \Leftrightarrow \left\|f\right\|_p < \infty \tag{6}$$

1.3 Properties of norms

Hölder's inequality

$$\left\|fg\right\|_1 \le \left\|f\right\|_p \left\|g\right\|_q \tag{7}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

1.4 Matrix properties

Singular A matrix is *singular* iff its determinant is zero.

Skew-symmetry A matrix A is *skew-symmetric* iff

$$-A = A^T. \tag{8}$$

Jacobian The *Jacobian* matrix is defined by

$$J = \frac{\mathrm{d}f}{\mathrm{d}x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}. \tag{9}$$

Hurwitz A matrix A is *Hurwitz* if all eigenvalues of A satisfy $\Re\lambda_i < 0$.

Positive definite A matrix being *positive definite* is equivalent to

- all its eigenvalues being positive,
- all its leading principal minors being positive.

In addition, we have

$$\lambda_{\min}(H)x^T x \le x^T H x \le \lambda_{\max}(H)x^T x \tag{10}$$

for a positive definite $x^T Hx$.

2 Seconds-order systems

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2 \tag{11}$$

2.1 Behaviour near equilibria

Linearise and find eigenvalues at each equilibrium:

- Real λ
  - $\lambda_1 < \lambda_2 < 0 \implies$ Stable node.
  - $0 < \lambda_1 < \lambda_2 \implies$ Unstable node.
  - $\lambda_1 < 0 < \lambda_2 \implies$ Saddle point.
- Complex $\lambda_{1,2} = \alpha \pm \beta\imath$
  - $\alpha = 0 \implies$ Center.
  - $\alpha < 0 \implies$ Stable focus.
  - $\alpha > 0 \implies$ Unstable focus.

2.2 Periodic orbits

Lemma 2.1 (Poincaré-Bendixson criterion) Consider (11). Let M be a bounded, closed subset of the plane such that

- M contains no equilibria, or *one* equilibrium for which the eigenvalues of the Jacobian has positive real parts,
- All trajectories in M stay in M.

Then, M contains a periodic orbit.

Lemma 2.2 (Bendixson (negative) criterion) If

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \tag{12}$$

is not zero and does not change sign on a simply connected region $\mathbb{D}$, then (11) has no periodic orbits in $\mathbb{D}$.

Corollary 2.1 (The index method)

- Nodes, foci, and centers have index 1.
- Saddle points have index -1.

The sum of indices of all equilibria within a periodic orbit is always 1.

3 Lyapunov stability

Lyapunov function (LF) $V(x)$ is an LF iff

- $V \in C^1$
- $V(0) = 0$
  $V(x) > 0 \quad \forall \quad x \in \mathbb{D} \backslash \{0\}$
- $\dot{V}(0) = 0$
  $\dot{V}(x) \leq 0 \quad \forall \quad x \in \mathbb{D} \backslash \{0\}$

Strict Lyapunov function (SLF) $V(x)$ is an SLF iff it is an LF and

- $\dot{V}(x) < 0 \quad \forall \quad x \in \mathbb{D} \backslash \{0\}$

3.1 Autonomous systems

Theorem 4.1 (Direct Lyapunov method)

- If $\exists$ an LF for the origin, then the origin is stable.
- If $\exists$ an SLF for the origin, then the origin is asymptotically stable.

Theorem 4.2 (GAS) If $\exists$ an SLF V for the origin and V is radially unbounded, then the origin is globally asymptotically stable.

3.2 Invariance principle

Invariant set A set M is an *invariant set* w.r.t. $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M \quad \forall t \in \mathbb{R}. \tag{13}$$

(Any solution in M stays in M for all future and past.)

Theorem 4.4 (LaSalle's theorem) If $\exists\, V : \mathbb{D} \to \mathbb{R}$ such that

- $V \in C^1$,
- $\exists\, c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subseteq \mathbb{D}$ is bounded,
- $\dot{V}(x) \leq 0 \quad \forall \quad x \in \Omega_c$.

Let $E = \{x \in \Omega_c | \dot{V}(x) = 0\}$. Let M be the largest invariant set in E. Then

$$x(0) \in \Omega_c \implies x(t) \xrightarrow{t \to \infty} M. \tag{14}$$

Corollary 4.1 Let $x^* = 0$ for $\dot{x} = f(x)$. If for an LF $V(x)$ we have $\dot{V}(x) \leq 0$ on D: Let $S = \{x \in D | \dot{V}(x) = 0\}$ and only $x(t) \equiv 0$ can stay in S, then $x = 0$ AS.

