LE Reviewer

Counting, Combinatorics, and Generating Functions

Vinluan, Ieuan David R.

Contents

1 Basic Counting Principles

1.1 Addition Rule

Definition. If a task can be done in n_1 ways OR n_2 ways, and these ways are disjoint, then there are $n_1 + n_2$ ways to do the task.

Generally, If there are k disjoint sets with cardinalities $|A_1|, |A_2|, ..., |A_k|$, then we have:

$$|A_1 \cup A_2 \cup ... \cup A_k| = |A_1| + |A_2| + ... + |A_k|$$

Example. A student must choose one class to take next semester. They can choose to take either a math class (5 options) or a history class (3 options). Total choices: $5 + 3 = 8$.

Always remember that the sets must be **mutually exclusive** (disjoint)!

1.2 Multiplication Rule (Product Rule)

Definition. If a task consists of a sequence of k steps, where:
- Step 1 can be done in n_1 ways
- Step 2 can be done in n_2 ways (for each way of doing step 1)
- Step k can be done in n_k ways

Then the total number of ways to complete the task is given by:

$$n_1 \times n_2 \times ... \times n_k$$

Example. A password consists of 2 letters followed by 3 digits.
- Letters: $26 \times 26 = 676$ ways
- Digits: $10 \times 10 \times 10 = 1000$ ways
- Total: $676 \times 1000 = 676\ 000$ passwords

1.3 Inclusion-Exclusion Principle

Given n sets $A_1, A_2, ..., A_n$, the cardinality of their union is given by:

$$n\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < ... < i_k \le n} n\left(A_{i_1} \cap ... \cap A_{i_k}\right) \right).$$

For two sets,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The general idea is to add individual sets, subtract pairwise intersections, add three-way intersections, subtract four-way, and so on.

Example. In a class of 30 students:

- 18 play basketball
- 15 play soccer
- 8 play both sports.

How many play at least one sport?

$$|B \cup S| = 18 + 15 - 8 = 25 \text{ students}$$

2 Pigeonhole Principle

2.1 Basic Pigeonhole Principle

Definition. If n items are placed into k containers, and $n > k$, then at least one container must contain more than one item.

Example 1. In a group of 13 people, at least 2 must share the same birth month.

- Pigeons: 13 people
- Holes: 12 months
- Since $13 > 12$, at least one month contains at least 2 people

Example 2. Among 5 points placed inside a unit square, at least 2 are within distance $\sqrt{2}/2$ of each other.

- Divide the square into 4 quadrants (each $1/2 \times 1/2$)
- By pigeonhole, at least one quadrant contains at least 2 points
- Maximum distance in a quadrant:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

2.2 Generalized Pigeonhole Principle

Definition. If n objects are placed into k boxes, then at least one box contains at least $\lceil n/k \rceil$ objects

Example. If 100 students take an exam graded 0-10, at least $\lceil 100/11 \rceil = 10$ students must receive the same grade.

3 Permutations and Combinations

3.1 Permutations

Definition. An arrangement of objects where **order matters**.

3.1.1 Permutations of n distinct objects

Number of ways to arrange n distinct objects:

$$P(n) = n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

Example. Arrangements of letters A, B, C: $3! = 6$

ABC, ACB, BAC, BCA, CAB, CBA

3.1.2 r-Permutations

Number of ways to arrange r objects from n distinct objects:

$$P(n, r) = \frac{n!}{(n-r)!} = n \times (n-1) \times ... \times (n-r+1)$$

Example. Choose and arrange 2 letters from $\{A, B, C, D\}$:

$$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$$

3.1.3 Circular Permutations

Given n distinct elements, the number of ways to arrange them in a circle is $(n-1)!$. Cyclical shifts of a permutation of n elements in a row are all considered the same when they are in a circle.

