Summary of TTK4215: System Identification and Adaptive Control

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Contents

1 Todo

- Canoncal forms
- Unbounded input (necessary shit for proofs or whatever)
- PE (p177)
- APPC (brief how-to)
- MRAC (brief how-to)
- Richness of input and number of parameters.

2 Preliminaries

2.1 Norms

We say $x \in \mathcal{L}_p$ when $||x||_p$ exists.

General p-norm

$$||x||_p = \left(\int_0^\infty |x(t)|^p \, \mathrm{d}t\right)^{1/p} \tag{1}$$

$\mathcal{L}_\infty$-norm

$$||x||_\infty = \sup_{t \geq 0} |x(t)| \tag{2}$$

2.2 Models for dynamic systems

$$\dot{\mathbf{x}} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u} \tag{3}$$

$$\mathbf{y} = \boldsymbol{C}^{\mathrm{T}}\mathbf{x} \tag{4}$$

Controllability

$$\boldsymbol{P}_c \triangleq \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{AB} \\ \vdots \\ \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \tag{5}$$

If $\boldsymbol{P}_c$ is nonsingular, the system is controllable, and can be transformed to the *controllability canonical form* by

$$\mathbf{x}_c = \boldsymbol{P}_c^{-1}\mathbf{x} \tag{6}$$

Properness A transfer function $\boldsymbol{G}(s) = \frac{\boldsymbol{N}(s)}{\boldsymbol{D}(s)}$ is

- *proper* if $\deg(\boldsymbol{N}) \leq \deg(\boldsymbol{D})$,
- *biproper* if $\deg(\boldsymbol{N}) = \deg(\boldsymbol{D})$,
- *stricly proper* if $\deg(\boldsymbol{N}) < \deg(\boldsymbol{D})$.

2.3 Transfer function properties

Consider the polynomial

$$X(s) = \alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0 \tag{7}$$

and the transfer function

$$G(s) = \frac{Z(s)}{R(s)}. \tag{8}$$

Monic: $X(s)$ is *monic* iff $\alpha_n = 1$.

Hurwitz: $X(s)$ is *Hurwitz* if all roots of $X(s) = 0$ are in the left half plane.

Minimum phase: A system defined by the t.f. $G(s)$ is *minimum phase* iff $Z(s)$ is Hurwitz.

Stability: A system defined by the t.f. $G(s)$ is *stable* if $R(s)$ is Hurwitz.

Coprime: Two polynomials are *coprime* if they have no common factors other than a constant.

2.4 (Strictly) positive real transfer functions

2.5 Positive Real (PR)

Definition 3.5.1 A rational function $G(s)$ of the complex variable $s = \sigma + j\omega$ is called PR if

- $G(s)$ is real for real s
- $\Re[G(s)] \geq 0 \quad \forall \quad \sigma > 0$

Lemma 3.5.1 A rational proper transfer function $G(s)$ is PR iff

- $G(s)$ is real for real s
- $G(s)$ is analytic in $\Re[s] \geq 0$, and the poles on the $j\omega$-axis are simple and such that the associated residues are real and positive.
- For all real value ω for which $s = j\omega$ is not a pole of $G(s)$, one has $\Re[G(j\omega)]$

2.6 Strict Positive Real (SPR)

Definition 3.5.2 Assume $G(s)$ is not identically zero for all s. Then $G(s)$ is SPR if $G(s - \epsilon)$ is PR for some $\epsilon > 0$.

