Empirical Asset Pricing
Assignment 01

Morteza Aghajanzadeh* Ge Song†

January 28, 2024

Question 1

(a) Let's define the variables that we need to use in the estimation.

$$f(v_t,\theta) = \begin{bmatrix} R_{t1}-\mu_1 \\ R_{t2}-\mu_2 \\ (R_{t1}-\mu_1)^2-\sigma_1^2 \\ (R_{t2}-\mu_2)^2-\sigma_2^2 \end{bmatrix} \quad , \quad \theta = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \sigma_1^2 \\ \sigma_2^2 \end{bmatrix}$$

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f(v_t,\theta)$$

We know from the lecture that we need to calculate the $\frac{\partial f}{\partial \theta'}$ to get the $\hat{D_T}$:

$$\frac{\partial f(v_t,\theta)}{\partial \theta'} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2(R_{t1}-\mu_1) & 0 & -1 & 0 \\ 0 & -2(R_{t2}-\mu_2) & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \hat{D_T} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial f(v_t,\theta)}{\partial \theta'} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -I$$

We also know that $A_T = I$ and $A_T g_T(\theta) = 0$. Therefore, we can calculate the $\hat{\theta}$:

$$A_T g_T(\theta) = 0 \Rightarrow \quad g_T(\theta) = 0$$

$$g_T(\theta) = \begin{bmatrix} \frac{\sum R_{t1}}{T} - \mu_1 \\ \frac{\sum R_{t2}}{T} - \mu_2 \\ \frac{\sum (R_{t1}-\mu_1)^2}{T} - \sigma_1^2 \\ \frac{\sum (R_{t2}-\mu_2)^2}{T} - \sigma_2^2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{\sum_{t=1}^{T} R_{t1}}{T} \\ \frac{\sum_{t=1}^{T} R_{t2}}{T} \\ \frac{\sum_{t=1}^{T} (R_{t1}-\hat{\mu_1})^2}{T} \\ \frac{\sum_{t=1}^{T} (R_{t2}-\hat{\mu_2})^2}{T} \end{bmatrix} = \begin{bmatrix} \hat{\mu_1} \\ \hat{\mu_2} \\ \hat{\sigma_1^2} \\ \hat{\sigma_2^2} \end{bmatrix} = \hat{\theta}$$

Our calculated $\hat{\theta}$ based on the given data is:

$$\hat{\theta} = \begin{bmatrix} 0.0162 \\ 0.0045 \\ 0.0212 \\ 0.0167 \end{bmatrix}$$

*Department of Finance, Stockholm School of Economics. Email: morteza.aghajanzadeh@phdstudent.hhs.se
†Department of Finance, Stockholm School of Economics. Email: ge.song@phdstudent.hhs.se

```
    mu_1 = sum(df['Stock1'])/len(df['Stock1'])
    mu_2 = sum(df['Stock2'])/len(df['Stock2'])
    sigma_1 = sum((df.Stock1 - mu_1)**2)/(len(df.Stock1))
    sigma_2 = sum((df.Stock2 - mu_2)**2)/(len(df.Stock2))
```

Listing 1: Python code for calculating $\hat{\theta}$

(b) Still we assume that there is no serial correlation in the moments. Therefore, we can calculate the $\hat{S_T}$ as follows:

$$
\begin{aligned}
\hat{S_T} =& \frac{1}{T}\sum_{t=1}^{T} f(v_t,\theta)f(v_t,\theta)' \\
=& \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} (R_{t1}-\mu_1)^2 & (R_{t1}-\mu_1)(R_{t2}-\mu_2) & (R_{t1}-\mu_1)^3-\sigma_1^2(R_{t1}-\mu_1) & (R_{t1}-\mu_1)(R_{t2}-\mu_2)^2-\sigma_2^2(R_{t1}-\mu_1) \\ (R_{t1}-\mu_1)(R_{t2}-\mu_2) & (R_{t2}-\mu_2)^2 & (R_{t1}-\mu_1)^2(R_{t2}-\mu_2)-\sigma_1^2(R_{t2}-\mu_2) & (R_{t2}-\mu_2)^3-\sigma_2^2(R_{t2}-\mu_2) \\ (R_{t1}-\mu_1)^3-\sigma_1^2(R_{t1}-\mu_1) & (R_{t1}-\mu_1)^2(R_{t2}-\mu_2)-\sigma_1^2(R_{t2}-\mu_2) & (R_{t1}-\mu_1)^4-2\sigma_1^2(R_{t1}-\mu_1)^2+\sigma_1^4 & (R_{t1}-\mu_1)^2(R_{t2}-\mu_2)^2-\sigma_1^2(R_{t1}-\mu_1)\sigma_2^2(R_{t2}-\mu_2) \\ (R_{t1}-\mu_1)(R_{t2}-\mu_2)^2-\sigma_2^2(R_{t1}-\mu_1) & (R_{t2}-\mu_2)^3-\sigma_2^2(R_{t2}-\mu_2) & (R_{t1}-\mu_1)^2(R_{t2}-\mu_2)^2-\sigma_1^2(R_{t1}-\mu_1)\sigma_2^2(R_{t2}-\mu_2) & (R_{t2}-\mu_2)^4-2\sigma_2^2(R_{t2}-\mu_2)^2+\sigma_2^4 \end{bmatrix} \\
=& \begin{bmatrix} \hat{\sigma_1}^2 & Cov(\hat{R}_1,\hat{R}_2) & \hat{m}_1^{(3)} & \hat{k}_{12}^{(2)} \\ Cov(\hat{R}_1,\hat{R}_2) & \hat{\sigma_2}^2 & \hat{k}_{21}^{(2)} & \hat{m}_2^{(3)} \\ \hat{m}_1^{(3)} & \hat{k}_{12}^{(2)} & \hat{m}_1^{(4)}-\hat{\sigma}_1^4 & \hat{k}_{12}^{(3)} \\ \hat{k}_{21}^{(2)} & \hat{m}_2^{(3)} & \hat{k}_{12}^{(3)} & \hat{m}_2^{(4)}-\hat{\sigma}_2^4 \end{bmatrix}
\end{aligned}
$$

Here the definition of $\hat{m}_1^{(j)}$ is the one in the lecture notes. For the $\hat{k}_{mn}^{(j)}$, I do not have a good way to write it in the matrix form. Therefore, I just write it as a way to sum up the terms in the matrix.

As we know that two process are independent, and normally distributed, we can calculate the $\hat{S_T}$ as follows:

$$
\hat{S_T} = \begin{bmatrix} \hat{\sigma_1}^2 & 0 & 0 & 0 \\ 0 & \hat{\sigma_2}^2 & 0 & 0 \\ 0 & 0 & 2\hat{\sigma_1}^4 & 0 \\ 0 & 0 & 0 & 2\hat{\sigma_2}^4 \end{bmatrix}
$$

Our calculated $\hat{S_T}$ based on the given data is:

$$
\hat{S_T} = \begin{bmatrix} 0.0212 & 0 & 0 & 0 \\ 0 & 0.0167 & 0 & 0 \\ 0 & 0 & 0.0347 & 0 \\ 0 & 0 & 0 & 0.0011 \end{bmatrix}
$$

```
theta = np.array([mu_1,mu_2, sigma_1, sigma_2])
def f_v(theta,x):
    mu_1 = theta[0]
    mu_2 = theta[1]
    sigma_1 = theta[2]
    sigma_2 = theta[3]
    x_1 = x[0]
    x_2 = x[1]
    f = np.array([x_1-mu_1,x_2-mu_2,(x_1-mu_1)**2-sigma_1,(x_2-mu_2)**2-
        sigma_2]).reshape(len(theta),1)
    return f
def s(theta,x):
    f = f_v(theta,x)
    return f @ f.T
x = np.array([df[['Stock1','Stock2']]])[0]
s_hat = sum([s(theta,i) for i in x])/len(x)
# set non-diagonal elements to zero
```

```
s_hat = s_hat * np.eye(4)
```

Listing 2: Python function for calculating standard error of estimation

(c) Now we want to adjust the standard errors by Newey-West estimator. Therefore, we need to calculate the $\hat{\Gamma_1}$ by using the fact that two distributions are independent:

$$\hat{\Gamma_1} = \begin{bmatrix} \frac{\sum_2^T (R_t^1-\mu_1)(R_{t-1}^1-\mu_1)}{T-1} & 0 & \frac{\sum_2^T (R_t^1-\mu_1)(R_{t-1}^1-\mu_1)^2}{T-1} & 0 \\ 0 & \frac{\sum_2^T (R_t^2-\mu_2)(R_{t-1}^2-\mu_2)}{T-1} & 0 & \frac{\sum_2^T (R_t^2-\mu_2)(R_{t-1}^2-\mu_2)^2}{T-1} \\ \frac{\sum_2^T (R_{t-1}^1-\mu_1)(R_{t-1}^1-\mu_1)^2}{T-1} & 0 & \frac{\sum_2^T (R_{t-1}^1-\mu_1)^2(R_{t-1}^1-\mu_1)^2}{T-1} + \sigma_1^4 & 0 \\ 0 & \frac{\sum_2^T (R_{t-1}^2-\mu_2)(R_{t-1}^2-\mu_2)^2}{T-1} & 0 & \frac{\sum_2^T (R_{t-1}^2-\mu_2)^2(R_{t-1}^2-\mu_2)^2}{T-1} + \sigma_2^4 \end{bmatrix}$$

and then we can calculate the $\hat{S_T}$ as follows:

$$\hat{S_T} = \begin{bmatrix} \hat{\sigma_1}^2 & 0 & 0 & 0 \\ 0 & \hat{\sigma_2}^2 & 0 & 0 \\ 0 & 0 & 2\hat{\sigma_1}^4 & 0 \\ 0 & 0 & 0 & 2\hat{\sigma_2}^4 \end{bmatrix} + \frac{1}{2}(\hat{\Gamma}_1 + \hat{\Gamma}_1')$$

Our calculated $\hat{S_T}$ based on the given data is:

$$\hat{S_T} = \begin{bmatrix} 0.0162 & 0 & 0 & 0 \\ 0 & 0.0160 & 0 & 0 \\ 0 & 0 & 0.04 & 0 \\ 0 & 0 & 0 & 0.0013 \end{bmatrix}$$

```
theta = np.array([mu_1,mu_2, sigma_1, sigma_2])
lag = 1
def gamma(theta,x,lag):
    gamma = {}
    for i in range(1,lag +1):
        lag = np.array(df[['Stock1','Stock2']].shift(i).dropna())
        tempt = []
        for num,j in enumerate(x[i:]):
            tempt.append(f_v(theta,j) @ f_v(theta,lag[num]).T)
        gamma[i] = sum(tempt)/len(tempt)
    gamma = [gamma[i] for i in gamma]
    return sum(gamma)

def s_newywest(theta,x):
    gamma_hat = gamma(theta,x,lag)
    return sum([s(theta,i) for i in x])/len(x) + 0.5 * (gamma_hat +
        gamma_hat.T)
s_hat_newywest = s_newywest(theta,x)
s_hat_newywest * np.eye(4)
```

Listing 3: Python function for calculating Newy-West standard error

(d) Now we want to compare the Sharpe ratio of two stocks. Therefore, we need to test the hypothesis that:

$$\begin{cases} H_0 : \frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} \\ H_1 : \frac{\mu_1}{\sigma_1} \neq \frac{\mu_2}{\sigma_2} \end{cases} \Rightarrow \begin{cases} H_0 : \mu_1\sigma_2 - \mu_2\sigma_1 = 0 \\ H_1 : \mu_1\sigma_2 - \mu_2\sigma_1 \neq 0 \end{cases}$$

now we can define the $R(\theta)$ as follows:

$$R(\theta) = \mu_1\sigma_2 - \mu_2\sigma_1$$

and then rewrite the hypothesis as follows:

$$H_0 : R(\theta) = 0$$
$$H_1 : R(\theta) \neq 0$$

Now we can use the Delta method to find the distribution of $R(\hat{\theta})$:

$$\sqrt{T}(R(\hat{\theta}) - R(\theta)) \xrightarrow{d} N(0, \frac{\partial R(\theta)}{\partial \theta'} V_\theta \frac{\partial R(\theta)}{\partial \theta'})$$
$$\sqrt{T}(R(\hat{\theta})) \xrightarrow{d} N(0, \frac{\partial R(\theta)}{\partial \theta'} V_\theta \frac{\partial R(\theta)}{\partial \theta'})$$

where V_θ is the variance of $\hat{\theta}$. Now we can calculate the $\frac{\partial R(\theta)}{\partial \theta'}$ as follows:

$$\frac{\partial R(\theta)}{\partial \theta'} = \begin{bmatrix} \sigma_2 & -\sigma_1 & -\mu_2 & \mu_1 \end{bmatrix}$$

and we know that $V_\theta = \hat{S_T}$. Therefore, we can find the distribution of $R(\hat{\theta})$ as follows:

$$\frac{\partial R(\theta)}{\partial \theta'} V_\theta \frac{\partial R(\theta)}{\partial \theta'} = \begin{bmatrix} \sigma_2 & -\sigma_1 & -\mu_2 & \mu_1 \end{bmatrix} \hat{S_T} \begin{bmatrix} \sigma_2 \\ -\sigma_1 \\ -\mu_2 \\ \mu_1 \end{bmatrix} = \hat{V}_T$$

