Resource Reallocation with Carbon Emission Policies

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Abstract

Governments worldwide are implementing policies to mitigate carbon emissions, necessitating significant economic resource reallocation. This study investigates the economic consequences of these shifts, focusing on quantifying reallocation costs and delineating conditions for optimal resource distribution. Utilizing a model of a closed economy characterized by monopolistic competition and firm heterogeneity, the study distinguishes between 'green' (eco-friendly) and 'brown' (fossil fuel-dependent) capital in the production function. The emission function has been extended to incorporate variable parameters that reflect the environmental impacts of these resources. This analysis aims to provide critical insights into the dynamics of resource allocation under environmental policy constraints, underscoring the trade-offs between economic output and environmental sustainability. The results are intended to guide policymaking by clarifying the economic costs associated with transitioning to a low-carbon economy and outlining strategies to balance environmental and economic objectives optimally.

Environment and Technology

- Total real output is:

$$Y = \Pi_1^S Y_s^{\lambda_s}, \quad \text{where} \quad \sum^S \lambda_s = 1 \tag{1}$$

- The real output in each sector s is:

$$Y_s = \left(\sum_{i=1}^{I} Y_{si}^{\frac{\sigma_s - 1}{\sigma_s}} \right)^{\frac{\sigma_s}{\sigma_s - 1}} \tag{2}$$

- The real output for firms i in sector s is:

$$Y_{si} = \hat{A}_{si} \hat{K}_{si}^{\beta_s} L_{si}^{1-\beta_s} \quad , \qquad \hat{K} = (\alpha_s G_{si}^{\frac{\gamma_s - 1}{\gamma_s}} + (1-\alpha_s) B_{si}^{\frac{\gamma_s - 1}{\gamma_s}})^{\frac{\gamma_s}{\gamma_s - 1}} \tag{3}$$

- The firm's emission is:

$$E_{si} = \tilde{A}_{si} B_{si} \tag{4}$$

- The nominal profit for firms:

$$\pi_{si} = (1+\tau_s^p) P_s Y_s - ([(1+\tau_{G_s}) r_{si} G_{si} + (1+\tau_{B_s}) r_{si} B_{si} + (1+\tau_{l_s}) w_{si} l_{si}] + \tau_E E_{si}) \tag{5}$$

Optimal Allocation

- To maximize profit, the firm follows a two-step process. First, it determines the optimal combination of capital and labor. Then, it selects the appropriate price level.

$$\max_{G_{si},B_{si},L_{si}} \quad -Cost \quad \text{s.t.} \qquad \hat{A}_{si}\hat{K}_{si}^{\beta_s}L_{si}^{1-\beta_s} = \bar{Y}_{si}$$

$$\frac{G_{si}}{B_{si}} = z_{si}^{k} = \left(\frac{\alpha_s}{1-\alpha_s}\frac{(1+\tau_s^B)r_{si} + \tau_s^E\tilde{A}}{(1+\tau_s^G)r_{si}}\right)^{\gamma_s} \tag{6}$$

$$\frac{L_{si}}{\hat{K}_{si}} = z_{si}^{l} = \frac{1-\beta}{\beta}\frac{1}{\alpha}\left(\alpha_s + (1-\alpha_s)z_{si}^{k\,-\frac{\gamma_s-1}{\gamma_s}}\right)^{\frac{1}{\gamma_s-1}}\frac{(1+\tau_s^G)r_{si}}{(1+\tau_s^W)w_{si}} \tag{7}$$

- Let's define the use the optimal ratios ($z_{si}^k \equiv (\frac{G_{si}}{B_{si}})*$ and $z_{si}^l \equiv (\frac{L_{si}}{\hat{K}_{si}})^*$)

$$\begin{aligned}\hat{K}_{si} &= B_{si}(\alpha_s z_{si}^{k\,\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s))^{\frac{\gamma_s}{\gamma_s-1}} \\ &= G_{si}(\alpha_s + (1-\alpha_s)z_{si}^{k\,\frac{1-\gamma_s}{\gamma_s}})^{\frac{\gamma_s}{\gamma_s-1}}\end{aligned} \tag{8}$$

- Optimal capital level for given output is (see appendix 1.1 for proof):

$$G_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}}\left(\alpha_s + (1-\alpha_s)z_{si}^{k\,-\frac{\gamma_s-1}{\gamma_s}}\right)^{\frac{\gamma_s}{1-\gamma_s}} z_{si}^{l\ (\beta_s-1)}$$

$$B_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}}\left(\alpha_s z_{si}^{k\,\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s)\right)^{\frac{\gamma_s}{1-\gamma_s}} z_{si}^{l\ (\beta_s-1)}$$

$$L_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}} z_{si}^{l\ \beta_s}$$

- The emission level for given output is (see appendix 1.2 for proof):

$$E_{si} = \frac{\tilde{A}_{si}}{\hat{A}_{si}}\left(\alpha_s z_{si}^{k\,\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s)\right)^{\frac{\gamma_s}{1-\gamma_s}} z_{si}^{l\ \beta_s-1}\bar{Y}_{si} = \psi_{si}\bar{Y}_{si} \tag{9}$$

- The cost of production is (see appendix 1.3 for details of the definition):

$$\begin{aligned}\Rightarrow C(\bar{Y}_{si}) &= \left[(1+\tau_{G_{si}})r_{si}^G G_{si} + (1+\tau_{B_{si}})r_{si}^K B_{si} + (1+\tau_{l_{si}})w_{si}l_{si}\right] + \tau_E E_{si} \\ &= C_{si}\bar{Y}_{si}\end{aligned}$$

- The optimal price level is (see appendix 1.5 for proof):

$$P_{si} = \frac{1}{1+\tau_{si}^p}\frac{\sigma_s}{\sigma_s - 1}C_{si} \tag{10}$$

$$C_{si} = \frac{1}{\hat{A}_{si}} {z_{si}^l}^{\beta_s} \left[r_{si}^G (\alpha_s + (1-\alpha_s) {z_{si}^k}^{-\frac{\gamma_s - 1}{\gamma_s}})^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{-1} \right.$$
$$+ (r_{si}^B + \tau_E \tilde{A}_{si})(\alpha_s {z_{si}^k}^{\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s))^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{-1}$$
$$\left. + w_{si} \right]$$

- Then the optimal output is:

$$Y_{si} = \left(\frac{P_s}{P_{si}}\right)^{\sigma_s} Y_s = P_s^{\sigma_s} Y_s \frac{1}{P_{si}^{\sigma_s}} = \kappa_s \frac{1}{P_{si}^{\sigma_s}} \tag{11}$$

- The optimal labor level is:

$$L_{si} = \frac{\kappa_s}{\hat{A}_{si} P_{si}^{\sigma_s}} {z_{si}^l}^{\beta_s} \tag{12}$$

