1 Problem set 1 – Ph.D. course in Household Finance

Your task is to solve and simulate the buffer-stock savings model of Carroll (1997). The deadline is November 14 at 23:59 and should be submitted individually on the course website in Canvas.

1.1 Model setup

Individuals start their economic life at the age of $t = 25$ and we follow them until the end of their lives at $T = 100$. One time period is a year. The life cycle consists of two phases: a working phase and a retirement phase with exogenous retirement at age 65.

Stochastic process for disposable income We consider the income process of Carroll and Samwick (1997). A working-age individual i, at age $t = 25, 26, ..., 64$, receives disposable income Y_{it}. It has a deterministic hump-shaped life-cycle trend, g_t, a permanent income component z_{it}, and a transitory idiosyncratic income shock, ω_{it}. Disposable income cannot be less than $\underline{Y}$ which is a parsimonious way to account for welfare and transfers. Let $y_{it} = \ln(Y_{it})$. Then for $t \leq 64$:

$$\begin{aligned} y_{it} &= g_t + z_{it} + \omega_{it}, & (1) \\ z_{it} &= \rho z_{it-1} + \eta_{it}, & (2) \\ y_{it} &\geq \ln(\underline{Y}). & (3) \end{aligned}$$

The random variable η_{it} is an idiosyncratic shock to permanent income, which is distributed $\mathrm{N}\left(-\sigma_\eta^2/2, \sigma_\eta^2\right)$. There is income heterogeneity already at age 25 through the initial persistent shock, z_{i25}, which is distributed $\mathrm{N}\left(-\sigma_z^2/2, \sigma_z^2\right)$. The random variable ω_{it} is a transitory income shock. It is distributed

$$\omega_{it} \sim N\left(-\sigma_\omega^2/2, \sigma_\omega^2\right). \qquad (4)$$

Upon retirement, which happens at $t = 65$, individuals have a safe pension income. It is modeled as a deterministic replacement rate, λ, relative to permanent labor income at 64:

$$\begin{aligned} Y_{i,t} &= \lambda Y_{i,64}^p \\ &= \lambda \exp(g_{64} + z_{i64}), \quad t \geq 65. \qquad (5) \end{aligned}$$

Preferences Individuals have CRRA preferences over consumption:

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \qquad (6)$$

where γ is the coefficient of risk aversion.

Budget constraints and laws of motions Individuals choose consumption (C_t) and savings (A_t) in every period subject to a no-borrowing constraint:

$$C_t + A_t = X_t, \tag{7}$$
$$A_t \geq 0, \tag{8}$$

where X_t denotes cash on hand. Its law of motion is:

$$X_{25} = \hat{A}_{25} + Y_{25}, \tag{9}$$
$$X_{t+1} = A_t R + Y_{t+1}, \quad t = 25, ..., 99 \tag{10}$$

where initial wealth is distributed

$$\ln(\hat{A}_{25}) \sim N(\mu_a - \sigma_A^2/2, \sigma_A^2). \tag{11}$$

The optimization problem Let $\mathcal{S}_t = \{Z_t, X_t\}$ denote the state variables for $t = 1, ..., T$. The optimization problem is given by:

$$V_t(\mathcal{S}_t) = \max_{C_t, A_t} u(C_t) + \beta E_t [V_{t+1}], \tag{12}$$

subject to (1)-(4) and (5)-(10), and where $V_{T+1} = 0$. The parameter β is the discount factor.

1.2 Parameter values

Let the numeraire of the model be SEK 10,000. The parameter values are:

- A life-cycle is from 25 to 100 years ($T = 75$)
- 40 years of working life ($T_w = 40$)
- 35 years of retirement ($T_r = 35$)
- The life-cycle profile for e^{g_t} is provided in the Excel file Income_profile.xlsx on Canvas. The unit is SEK 10,000.
- $\underline{Y} = 4.8$
- $\lambda = 0.80$
- $R = 1.02$
- $\beta = 0.945$
- $\gamma = 2$
- $\rho = 1$

- $\sigma_\omega = 0.10$
- $\sigma_\eta = 0.0713$
- $\sigma_z = 0.589$
- $\mu_A = 1.916$
- $\sigma_A = 2.129$

1.3 Assignment (total of 25 points)

1.3.1 Part 1 (20 points)

- Solve the model above in a programming language of your choice, using dynamic programming. In particular, use the endogenous grid point method of Carroll (2006). Report policy functions in a graph, akin to the lecture notes.
- Simulate the model with 1,000 individuals. Display the average life-cycle profiles for Y_t, A_t and C_t. In another graph, report simulated paths for a particular consumer.

