# Big Data Asset Pricing

## Lecture 1: A Primer on Asset Pricing

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# Overview of the Course: Big Data Asset Pricing

**Lectures**

- Quickly getting to the research frontier
  1. **A primer on asset pricing**
  2. A primer on empirical asset pricing
  3. Working with big asset pricing data (videos)
- Twenty-first-century topics
  4. The factor zoo and replication
  5. Machine learning in asset pricing
  6. Asset pricing with frictions

**Exercises**

1. Beta-dollar neutral portfolios
2. Construct value factors
3. Factor replication analysis
4. High-dimensional return prediction
5. Research proposal

# This Lecture: A Primer in Asset Pricing

- Fundamentals of asset pricing
- $m$-pricing implies $\beta$-pricing
- Projecting $m$ on assets: mimicking portfolio
- Hansen-Jagannathan bound
- $\beta$-pricing implies $m$-pricing
- $\beta$-pricing in equilibrium
- Is $\beta$-pricing the same as market efficiency or rationality?
- Multi-factor $\beta$-pricing: factor models

# The Idea of These Primers and Words of Warning

- The idea of primers in asset pricing and empirical asset pricing: Brief overview so we
  - see clearly how the pieces fit together
  - don't lose sight of the end goal
  - can move on to the research frontier in the next lectures

- NB: The material is highly compressed
  - These notes cover so much ground that it might be a full course
  - In-depth understanding of this material is important
  - So read and listen carefully – and ask lots of questions
  - A few places the text is not fully self-contained, so I refer to the references in the end (e.g., certain test statistics)

# Fundamentals of Asset Pricing

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**Question:** What determines prices and expected returns?

**Answer (asset pricing without frictions):**

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**Question:** What determines prices and expected returns?

**Answer (asset pricing without frictions):**
State prices

- Based on a state space and its natural probabilities, there are
  - 3 equivalent ways of "pricing" = computing prices and risk premia:

1. **state prices**, $\psi$
2. **state price deflator**, $m$
3. **risk-neutral probabilities**, $Q$, and the risk-free rate, $r^f$

- Equivalence:
  - to each other
  - to no arbitrage = law of one price (under technical conditions)

$$\psi \text{ exists} \Leftrightarrow m \text{ exists} \Leftrightarrow Q \text{ exists} \Leftrightarrow \text{no arbitrage}$$

# Fundamentals of Asset Pricing, continued

- Mickey Mouse model
  - Asset with payoff $d_\omega$ next time period
  - Each $\omega$ is a discrete state with probability $Pr_\omega$
  - Want to find price, $p$, and expected return, $E(r) = E(d)/p - 1$
- **Three approaches – I focus on $m$ from next slides:**

1. $\psi_\omega$ = state prices = Arrow securities = price of \$1 in state $\omega$

$$p = \sum_\omega \psi_\omega d_\omega$$

2. $m_\omega = \frac{\psi_\omega}{Pr_\omega}$ = state price deflator (spd) = state price density
   = pricing kernel = stochastic discount factor (sdf)

$$p = \sum_\omega Pr_\omega m_\omega d_\omega = E(md)$$

3. $q_\omega = \frac{\psi_\omega}{\sum_j \psi_j} = \psi_\omega(1 + r^f)$ = risk-neutral probabilities
   = equivalent martingale measure

$$p = \sum_\omega q_\omega \frac{d_\omega}{1 + r^f} = E^Q\left(\frac{d}{1 + r^f}\right)$$

# Fundamentals of Asset Pricing, continued

- **Broader framework <u>used from now on</u>:**
  - Discrete-time economy, $t = 1, 2, \ldots$
  - Riskless rate, $r_t^f$ paid at time $t$ (some textbooks call this $r_{t-1}^f$)
  - Risky asset $i \in \{1, \ldots, N\}$ with, at time $t$,
    - dividend, $d_t^i$
    - price, $p_t^i$
    - excess return, $r_t^i$
- **Connection between prices and excess returns**:

$$r_{t+1}^i = \frac{d_{t+1}^i + p_{t+1}^i}{p_t^i} - 1 - r_{t+1}^f$$

- **Terminology**:
  - risk premium = expected excess return = $E_t(r_{t+1}^i)$
  - discount rate: same or $r_{t+1}^f + E_t(r_{t+1}^i)$ or $1 + r_{t+1}^f + E_t(r_{t+1}^i)$