Technology

We need to find an expression for the technologies ($\hat{A}_{si}$ and $\tilde{A}_{si}$) based on observable variables. (see appendix 1.6 and 1.7 for proof)

$$\hat{A}_{si} = \nu_s \frac{(P_{si} Y_{si})^{\frac{\sigma_s}{\sigma_s - 1}}}{\hat{K}_{si}^{\beta_s} L_{si}^{1-\beta_s}}, \quad \text{where} \quad \nu_s = \frac{1}{P_s (P_s Y_s)^{\frac{1}{\sigma_s - 1}}}$$

$$\tilde{A}_{si} = \frac{E_{si}}{B_{si}}$$

Wedges

- The marginal nominal product of each input should be equal to the marginal cost of that for the maximizing firm (see appendix 1.9 for proof).

$$\alpha_s \beta_s \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si} Y_{si}}{\hat{K}_{si}} (\frac{\hat{K}_{si}}{G_{si}})^{\frac{1}{\gamma_s}} = \frac{\partial Cost_{si}}{\partial G_{si}}$$

$$(1-\alpha_s)\beta_s \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si} Y_{si}}{\hat{K}_{si}} (\frac{\hat{K}_{si}}{B_{si}})^{\frac{1}{\gamma_s}} = \frac{\partial Cost_{si}}{\partial B_{si}}$$

$$(1-\beta_s) \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si} Y_{si}}{L_{si}} = \frac{\partial Cost_{si}}{\partial L_{si}}$$

Estimation

I will follow the the Kmenta (1967),Blundell and Bond (2000),Davis, Fisher, and Whited (2014) to estimate the parameters of the model. The production function is:

$$
\begin{aligned}
Y_{it} &= \hat{A}_{it}(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}})^{\frac{\beta\gamma}{\gamma-1}} L_{it}^{1-\beta} \\
\ln Y_{it} &= \ln \hat{A}_{it} + \frac{\beta\gamma}{\gamma-1}\ln(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}}) + (1-\beta)\ln L_{it}
\end{aligned}
$$

as I cannot measure the production directly, I will use the total revenue in the place of the production. So, the equation becomes:

$$
\begin{aligned}
\ln \nu_t (P_{it}Y_{it})^{\frac{\sigma}{\sigma-1}} &= \ln \hat{A}_{it} + \frac{\beta\gamma}{\gamma-1}\ln(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}}) + (1-\beta)\ln L_{it} \\
\ln P_{it}Y_{it} &= \frac{\sigma-1}{\sigma}(\ln \hat{A}_{it} + \frac{\beta\gamma}{\gamma-1}\ln(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}}) + (1-\beta)\ln L_{it}) - \ln \nu_s
\end{aligned}
$$

where ν_s is the price of the output of the sector s.

I take the first-order approximation of the production function around the $\gamma_s = 1$. This requires to take the limit of the third term in the production function as γ_s approaches 1. The limit is:

$$
\lim_{\gamma_s \to 1} \frac{\gamma}{\gamma-1}\ln(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}}) = \alpha \ln G_{it} + (1-\alpha)\ln B_{it}
$$

and the limit of the derivative of this same term

$$
\lim_{\gamma_s \to 1} \frac{\partial}{\partial \gamma_s}\left(\frac{\gamma}{\gamma-1}\ln(\alpha G_{it}^{\frac{\gamma-1}{\gamma}} + (1-\alpha)B_{it}^{\frac{\gamma-1}{\gamma}})\right) = \frac{\alpha(1-\alpha)}{2}(\ln G_{it} - \ln B_{it})^2
$$

The first-order approximation of the production function is:

$$
\begin{aligned}
\ln P_{it}Y_{it} \sim & \frac{\sigma}{\sigma-1}(\ln \hat{A}_{it}) + \frac{\sigma}{\sigma-1}\beta\alpha \ln G_{it} + \frac{\sigma}{\sigma-1}\beta(1-\alpha)\ln B_{it} + \frac{\sigma}{\sigma-1}(1-\beta)\ln L_{it} \\
& + \frac{\sigma}{\sigma-1}\beta\frac{\alpha(1-\alpha)(\gamma_s-1)}{2}(\ln G_{it} - \ln B_{it})^2 - \ln \nu_t
\end{aligned}
$$

- IV:

- Firm level:
  I assume that the $\hat{A}_{it} \equiv e^a e^{u_{it}}$, where a is the common component of productivity and u_{it} is the unpredictable component. We can rewrite the production function as:

$$
\begin{aligned}
\ln P_{it}Y_{it} \sim & \frac{\sigma}{\sigma-1}(a + u_{it}) + \frac{\sigma}{\sigma-1}\beta\alpha \ln G_{it} + \frac{\sigma}{\sigma-1}\beta(1-\alpha)\ln B_{it} + \frac{\sigma}{\sigma-1}(1-\beta)\ln L_{it} \\
& + \frac{\sigma}{\sigma-1}\beta\frac{\alpha(1-\alpha)(\gamma_s-1)}{2}(\ln G_{it} - \ln B_{it})^2 - \ln \nu_t
\end{aligned}
$$

To estimate this approximation, I assume that the error term can be decomposed as $u_{it} = f_i + \epsilon_{it}$ where f_i varies across firm. Then I can use OLS with firm fixed effects within each Industry and appropriately defined coefficients:

$$\ln P_{it}Y_{it} = \beta_0^Y + \beta_G^Y \ln G_{it} + \beta_B^Y \ln B_{it} + \beta_L^Y \ln L_{it} + \beta_{GB}^Y (\ln G_{it} - \ln B_{it})^2 + \mu_i + \mu_t + \epsilon_{it}$$

The industry effects will be absorbed by the constant term as the estimation is done within each industry.

- AR(1):
  Alternatively, I can assume that the error term is $u_{it} = z_{it} + \epsilon_{it}$ where z_{it} is a predictable component that follows the AR(1) process $z_{it} = \rho z_{it-1} + \nu_{it}$. Then I will rewrite the approximation of the production function as:

$$\begin{aligned}
\ln P_{it}Y_{it} - \rho \ln P_{it-1}Y_{it-1} = {} & \beta_0^Y (1-\rho) + \beta_G^Y (\ln G_{it} - \rho \ln G_{it-1}) \\
& + \beta_B^Y (\ln B_{it} - \rho \ln B_{it-1}) + \beta_L^Y (\ln L_{it} - \rho \ln L_{it-1}) \\
& + \beta_{GB}^Y [(\ln G_{it} - \ln B_{it})^2 - \rho (\ln G_{it-1} - \ln B_{it-1})^2] \\
& + (z_{it} - \rho z_{it-1}) + (\epsilon_{it} - \rho \epsilon_{it-1})
\end{aligned}$$

  where $(z_{it} - \rho z_{it-1}) + (\epsilon_{it} - \rho \epsilon_{it-1}) = \nu_{it} + (\epsilon_{it} - \rho \epsilon_{it-1})$ is the zero mean error term. Then I can use GMM to estimate the four coefficients with five instruments, namely, $\ln G_{it-1}$, $\ln B_{it-1}$, $\ln L_{it-1}$, $(\ln G_{it-1} - \ln B_{it-1})^2$, and $\ln P_{it-1}Y_{it-1}$.

