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# Informed manipulation ☆

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**Abstract**

In asymmetric information models of financial markets, prices imperfectly reveal the private information held by traders. Informed insiders thus have an incentive not only to trade less aggressively but also to manipulate the market by trading in the wrong direction and undertaking short-term losses, thereby increasing the noise in the trading process. In this paper we show that when the market faces uncertainty about the existence of the insider in the market and when there is a large number of trading periods before all private information is revealed, long-lived informed traders will manipulate in every equilibrium.


*JEL classification:* D82; G12

*Keywords:* Insider trading; Manipulation

## 1. Introduction

In asymmetric information models of financial markets, prices imperfectly reveal the private information held by traders. An informed trader is thus hurt precisely because his trading partially reveals the private information that allowed him to trade profitably in the first place. Kyle [15] shows that informed insiders strategically

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choose to trade less aggressively in a situation where their trading affects the equilibrium price than in the situation where it does not. In other words, the absolute value of the informed trader's position on an asset is likely to be smaller in a strategic context when the insider is informationally large in the market.

However, in the linear equilibrium of Kyle's model, the informed trader's trading strategy is monotonic in his information in the sense that he buys the asset when the asset is undervalued given his information (i.e., when the value of the asset given his information is more than the (expected) price in the market for buying the asset), and sells the asset when the asset is overvalued (when the value of the asset given his information is less than the (expected) price in the market for selling the asset). To quote from [15], the unique linear equilibrium of the model, "rules out a situation in which the insider can make unbounded profits by first destabilizing prices with unprofitable trades made at the *n*th auction, then recouping the losses and much more with profitable trades at future auctions" (p. 1323).

In this paper we consider precisely this kind of manipulative strategic trading by informed insiders. In a dynamic setting, not only might informed traders want to hide their trades by trading less aggressively, they might also find it in their interest to confuse other market participants by trading in the "wrong" direction for short-term losses but long-term profits. For example, an insider who knows that the prospects of a certain asset are not good might actually start buying the asset in order to drive its price up and then sell it without its price falling too fast.

We investigate this possibility of manipulative trading in a Glosten–Milgrom [9] type of bid–ask model[1] where, in every period, competitive market makers first post bid and ask prices and then the trader trades, so that the trader knows the price at which any order submitted in that period will be executed. We show that when the market faces uncertainty about the existence of an informed trader in the market, and when the number of periods of trading before all private information is revealed is sufficiently long, every equilibrium involves manipulation by the informed trader. We also provide a three-period simplified example of the model, where the equilibrium involves manipulation.

We prove our result by obtaining a bound on the number of periods of trading, above which all equilibria must necessarily be manipulative. While this bound depends on the prior probability of noise trading in the market, it is independent of the precise specification of noise trader strategies, allowing in particular for history-dependent strategies.

Manipulative trading benefits the trader whenever he is informed—his profits are higher than the benchmark where he is replaced by many short-lived informed traders, who buy when they have good news and sell when they have bad news. Since market makers earn zero expected profits, the gains of the informed trader are transformed into losses for the noise traders. By creating liquidity, manipulation by one type of the informed trader makes prices less informative and less responsive to trades and so reduces the adverse selection pressure on all types of informed traders, thereby increasing their profits.

[1] In [7] we prove the identical result for a Kyle type of market order model.

In Section 1.1, we discuss and relate this paper to the literature on manipulation. In Sections 2 and 3, we introduce a simple bid–ask model in the spirit of Glosten and Milgrom [9] and prove that, when information is sufficiently long-lived, in every equilibrium, some type of the informed trader will manipulate. In Section 4, we provide a simple three-period example that illustrates our result. Section 5 concludes and the appendix contains some of the proofs.

### 1.1. Review of the literature

The seminal paper on strategic trading by an informed insider with long-lived private information is by Kyle [15]. Kyle demonstrates the existence of a unique linear equilibrium where the insider trades in the direction of his information. In our bid–ask model we show that the dynamic trading strategy of the insider is very different from that in Kyle's model as it involves manipulation. Chakraborty and Yılmaz [7] consider a discrete Kyle-type of market order version of our model and show that the identical result on the necessity of manipulation for long horizons obtains in that model as well.

A number of other authors have considered the definition and possibility of manipulation. Jarrow [11] formulates sufficient conditions for manipulation to be unprofitable. These sufficient conditions are properties of the "reduced-form" price function.[2] We set up a model of strategic trading where equilibrium prices at each date are equal to the expected value of the asset being traded, given the market's expectations about the insider's strategy. However, whereas [11] looks at manipulation by an uninformed trader, in this paper we look at informed manipulation.

Allen and Gale [1] propose a classification scheme for models of manipulation. They also provide a model of strategic trading in which some equilibria involve manipulation. Using the classification scheme proposed by Allen and Gale [1], both our model and their model are examples of pure trade-based manipulation, where the informed trader does not announce any information (information-based manipulation) or take any actions (action-based manipulation), except for those that involve trading the asset. However, unlike our model, they consider manipulation by an uninformed trader. In their model, manipulation by the informed trader is not profitable, although manipulation by an uninformed trader *is* profitable in the presence of certain restrictions on the strategy of the informed trader.

Allen and Gorton [2] also consider a model of pure trade-based uninformed manipulation in which an asymmetry in buys and sells in noise traders' trades creates the possibility of manipulation. In every equilibrium of their model, the uninformed manipulator makes zero profits.

Brunnermeier [5] considers manipulation in a Kyle-type of market order model. Since an informed trader, apart from possessing private information about the fundamentals, also has private information regarding (a) the extent to which his

[2] A reduced-form price function is one in which the response of the market to the large trader's trades has already been incorporated, so that it is a function only of the large trader's trades.

information is already reflected in the current price and (b) future public announcements about the fundamentals, he has an incentive to trade very aggressively early on, and then partially unwind his position later for speculative profits. Brunnermeier's work is different from the present paper (as well as from a number of other papers in the literature) because it does not rely on the possibility of the insider's trades being revealed to the market. On the contrary, it utilizes the fact that the insider has better information about his own trades than does the market and also the fact that an informed trader's private information about the asset is also an imperfect signal of any future public announcement about the value of the asset. However manipulation, in the sense of undertaking trades that involve short-term losses, does not occur in [5].

Kumar and Seppi [13], in a situation with multiple risky assets, show how an uninformed trader makes profits when an information event takes place. The manipulator, by a sequence of trades in the spot and futures markets, is able to profit knowing that informed traders are about to trade based on the private information they receive. This example requires a "cash-settled" futures contract along with the underlying asset. Kyle [14] presents another model of market manipulation in futures markets: a large trader can undetectably acquire a large position and then manipulate by implementing a profitable squeeze.

There are a number of papers that consider the nature and possibility of non-trade-based manipulation. Benabou and Laroque [4] consider an information-based, reputational model of manipulation in which an insider sometimes tells the truth to build a reputation and sometimes lies and profitably manipulates. Both Benabou and Laroque [4] and the present paper are therefore related to the large literature on reputation in repeated games. However, there are at least two important differences. First, Benabou and Laroque allow for costless messages with which the manipulator builds (or loses) a reputation, while they assume that the insider is a price-taker and can trade without being detected so that only his announcements (but not his trades) affect prices. In contrast, we do not consider the possibility of messages—the only way that the manipulator can build a reputation is by undertaking potentially costly (or profitable) trades. Second, Benabou and Laroque consider a situation where the private information about the asset is revealed at the end of every period so that the only reputation to build is that about the rationality of the dynamic trader. In contrast, in our model the private information of the asset is not revealed till the end of play so that the updating process is not only on the type (rationality) of the dynamic trader but also on the underlying value of the asset.

Bagnoli and Lipman [3] develop a model of action-based manipulation where the manipulator pools with someone who can take an action that alters the value of the firm. In their model, the manipulator takes a position, announces a takeover bid, and then unwinds his position. Vila [18] presents another model of action-based manipulation; here the manipulator pools with someone who buys stock prior to a takeover bid in which the value of the firm will be increased.