# Fundamentals of Asset Pricing: *m*-Pricing

- **Price** of any risky asset $i \in \{1, ..., N\}$:

$$p_t^i = \mathrm{E}_t\left[\frac{M_{t+1}}{M_t}\left(d_{t+1}^i + p_{t+1}^i\right)\right] = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty}\frac{M_s}{M_t}d_s^i\right]$$

$$= \mathrm{E}_t\left[m_{t+1}\left(d_{t+1}^i + p_{t+1}^i\right)\right] \quad = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty} m_{t,s}d_s^i\right]$$

- **Notation**: $m_{t,s} = \frac{M_s}{M_t}$ and $m_{t+1} = m_{t,t+1} = \frac{M_{t+1}}{M_t}$
  - People denote both $M$ and $m$ as state price deflators
  - In 1-period models, usually no need to distinguish
- **Expected returns**: $1 = \mathrm{E}_t[m_{t+1}\left(1 + r_{t+1}^f + r_{t+1}^i\right)]$ **i.e.,**
  - **risk-free**: $1 + r_{t+1}^f = \frac{1}{\mathrm{E}_t[m_{t+1}]}$ **and risky** $0 = \mathrm{E}_t[m_{t+1}\, r_{t+1}^i]$
- **Covariances matter**: $\mathrm{E}_t[r_{t+1}^i] = (1 + r_{t+1}^f)\mathrm{Cov}_t[-m_{t+1}, r_{t+1}^i]$
  - Intuition?

# Deeper Answer

**Question:** What determines prices and expected returns?
**Deeper answer from asset pricing without frictions:**
$m$ **+ how it is determined**

- 3 frameworks for determining $m$ (see book by Duffie (2010)):

| | Condition | $m$ | Focus of |
|---|---|---|---|
| A. | No arbitrage | Must exist, can be inferred from prices | Derivatives |
| B. | Agent optimality | Agent's marginal utility | Institutional |
| C. | Equilibrium | Representative agent's marginal utility | Macro-finance |

- So state price deflators depend on
  - A. related assets (e.g., underlying stock for option) or factors
  - B. agent's marginal utility of consumption
    - $M_t = \beta^t u'(c_t)$ and $m_{t+1} = \beta u'(c_{t+1})/u'(c_t)$
  - C. aggregate consumption and all utilities aggregated with Pareto weights, $\lambda$
    - $M_t = \beta^t u'_\lambda(c_t)$ and $m_{t+1} = \beta u'_\lambda(c_{t+1})/u'_\lambda(c_t)$

# Asset Pricing with Frictions

**Question:** What determines prices and expected returns?
**Answer from asset pricing with frictions:**

- $m$ **may not exist**
  - "No-arbitrage condition" can break for "paper profits"
- We can still look at implications of
  A. no arbitrage *net of frictions*
  B. agent optimality
  C. equilibrium
- We come back to this in the last lecture

# State Prices and Betas

See Cochrane (2009) ch. 6, although my derivations are different

# $m$-Pricing Implies $\beta$-Pricing

- Recall: $m$-pricing means that, for all $i$,

$$1 = \mathrm{E}_t[m_{t+1}\,(1 + r^f_{t+1} + r^i_{t+1})]$$

or equivalently that

$$0 = \mathrm{E}_t[m_{t+1}\, r^i_{t+1}] \quad \text{and} \quad \mathrm{E}_t[m_{t+1}] = \frac{1}{1 + r^f_{t+1}}$$

- As we already saw, $m$-pricing implies

$$\mathrm{E}_t[r^i_{t+1}] = (1 + r^f_{t+1})\mathrm{Cov}_t[-m_{t+1}, r^i_{t+1}]$$

which is the same as

$$\boxed{\mathrm{E}_t[r^i_{t+1}] = \beta^i_t \lambda_t}$$

where $\beta^i_t = \frac{\mathrm{Cov}(-m_{t+1}, r^i_{t+1})}{\mathrm{Var}_t(m_{t+1})}$ and $\lambda_t = (1 + r^f_{t+1})\mathrm{Var}_t(m_{t+1}) > 0$