Theorem 3.5.2 (Necessary and sufficient conditions.) Assume that a rational function $G(s)$ of the complex variable $s = \sigma + j\omega$ is real for real s and is not identically zero for all s. Let n^* be the relative degree of $G(s) = Z(s)/R(s)$ with $|n^*| \leq 1$. Then, $G(s)$ is SPR iff

- $G(s)$ is analytic for $\sigma \geq 0$
- $\Re[G(j\omega)] > 0 \quad \forall \quad \omega$
- When $n^* = 1, \quad \lim_{|\omega| \to \infty} \omega \Re[G(j\omega)] > 0$
- When $n^* = -1, \quad \lim_{|\omega| \to \infty} \frac{G(j\omega)}{j\omega} > 0$

Corollary 3.5.1

- $G(s)$ is PR/SPR iff $1/G(s)$ is PR/SPR
- If $G(s)$ is SPR, then, $|n^*| \leq 1$, and the zeros and poles of $G(s)$ lie in $\Re[s] < 0$.
- When $n^* = 1, \quad \lim_{|\omega| \to \infty} \omega \Re[G(j\omega)] > 0$
- When $n^* = -1, \quad \lim_{|\omega| \to \infty} \frac{G(j\omega)}{j\omega} > 0$

KYP Lemma (3.5.2) Given a square matrix A with eigenvalues $\Re(\lambda) \leq 0$, a vector B such that (A, B) controllable, a vector C, and scalar $d \geq 0$, then the t.f.

$$G(s) = d + C^{\mathrm{T}}(s\mathrm{I} - A)^{-1}B \tag{9}$$

is PR iff $\exists$ a symmetric pos. def. matrix P and a vector q such that

$$A^{\mathrm{T}}P + PA = -qq^{\mathrm{T}} \tag{10}$$

$$PB - C = \pm\sqrt{2d} \cdot q. \tag{11}$$

LKY Lemma (3.5.3) Given a stable matrix A, a vector B such that (A, B) controllable, a vector C and a scalar $d \geq 0$, then the t.f.

$$G(s) = d + C^{\mathrm{T}}(s\mathrm{I} - A)^{-1}B \tag{12}$$

is SPR iff for any pos. def. matrix L, $\exists$ a symmetric pos. def. matrix P, a scalar $\nu > 0$ and a vector q such that

$$A^{\mathrm{T}}P + PA = -qq^{\mathrm{T}} - \nu L \tag{13}$$

$$PB - C = \pm q\sqrt{2d}. \tag{14}$$

MKY Lemma (3.5.4) Given a stable matrix A, vectors B, C, and a scalar $d \geq 0$, we have: If

$$G(s) = d + C^{\mathrm{T}}(s\mathrm{I} - A)^{-1}B \tag{15}$$

is SPR, then for any $L = L^{\mathrm{T}} > 0$, $\exists$ a scalar $\nu > 0$, a vector q and a $P = P^{\mathrm{T}} > 0$ such that

$$A^{\mathrm{T}}P + PA = -qq^{\mathrm{T}} - \nu L \tag{16}$$

$$PB - C = \pm q\sqrt{2d}. \tag{17}$$

3 Parametric models

3.1 Linear

$$z = {\theta^*}^{\mathrm{T}}\phi \tag{18}$$

$$y = {\theta_\lambda^*}^{\mathrm{T}}\phi \tag{19}$$

3.2 Bilinear

$$y = k_0({\theta^*}^{\mathrm{T}}\phi + z_0) \tag{20}$$

4 Parameter estimation

4.1 SPR Lyapunov method

Based on choosing an adaptive law so that a *Lyapunov-like* function guarantees $\tilde{\theta} \to 0$. The parametric model $z = W(s){\theta^*}^{\mathrm{T}}\psi$ is rewritten $z = W(s)L(s){\theta^*}^{\mathrm{T}}\phi$, with $L(s)$ a proper stable t.f., and $W(s)L(s)$ a proper SPR t.f.

