Summary of TTK4115

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1 Matrix stuff

Nullity $\text{nul}(\boldsymbol{A})$ = No. of columns of $\boldsymbol{A} - \text{rank}(\boldsymbol{A})$

Positive definite A symmetric $n \times n$ real matrix $\boldsymbol{M}$ is *positive definite* if all its eigenvalues are positive. (Or if $\mathbf{z}^{\mathrm{T}}\boldsymbol{M}\mathbf{z} > 0$ for every non-zero vector $\mathbf{z}$ of n real numbers.) It is positive *semidefinite* if all eigenvalues are positive or zero.

Singularity A square matrix is singular if it is not invertible, i.e. if its determinant is 0.

Matrix exponential (diagonal)

$$\boldsymbol{A} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix} \quad \Longrightarrow \quad e^{\boldsymbol{A}t} = \begin{bmatrix} e^{a_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{a_n t} \end{bmatrix}$$

Matrix exponential (Cayley-Hamilton Method)

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{n-1} \alpha_k \boldsymbol{A}^k \quad \text{with } \alpha_0 \cdots \alpha_{n-1} \text{ determined by} \quad e^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k$$

Matrix exponential (Laplace method)

$$e^{\boldsymbol{A}t} = \mathscr{L}\left\{(s\mathbf{I} - \boldsymbol{A})^{-1}\right\}$$

Matrix exponential (Jordan form)

$$e^{\boldsymbol{A}t} = \boldsymbol{Q} e^{\bar{\boldsymbol{A}}t} \boldsymbol{Q}^{-1} \quad \text{where} \quad \bar{\boldsymbol{A}} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$$

Matrix exponential (series expansion)

$$e^{\boldsymbol{A}} = \mathbf{I} + \boldsymbol{A} + \frac{1}{2}\boldsymbol{A}^2 + \frac{1}{6}\boldsymbol{A}^3 + \cdots + \frac{1}{(q-1)!}\boldsymbol{A}^{q-i} \quad \text{where} \quad \boldsymbol{A}^q = \mathbf{0}$$

Controllability matrix

$$C = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Observability matrix

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Minimal realisation Given a transfer function; a state-space model that is controllable and observable, and has the same input-output behaviour as the function, is minimal.

2 Eigen stuff

Eigenvalues Values of λ such that $\Delta(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$.

Eigenvectors Vectors $\mathbf{v}$ such that $(\boldsymbol{A} - \lambda\mathbf{I})\mathbf{v} = 0$.

3 Stability

Asymptotic stability Occurs if all poles have strictly negative real parts.

Instability Occurs if one or more poles have positive real parts.

Marginal stability Occurs when the real part of every pole is non-positive, at least one pole has zero real value, and there are no repeated poles on the imaginary axis.

BIBO stability If bounded input $\rightarrow$ bounded output. Defined for the zero-state response (initially relaxed system). See Section 4.

Lyapunov stability If every finite initial state gives a finite response. I.e. the zero-input response.

4 BIBO Stability

4.1 BIBO Stability for Continuous Systems

A continuous system is BIBO stable *iff*:

- (SISO) $g(t)$ is absolutely integrable in $[0, \infty)$ or $\int_0^\infty |g(t)|\,\mathrm{d}t \leq M < \infty$ for some constant M.
- (SISO/MIMO) Every pole of every transfer function in $\hat{G}(s)$ or $\hat{g}(s)$ has a negative real part.

4.2 BIBO Stability for Discrete Systems

A discrete system is BIBO stable iff:

- Every pole of every transfer function in $\hat{G}(s)$ or $\hat{g}(s)$ has magnitude less than 1.

4.3 Lyapunov Stability for Linear Systems

An LTI system $\mathbf{x} = \boldsymbol{A}\mathbf{x}$ is stable if there exists a *symmetric* positive definite matrix $\boldsymbol{P}$ that satisfies the Lyapunov Equation

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{M}\boldsymbol{P} = -\boldsymbol{N}$$

Where $\boldsymbol{N}$ is an arbitrary positive definite matrix.

5 Discretisation

$$\boldsymbol{A}_d = e^{\boldsymbol{A}T}, \qquad \boldsymbol{B}_d = \int_0^T e^{\boldsymbol{A}\tau}\,\mathrm{d}\tau\boldsymbol{B}, \qquad \boldsymbol{C}_d = \boldsymbol{C}, \qquad \boldsymbol{D}_d = \boldsymbol{D}$$

