Summary of TTK18

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1 Linear matrix inequalities for piecewise affine systems

1.1 Piecewise affine systems

A piecewise affine system is a system where the state space X is divided into non-overlapping partitions X_i with distinct models in each partition:

- Partitions containing the origin are linear:

$$\begin{aligned} x &= A_i x \\ x_{k+1} &= A_i x_k \end{aligned} \tag{1}$$

- Partitions not containing the origin are affine:

$$\begin{aligned} x &= A_i x + a_i \\ x_{k+1} &= A_i x_k + a_i \end{aligned} \tag{2}$$

The partitions are indexed, with an index set

$$I = I_0 \cup I_1 \tag{3}$$

where I_0 are the parititions containing the origin, and I_1 are the partitions not containing the origin.

1.2 Lyapunov stability

Sometimes you can find a Lyapunov function for the whole PWA system:

- If $a_i = 0 \forall i$ and $\exists P = P^T > 0$ such that $A_i^T P + PA_i < 0 \forall i \in I$, then the origin is exponentially stable.

- if $a_i = 0 \forall i$ and $\exists R_i > 0 \forall i \in I$ such that $\sum_{i \in I}(A_i^T R_i + R_i A_i) > 0$, then a common Lyapunov function can be found.

When a common function cannot be found, we must look for one that depends on the partition.

1.2.1 Notation

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \tag{4}$$

Each partition is a polyhedron, so we can write

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}, \quad \bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix} \tag{5}$$

where $e_i = 0$ and $f_i = 0$ for all $i \in I_0$ such that

$$\bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \forall x \in X_i \forall i \in I \tag{6}$$

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} \forall\, x \in X_i \cap X_j, \forall i, j \in I \tag{7}$$

That is, $E_i x + e_i \geq 0$ for all values of x, and $F_i x + f_i = F_j x + f_j$ for all x along borders between partitions.

1.2.2 Lyapunov function

$$V(x) = \begin{cases} x^T P_i x & x \in X_i, i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} & x \in X_i, i \in I_1 \end{cases} \tag{8}$$

where[1]

$$P_i = F_i^T T F_i, \quad \bar{P}_i = \bar{F}_i^T T \bar{F}_i \tag{9}$$

1.2.3 Relaxing the LF

While $V(x) > 0$ is required for all x, we only need $P_i > 0$ for $x \in X_i$. The S-procedure can be used to alter the LF to reflect this[2]:

$$V(x) = \begin{cases} x^T E_i^T U_i E_i x & x \in X_i, i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{E}_i^T U_i \bar{E}_i^T \begin{bmatrix} x \\ 1 \end{bmatrix} & x \in X_i, i \in I_1 \end{cases} \tag{10}$$

[1] TODO: wtf is the T matrix?

[2] TODO: Double check that this actually is the Lyapunov function.

where U_i has only non-negative elements. The condition $\dot{V}(x) < 0$ only needs to hold within each partition, as well.

1.2.4 Overall LF design

Find symmetric matrices T, U_i, W_i (U_i and W_i with no negative elements), such that

$$\begin{aligned} &\begin{cases} A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \\ P_i - E_i^T W_i E_i > 0 \end{cases} \quad i \in I_0 \\ &\begin{cases} \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i < 0 \\ \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i > 0 \end{cases} \quad i \in I_1 \end{aligned} \tag{11}$$

with (9) still holding:

$$P_i = F_i^T T F_i, \quad \bar{P}_i = \bar{F}_i^T T \bar{F}_i \tag{12}$$

1.3 Stability of disrete-time PWA systems