$$\begin{aligned}
\ln P_{it}Y_{it} = {} & \beta_0^Y (1-\rho) + \beta_G^Y \ln G_{it} - \rho \beta_G^Y \ln G_{it-1} \\
& + \beta_B^Y (\ln B_{it} - \rho \ln B_{it-1}) + \beta_L^Y (\ln L_{it} - \rho \ln L_{it-1}) \\
& + \beta_{GB}^Y [(\ln G_{it} - \ln B_{it})^2 - \rho (\ln G_{it-1} - \ln B_{it-1})^2] \\
& + \rho \ln P_{it-1}Y_{it-1} + (z_{it} - \rho z_{it-1}) + (\epsilon_{it} - \rho \epsilon_{it-1})
\end{aligned}$$

Now given the estimates of the equation above, I can estimate the parameters of the production function. First two parameters are given by the following ratios:

$$\begin{aligned}
\frac{\beta_G}{\beta_B} &= \frac{\alpha}{1-\alpha} \to \alpha \checkmark \\
\frac{\beta_G}{\beta_L} &= \frac{\alpha\beta}{1-\beta} \to \beta \checkmark
\end{aligned}$$

Now, I know the the α and β parameters. I can use the following ratios to estimate the σ:

$$\begin{aligned}
(1-\beta_s)\frac{\sigma_s - 1}{\sigma_s}\frac{P_{si}Y_{si}}{L_{si}} &= \frac{\partial Cost}{\partial L} = (1+\tau_s^W) w_{si} \\
\Rightarrow \frac{(1+\tau_s^W) w_{si} L}{P_{it}Y_{it}} &= \frac{WL}{P_{it}Y_{it}} = (1-\beta)\frac{\sigma - 1}{\sigma} \to \sigma \checkmark
\end{aligned}$$

Finally, Given the identified parameters, I can estimate the γ using the following ratio:

$$\frac{\beta_{GB}}{\beta_B \beta_G} = \frac{\sigma}{\sigma - 1}\frac{(\gamma - 1)}{2\beta} \to \gamma \checkmark$$

Given the estimated parameters of the production function, I set the production productivity to match the total revenue in the sector. Then I can estimate the emission inefficiency to match the emission intensity of the sector. The estimated parameters are given in the Table 1.

Table 1: Parameters to be estimated

Parameter	**Source/Moment**
Panel A: Estimation	
α_s	β_G/β_B
β_s	β_G/β_L
σ_s	WL/PY
γ_s	$\beta_{GB}/\beta_B\beta_G$
$Mean(\log(\hat{A}_{si}))$	$Mean(L_{si})$
$Sd(\log(\hat{A}))$	$Sd(L_{si})$
$Mean(\log(\tilde{A}_{si}))$	$Mean(E/PY)$
$Sd(\log(\tilde{A}))$	$Sd(E/PY)$
$Corr(\log(\hat{A}), \log(\tilde{A}))$	$Corr(PY, E/PY)$
Panel B: Additional Moments	
$\left(\frac{\alpha_s}{1-\alpha_s}\frac{(1+\tau_s^B)r_{si}+\tau_s^E\tilde{A}}{(1+\tau_s^G)r_{si}}\right)^{\gamma_s}$	$z_k = G/B$
$\frac{1-\beta}{\beta}\frac{1}{\alpha}\left(\alpha_s + (1-\alpha_s){z_{si}^k}^{-\frac{\gamma_s-1}{\gamma_s}}\right)^{\frac{1}{\gamma_s-1}} \frac{(1+\tau_s^G)r_{si}}{(1+\tau_s^W)w_{si}}$	$z_l = L/K$
$\gamma\frac{\tilde{A}}{r^B+\tau_E\tilde{A}}z_k$	$\partial z_k/\partial\tau_E$
$\frac{1-\alpha}{\alpha {z_{si}^k}^{\frac{\gamma_s-1}{\gamma_s}}+(1-\alpha)}\frac{\tilde{A}}{r^B+\tau_E\tilde{A}}z_l$	$\partial z_l/\partial\tau_E$
$\frac{\tilde{A}}{r^B+\tau_E\tilde{A}}\left[-\frac{\gamma\alpha}{\circ} - (\frac{w}{\frac{r}{z_l}+w})\frac{1-\alpha}{\triangle}\right]$	$\partial\ln\epsilon/\partial\tau_E$
$\dfrac{\partial\ln\epsilon/\partial\tau_E}{\dfrac{1}{\tau_E}+\partial\ln\epsilon/\partial\tau_E}$	$\Delta(\frac{E}{PY})/\Delta(\frac{C}{PY})$
Panel C: Calibration	
r	5%
w	500 TSEK

Additional Moments

1.

$$\ln z_k = \gamma(\ln \frac{\alpha}{1-\alpha} + \ln(r^B + \tau_E \tilde{A}) - \ln r^G) = \gamma(\ln \frac{\alpha}{1-\alpha} - \ln r^G) + \gamma \ln(r^B + \tau_E \tilde{A})$$

$$\Rightarrow \frac{\partial z_k / \partial \tau_E}{z_k} = \gamma \frac{\tilde{A}}{r^B + \tau_E \tilde{A}} > 0 \Bigg|_{\partial \tilde{A}/\partial \tau_E = 0}$$

Regression:

$$\ln z_k = \mu_i + \mu_t + \beta^{z_k}_{\tau_E} \tau_E + \epsilon_{z_k}$$

2.

$$\ln z_l = \ln \frac{1-\beta}{\beta} - \ln \alpha + \frac{1}{\gamma_s} \ln(\frac{G}{K}) + \ln r^G - \ln W$$

$$\Rightarrow \frac{\partial z_l / \partial \tau_E}{z_l} = \frac{1}{\gamma_s} \frac{\partial \ln(\frac{G}{K})}{\partial \tau_E} = \frac{1}{\gamma_s} \frac{\gamma(1-\alpha)}{\alpha z_k^{\frac{\gamma-1}{\gamma}} + (1-\alpha)} \cdot \frac{\tilde{A}}{r^\beta + \tau_E \tilde{A}}$$

$$\Rightarrow \frac{\partial z_l / \partial \tau_E}{z_l} = \frac{1-\alpha}{\alpha z_{si}^{k\,\frac{\gamma_s - 1}{\gamma_s}} + (1-\alpha)} \frac{\tilde{A}}{r^B + \tau_E \tilde{A}} > 0 \Bigg|_{\partial \tilde{A}/\partial \tau_E = 0}$$

Regression:

$$\ln z_l = \mu_i + \mu_t + \beta^{z_l}_{\tau_E} \tau_E + \epsilon_{z_l}$$

3.