3.1.4 Permutations with Repetition

If there are n objects in total, with n_1 of type 1, n_2 of type 2, ..., n_k of type k:

$$P = \frac{n!}{n_{1!} \cdot n_{2!} \cdot ... \cdot n_{k!}}$$

Example. Arrangements of MISSISSIPPI:
- M: 1, I: 4, S: 4, P: 2
- Total:

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$$

3.2 Combinations

Definition. A selection of objects where **order does not matter**.

3.2.1 r-Combinations

Number of ways to choose r objects from n distinct objects:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example. Choose 2 letters from $\{A, B, C, D\}$:

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

Selections: AB, AC, AD, BC, BD, CD

3.2.2 Derangements

A derangement is a permutation of n elements where no element appears in its original position. This is denoted as D_n and is given by the formula

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

4 Pascal's Formulas and Binomial Theorem

Each entry is $\binom{n}{r}$ where n is the row and r is the position.

4.1 Pascal's Identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof. Consider choosing a subset of size r from a set S with n items. Fix one specific item $x \in S$:

- Case 1: we choose x. Then, choose $r-1$ more from remaining $n-1$ items:

$$\binom{n-1}{r-1}$$

- Case 2: we don't choose x. Then, choose r from remaining $n-1$ items $\to$

$$\binom{n-1}{r}$$

- Thus, in total:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

4.2 Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Proof. Consider choosing a subset of size r from two sets A, B of sizes m and n, respectively.

- Suppose k come from A and $r-k$ come from B
- There are

$$\binom{m}{k}\binom{n}{r-k}$$

  ways to choose a subset
- Sum up over all possible k:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

4.3 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

4.3.1 Special Cases

Setting $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

This gives the total number of subsets of an n-element set.

Setting $x = 1, y = -1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

This shows that the number of even-sized subsets equals odd-sized subsets.

Setting $y = 1$:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

5 Recurrence Relations

5.1 Definition

A **recurrence relation** (or recurrence) expresses a sequence term a_n in terms of previous terms.

Example. Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$

The equations $F_0 = 0, F_1 = 1$ are called **initial conditions**.

5.2 Common Recurrence Relations

5.2.1 Arithmetic Sequence

$$a_n = a_{n-1} + d$$

Closed form: $a_n = a_0 + nd$

5.2.2 Geometric Sequence

$$a_n = r \cdot a_{n-1}$$

Closed form: $a_n = a_0 \cdot r^n$

5.3 Linear Homogeneous Recurrences

A recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

is called a **linear homogeneous recurrence of order *k*** with constant coefficients.

5.3.1 Solving Second-Order Linear Homogeneous Recurrences

For $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:

Step 1: Write the **characteristic equation**:

$$r^2 = c_1 r + c_2$$

or equivalently: $r^2 - c_1 r - c_2 = 0$

Step 2: Solve for roots r_1, r_2

Step 3: General solution depends on the roots:

Case 1: Two distinct real roots $r_1 \neq r_2$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Case 2: One repeated root $r_1 = r_2 = r$

$$a_n = \alpha_1 r^n + \alpha_2 n r^n$$

Step 4: Use initial conditions to find α_1, α_2

5.4 Linear Non-Homogeneous Recurrences

Form: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$, where $f(n)$ is the **non-homogeneous term**.

5.4.1 Solving

Step 1: Solve the associated homogeneous recurrence to get $a_n^{(h)}$

Step 2: Find a particular solution $a_n^{(p)}$ based on the form of $f(n)$:

Form of $f(n)$	**Trial Solution for $a_n^{(p)}$**
c (constant)	A
$c \cdot n$	$An + B$
$c \cdot n^2$	$An^2 + Bn + C$
$c \cdot s^n$	$A \cdot s^n$

Important: If your guess for $a_n^{(p)}$ solves the homogeneous equation (i.e., is a root of the characteristic equation), multiply by n.

Step 3: General solution: $a_n = a_n^{(h)} + a_n^{(p)}$

Step 4: Use initial conditions to find constants

6 Generating Functions

6.1 Definition

The **ordinary generating function** (OGF) for a sequence $\{a_n\}_{n=0}^{\infty}$ is:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

The generating function encodes the entire sequence into a single function.