The literature on manipulation and disclosure laws explicitly considers the effect of the insider's trades being directly revealed to the market, owing to the presence of mandatory disclosure laws. Fishman and Hagerty [8] provide a one-period equilibrium model of profitable manipulation when an uninformed insider successfully misleads the market into thinking he is informed. The profitable opportunity arises owing to the mandatory disclosure of the insider's trades. John and Narayanan [12] and Huddart et al. [10] also look at the effect of mandatory disclosure laws on the insider's incentive to manipulate.

## 2. The model

We consider a market for one risky asset.[3] The long-term return or the fundamental value of the asset, $v$, is not known to all participants in the market. In particular, we assume $v \in V \equiv \{0, 1\}$, with the prior probability that $v = 1$ equal to $\lambda \in (0, 1)$.

One trader with private information repeatedly trades the asset. In each period, competitive uninformed market makers post-bid and ask prices, equal to the expected value of the asset conditional on the observed history of trades in equilibrium. The trader trades at those prices (possibly choosing not to trade). Trading happens for $T$ successive periods after which all private information is revealed.

The private information or type of the trader is denoted by $\theta \in \Theta \equiv \{0, 1, N\}$. When $\theta = 0$, the trader is informed and knows that the value of the asset is $v = 0$. When $\theta = 1$, the trader is informed and knows that the value of the asset is $v = 1$. When $\theta = N$, the trader is a noise trader and his trading is driven by exogenous (e.g., liquidity) motives. The existence of this last type of trader is meant to capture the notion that the market faces uncertainty regarding the existence of an informed trader who trades on the basis of his information.[4] Although there are three types of the trader, only one is actually chosen by nature for any given play of the game. We suppose that the prior distribution of $\theta$ is specified by

$$\Pr[\theta = 0 \mid v = 0] = \Pr[\theta = 1 \mid v = 1] \equiv \mu \in (0, 1), \tag{1}$$

while

$$\Pr[\theta = N \mid v = 0] = \Pr[\theta = N \mid v = 1] \equiv 1 - \mu. \tag{2}$$

That is, with probability $\mu$, all the trades have been submitted by the informed trader (whose trades are correlated with the fundamental value of the asset); while with probability $1 - \mu$, all the trades have been submitted by a noise trader (whose trades are uncorrelated with the fundamental value of the asset).

Let $t = 0, 1, \ldots,$ index time. The timing structure of the $T$-period trading game is as follows:

1. At date 0, nature chooses $v$ and $\theta$. The trader observes $\theta$.
2. In successive periods indexed by $t = 1, \ldots, T$, having observed the trades and prices up to date $t - 1$, the competitive market makers post a price

[3] We also assume that there is one riskless asset whose gross rate of return is normalized to 1.

[4] In the model that we develop here (with only one trader) the existence of the type $\theta = N$ is also necessary for profitable trade, and it prevents the market from collapsing.

for each possible trade, and the trader chooses his trade (possibly not trading).
3. In period $T+1$, the realization of $v$ is publicly disclosed.

## 2.1. Prices, strategies and payoffs

In each period, the trader can submit a demand equal to a buy order of size 1, or a sell order of size 1 or can choose not to trade. Let $E = \{-1, 0, 1\}$ denote the set of possible trades that can occur in any period, with $e$ its generic element. Thus, $e = 1$ denotes a buy order, $e = -1$ a sell order and $e = 0$, no trade. The restriction to unit-sized buy and sell orders is purely for simplicity, and all results generalize to the case where a finite number of possible trade sizes can be submitted. Let $E^t$ denote the $t$-fold Cartesian product of $E$; this is the set of possible $t$-period histories of trades. Let $e^t$ denote the generic element $\{e_1, \dots, e_t\}$ of $E^t$. Given a $t$-period history of trades $e^t$, we will denote the $(t+1)$-period history that is generated by a $(t+1)$th period trade $e_{t+1}$ as $\{e^t, e_{t+1}\}$. Let $E^0$ the null history of trades with $e^0$ its unique element. Denote by $\Delta(E)$, the set of probability distributions on $E$.

In every period $t$, market makers post an ask price and a bid price. The trader can choose to buy the asset at the ask price or sell the asset at the bid price or choose not to trade. Let $p_{1,t}$ be the ask price posted in period $t$ and $p_{-1,t}$, the bid price.[5] Let $p_t \equiv (p_{1,t}, p_{-1,t}) \in P \equiv [0,1]^2$. Denote by $P^t$ the set of possible $t$-period histories of bid and ask prices (i.e., the $t$-fold product of $P$) with $p^t$ denoting its generic element. Denote by $P^0$ the null history of prices with $p^0$ its unique element.

Let $H^t = (P \times E)^t$ denote the set of $t$-period histories of prices and trades, with $h^t$ its generic element. Let $H^0$ be the null history set with $h^0$ its unique element. Given a history $h^t$, the $(t+1)$ period history generated by prices $p_{t+1}$ and a trade $e_{t+1}$ will be denoted as $\{h^t, p_{t+1}, e_{t+1}\}$. Let $\mathbf{e}(h^t)$ be the history of trades associated with the $t$-period history $h^t \in H^t$ of trades and prices, $t = 0, \dots, T$. Finally, let $\bar{H} = \bigcup_{t=0}^{T-1} H^t$.

For each type of the trader, a trading strategy specifies a probability distribution over trades in period $t+1$, given an observed history of past prices and trades $h^t$, as well as the ask and bid prices $p_{t+1}$ that are posted in period $t+1$. A strategy for the trader is thus defined as a function $\sigma : \bar{H} \times P \to \Delta(E)$. Let $\sigma(e \mid h^t, p_{t+1})$ be the probability that $\sigma$ assigns to action $e$ given a history $h^t$ and $(t+1)$th period prices $p_{t+1}$, and let $\sigma(h^t, p_{t+1})$ be the trio $\{\sigma(e|h^t, p_{t+1})\}_{e \in E}$. Denote by $\Sigma$ the set of trading strategies for the trader. Each possible type $\theta \in \Theta$ of the trader chooses a trading strategy in $\Sigma$. We will denote by $\sigma_\theta \in \Sigma$, the trading strategy chosen by type $\theta$ of the trader.

In our model $\sigma_N$ is exogenous, and we impose the following two conditions on it:

1. $\exists c > 0$ s.t. $\min_{e \in E} \sigma_N(e \mid h^t, p_{t+1}) > c$ for all $h^t \in \bar{H}, p_{t+1} \in P, t = 0, \dots, T-1$.
2. For all $\hat{h}^t, \tilde{h}^t \in \bar{H}$, such that $\mathbf{e}(\hat{h}^t) = \mathbf{e}(\tilde{h}^t), \sigma_N(\hat{h}^t, p_{t+1}) = \sigma_N(\tilde{h}^t, p_{t+1})$ for all $p_{t+1}$. (N)

[5] It will be notationally convenient to define $p_{e,t}$ for all trades $e \in E$. In what follows, we will adopt the innocuous convention that $p_{0,t} \equiv 0$ for all $t$.

The first condition states that when $\theta = N$, the trader buys, sells, and does not trade with probability bounded away from zero for all histories. It is necessary to prevent profitable trading from stopping after some histories. The second condition states that $\sigma_N$ is independent of current and past prices, although it could depend on the history of trades.[6]

A price rule is defined as a function $\mathbf{p} : \bar{H} \to P$ specifying a bid and an ask price that will be posted by the market makers after every history. Denote by $\mathbf{p}_1(h^t)$, the ask price and $\mathbf{p}_{-1}(h^t)$ the bid price that is chosen in period $(t+1)$ after $h^t \in \bar{H}$. Let $\mathbf{P} = \{\mathbf{p} \mid \mathbf{p} : \bar{H} \to P\}$ be the set of possible price rules.

Given a price rule $\mathbf{p} \in \mathbf{P}$, we define the set of $t$-period histories that are *consistent with* $\mathbf{p}$, inductively as follows:

$$H^t(\mathbf{p}) \equiv H^{t-1}(\mathbf{p}) \times \{\mathbf{p}(h^{t-1}) | h^{t-1} \in H^{t-1}(\mathbf{p})\} \times E, t = 1, \ldots, T, \tag{3}$$

with $H^0(\mathbf{p}) = H^0$. Note that given any price rule $\mathbf{p}$, the set $H^t(\mathbf{p})$ allows for all possible $t$-period histories of trades $e^t \in E^t$ to occur, but does not allow for past prices that are different from those chosen according to the rule $\mathbf{p}$.