6 Similarity transform

A linear change of coordinates where the original object is expressed with respect to a different basis. The representation of $\mathbf{x}$ with respect to the basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ is $\bar{\mathbf{x}}$ and with $\boldsymbol{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$, the similarity transform is

$$\mathbf{x} = \boldsymbol{Q}\bar{\mathbf{x}}.$$

The system originally expressed as

$$\mathbf{x} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u}$$

$$\mathbf{y} = \boldsymbol{C}\mathbf{x} + \boldsymbol{D}\mathbf{u}$$

is transformed to

$$\bar{\mathbf{x}} = \bar{\boldsymbol{A}}\bar{\mathbf{x}} + \bar{\boldsymbol{B}}\mathbf{u}$$

$$\mathbf{y} = \bar{\boldsymbol{C}}\bar{\mathbf{x}} + \bar{\boldsymbol{D}}\mathbf{u}$$

where

$$\bar{\boldsymbol{A}} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}, \qquad \bar{\boldsymbol{B}} = \boldsymbol{Q}^{-1}\boldsymbol{B}, \qquad \bar{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{Q}, \quad \bar{\boldsymbol{D}} = \boldsymbol{D}$$

7 Jordan canonical form

General strategy:

1. Find all eigenvectors corresponding to an eigenvalue of $\boldsymbol{A}$.
2. The number of L.I. eigenvectors is the number of Jordan blocks.
3. For each eigenvector $\mathbf{q}$, solve $(\lambda\mathbf{I} - \boldsymbol{A})\mathbf{v} = \mathbf{q}$ for the vector $\mathbf{v}$.

8 Statistics

Expected value $\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}t$

Variance $\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$

Autocorrelation $R_X(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$

Wide-sense stationary process $X(t)$ is WSS if its mean and autocorrelation functions are time invariant: $\mathrm{E}[X(t)] = \nu$ and $R_X(t_1, t_2) = f(t_2 - t_1)$.

Spectral density function $S_X(\jmath\,\omega) = \mathcal{F}\left\{R_X(\tau)\right\}$

Gauss–Markov process A stationary Gaussian process $X(t)$ that has an exponential autocorrelation is called a *Gauss–Markov* process.

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$S_X(\jmath\,\omega) = \frac{2\sigma^2\beta}{\beta^2 + \omega^2} \quad \text{or} \quad S_X(s) = \frac{2\sigma^2\beta}{\beta^2 - s^2}$$

9 Linear-quadratic regulator

A system is given as

$$\mathbf{x} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u}$$

with a state feedback $\mathbf{u} = -\boldsymbol{K}\mathbf{x}$ chosen to minimise the cost function

$$J = \int_0^\infty \mathbf{x}^\mathrm{T}\boldsymbol{Q}\mathbf{x} + \mathbf{u}^\mathrm{T}\boldsymbol{R}\mathbf{u}\,\mathrm{d}t$$

where $\boldsymbol{Q}$ is symmetric and positive semidefinite, $\boldsymbol{R}$ is symmetric and positive definite, and $\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^\mathrm{T}\boldsymbol{P}$. The matrix $\boldsymbol{P}$ is found by solving

$$\boldsymbol{A}^\mathrm{T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\mathrm{T}\boldsymbol{P} + \boldsymbol{Q} = \mathbf{0}$$

The relative values of the elements of Q and R enforce tradeoffs between the magnitude of the control action and the speed of the response. The equilibrium can be shifted from $\mathbf{0}$ to $\mathbf{x}_{\mathrm{eq}}$ by instead using $\mathbf{u} = \boldsymbol{P}\mathbf{x}_{\mathrm{eq}} - \boldsymbol{K}\mathbf{x}$.

10 Kalman filter

The system is given as:

$$\mathbf{x}_{k+1} = \boldsymbol{A}\mathbf{x}_k + \boldsymbol{B}\mathbf{u}_k + \mathbf{w}_k$$

$$\mathbf{z}_k = \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k$$

The Kalman measurement update equations are:

$$\boldsymbol{K}_k = \boldsymbol{P}_k^-\boldsymbol{H}_k^\mathrm{T}(\boldsymbol{H}_k\boldsymbol{P}_k^-\boldsymbol{H}_k^\mathrm{T} + \boldsymbol{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \boldsymbol{K}_k(\mathbf{z}_k - \boldsymbol{H}_k\hat{\mathbf{x}}_k^-)$$

$$P_k = (\mathbf{I} - K_k H_k) P_k^- (\mathbf{I} - K_k H_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}}$$

And the time update equations are:

$$\hat{\mathbf{x}}_{k+1}^- = A\hat{\mathbf{x}}_k + B\mathbf{u}_k$$

$$P_{k+1}^- = A P_k A^{\mathrm{T}} + Q_k$$

11 Extended Kalman filter

The system is given as:

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k$$

The Kalman measurement update equations are:

$$K_k = P_k^- C_k^{\mathrm{T}} (C_k P_k^- C_k^{\mathrm{T}} + R)^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + K_k \left(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)\right)$$

$$P_k = (\mathbf{I} - K_k C_k) P_k^- (\mathbf{I} - K_k C_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}}$$

And the time update equations are:

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{f}(\hat{\mathbf{x}}_k, \mathbf{u}_k)$$

$$P_{k+1}^- = A_k P_k A_k^{\mathrm{T}} + Q_k$$

Where:

$$A_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k = \hat{\mathbf{x}}_k} \qquad \text{and} \qquad C_k = \left.\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{x}_k}\right|_{\mathbf{x}_k = \hat{\mathbf{x}}_k^-}$$