1.3.2 Useful sources

- For the Tauchen or Roewenherst algorithms:
  - Giulio Fella: `https://giuliofella.net/`
  - Martin Flodén: A Note on the Accuracy of Markov-Chain Approximations to Highly Persistent AR(1)-Processes, Economics Letters, June 2008, 99, 516-520. Matlab code: `https://martinfloden.net/files/ar1_processes_matlab_code.zip`
- Lecture 2 of Ben Moll's first-year course: `https://benjaminmoll.com/wp-content/uploads/2021/04/Lecture2_EC442_Moll.pdf`
- Ben Moll and Greg Kaplan have a lot of code: `http://benjaminmoll.com/ha_codes/`
  - Have a look at the readme.txt file and egp_IID_lifecycle.m if you want
- Lot's of other code, e.g. Chris Caroll: `http://econ.jhu.edu/People/CCarroll/EndogenousArchive.zip`

1.3.3 Some concrete tips

- Separate model parameters from technical parameters – you want flexibility also on the technical ones
- Construct the objects necessary for integration just once, before the solution algorithm starts

- Define subroutines that represent the laws of motions for X and z: $X_{t+1}(a, z_{j,t})$ and $z_{t+1}(z_{j,t})$. Notice that they return vectors $(N_z \cdot N_q) \times 1$ and each outcome is associated with probability $w_i \cdot \pi(z_{k,t+1}|z_{j,t})$.
- The simulation:
  - Use the same set of shocks every time to be able to debug efficiently. Draw them once, then save them and then load them every time.
  - Re-use as much of the code from the solution as possible, for instance the law of motions for the state variables.
  - Make the problem minimalistic first: no need to start with large T, or with dense grids, or with income uncertainty.

1.3.4 Part 2 (5 points)

Extend the model so that it resembles Cocco et al. (2005).

- Define $A_t = S_t + B_t$, where S_t is a stock market index and B_t is a risk-free bond. The bond earns the return $R^f = (1 + r^f)$. The stock market index has a stochastic return which is log-normally distributed:

$$R_t = \exp(\ln(R^f) + \mu + \varepsilon_t) \tag{13}$$

  where μ is the equity premium and $\varepsilon_t \sim N(-\sigma_\varepsilon^2/2, \sigma_\varepsilon^2)$. The choice of the equity share, $\alpha_t = \frac{S_t}{S_t+B_t}$, becomes a second control variable with the constraint:

$$\alpha_t \in [0, 1]. \tag{14}$$

- The law of motion for cash-in-hand becomes $X_{t+1} = Y_{t+1} + A_t(R^f + \alpha_t(R_{t+1} - R^f))$.
- To solve this extended model, you can still use the endogenous grid point method in that you use a grid for A. For a give choice of A_t the individual strives to satisfy the Euler equation for the excess return:

$$0 = E_t\left[\beta C_{t+1}^{-\gamma}(R_{t+1} - R^f)\right] \text{, or} \tag{15}$$

$$0 = E_t\left[M_{t+1}(R_{t+1} - R^f)\right] \tag{16}$$

  where $M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is the individual's stochastic discount factor. This optimality conditions determines the choice of α_t. You can search for the optimal choice of α_t in a number of ways. For instance, on pages 94–95 in Judd's book there is a description of the bracketing algorithm. You can also use Golden section search `https://en.wikipedia.org/wiki/Golden-section_search`, or attempt to use Matlab's function 'fmincon' (if you have the Optimizer toolbox).
- Since the model is on annual frequency, a reasonable calibration is $\mu = 0.04$ and $\sigma_\varepsilon = 0.18$.

References

Carroll, C. (2006, September). The method of the endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters 91*(3), 312–320.

Carroll, C. D. (1997, 02). Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis*. *The Quarterly Journal of Economics 112*(1), 1–55.

Carroll, C. D. and A. A. Samwick (1997). The nature of precautionary wealth. *Journal of Monetary Economics 40*, 41–71.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout (2005). Consumption and portfolio choice over the life cycle. *Review of Financial Studies 18*(2), 491–533.