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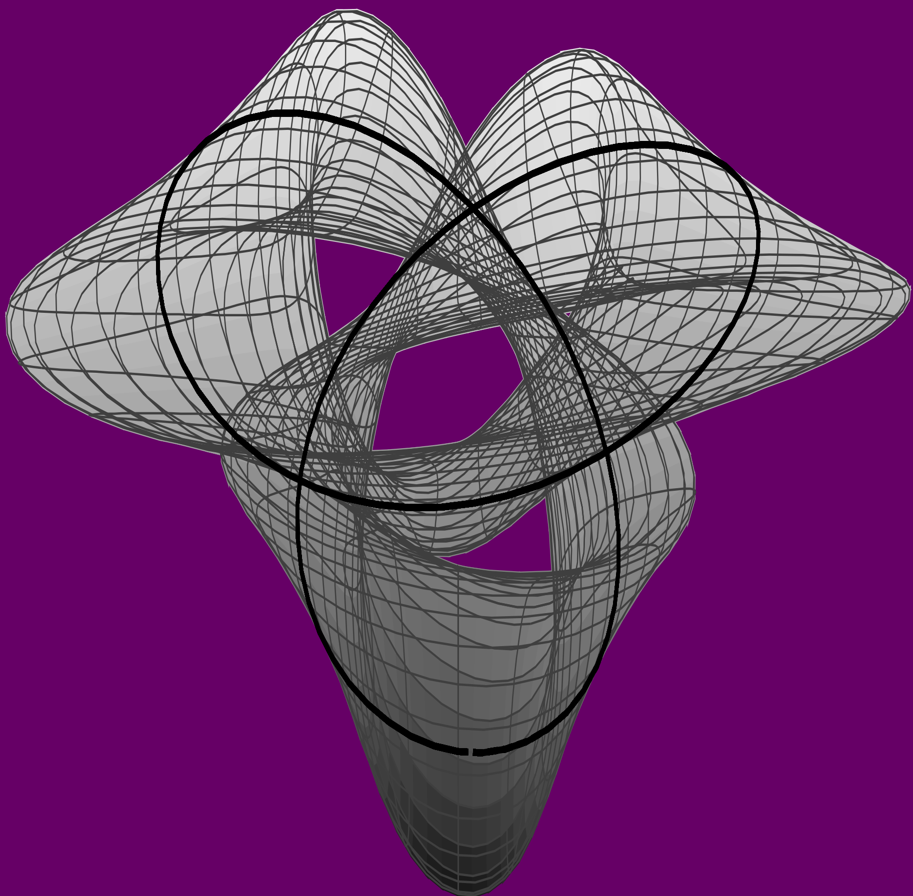
Mathematics of Curves and Surfaces



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Chapter 1

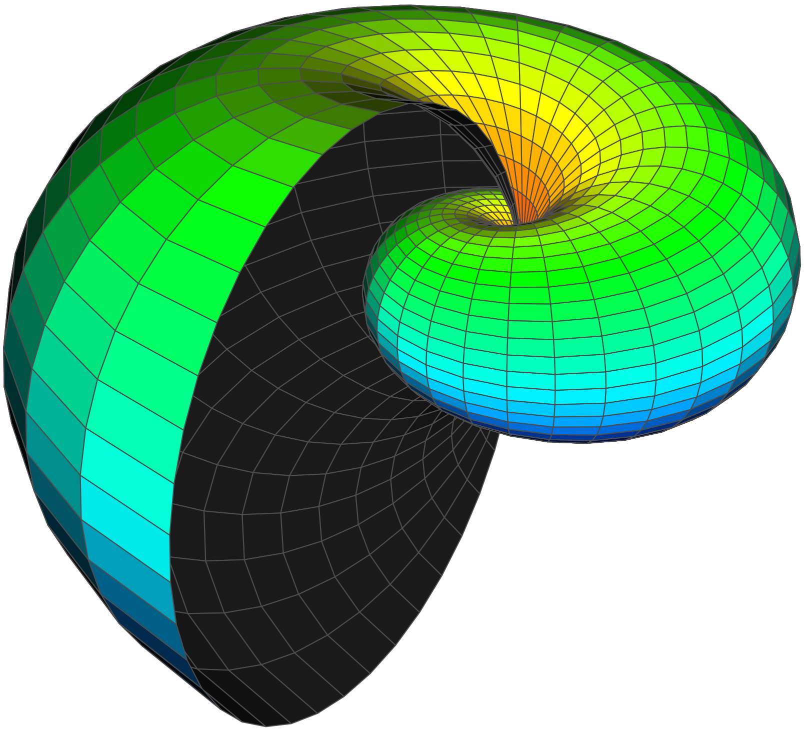
Introduction

I began writing this document as a side effect of trying to understand General Relativity, which makes heavy use of differential geometry.

Here I will show some examples of 2D surfaces embedded in 3D euclidian spaces, how to create curves on these surfaces and how to evaulate the length of these curves. The necessary mathematics is introduced but the reader is assumed to be familiar with vectors, matrices and differential calculus. Several examples are given with detailed calculations.

Chapter 2 gives some basic definitions of vectors and their properties. We then do a quick brush up on functions of one variable in 2-dimensions in chapter 3, how to find the areas and lengths of these curves, etc. Chapter 4 describes a number of different coordinate systems in 3-dimentional space and how to transform from one coordinate system to another. Following this is a short description of maps and projections in chapter 5. In chapter 6 we show how a surface can be described as a parametrization of the three dimentional coordinates $\mathbf{r}$ by two independent variables, and how curves can be described as the parametrization of $\mathbf{r}$ by one variable. In section 8 a number of detailed examples of calculations based on the metric tensor are given. A brief description of parallel transport of vectors along curves is given in chapter 9. Chapter 10 extends the previous ideas to higher dimensions.

The document is typeset with LaTeXa lot of different packages: *amsmath, graphicx, listings, amsthm, fancyhdr*,...the list goes on. The figures are created by tools with varying support for integration with LaTeX such as **gnuplot**, Ti*k*Z *pstricks* and *pst-3dplot* packages. Where numerical results are required, such as difficult integrals, or simplification of formulae **WolframAlpha.com** has been used as well as Romberg integration routines written in C and C++. To reduce the occurence of manual errors, the software package Maxima was used for symbolic manipulation of algebraic expressions. Appendix C gives details on how to use these tools in conjunction with LaTeX.



Mathematical surface of a logaritmic spiral very similar to the one created by the *Nautilus pompilius* cephalopod.

Chapter 2

Vectors

A vector is an "arrow" in space given by a starting point and and endpoint. A vector has a length and a direction. Since vectors are an important part of the geometry of surfaces, the will be introduced briefly here.

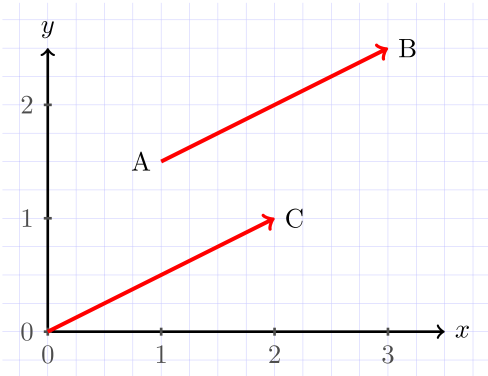


Figure 2.1: The vector from A to B is equivalent to the vector from the origin to C, since both have an x-component of 2 and a y-component of 1.

In figure 2.1 is shown three points A,B and C. A vector is drawn from A to B and another from the origin O to C. These vectors are identical:

$$\overline{AB} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} 3 - 1 \\ 2.5 - 1.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\overline{OC} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} 2 - 0 \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Two vectors can be added by adding the x, and y components, and the same holds for subtraction. A vector can be multiplied (or divided) by a factor by multiplying (or dividing) each component of the vector by that factor:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

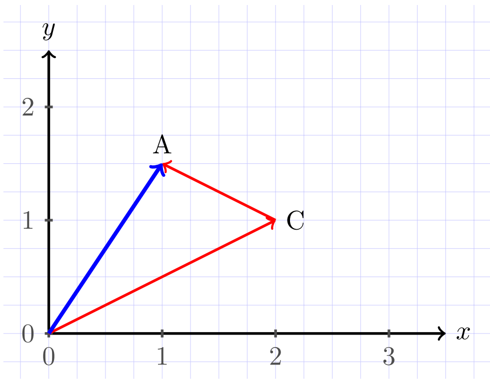


Figure 2.2: The vector OA is equal to the sum of the vectors $OC + CA$.

$$a \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ az \end{pmatrix}$$

2.1 The dot product

The dot product of two vectors $\mathbf{p}, \mathbf{q}$ is defined as

$$\mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \cdot \begin{pmatrix} q_x \\ q_y \end{pmatrix} = p_x q_x + p_y q_y$$

This is a number (scalar). The value of the dot product is related to the angle α between the two vectors in the following way.

$$\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos \alpha$$

as a consequence, if two vectors are perpendicular their dot product is zero, and the dot product of a vector by itself is $\mathbf{p} \cdot \mathbf{p} = p_x^2 + p_y^2 = |\mathbf{p}|^2$

2.2 The cross product

The cross product of two vectors produces a third vector which is perpendicular to the two vectors. If

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \text{ then } \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

Another property of the cross product is that the length of the resultant vector is equal to the area spanned by the two vectors.

2.3 Formulas for vector products

Some formulas for dot and cross products and their combinations

$$\begin{aligned}
\mathbf{p} \times \mathbf{q} &= -(\mathbf{q} \times \mathbf{p}) \\
\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) &= (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{r} \\
(\mathbf{p} \times \mathbf{q}) \cdot (\mathbf{p} \times \mathbf{q}) &= (\mathbf{p} \cdot \mathbf{p})(\mathbf{q} \cdot \mathbf{q}) - (\mathbf{p} \cdot \mathbf{q})^2 \\
\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) &= \mathbf{q}(\mathbf{p} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{p} \cdot \mathbf{q})
\end{aligned}$$

Example 2.1. What are the dot products of combinations of these vectors?

$$\mathbf{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Answer:

$$\mathbf{a} \cdot \mathbf{b} = (-1) \cdot 2 + 4 \cdot (1/2) = 0$$

$$\mathbf{a} \cdot \mathbf{c} = (-1) \cdot 1 + 4 \cdot (-1) = -5$$

$$\mathbf{b} \cdot \mathbf{c} = 2 \cdot 1 + (1/2) \cdot (-1) = 1.5$$

so only the pair **a** and **b** are orthogonal.

Example 2.2. What are the length of the vectors **a**, **b**, and **c** in the previous example?

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{a} = \quad |\mathbf{a}|^2 &= \quad (-1) \cdot (-1) + 4 \cdot 4 = 17 \\
\mathbf{b} \cdot \mathbf{b} = \quad |\mathbf{b}|^2 &= \quad 2 \cdot 2 + (1/2) \cdot (1/2) = 4.25 \\
\mathbf{c} \cdot \mathbf{c} = \quad |\mathbf{c}|^2 &= \quad 1 \cdot 1 + (-1) \cdot (-1) = 2
\end{aligned}$$

so the lengths of **a**, **b**, and **c** are $\sqrt{17}$, $\sqrt{4.25}$ and $\sqrt{2}$.

Example 2.3. If the two vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{then} \qquad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Chapter 3

2D Curves

In this chapter we will start by showing the familiar functions of one variable $y = f(x)$. We will quickly find out that these are somewhat limited in the variety of shapes they can express, which leads onwards to the more general parametric 2D curves.

3.1 Functions of one variable

A function of one variable, $y = f(x)$ simply connects pairs of numbers: For each x there is a corresponding y. Functions of one variable are suitable for representing simple relations where the x values are increasing continuously (time, days, months, ...). This means that we cannot use this method to draw geometry such as circles, ellipses and other closed or self intersecting shapes shown in figures 3.3 and 3.4.

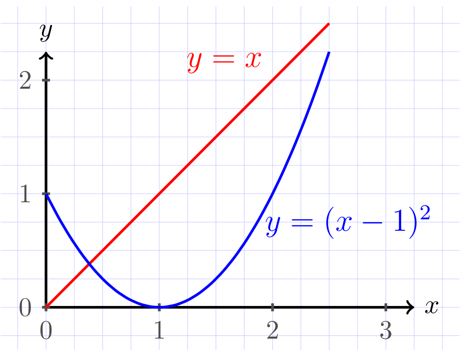


Figure 3.1: Two simple functions of the single variable x.