The one-period payoff of the informed type $\theta \in \{0, 1\}$ of the trader from any trade $e$ at any transaction price $\hat{p}$ is $(\theta - \hat{p})e$. The informed types of the trader maximize the expected undiscounted sum of their per-period profits. For any strategy $\sigma_\theta$, and a price rule $\mathbf{p}$, let $U_t(\theta, \sigma_\theta, \mathbf{p} | h^{t-1}, p_t)$ be the expected continuation payoff for type $\theta \in \{0, 1\}$ of the informed trader, at the *beginning of period* $t = 1, \ldots, T$, given a history $h^{t-1} \in H^{t-1}$, and given the period $t$ posted prices $p_t \in P$. Let $W_t(\theta, \sigma_\theta, \mathbf{p} | h^t)$ be the expected continuation payoff for type $\theta \in \{0, 1\}$ of the informed trader, at the *end of period* $t = 1, \ldots, T$, given a history $h^t \in H^t$, before the informed trader sees the prices posted in period $t + 1$.

The functions $U_t$ and $W_t$ are recursively defined as follows:

$$W_T(\theta, \sigma_\theta, \mathbf{p} | h^T) \equiv 0 \quad \text{for all } h^T \in H^T, \tag{4}$$

$$\begin{aligned} U_t(\theta, \sigma_\theta, \mathbf{p} | h^{t-1}, p_t) \equiv \sum_{e \in E} & \sigma_\theta(e | h^{t-1}, p_t) \\ & \times [(\theta - p_{e,t})e + W_t(\theta, \sigma_\theta, \mathbf{p} | \{h^{t-1}, p_t, e\})], \end{aligned} \tag{5}$$

for all $h^{t-1} \in H^{t-1}$, $p_t \in P$ and $t = 1, \ldots, T$; and,

$$W_t(\theta, \sigma_\theta, \mathbf{p} | h^t) \equiv U_{t+1}(\theta, \sigma_\theta, \mathbf{p} | h^t, \mathbf{p}(h^t)) \text{ for all } h^t \in H^t, t < T. \tag{6}$$

Expression (4) is a boundary condition. Expression (5) states that the expected payoff in period $t$, given a history $h^{t-1}$ and period $t$ posted prices $p_t$ (that are not necessarily equal to $\mathbf{p}(h^{t-1})$), is equal to the expected value of the $t$th period trade at the prices $p_t$ plus the expected end of period payoff given that a $t$-period history $\{h^{t-1}, p_t, e\}$ has occurred. Expression (6) states that the informed trader's end of period $t$ expected payoff is computed given that the prices in the next period will equal $\mathbf{p}(h^t)$. Starting with (4) and then successively working through (5) and (6)

[6] Notice that these restrictions on $\sigma_N$ still allow for noise trading strategy that is standard in the literature. See [17].

backwards in time, an expected continuation payoff is specified for every decision node $h^{t-1} \in H^{t-1}$, $p_t \in P$, for each type $\theta \in \{0, 1\}$, strategy $\sigma_\theta$ and price rule $\mathbf{p}$.

### 2.2. Equilibrium

In our definition of equilibrium, we directly impose the notion that the market makers face competition and choose prices to equal the expected value of the asset conditional on the observed history of trades up to and including the trade in that period. Given such competitive pricing, equilibrium also requires that type $\theta \in \{0, 1\}$ of the informed trader chooses a strategy $\sigma_\theta$ that maximizes his expected payoff after every history.

**Definition 1.** An equilibrium is a triple $\{\sigma_0, \sigma_1, \mathbf{p}\}$ such that the following hold.

1. The informed types' strategies are sequentially rational given $\mathbf{p}$: for all $\theta \in \{0, 1\}$, for all $t \in \{1, \ldots, T\}$, histories $h^{t-1} \in \bar{H}$ and prices $p_t \in P$,

$$\sigma_\theta \in \arg \max_{\sigma \in \Sigma} U_t(\theta, \sigma, \mathbf{p} \mid h^{t-1}, p_t).$$

2. The price rule $\mathbf{p}$ is competitive along consistent histories: for all $t \in \{1, \ldots, T\}$ and histories $h^{t-1} \in H^{t-1}(\mathbf{p})$,

$$\mathbf{p}_e(h^{t-1}) = E[v | h^{t-1}, e_t = e] \text{ for all } e \in \{-1, 1\}.$$

The expectation in Definition 1 is derived using the priors $\lambda$ and $\mu$, the noise trader strategy $\sigma_N$ as well as the equilibrium strategies $\sigma_0$ and $\sigma_1$ used by the informed types of the traders. Since $\sigma_N$ satisfies condition (N), the expectation is well-defined for all histories consistent with $\mathbf{p}$. Our definition of equilibrium is equivalent to that used elsewhere in the literature (e.g., [9]). We require the informed trader's strategy to be sequentially rational after every history, whether or not a such a history is consistent with prices being chosen according to the rule $\mathbf{p}$. In contrast, our equilibrium notion imposes competitive pricing only for those histories that are consistent with $\mathbf{p}$, with prices being completely unrestricted otherwise. This is weaker than requiring competitive pricing not only on the path of play, but also for histories where prices have not been chosen competitively in the past. We show below that requiring competitive pricing only for consistent histories (together with sequential rationality) is enough to rule out nonmanipulative equilibria for large $T$.

### 2.3. Manipulative strategies

An informed trader, when selecting his trade in any period, must balance the short-term profit from the trade with the long-term effect his trade has on future prices and hence on future profits. We say that a strategy is *manipulative* if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit. If such a strategy is used in equilibrium, then it must be to manipulate the market prices, which will enable him to recoup the short-term losses (and more) in the future.

**Definition 2.** Given a price rule $\mathbf{p}$, a strategy $\sigma \in \Sigma$ is nonmanipulative for type $\theta \in \{0,1\}$ if, for all $t = 1, \ldots, T$, $h^{t-1} \in H^{t-1}(\mathbf{p})$, $e \in \{-1, 1\}$,

$$\sigma(e \,|h^{t-1}, \mathbf{p}(h^{t-1})) > 0 \Rightarrow [\theta - \mathbf{p}_e(h^{t-1})]e \geqslant 0.$$

Otherwise $\sigma$ is manipulative for $\theta$.

Note that this definition of nonmanipulative strategies implies a joint restriction on the strategy of the informed trader and the price rule.[7] For any price rule $\mathbf{p}$, let $\Sigma_\theta^{\mathrm{nm}}(\mathbf{p})$ be the set of nonmanipulative strategies for type $\theta \in \{0,1\}$ and let

$$\Sigma^{\mathrm{nm}}(\mathbf{p}) = \Sigma_0^{\mathrm{nm}}(\mathbf{p}) \times \Sigma_1^{\mathrm{nm}}(\mathbf{p}) \subset \Sigma \times \Sigma \tag{7}$$

be the set of pairs of nonmanipulative strategies, one for each type.

We will show below that in any equilibrium, the price $\mathbf{p}_e(h^{t-1}) \in (0,1)$ for all $e \in \{-1,1\}$ and $h^{t-1} \in H^{t-1}(\mathbf{p})$, as a direct consequence of assumption (N) and the definition of equilibrium. For such prices, note that the conditions needed for the strategy of the informed trader to qualify as nonmanipulative are very weak. For example, not buying with probability 1, even if the price is lower than its expected value is not considered manipulative for type $\theta = 1$: choosing the timing of trading is not manipulative. Consequently, the myopic or one-shot optimal behavior of always buying with good news and always selling with bad news is not the only possible nonmanipulative pair of strategies for types $\theta \in \{0,1\}$.[8] Finally, note that manipulative strategies are not necessarily mixed strategies and that mixed strategies are not necessarily manipulative. For $\theta = 1$, the pure strategy of selling just once and buying after that is manipulative, and the mixed strategy of buying and not trading every period with positive probability is nonmanipulative.