$$Y_i = \kappa_s p_i^{-\sigma_s} \Rightarrow P_i Y_i = \kappa_s p_i^{1-\sigma_s} \Rightarrow \ln P_i Y_i = \ln \kappa_s + (1-\sigma_s) \ln p_i$$

$$
\begin{aligned}
\Rightarrow \frac{\partial P_iY_i/\partial\tau_E}{P_iY_i} &= (1-\sigma_s)\frac{\partial p_i/\partial\tau_E}{p_i} = (1-\sigma_s)\frac{\partial \frac{\sigma_s}{\sigma_s-1}C_{si}/\partial\tau_E}{p_i} \\
&= \frac{-\sigma_s}{p_i}\frac{\partial}{\partial\tau_E}\frac{1}{\hat{A}_{si}}\left[\underbrace{r^G(\frac{G}{K})z_L^{\beta-1} + r^B(\frac{B}{K})z_L^{\beta-1}}_{r=r^G(\frac{G}{K})+r^B(\frac{B}{K})} + wz_L^\beta\right] \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}\frac{\partial}{\partial\tau_E}z_L^\beta(\frac{r}{z_l}+w) \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}\left[(\frac{r}{z_l}+w)\beta z_L^{\beta-1}\frac{\partial z_L}{\partial\tau_E} + z_L^\beta(\frac{\partial r/\partial\tau_E z_l - r\partial z_l/\partial\tau_E}{z_l^2})\right] \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}z_L^\beta\left[(\frac{r}{z_l}+w)\beta\frac{\partial z_l/\partial\tau_E}{z_l} + \frac{\partial r/\partial\tau_E}{z_l} - \frac{r}{z_l}\frac{\partial z_l/\partial\tau_E}{z_l}\right] \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}z_L^\beta\left[((\frac{r}{z_l}+w)\beta - \frac{r}{z_l})\frac{\partial z_l/\partial\tau_E}{z_l} + \frac{\partial r/\partial\tau_E}{z_l}\right] \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}z_L^\beta\left[((\frac{r}{z_l}+w)\beta - \frac{r}{z_l})\frac{\partial z_l/\partial\tau_E}{z_l} + \underbrace{r^G(\frac{G}{K})\frac{\partial\ln(\frac{G}{K})/\partial\tau_E}{z_l} + r^B(\frac{B}{K})\frac{\partial\ln(\frac{B}{K})/\partial\tau_E}{z_l}}_{0}\right] \\
&= \frac{-\sigma_s}{\hat{A}_{si}p_i}z_L^\beta\frac{\tilde{A}}{r^\beta+\tau_E\tilde{A}}((\frac{r}{z_l}+w)\beta - \frac{r}{z_l})\frac{1-\alpha}{\triangle}
\end{aligned}
$$

4.

$$
\begin{aligned}
&\frac{E}{\nu_s(PY)^{\frac{\sigma_s}{\sigma_s-1}}} = \frac{\tilde{A}}{\hat{A}}(\frac{B}{K})z_l{}^{\beta-1}, \quad \epsilon \equiv \frac{E}{PY}, \quad c \equiv \frac{\tau_E E}{PY} \\
&\Rightarrow \ln\epsilon = \ln\frac{\tilde{A}}{\hat{A}} + \ln(\frac{B}{K}) + (\beta-1)\ln z_l + \frac{1}{\sigma-1}\ln PY + \ln\nu_s \\
&\Rightarrow \ln c = \ln\tau_E + \ln\frac{\tilde{A}}{\hat{A}} + \ln(\frac{B}{K}) + (\beta-1)\ln z_l + \frac{1}{\sigma-1}\ln PY + \ln\nu_s \\
&\Rightarrow \frac{\partial\ln\epsilon}{\partial\tau_E} = \frac{\partial\ln\frac{B}{K}}{\partial\tau_E} + (\beta-1)\frac{\partial\ln z_l}{\partial\tau_E} + \frac{1}{\sigma-1}\frac{\partial\ln PY}{\partial\tau_E} \\
&\Rightarrow \frac{\partial\ln\epsilon}{\partial\tau_E} = \frac{\partial\ln\frac{B}{K}}{\partial\tau_E} + (\beta-1)\frac{\partial\ln z_l}{\partial\tau_E} - \frac{\sigma}{\sigma-1}\frac{z_L^\beta}{\hat{A}_{si}p_i}\frac{\tilde{A}}{r^\beta+\tau_E\tilde{A}}((\frac{r}{z_l}+w)\beta - \frac{r}{z_l})\frac{1-\alpha}{\triangle}
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial \ln \epsilon}{\partial \tau_E} &= -\frac{\gamma\alpha}{\circ} \cdot \frac{\tilde{A}}{r^\beta + \tau_E \tilde{A}} + (\beta - 1)\frac{1-\alpha}{\triangle}\frac{\tilde{A}}{r^B + \tau_E \tilde{A}} - \frac{\sigma}{\sigma - 1}\frac{z_L^\beta}{\hat{A}_{si} p_i}\frac{\tilde{A}}{r^\beta + \tau_E \tilde{A}}((\frac{r}{z_l} + w)\beta - \frac{r}{z_l})\frac{1-\alpha}{\triangle} \\
&= \frac{\tilde{A}}{r^B + \tau_E \tilde{A}}\left[-\frac{\gamma\alpha}{\circ} + (\beta - 1)\frac{1-\alpha}{\triangle} - \frac{\sigma}{\sigma - 1}\frac{z_L^\beta}{\hat{A}_{si} p_i}((\frac{r}{z_l} + w)\beta - \frac{r}{z_l})\frac{1-\alpha}{\triangle}\right] \\
&= \frac{\tilde{A}}{r^B + \tau_E \tilde{A}}\left[-\frac{\gamma\alpha}{\circ} + (\beta - 1)\frac{1-\alpha}{\triangle} - \frac{1}{\frac{r}{z_l} + w}((\frac{r}{z_l} + w)\beta - \frac{r}{z_l})\frac{1-\alpha}{\triangle}\right] \\
&= \frac{\tilde{A}}{r^B + \tau_E \tilde{A}}\left[-\frac{\gamma\alpha}{\circ} - (\frac{w}{\frac{r}{z_l} + w})\frac{1-\alpha}{\triangle}\right]\Bigg|_{\partial \ln(\hat{A}/\tilde{A})/\partial \tau_E = 0}
\end{aligned}$$

$$\begin{aligned}
&\ln c = \ln \tau_E + \ln \epsilon \Rightarrow \frac{\partial \ln c}{\partial \tau_E} = \frac{1}{\tau_E} + \frac{\partial \ln \epsilon}{\partial \tau_E} \\
&\Rightarrow \frac{\partial \ln \epsilon}{\partial \ln c} = \frac{\partial \ln \epsilon / \partial \tau_E}{\partial \ln c / \partial \tau_E} = \frac{\partial \ln \epsilon / \partial \tau_E}{\frac{1}{\tau_E} + \partial \ln \epsilon / \partial \tau_E}
\end{aligned}$$