Now we can calculate the test statistic as follows:

$$T R(\hat{\theta})' \hat{V}_T^{-1} R(\hat{\theta}) \xrightarrow{d} \chi_1^2$$
$$\frac{T(\mu_1\sigma_2 - \mu_2\sigma_1)^2}{\hat{S}_T} \xrightarrow{d} \chi_1^2$$

Let's calculate the test statistic:

$$\hat{V}_T = \begin{bmatrix} 0.0167 & -0.0212 & -0.0045 & 0.0162 \end{bmatrix} \begin{bmatrix} 0.0212 & 0 & 0 & 0 \\ 0 & 0.0167 & 0 & 0 \\ 0 & 0 & 0.0347 & 0 \\ 0 & 0 & 0 & 0.0011 \end{bmatrix} \begin{bmatrix} 0.0167 \\ -0.0212 \\ -0.0045 \\ 0.0162 \end{bmatrix} = 0.001449$$

(a) $f(v_t, \theta) =$

(b)

(c)

(d) $H_0 : \mu_1 * \sigma_2 - \mu_2 * \sigma_1 = 0$
$R(\theta) = \mu_1 * \sigma_2 - \mu_2 * \sigma_1$
$\theta =$

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \sigma_1 \\ \sigma_2 \end{bmatrix}$$

$\frac{\partial R(\theta)}{\theta'} =$

$$\begin{bmatrix} \sigma_2 & -\sigma_1 & -\mu_2 & \mu_1 \end{bmatrix}$$

$T(R(\hat{\theta_T})'[\frac{\partial R(\theta)}{\theta'} \hat{V} \frac{\partial R(\theta)}{\theta'}']^{-1}(R(\hat{\theta}) \sim \chi^2$
Sharpe Ratio Stock 1: 0.1113
Sharpe Ratio Stock 2: 0.0352
T-test Statistic: 1.4645
p-value: 0.1433 Fail to reject the null hypothesis: No significant difference in Sharpe ratios.

(e) After introducing the correlation matrix:
Sharpe Ratio Stock 1: 0.1113
Sharpe Ratio Stock 2: 0.0352
Sample Correlation: 0.4187
T-test Statistic: 12.5329
Degrees of Freedom: 1198
p-value: 0.0000
Reject the null hypothesis: Stocks have different Sharpe ratios.

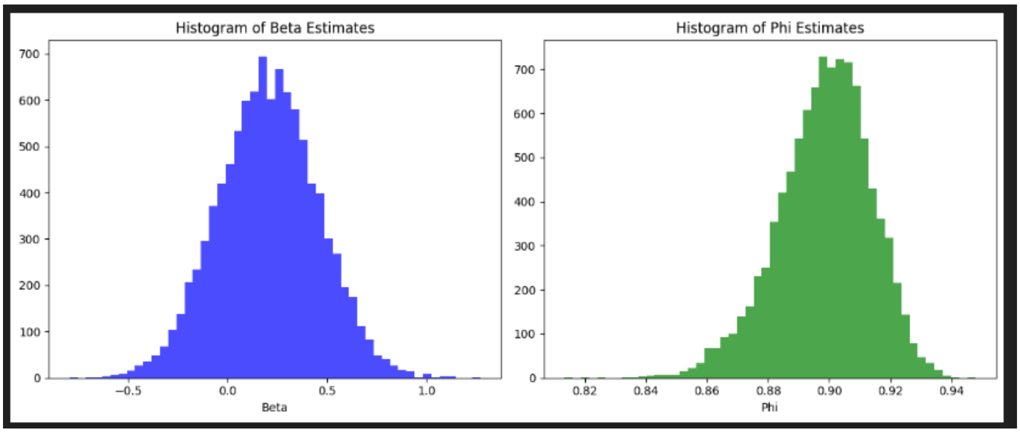
(f) Observed Sharpe Ratio Difference: 0.0761
95% Confidence Interval: [-0.0325, 0.1561]
The sharp ratio difference is not significant at 5% level.

Question 2

(a) Optional

Question 3

(a) N=840



(b) N=240
Looks more skewed to the right side for $\hat{\phi}$