The length of a curve specified as y=f(x) is

$$\int_a^b \sqrt{(dx)^2 + (\frac{\partial y}{\partial x}dx)^2} = \int_a^b \sqrt{1 + (y')^2}dx$$

The formula can be understood in terms of Pythagoras formula for small changes in x and y: If we increment x by dx from x_0 to x_0+dx the corresponding change in y, dy is the rate that y changes with x, $\partial y/\partial x$ evaluated at x_0, multiplied with dx. The total length is then the hypotenuse of the right angle triangle with sides dx and dy, which is $\sqrt{(dx)^2 + (dy)^2}$. This is illustrated in figure 3.2. Integrating this formula over the interval of interest gives the total length of the curve.

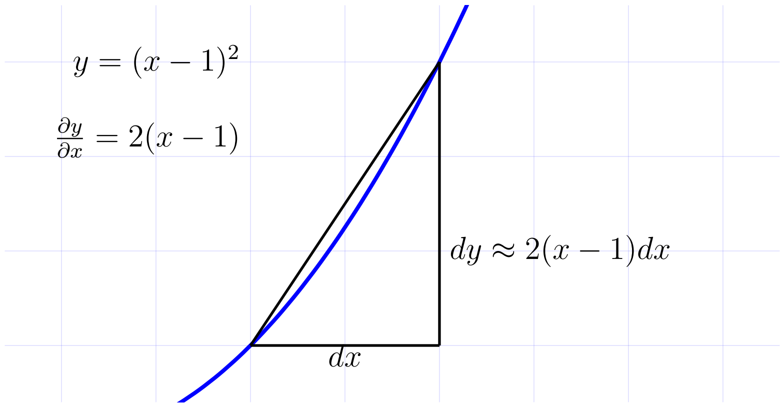


Figure 3.2: The length of a small curve segment can be found by Pythagoras' equation. The approximation becomes better as dx gets smaller.

3.2 Parametric Curves

We now introduce parametric curves. Instead of letting y vary as a function $f(x)$ we now let both x and y depend on t. This can be visualized by imagining an 'o' drawn with a pen. As the pen draws a circle (counter clockwise, starting from the top) you can imagine how the x and y coordinates vary: First x decreases while y barely changes, then both x and y are decreasing. When we reach the bottom of the 'o' x is increasing while y only moves slightly, etc. As it takes some time to draw the 'o' each point is drawn at a certain moment of time, or simply that $\mathbf{r}(t) = \Big(x(t), y(t)\Big)$. Let's give some examples.

The straight line $y = ax + b$ is parametrized as $(t, at + b)$
The curve given by $y = \sin(x)$ is parametrized by $(t, \sin(t))$
And the general case $y = f(x)$ is parametrized as $(t, f(t))$.

So far we have just covered the functions mentioned earlier, but now we can also let the x-coordinate vary. For example $(\sin(t), \cos(t))$ draws a circle with radius 1 and center (0,0). For further examples see figures 3.3 and 3.4.

The length of curves specified as (x,y)=(f(t),g(t)) is given in a similar way as for the simple curves mentioned in section 3.1 as we again use Pythagoras. Using the relation

$$\Big(\frac{\partial x}{\partial t}dt\Big)^2 + \Big(\frac{\partial y}{\partial t}dt\Big)^2 = \Big(\dot{f}(t)dt\Big)^2 + \Big(\dot{g}(t)dt\Big)^2$$

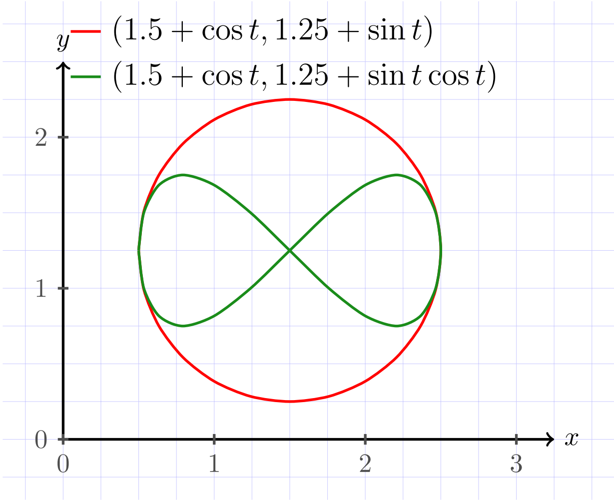


Figure 3.3: Circle with radius 1 and center $(1.5, 1.25)$ (red) and a self intersecting curve (green).

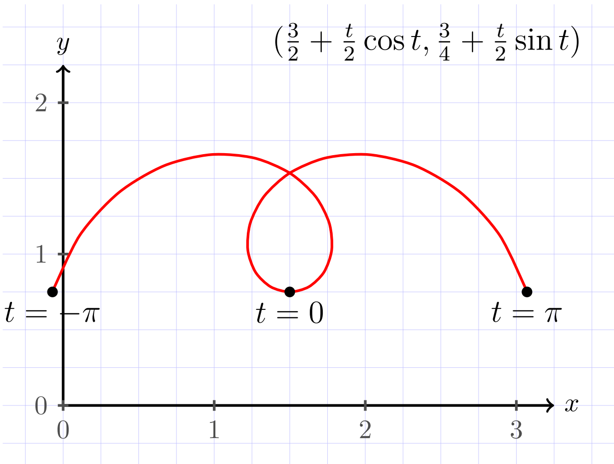


Figure 3.4: Self intersecting curve showing the points created at $t = -\pi, 0$ and π respectively. $t \in [-\pi; \pi]$

we get for the length of the curve

$$\mathcal{L} = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{(\dot{f})^2 + (\dot{g})^2} dt \tag{3.1}$$

Finally we should mention the parameter t. If no limits are specified then we assume that t can take the values from $-\infty < t < \infty$. Sometimes, as in the case of the circle this would create endless loops around the same curve. In such cases an interval can be specified as $t \in [0; 2\pi]$, meaning that t runs from 0 to 2π.

Much more can be said about parametric curves: The speed of traversal, slopes, asymptotic behavior, turning points, intersection with axes, ...

3.3 Tangents

We have seen that a function has associated with it a y-value for every x-value. But curves have other properties. For example the slope varies with x and this slope is called the tangent of the curve, and we can calculate the slope when we know the function. For functions of the type $y = f(x)$, the slope is defined naturally as the small changes in the y-direction caused by a small change in x. This is called the derivative of the function. There are several ways of denoting the derivative of a function $y = f(x)$:

$$slope = \frac{\partial f(x)}{\partial x} = f'(x) = f' = y' = \frac{\partial y}{\partial x}$$

and if the variable is measuring time, t as in $y = g(t)$ the derivatime is sometimes denoted $\dot{g}(t)$ or $\dot{g}$.

When the function is parametrized $(x, y) = (f(t), g(t))$, we differentiate each function and obtain a tangent vector.

$$\mathbf{v}_t = \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix}$$

3.4 Curvature

The formula for the curvature of a curve of the form $y = f(x)$ is

$$\kappa = \frac{1}{R^2} = \frac{|y''|}{(1 + y'^2)^{3/2}} \tag{3.2}$$

If the curve is parametrized as $(x, y) = (f(t), g(t))$ then the formula for the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} \tag{3.3}$$

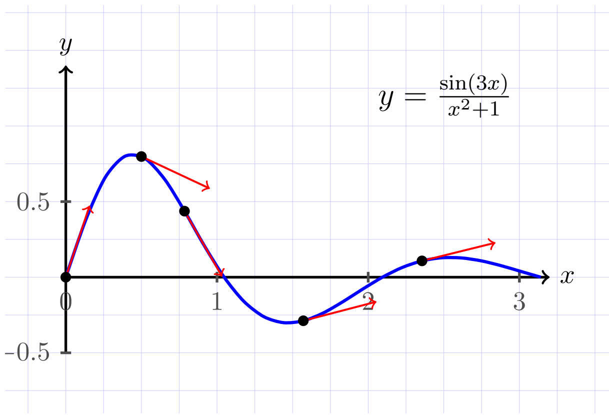


Figure 3.5: A curve and its tangent vectors at selected points.

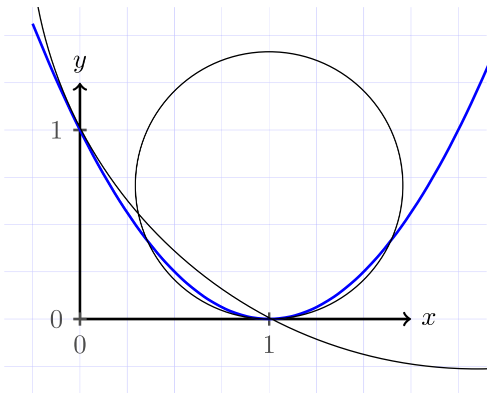


Figure 3.6: Curvature of the function $y = (x-1)^2$ at $x = 0$ and $x = 1$ respectively. The radius of curvature is indicated.

Example 3.1. In figure 3.6, $y = (x-1)^2$, $y' = 2x - 2$ and $y'' = 2$. Using equation 3.2 we obtain curvature of y as a function of x.

$$\kappa(x) = \frac{2}{(4x^2 - 8x + 5)^{3/2}}$$

Evaluating this at $x = 0$ and $x = 1$ we get

$$\kappa(0) = \frac{2}{5^{3/2}} \approx 0.179, \qquad \kappa(1) = 2$$

Using the relation $\kappa = 1/R^2$ we can calculate the radius of curvature at $x = 0$ and $x = 1$

$$R(0) \approx 2.364, \qquad R(1) \approx 0.707$$

These are the values used for the two circles in the figure.

3.5 Curves in higher dimensions

Some of the previous discussions of parametric functions can be generalized to higher dimensions. For example the trajectory of an object as it travels through 3-dimensional space can be represented as the position of its center of mass as a function of time

$$(x, y, z) = \big(X(t), Y(t), Z(t)\big)$$

whose tangent vector is

$$\mathbf{v}_t = \big(X'(t), Y'(t), Z'(t)\big)$$

or if several physical values are measured for a process as a function of time, these can be viewed as a parametric curve in n-dimenstions

$$\mathbf{p}(t) = \big(x_1(t), x_2(t), \ldots, x_n(t)\big)$$

where we have replaced the x, y, z coordinates by $x_1, \ldots, x_n$.

A straight forward extension of equation 3.1 for the length of a parametric curve in 2 dimensions gives, the length of an n-dimensional curve

$$\mathcal{L} = \int_a^b \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \ldots + \dot{x}_n^2} dt$$

Chapter 4

Coordinate systems

4.1 Basis vectors

The principal use of a coordinate system is to provide directions i space. A point is clearly located somewhere, but how to define "somewhere"?

Assume that somehow we have obtained a number of directions (one for the line, two for the plane, three for space, ... In the following we assume the dimensionality 2. We shall call these directions basis vectors $\mathbf{e}_i$. Every point in space can be described uniquely by two numbers λ_i associated with the basis vectors

$$\mathbf{r} = \lambda^1\, \mathbf{e}_1 + \lambda^2\, \mathbf{e}_2 \equiv \lambda^i\, \mathbf{e}_i$$

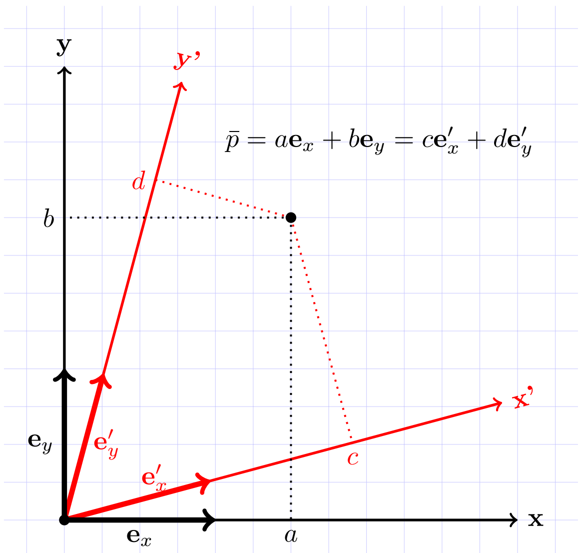


Figure 4.1: The same point will have different coordinates when the basis vectors are different.

where the summation over all values of i is implicitly assumed.

The basis vectors can be calculated as $\mathbf{e}_i = \partial\mathbf{r}/\partial u_i$.

4.2 The metric tensor

In order to calculate lengths of curves, areas, angles and curvatures, a mathematical tool has shown to be handy. It is called the metric tensor. The metric tensor can be derived as the dot product of all possible combinations of the basis vectors. This is written as $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

Don't worry about the word tensor - so far we can just regard it as a matrix, shown here for dimension 3.

