Exercises for Empirical Asset Pricing

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Not for Distribution

Please Do Not Ruin this Exercise for Future Students

Solution to Exercise 1: Beta-Dollar-Neutral

1. The interpretation of x is a beta-dollar-neutral portfolio that is as similar as possible to the original portfolio, y.

2. Define a vector of Lagrange multipliers as $2\lambda \in \mathbb{R}^{2\times 1}$ and consider the Lagrangian:

   $$L = (x-y)'(x-y) - 2(x'B - 0)\lambda$$

   The first-order condition is

   $$0 = (x-y) - B\lambda$$

   so

   $$x = y + B\lambda$$

   To satisfy the constraint, we must have

   $$0 = x'B = y'B + \lambda' B'B$$

   so

   $$\lambda = -(B'B)^{-1}B'y$$

   where $B'B$ has full rank.[1] So, we have

   $$x = (I - B(B'B)^{-1}B')y$$

   So, we see that x is the projection of y on the space orthogonal to B.

3. This strategy y is a form of betting-against-beta strategy because it goes long stocks with low betas while shorting those with high betas. This strategy seeks to exploit that the security market line is too low empirically – that is, stocks sorted by beta have delivered similar average returns despite having different betas. In other words, low-beta stocks have positive CAPM alphas.

   The notional and market exposures of the strategy are:

   $$\begin{aligned} \text{notional exposure} =& y'\vec{1} = (a\vec{1} - \beta)'\vec{1} = N(a - \bar{\beta}) \\ \text{market exposure} =& y'\beta = (a\vec{1} - \beta)'\beta = N(a\bar{\beta} - \beta'\beta/N) \end{aligned}$$

[1] To see the full rank, note that

$$B'B = \begin{pmatrix} 1' \\ \beta' \end{pmatrix} (1, \beta) = N \begin{pmatrix} 1 & \bar{\beta} \\ \bar{\beta} & \beta'\beta/N \end{pmatrix}$$

So, this matrix has full rank as long as the determinant is not zero, that is, $0 \neq 1 \times \beta'\beta/N - \bar{\beta}^2$. This holds, since by Jensen's inequality $\bar{\beta}^2 < \beta'\beta/N$ (except if all betas are the same, in which case beta and dollar neutrality is the same, so we would not have this problem).

When $\bar{\beta} = 1$, we see that $a = 1$ means that the strategy is dollar-neutral (no notional exposure) while $a = \beta'\beta/N$ means that the strategy is beta neutral. Clearly, no choice of a makes the strategy both dollar and beta neutral.

The solution to part 2 with this strategy is

$$x = (I - B(B'B)^{-1}B')B \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (B - B) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

So once we make y beta-dollar neutral, there is nothing left.

4. Consider the cross-sectional regression of y on B:

$$y = B\theta + \varepsilon$$

The OLS estimate of the regression coefficient is $\hat{\theta} = (B'B)^{-1}B'y$. We see that $B\hat{\theta} = B(B'B)^{-1}B'y$ is the projection of y on the span of B with the projection matrix $B(B'B)^{-1}B'$.

The estimated residual is $\hat{\varepsilon} = y - B\hat{\theta} = (I - B(B'B)^{-1}B')y$, which is the projection of y on the space orthogonal to B with the projection matrix $I - B(B'B)^{-1}B'$.

We see that $\hat{\varepsilon}$ is the same as the solution to 2. Recall that the solution to 2 is to the "closest" portfolio to y that is orthogonal to B (i.e., beta-dollar neutral) — i.e., the projection of y on the space of portfolio orthogonal to B, which is exactly the same as $\hat{\varepsilon}$.

5. A strategy y can be decomposed into two components, the projection on B and the projection on the orthogonal complement. The former is the part that cannot be made beta-dollar-neutral, the latter is the part that already is beta-dollar-neutral. Hence, a strategy can be made beta-dollar-neutral if the orthogonal part is not trivial, $x = (I - B(B'B)^{-1}B')y$. More specifically, if $E_t(x'r_{t+1}) > 0$.