$$z = W(s)L(s){\theta^*}^{\mathrm{T}}\phi \tag{21}$$

$$\hat{z} = W(s)L(s)\theta^{\mathrm{T}}\phi \tag{22}$$

$$\epsilon = z - \hat{z} - W(s)L(s)\epsilon n_s^2 \tag{23}$$

$$\dot{\theta} = \Gamma\epsilon\phi \tag{24}$$

4.2 Gradient method

$$z = {\theta^*}^{\mathrm{T}}\phi \tag{25}$$

$$\hat{z} = \theta^{\mathrm{T}}\phi \tag{26}$$

$$\epsilon = \frac{z - \hat{z}}{m^2} \tag{27}$$

4.2.1 Instantaneous cost

$$\dot{\theta} = \Gamma\epsilon\phi \tag{28}$$

4.2.2 Integral cost

$$\dot{\theta} = -\Gamma(R\theta + Q) \tag{29}$$

$$\dot{R} = -\beta R + \frac{\phi\phi^{\mathrm{T}}}{m^2} \tag{30}$$

$$\dot{Q} = -\beta Q - \frac{z\phi}{m^2} \tag{31}$$

4.3 With projection

$$\dot{\theta} = \begin{cases} \Gamma\epsilon\phi & \text{if } \theta \in \mathcal{S}^0 \\ \Gamma\epsilon\phi - \Gamma\frac{\nabla g \nabla g^{\mathrm{T}}}{\nabla g^{\mathrm{T}}\Gamma\nabla g}\Gamma\epsilon\phi & \text{otherwise} \end{cases} \tag{32}$$

4.4 Least squares

$$z = {\theta^*}^{\mathrm{T}} \phi \tag{33}$$
$$\hat{z} = \theta^{\mathrm{T}} \phi \tag{34}$$
$$\epsilon = \frac{z - \hat{z}}{m^2} \tag{35}$$

4.4.1 Pure least squares

$$\dot{\theta} = P \epsilon \phi \tag{36}$$
$$\dot{P} = -P \frac{\phi \phi^{\mathrm{T}}}{m^2} P \tag{37}$$

4.4.2 With covariance resetting

$$\dot{\theta} = P \epsilon \phi \tag{38}$$
$$\dot{P} = -P \frac{\phi \phi^{\mathrm{T}}}{m^2} P, \quad P(t_r^+) = P_0 = \rho_0 I \tag{39}$$

4.4.3 With forgetting

$$\dot{\theta} = P \epsilon \phi \tag{40}$$
$$\dot{P} = \begin{cases} \beta P - P \frac{\phi \phi^{\mathrm{T}}}{m^2} P & \text{if } ||P(t)|| \leq R_0 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

5 Model reference adaptive control (MRAC)

MRAC requires a plant and a reference model. A controller is made so that the controller and plant together behave similar to the reference model. An adaptive algorithm estimates the controller parameters θ. There are two main categories:

- *Direct*, where θ is equal to the controller gains.
- *Indirect*, where the controller gains are a function of θ.

Huge drawback: Requires plant of minimum phase. Also requires known relative degree and bounded plant order.