6.2 Common Generating Functions

6.2.1 Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

6.2.2 Variations

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

6.3 Operations on Generating Functions

6.3.1 Addition

If $G(x) = \sum a_n x^n$ and $H(x) = \sum b_n x^n$, then:

$$G(x) + H(x) = \sum (a_n + b_n) x^n$$

6.3.2 Scalar Multiplication

$$c \cdot G(x) = \sum (c \cdot a_n) x^n$$

6.3.3 Multiplication by x^k

$$x^k G(x) = \sum a_n x^{n+k}$$

This shifts the sequence by k positions.

6.3.4 Differentiation

$$G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Equivalently: $xG'(x) = \sum_{n=1}^{\infty} n a_n x^n$

6.3.5 Integration

$$\int G(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

6.3.6 Convolution

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences, and let their OGFs be $F(x)$ and $G(x)$, respectively. Let the product $F(x)G(x)$ be the OGF of $\{c_n\}_{n=0}^{\infty}$. We have

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

The coefficient c_n counts the number of ways to achieve a total of n by:

- first choosing something worth k in a_k ways,
- then choosing something worth $n-k$ in b_{n-k} ways,
- and summing over all possible values of k.

6.4 Using Generating Functions to Solve Recurrences

6.4.1 General Method

Step 1: Let $G(x) := \sum_{n=0}^{\infty} a_n x^n$

Step 2: Multiply the recurrence by x^n and sum over appropriate values of n

Step 3: Express the result in terms of $G(x)$ and initial conditions

Step 4: Solve for $G(x)$

Step 5: Expand $G(x)$ as a power series to find a_n

Example 1. Solve the recurrence: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = 0, F_1 = 1$.

Solution.

Step 1: Let $G(x) := \sum_{n=0}^{\infty} F_n x^n$

Step 2: Multiply recurrence by x^n and sum from $n = 2$ to ∞:

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

Step 3: Express in terms of $G(x)$:

- LHS:

$$G(x) - F_0 - F_1 x = G(x) - x$$

- First sum on RHS:

$$x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} = x(G(x) - F_0) = xG(x)$$

- Second sum on RHS:

$$x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = x^2 G(x)$$

Thus, $G(x) - x = xG(x) + x^2 G(x)$.

Step 4: Solve for $G(x)$:

$$G(x)(1 - x - x^2) = x$$

$$G(x) = \frac{x}{1 - x - x^2}$$

Step 5: Use partial fractions. Factor denominator:

$$1 - x - x^2 = -(x - \varphi)(x - \hat{\varphi})$$

where $\varphi = \left(1 + \sqrt{5}\right)/2$ and $\hat{\varphi} = \left(1 - \sqrt{5}\right)/2$

$$G(x) = \frac{x}{(1 - \varphi x)(1 - \hat{\varphi} x)} = \frac{A}{1 - \varphi x} + \frac{B}{1 - \hat{\varphi} x}$$

Solving: $A = 1/\sqrt{5}$, and $B = -1/\sqrt{5}$.

Expand using geometric series:

$$G(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \varphi^n x^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \hat{\varphi}^n x^n$$

Therefore:

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n)$$

Example 2.

Solve the recurrence: $a_n = 3a_{n-1} + 2$ for $n \geq 1$, with $a_0 = 1$.

