1 Linear matrix inequalities for piecewise affine systems

1.1 Piecewise affine systems

A piecewise affine system is a system where the state space X is divided into non-overlapping partitions X_i with distinct models in each partition:

- Partitions containing the origin have a linear model:

$$\begin{aligned} x &= A_i x \\ x_{k+1} &= A_i x_k \end{aligned} \tag{1}$$

- Partitions not containing the origin have affine models:

$$\begin{aligned} x &= A_i x + a_i \\ x_{k+1} &= A_i x_k + a_i \end{aligned} \tag{2}$$

The partitions are indexed, with an index set

$$I = I_0 \cup I_1 \tag{3}$$

where I_0 are the paritiitions containing the origin, and I_1 are the partitions not containing the origin.

1.2 Lyapunov stability

Sometimes you can find a Lyapunov function for the whole PWA system:

- If $a_i = 0 \forall i$ and $\exists P = P^T > 0$ such that $A_i^T P + PA_i < 0 \forall i \in I$, then the origin is exponentially stable.
- if $a_i = 0 \forall i$ and $\exists R_i > 0 \forall i \in I$ such that $\sum_{i \in I}(A_i^T R_i + R_i A_i) > 0$, then a common Lyapunov function can be found.

When a common function cannot be found, we must look for one that depends on the partition.

1.2.1 Notation

$$\begin{aligned} \bar{A}_i &= \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \\ \bar{E}_i &= \begin{bmatrix} E_i & e_i \end{bmatrix} \\ \bar{F}_i &= \begin{bmatrix} F_i & f_i \end{bmatrix} \end{aligned} \tag{4}$$

where

$$e_i = 0 \text{and} f_i = 0 \forall i \in I_0 \tag{5}$$

$$\bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \forall x \in X_i \forall i \in I \tag{6}$$

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} \forall x \in X_i \cap X_j, \forall i, j \in I \tag{7}$$

1.2.2 Lyapunov function

$$V(x) = \begin{cases} x^T P_i x & x \in X_i, i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} & x \in X_i, i \in I_1 \end{cases} \tag{8}$$

where

$$P_i = F_i^T T F_i, \bar{P}_i = \bar{F}_i^T T \bar{F}_i \tag{9}$$