Regression:

$$\ln \epsilon = \mu_i + \mu_t + \beta \ln c + \varepsilon_\epsilon$$

5. Regression:

$$\begin{aligned}
&\Rightarrow \ln \epsilon \sim \ln \frac{\tilde{A}}{\hat{A}} + \alpha \ln z_k + \frac{\alpha(1-\alpha)(\gamma - 1)}{2}(\ln z_k)^2 + (1 - \beta)\ln z_l + \frac{1}{\sigma - 1}\ln PY + \ln \nu_s \\
&\Rightarrow \ln \epsilon = \mu_i + \mu_t + \beta_{z_k}^\epsilon \ln z_k + \beta_{z_k^2}^\epsilon (\ln z_k)^2 + \beta_{z_l}^\epsilon \ln z_l + \beta_R^\epsilon \ln PY
\end{aligned}$$

$$\begin{aligned}
&\beta_{z_k}^\epsilon = \alpha \to \alpha ✓ \\
&\beta_{z_k^2}^\epsilon = \frac{\alpha(1-\alpha)(\gamma - 1)}{2} \to \gamma ✓ \\
&\beta_{z_l}^\epsilon = 1 - \beta \to \beta ✓ \\
&\beta_R^\epsilon = \frac{1}{\sigma - 1} \to \sigma ✓
\end{aligned}$$

Additional Derivatives

$$Y = \hat{A}K^{\beta}L^{1-\beta}$$

$$\frac{\partial Y}{\partial G} = \beta\frac{Y}{K}\frac{\partial K}{\partial G} = \beta\frac{Y}{K}\frac{\gamma_s}{\gamma_s - 1}K^{\frac{1}{\gamma_s}}\alpha\frac{\gamma_s - 1}{\gamma_s}G^{\frac{-1}{\gamma_s}}$$

$$\frac{\partial Y}{\partial B} = \beta\frac{Y}{K}\frac{\partial K}{\partial B} = \beta\frac{Y}{K}\frac{\gamma_s}{\gamma_s - 1}K^{\frac{1}{\gamma_s}}(1-\alpha)\frac{\gamma_s - 1}{\gamma_s}B^{\frac{-1}{\gamma_s}}$$

$$F.O.C \Rightarrow \lambda = \frac{\circ\alpha G^{\frac{-1}{\gamma_s}}}{r^G} = \frac{\circ(1-\alpha)B^{\frac{-1}{\gamma_s}}}{r^B}$$

$$\Rightarrow \frac{G}{B} = \left(\frac{\alpha}{1-\alpha}\frac{r^B}{r^G}\right)^{\gamma_s}$$

then given z_k and using equations from 8, I have:

$$\frac{\partial Y}{\partial L} = (1-\beta)\frac{Y}{L}$$

$$F.O.C \Rightarrow \lambda = \frac{\beta\frac{Y}{K}\frac{\gamma_s}{\gamma_s - 1}(K/G)^{\frac{1}{\gamma_s}}\alpha\frac{\gamma_s - 1}{\gamma_s}}{r^G} = \frac{(1-\beta)\frac{Y}{L}}{W}$$

$$\Rightarrow \frac{L}{K} = \frac{1-\beta}{\beta}\frac{1}{\alpha}(K/G)^{\frac{1}{\gamma_s}}\frac{r^G}{W}$$

$$= \frac{1-\beta}{\beta}\frac{1}{\alpha}(\alpha_s + (1-\alpha_s)z_k^{-\frac{\gamma_s - 1}{\gamma_s}})^{\frac{1}{\gamma_s - 1}}\frac{r^G}{W}$$

$$\frac{\partial \ln\left(\frac{G}{k}\right)}{\partial \tau_E} = \frac{\gamma(1-\alpha)}{\alpha z_k^{\frac{\gamma-1}{\gamma}} + (1-\alpha)} \cdot \frac{\tilde{A}}{r^{\beta} + \tau_E\tilde{A}} \equiv \frac{\gamma(1-\alpha)}{\circ}\star$$

■

$$\frac{\partial \ln\left(\frac{B}{k}\right)}{\partial \tau_E} = \frac{-\gamma\alpha}{\alpha + (1-\alpha)z_k^{-\frac{\gamma-1}{\gamma}}} \cdot \frac{\tilde{A}}{r^{\beta} + \tau_E\tilde{A}} \equiv \frac{-\gamma\alpha}{\triangle}\star$$

■

$$z_l^{-1}r^G(\frac{G}{K})\frac{\partial \ln(\frac{G}{K})}{\partial \tau_E}$$

$$= \frac{\beta}{1-\beta}\alpha(\frac{G}{K})^{\frac{\gamma-1}{\gamma}}w\frac{\partial \ln(\frac{G}{K})}{\partial \tau_E}$$

$$= \frac{\beta}{1-\beta}\alpha w\frac{1}{\circ}\frac{\gamma(1-\alpha)}{\triangle}\frac{\tilde{A}}{r^{\beta} + \tau_E\tilde{A}}$$

$$= \gamma\frac{\beta}{1-\beta}\frac{\alpha(1-\alpha)}{\circ\triangle}w\frac{\tilde{A}}{r^{\beta} + \tau_E\tilde{A}}$$

■

$$
\begin{aligned}
& r^B(\frac{B}{K})z_l^{-1}\frac{\partial \ln(\frac{B}{K})}{\partial \tau_E} \\
&= \frac{\beta}{1-\beta}(1-\alpha)(\frac{B}{K})^{\frac{\gamma-1}{\gamma}}w\frac{\partial \ln(\frac{B}{K})}{\partial \tau_E} \\
&= \frac{\beta}{1-\beta}(1-\alpha)w\frac{1}{\triangle}\frac{-\gamma\alpha}{\circ}\frac{\tilde{A}}{r^\beta+\tau_E\tilde{A}} \\
&= -\gamma\frac{\beta}{1-\beta}\frac{\alpha(1-\alpha)}{\circ\triangle}w\frac{\tilde{A}}{r^\beta+\tau_E\tilde{A}}
\end{aligned}
$$

■

$$
\begin{aligned}
\frac{\sigma}{\sigma-1}\frac{z_L^\beta}{\hat{A}_{si}p_i} &= \frac{\sigma}{\sigma-1}\frac{z_L^\beta}{\hat{A}_{si}\frac{\sigma-1}{\sigma}C_{si}} \\
&= \frac{1}{\frac{r}{z_l}+w}
\end{aligned}
$$

