

Empirical Asset Pricing

Assignment 01

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Question 1

(a)

(b)

(c)

(d) Let's define the variabls as follows:

$$f(v_t, \theta) = \begin{bmatrix} \Delta c_t - \mu_c \\ r_{m,t} - \mu_m \\ r_{m,t} - r_{f,t} + \frac{1}{2}(r_{m,t} - \mu_m)^2 - \gamma(r_{m,t} - \mu_m)(\Delta c_t - \mu_c) \\ r_{f,t} + \ln(\delta) - \gamma\Delta c_t + \frac{1}{2}\gamma^2(\Delta c_t - \mu_c)^2 \end{bmatrix}, \quad \theta = \begin{bmatrix} \mu_c \\ \mu_m \\ \gamma \\ \delta \end{bmatrix}$$

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^T f(v_t, \theta)$$

As we can see, the system is exactly identified, since the number of parameters is equal to the number of moments. So, we can use the GMM estimator to estimate the parameters.

$$\begin{aligned} g_T(\theta) &= 0 \\ \Rightarrow \frac{1}{T} \sum_{t=1}^T f(v_t, \theta) &= 0 \\ \Rightarrow \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \Delta c_t - \mu_c \\ r_{m,t} - \mu_m \\ r_{m,t} - r_{f,t} + \frac{1}{2}(r_{m,t} - \mu_m)^2 - \gamma(r_{m,t} - \mu_m)(\Delta c_t - \mu_c) \\ r_{f,t} + \ln(\delta) - \gamma\Delta c_t + \frac{1}{2}\gamma^2(\Delta c_t - \mu_c)^2 \end{bmatrix} &= 0 \\ \Rightarrow \begin{bmatrix} \frac{1}{T} \sum_{t=1}^T \Delta c_t - \mu_c \\ \frac{1}{T} \sum_{t=1}^T r_{m,t} - \mu_m \\ \frac{1}{T} \sum_{t=1}^T r_{m,t} - r_{f,t} + \frac{1}{2}(r_{m,t} - \mu_m)^2 - \gamma(r_{m,t} - \mu_m)(\Delta c_t - \mu_c) \\ \frac{1}{T} \sum_{t=1}^T r_{f,t} + \ln(\delta) - \gamma\Delta c_t + \frac{1}{2}\gamma^2(\Delta c_t - \mu_c)^2 \end{bmatrix} &= 0 \\ \Rightarrow \begin{bmatrix} \mathbb{E}[\Delta c_t] - \mu_c \\ \mathbb{E}[r_{m,t}] - \mu_m \\ \mathbb{E}[r_{m,t} - r_{f,t}] + \frac{1}{2}\mathbb{E}[(r_{m,t} - \mu_m)^2] - \gamma\mathbb{E}[(r_{m,t} - \mu_m)(\Delta c_t - \mu_c)] \\ \mathbb{E}[r_{f,t}] + \ln(\delta) - \gamma\mathbb{E}[\Delta c_t] + \frac{1}{2}\gamma^2\mathbb{E}[(\Delta c_t - \mu_c)^2] \end{bmatrix} &= 0 \end{aligned}$$

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$$\Rightarrow \begin{bmatrix} \mu_c - \mathbb{E}[\Delta c_t] \\ \mu_m - \mathbb{E}[r_{m,t}] \\ \gamma - \frac{\mathbb{E}[r_{m,t} - r_{f,t}] + \hat{\sigma}_m^2/2}{\hat{\sigma}_{mc}} \\ \delta - \exp(-\mathbb{E}[r_{f,t}] + \gamma \mathbb{E}[\Delta c_t] - \frac{1}{2} \gamma^2 \sigma_c^2) \end{bmatrix} = 0$$

$$\rightarrow \theta = \begin{bmatrix} \mathbb{E}[\Delta c_t] \\ \mathbb{E}[r_{m,t}] \\ \frac{\mathbb{E}[r_{m,t} - r_{f,t}] + \hat{\sigma}_m^2/2}{\hat{\sigma}_{mc}} \\ \exp(-\mathbb{E}[r_{f,t}] + \gamma \mathbb{E}[\Delta c_t] - \frac{1}{2} \gamma^2 \sigma_c^2) \end{bmatrix}$$

where $\hat{\sigma}_m^2$ is the sample variance of $r_{m,t}$ and $\hat{\sigma}_{mc}$ is the sample covariance of $r_{m,t}$ and Δc_t .

As we can see it is the same method as we used for estimating in the first method of previous question. The only difference is that now we can estimate the variance of the estimator by using the equation for the variance of the GMM estimator.

$$\hat{S}_T = \frac{1}{T} \sum_{t=1}^T f(v_t, \hat{\theta}) f(v_t, \hat{\theta})$$

where $\hat{\theta}$ is the estimated parameter vector.

Calculate the variance of the estimator analytically is a bit more complicated, but it is possible. I will only drive the variance of the estimator numerically.

(e) Now we change the target moments to be the following:

$$f(v_t, \theta) = \begin{bmatrix} \exp(\ln(\delta) - \gamma \Delta c_t + r_{m,t}) - 1 \\ \exp(\ln(\delta) - \gamma \Delta c_t + r_{f,t}) - 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^T f(v_t, \theta)$$

again the system is exactly identified, since the number of parameters is equal to the number of moments. So, we can use the GMM estimator to estimate the parameters.

$$g_T(\theta) = 0$$

$$\Rightarrow \frac{1}{T} \sum_{t=1}^T f(v_t, \theta) = 0$$

$$\Rightarrow \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \exp(\ln(\delta) - \gamma \Delta c_t + r_{m,t}) - 1 \\ \exp(\ln(\delta) - \gamma \Delta c_t + r_{f,t}) - 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \frac{1}{T} \sum_{t=1}^T \exp(\ln(\delta) - \gamma \Delta c_t + r_{m,t}) - 1 \\ \frac{1}{T} \sum_{t=1}^T \exp(\ln(\delta) - \gamma \Delta c_t + r_{f,t}) - 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \mathbb{E}[\exp(\ln(\delta) - \gamma \Delta c_t + r_{m,t})] - 1 \\ \mathbb{E}[\exp(\ln(\delta) - \gamma \Delta c_t + r_{f,t})] - 1 \end{bmatrix} = 0$$

As our target moments are non-linear, we can not use the same method as we used for the linear moments. So we need to use the numerical optimization methods to estimate the parameters.