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

We here give some examples from two dimensional space on how the metric tensor is invovled in evaluating different geometrical properties.

Lengths

Let's say we have a point p given as a linear combination of the two basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. $p = a\mathbf{e}_1 + b\mathbf{e}_2$

The dot product of p with itself is the square of the length:

$$|p|^2 = p \cdot p = a^2\mathbf{e}_1 \cdot \mathbf{e}_1 + 2ab\ \mathbf{e}_1 \cdot \mathbf{e}_2 + b^2\mathbf{e}_2 \cdot \mathbf{e}_2 = a^2 g_{11} + 2ab\ g_{12} + b^2 g_{22}$$

Angles

For two vectors p, q defined as $p = a\mathbf{e}_1 + b\mathbf{e}_2$, $q = c\mathbf{e}_1 + d\mathbf{e}_2$

The dot product of p and q is related to the angle α between them as

$$p \cdot q = |p||q|\cos(\alpha)$$

working out the dot product from the definitions of p and q gives

$$|p||q|\cos(\alpha) = ac\ \mathbf{e}_1 \cdot \mathbf{e}_1 + (ad + bc)\mathbf{e}_1 \cdot \mathbf{e}_2 + bd\ \mathbf{e}_2 \cdot \mathbf{e}_2 = A\ g_{11} + B\ g_{12} + C\ g_{22}$$

Areas

For two vectors p, q defined as $p = a\mathbf{e}_1 + b\mathbf{e}_2$, $q = c\mathbf{e}_1 + d\mathbf{e}_2$

The length of the cross product is equal to the area spanned by the two vectors, or

$$A^2 = p \times q \cdot p \times q = (a\mathbf{e}_1 + b\mathbf{e}_2) \times (c\mathbf{e}_1 + d\mathbf{e}_2) \cdot (a\mathbf{e}_1 + b\mathbf{e}_2) \times (c\mathbf{e}_1 + d\mathbf{e}_2)$$

this can be simplified, as $\mathbf{a} \times \mathbf{a} = 0$, and $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, to

$$A^2 = (ad - bc)(\mathbf{e}_1 \times \mathbf{e}_2) \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$$

$$= (ad - bc)[(\mathbf{e}_1 \cdot \mathbf{e}_1)(\mathbf{e}_2 \cdot \mathbf{e}_2) - \mathbf{e}_1 \cdot \mathbf{e}_2] = (ad - bd)[g_{11}^2 g_{22}^2 - g_{12}]$$

Curvature

The curvature of a surface with a metric tensor g_{ij} is given as

$$K = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{11}\right) - \frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{12}\right)\right]$$

where

$$\Gamma^2_{1j} = \frac{1}{2}g^{k2}\left(\frac{\partial g_{1k}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^1} - \frac{\partial g_{1j}}{\partial u^k}\right)$$

and

$$g = g_{11}\; g_{22} - g_{12}^2$$

This formula was discovered by Carl Friedrich Gauss, and also has an equivalent in higher dimensions. Even if we don't want to calculate the curvature we can sometimes recognize a flat surface: If the components of the metric tensor are all constant, all derivatives are zero and therefore the curvature is zero and the surface flat.

Summary

To summarize: For the purpose of calculating all sorts of geometrical values we need to know the basis vectors, and from them the metric tensor. These considerations are also valid for higher dimensions.

4.3 Cartesian coordinates

The cartesian coordinates, named after Rene Descartes, are our well known every day coordinates x, y and z.

$$\mathbf{r} = (x, y, z)$$

The coordinates can have any value.

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < z < \infty$$

The basis vectors are calculated as

$$\mathbf{e}_1 = \partial\mathbf{r}/\partial u_1 = \partial\mathbf{r}/\partial x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2 = \partial\mathbf{r}/\partial u_2 = \partial\mathbf{r}/\partial y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_3 = \partial\mathbf{r}/\partial u_3 = \partial\mathbf{r}/\partial z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

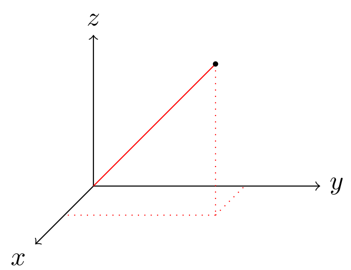


Figure 4.2: The Cartesian coordinate system. Coordinates are given by x, y, z.

and the metric tensor is

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the metric tensor is constant and that all elements outside the diagonal are zero. This means that the distance between two points differing by (dx, dy, dz) is

$$\Delta s = \sqrt{g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2} = \sqrt{dx^2 + dy^2 + dz^2}$$

4.4 Cylindrical coordinates

Cylindrical coordinates are defined as

$$\mathbf{r}(\rho, \theta, z) = (\rho \sin\theta, \rho\cos\theta, z)$$

With the following ranges for the coordinates

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad -\infty < z < \infty$$

$$\mathbf{e}_1 = \partial\mathbf{r}/\partial u_1 = \partial\mathbf{r}/\partial\rho = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2 = \partial\mathbf{r}/\partial u_2 = \partial\mathbf{r}/\partial\theta = \begin{pmatrix} \rho\cos\theta \\ -\rho\sin\theta \\ 0 \end{pmatrix}$$

$$\mathbf{e}_3 = \partial\mathbf{r}/\partial u_3 = \partial\mathbf{r}/\partial z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that

$$\Delta s^2 = g_{11}d\rho^2 + g_{22}d\theta^2 + g_{33}dz^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$$

4.5 Spherical coordinates

$$\mathbf{r} = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

Where the conventional ranges for the coordinates are

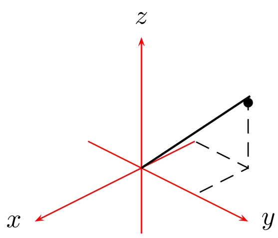
$$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$$

The basis vectors are

$$\mathbf{e}_1 = \partial\mathbf{r}/\partial r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$\mathbf{e}_2 = \partial\mathbf{r}/\partial\theta = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}$$

$$\mathbf{e}_3 = \partial\mathbf{r}/\partial\phi = \begin{pmatrix} -r\sin\theta\sin\phi \\ -r\sin\theta\cos\phi \\ 0 \end{pmatrix}$$



For this coordinate system the metric tensor can be calculated to

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\Delta s^2 = g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

4.6 Coordinate transformations

Since a point in space can be represented using any regular coordinate system we might find it useful to change coordinate system, depending on the purpose. As a natural consequence the need for transforming between coordinate systems arise.

There exist general methods to transform back and forth between different coordinate systems but before revealing these a few select cases will be shown.

Translated Cartesian

The simpelst example of a coordinate system transformation is the translated cartesian coordinate system. The relations between the two coordinate systems are simply

$$u_i = (x, y, z) = (x' + a, y' + b, z' + c)$$

and

$$u'_i = (x', y', z') = (x - a, y - b, z - c)$$

Rotated Cartesian to Cartesian

Now lets look at transformation between two cartesian coordinate systems. One is rotated α degrees relative to the other.

$$u_i = (x, y, z) = (r\cos\theta, r\sin\theta, z)$$

$$u'_i = (x', y', z') = (r'\cos\theta', r'\sin\theta', z')$$

Where $r' = r, z' = z, \theta' = \theta - \alpha$

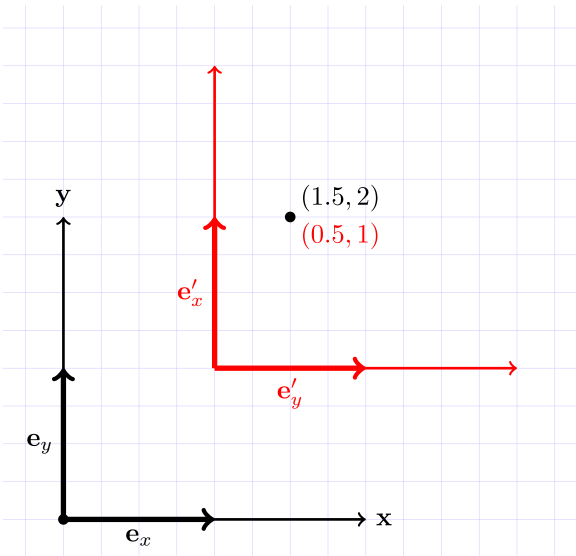
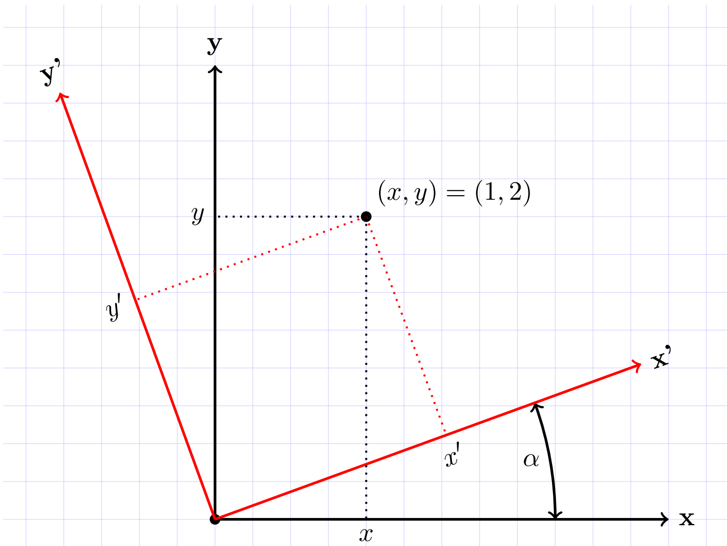


Figure 4.3: Translated (red) cartesian coordinate system



Making use of the trigonometric identities

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

We get the following transformations between the coordinate systems.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

This seems right: When α is 0 then $(x, y, z) = (x', y', z')$.

Cartesian to Spherical

For example, the connection between Cartesian and Spherical coordinates is given as

$$u_i = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

and

$$u'_i = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) \end{pmatrix}$$

The value for ϕ is only valid when $x \geq 0$, if not then the above value should be subtracted from π.

General Transformations

I'm too tired now...

Example 4.1. In the normal cartesian coordinate system $(x, y, z) = (5, 3, \pi)$, what are the coordinates in a coordinate system rotated 30° counter clockwise?

$(x', y', z') = (5\cos(30) + 3\sin(30), 3\cos(30) - 5\sin(30), \pi) \approx (5.83, 0.098, \pi)$

Chapter 5

Maps

As a short digression we will here describe how coordinate transformations have some practical applications within navigation. A position on the surface of the earth is given by a pair of coordinates called the latitude and longitude. The process of creating a correspondance between points on the (spherical) earth and on a piece of paper (a map) is called a projection.

5.1 Mercator projection

Given the latitude and longitude as ϕ and λ on the sphere the Mercator projection provides the following formula for the projection

$$\mathbf{r}' = (x', y') = \Big(\lambda - \lambda_0,\ \ln\big(\tan(\frac{\pi}{4} + \frac{\phi}{2})\big)\Big)$$

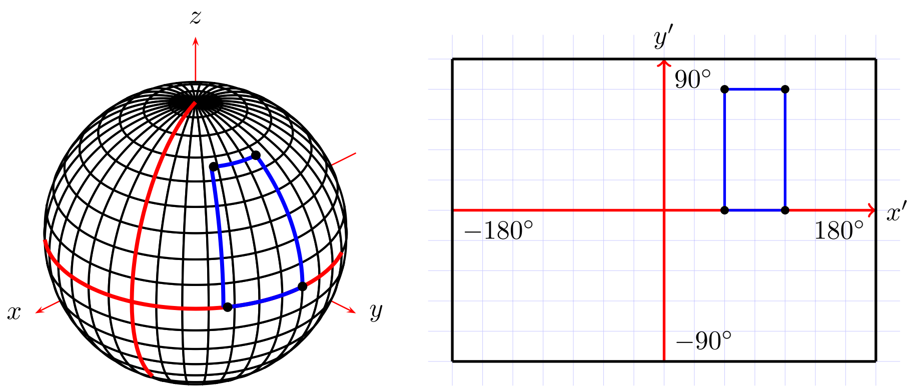


Figure 5.1: Mercator projection. Note how the upper horizontal line is stretched so that it has the same length as the lower. This is why areas are not preserved in the Mercator projection.

$\boxed{\text{Add points and definitions to maps}}$

Example 5.1. What are the coordinates in the Mercator transform of the point on equator on the Greenwich meridian $(0°, 0°)$? How about the location of Copenhagen $(12°45', 55°40')$?