Solution. Let

$$G(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Multiply recurrence by x^n and sum from $n = 1$ to ∞:

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n$$

LHS:

$$G(x) - a_0 = G(x) - 1$$

First term of RHS:

$$3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 3xG(x)$$

Second term on right:

$$2 \sum_{n=1}^{\infty} x^n = \frac{2x}{1 - x}$$

So:

$$G(x) - 1 = 3xG(x) + \frac{2x}{1 - x}$$

Solve for $G(x)$:

$$
\begin{aligned}
G(x)(1-3x) &= 1 + \frac{2x}{1-x} \\
&= \frac{1-x+2x}{1-x} \\
&= \frac{1+x}{1-x}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
G(x) &= \frac{1+x}{(1-x)(1-3x)} \\
&= \frac{2}{1-3x} - \frac{1}{1-x}
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
G(x) &= \frac{2}{1-3x} - \frac{1}{1-x} \\
&= 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^n (2 \cdot 3^n - 1)
\end{aligned}
$$

Therefore:

$$a_n = 2 \cdot 3^n - 1$$

6.5 Solving Counting Problems with Generating Functions

Example. In how many ways can we make change for n cents using pennies, nickels, and dimes?

Solution. Let a_n = number of ways to make n cents.

Generating function:

- Pennies (1¢): can use 0, 1, 2, 3, ...

$$(1 + x + x^2 + x^3 + ...) = \frac{1}{1-x}$$

- Nickels (5¢): can use 0, 1, 2, 3, ...

$$(1 + x^5 + x^{10} + x^{15} + ...) = \frac{1}{1-x^5}$$

- Dimes (10¢): can use 0, 1, 2, 3, ...

$$(1 + x^{10} + x^{20} + ...) = \frac{1}{1-x^{10}}$$

Total generating function:

$$G(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}$$

The coefficient of x^n in the expansion of $G(x)$ gives a_n.

Example. How many ways are there to distribute 10 identical candies to 3 children such that each child gets at least 1 candy?

Solution. Give 1 candy to each child first. Now distribute remaining 7 candies with no restrictions.

Generating function for each child:

$$1 + x + x^2 + ... = \frac{1}{1 - x}$$

For 3 children:

$$G(x) = \frac{1}{(1 - x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$$

Coefficient of x^7:

$$\binom{7+2}{2} = \binom{9}{2} = 36$$

So there are **36 ways**.

7 Key Formulas

7.1 Counting

- Addition Rule:

$$|A \cup B| = |A| + |B|$$

- Multiplication Rule:

$$n_1 \times n_2 \times ... \times n_k$$

- Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Pigeonhole: n objects in k boxes $\Rightarrow$ at least $\lceil n/k \rceil$ in one box

7.2 Permutations & Combinations

- Permutations:

$$P(n, r) = \frac{n!}{(n-r)!}$$

- Permutations with repetition:

$$n^r$$

- Permutations with repeated elements:

$$\frac{n!}{n_1!n_2!...n_k!}$$

- Combinations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Combinations with repetition:

$$\binom{n+r-1}{r}$$

7.3 Pascal & Binomial

- Pascal's Identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

- Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

7.4 Recurrences

- Characteristic equation for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$: $r^2 - c_1 r - c_2 = 0$
- Distinct roots: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- Repeated root: $a_n = \alpha_1 r^n + \alpha_2 n r^n$

7.5 Generating Functions

- Geometric:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- Power of $(1-x)$ as denominator:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

8 Exercises

Instructions. Answer the following correctly, completely, and precisely.

1. Let $n := 2^{12} \cdot 3^5 \cdot 4^3$. How many positive factors does n have?
2. Let $S(x)$ denote the sum of the digits of x. Let $A := \{x \mid 1 \leq x < 1000, S(x) = 12\}$. Give $|A|$.
3. A palindrome is a number that reads the same when it is read backwards. For instance, 121, 1001, and 1 are palindromes, but 31 and 23 are not.