Corollary 4.2 If corollary 4.1 holds with $D = \mathbb{R}^n$, then $x = 0$ GAS.

3.3 Linear systems and linearisation

Theorem 4.7 (Lyapunov's indirect method) Let $x^* = 0$ for $\dot{x} = f(x)$ where $f : \mathbb{D} \to \mathbb{R}^n$ satisfies $f \in \mathbb{C}^1$ and $\mathbb{D}$ is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=0} \tag{15}$$

and λ_i be the eigenvalues of A. Then

1. $\Re\lambda_i < 0$ for all $\lambda_i \implies x = 0$ AS.
2. $\Re\lambda_i > 0$ for any $\lambda_i \implies x = 0$ unstable.

3.4 Comparison functions

Class $\mathcal{K}$ function A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ iff

- it is stricly increasing,
- $\alpha(0) = 0$.

Class $\mathcal{K}_\infty$ function A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}_\infty$ iff

- it is of class $\mathcal{K}$,
- $a = \infty$,
- $\alpha(r) \xrightarrow{r\to\infty} \infty$.

Class $\mathcal{KL}$ function A continunous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{KL}$ if for each fixed s

- $\beta(r, s)$ is a class $\mathcal{K}$ function w.r.t. r,

and for each fixed r

- $\beta(r, s)$ is decreasing w.r.t. s,
- $\beta(r, s) \xrightarrow{s\to\infty} 0$.

3.5 Nonautonomous systems

$$\dot{x} = f(t, x), \quad f : [0, \infty) \times \mathbb{D} \to \mathbb{R}^n \tag{16}$$

Decrescentness $V(t, x)$ is *decrescent* iff

$$\left.\begin{array}{l} V(t, 0) = 0 \\ V(t, x) \leq W_2(x) \end{array}\right\} \forall\, t \geq 0, \text{ for some pos. def. } W_2(x). \tag{17}$$

Theorem 4.8–4.9 Let $V : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ and $V \in C^1$ for (16). Then $x^* = 0$ is

	S	US	UAS	GUAS
V	PD	PD, decr.	PD, decr.	PD, decr., RU
$\dot{V}$	NSD	NSD	ND	ND
$\forall x \in$	$\mathbb{D}$	$\mathbb{D}$	$\mathbb{D}$	$\mathbb{R}^n$

(PD = positive definite, decr. = decrescent, RU = radially unbounded, NSD = negative semidefinite, ND = negative definite.)

Theorem 4.10 (exponential stability) If $\exists\, a, k_1, k_2, k_3 > 0$ such that

- $V \in C^1$
- $k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a \quad \forall\, x \in \mathbb{D}$
- $\dot{V}(x) \leq -k_3\|x\|^a \quad \forall\, x \in \mathbb{D}$

then $x = 0$ ES. If $\mathbb{D} = \mathbb{R}^n$, $x = 0$ is GES.

3.6 Converse theorems

Corollary 4.3 The origin of $\dot{x} = f(x)$ is ES iff A is Hurwitz, where

$$A = \left[\frac{\partial f}{\partial x}\right]\Bigg|_{x=0}. \tag{18}$$

3.7 Input-to-state stability

$$\Sigma: \quad \dot{x} = f(t, x, u) \tag{19}$$

Theorem 4.19 Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be cont. diff.able such that

$$\alpha_1\left(||x||\right) \leq V(t, x) \leq \alpha_2\left(||x||\right) \tag{20}$$

$$\dot{V} \leq -W_3(x) \quad \forall \quad ||x|| \geq \rho\left(||u||\right) > 0 \tag{21}$$

$\forall (t, x, u) \in [0, \infty) \times R^n \times R^m$, where

- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho \in \mathcal{K}$, and
- $W_3(x)$ is a cont. pos. def. function on $\mathbb{R}^n$.

Then, the system is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

ISS vs. 0-GUAS

$$\Sigma \text{ ISS} \implies \Sigma \text{ 0-GUAS} \tag{22}$$

$$\neg(\Sigma \text{ 0-GUAS}) \implies \neg(\Sigma \text{ ISS}) \tag{23}$$

Lemma 4.6 (ISS vs. 0-GES) For (19) with $f \in C^1$ and f globally Lipschitz in (x, u), then

$$\Sigma \text{ 0-GES} \implies \Sigma \text{ ISS}. \tag{24}$$

Lemma 4.7 The cascade of a GUAS and an ISS system is ISS. (Output of GUAS is the input of ISS.)