## 3. The main result

In the model just described, if the private information of the dynamic trader is sufficiently long lived (i.e., if $T$ is sufficiently large) then every equilibrium involves

[7] To see this note that when $\mathbf{p}_{-1}(h^{t-1})$ is equal to 1 for all $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$, all strategies for type $\theta = 1$ are nonmanipulative. On the other hand if $\mathbf{p}_{-1}(h^{t-1}) < 1$ for some $h^{t-1} \in H^{t-1}(\mathbf{p})$, then a nonmanipulative strategy for type $\theta = 1$ would prescribe not selling after $h^{t-1}$ given $p_t = \mathbf{p}(h^{t-1})$.

[8] This would be part of the equilibrium if the dynamic trader were replaced with many identical one-period traders. Our main result is thus stronger than the claim that the equilibrium strategy of the dynamic informed trader will be different from that generated by many such short-lived traders.

manipulative trading. We prove our result with the help of the following two lemmas.

**Lemma 1.** *For all* $T, \mu, \lambda$ *and* $\sigma_N$ *satisfying* (N), *an equilibrium* $\{\sigma_0, \sigma_1, \ \mathbf{p}\}$ *exists.*

**Proof.** See the appendix. $\square$

The existence of an equilibrium in our model is not completely straightforward. We prove existence by exploiting the finiteness of the set $E^t$ of histories of trades, with the help of trading strategies that are independent of current and past prices.

The next lemma provides a bound on the expected payoff for each type of the informed trader in any candidate equilibrium involving nonmanipulative strategies.

**Lemma 2.** *If* $\{\sigma_0, \sigma_1, \ \mathbf{p}\}$ *is an equilibrium then* $\mathbf{p}_e(h^{t-1}) \in (0,1)$ *for all* $e \in \{-1, 1\}$, $h^{t-1} \in H^{t-1}(\mathbf{p})$ *and* $t = 1, \ldots, T$. *If in addition*, $(\sigma_0, \sigma_1) \in \Sigma^{\mathrm{nm}}(\mathbf{p})$, *then*

1. $\mathbf{p}_1(h^{t-1}) \geqslant \lambda \geqslant \mathbf{p}_{-1}(h^{t-1})$ *for all* $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$, *where the first* (*respectively*, *second*) *inequality holds with equality if* $h^{t-1}$ *contains at least one sell* (*respectively*, *buy*);
2. *the expected* $T$*-period payoffs of types* $\theta = 1$ *and* $\theta = 0$ *are bounded above as follows*:

$$U_1(0, \sigma_0, \mathbf{p} \mid h^0, \mathbf{p}(h^0)) < k_0 \lambda T,$$

$$U_1(1, \sigma_1, \mathbf{p} \mid h^0, \mathbf{p}(h^0)) < k_1 (1 - \lambda) T,$$

*where* $k_0, k_1 \in (0,1)$ *and do not depend on* $\sigma_0, \sigma_1$ *or* $\sigma_N$.

**Proof.** See the appendix. $\square$

In an equilibrium with nonmanipulative strategies, the type $\theta = 1$ does not sell and type $\theta = 0$ does not buy with positive probability. Consequently, from the second condition in the definition of equilibrium, all ask prices are weakly greater than $\lambda$ and all bid prices weakly less than $\lambda$. The important part of the last result is that the bound obtained on payoffs is uniform, i.e., does not depend on $\sigma_0, \sigma_1$ or $\sigma_N$.

We can now present our main result.

**Proposition 3.** *There exists* $\bar{\mathbf{T}}$ *such that for all* $T > \bar{\mathbf{T}}$, *and any* $\sigma_N$ *satisfying condition* (N), *every equilibrium* $\{\sigma_0, \sigma_1, \mathbf{p}\}$ *has* $(\sigma_0, \sigma_1) \notin \Sigma^{\mathrm{nm}}(\mathbf{p})$.

**Proof.** Suppose that $\{\sigma_0, \sigma_1, \ \mathbf{p}\}$ is an equilibrium with $(\sigma_0, \sigma_1) \in \Sigma^{\mathrm{nm}}(\mathbf{p})$. From Definition 2 we obtain that type $\theta = 1$ is not supposed to sell and type $\theta = 0$ not supposed to buy, for histories consistent with $\mathbf{p}$.

Consider the following deviation strategy of type $\theta = 0$: suppose that $\theta = 0$ buys in period 1 and sells from then on. From Lemma 2, $\mathbf{p}_1(h^0) \in (0, 1)$ so that the loss from the first period buy is at most $-1$. Further, the prices at which $\theta = 0$ can sell from period 2 onwards are equal to $\lambda$. Therefore $\theta = 0$'s $t$-period profit from this strategy is at least $-1 + \lambda(T - 1)$. From Lemma 2, it is seen that there exists $\bar{\mathbf{T}} = \frac{1}{1-k_0}\frac{1+\lambda}{\lambda}$ such that for all $T > \bar{\mathbf{T}}$, the deviation strategy above gives higher payoffs than $U_1(0, \sigma_0, \mathbf{p}|h^0, \mathbf{p}(h^0))$. Since $k_0$ does not depend on $\sigma_0, \sigma_1$ or $\sigma_N$, there exists no equilibrium $\{\sigma_0, \sigma_1, \mathbf{p}\}$ with $(\sigma_0, \sigma_1) \in \Sigma^{\mathrm{nm}}(\mathbf{p})$, for any $T > \bar{\mathbf{T}}$ and any $\sigma_N$ satisfying condition (N). Since an equilibrium exists by Lemma 1, this completes the proof. $\square$

It might be asked whether the result holds when there is more than one long-lived informed trader in the market. Intuition suggests that competition between informed traders might make it difficult for any one trader to trade manipulatively and recoup the losses, before the other trader reveals the information through his trades. The general answer to this question depends on the structure of information held jointly by the two insiders. In particular, suppose that the types of the traders are perfectly correlated, so that either there are two informed traders (with identical information) or none. Suppose, for simplicity, that each trader gets to trade in any period with probability $\frac{1}{2}$, independent across periods, and the market maker does not know which trader has traded in any period. One can show that in such a case, our manipulation result goes through unchanged.[9] The intuition is identical to that above. In a candidate nonmanipulative equilibrium, each trader by trading manipulatively can convince the market that no information exists in the market. They can then trade in the future at a price equal to the ex ante expected value, which benefits both traders for a long horizon, compared to their payoffs from trading nonmanipulatively.

## 4. An example

In this section we provide a three-period example of a version of our main result. In the example, we will rule out the no trade option and will allow traders to only buy or sell in each period. For this case, *only* the strategy of buying always given good news and selling always given bad news can be considered nonmanipulative, for histories consistent with the equilibrium prices.

We assume that a noise trader buys and sells with equal probabilities for all histories and choose $\mu = 0.90$ and $\lambda = 0.25$. For these parameter values, prices will be quite sensitive to trades. As a result, the informed trader of type $\theta = 1$ will manipulate even with such a short horizon.

We first show that nonmanipulation is not an equilibrium. In any equilibrium, all bid and ask prices along any history that can arise in equilibrium must lie in the

[9] See [6].

interval $(0,1)$. Thus, in a candidate nonmanipulative equilibrium, type $\theta = 1$ does not sell and type $\theta = 0$ does not buy, along the path of play. As a result, in such a candidate equilibrium, the first period equilibrium bid and ask prices (equal to the expected value of the asset given the trade) are given by

$$p_{-1,1} = \frac{\frac{1}{2}(1-\mu)\lambda}{\frac{1}{2}(1-\mu)\lambda + (\mu + \frac{1}{2}(1-\mu))(1-\lambda)} = 0.017241,$$

$$p_{1,1} = \frac{(\mu + \frac{1}{2}(1-\mu))\lambda}{(\mu + \frac{1}{2}(1-\mu))\lambda + \frac{1}{2}(1-\mu)(1-\lambda)} = 0.86364.$$