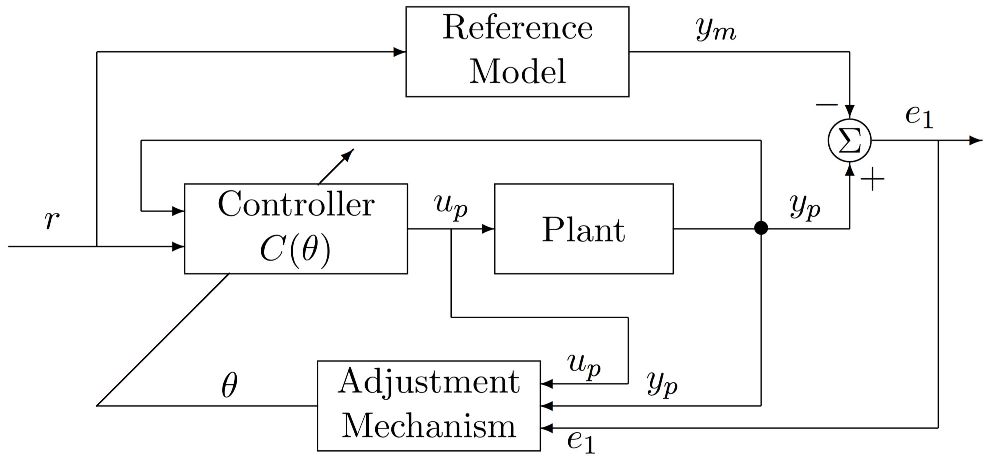

Figure 1: MRAC structure

5.1 How to... MRAC

Given a plant $y(s) = \frac{b}{s+a}u(s)$ and a reference model $y_m(s) = \frac{b_m}{s+a_m}r$ where $r \in L_\infty$. Then the optimal ideal controller has the structure $u^* = -\theta_1^* y + \theta_2^* r$, where θ_1^* and θ_2^* is optimal controller parameters. The optimal parameters can be found by comparison between y and y_m:

$$\begin{aligned} y &= \frac{b}{s+a}(-\theta_1^* y + \theta_2^* r) \\ y(1 + \frac{b\theta_1^*}{s+a}) &= \frac{b\theta_2^*}{s+a} r \\ y &= \frac{b\theta_2^*}{s+a+b\theta_1^*} r \end{aligned} \tag{42}$$

which gives us $\theta_1^* = \frac{a_m - a}{b}$ and $\theta_2^* = \frac{b_m}{b}$. The closed loop differential equations is

$$\begin{aligned} \dot{y} &= -ay + bu \\ &= -ay + b(-\theta_1 y + \theta_2 r) \\ &= (-a - b\theta_1)y + b\theta_2 r \\ \dot{y_m} &= -a_m y_m + b_m r \end{aligned} \tag{43}$$

We now define $e \triangleq y - y_m$, $\tilde{\theta_1} \triangleq \theta_1 - \theta_1^*$, $\tilde{\theta_2} \triangleq \theta_2 - \theta_2^*$ and derive the error dynamics.

$$
\begin{aligned}
\dot{e} &= \dot{y} - \dot{y_m} \\
&= (-a - b\theta_1)y + b\theta_2 r - (-a_m y_m + b_m r) \\
&= -a_m y + a_m y_m + (-a + a_m - b\theta_1)y + b\theta_2 r - b_m r \\
&= -a_m e + (-a + a_m - b(\tilde{\theta}_1 + \frac{a_m - a}{b}))y + b(\tilde{\theta}_2 + \frac{b_m}{b})r - b_m r \\
&= -a_m e - \tilde{\theta}_1 by + \tilde{\theta}_2 br
\end{aligned} \tag{44}
$$

We now use a Lyapunov (ish?) function $V = \frac{1}{2}e^2 + \frac{b}{2\gamma_1}\tilde{\theta}_1^2 + \frac{b}{2\gamma_2}\tilde{\theta}_2^2$.

$$
\begin{aligned}
\dot{V} &= e\dot{e} + \frac{b}{\gamma_1}\tilde{\theta}_1\dot{\tilde{\theta}}_1 + \frac{b}{\gamma_2}\tilde{\theta}_2\dot{\tilde{\theta}}_2 \\
&= e(-a_m e - \tilde{\theta}_1 by + \tilde{\theta}_2 br) + \frac{b}{\gamma_1}\tilde{\theta}_1\dot{\theta}_1 + \frac{b}{\gamma_2}\tilde{\theta}_2\dot{\theta}_2 \\
&= -a_m e^2 + \frac{b}{\gamma_1}\tilde{\theta}_1(\dot{\theta}_1 - \gamma_1 ye) + \frac{b}{\gamma_2}\tilde{\theta}_2(\dot{\theta}_2 + \gamma_2 re)
\end{aligned} \tag{45}
$$

The update laws are selected such that the derivative Lyapunov function are negative semi-definite.

$$
\begin{aligned}
(\dot{\theta}_1 - \gamma_1 ye) = 0 \Rightarrow \dot{\theta}_1 = \gamma_1 ye \\
(\dot{\theta}_2 + \gamma_2 re) = 0 \Rightarrow \dot{\theta}_2 = \gamma_2 re
\end{aligned} \tag{46}
$$

It can now be shown that $e, \tilde{\theta}_1, \tilde{\theta}_2, r, y_m, y, \dot{e} \in L_\infty$ and $e \in L_2$, which leads to $\lim_{t\to\infty} e(t) \to 0$ from Lemma 3.2.5.