4 Passivity

$$\dot{x} = f(x, u) \tag{25}$$

$$y = h(x, u) \tag{26}$$

4.1 Memoryless functions

Definition 6.1 The system $y = h(t, u)$ is

- passive if $u^T y \geq 0$,
- lossless if $u^T y = 0$,
- input-feedforward passive if $u^T y \geq u^T \phi(u)$ for some $\phi(u)$,
- input strictly passive if it is IFP and $u^T \phi(u) > 0 \; \forall \; y \neq 0$,
- output-feedback passive if $u^T y \geq y^T \rho(y)$ for some $\rho(y)$,
- output strictly passive if it is OFP and $y^T \rho(y) > 0 \; \forall \; y \neq 0$.

4.2 State models

Definition 6.3 The system (25)–(26) with storage function $V(x) \geq 0$ is

- passive if $u^T y \geq \dot{V}$,
- lossless if $u^T y = \dot{V}$,
- input-feedforward passive if $u^T y \geq \dot{V} + u^T \phi(u)$ for some function ϕ,
- input strictly passive if it is IFP with $u^T \phi(u) > 0 \; \forall \; u \neq 0$,
- output-feedback passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ,
- output strictly passive if it is OFP with $y^T \rho(y) > 0 \; \forall \; y \neq 0$,
- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some pos. def ψ.

4.3 $\mathcal{L}_2$ and Lyapunov stability

Lemma 6.5 (finite-gain $\mathcal{L}_2$ stable) If a system is output strictly passive with $\rho(y) = \delta y$ with $\delta > 0$ then it is finite-gain $\mathcal{L}_2$ stable with gain $\gamma \leq \delta^{-1}$.

Definition 6.5 (zero-state observability) The system (25)–(26) is zero-state observable if only the solution $x(t) \equiv 0$ of $\dot{x} = f(x, 0)$ can stay in $S = \{x \in \mathbb{R}^n | h(x, 0) = 0\}$.

4.4 Feedback systems

Theorem 6.1 The feedback connection of two passive systems is passive, with $V = V_1 + V_2$.

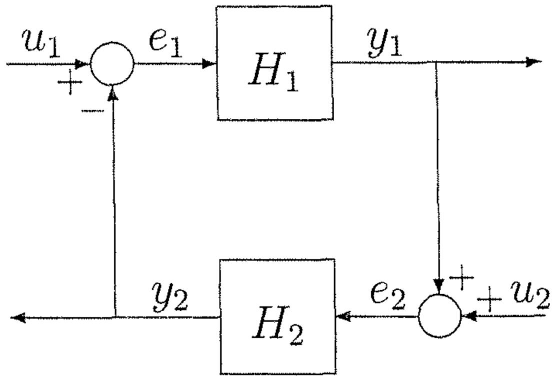

Figure 1: Feedback connection

Theorem 6.2 ($\mathcal{L}_2$-stability of feedback connection) If H_1 and H_2 satisfy

$$e_i^T y_i \geq \dot{V}_i + \epsilon_i e_i^T e_i + \delta_i y_i^T y_i \tag{27}$$

and

$$\epsilon_1 + \delta_2 > 0 \text{ and } \epsilon_2 + \delta_1 > 0 \tag{28}$$

then the feedback connection is finite-gain $\mathcal{L}_2$-stable.

5 Stability of perturbed systems

We consider perturbed systems on the form

$$\dot{x} = f(t,x) + g(t,x) \tag{29}$$

with *nominal* systems

$$\dot{x} = f(t,x). \tag{30}$$

5.1 Vanishing perturbation

Lemma 9.1

- The origin is an ES equilibrium of the nominal system.
- $V(t,x)$ is an LF of the nominal system, and satisfies

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \tag{31}$$

  and

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| . \tag{32}$$

- The perturbation $g(t, x)$ satisfies

$$\left\|g(t, x)\right\| \leq \gamma \|x\|, \quad \gamma < \frac{c_3}{c_4}. \tag{33}$$

Then $x^* = 0$ of the perturbed system is ES. If the assumptions hold globally, $x = 0$ is GES.