Assuming that we center the map on the Greenwich meridian we set $\lambda_0 = 0$. This gives $(x', y') = (0, \ln(\tan(\pi/4))) = (0, 0)$.

For the location of Copenhagen we first represent the coordinates in decimal degrees. Noting that $60' = 1°$, gives $(12.75°, 55.66°)$, so the longitude is $12.75°$. For the latitude we get $\ln(\tan(45° + 55.66°/2)) = 1.17$. This a somewhat arbitrary scale where equator is 0 and $45°$ is at 0.88. Since the North pole will be at infinity in this projection some cutoff latitude must be employed.

Chapter 6

Surfaces

A surface can be created as a function of two variables $z = f(x, y)$ analogous to the curves described earlier. However this representation suffers from the same problems, namely that only simple geometries can be described. In stead we will move directly to the parametric description of surfaces.

A surface can be parametrized by two parameters u,v.

$$\mathbf{r} = (x, y, z) = \Big(x(u, v), y(u, v), z(u, v)\Big)$$

This representation is much more flexible: x, y and z are free to move in arbitrarily complicates ways to create closed and convoluted surfaces. One example is shown on the front page. This is a surface folded like a trefoil knot.

6.1 Sphere

In order to create a 2D surface we must place some restrictions on the three coordinates. For example by fixing r to r_0 and letting θ and ϕ run through their allowed ranges produces a sphere, and we say that the surface is parametrized by the two coordinates θ and ϕ.

$$\mathbf{r}(\theta, \phi) = r_0(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$$

$$\mathbf{e}_1 = \mathbf{e}_\theta = \partial\mathbf{r}/\partial\theta = \begin{pmatrix} \cos\theta\sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta \end{pmatrix}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \partial\mathbf{r}/\partial\phi = \begin{pmatrix} \sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ 0 \end{pmatrix}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

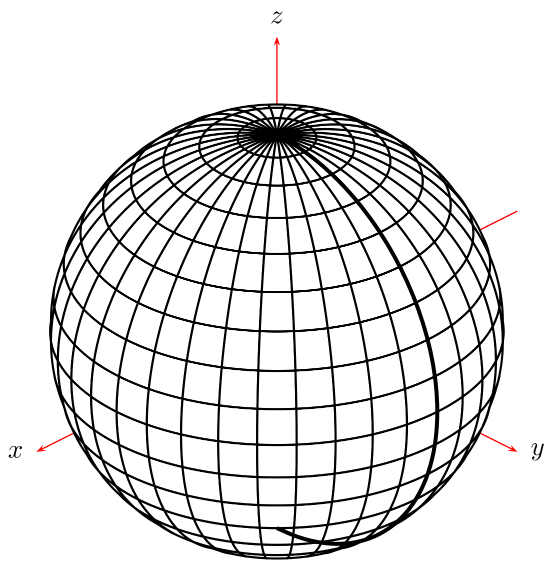


Figure 6.1: Sphere with a curve going from north to south along a great circle.

6.2 Curves on a sphere

Let us now describe curves on a surface. A curve can be parametrized by a single parameter, t. This looks as follows in the case of spherical coordinates (note that r is constant)

$$\mathbf{r}(t) = r_0 \Big(\sin\Big(\theta(t)\Big) \cos\Big(\phi(t)\Big), \sin\Big(\theta(t)\Big) \sin\Big(\phi(t)\Big), \cos\Big(\theta(t)\Big)\Big)$$

Letting $\phi(t) = 0$, $\theta(t) = t$ and $0 \leq t \leq \pi$ gives the following simple curve:

$$\mathbf{r}(t) = r_0(\sin(t), 0, \cos(t))$$

This is a great circle from the north pole to the south pole. The length of this curve is simply πr_0.

By chosing the following parametrization $\theta(t) = t, \quad \phi(t) = 2t - \pi$ we obtain the following cute winding curve shown in figure 6.2

$$l = r_0 \int_0^\pi \sqrt{g_{11}\dot{\theta}^2 + g_{22}\dot{\phi}^2 dt} = r_0 \int_0^\pi \sqrt{1 + 4\sin^2 t} dt \approx r_0 \cdot 5.27037\ldots$$

6.3 Cylinder

We can parametrize the cylinder by θ and z by setting $\rho = \rho_0$,

$$\mathbf{r}(\theta, z) = (\rho_0 \sin\theta, \rho_0 \cos\theta, z)$$

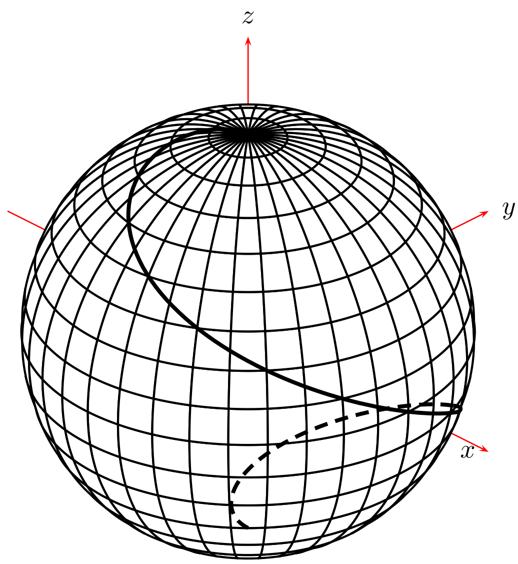


Figure 6.2: Sphere with a curve winding down from north to south.

$$\mathbf{e}_1 = \mathbf{e}_\theta = \partial \mathbf{r}/\partial\theta = \begin{pmatrix} \rho_0 \cos\theta \\ -\rho_0 \sin\theta \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2 = \mathbf{e}_z = \partial \mathbf{r}/\partial z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} \rho_0^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the metric tensor for the cylinder surface is constant the surface has zero curvature.

6.4 Curves on a cylinder

We parametrize a curve on the cylinder by letting ρ, θ, z depend on a single parameter, and thus get the coordinates as a function of t.

$$\mathbf{r}(t) = (\rho_0 \sin\theta(t), \rho_0 \cos\theta(t), z(t))$$

If we let $\theta(t) = bt$, nd $z(t) = at$, where we can think of b as the rotation speed and a as the speed along the z axis, we obtain

$$\mathbf{r}(t) = (\rho_0 \sin(bt), \rho_0 \cos(bt), at)$$

which, with $a = 1, b = 2$, gives the curve shown in figure 6.3. The length of this curve is

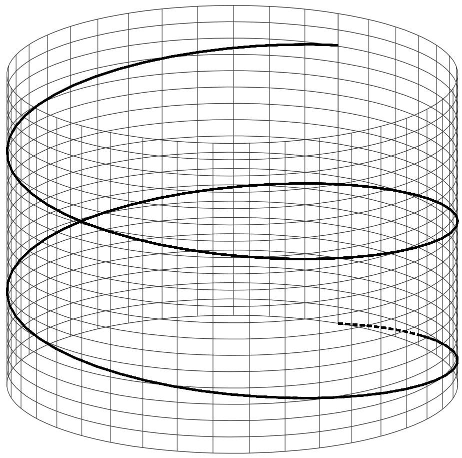


Figure 6.3: Cylinder surface with a spiral curve.

$$l = \int_0^{2\pi} \sqrt{g_{11}\dot{\theta}^2 + g_{22}\dot{z}^2}dt = \int_0^{2\pi} \sqrt{b^2\rho_0^2 + a^2}dt = 2\pi\sqrt{a^2 + b^2\rho_0^2}$$

This seems reasonable: When $a = 0$ the curve degenerates to a circle with the radius ρ_0, and the length is simply the length of the circle, $2\pi\rho_0$, times the number of windings b.

6.5 Paraboloid

A well known surface is the paraboloid, which is the same shape as a satellite disk. Its parametrization is

$$\mathbf{r}(u, v) = (u, v, -(u^2 + v^2))$$

$$\mathbf{e}_1 = \mathbf{e}_u = \partial\mathbf{r}/\partial u = \begin{pmatrix} 1 \\ 0 \\ -2u \end{pmatrix}$$

$$\mathbf{e}_2 = \mathbf{e}_v = \partial\mathbf{r}/\partial v = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 + 4u^2 & 4uv \\ 4uv & 1 + 4v^2 \end{pmatrix}$$

This is the first case where $g_{12} = g_{21}$ is not zero.

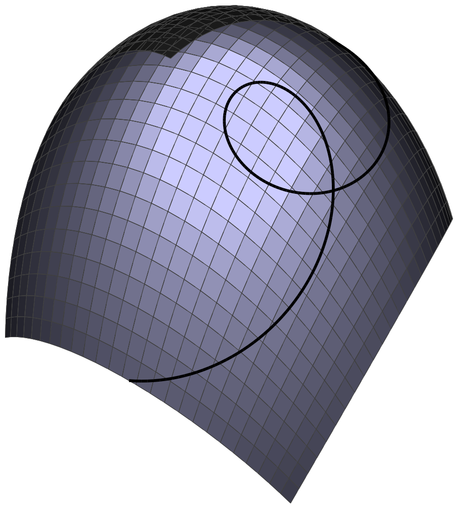


Figure 6.4: Paraboloid surface.

6.6 Curves on the paraboloid

The curve on the paraboloid surface shown in figure 6.4 is parametrized as

$$u(t) = t\sin(t), \quad v(t) = t\cos(t), \quad -\pi \leq t \leq \pi$$

$$\begin{aligned} f(t) &= g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2 \\ &= (1 + 4t^2\sin^2 t)(\sin(t) + t\cos(t))^2 \\ &+ 8t^2\sin(t)\cos(t)(\cos(t) - t\sin(t))(\sin(t) + t\cos(t)) \\ &+ (1 + 4t^2\cos^2 t)(\cos(t) - t\sin(t))^2 \end{aligned}$$

and the length of the curve is

$$= \int_{-\pi}^{\pi} \sqrt{f(t)}dt \approx 23.47564\ldots$$

6.7 Hyperbolic paraboloid

The hyperbolic paraboloid is a surface, shown in figure 6.5, which can be parametrized by u,v in the following manner

$$\mathbf{r}(u, v) = (u, v, u^2 - v^2)$$

$$\mathbf{e}_1 = \mathbf{e}_u = \partial\mathbf{r}/\partial u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}$$

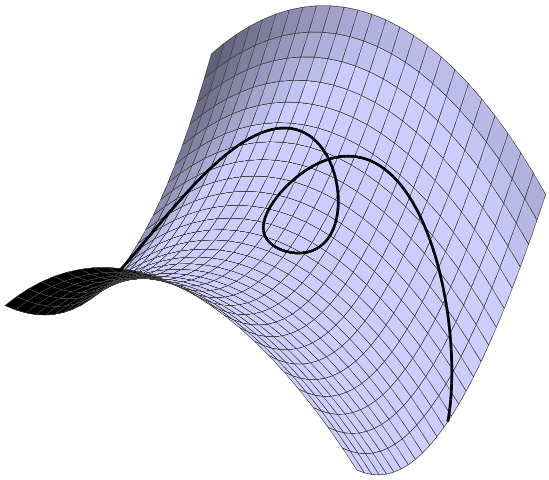


Figure 6.5: Hyperbolic paraboloid surface.

$$\mathbf{e}_2 = \mathbf{e}_v = \partial \mathbf{r}/\partial v = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{pmatrix}$$

The metric tensor is similar to the one for the paraboloid surface, differing only in the sign of the cross terms.