   Let $S(x)$ denote the sum of the digits of x. A positive 5-digit palindrome y is chosen at random. If the probability that $S(y) = 10$ is given by a/b where a and b are relatively prime positive integers, find $a + b$.
4. Suppose $|A| = 23$ and $|B| = 34$. Give the lower and upper bounds of $|A \cup B|$.
5. Let S be the set containing 6-letter words made from the first 11 letters of the alphabet without repeated letters. Let T be the set containing all elements of S that do not contain the sub-word `bed`. Give $|T|$.
6. Prove or disprove the following statement: for all integers $n > 1$, in a group of n people, there exist two people who are friends with the same number of people in the group. Note that friendship is a symmetrical relation (i.e., if person X is friends with person Y, then person Y is friends with person X).
7. There is a row of 35 chairs. Prove or disprove the following statement: if 28 people are to occupy the chairs, there will always exist 4 consecutive occupied chairs.
8. Prove or disprove the following statement: in any sequence of $n^2 + 1$ distinct real numbers, there exists an increasing or decreasing subsequence of length $n + 1$.
9. Given an integer $n \geq 2$ and a set S consisting of n distinct odd integers in the range $[3, 2^X]$, prove or disprove the following statement: if $n \geq X$, then there always exists two distinct integers $x, y \in S$ such that $y > x$ and $y \bmod x$ is even.
10. Let S be a set of n integers, with $n \geq 2$. Prove or disprove the following statement: there exist a pair of integers $x, y \in S$ such that $x - y$ is divisible by $n - 1$.
11. How many permutations of the string `COOKIEZI` are there such that there exists some `I` that comes before some `E`?
12. Let A be the multiset $\{1, 1, 3, 4, 5\}$. How many permutations of A are lexicographically greater than the permutation $(3, 1, 4, 1, 5)$?
13. How many ways are there to seat 6 men and 6 women in a row so that no two women sit next to each other?
14. Let n be a positive integer. How many permutations p of the integers from 1 to n are there such that the following hold:

- $a_i \neq i$ for all $1 \leq i \leq n$, and
- $a_1 \neq 2$.

You may express your answer in terms of D_n, the number of derangements for a sequence of n elements.

15. For a permutation p of the integers from 1 to n, a fixed point is defined as an index i such that $p_i = i$. How many permutations of $(1, 2, ..., n)$ contain exactly one fixed point?
16. Find the coefficient of x^7 in $\left(1 + 2x + 3x^2\right)^{10}$.
17. Simplify the expression
$$\sum_{k=0}^{n} \binom{n}{k} 99^k.$$
18. Chu Shih-chieh's identity is
$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$
for $n, r \in \mathbb{N}$ and $n \geq r$. Use a combinatorial proof to verify this.
19. Use a combinatorial proof to show that
$$\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$$
for all $n \in \mathbb{N}$.
20. Use a combinatorial proof to show that
$$\sum_{k=0}^{n} k^3 \binom{n}{k} = 2^{n-1}n + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}$$
for all $n \in \mathbb{N}$.
21. Consider an $n \times n$ grid. We denote by (i, j) the cell on the ith row from the bottom and the jth column from the left. An ant starts on $(1, 1)$ and makes its way to (n, n) by only moving up or right. Let $a_{i,j}$ be the number of ways to reach the (i, j) by only moving up or right. Give a recurrence relation for $a_{i,j}$ and give the closed form of $a_{n,n}$.
22. Consider the sequence
$$(1, -1, 1, -1, 1, -1, ...).$$
Let a_i denote the ith term of the sequence, starting from $i = 0$. Give a recurrence relation for a_i and its closed form.
23. Consider a staircase. You may climb 1, 2, or 3 steps at a time, but you cannot take consecutive 2-steps. Let a_n be the number of ways to reach step n. Give the recurrence.
24. Give the closed form: $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$, where $a_0 = 0,\ a_1 = 1,\ a_2 = 2,$ and $a_3 = 3$.
25. Give the closed form: $a_n = 13a_{n-1} - 40a_{n-2} + 2^n$, with $a_0 = 0, a_1 = 0$.
26. We know that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Use generating functions to prove this identity.

27. Find the generating function for $a_n = n$. Hint: start from the generating function of the sequence $b_n = 1$ for all $n \geq 0$.