# Portfolios

To make tradable factors (considered next), investments, and many other things in asset pricing, we need portfolios

- Portfolio weights can be measured as
  - fractions of wealth invested in each asset, $x_t^i$
  - money (say, dollars) invested in each asset, $x_t^{\$,i}$
  - shares invested in each asset, $\bar{x}_t^i$
- Connection: $x_t^i w_t = x_t^{\$,i} = \bar{x}_t^i p_t^i$
  - Each can be useful in different circumstances
- **I focus on fractions, $x_t^i$, from now on**
- Here, $w_t$ is the investor's wealth, which evolves as

$$w_{t+1} = \sum_i x_t^i w_t (1 + r_{t+1}^f + r_{t+1}^i) + (w_t - \sum_i x_t^i w_t)(1 + r_{t+1}^f)$$
$$= w_t \sum_i x_t^i \, r_{t+1}^i + w_t (1 + r_{t+1}^f) = w_t (1 + r_{t+1}^f + x_t' r_{t+1})$$

- **Portfolio's excess return, $\frac{w_{t+1}}{w_t} - 1 - r_{t+1}^f$, is super simple:**

$$\boxed{x_t' r_{t+1}}$$

# Portfolios and Excess Returns

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$$x_t' r_{t+1}$$

- What is the interpretation of
  - $\sum_i x_t^i = 1$?
  - $\sum_i x_t^i < 1$?
  - $\sum_i x_t^i > 1$?

# Portfolios and Excess Returns

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- Any linear combination of excess returns is an excess return
  - An excess return can be seen as a self-financing strategy (long \$1 in the risky asset, financing by borrow \$1)$\rightarrow$so are linear combinations
- Example:
  - If we run regression $r_t^i = \alpha + \beta^i r_t^{Mkt} + \varepsilon_t$
  - then $r_t^i - \beta^i r_t^{Mkt} = \alpha + \varepsilon_t$ is an excess return of a hedged position
- Side note: compounding excess returns:
  - $\prod_t (1 + r_t^f + r_t^i) - \prod_t (1 + r_t^f)$
  - Not $\prod_t (1 + r_t^i)$
  - Fine to use $\sum_t r_t^i$ for illustration (but it is not a cumulative return)

# $m$-Pricing Implies $\beta$-Pricing with Tradable Factor

- If there is $m$-pricing, there is no arbitrage
- When there is no arbitrage
  - we can construct a mean-variance frontier, where
  - the tangency portfolio, $r^*$, is risky, $\mathrm{Var}_t(r^*_{t+1}) > 0$
    - (if there is arbitrage, $\mathrm{Var}_t(r^*_{t+1}) = 0$, so we cannot compute $\beta^i_t = \frac{\mathrm{Cov}(r^*_{t+1}, r^i_{t+1})}{\mathrm{Var}_t(r^*_{t+1})}$)

**Result**

*For any $a > 0$, the portfolio, $x_t = a\mathrm{Var}_t(r_{t+1})^{-1}E_t(r_{t+1})$, is mean-variance efficient. The "tangency portfolio" corresponds to a choice of $a$ such that portfolios weights add up to one, $1'x_t = 1$. For any $a$, the portfolio return, $r^*_{t+1} = x'_t r_{t+1}$, yields beta-pricing:*

$$E_t[r^i_{t+1}] = \beta^i_t \lambda_t$$

*for all $i$, where $\beta^i_t = \frac{Cov(r^*_{t+1}, r^i_{t+1})}{\mathrm{Var}_t(r^*_{t+1})}$ and $\lambda_t = E_t(r^*_{t+1})$*