6.8 Curves on the hyperbolic paraboloid

The curve shown in figure 6.5 is created by the same parametrization as for the paraboloid in figure 6.4.

$$u(t) = t\sin(t), \quad v(t) = t\cos(t), \quad -\pi \le t \le \pi$$

$$\begin{aligned} g(t) &= g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2 \\ &= (1+4t^2\sin^2 t)(\sin(t) + t\cos(t))^2 \\ &- 8t^2\sin(t)\cos(t)(\cos(t) - t\sin(t))(\sin(t)+t\cos(t)) \\ &+ (1+4t^2\cos^2 t)(\cos(t) - t\sin(t))^2 \end{aligned}$$

and the length of the curve is

$$= \int_{-\pi}^{\pi} \sqrt{g(t)}dt \approx 35.57158\ldots$$

6.9 Modulated sphere

A slightly more complicated surface can be created by allowing r to depend on θ and ϕ, For example as $r(\theta, \phi) = a\sin\omega\theta + b$. When $a = 0.1, \omega = 10, b = 1$, the result is the following parametrization of the "modulated sphere" shown in figure 6.6

$$\mathbf{r}(\theta, \phi) = (r(\theta, \phi)\sin\theta\sin\phi, r(\theta, \phi)\sin\theta\cos\phi, r(\theta, \phi)\sin\theta)$$

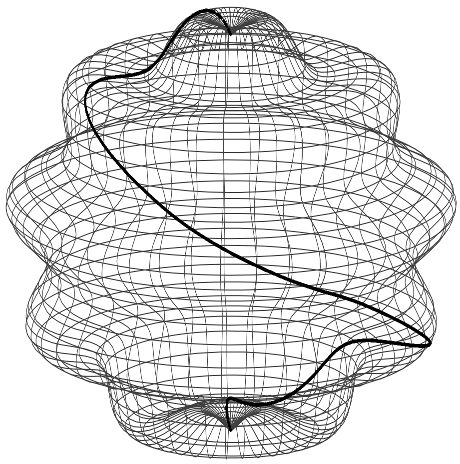


Figure 6.6: Modulated sphere ($a = 0.1, \omega = 10, b = 1$)

Now it gets a little complicated..

$$\mathbf{e}_1 = \mathbf{e}_\theta = \partial\mathbf{r}/\partial\theta = \begin{pmatrix} [a\cos\theta\sin(\omega\theta) + a\omega\sin\theta\cos(\omega\theta) + b\cos\theta]\sin\phi \\ [a\cos\theta\sin(\omega\theta) + a\omega\sin\theta\cos(\omega\theta) + b\cos\theta]\cos\phi \\ -a\sin\theta\sin(\omega\theta) + a\omega\cos\theta\cos(\omega\theta) - b\sin\theta \end{pmatrix}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \partial\mathbf{r}/\partial\phi = \begin{pmatrix} (a\sin\omega\theta + b)\sin\theta\cos\phi \\ -(a\sin\omega\theta + b)\sin\theta\sin\phi \\ 0 \end{pmatrix}$$

..and after some manipulations, where we primarily use the trigonometric identity $\sin^2 a + \cos^2 a = 1$, we get

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} f(\theta) & 0 \\ 0 & (a\sin\omega\theta + b)^2\sin^2\theta \end{pmatrix}$$

Where

$$\begin{aligned} f(\theta) = &\Big((a\sin(\omega\theta) + b)\cos\theta + a\omega\sin\theta\cos(\omega\theta)\Big)^2 \\ &+ \Big(-(a\sin(\omega\theta) + b)\sin\theta + a\omega\cos\theta\cos(\omega\theta)\Big)^2 \\ &= (a\sin\omega\theta + b)^2 + a^2\omega^2\cos^2(\omega\theta) \end{aligned}$$

Note that already at this stage the formulae gets so complicated that there is great risk in not getting it right by hand calculation, and no chance of analytic integration. But we can check that this is plausible by noting that in the limit of no deformation $(a, b) \to (0, 1)$, $f = 1$ and we recover the metric tensor for the sphere.

6.10 Curves on the modulated sphere

Using the same parametrization, $\theta(t) = t, \phi(t) = 2t - \pi$, as for the winding curve on the sphere we get a complicated curve as shown in figure 6.6. Since

$$g_{11}\dot{\theta}^2 + g_{22}\dot{\phi}^2 = f(t) + 4(a \sin \omega t + b)^2 \sin^2 t$$

we obtain the formula for the curve length as

$$l = \int_0^\pi \sqrt{f(t) + 4(a \sin \omega t + b)^2 \sin^2 t} dt \approx 5.73653 \ldots$$

Chapter 7

Curvature

In this chapter we will give the formulae for working with curvatures. This part is filled with symbols which makes it somewhat hard to comprehend. Just consider the symbols as functions which can be calculated from the metric tensor.

7.1 Gaussian Curvature

Gauss' curvature formula describes the curvature of a two dimensional surface parametrized by (u, v).

$$K = -\frac{1}{g_{11}}\left(\frac{\partial}{\partial u}\Gamma^2_{12} - \frac{\partial}{\partial v}\Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right)$$

7.2 Riemann curvature tensor

The Riemann curvature tensor is a tensor of the fourth order, which is defined in the following way

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

7.3 Ricci curvature tensor

The Ricci tensor is related to the Riemann curvature tensor by contracting two indices

$$R_{\sigma\nu} = R^\rho{}_{\sigma\rho\nu} = \Gamma^\rho_{\nu\sigma,\rho} - \Gamma^\rho_{\rho\sigma,\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}$$

Example 7.1. For the spherical surface with unit length the only nonzero Christoffel symbols are

$$\begin{aligned}\Gamma^2_{12} &= \frac{\cos\theta}{\sin\theta}\\ \Gamma^1_{22} &= -\cos\theta\sin\theta\end{aligned}$$

of which only Γ^2_{12} appears in Gauss' formula. Noting that $g_{11} = 1$, the Gaussian curvature is then

$$K = -\frac{d}{d\theta}\frac{\cos\theta}{\sin\theta} - \left(\frac{\cos\theta}{\sin\theta}\right)^2 = 1$$

which shows that the sphere has a constant, positive curvature.

Example 7.2. A for the hyperbolic paraboloid, the nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{11} &= \frac{4u}{4v^2+4u^2+1} \\
\Gamma^2_{11} &= -\frac{4v}{4v^2+4u^2+1} \\
\Gamma^1_{22} &= -\frac{4v}{4v^2+4u^2+1} \\
\Gamma^2_{22} &= \frac{4v}{4v^2+4u^2+1}
\end{aligned}$$

the gaussian formula reduces to

$$K = \frac{1}{g_{11}}\left(\frac{d}{dv}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{22}\right) = -\frac{4}{(4v^2+4u^2+1)^2}$$

The Gaussian curvature is now a function of u, and v, and we see that this surface has negative curvature everywhere with a minimum occuring at $(u, v) = (0, 0)$, where it has the value -4.

Chapter 8

Tensor Examples

In this section we will give some examples of the Christoffel symbols (of both kinds) derived from specific coordinate systems. It will be demonstrated that depending on the metric tensor components there can be few or many of either kind.

When working with tensors and their derivatives, such as the ones needed for calculating the Christoffel symbols, it is very time consuming to do these calculations by hand. To demonstrate the complexity of this algebraic manipulation let us calculate Γ^2_{12} for the spherical coordinates described in section 4.5.

$$\Gamma^2_{12} = \frac{1}{2} g^{2m} \left(\frac{\partial g_{m1}}{\partial u^2} + \frac{\partial g_{m2}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^m} \right)$$

since $u^1 = r, u^2 = \theta$, and $u^3 = \phi$ we get, when summing over $m = 1, 2, 3$

$$\begin{aligned} \Gamma^2_{12} &= \tfrac{1}{2} g^{21} \left(\tfrac{\partial g_{11}}{\partial \theta} + \tfrac{\partial g_{12}}{\partial r} - \tfrac{\partial g_{12}}{\partial r} \right) + \tfrac{1}{2} g^{22} \left(\tfrac{\partial g_{21}}{\partial \theta} + \tfrac{\partial g_{22}}{\partial r} - \tfrac{\partial g_{12}}{\partial \theta} \right) \\ &\quad + \tfrac{1}{2} g^{23} \left(\tfrac{\partial g_{31}}{\partial \theta} + \tfrac{\partial g_{32}}{\partial r} - \tfrac{\partial g_{12}}{\partial \phi} \right) \\ &= \tfrac{1}{2} g^{22} \tfrac{\partial g_{22}}{\partial r} = \tfrac{1}{2} \tfrac{1}{r^2} (2r) = \tfrac{1}{r} \end{aligned}$$

It turns out that all but one derivative of the metric tensor are zero, but this is not always the case! Note that there are N^3 Christoffel symbols, so in three dimensions we need to do this 27 times performing $9 * 27 = 243$ derivatives!

Clearly the possibility for errors is significant, so why not let a symbolic math program handle the tiresome and error prone calculations? After all this has been done by professionals since 1966 [6]. The following examples were created using Maxima [10]. For source code examples see section C.5.

The Christoffel symbol of the first kind is defined as

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

and the Christoffel symbol of the second kind is

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{mi}}{\partial u^j} + \frac{\partial g_{mj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^m} \right)$$

In the following we will calculate the Christoffel symbols of the first and second kind for a number of coordinate systems using these formulae.

8.1 Cylindrical Coordinates

From section 4.4 we have

$$\mathbf{r}(\rho, \phi, z) = (\rho\cos(\phi), \rho\sin(\phi), z)$$

and

$$\Gamma_{122} = \rho, \qquad \Gamma_{221} = -\rho$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma^2_{12} = \frac{1}{\rho}, \qquad \Gamma^1_{22} = -\rho$$

8.2 Spherical Coordinates

Of the 27 potential Christoffel symbols of the first kind, only six are nonzero and only three needs to be calculated due to symmetry properties.

$$\begin{aligned} \Gamma_{122} &= r \\ \Gamma_{133} &= r\cos^2\theta \\ \Gamma_{221} &= -r \end{aligned} \qquad \begin{aligned} \Gamma_{233} &= -r^2\cos\theta\sin\theta \\ \Gamma_{331} &= -r\cos^2\theta \\ \Gamma_{332} &= r^2\cos\theta\sin\theta \end{aligned}$$

The inverse of g_{ij} is

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\cos^2\theta} \end{pmatrix}$$

From this we can calculate the Christoffel symbols of second kind.

$$\begin{aligned} \Gamma^2_{12} &= \frac{1}{r} \\ \Gamma^3_{13} &= \frac{1}{r} \\ \Gamma^1_{22} &= -r \end{aligned} \qquad \begin{aligned} \Gamma^3_{23} &= -\frac{\sin\theta}{\cos\theta} \\ \Gamma^1_{33} &= -r\cos^2\theta \\ \Gamma^2_{33} &= \cos\theta\sin\theta \end{aligned}$$

8.3 Ellipsoidal Coordinates

If we take the innocent spherical coordinates and change them in the following way

$$\mathbf{r}(u, v, w) = (au \sin v \cos w, bu \sin v \sin w, cu \cos v)$$

we get the ellipsoidal coordinates where r has been replaced by u, θ by v and ϕ by w we get for the metric tensor $g_{ij} =$

$$\begin{pmatrix} \sin^2 v(b^2 \sin^2 w + a^2 \cos^2 w) + c^2 \cos^2 v & u \cos v \sin v(b^2 \sin^2 w + a^2 \cos^2 w - c^2) & \left(b^2 - a^2\right) u \sin^2 v \cos w \sin w \\ u \cos v \sin v(b^2 \sin^2 w + a^2 \cos^2 w - c^2) & u^2 \cos^2 v(b^2 \sin^2 w + a^2 \cos^2 w) + c^2 u^2 \sin^2 v & \left(b^2 - a^2\right) u^2 \cos v \sin v \cos w \sin w \\ \left(b^2 - a^2\right) u \sin^2 v \cos w \sin w & \left(b^2 - a^2\right) u^2 \cos v \sin v \cos w \sin w & a^2 u^2 \sin^2 v \sin^2 w + b^2 u^2 \sin^2 v \cos^2 w \end{pmatrix}$$

... and as you can see the page is barely wide enough to hold the result even with a tiny font! But from Maxima we get the following 15 Christoffel symbols of the first kind

$$\begin{aligned}
\Gamma_{121} &= \cos v \sin v \left(b^2 \sin^2 w + a^2 \cos^2 w - c^2\right) \\
\Gamma_{122} &= u \left(b^2 \cos^2 v \sin^2 w + a^2 \cos^2 v \cos^2 w + c^2 \sin^2 v\right) \\
\Gamma_{123} &= (b - a)\ (b + a)\ u \cos v \sin v \cos w \sin w \\
\Gamma_{131} &= (b - a)\ (b + a) \sin^2 v \cos w \sin w \\
\Gamma_{132} &= (b - a)\ (b + a)\ u \cos v \sin v \cos w \sin w \\
\Gamma_{133} &= u \sin^2 v \left(a^2 \sin^2 w + b^2 \cos^2 w\right) \\
\Gamma_{221} &= -u \left(b^2 \sin^2 v \sin^2 w + a^2 \sin^2 v \cos^2 w + c^2 \cos^2 v\right) \\
\Gamma_{222} &= -u^2 \cos v \sin v \left(b^2 \sin^2 w + a^2 \cos^2 w - c^2\right) \\
\Gamma_{223} &= -(b - a)\ (b + a)\ u^2 \sin^2 v \cos w \sin w \\
\Gamma_{231} &= (b - a)\ (b + a)\ u \cos v \sin v \cos w \sin w \\
\Gamma_{232} &= (b - a)\ (b + a)\ u^2 \cos^2 v \cos w \sin w \\
\Gamma_{233} &= u^2 \cos v \sin v \left(a^2 \sin^2 w + b^2 \cos^2 w\right) \\
\Gamma_{331} &= -u \sin^2 v \left(b^2 \sin^2 w + a^2 \cos^2 w\right) \\
\Gamma_{332} &= -u^2 \cos v \sin v \left(b^2 \sin^2 w + a^2 \cos^2 w\right) \\
\Gamma_{333} &= -(b - a)\ (b + a)\ u^2 \sin^2 v \cos w \sin w
\end{aligned}$$