28. The sequence of triangular numbers T_n is given by

$$T_n = \frac{n(n+1)}{2}.$$

Give the generating function for T_n.

29. Four fair six-sided dice are rolled. Use generating functions to count how many ways there are to achieve a total of 12.

30. We know that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Use generating functions to prove this identity. Hint: start from the identity

$$\frac{1}{1-x} = \sum_{k\geq 0} x^k$$

and differentiate r times.

9 Solutions

1. We note that $n = 2^{12} \cdot 3^5 \cdot 4^3 = 2^{18} \cdot 3^5$. All factors of n are of the form $2^a \cdot 3^b$, where a and b are integers and $0 \leq a \leq 18$ and $0 \leq b \leq 5$ hold. Since there are 19 possible values for a and 6 possible values for b, there are $19 \cdot 6 = 114$ total factors.

2. Let the digits of some $x \in A$ be d_1, d_2, and d_3. We must have $d_1 + d_2 + d_3 = 12$.

   Without restrictions, there are

$$\binom{12+3-1}{12} = \binom{14}{12}$$

   ways to assign values to d_1, d_2, and d_3. However, we must remove the cases where at least one of them is greater than 9.

   Note that since $d_1 + d_2 + d_3 = 12$, at most one of them can be greater than 9. Thus, we can fix which of the digits is greater than 9.

   Suppose $d_1 > 9$. Then, let $d_1' := d_1 - 10$. Thus, we have:

$$\begin{aligned} d_1 + d_2 + d_3 &= d_1' + 10 + d_2 + d_3 = 12 \\ d_1' + d_2 + d_3 &= 2 \end{aligned}$$

   There are

$$\binom{2+3-1}{2} = 6$$

solutions to this. Thus, we have 6 solutions where $d_1 > 9$. Note that by symmetry, we also have 6 solutions where $d_2 > 9$ and 6 solutions where $d_3 > 9$.

Thus, there are

$$\binom{14}{12} - 6 - 6 - 6 = 73$$

valid numbers.

3. We first note that there are $9 \cdot 10 \cdot 10 = 900$ positive 5-digit palindromes.

   Consider some positive 5-digit palindrome whose digit sum is 10. Let x be the first and fifth digit, y be the second and fourth digit, and z be the third digit. Then, we have

   $$2x + 2y + z = 10,$$

   which implies z is even. Then, we have

   $$x + y = 5 - \frac{z}{2}.$$

   Note that $x > 0$, $y \geq 0$, and $z \in \{0, 2, 4, 6, 8\}$. For each z, there are $5 - z/2$ valid pairs (x, y). Thus, over all z, we have $5 + 4 + 3 + 2 + 1 = 15$ positive 5-digit palindromes whose digit sum is 10.

   The final probability is

   $$\frac{15}{900} = \frac{1}{60},$$

   and the final answer is $1 + 60 = 61$.

4. We note that $|A \cup B| = |A| + |B| - |A \cap B|$. We have

   $$0 \leq |A \cap B| \leq 23.$$

   Thus, $34 \leq |A \cup B| \leq 57$.

5. We proceed with complementary counting. First, we have $|S| = P(11, 6)$. Then, we count how many elements of S contain `bed`. Note that we have 8 choices for the rest of the letters, and then we must permute the 4 elements (treating `bed` as a single unit). Thus, there are

   $$\binom{8}{3} \cdot 4!$$

   elements of S that contain `bed`. Thus, the final answer is

   $$\frac{11!}{5!} - \binom{8}{3} \cdot 4!.$$

6. The statement is true.

   Proof. Let $f(x)$ be the number of friends of some person x in the group. Thus, for all x, we have $f(x) \in [0, n-1]$. Consider two cases.

   Case 1. There is a person with no friends in the group.

   Then, it is impossible for someone else to have $n - 1$ friends in the group; thus, for all x, $f(x) \in [0, n-2]$. We have n people and $n - 1$ possible values for f.

Case 2. There is a person who is friends with everyone in the group.