Similarly, along a consistent history of all buys, the next two ask prices are given by $p_{1,2} = 0.925$ and $p_{1,3} = 0.96053$. Thus, if type $\theta = 1$ of the informed trader trades according to the nonmanipulative strategy by buying in every period, his profit will be equal to

$$3 - 0.86364 - 0.925 - 0.96053 = 0.25084.$$

On the other hand, if he deviates and sells in the first period he makes a loss of $1 - p_{-1,1}$ in the first period. However, when he buys from the next period onwards the market puts probability 1 on the trades being driven by noise, as all histories of trades are consistent with type $\theta = N$ trading. Thus, along the history of a sell followed by buys, the equilibrium ask prices are given by $p'_{1,2} = p'_{1,3} = \lambda = 0.25$. Therefore, if he deviates and sells in the first period and then buys in the next two, his profit will be

$$-1 + 0.017241 + 2(1-\lambda) = 0.51724.$$

Since this deviation is profitable for type $\theta = 1$, there does not exist a nonmanipulative equilibrium.[10]

We now look for a manipulative equilibrium. We look for an equilibrium of the following form:

$$\sigma_1(1|h^{t-1}, p_t) = \begin{cases} x & \text{if } t = 1 \text{ and } p_1 = \mathbf{p}(h^0), \\ 1 & \text{otherwise}, \end{cases} \tag{8}$$

for some $x \in (0,1)$; and $\sigma_0(-1|h^{t-1}, p_t) = 1$ for all $h^{t-1}, p_t,\ t \geqslant 1$. In words, type 1 buys in the first period with probability $x \in (0,1)$ and sells with probability $1-x$, given the posted first period equilibrium prices. He buys with probability 1 otherwise, regardless of posted prices, the past history or trades. In contrast, the informed trader of type 0 does not manipulate (sells always) in any period, for any history and any posted price.

In equilibrium, the prices must equal the expected value of the asset for every consistent history in $H^{t-1}(\mathbf{p})$.[11] Given that the informed trader trades according to

[10] It is readily checked that type 0 of the informed trader will not manipulate given the prices above and his profit from selling always will equal 0.03 0715.

[11] Without loss of generality, we set $p_1(h^{t-1}) = 1$ and $p_{-1}(h^{t-1}) = 0$, for every *inconsistent* history $h^{t-1} \in H^{t-1} \backslash H^{t-1}(\mathbf{p}),\ t = 1, 2, 3.$

the strategy above, the equilibrium ask prices, along the consistent history of all buys, must equal

$$p_{1,t} = \frac{(x\mu + \frac{1}{2^t}(1-\mu))\lambda}{(x\mu + \frac{1}{2^t}(1-\mu))\lambda + \frac{1}{2^t}(1-\mu)(1-\lambda)},$$

for $t = 1, 2, 3$; whereas the first period bid price will equal

$$p_{-1,1} = \frac{((1-x)\mu + \frac{1}{2^t}(1-\mu))\lambda}{((1-x)\mu + \frac{1}{2^t}(1-\mu))\lambda + (\mu + \frac{1}{2^t}(1-\mu))(1-\lambda)}.$$

The next two ask prices along the consistent history of a sell followed by two buys will equal, for $t = 2, 3$,

$$p'_{1,t} = \frac{((1-x)\mu + \frac{1}{2^t}(1-\mu))\lambda}{((1-x)\mu + \frac{1}{2^t}(1-\mu))\lambda + \frac{1}{2^t}(1-\mu)(1-\lambda)}.$$

In such a candidate equilibrium, type 1 must be indifferent between buying and selling in the first period, i.e., $x$ must be such that

$$3 - p_{1,1} - p_{1,2} - p_{1,3} = 1 + p_{-1,1} - p'_{1,2} - p'_{1,3}.$$

Computations indicate that such an equilibrium exists with

$$x \cong 0.98357. \tag{9}$$

Below we provide all the prices along the two 3-period consistent histories of three buys and a sell followed by two buys, to which the strategy of type 1 attaches positive probability:

| | Three buys | Sell and two buys |
|---|---|---|
| $t = 1$ | $p_{1,1} = 0.86178$ | $p_{-1,1} = 0.022227$ |
| $t = 2$ | $p_{1,2} = 0.92387$ | $p'_{1,2} = 0.34661$ |
| $t = 3$ | $p_{1,3} = 0.9599$ | $p'_{1,3} = 0.42117$ |

The equilibrium profit from manipulating of type $\theta = 1$ is equal to 0.25445. It can be easily verified that selling always is a best response of $\theta = 0$ and that his equilibrium profit is equal to 0.035701.

Notice that both types of the informed trader are better off in the equilibrium, compared to the nonmanipulative benchmark case above. Type 1 of the informed trader's profit in equilibrium is equal to 0.25445 whereas his profit from always buying in the nonmanipulative benchmark was seen above to be equal to 0.25084, a rise of 1.44%. Since, in this example, the nonmanipulative benchmark can be understood as the case corresponding to the dynamic informed trader being replaced by a sequence of short-lived myopic traders, this rise in the informed trader's profit can be interpreted as the returns to having a long horizon. By manipulating, type 1 of informed trader with a long horizon "creates his own noise" and lowers all ask prices along the history of all buy orders, giving him higher profits. Type $\theta = 0$, is also better off compared to the nonmanipulative benchmark. His profit in the nonmanipulative benchmark is 0.03 0715 whereas his equilibrium profit is equal to 0.035701, a rise of 16.23%. Since type $\theta = 1$ manipulates, a first period sell can come from the informed trader with good news with positive probability. The first period bid price is thus higher than in the nonmanipulative case. This raises the profits of type 0 from selling in the first and every subsequent period, even though he does not manipulate.[12]

## 5. Conclusion

The primary contribution of this paper is to show that in Glosten–Milgrom [9] type of models with one insider trading repeatedly, if the number of periods is large enough, then the equilibrium will necessarily involve manipulation, as long as the market faces uncertainty about the existence of the informed trader. If there are enough periods of trading left to recoup his initial losses, the insider will undertake unprofitable trades to try and build a "reputation" of not trading on information, i.e., of being a noise trader. Manipulation endogenously creates liquidity and raises the informed trader's profits.

The result is robust to the precise specification of the market structure. For example, suppose that each period the trader submits an order $x_t$ from some finite set $X$, but the market maker only observes $h_t = x_t + y_t$ where $y_t$ belongs to some finite set $Y$. The term $y_t$ can be interpreted as the demand coming from noise traders, with the sum $x_t + y_t$ being the aggregate market order. This model corresponds to a discrete version of the market order model introduced by Kyle [15], where the trader faces uncertainty about the price at which his trades will be executed. In such a market, if the insider deviates from a candidate nonmanipulative equilibrium by manipulating repeatedly, then this deviation will ultimately be noticed by the market maker with probability 1, so that he will be able to trade thereafter at the ex ante expected price $\lambda$. In [7], we are able to show that this intuition is correct, and the identical result on the necessity of manipulation for long horizons obtains also in the market order model.

## Appendix A

### *A.1. Preliminaries*

To facilitate the proofs of Lemmas 1 and 2, we will start with defining the notion of price-independent strategies (or, p.i.-strategies for short) and associated objects.

[12] In the working paper version of this paper we provide a four-period example with $\mu = 0.9$ and $\lambda = 0.5$, where both types of the informed trader manipulate. Type 1 manipulates by selling with positive probability in the first period, *and* in the second period following a first period buy. Type 0 manipulates by buying with positive probability in the first period, and in the second period following a first period sell. Compared to the myopic, nonmanipulative benchmark, the profits of each type of the informed trader from manipulation are higher by about 16.52%, in the four-period example.

Let $\bar{E} = \bigcup_{t=0}^{T-1} E^t$ and $\hat{E} = \bar{E} \cup E^T$. A p.i.-strategy is defined as a function $s : \bar{E} \to \Delta(E)$, with $s(e|e^t)$ being the probability that $s$ attaches to the trade $e$ after a trading history $e^t$, and $s(e^t) = \{s(e|e^t)\}_{e \in E}$. Let $S$ be the set of p.i.-strategies. Define the metric $d : S \times S \to \mathbb{R}$ on $S$ as follows:

$$d(s, s') = \max_{e^t \in \bar{E}} \max_{e \in E} |s(e|e^t) - s'(e|e^t)|. \tag{A.1}$$

The set $S$ is a convex, compact subset of a finite-dimensional Euclidean space with respect to this metric.[13] Thus, $S \times S$ is also convex and compact with respect to the product topology generated from (A.1). Recall from condition (N) that $\sigma_N$ can be written as a p.i.-strategy, henceforth to be denoted by $s_N$. We will denote by $s_\theta \in S$, a p.i.-strategy assigned to type $\theta \in \{0, 1\}$ of the trader.