You can verify that this seems reasonable by setting $a = b = c = 1$ in which case many of these become zero and we are left with the six symbols from the Spherical coordinates. The formula for the inverse of the metric tensor is not given as it turns out to be quite big, but using Maxima we can calculate the Christoffel symbols of the second kind and there turns out to be six of these just as in the case of the Spherical coordinates.

$$\begin{aligned}
\Gamma^2_{12} &= \tfrac{1}{u} & \Gamma^3_{23} &= \tfrac{\cos v}{\sin v} \\
\Gamma^3_{13} &= \tfrac{1}{u} & \Gamma^1_{33} &= -u \sin^2 v \\
\Gamma^1_{22} &= -u & \Gamma^2_{33} &= -\cos v \sin v
\end{aligned}$$

8.4 Other Coordinates

This coordinate system is taken from [4] p. 11. It probably does not have any practical applications, but it is the first example of a metric tensor where all components are nonzero.

$$\mathbf{r}(u, v, w) = (u + v, u - v, 2uv + w)$$

$$g_{ij} = \begin{pmatrix} 4v^2 + 2 & 4uv & 2v \\ 4uv & 4u^2 + 2 & 2u \\ 2v & 2u & 1 \end{pmatrix}$$

There are only three Christoffel symbols of the first kind

$$\begin{aligned} \Gamma_{121} = & \ 4v \\ \Gamma_{122} = & \ 4u \\ \Gamma_{123} = & \ 2 \end{aligned}$$

The inverse of the metric tensor is

$$g^{ij} = \begin{pmatrix} 1/2 & 0 & -v \\ 0 & 1/2 & -u \\ -v & -u & 2v^2 + 2u^2 + 1 \end{pmatrix}$$

This means that we are left with only one Christoffel symbol of the second kind

$$\Gamma^3_{12} = \ 2$$

8.5 Geodesics

Why bother with all these Christoffel symbols? The reason is that in order to find the equations of motions in General Relativity we need to find the possible patterns of movements (straight lines) and these are called Geodesics. There is a connection between Geodesics and the Christoffel symbols given by

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0$$

Now let us take the Spherical coordinate system and find the Geodesic curves. First by noting that $u^i = u^1 = r$,

$$\frac{d^2r}{dt^2} + \Gamma^1_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0 \tag{8.1}$$

$$\frac{d^2r}{dt^2} + \Gamma^1_{22}(\dot{\theta})^2 + \Gamma^1_{33}(\dot{\phi})^2 = 0 \tag{8.2}$$

here $u^i = u^2 = \theta$,

$$\frac{d^2\theta}{dt^2} + \Gamma^2_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0$$

$$\frac{d^2\theta}{dt^2} + \Gamma^2_{12}\dot{r}\dot{\theta} + \Gamma^2_{33}(\dot{\phi})^2 = 0$$

finally by setting $u^i = u^3 = \phi$, we get

$$\frac{d^2\phi}{dt^2} + \Gamma^3_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0$$
$$\frac{d^2\phi}{dt^2} + \Gamma^3_{13}\dot{r}\dot{\phi} + \Gamma^3_{23}\dot{\theta}\dot{\phi} = 0$$

so the equations of motion are given by the three second-order differential equations

$$\ddot{r} - r\dot{\theta}^2 - r\cos^2\theta\dot{\phi}^2 = 0$$
$$\ddot{\theta} + \frac{1}{r}\dot{r}\dot{\theta} + \cos\theta\sin\theta(\dot{\phi})^2 = 0$$
$$\ddot{\phi} + \frac{1}{r}\dot{r}\dot{\phi} - \frac{sin\theta}{\cos\theta}\dot{\theta}\dot{\phi} = 0$$

This seems complicated, but let us see what it means in a simple case, where θ is constant at θ_0 and $r = 1$. In this case the equations reduce to

$$-cos^2\theta_0\dot{\phi}^2 = 0 \tag{8.3}$$
$$\ddot{\phi} = 0 \tag{8.4}$$

from the first equation we get $\theta_0 = \pi/2$, and from the second equation we get $\phi(t) = at + b$.

This means that the geodesic curves of constant θ and r are great circles on the equator.

$\boxed{\text{Find a better (simpler) example?}}$

8.6 Higher dimensional spaces

Schwarzchild metric (4d)

An important metric in General Relativity is the Schwarzchild metric which is a topic in section 10. When the four coordinates are t, r, θ, ϕ, and $a = GM/c^2$ and spherical coordinates are used, its metric tensor is

$$g_{ij} = \begin{pmatrix} 1-\frac{a}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{a}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

The Christoffel symbols of the first kind are

$$\begin{aligned} \Gamma_{112} &= \tfrac{ar-a^2}{2r^3} \\ \Gamma_{121} &= \tfrac{a}{2r^2-2ar} \\ \Gamma_{222} &= -\tfrac{a}{2r^2-2ar} \\ \Gamma_{233} &= \tfrac{1}{r} \\ \Gamma_{244} &= \tfrac{1}{r} \end{aligned} \qquad \begin{aligned} \Gamma_{332} &= a-r \\ \Gamma_{344} &= \tfrac{\cos\theta}{\sin\theta} \\ \Gamma_{442} &= (a-r)\sin^2\theta \\ \Gamma_{443} &= -\cos\theta\sin\theta \end{aligned}$$

and the inverse metric tensor is

$$g^{ij} = \begin{pmatrix} \frac{1}{1-\frac{a}{r}} & 0 & 0 & 0 \\ 0 & \frac{a}{r}-1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

and the Christoffel symbols of the second kind are given as

$$\begin{aligned} \Gamma^2_{11} &= -\frac{a}{2r^2} & \Gamma^4_{24} &= -r\sin^2\theta \\ \Gamma^1_{12} &= \frac{a}{2r^2} & \Gamma^2_{33} &= r \\ \Gamma^2_{22} &= \frac{a}{2\left(1-\frac{a}{r}\right)^2 r^2} & \Gamma^4_{34} &= -r^2\cos\theta\sin\theta \\ \Gamma^3_{23} &= -r & \Gamma^2_{44} &= r\sin^2\theta \\ & & \Gamma^3_{44} &= r^2\cos\theta\sin\theta \end{aligned}$$

Kaluza-Klein metric (5d)

A space of five dimensions were contrieved in a unification theory by Kaluza and Klein. Several variations of the theme exist, but the following is an example from [11] where the metric tensor is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 & 0 \\ 0 & 0 & -A^2(t)r^2 & 0 & 0 \\ 0 & 0 & 0 & -A^2(t)r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & -B^2(t) \end{pmatrix}$$

the inverse of the metric tensor is

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{A^2(t)} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{A^2(t)r^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{A^2(t)r^2\sin^2\theta} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{B^2(t)} \end{pmatrix}$$

and the Christoffel symbols of the second kind are

$$\begin{aligned} \Gamma^2_{12} &= \frac{\dot{A}}{A} & \Gamma^1_{33} &= r^2 A\dot{A} \\ \Gamma^3_{13} &= \frac{\dot{A}}{A} & \Gamma^2_{33} &= -r \\ \Gamma^4_{14} &= \frac{\dot{A}}{A} & \Gamma^4_{34} &= \frac{\cos\theta}{\sin\theta} \\ \Gamma^5_{15} &= \frac{\dot{B}}{B} & \Gamma^1_{44} &= r^2\sin^2\theta A\dot{A} \\ \Gamma^1_{22} &= A\dot{A} & \Gamma^2_{44} &= -r\sin^2\theta \\ \Gamma^3_{23} &= \frac{1}{r} & \Gamma^3_{44} &= -\cos\theta\sin\theta \\ \Gamma^4_{24} &= \frac{1}{r} & \Gamma^1_{55} &= B\dot{B} \end{aligned}$$

which is not too bad considering that there could have been $5^3 = 125$!. Note that the symbols depend on the choice of the functions $A(t)$ and $B(t)$ and their derivatives. The functions would have to be found as a solution to the Einstein field equations, but one could assume a form of the functions and then calculate the consequences for cosmology.

Chapter 9

Parallel Transport

In this chapter we will discuss parallel transport of vectors along a curve. This turns out to be an important concept because it is closely related to Geodesics and the possible solutions to the equations of motion in spacetime. We will start by going through an example from [4] to illustrate the ideas. We wish to make a parallel transport of a vector pointing 'south' along a curve of constant latitude θ_0.

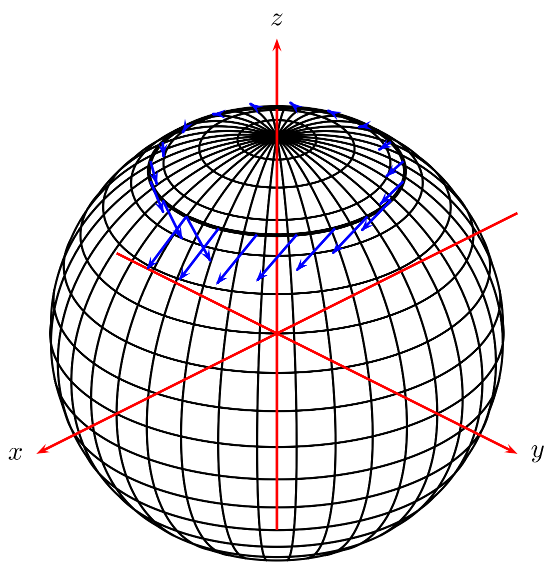


Figure 9.1: Parallel transport of a vector around a circle at constant latitude θ_0 on the surface of a sphere, following the example in [4] p. 68. Here $\alpha = 0, \theta_0 = 0.6$.

$$\lambda = \lambda^1 \mathbf{e}_1 + \lambda^2 \mathbf{e}_2$$

where ...

$$\mathbf{v} = \mathbf{r}(r_0, \theta, \phi) + \left[\lambda^1 \mathbf{e}_1 + \lambda^2 \mathbf{e}_2\right] t$$

setting $r = 1$, we get a parametric representation of the transported vector as a function of t for a given position on the sphere (θ, ϕ).

For transport on a circle at latitude θ_0

$$v(t)_{\theta,\phi} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + \left[\cos(\alpha - \phi\cos\theta) \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} + \frac{\sin(\alpha - \phi\cos\theta)}{\sin\theta} \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix} \right] t$$

Chapter 10

Curves in higher dimensions

There is no reason to stop at 3D as mathematics do not place any limits on curves.

In fact General Relativity operates with the notion of spacetime which is a four dimensional construction, and in other areas of physics even higher dimensions are used. String theory which tries to unite the four fundmental forces operates with dimensions of 10 and upwards. Moving to higher dimensions, however, presents some difficulties as 1) the metric tensor has higher dimensionality and hence more terms and cross terms to evaluate. 2) For dimensions higher than 3 graphical visualization is impossible.

10.1 Spacetime

Spacetime is the incorporation of time as a fourth coordinate into three dimensional space.

$$u_\nu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

With the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and a the distance between nearby points is $\Delta s^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

Note that since all elements of g are constant, all Christoffel symbols are 0, this space is flat!

10.2 Schwarzchild Metric

The Schwarzchild metric is a specific solution to Einsteins field equations for gravitation around a massive object in empty space given spherical symmetry. Examples of the derivation of this metric are given in [4] and [8].