# $m$-Pricing Implies $\beta$-Pricing with Tradable Factor, cont.

- **Proof:**
  - Look for tangency portfolio $x \in \mathbb{R}^N$ with excess return $r^x_{t+1} = x' r_{t+1}$ that maximizes the following, for any $\gamma > 0$

$$\max_x \mathrm{E}_t(x' r_{t+1}) - \frac{\gamma}{2}\mathrm{Var}_t(x' r_{t+1})$$

  - First order condition: $0 = \mathrm{E}_t(r_{t+1}) - \gamma \mathrm{Var}_t(r_{t+1})x$
  - Solution: $x = \frac{1}{\gamma}\mathrm{Var}_t(r_{t+1})^{-1}\mathrm{E}_t(r_{t+1})$
  - Note that $\mathrm{E}_t(r_{t+1}) = \gamma \mathrm{Var}_t(r_{t+1})x$ so

$$\mathrm{E}_t(r^i_{t+1}) = \beta^i_t \gamma \mathrm{Var}_t(r^x_{t+1})$$

    where $\beta^i_t = \frac{e_i'\mathrm{Var}_t(r_{t+1})x}{\mathrm{Var}_t(r^x_{t+1})} = \frac{\mathrm{Cov}_t(e_i' r_{t+1}, r_{t+1}' x)}{\mathrm{Var}_t(r^x_{t+1})} = \frac{\mathrm{Cov}_t(r^i_{t+1}, r^x_{t+1})}{\mathrm{Var}_t(r^x_{t+1})}$
  - Using this relation for $x$ yields $\mathrm{E}_t(r^x_{t+1}) = \gamma \mathrm{Var}_t(r^x_{t+1})$ so we have the desired result

$$\mathrm{E}_t(r^i_{t+1}) = \beta^i_t \mathrm{E}_t(r^x_{t+1})$$

# $m$-Pricing Implies $\beta$-Pricing with Tradable Factor, cont.

- The tangency portfolio in the mean-variance diagram
  - How do you draw this?
  - What happens when you vary the risk aversion $\gamma$ from the proof?
  - Which $\gamma$ corresponds to a notional exposure of 1?
  - Why does the beta relation look the same for different $\gamma$'s?
- Note: in an equilibrium model, $\gamma$ clearly affects equilibrium prices and expected returns (but, again, the beta relation still looks the same)

# Projecting $m$ on Assets: Mimicking Portfolio

Alternative characterization of the tradable factor with $\beta$-pricing

- The projection of $-m$ on the tradable securities
- I.e., the regression of $-m$ on $r$:

$$-m_{t+1} = a_t + b_t' r_{t+1} + \varepsilon_{t+1}$$

where $\mathrm{E}_t(\varepsilon_{t+1}) = 0$ and $\mathrm{E}_t(\varepsilon_{t+1} r_{t+1}) = 0$

- General formula for regression coefficients $a_t$ and $b_t$:

$$\begin{aligned} b_t =& \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{Cov}_t(r_{t+1}, -m_{t+1}) \\ a_t =& \mathrm{E}_t(-m_{t+1}) - b' \mathrm{E}_t(r_{t+1}) \end{aligned}$$

- Using properties of $m_{t+1}$:

$$\begin{aligned} b_t =& \frac{1}{1 + r_{t+1}^f} \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}] \\ a_t =& -\left( \frac{1}{1 + r_{t+1}^f} + b_t' \mathrm{E}_t(r_{t+1}) \right) \end{aligned}$$

# Projecting $m$ on Assets: Mimicking Portfolio, cont.

We have shown:

**Result**

*The regression of $-m_{t+1}$ on the assets, $r^b_{t+1} = b_t' r_{t+1}$, is proportional to the tangency portfolio since $b_t = \frac{1}{1+r^f_{t+1}} \mathrm{Var}_t(r_{t+1})^{-1} E_t[r_{t+1}]$.*

**Result**

*The regression of $m$ on the assets, $r^b_{t+1} = b_t' r_{t+1}$, has $\beta$-pricing (because all mean-variance efficient portfolios have $\beta$-pricing)*

**Result**

*Any state price deflator can be written*

$$m_{t+1} = \frac{1}{1+r^f_{t+1}} \left(1 - E_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1}(r_{t+1} - E_t[r_{t+1}])\right) + \varepsilon_{t+1}$$