For all $t' \leqslant t$, denote by $e^t_{t'}$ the $t'$th element of $e^t$ and $e^{t'(t)}$ the first $t'$ elements. Any p.i.-strategy $s \in S$ generates a probability distribution over histories of trades. For a given $s$, and any $e^t, e^{t'} \in \hat{E}$ with $t \geqslant t'$, let $q(e^t \mid e^{t'}, s)$ be the probability that $e^t$ is generated by $s$ given $e^{t'}$:

$$q(e^t \mid e^{t'}, s) = \begin{cases} 1 & \text{if } e^{t'} = e^t, \\ \prod_{t''=t'}^{t-1} s(e^t_{t''+1} \mid e^{t''(t)}) & \text{if } t' < t, e^{t'(t)} = e^{t'}, \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

Let $Z = \{z : \hat{E} \to [0, 1]\}$ be the set of functions mapping histories of trades in $\hat{E}$ to the unit interval. For a fixed $s_N$, let $\bar{v} : S \times S \to Z$ be the mapping that takes a pair $(s_0, s_1) \in S \times S$ to a function $\bar{v}_{s_0,s_1} \in Z$, with $\bar{v}_{s_0,s_1}(e^t) \in [0, 1]$ specifying the expected value of the asset (derived from $s_0, s_1$ and $s_N$), after each trading history $e^t \in \hat{E}$. Thus,

$$\bar{v}_{s_0,s_1}(e^t) \equiv \frac{\{\mu q(e^t \mid e^0, s_1) + (1 - \mu) q(e^t \mid e^0, s_N)\}\lambda}{(1 - \mu) q(e^t \mid e^0, s_N) + \mu q(e^t \mid e^0, s_1)\lambda + \mu q(e^t \mid e^0, s_0)(1 - \lambda)}. \tag{A.3}$$

By condition (N), the mapping $\bar{v}$ is well-defined and $\bar{v}_{s_0,s_1}(e^t) \in (0, 1)$ for all $e^t \in \hat{E}$, $(s_0, s_1) \in S \times S$.

Finally, we define the following value-function for type $\theta \in \{0, 1\}$. Given any $(s_0, s_1) \in S \times S$, for $\theta \in \{0, 1\}$, $\hat{s} \in S$, $t = 1, \ldots, T - 1$, let

$$\begin{aligned} &u_t(\theta, \hat{s}, \bar{v}_{s_0,s_1}|e^{t-1}) \\ &\quad \equiv \sum_e \hat{s}(e|e^{t-1})[(\theta - \bar{v}_{s_0,s_1}(\{e^{t-1}, e\}))e + u_{t+1}(\theta, \hat{s}, \bar{v}_{s_0,s_1}|\{e^{t-1}, e\})], \end{aligned} \tag{A.4}$$

with $u_{T+1}(\theta, \hat{s}, \bar{v}_{s_0,s_1} \mid e^T) \equiv 0$. Observe from (A.2) that we can also write the value function in (A.4) as follows:

$$u_t(\theta, \hat{s}, \bar{v}_{s_0,s_1}|e^{t-1}) = \sum_{t'=t}^{T} \left[ \sum_{e^{t'} \in E^{t'}} q(e^{t'} \mid e^{t-1}, \hat{s})[\theta - \bar{v}_{s_0,s_1}(e^{t'})] e^{t'}_{t'} \right]. \tag{A.5}$$

[13] Since $E$ has three elements, $\bar{E}$ is a finite set. Further, $s$ specifies a probability, one for each element of $E$. Therefore, $S$ is a subset of a finite-dimensional Euclidean space. It is compact since it is closed and bounded.

**Proof of Lemma 1.** We will prove the existence of an equilibrium as follows. Fixing $s_N$, we will first define a best-reply correspondence $\xi^t : S \times S \rightrightarrows S \times S$ for types $\theta \in \{0, 1\}$ and show that it has a fixed point $(s_0^*, s_1^*)$. From this fixed-point $(s_0^*, s_1^*)$ and the mapping $\bar{v}$ we will then construct strategies $\sigma_0, \sigma_1$ and a price rule $\mathbf{p}$ that will constitute an equilibrium.

Pick any $(s_0, s_1) \in S \times S$. For $t \geqslant 1$, define the correspondence $\xi^t : S \times S \rightrightarrows S \times S$ as follows:

$$\xi_0^t(s_0, s_1) = \left\{ s \in S | s \in \arg\max_{\hat{s} \in S} u_{t'}(0, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{t'-1}) \forall e^{t'-1}, t \leqslant t' \leqslant T \right\}, \tag{A.6}$$

$$\xi_1^t(s_0, s_1) = \left\{ s \in S | s \in \arg\max_{\hat{s} \in S} u_{t'}(1, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{t'-1}) \forall e^{t'-1}, t \leqslant t' \leqslant T \right\}. \tag{A.7}$$

We proceed in steps. In step 1 we show that $\xi^1$ is a nonempty convex-valued correspondence. In step 2 we show that it has a closed graph which, together with step 1, imply the existence of a fixed-point $(s_0^*, s_1^*)$. In step 3, we construct an equilibrium triple $\{\sigma_0, \sigma_1, \mathbf{p}\}$ from the fixed point $(s_0^*, s_1^*)$ and the function $\bar{v}_{s_0^*, s_1^*}$.

*Step* 1 (Non-empty and convex-valued correspondence): We show, by induction on $t$, that $\xi^1$ is a nonempty, convex-valued correspondence. Pick any history of trades $e^{T-1}$ and, without loss of generality, consider the type $\theta = 0$. Then,

$$u_T(0, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{T-1}) = \sum_e \hat{s}(e|e^{T-1})[-\bar{v}_{s_0,s_1}(\{e^{T-1}, e\})e].$$

Since $\bar{v}_{s_0,s_1}(e^{T-1}) \in (0, 1)$, $\hat{s}(-1|e^{T-1}) = 1$ maximizes the last expression. Therefore, the set $\xi_0^T(s_0, s_1)$ is nonempty as it contains all $s \in S$ with $s(-1|e^{T-1}) = 1$ for all $e^{T-1}$. Further, $\xi_0^T(s_0, s_1)$ is a convex set.

Suppose, as part of the inductive hypothesis that $\xi_0^t(s_0, s_1)$ is a nonempty convex set. We show that then $\xi_0^{t-1}(s_0, s_1)$ is also a nonempty convex set. Pick any $e^{t-2}$ and $s' \in \xi_0^t(s_0, s_1)$ and let $\hat{s}$ be such that

$$\hat{s}(e^{t'-1}) = s'(e^{t'-1}) \text{ for all } e^{t'-1}, t' \geqslant t.$$

Then, from (A.4),

$$\begin{aligned} &u_{t-1}(0, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{t-2}) \\ &\quad = \sum_e \hat{s}(e|e^{t-2})[-\bar{v}_{s_0,s_1}(\{e^{t-2}, e\})e + u_t(0, s', \bar{v}_{s_0,s_1} \mid \{e^{t-2}, e\})]. \end{aligned} \tag{A.8}$$

For fixed $s'$, the expression on the right-hand side above is linear in the probabilities $\{\hat{s}(e|e^{t-2})\}_{e \in E}$. Therefore the expression in (A.8) has a maximum so that the set $\xi_0^{t-1}(s_0, s_1)$ is nonempty. Furthermore, since $\xi_0^t(s_0, s_1)$ is a convex set by the inductive hypothesis, it is immediate that $\xi_0^{t-1}(s_0, s_1)$ is a convex set.

By induction, the set $\xi_0^1(s_0, s_1)$ is thus nonempty and convex. Analogously, the $\xi_1^1(s_0, s_1)$ is nonempty and convex. Since the pair $(s_0, s_1)$ was arbitrary, this shows that the correspondence $\xi^1$ is nonempty and convex valued.