1 Definitions and Proofs

1.1 Optimal level for given output

$$
\bar{Y}_{si} = \hat{A}_{si}\hat{K^*}_{si}^{\beta_s}L^*{}_{si}^{1-\beta_s} = \hat{A}_{si}\hat{K^*}_{si}(\frac{L^*}{\hat{K^*}})_{si}^{(1-\beta_s)} = \hat{A}_{si}\hat{K^*}_{si}{z_{si}^l}^{(1-\beta_s)}
$$

we can rewrite the optimal capital level as:

$$
\begin{aligned}
\hat{K}_{si} &= B_{si}(\alpha_s {z_{si}^k}^{\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s))^{\frac{\gamma_s}{\gamma_s-1}} \\
&= G_{si}(\alpha_s + (1-\alpha_s){z_{si}^k}^{\frac{1-\gamma_s}{\gamma_s}})^{\frac{\gamma_s}{\gamma_s-1}}
\end{aligned}
$$

and then drive the optimal level of each type of capital:

$$
G_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}}\left(\alpha_s + (1-\alpha_s){z_{si}^k}^{-\frac{\gamma_s-1}{\gamma_s}}\right)^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{(\beta_s-1)}
$$

$$
B_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}}\left(\alpha_s {z_{si}^k}^{\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s)\right)^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{(\beta_s-1)}
$$

$$
L_{si}^* = \frac{\bar{Y}_{si}}{\hat{A}_{si}}{z_{si}^l}^{\beta_s}
$$

1.2 Emission optimal level

First we need to convert the production capital ($\hat{K}_{si}$) to the emission capital ($\tilde{K}_{si}$):

$$
\tilde{K} = (\mu_s G_{si}^{\frac{\eta_s-1}{\eta_s}} + (1-\mu_s)B_{si}^{\frac{\eta_s-1}{\eta_s}})^{\frac{\eta_s}{\eta_s-1}} = G_{si}(\mu_s + (1-\mu_s){z_{si}^k}^{(1-\eta_s)})^{\frac{\eta_s}{\eta_s-1}}
$$

$$
\hat{K} = (\alpha_s G_{si}^{\frac{\gamma_s-1}{\gamma_s}} + (1-\alpha_s)B_{si}^{\frac{\gamma_s-1}{\gamma_s}})^{\frac{\gamma_s}{\gamma_s-1}} = G_{si}(\alpha_s + (1-\alpha_s){z_{si}^k}^{(1-\gamma_s)})^{\frac{\gamma_s}{\gamma_s-1}}
$$

$$\Rightarrow \phi_{si} \equiv \frac{\tilde{K}}{\hat{K}} = \frac{(\mu_s + (1-\mu_s) {z_{si}^k}^{(1-\eta_s)})^{\frac{\eta_s}{\eta_s - 1}}}{(\alpha_s + (1-\alpha_s) {z_{si}^k}^{(1-\gamma_s)})^{\frac{\gamma_s}{\gamma_s - 1}}} = \frac{(\mu_s {z_{si}^k}^{(\eta_s - 1)} + (1-\mu_s))^{\frac{\eta_s}{\eta_s - 1}}}{(\alpha_s {z_{si}^k}^{(\gamma_s - 1)} + (1-\alpha_s))^{\frac{\gamma_s}{\gamma_s - 1}}}$$

As we can see the ϕ_{si} is a function of z_{si}^k, μ_s, α_s and γ_s. Now we can calculate the optimal emission level (E_{si}) as follows:

$$\begin{aligned}
E_{si} &= \tilde{A}_{si} \tilde{K}_{si}^{\theta_s} L_{si}^{1-\theta_s} = \tilde{A}_{si} \tilde{K}_{si} (\frac{L_{si}}{\tilde{K}_{si}})^{1-\theta_s} \\
&= \tilde{A}_{si} (\phi_{si} \hat{K}_{si}) (\frac{L_{si}}{\phi_{si} \hat{K}_{si}})^{1-\theta_s} \\
&= \tilde{A}_{si} {\phi_{si}}^{\theta_s} \hat{K}_{si} {z_{si}^l}^{1-\theta_s} \\
&= \tilde{A}_{si} {\phi_{si}}^{\theta_s} {z_{si}^l}^{\beta_s - \theta_s} \hat{K}_{si} {z_{si}^l}^{1-\beta_s} \\
&= \frac{\tilde{A}_{si}}{\hat{A}_{si}} (\frac{\phi_{si}}{z_{si}^l})^{\theta_s} {z_{si}^l}^{\beta_s} \bar{Y}_{si}
\end{aligned}$$

1.3 Cost Minimization function

$$C(\bar{F}_{si}) = C_{si} \bar{F}_{si}$$

$$\begin{aligned}
\text{where} \quad C_{si} = &(1+\tau_{G_{si}}) r_{si}^G \frac{1}{\hat{A}_{si}} \left(\alpha_s + (1-\alpha_s) {z_{si}^k}^{-\frac{\gamma_s - 1}{\gamma_s}} \right)^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{(\beta_s - 1)} \\
&+ (1+\tau_{B_{si}}) r_{si}^B \frac{1}{\hat{A}_{si}} \left(\alpha_s {z_{si}^k}^{\frac{\gamma_s - 1}{\gamma_s}} + (1-\alpha_s) \right)^{\frac{\gamma_s}{1-\gamma_s}} {z_{si}^l}^{(\beta_s - 1)} \\
&+ (1+\tau_{l_{si}}) w_{si} \frac{1}{\hat{A}_{si}} {z_{si}^l}^{\beta_s} \\
&+ \tau_E \frac{\tilde{A}_{si}}{\hat{A}_{si}} (\frac{\phi_{si}}{z_{si}^l})^{\theta_s} {z_{si}^l}^{\beta_s}
\end{aligned}$$

1.4 Sector Price

We need to solve the sector price P_s as function of firm price P_{si}, where P_s is defined as the price of acquiring a unit of the sector benefit:

$$\begin{aligned}
&\min_{F_{si}} \quad \left\{ \sum_i P_{si} F_{si} \right\} \\
&\text{s.t.} \quad \left(\sum_i F_{si}^{\frac{\sigma_s - 1}{\sigma_s}} \right)^{\frac{\sigma_s}{\sigma_s - 1}} = \bar{F}_s
\end{aligned}$$

$$F.O.C \Rightarrow P_{si}^{\sigma_s} F_{si} = P_s^{\sigma_s} F_s$$

1.5 Firm Price

We need to solve the firm price P_{si} as function of sector price P_s, where P_{si} is defined as the price of acquiring a unit of the firm benefit:

$$\max_{P_{si}} \quad \pi_{si} = (1+\tau_{si}^p)P_{si}Y_{si} - C_{si}Y_{si}$$
$$\max \quad (1+\tau_{si}^p)P_{si}(\frac{P_s}{P_{si}})^{\sigma_s}Y_s - C_{si}(\frac{P_s}{P_{si}})^{\sigma_s}Y_s$$
$$\max \quad (1+\tau_{si}^p)P_{si}^{1-\sigma_s}P_s^{\sigma_s}Y_s - C_{si}P_{si}^{-\sigma_s}P_s^{\sigma_s}Y_s$$
$$\max \quad P_s^{\sigma_s}Y_s\left((1+\tau_{si}^p)P_{si}^{1-\sigma_s} - C_{si}P_{si}^{-\sigma_s}\right)$$
$$\max \quad (1+\tau_{si}^p)P_{si}^{1-\sigma_s} - C_{si}P_{si}^{-\sigma_s}$$
$$\Rightarrow 0 = P_{si}^{-\sigma_s-1}\left[(1+\tau_{si}^p)(1-\sigma_s)P_{si} + \sigma_s C_{si}\right]$$
$$\Rightarrow P_{si} = \frac{1}{1+\tau_{si}^p}\frac{\sigma_s}{\sigma_s-1}C_{si}$$