*and the minimum-variance sdf has $\varepsilon_{t+1} = 0$.*

# Projecting $m$ on Assets: Mimicking Portfolio, cont.

- The minimum variance sdf is the projection of the "true" state price on the marketable space, meaning that this $m$ is a function of returns:

$$m_{t+1} = \frac{1}{1+r_{t+1}^f}\left(1 - \mathsf{E}_t[r_{t+1}]'\mathrm{Var}_t(r_{t+1})^{-1}(r_{t+1} - \mathsf{E}_t[r_{t+1}])\right)$$

- If the market is complete, this is the unique state price
  - Otherwise, there are many $m$'s that price the assets (i.e., many choices of $\varepsilon_{t+1}$ uncorrelated with $r_{t+1}$)

- "Mimicking portfolio": $\frac{1}{1+r_{t+1}^f}\mathsf{E}_t[r_{t+1}]'\mathrm{Var}_t(r_{t+1})^{-1}r_{t+1}$
  - The part of $-m_{t+1}$ that yields beta pricing

# Hansen-Jagannathan Bound

- Sharpe ratio and any asset $i$: $SR_t(r^i_{t+1}) = \frac{\mathrm{E}_t(r^i_{t+1})}{\sigma_t(r^i_{t+1})}$

**Result (Hansen and Jagannathan (1991))**

*The Sharpe ratios of the tangency portfolio, $r^*$, and any other portfolio, $r^j$, are bounded by the volatility of the sdf:*

$$SR_t(r^j_{t+1}) \le SR_t(r^*_{t+1}) = \min_{m:\ m \text{ prices } r_{t+1}} \frac{\sigma_t(m)}{E_t(m)} \le \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} = (1+r^f_{t+1})\sigma_t(m_{t+1})$$

- Proof
  - The variance of $m$ using the projection from previous slides:

$$\mathrm{Var}_t(m_{t+1}) = (\frac{1}{1+r^f_{t+1}})^2 \mathrm{E}_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}] + \mathrm{Var}_t(\varepsilon_{t+1})$$

  - Recall: $\mathrm{E}_t(m_{t+1}) = \frac{1}{1+r^f_{t+1}}$ so

$$\frac{\sigma_t(m_{t+1})}{\mathrm{E}_t(m_{t+1})} = \sqrt{\mathrm{E}_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}] + (1+r^f_{t+1})^2 \mathrm{Var}_t(\varepsilon_{t+1})}$$

  - SR of tangency:

$$\frac{\mathrm{E}_t(r^*_{t+1})}{\sigma_t(r^*_{t+1})} = \frac{b_t' \mathrm{E}_t(r_{t+1})}{\sqrt{b_t' \mathrm{Var}_t(r_{t+1}) b_t}} = \frac{\mathrm{E}_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}]}{\sqrt{\mathrm{E}_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}]}} = \sqrt{\mathrm{E}_t[r_{t+1}]' \mathrm{Var}_t(r_{t+1})^{-1} \mathrm{E}_t[r_{t+1}]}$$

# $\beta$-Pricing Implies $m$-Pricing

**Result**

*If there exists portfolio with excess return $r^*$ s.t. for all $i$*

$$E_t[r^i_{t+1}] = \beta^i_t \lambda_t$$

*where* $\beta^i_t = \frac{Cov(r^*_{t+1}, r^i_{t+1})}{\mathrm{Var}_t(r^*_{t+1})}$ *and* $\lambda_t = E_t(r^*_{t+1})$, *then*

$$m_{t+1} = \frac{1}{1 + r^f_{t+1}} \left( 1 - (r^*_{t+1} - E_t(r^*_{t+1})) \frac{E_t(r^*_{t+1})}{\mathrm{Var}_t(r^*_{t+1})} \right)$$

*satisfies* $0 = E_t[m_{t+1}\, r^i_{t+1}]$ *and* $E_t(m_{t+1}) = 1/(1 + r^f_{t+1})$

- Proof:
  - (Inspection) Just check
  - (Constructive) Assume $m = a + br^*$ and solve for $a$ and $b$

# Equilibrium $\beta$-Pricing and Market Efficiency

# $\beta$-Pricing in Equilibrium: CAPM

- If all investors want to maximize their Sharpe ratio
  - Each chooses a combination of the tangency portfolio and $r^f$
  - Therefore the market portfolio is the tangency portfolio, $r^{MKT} = r^*$
- So, we see