*Step* 2 (Closed graph and fixed point): Next, we show that $\xi^1$ has a closed graph. We begin by showing that for all $e^{t-1}$, the value function $u_t(1, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{t-1})$ is continuous in $\hat{s}$. From (A.5), this is equivalent to showing that $q(e^{t'}|e^{t-1}, \hat{s})$ is continuous in $\hat{s}$ for all $e^{t'}$, $t' \geqslant t$. But for a fixed $T$, this is immediate from (A.2) and the definition of the metric in (A.1). Analogously, $u_t(0, \hat{s}, \bar{v}_{s_0,s_1} \mid e^{t-1})$ is continuous in $\hat{s}$ for each $e^{t-1}$ and $t \geqslant 1$.

Similarly, it is easily seen from (A.3) that $\bar{v}_{s_0,s_1}(e^t)$ is continuous in $s_0$ and $s_1$ for each $e^t$, as $q(e^t|e^0, s)$ is continuous in $s$. Therefore, from (A.4) it follows that for $\theta \in \{0, 1\}$, $u_t(\theta, \hat{s}, \bar{v}_{s_0,s_1}|e^{t-1})$ is continuous in $s_0$ and $s_1$.

To show that $\xi^1$ has a closed graph, we want to prove the following: If $(s_0^n, s_1^n, \hat{s}_0^n, \hat{s}_1^n) \to (s_0', s_1', \hat{s}_0, \hat{s}_1)$ with $(\hat{s}_0^n, \hat{s}_1^n) \in \xi^1(s_0^n, s_1^n)$ for all $n$, then $(\hat{s}_0, \hat{s}_1) \in \xi^1(s_0', s_1')$. Suppose not, i.e., there exists a sequence as above but, without loss of generality, $\hat{s}_1 \notin \xi_1^1(s_0', s_1')$. Then there exists $t \in \{1, \dots, T\}$, $e^{t-1}$, $\varepsilon > 0$ and $\tilde{s} \in S$ such that

$$u_t(1, \tilde{s}, \bar{v}_{s_0',s_1'} \mid e^{t-1}) > u_t(1, \hat{s}_1', \bar{v}_{s_0',s_1'} \mid e^{t-1}) + 3\varepsilon.$$

For $n$ large enough, by continuity, we must have

$$\begin{aligned} u_t(1, \tilde{s}, \bar{v}_{s_0^n,s_1^n} \mid e^{t-1}) &> u_t(1, \tilde{s}, \bar{v}_{s_0',s_1'} \mid e^{t-1}) - \varepsilon, \\ &> u_t(1, \hat{s}_1, \bar{v}_{s_0',s_1'} \mid e^{t-1}) + 2\varepsilon, \\ &> u_t(1, \hat{s}_1^n, \bar{v}_{s_0',s_1'} \mid e^{t-1}) + \varepsilon, \\ &> u_t(1, \hat{s}_1^n, \bar{v}_{s_0^n,s_1^n} \mid e^{t-1}). \end{aligned}$$

But this contradicts the fact that $\hat{s}_1^n \in \xi_1^1(s_0^n, s_1^n)$ for all $n$.

Since $S \times S$ is compact and convex and $\xi^1$ is a nonempty, convex-valued correspondence with a closed graph, it has a fixed point $(s_0^*, s_1^*)$, by the Kakutani–Fan–Glicksberg theorem.[14]

*Step* 3 (Construction of equilibrium): We now construct an equilibrium $\{\sigma_0, \sigma_1, \mathbf{p}\}$ from the fixed point $(s_0^*, s_1^*)$ and $\bar{v}_{s_0^*,s_1^*}$. We begin by constructing the price rule $\mathbf{p}$ and associated consistent histories $H^t(\mathbf{p})$.

Let $H^0(\mathbf{p}) = H^0$. For $t = 1, \dots, T$, if $h^{t-1} \in H^{t-1}(\mathbf{p})$, let

$$\mathbf{p}_e(h^{t-1}) = \bar{v}_{s_0^*,s_1^*}(\{\mathbf{e}(h^{t-1}), e\}) \text{ for all } e \in \{-1, 1\},$$

and so define $H^t(\mathbf{p})$ from (3); and if $h^{t-1} \notin H^{t-1}(\mathbf{p})$, let

$$\mathbf{p}_1(h^{t-1}) = 1 \quad \text{and} \quad \mathbf{p}_{-1}(h^{t-1}) = 0.$$

This defines a price rule $\mathbf{p} : \bar{H} \to [0, 1]$.

For $\theta \in \{0, 1\}$ and $t = 1, \dots, T$, let

$$\sigma_\theta(e|h^{t-1}, p_t) = s_\theta^*(e|\mathbf{e}(h^{t-1})) \text{ for all } e \in E,$$

[14] See [16].

if $h^{t-1} \in H^{t-1}(\mathbf{p})$ and $p_t = \mathbf{p}(h^{t-1})$; and let

$$\sigma_1(1|h^{t-1}, p_t) = 1 = \sigma_0(-1|h^{t-1}, p_t),$$

otherwise. This defines a strategy $\sigma_\theta : \bar{H} \times P \to \Delta(E)$ for each type $\theta \in \{0, 1\}$.

The triple $\{\sigma_0, \sigma_1, \mathbf{p}\}$ thus constructed is an equilibrium. To see this, consider the second condition in Definition 1 first. Pick any $e \in \{-1, 1\}$ and $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$. Given the price rule $\mathbf{p}$, $E[v|h^{t-1}, e_t = e]$ depends only on the relative likelihoods that the history of trades $\{\mathbf{e}(h^{t-1}), e\}$ has been generated by $\sigma_0, \sigma_1$ and $\sigma_N$ (equivalently, $s_N$) Recall that $\bar{v}_{s_0^*, s_1^*}(e^t)$ is the expected value of the asset given $e^t$, derived from $s_0^*, s_1^*$ and $s_N$, for all $e^t$. By construction, $\sigma_\theta$ generates the same probability distribution over trades as $s_\theta^*$, for $\theta \in \{0, 1\}$, $h^{t-1} \in H^{t-1}(\mathbf{p})$ and $p_t = \mathbf{p}(h^{t-1})$. It then follows that

$$E[v|h^{t-1}, e_t = e] = \bar{v}_{s_0^*, s_1^*}(\{\mathbf{e}(h^{t-1}), e\}) = \mathbf{p}_e(h^{t-1}) \text{ for } e \in \{-1, 1\}.$$

As for the sequential rationality condition in Definition 1, consider first histories that are consistent with $\mathbf{p}$. By construction, for all $\theta \in \{0, 1\}$, $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$,

$$U_t(\theta, \sigma_\theta, \mathbf{p}|h^{t-1}, \mathbf{p}(h^{t-1})) = u_t(\theta, s_\theta^*, \bar{v}_{s_0^*, s_1^*}|\mathbf{e}(h^{t-1})).$$

Suppose that there exists $\sigma' \neq \sigma_\theta$ such that

$$U_t(\theta, \sigma', \mathbf{p}|h^{t-1}, \mathbf{p}(h^{t-1})) > U_t(\theta, \sigma_\theta, \mathbf{p}|h^{t-1}, \mathbf{p}(h^{t-1})), \tag{A.9}$$

for some $h^{t-1} \in H^{t-1}(\mathbf{p})$ and $\theta \in \{0, 1\}$, so that $\sigma_\theta$ is not sequentially rational. Let $s' \in S$ be such that $s'(e|\mathbf{e}(h^{t-1})) = \sigma'(e|h^{t-1}, \mathbf{p}(h^{t-1}))$ for all $e \in E$, $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$. Then,

$$U_t(\theta, \sigma', \mathbf{p}|h^{t-1}, \mathbf{p}(h^{t-1})) = u_t(\theta, s', \bar{v}_{s_0^*, s_1^*}|\mathbf{e}(h^{t-1})),$$

for all $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$. Consequently,

$$u_t(\theta, s', \bar{v}_{s_0^*, s_1^*}|\mathbf{e}(h^{t-1})) > u_t(\theta, s_\theta^*, \bar{v}_{s_0^*, s_1^*}|\mathbf{e}(h^{t-1})),$$

for $h^{t-1} \in H^{t-1}(\mathbf{p})$ such that (A.9) holds, contradicting the fact that $s_\theta^* \in \xi_\theta^1(s_0^*, s_1^*)$. Thus $\sigma_\theta$ must be sequentially rational for all histories $h^{t-1} \in H^{t-1}(\mathbf{p})$ with $p_t = p(h^{t-1})$, for each $\theta \in \{0, 1\}$. On the other hand, if $p_t \neq \mathbf{p}(h^{t-1})$ or $h^{t-1} \notin H^{t-1}(\mathbf{p})$, sequential rationality of $\sigma_\theta$ is immediate, from the construction of $\sigma_\theta$ and $\mathbf{p}$. This concludes the proof of existence. $\square$