1.6 Production Technology

$$P_{si}^{\sigma_s} = P_s^{\sigma_s}Y_sY_{si}^{-1} \Rightarrow (P_{si}Y_{si})^{\sigma_s} = P_s^{\sigma_s}Y_sY_{si}^{\sigma_s-1} \Rightarrow (P_{si}Y_{si})^{\sigma_s} = (P_sY_s)(P_sY_{si})^{\sigma_s-1}$$
$$\Rightarrow (P_{si}Y_{si})^{\frac{\sigma_s}{\sigma_s-1}} = (P_sY_s)^{\frac{1}{\sigma_s-1}}P_sY_{si}$$
$$\Rightarrow Y_{si} = \frac{(P_{si}Y_{si})^{\frac{\sigma_s}{\sigma_s-1}}}{P_s(P_sY_s)^{\frac{1}{\sigma_s-1}}}$$
$$\Rightarrow \hat{A}_{si} = \frac{1}{P_s(P_sY_s)^{\frac{1}{\sigma_s-1}}}\frac{(P_{si}Y_{si})^{\frac{\sigma_s}{\sigma_s-1}}}{\hat{K}_{si}^{\beta_s}L_{si}^{1-\beta_s}} = \nu_s\frac{(P_{si}Y_{si})^{\frac{\sigma_s}{\sigma_s-1}}}{\hat{K}_{si}^{\beta_s}L_{si}^{1-\beta_s}}$$

1.7 Emission Technology

$$E_{si} = \tilde{A}_{si}\tilde{K}_{si}^{\theta_s}L_{si}^{1-\theta_s} \Rightarrow \tilde{A}_{si} = \frac{E_{si}}{\tilde{K}_{si}^{\theta_s}L_{si}^{1-\theta_s}}$$

1.8 Reallocation

It is the social planner's problem to reallocate resources, therefore there is no need to price the inputs. The social planner's problem is to maximize the total real output subject to the resource constraint. Firms in the economy will produce the optimal level of output given the zero cost of inputs. The social planner's problem is:

$$\begin{aligned} \max \quad & \left(\sum_i \hat{Y}_{si}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ s.t. \quad & \sum_i \hat{L}_{si} = L_s \\ & \sum_i \hat{E}_{si} = E_s \\ & \sum_i \hat{G}_{si} + \hat{B}_{si} = K_s \end{aligned} \tag{13}$$

Now I will write the lagrangian for the social planner's problem. The social planner's problem is to maximize the total output in the economy subject to the resource constraint. The lagrangian is:

$$\mathcal{L}_s = \left(\sum_i \hat{Y}_{si}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} + \lambda_L(L_s - \sum_i \hat{L}_{si}) + \lambda_E(E_s - \sum_i \hat{E}_{si}) + (K_s - \sum_i \hat{G}_{si} + \hat{B}_{si})$$

where λ_L, λ_E are the lagrange multipliers for the labor and emission constraints, respectively that are normalized by the capital lagrange multiplier. This results in the fact that the relative marginal cost of labor is λ_L and the relative marginal cost of acquiring one unit of capital is 1. Now I can use the results from the firm's problem and the optimal input ratios to derive the optimal output and emissions for the social planner. The optimal input ratios are:

$$z_s^k = (\frac{\alpha_s}{1-\alpha_s})^{\gamma_s} \tag{14}$$

$$z_s^l = \frac{1-\beta_s}{\beta_s}\frac{1}{1-\alpha_s}(\alpha_s z_s^{k(\gamma_s-1)} + (1-\alpha_s))^{\frac{1}{1-\gamma_s}}\lambda_L \tag{15}$$

Then I can rewrite the optimal output and emissions for firm under the social planner's problem as:

$$\begin{aligned} \hat{Y}_{si} &= \hat{A}_{si} {z_s^l}^{-\beta} \hat{L}_{si} \\ \hat{E}_{si} &= \frac{\tilde{A}_{si}}{\hat{A}_{si}}(\frac{\phi_s}{z_s^l})^{\theta_s} {z_{si}^l}^{\beta_s} \hat{Y}_{si} = \tilde{A}_{si}(\frac{\phi_s}{z_s^l})^{\theta_s}\hat{L}_{si} \end{aligned} \tag{16}$$

and rewrite the lagrangian as:

$$\begin{aligned} \mathcal{L}_s &= \left(\sum_i \hat{A}_{si} {z_s^l}^{-\beta} \hat{L}_{si}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} + \lambda_L(L_s - \sum_i \hat{L}_{si}) + \lambda_E(E_s - \sum_i \tilde{A}_{si}(\frac{\phi_s}{z_s^l})^{\theta_s}\hat{L}_{si}) \\ \frac{\partial \mathcal{L}_s}{\partial \hat{L}_{si}} &= \left(\sum_i \hat{A}_{sk} {z_s^l}^{-\beta} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{si} {z_s^l}^{-\beta})^{\frac{\sigma-1}{\sigma}} \hat{L}_{si}^{-\frac{1}{\sigma}} - \lambda_L - \lambda_E \tilde{A}_{si}(\frac{\phi_s}{z_s^l})^{\theta_s} = 0 \\ \frac{\partial \mathcal{L}_s}{\partial \hat{L}_{sj}} &= \left(\sum_i \hat{A}_{sk} {z_s^l}^{-\beta} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{sj} {z_s^l}^{-\beta})^{\frac{\sigma-1}{\sigma}} \hat{L}_{sj}^{-\frac{1}{\sigma}} - \lambda_L - \lambda_E \tilde{A}_{sj}(\frac{\phi_s}{z_s^l})^{\theta_s} = 0 \end{aligned}$$

There are two possible cases for the optimal allocation of resources. In the first case, the social planner only cares about maximizing total output and does not consider emissions, resulting in $\lambda_E = 0$. In this scenario, the social planner will allocate all available resources to the firm with the highest productivity:

$$\lambda_L = \left(\sum_i \hat{A}_{sk} {z_s^l}^{-\beta} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{si} {z_s^l}^{-\beta})^{\frac{\sigma-1}{\sigma}} \hat{L}_{si}^{-\frac{1}{\sigma}} = \left(\sum_i \hat{A}_{sk} {z_s^l}^{-\beta} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{sj} {z_s^l}^{-\beta})^{\frac{\sigma-1}{\sigma}} \hat{L}_{sj}^{-\frac{1}{\sigma}}$$

$$\Rightarrow (\frac{\hat{A}_{si}^{\sigma-1}}{\hat{L}_{si}})^{\frac{1}{\sigma}} = (\frac{\hat{A}_{sj}^{\sigma-1}}{\hat{L}_{sj}})^{\frac{1}{\sigma}} \Rightarrow \frac{\hat{L}_{si}}{\hat{L}_{sj}} = \left(\frac{\hat{A}_{si}}{\hat{A}_{sj}}\right)^{\sigma-1}$$