5.2 Nonvanishing perturbation

6 Perturbation theory and averaging

$$\dot{x} = f(x) + \epsilon g(t, x, \epsilon) \tag{34}$$

6.1 Periodic perturbation of autonomous systems

Definition of $P_\epsilon(x)$: $\phi(t; t_0, x_0, \epsilon)$ is the solution of (34) that starts at (t_0, x_0). $P_\epsilon(x)$ is

$$P_\epsilon(x) = \phi(T; 0, x, \epsilon) \tag{35}$$

Lemma 10.1 The system (34) has a T-periodic solution iff

$$x = P_\epsilon(x) \tag{36}$$

has a solution.

6.2 Averaging

7 Feedback linearization

Consider a class of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \\ u &= \alpha(x) + \beta(x)v \\ z &= T(x) \end{aligned} \tag{37}$$

where u is a state feedback conboller, and T is a change of variables.

To be able to cancel nonlinearities with feedback the input and non-linearities must appear together as a sum $\lambda(x) + u$ or as a product $\lambda(x)u$, where the matrix $\lambda(x)$ is non-singular in the domain of interest, and $u = \beta(x)v$, $\beta(x) = \lambda^{-1}$.

Definition 13.1 A nonlinear system as (37) where $f : D \rightarrow R^n$ and $G : D \rightarrow R^{n \times p}$ are sufficiently smooth on a domain $D \subseteq R^n$, is said to be feedback linearizable (or input-state linearizable) if there exists a diffeomorphism $T : D \rightarrow R^n$ such that $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms (37) into the form

$$\dot{z} = Az + B\lambda(x)[u - \alpha(x)] \tag{38}$$

with (A, B) controllable and $\lambda(x)$ nonsingular $\forall x \in D$.

7.1 Input-output linearization

Consider (37) which satisfies Def. 13.1. The derivative $\dot{y}$ is given by

$$\dot{y} = \frac{\partial h}{\partial x}\left(f(x) + g(x)\right) \triangleq L_f h(x) + L_g h(x) u \tag{39}$$

where $L_f h(x) \triangleq \frac{\partial h}{\partial x} f(x)$ is the *Lie Derivative* of h w.r.t. f.

Relative degree The relative degree is the number of times y must be differentiated until $u \in D_0 \subseteq D$ appears. A system must have a well defined relative degree to be input-output linearizable. (It must also be minimum phase.)

Diffeomorphism Wikipedia: *In mathematics, a diffeomorphism is an isomorphism of smooth manifolds. It is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are smooth.*

Theorem 13.1 Consider (37) with relative degree $\rho \leq n$ in D. If $\rho = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{40}$$

restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in D$, a neighborhood N of x_0 and smooth function $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n - \rho, \forall x \in D_0 \tag{41}$$

is satisfied $\forall x \in N$ and the map

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{42}$$

restricted to N, is a diffeomorphism on N.

Method

1. Set system on following form $\dot{x} = f(x) + g(x)u$
2. Find the relative degree ρ, ($\rho = n \Rightarrow$ no internal dynamics)
3. Write the system in normal form (external and internal dynamics)
4. Choose u to cancel the nonlinearities
5. Analyze the zero-dynamics
6. Choose v to solve the control problem

7.2 Full-state linearization

7.3 State feedback control

8 Nonlinear design tools

8.1 Backstepping

General idea Start by selecting a state x_i, where i is some index. Given $\dot{x}_i = f_i(x)$, consider one of the other states present in f_i as the input. Let's call this state x_j. Find an expression $x_j = \phi_j(x)$ that stabilizes x_i. (Using a Lyapunov function $V(x_i)$.) Then, define $z_j = x_j - \phi_j(x)$, and rewrite the system in terms of x_i and z_j. Now, considering $\dot{z}_j$, use the same method to find an expression for a state present in $\dot{z}_j$ to stabilize z_j and x_i. (Now with a Lyapunov function $V(x_i, z_j)$.) Keep going until you run out of states to stabilize.

8.2 Passivity-based control

$$\dot{x} = f(x, u) \tag{43}$$
$$y = h(x) \tag{44}$$

Theorem 14.4 If (43)–(44) is

- passive with an RU, pos. def. storage function,
- zero-state observable

then $x = 0$ can be globally stabilized by $u = -\phi(y)$, with ϕ locally Lipschitz with $\phi(0) = 0$, $y^T \phi(y) > 0 \forall y \neq 0$.