**Result**

*If all investors maximize SR, the market portfolio, $r^{MKT}$, has beta pricing*

$$E_t[r^i_{t+1}] = \beta^i_t \lambda_t$$

*where* $\beta^i_t = \frac{Cov(r^i_{t+1}, r^{MKT}_{t+1})}{\mathrm{Var}_t(r^{MKT}_{t+1})}$ *and* $\lambda_t = E_t(r^{MKT}_{t+1})$

*Further, the following sdf prices all assets*

$$m_{t+1} = \frac{1}{1 + r^f_{t+1}} \left( 1 - (r^{MKT}_{t+1} - E_t(r^{MKT}_{t+1})) \frac{E_t(r^{MKT}_{t+1})}{\mathrm{Var}_t(r^{MKT}_{t+1})} \right)$$

# Is $\beta$-Pricing the Same as Market Efficiency/ Rationality? I

- Rational investors imply the existence of $m$- and $\beta$-pricing
- Is the reverse also true? I.e., does $\beta$-pricing imply rationality?

# Is $\beta$-Pricing the Same as Market Efficiency/ Rationality? I

- Rational investors imply the existence of $m$- and $\beta$-pricing
- Is the reverse also true? I.e., does $\beta$-pricing imply rationality?
- No
  - As long as there is no arbitrage, there is $\beta$-pricing
  - Recall that the tangency portfolio always has $\beta$-pricing
  - For example:
    - Suppose behavioral investors do crazy things that mess up market prices
    - Rational investors (arbitrageurs) eliminate pure arbitrage and partly correct prices, but their trades are limited by risk aversion and other constraints
    - Equilibrium prices remain partly inefficient: irrational trades create noise in prices and information is only partially incorporated into prices
    - Nevertheless, the tangency portfolio has $\beta$-pricing
- So testing rationality requires $\beta$-pricing for an economically meaningful factor, a structural model, or other types of tests
- Same goes for multi-factor models – see discussion of Fama-French factors

# Is $\beta$-Pricing the Same as Market Efficiency/ Rationality? II

- Rational pricing also implies that SRs are bounded (Hansen-Jagannathan bound)
- In the reverse also true? I.e., bounded SR imply rationality?
- No
  - Bounded SR just means that arbitrage has removed the best investments, i.e., the tangency portfolio is not so attractive that arbitrageurs trade even more
  - Can be true whether or not prices are affected by behavioral effects
  - If we think that near-arbitrage (=high SR) cannot exist, then factors that are important for expected returns must also be important for the variance-covariance matrix
  - Tests of characteristics-vs-covariances are not tests of rationality, but tests of whether near-arbitrage exists (before transaction costs and funding constraints)
  - See Kozak et al. (2018) for further results

# Multi-Factor $\beta$-Pricing: Factor Models

# What is a Factor Model?

- Excess returns of $N$ assets, $r_t^i$, and $K$ factors, $f_t^k$,

$$r_t^i = \alpha^i + \sum_k \beta^{i,k} f_t^k + \varepsilon_t^i = \alpha^i + \beta^i f_t + \varepsilon_t^i$$

where $\mathrm{E}(\varepsilon_t^i) = 0$ and $\mathrm{E}(f_t^k \varepsilon_t^i) = 0$. In vector form:

$$\underbrace{r_t}_{N\times 1} = \underbrace{\alpha}_{N\times 1} + \underbrace{\beta}_{N\times K} \underbrace{f_t}_{K\times 1} + \underbrace{\varepsilon_t}_{N\times 1}$$

- NB: This is just a statistical model – the economics comes from testing properties of the model (the parameters, $\alpha, \beta$, the factors and their explanatory power for returns, the risks, etc.)

# Why Multi-Factor Models?

We can always use a one-factor model for $\beta$-pricing

- As shown above, the tangency portfolio, $r^*$, gives $\beta$-pricing
- So why do people often use multi-factor models?