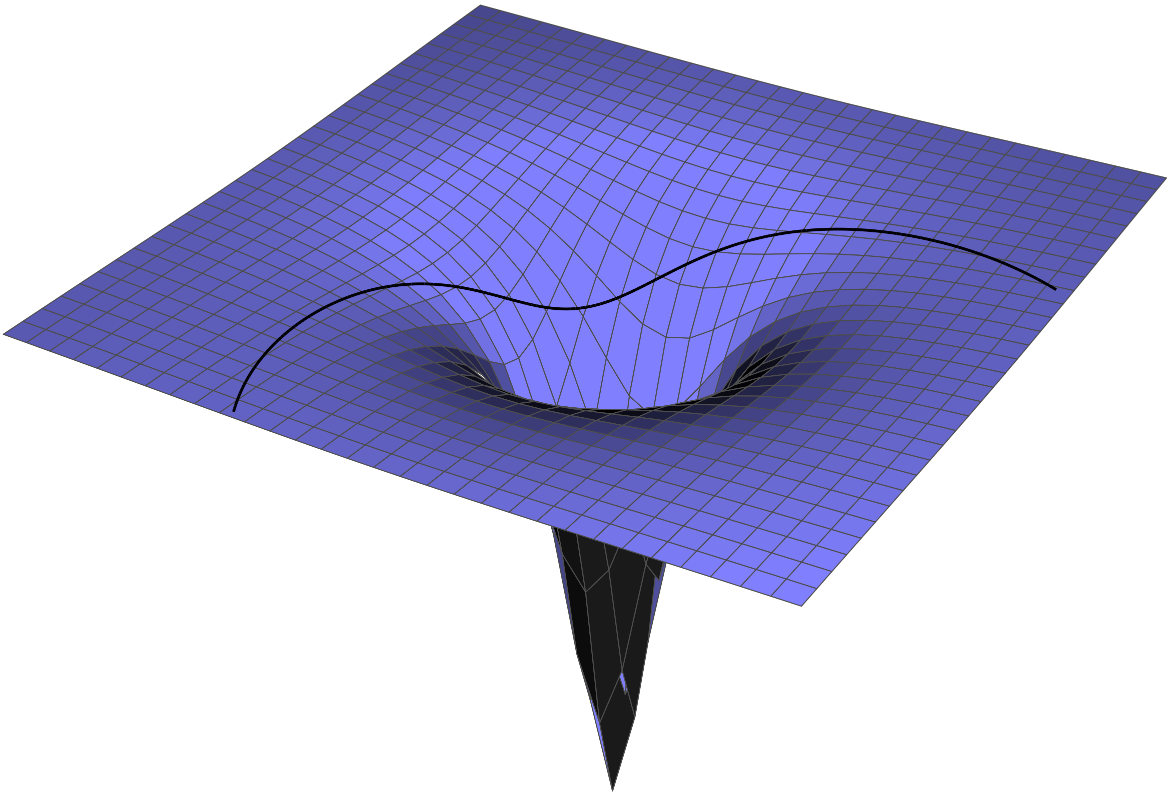


Figure 10.1: Curvature of space near a massive object.

$$g_{\mu\nu} = \begin{pmatrix} 1 - 2MG/c^2 r & 0 & 0 & 0 \\ 0 & -(1 - 2MG/c^2 r)^{-1} & 0 & 0 \\ 0 & & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

the Christoffel symbols for this metric are given in section 8.6.

10.3 Kerr Metric

For a rotating massive object the metric tensor is a further generalization of the Schwarzchild metrix, where

$$g_{\mu\nu} = \begin{pmatrix} \left(1 - \frac{r_s r}{\rho^2} c^2\right) & 0 & 0 & \frac{r_s r \alpha \sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{r_s r \alpha \sin^2\theta}{\rho^2} & 0 & 0 & \left(r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2} \sin^2\theta\right) \sin^2\theta \end{pmatrix}$$

where

$$r_s = \frac{2GM}{c^2} \qquad \rho^2 = r^2 + \alpha^2 \cos^2\theta$$
$$\alpha = \frac{J}{Mc} \qquad \Delta = r^2 - r_s r + \alpha^2$$

In the limit of no rotation (where $J = 0$) the metric reduces to the Schwarzchild metric.

10.4 Robertson-Walker Metric

Friedmann Lemaître Einstein field equations

Or Friedmann-Lemaître-Robertson-Walker metric as it probably should be named due to its many contributors. This is another exact solution of the Einstein Field equations in the context cosmology, where a homogenous and isotropic, expanding (or contracting) universe is described. The metric tensor is given as

$$g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -A(t)^2 \frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & -A(t)^2 r^2 & 0 \\ 0 & 0 & 0 & -A(t)^2 r^2 \sin^2\theta \end{pmatrix}$$

where $A(t)$ is a scale factor affecting the spatial components.

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & -\frac{1+kr^2}{A^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2 A^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{A^2 r^2 \sin^2\theta} \end{pmatrix}$$

and

$$\Gamma^2_{12} = \frac{\dot{A}}{A}$$
$$\Gamma^3_{13} = \frac{\dot{A}}{A}$$
$$\Gamma^4_{14} = \frac{\dot{A}}{A}$$
$$\Gamma^1_{22} = \frac{A\dot{A}}{c^2(1+kr^2)}$$
$$\Gamma^2_{22} = -\frac{kr}{1+kr^2}$$
$$\Gamma^3_{23} = \frac{1}{r}$$
$$\Gamma^4_{24} = \frac{1}{r}$$
$$\Gamma^1_{33} = \frac{r^2 A\dot{A}}{c^2}$$
$$\Gamma^2_{33} = -r(1+kr^2)$$
$$\Gamma^4_{34} = \frac{\cos\theta}{\sin\theta}$$
$$\Gamma^1_{44} = \frac{r^2 A\dot{A}\sin^2\theta}{c^2}$$

Appendix A

Notation and Symbols

Notation	**Meaning**
a	The scalar a
$\mathbf{p}$	The vector $\mathbf{p}$
$\mathbf{e}_i$	Basis vectors in any coordinate system, i=1...N
u^i	Coordinates in any coordinate system, i=1...N
$\lvert\mathbf{p}\rvert$	Length of vector $\mathbf{p}$
$\dot{f}$	Differentiation of f with respect to time t
$\ddot{f}$	Second order differentiation of f with respect to time t
$\mathbf{p} \cdot \mathbf{q}$	Dot product of vectors $\mathbf{p}$ and $\mathbf{q}$
$\mathbf{p} \times \mathbf{q}$	Cross product of vectors $\mathbf{p}$ and $\mathbf{q}$
g_{ij}	Components of the metric tensor
g^{ij}	Components of the inverse of the metric tensor
Γ_{ijk}	Christoffel symbol of the first kind
Γ^k_{ij}	Christoffel symbol of the second kind
R_{ij}	Ricci tensor
R_{ijkl}	Riemann tensor
R^i_{jkl}	Riemann tensor

Appendix B

People of Geometry

This is just a brief collection of names of people who have been influential in either pure geometry or the geometry of spacetime.

Pythagoras (570-495 BC):*Pythagoras' theorem.* Attributed as the discoverer of Pythagora's theorem relating the side of a right angled triangle to its diagonal. Was influential in creating a natural philosophical school whose ideas has reverbated for millennia.

Euclid (ca. 300 BC):*Geometry and number theory.* Also named the father of geometry, Euclud wrote a treatise "Elements" on geometry that kept inspiring mathematicians for two millennia. Euclids geometry was eventually extended to curved spaces.

Nicolaus Copernicus (1473-1543):*Astronomy, heliocentric model.* Copernicus was a renaissance astronomer who challenged the current view that earth was the center of the universe by suggesting that a heliocentric model better fitted observation and simplified the apparent movements of the planets in the night sky.

Galileo Galilei (1564-1642):*Investigation of motion.* Galileo has been called the father of modern science due to his introduction of systematic investigations. He performed a number of scientific studies of motion by rolling balls on an inclined plane and concluded that objects of different mass or density would fall at the same speed were it not for wind resistance, friction etc. A test of his claim was actually performed on the moon by one of the Apollo missions.Galileo built a telescope and was the first to observe Saturns rings.

Johannes Kepler (1571-1630):*Laws of planetary motion.* Kepler was the first to describe the laws of planetary motion based on observational astronomy. Among Keplers other contributions were a conjecture about the optimal packing of spheres which were only recently proved - nearly 400 years after the conjecture was made.

Isaac Newton (1643-1727):*Invention of differential calculus, equations of motion.* Newton was in many respects the father of celestial mechanics. He was the first to formulate the principles of gravitational attraction and from this deduce the equations of motions for planets orbiting the sun. In the process he also invented differential calculus. Newtons physics were unchallenged for nearly 300 years when they were extended by Einstein. For many practical purposes - even putting a man on the moon, Newtons formulae are still adequate. Newtons main work is the Principia which deals with optics, mathematics and physics.

Leonhard Euler (1707-1783):*Mathematical foundations and geometry.* Euler developed much of the modern symbols used in physics and contributed in every field of pure and applied mathematics. Euler was by any standards the most productive mathematician the world has produced. For an excellent book on some of his contributions put in historical perspective see [2].

Karl Friedrich Gauss (1777-1855):*Curvature and geometric measurements.* Gauss made numerous contributions to pure and applied mathematics. He developed methods for accurate surveying and investigated the curvature of surfaces. He discovered that curvature of a surface is an intrinsic property of the surface, and thus can be measured on the surface and provided a formula for calculating the curvature of a surface from the metric tensor. Gauss is sometimes referred to as the prince of mathematics. The book [5] is a very read-worthy fiction of Gauss' life and carreer.

Carl Gustav Jacob Jacobi (1804-1851):*Transformation matrix.* Jacobi was a matematician and contributed much to the field. In relation to General Relativity it is the Jacobian matrix involved in coordinate transforms that are the most significant.

Bernhard Riemann (1826-1866):*Definition of Manifolds.* Riemann extended Gauss' work on differential Geometry to higher dimensions, and founded the field of Riemannian Geometry. The Riemann curvature tensor is names after him. Perhaps his most well known contribution is to analytic number theory, where Riemanns zeta function was introduced and his famous conjecture was made.

Elwin Bruno Christoffel (1829-1900):*Tensor algebra.* Christoffel contributed to the development of Tensor algebra which is so fundamental to General Relativity. The Christoffel symbols named after him are essential in tensor analysis.

Gregorio Ricci-Curbastro (1853-1925):*Tensor algebra.* Riccis contribitions were in the fields of geometry, dynamics and tensor analysis.

Hermann Minkowsky (1864-1909):*Spacetime.* Minkowsky made the connection between time and space, combining them in what is now called spacetime. This was one of the steps in the direction of General Relativity.

Karl Schwarzchild (1873-1916):*Solution to the EFE.* Shortly after Einsteins new theory was presented, Schwarzchild solved the equations for empty space outside a single nonrotating, chargless mass. This achievement was made while Schwarzchild was a soldier in the trenches of World War I.

Albert Einstein (1879-1955):*General Relativity.* Einstein made four big contributions to modern physics, but the most profound and intellectual was the formulation of the equations of motion extending Newtons formula to massive objects and high speeds. The discovery, or rather the construction of the formula was a monumental task of mathematics. Einsteins theory is applied in modern GPS receivers.

Roy Kerr (1934-):*Solution to the EFE.* About 45 years after the first exact solution to the Einstein Field Equations (EFE) Kerr found a solution for a rotating mass.

Kip Thorne (1940-):*Authority on General Relativity.* Thorne co-authored the textbook on Gravitation that is still the most comprehensive textbooks on the subject. Thorne has been involved in validation of Einsteins theory by the gravitational wave experiment LIGO.