Then, it is impossible for someone else to have no friends in the group; thus, for all x, $f(x) \in [1, n-1]$. We have n people and $n-1$ possible values for f.

Since in all cases, there are n people and $n-1$ possible values for $f(x)$, there exists two people a and b such that $f(a) = f(b)$. Thus, two people must have the same number of friends in the group. ■

7. The statement is true.

   Proof. There are 7 empty chairs and, therefore, 8 segments of consecutive occupied seats. Since there are 28 people, one segment must have at least $\lceil 28/8 \rceil = 4$ people. Thus, there always exists 4 consecutive occupied chairs. ■

8. The statement is true.

   Proof. Let the sequence be $a_1, a_2, ..., a_{n^2+1}$. We proceed with a proof by contradiction.

   Suppose there exists a sequence of $n^2 + 1$ distinct real numbers where length of the longest monotone subsequence is at most n. Let x_i be the length of the longest increasing subsequence ending at a_i, and let y_i be the length of the longest decreasing subsequence ending at a_i. Since $1 \leq x_i, y_i \leq n$, there are $n \cdot n = n^2$ distinct pairs (x_i, y_i).

   There are $n^2 + 1$ elements, so by the pigeonhole principle, there exist two elements a_i and a_j such that $(x_i, y_i) = \left(x_j, y_j\right)$.

   Without loss of generality, suppose $i < j$. Since $a_i \neq a_j$, we can consider two cases.

   If $a_i < a_j$, then we can extend the longest increasing subsequence ending at a_i by appending a_j. Thus, we cannot have $x_i = x_j$.

   Similarly, if $a_i > a_j$, then we can extend the longest decreasing subsequence ending at a_i by appending a_j. Thus, we cannot have $y_i = y_j$.

   Since both cases lead to contradictions, the initial assumption was false. Thus, the original statement is true. ■

9. The statement is true.

   Proof. Consider two odd integers x and y, where $x < y$ and $y \bmod x$ is even. Note that we have

$$y = \left\lfloor \frac{y}{x} \right\rfloor \cdot x + (y \bmod x).$$

   Since y and x are odd and $y \bmod x$ is even, $\lfloor y/x \rfloor$ should be odd.

   Let $s := \min_{x \in S} x$ Partition the range $\left[3, 2^X\right]$ into at most $X - 1$ subranges: $[s, 2s)$, $[2s, 4s)$, and so on, until $\left[2^{X-2}s, 2^{X-1}s\right)$. Note that $2^X < 2^{X-1}s$ because $s > 2$; thus, the union of these ranges fully covers $\left[3, 2^X\right]$.

   Claim. If two odd integers a, b with $a < b$ are in the same subrange, then $b \bmod a$ is even.

   Proof of claim. Note that because a and b are in the same subrange, $b < 2a$. Since $a < b < 2a$, we must have

$$1 < \frac{b}{a} < 2,$$

which implies

$$\left\lfloor \frac{b}{a} \right\rfloor = 1.$$

Since a and b are both odd and $\lfloor b/a \rfloor$ is odd, $b \bmod a$ is even. ■

Now, since $n \geq X$, by the pigeonhole principle, at least two elements of S will be in the same subrange. Let these two elements be x and y, and without loss of generality, suppose $x < y$. Since they are in the same subrange, $y \bmod x$ is even. Thus, we can always find two distinct integers $x, y \in S$ such that $y > x$ and $y \bmod x$ is even. ■

10. The statement is true.

    Proof. Note that there are $n - 1$ possible remainders on division by $n - 1$. Since there are n integers in S, at least two must have the same remainder on division by $n - 1$. Let these two integers be a and b. Then, we must have

    $$a \equiv b \pmod{n-1} \Rightarrow a - b \equiv 0 \pmod{n-1}.$$

    As desired. ■

11. TODO
12. TODO
13. TODO
14. TODO
15. TODO
16. TODO
17. TODO

End