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1. "Building" mimicking portfolio from *observable* characteristics

- Start with factors constructed based on characteristics, $f_t^1, \dots, f_t^K$, such as market, size, and value (Fama-French)
- Combining these can yield a higher Sharpe ratio
- Say that the highest SR arises with weights $\sum_k b_k f_t^k$
- If this combination has the highest SR among *all* portfolios, then we have beta pricing with $r_t^* = \sum_k b_k f_t^k$, i.e.,

$$\mathrm{E}_t[r_{t+1}^i] = \frac{\mathrm{Cov}(r_{t+1}^*, r_{t+1}^i)}{\mathrm{Var}_t(r_{t+1}^*)}\lambda_t = \frac{\mathrm{Cov}(\sum_k b_k f_{t+1}^k, r_{t+1}^i)}{\mathrm{Var}_t(r_{t+1}^*)}\lambda_t = \sum_k \beta_t^{i,f^k}\lambda_t^k$$

where $\lambda_t^k = \frac{\mathrm{Var}_t(f_{t+1}^k) b_k \lambda_t}{\mathrm{Var}_t(r_{t+1}^*)}$

- If so, $\sum_k b_k f_t^k$ may proxy for marginal utilities etc.

# Why Multi-Factor Models, continued

2. APT (Ross (1976))
   - Notice that the variance-covariance matrix, $\mathrm{Var}_t(r_{t+1})$, is generated by a few factors plus (relatively) idiosyncratic risk
   - Idiosyncratic risk: largely diversified away w/many assets
   - So to rule out near-arbitrage, expected returns must line up with the factors that generate the variance
3. ICAPM (Merton (1973))
   - Investors are forward looking and think long term
   - They like stocks that pay off when the market in general is low (like CAPM), but also when future investment opportunities are bad (depending on parameters), for example when
     - the highest Sharpe ratio is low, $SR_{t+1}(r^*_{t+2})$
     - the real interest rate is low, $r^f_{t+2}$

# Factor Models: Statistical and Economic Implications

See Cochrane (2009) ch. 12 for more details and test statistics

# Statistical Implications of a Factor Model

Factor model: $r_t = \alpha + \beta f_t + \varepsilon_t$

- Expected returns

$$\boxed{\mathsf{E}(r_t) = \alpha + \beta \mathsf{E}(f_t)}$$

- Variance-covariance matrix

$$\boxed{\mathrm{Var}(r_t) = \beta \mathrm{Var}(f_t) \beta' + \mathrm{Var}(\varepsilon_t^i)}$$

- Regression coefficients

$$\boxed{\beta = \Sigma_{rf} \Sigma_{ff}^{-1} \quad \text{and} \quad \alpha = \mathsf{E}(r_t) - \beta \mathsf{E}(f_t)}$$

where $\Sigma_{ff} = \mathrm{Var}(f_t)$ and $\Sigma_{rf} = \mathsf{Cov}(r_t, f_t)$

- For asset $i$ $\beta^i = \Sigma_{r^i f} \Sigma_{ff}^{-1}$ and $\alpha^i = \mathsf{E}(r_t^i) - \beta^i \mathsf{E}(f_t)$
- If $r, f$ jointly normal, the regression is the conditional mean

$$\mathsf{E}(r_t | f_t) = \mathsf{E}(r_t) + \Sigma_{rf} \Sigma_{ff}^{-1} (f_t - \mathsf{E}(f_t)) = \alpha + \beta f_t$$

but, more generally, it is just the linear projection

# Statistical Implications: Deriving Regression Coefficients

- Factor model

$$r_t = \alpha + \beta f_t + \varepsilon_t$$

- Subtract the mean on both sides

$$r_t - \mathrm{E}(r_t) = \beta(f_t - \mathrm{E}(f_t)) + \varepsilon_t$$

- Multiply both sides by $(f_t - \mathrm{E}(f_t))'$

$$(r_t - \mathrm{E}(r_t))(f_t - \mathrm{E}(f_t))' = \beta(f_t - \mathrm{E}(f_t))(f_t - \mathrm{E}(f_t))' + \varepsilon_t(f_t - \mathrm{E}(f_t))'$$