Appendix C

Useful tools

C.1 Creating surfaces with gnuplot

Creating a parametrized surface with gnuplot is very easy. The following listing generates the paraboloid surface in figure C.1, with the command

> $ **gnuplot parabola**
>
> assuming that the listing below is in a file called *parabola*

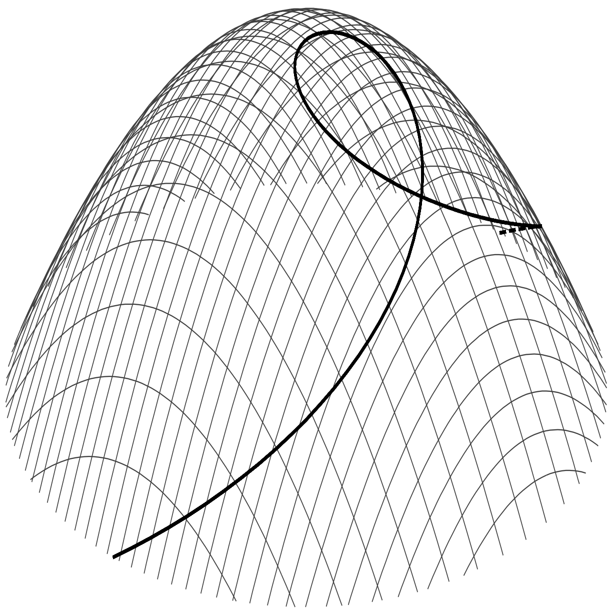


Figure C.1: Surface plotting with Gnuplot.

```
set isosamples 40,20
set parametric
set surface
set urange [-pi:pi]
set vrange [-pi:pi]
set zrange [-pi*pi:0]
set view 40, 40

set size 1,1

set border 0
set nokey
set noxtics
set noytics
set noztics

x(u,v)= u
y(u,v)= v
z(u,v)= -(u*u + v*v)

cx(u,v) = u*sin(u)
cy(u,v) = u*cos(u)
cz(u,v) = z(cx(u,v),cy(u,v))

set term x11
splot x(u,v), y(u,v), z(u,v)  lt rgb "#404040", \
cx(u,v), cy(u,v), cz(u,v) lt rgb "black" lw 3

pause -1

set term epslatex
set output "parab.eps"
replot
```

C.2 Surfaces with Google Sketchup

Google Sketchup is a 3D modelling tool provided free of charge. One cool feature of Sketchup is the API which makes it easy to write custom plugins for creating 3D graphics. In Sketchup we have fine control over colors, transparency, shadows, materials and because Sketchup is Polygon based it can also do hidden line removal. Put the following Ruby script in the Plugins folder and it will automatically run when you start Sketchup. The result is shown in figure C.2.

```
model    = Sketchup.active_model
entities = model.active_entities
materials= model.materials

def myz(x,y)
   return (x*x + y*y)/20.0
end

m1 = materials.add "my_mat_1"
```

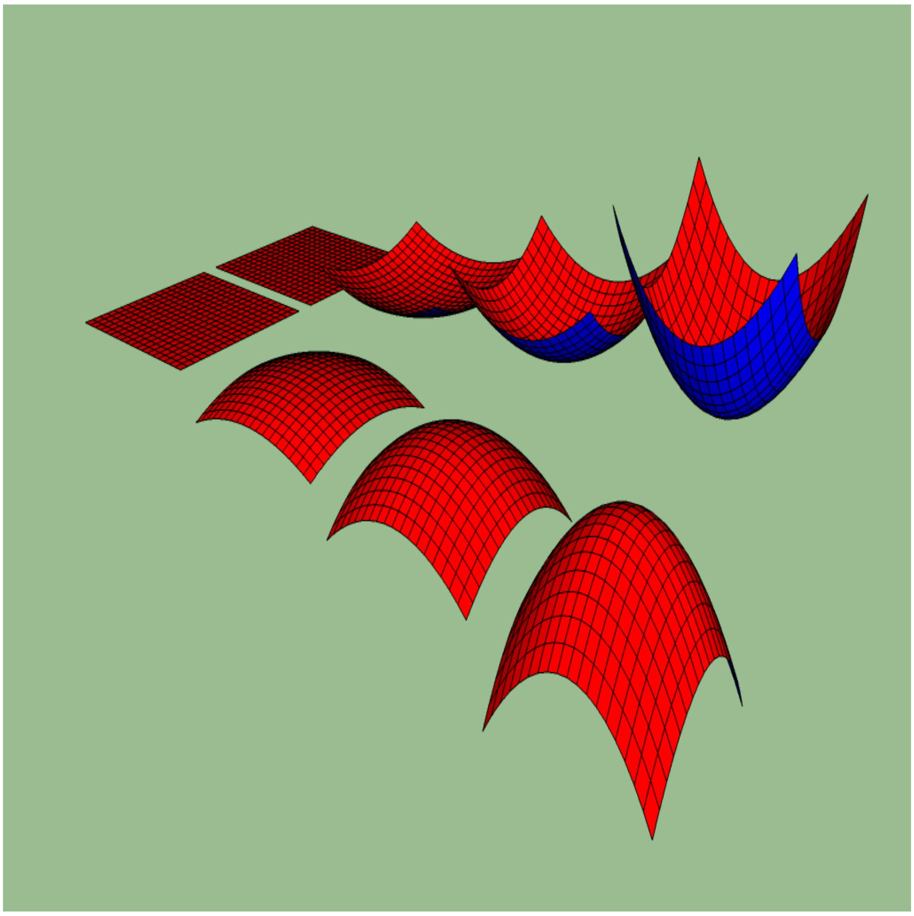


Figure C.2: Plots generated by Google Sketchup plugin.

```
m2 = materials.add "my_mat_2"
m1.color = "blue"
m2.color = "red"

def parab(x,y,v,entities)
  for i in -9...9
    for j in -9...9
      mx=(1.0*i)
      my=(1.0*j)
      p1= [x+mx  , y+my  , myz(mx  ,my  )/v ]
      p2= [x+mx+1, y+my  , myz(mx+1,my  )/v ]
      p3= [x+mx+1, y+my+1, myz(mx+1,my+1)/v ]
      p4= [x+mx  , y+my+1, myz(mx  ,my+1)/v ]

      f=entities.add_face(p1,p2,p3,p4)
      f.material="my_mat_2"
      f.back_material="my_mat_1"
    end
```

```
  end
end

parab( 0,  0, 0.5,entities)
parab(20,  0,   1,entities)
parab(40,  0,   2,entities)
parab(60,  0, 800,entities)
parab( 0, 20,-0.5,entities)
parab(20, 20,  -1,entities)
parab(40, 20,  -2,entities)
parab(60, 20,-800,entities)
```

C.3 LaTeX, TikZ and GNUPLOT

If you feel the need for more graphical consistency in your mathematical figures, you could use TikZ which is a large framework for drawing graphics within LaTeX.

This ended up being my favorite because embedding the graphics allows for a more coherent graphical representation with styles. Another advantage is that there is less hassle with including external plots and keeping these files up to date.

If The following lines of code are added to your document,

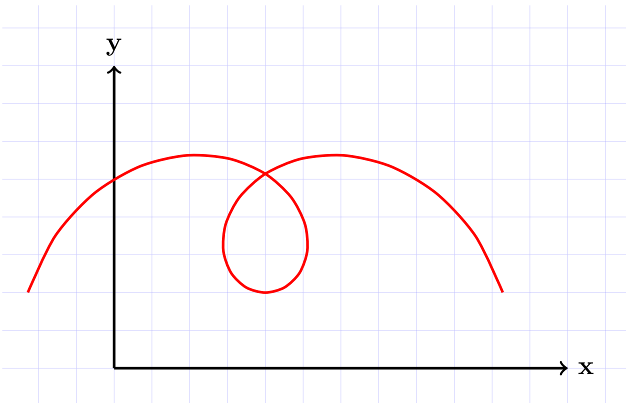
```
  \begin{tikzpicture}[scale=2]
    \draw[step=.25cm,mjcgrid] (-0.74,-0.24) grid (3.4,2.4);
    \pgfsetlinewidth{1pt}

    \coordinate (o)  at (0,0);
    \coordinate (ox) at (3,0);
    \coordinate (oy) at (0,2);
    \draw [->] (o) -- (ox);
    \draw [->] (o) -- (oy);

    \node [right]  at (ox) {$\mathbf{x}$};
    \node [above]  at (oy) {$\mathbf{y}$};

    \draw[scale=0.5,domain=-3.141:3.141,smooth,red]
          plot[parametric,prefix=pgffigs/mjc,id=par_test]
          function{2+t*cos(t),1+t*sin(t)};
  \end{tikzpicture}
```

this figure will be produced. The key “magic” is the **draw ... plot** commands. TikZ generates a gnuplot file, makes LaTeX call GNUPLOT, which in turn generates a file with a table of the computed values. These values are then read and presented in the graphical framework TikZ produces.



C.4 Numerical integration

For definite integrals of functions of one variable, Romberg integration was used. The program listed below is compiled and executed by the following commands

$ **gcc romberg.c**

$ **./a.out**

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

double sphere (double x) {
   return sqrt(1 + 4*sin(x)*sin(x));
}

double parab(double t){
  return sqrt(
      (1+4*t*t*sin(t)*sin(t)) * pow((sin(t) + t*cos(t)), 2)
     +(1+4*t*t*cos(t)*cos(t)) * pow((cos(t) - t*sin(t)), 2)
     + 8*t*t*sin(t)*cos(t)*(cos(t) - t*sin(t))* (sin(t) + t*cos(t))      );
}
double hyperb(double t){
  return sqrt(
      (1+4*t*t*sin(t)*sin(t)) * pow((sin(t) + t*cos(t)), 2)
     +(1+4*t*t*cos(t)*cos(t)) * pow((cos(t) - t*sin(t)), 2)
     - 8*t*t*sin(t)*cos(t)*(cos(t) - t*sin(t))* (sin(t) + t*cos(t))     );
}

double a=0.1;
double b=1;
double o=10;

double defsph (double t) {
    return sqrt( pow(a*sin(o*t)+b,2) +
                 pow(a*o*cos(o*t),2) +
                 pow( 2*(a*sin(o*t)+b)*sin(t) ,2)                      );
}

void romberg(double f(double), double a, double b, int n, double **R)
{
  int i, j, k;
  double h, sum;

  h = b - a;
  R[0][0] = 0.5 * h * (f(a) + f(b));
  printf(" R[0][0] = %f\n", R[0][0]);

  for (i = 1; i <= n; i++)
  {
     h *= 0.5;
     sum = 0;
     for (k = 1; k <= pow(2,i)-1; k+=2)
     {
       sum += f(a + k * h);
     }
```

```
      R[i][0] = 0.5 * R[i-1][0] + sum * h;
      printf(" R[%d][0] = %f\n", i, R[i][0]);
      for (j = 1; j <= i; j++)
      {
        R[i][j] = R[i][j-1] + (R[i][j-1] - R[i-1][j-1]) / (pow(4,j)-1);
        printf(" R[%d][%d] = %f\n", i,j, R[i][j]);
      }
    }
}

int main()
{
  int n = 10;
  int i;
  double **R;

  R = calloc((n+1), sizeof(double *));
  for (i = 0; i <= n; i++)
    R[i] = calloc((n+1), sizeof(double));

  printf("Sphere Winding curve\n");
  romberg(sphere,          0.0, 3.1415926, 8, R);
  printf("Deformed sphere\n");
  romberg(defsph,                0, 3.1415926, 8, R);
  printf("parab\n");
  romberg(parab,   -3.1415926, 3.1415926, 8, R);
  printf("hyperb\n");
  romberg(hyperb,  -3.1415926, 3.1415926, 8, R);

  return 0;
}
```

C.5 Symbolic algebra with Maxima

The following code written in Maxima, produces the output in section 8.2

```
    load(ctensor);

    ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
       r*sin(theta),[r,theta,phi]]);

    lg:trigsimp(lg);

    christof(lcs);

    ug:invert(lg);

    christof(mcs);
```

Bibliography

[1] M.V. Berry "***Principles of Cosmology and Gravitation***", Adam Hilger

[2] W. Dunham "***Euler, The master of us all***", The Mathematical Association of America

[3] A. Einstein "***Relativity, the special and general theory***", Crown Publishers Inc.

[4] J. Foster and D.J. Nightingale "***A Short Course in General Relativity***", Springer-Verlag New York Inc.; 2nd Revised edition

[5] D. Kehlmann "***Measuring the World***", Pantheon Books

[6] C.W. Misner, K.S. Thorne and J.A. Wheeler "***GRAVITATION***", W.H. Freeman and Company

[7] T. Oetiker et al. "***The not so Short Introduction to LaTeX 2ε***",

[8] T. Padman "***GRAVITATION Foundations and Frontiers***", Cambridge University Press

[9] W.H. Press, S.A. Teukolsky, V.T. Vetterling and B.P. Flanagan "***Numerical Recipes 3rd Edition: The Art of Scientific Computing***", Cambridge University Press

[10] W. Schelter et al. "***Maxima Computer Algebra System***", **www.maxima-project.org**

[11] R.K. Tiwari, F. Rahaman and S. Ray "***Five Dimensional Cosmological Models in General Relativity***", Int.J.Theor.Phys.49:2348-2357,2010

[12] T. Tantau "***The TikZ and PGF Packages***", Institut Für Teoretische Informatik, Universität zu Lübeck

Index

This pamphlet started as a self study in General Relativity, a subject along with Astronomy which have inspired me over the years. However, in order to (as far as I can) completely understand the subject of GR I really had to read up on the mathematics of curved surfaces.

'Not too bad.' - A. Nonymous

I then decided to wrap up my notes so I could pick up where I left at some later time. For typesetting I reverted back to one of my old favorites - LaTeX- and investigated what was new since I wrote my Ph.D. fifteen years ago. I then found the TikZ package which was very suitable for presenting some nice graphics. So here we are - enjoy!

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