In the second case, the social planner cares about the emissions, and the social planner will allocate all available resources to the firm with the lowest emissions:

$$\lambda_E = \frac{\left(\sum_i \hat{A}_{sk} z_s^{l^{-\beta}} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{si} z_s^{l^{-\beta}})^{\frac{\sigma-1}{\sigma}} \hat{L}_{si}^{-\frac{1}{\sigma}}}{\tilde{A}_{si}(\frac{\phi_s}{z_s^l})^{\theta_s}} = \frac{\left(\sum_i \hat{A}_{sk} z_s^{l^{-\beta}} \hat{L}_{sk}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (\hat{A}_{sj} z_s^{l^{-\beta}})^{\frac{\sigma-1}{\sigma}} \hat{L}_{sj}^{-\frac{1}{\sigma}}}{\tilde{A}_{sj}(\frac{\phi_s}{z_s^l})^{\theta_s}}$$

$$\Rightarrow \frac{\hat{A}_{si}^{\frac{\sigma-1}{\sigma}}}{\tilde{A}_{si}\tilde{L}_{si}^{\frac{1}{\sigma}}} = \frac{\hat{A}_{sj}^{\frac{\sigma-1}{\sigma}}}{\tilde{A}_{sj}\tilde{L}_{sj}^{\frac{1}{\sigma}}} \Rightarrow \frac{\tilde{L}_{si}}{\tilde{L}_{sj}} = \frac{\hat{A}_{si}^{\sigma-1}/\tilde{A}_{si}^{\sigma}}{\hat{A}_{sj}^{\sigma-1}/\tilde{A}_{sj}^{\sigma}}$$

as we know the optimal ratio, we can make sure that the allocation satisfies the budget constraint. The optimal allocation of resources is:

$$\hat{L}_{si} = \frac{\hat{A}_{si}^{\sigma-1}}{\sum_j \hat{A}_{sj}^{\sigma-1}} L_s$$
$$\tilde{L}_{si} = \frac{\hat{A}_{si}^{\sigma-1}/\tilde{A}_{si}^{\sigma}}{\sum_j \hat{A}_{sj}^{\sigma-1}/\tilde{A}_{sj}^{\sigma}} L_s$$

and we know that the social planner will decide about the green and brown capital allocation in the sector. The optimal allocation of green and brown capital is:

$$B_s = \frac{1}{1+z_s^k} K_s$$
$$G_s = \frac{z_s^k}{1+z_s^k} K_s$$

$$\hat{G}_{si} = \frac{\hat{A}_{si}^{\sigma-1}}{\sum_j \hat{A}_{sj}^{\sigma-1}} G_s$$
$$\tilde{G}_{si} = \frac{\hat{A}_{si}^{\sigma-1}/\tilde{A}_{si}^{\sigma}}{\sum_j \hat{A}_{sj}^{\sigma-1}/\tilde{A}_{sj}^{\sigma}} G_s$$

$$\hat{B}_{si} = \frac{\hat{A}_{si}^{\sigma-1}}{\sum_j \hat{A}_{sj}^{\sigma-1}} B_s$$
$$\tilde{B}_{si} = \frac{\hat{A}_{si}^{\sigma-1}/\tilde{A}_{si}^{\sigma}}{\sum_j \hat{A}_{sj}^{\sigma-1}/\tilde{A}_{sj}^{\sigma}} B_s$$

1.9 Wedges

From the 1.4 we know that $Y_{si} = (\frac{P_s}{P_{si}})^{\sigma_s} Y_s$ and then we can put it in the nominal output for firms:

$$P_{si}Y_{si} = P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{\sigma_s-1}{\sigma_s}}$$

$$\begin{aligned}
\frac{\partial P_{si}Y_{si}}{\partial G_{si}} &= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \frac{\partial Y_{si}}{\partial G_{si}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \beta_s \frac{Y_{si}}{\hat{K}_{si}} \frac{\partial \hat{K}_{si}}{\partial G_{si}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \beta_s \frac{Y_{si}}{\hat{K}_{si}} \alpha_s G_{si}^{\frac{-1}{\gamma_s}} \hat{K}_{si}^{\frac{1}{\gamma_s}}
\end{aligned}$$

From the 1.4 we know that $P_{si} = P_s^{\sigma_s} Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}}$ and then we can use it and then

$$\begin{aligned}
\frac{\partial P_{si}Y_{si}}{\partial G_{si}} &= \frac{\sigma_s - 1}{\sigma_s} P_{si} \beta_s \frac{Y_{si}}{\hat{K}_{si}} \alpha_s G_{si}^{\frac{-1}{\gamma_s}} \hat{K}_{si}^{\frac{1}{\gamma_s}} \\
&= \alpha_s \beta_s \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si}Y_{si}}{\hat{K}_{si}} (\frac{\hat{K}_{si}}{G_{si}})^{\frac{1}{\gamma_s}}
\end{aligned}$$

And then we can derive the marginal benefit of the brown capital:

$$\begin{aligned}
\frac{\partial P_{si}Y_{si}}{\partial B_{si}} &= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \frac{\partial Y_{si}}{\partial B_{si}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \beta_s \frac{Y_{si}}{\hat{K}_{si}} \frac{\partial \hat{K}_{si}}{\partial B_{si}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \beta_s \frac{Y_{si}}{\hat{K}_{si}} (1-\alpha_s) B_{si}^{\frac{-1}{\gamma_s}} \hat{K}_{si}^{\frac{1}{\gamma_s}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_{si} \beta_s \frac{Y_{si}}{\hat{K}_{si}} (1-\alpha_s) B_{si}^{\frac{-1}{\gamma_s}} \hat{K}_{si}^{\frac{1}{\gamma_s}} \\
&= (1-\alpha_s) \beta_s \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si}Y_{si}}{\hat{K}_{si}} (\frac{\hat{K}_{si}}{B_{si}})^{\frac{1}{\gamma_s}}
\end{aligned}$$

And finally, we can derive the marginal benefit of labor:

$$\begin{aligned}
\frac{\partial P_{si}Y_{si}}{\partial L_{si}} &= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} \frac{\partial Y_{si}}{\partial L_{si}} \\
&= \frac{\sigma_s - 1}{\sigma_s} P_s Y_s^{\frac{1}{\sigma_s}} Y_{si}^{\frac{-1}{\sigma_s}} (1-\beta_s) \frac{Y_{si}}{L_{si}} \\
&= (1-\beta_s) \frac{\sigma_s - 1}{\sigma_s} \frac{P_{si}Y_{si}}{L_{si}}
\end{aligned}$$

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