**Proof of Lemma 2.** Suppose that $\sigma_0, \sigma_1$, $\mathbf{p}$ is an equilibrium. Let $s_0, s_1 \in S$ be such that, for all $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \ldots, T$,

$$s_\theta(e|\mathbf{e}(h^{t-1})) = \sigma_\theta(e|h^{t-1}, \mathbf{p}(h^{t-1})) \text{ for all } e \in E, \theta \in \{0, 1\}. \tag{A.10}$$

Then, from Definition 1 and (A.3), for all $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \dots, T$,

$$\mathbf{p}_e(h^{t-1}) = \bar{v}_{s_0,s_1}(\{\mathbf{e}(h^{t-1}), e\}) \in (0,1), \text{ for all } e \in \{-1, 1\}, \tag{A.11}$$

showing that prices must lie in the interval $(0, 1)$ in any equilibrium. Furthermore, for $\theta \in \{0, 1\}$,

$$U_t(\theta, \sigma_\theta, \mathbf{p}|h^{t-1}, \mathbf{p}(h^{t-1})) = u_t(\theta, s_\theta, \bar{v}_{s_0,s_1}|\mathbf{e}(h^{t-1})). \tag{A.12}$$

Suppose now that $(\sigma_0, \sigma_1) \in \Sigma^{\mathrm{nm}}(\mathbf{p})$. From Definition 2 and (A.10) we obtain

$$s_1(-1|\mathbf{e}(h^{t-1})) = s_0(1|\mathbf{e}(h^{t-1})) = 0, \tag{A.13}$$

so that from (A.3) and (A.11), we have $\mathbf{p}_1(h^{t-1}) \geqslant \lambda \geqslant \mathbf{p}_{-1}(h^{t-1})$ for all $h^{t-1} \in H^{t-1}(\mathbf{p})$, $t = 1, \dots, T$. Further, if $h^{t-1}$ contains at least one sell order (respectively, buy order), then $\mathbf{p}_1(h^{t-1}) = \lambda$ (respectively, $\mathbf{p}_{-1}(h^{t-1}) = \lambda$).

We now establish the bound on payoffs for type $\theta = 0$. The case for $\theta = 1$ is entirely analogous. From (A.5), (A.11) and (A.12), we can write the date 0 expected profits of type $\theta = 0$ as

$$U_1(0, \sigma_0, \mathbf{p} \mid h^0, \mathbf{p}(h^0)) = \sum_{t=1}^{T} \left[ \sum_{e^t \in E^t} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t)(-e_t^t) \right], \tag{A.14}$$

where the term inside the square brackets is expected period-$t$ profit of $\theta = 0$.

Let

$$\mathrm{Supp}_t(s_0) = \{e^t \in E^t \mid q(e^t \mid e^0, s_0) > 0\},$$

$$A_t(s_0) = \{e^t \in Supp_t(s_0) \mid e_t^t = -1\},$$

and

$$A_t^*(s_0) = \{e^t \in A_t(s_0) \mid q(e^t \mid e^0, s_0) > q(e^t \mid e^0, s_N)\}.$$

Note that either $A_t(s_0)$ or $A_t^*(s_0)$ may be empty. From (A.11), if $A_t(s_0)$ is empty, then

$$\sum_{e^t \in E^t} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t)(-e_t^t) = 0.$$

So suppose that $A_t(s_0)$ is nonempty. Then,

$$\begin{aligned}
\sum_{e^t \in E^t} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t)(-e_t^t) &= \sum_{e^t \in A_t(s_0)} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t) \\
&= \sum_{e^t \in A_t(s_0) \setminus A_t^*(s_0)} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t) \\
&\quad + \sum_{e^t \in A_t^*(s_0)} q(e^t \mid e^0, s_0) \bar{v}_{s_0,s_1}(e^t).
\end{aligned}$$

From (A.11) and (A.13), for all $e^t \in A_t(s_0)$,

$$\bar{v}_{s_0,s_1}(e^t) = \frac{q(e^t \mid e^0, s_N)(1-\mu)\lambda}{q(e^t \mid e^0, s_N)(1-\mu) + q(e^t \mid e^0, s_0)\mu(1-\lambda)} \leqslant \lambda, \tag{A.15}$$

with the inequality being strict if $q(e^t \mid e^0, s_0) > 0$. Furthermore, if $e^t \in A_t^*(s_0)$,

$$\bar{v}_{s_0,s_1}(e^t) < \frac{(1-\mu)\lambda}{(1-\mu)+\mu(1-\lambda)} \equiv v^*(\mu,\lambda) < \lambda. \tag{A.16}$$

Then,

$$\begin{aligned}
&\sum_{e^t \in E^t} q(e^t \mid e^0, s_0)\bar{v}_{s_0,s_1}(e^t)(-e_t^t) \\
&\quad < \sum_{e^t \in A_t(s_0)\backslash A_t^*(s_0)} q(e^t \mid e^0, s_0)\lambda + \sum_{e^t \in A_t^*(s_0)} q(e^t \mid e^0, s_0) v^*(\mu,\lambda) \\
&\quad \leqslant \sum_{e^t \in A_t(s_0)\backslash A_t^*(s_0)} q(e^t \mid e^0, s_0)\lambda + \left(1 - \sum_{e^t \in A_t(s_0)\backslash A_t^*(s_0)} q(e^t \mid e^0, s_0)\right) v^*(\mu,\lambda) \\
&\quad \leqslant \sum_{e^t \in A_t(s_0)\backslash A_t^*(s_0)} q(e^t \mid e^0, s_N)\lambda + \left(1 - \sum_{e^t \in A_t(s_0)\backslash A_t^*(s_0)} q(e^t \mid e^0, s_N)\right) v^*(\mu,\lambda) \\
&\quad \leqslant \sum_{e^t \in A_t(s_0)} q(e^t \mid e^0, s_N)\lambda + \left(1 - \sum_{e^t \in A_t(s_0)} q(e^t \mid e^0, s_N)\right) v^*(\mu,\lambda) \\
&\quad \leqslant \sum_{e^t \in E_{-1}^t} q(e^t \mid e^0, s_N)\lambda + \left(1 - \sum_{e^t \in E_{-1}^t} q(e^t \mid e^0, s_N)\right) v^*(\mu,\lambda),
\end{aligned}$$

where $E_{-1}^t \equiv \{e^t \in E^t \mid e_t^t = -1\} \supset A_t(s_0)$. The first inequality above follows from (A.15) and (A.16) and the fact that either $A_t(s_0)\backslash A_t^*(s_0)$ or $A_t^*(s_0)$ is nonempty as $A_t(s_0)$ is nonempty; the third inequality follows from the definitions of $A_t(s_0)$ and $A_t^*(s_0)$; with the remaining inequalities following from the fact that $A_t^*(s_0) \subset A_t(s_0) \subset E_{-1}^t \subset E^t$ and the fact that $v^*(\mu,\lambda) < \lambda$.

Note that, by condition (N),

$$\sum_{e^t \in E_{-1}^t} q(e^t \mid e^0, s_N) = \sum_{e^{t-1} \in E^{t-1}} q(e^{t-1} \mid e^0, s_N) s_N(-1|e^{t-1}) < 1 - 2c \in (0,1).$$

Therefore,

$$\sum_{e^t \in E^t} q(e^t \mid e^0, s_0)\bar{v}_{s_0,s_1}(e^t)(-e_t^t) < k_0\lambda,$$

regardless of whether $A_t(s_0)$ is empty or not, where $k_0 \equiv (1-2c) + 2c\frac{v^*(\mu,\lambda)}{\lambda} \in (0,1)$. But then, from (A.14), $U_1(0, \sigma_0, \mathbf{p} \mid h^0, \mathbf{p}(h^0)) < k_0\lambda T$, completing the proof. $\square$

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