6 Adaptive pole placement control (APPC)

APPC is pretty cool, because it does not require a plant of minimum phase (as opposed to MRAC).

6.1 Indirect PPC

Given the system

$$
y_p = \frac{Z_p(s)}{R_p(s)} u_p \tag{47}
$$

where

- R_p is monic, of known degree n,
- Z_p, R_p are coprime,
- G_p strictly proper ($\deg Z_p < n$).

The objective is to choose u_p so that the closed-loop poles are the roots of a polynomial $A^*(s)$. Can be done by choosing P (degree $q+n-1$) and L (monic, degree $n-1$) such that

$$LQ_mR_p + PZ_p = A^* \tag{48}$$

is satisfied for A^* of degree $2n+q-1$. Q_m is chosen so that

$$Q_m y_p = 0, \tag{49}$$

and the control input is given by

$$Q_m L u_p = -P(y_p - y_m). \tag{50}$$

7 Robustness

7.1 Parameter drift

An unknown bounded disturbance added to the measurement y can lead to *parameter drift*, meaning $\theta \to \infty$ as $t \to \infty$. The equilibrium $\tilde{\theta}_e = 0$ can be made u.a.s. by making $u(t)$ PE. (But we don't always decide $u(t)$.)

8 Extremum seeking

The basic idea: Want to find the optimal plant input θ^* that maximises the output y. Add a slow periodic perturbation to our estimate $\hat{\theta}$. If increasing θ increases y, then y will oscillate in phase with θ. In the opposite case, y with be out of phase with θ. The DC component of y is removed with a high-pass filter, and the result multiplied with the perturbation signal. The product will have a positive DC component if y and θ are in phase, and negative DC if they are out of phase. This DC component is extracted with a low-pass filter, and is then a value indicating how far off θ is, and in what direction. This is integrated and multiplied with a gain k to form the estimate $\hat{\theta}$. Pretty clever.

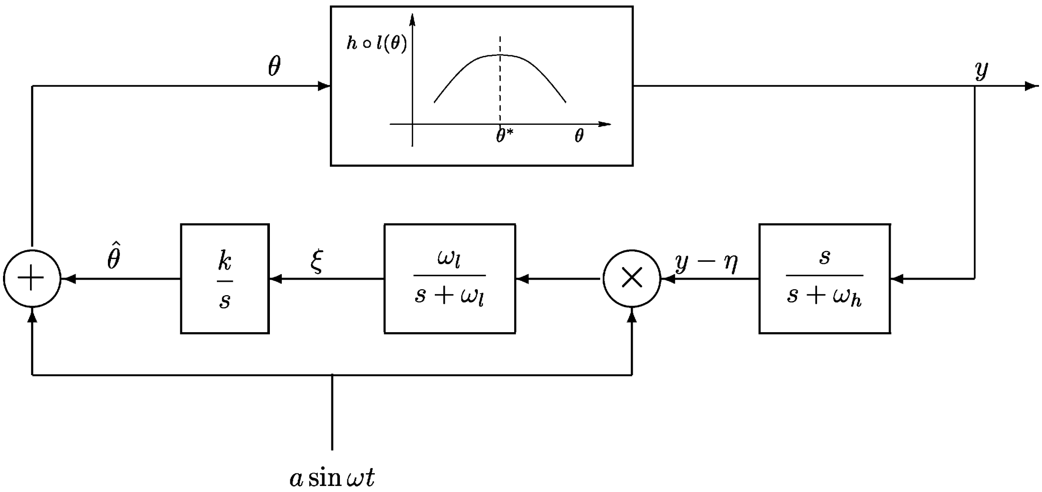

Figure 2: Block diagram for extremum seeking