- Taking the expected value

$$\Sigma_{rf} = \beta \Sigma_{ff}$$

- Conclusion

$$\beta = \Sigma_{rf}\Sigma_{ff}^{-1} \quad \text{and} \quad \alpha = \mathrm{E}(r_t) - \beta \mathrm{E}(f_t)$$

# Economic Implications of Factor Models

- Suppose that we have a single factor $f$
- Want to examine whether $f$ "prices" all the assets, $r^1, ..., r^N$ (unconditonal model)
  - That is, for all $i$, $\mathrm{E}[r_t^i] = \beta^i \lambda$ where $\beta^i = \frac{\mathrm{Cov}(r_t^i, f_t)}{\mathrm{Var}(f_t)}$
  - This restriction can also be written as

$$r_t^i = \beta^i \lambda + \beta^i (f_t - \mathrm{E}[f_t]) + \varepsilon_t$$

  - We run a time series regression of $r_t^i$ on $f_t$:

$$r_t^i = \alpha^i + \beta^i f_t + \varepsilon_t$$

- Slope is right, $\beta^i = \frac{\mathrm{Cov}(r_t^i, f_t)}{\mathrm{Var}(f_t)}$, and intercept is $\alpha^i = \beta^i(\lambda - \mathrm{E}[f_t])$

When the factor is

1. tradable, $\lambda = \mathrm{E}(f_t)$, so $\alpha^i = 0$ for all $i$
2. non-tradable, where $f_t = m_t$ and iid: $\lambda = (1 + r^f)\mathrm{Var}(m_t)$ (see slide "$m$-Pricing Implies $\beta$-Pricing"), so $\alpha^i$ need not be 0
3. non-tradable, where $f_t = a + bm_t$: even less reason for $\alpha^i = 0$

Here, 1 yields a time series test of model, while 2 and 3 mean that model must be tested using a cross-sectional regression (see next class)

# Economic Implications of Tradable Factors

- We have observed tradable factors $f$
- We can estimate factor model using a time series regression
- Want to examine whether $f$ "prices" all the assets, $r^1, ..., r^N$
  - I.e., there exists $b$ s.t. $r_t^* = \sum_k b_k f_t^k = b' f_t$ has $\beta$-pricing
- If $f$ prices the assets and we run time series regression $r_t = \alpha + \beta f_t + \varepsilon_t$, then we must have

$$\boxed{\alpha = 0}$$

  - This is obvious when there is 1 factor, see previous slide
    - With 1 tradable factor, this is just the definition of $\beta$-pricing
  - With multiple factors: see next slide
    - Intuitively, this is because expected excess returns are purely driven by factor exposures
    - Said differently, if you hedge out factor exposures, then the expected return is zero

# Multiple Tradable Factors—Why Alpha is Still Zero

Suppose there exists $b$ s.t. $r_t^* = \sum_k b_k f_t^k = b' f_t$ has $\beta$-pricing

- Then

$$\begin{aligned} \mathsf{E}(r_t^i) &= \frac{\mathrm{Cov}(b' f_t, r_{t+1}^i)}{\mathrm{Var}_t(b' f_t)} \mathsf{E}_t(b' f_t) \\ &= \frac{\mathsf{E}_t(b' f_t)}{\mathrm{Var}_t(b' f_t)} \Sigma_{r^i f} b \\ &= \frac{\mathsf{E}_t(b' f_t)}{\mathrm{Var}_t(b' f_t)} \Sigma_{r^i f} \Sigma_{ff}^{-1} \Sigma_{ff} b \\ &= \beta^i \bar{b} \end{aligned}$$

where $\bar{b} = \frac{\mathsf{E}_t(b' f_t)}{\mathrm{Var}_t(b' f_t)} \Sigma_{ff} b$ and using $\beta^i = \Sigma_{r^i f} \Sigma_{ff}^{-1}$

- Since $f^k$ is itself tradable with a beta equal to $e^k = (0, ..., 0, 1, 0, ..., 0)$, we conclude that $\bar{b}^k = E(f_t^k)$
- So: $\alpha^i + \beta^i \mathsf{E}(f_t) = \mathsf{E}(r_t^i) = \beta^i \bar{b} = \beta^i E(f_t)$
- In other words, we see that, for all $i$,

$$\alpha^i = 0$$

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