LE Reviewer

Counting, Combinatorics, and Generating Functions

Vinluan, Ieuan David R.

Contents

1 Basic Counting Principles

1.1 Addition Rule

Definition. If a task can be done in n_1 ways OR n_2 ways, and these ways are disjoint, then there are $n_1 + n_2$ ways to do the task.

Generally, If there are k disjoint sets with cardinalities $|A_1|, |A_2|, ..., |A_k|$, then we have:

$$|A_1 \cup A_2 \cup ... \cup A_k| = |A_1| + |A_2| + ... + |A_k|$$

Example. A student must choose one class to take next semester. They can choose to take either a math class (5 options) or a history class (3 options). Total choices: $5 + 3 = 8$.

Always remember that the sets must be **mutually exclusive** (disjoint)!

1.2 Multiplication Rule (Product Rule)

Definition. If a task consists of a sequence of k steps, where:
- Step 1 can be done in n_1 ways
- Step 2 can be done in n_2 ways (for each way of doing step 1)
- Step k can be done in n_k ways

Then the total number of ways to complete the task is given by:

$$n_1 \times n_2 \times ... \times n_k$$

Example. A password consists of 2 letters followed by 3 digits.
- Letters: $26 \times 26 = 676$ ways
- Digits: $10 \times 10 \times 10 = 1000$ ways
- Total: $676 \times 1000 = 676\ 000$ passwords

1.3 Inclusion-Exclusion Principle

Given n sets $A_1, A_2, ..., A_n$, the cardinality of their union is given by:

$$n\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < ... < i_k \le n} n\left(A_{i_1} \cap ... \cap A_{i_k}\right) \right).$$

For two sets,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The general idea is to add individual sets, subtract pairwise intersections, add three-way intersections, subtract four-way, and so on.

Example. In a class of 30 students:

- 18 play basketball
- 15 play soccer
- 8 play both sports.

How many play at least one sport?

$$|B \cup S| = 18 + 15 - 8 = 25 \text{ students}$$

2 Pigeonhole Principle

2.1 Basic Pigeonhole Principle

Definition. If n items are placed into k containers, and $n > k$, then at least one container must contain more than one item.

Example 1. In a group of 13 people, at least 2 must share the same birth month.

- Pigeons: 13 people
- Holes: 12 months
- Since $13 > 12$, at least one month contains at least 2 people

Example 2. Among 5 points placed inside a unit square, at least 2 are within distance $\sqrt{2}/2$ of each other.

- Divide the square into 4 quadrants (each $1/2 \times 1/2$)
- By pigeonhole, at least one quadrant contains at least 2 points
- Maximum distance in a quadrant:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

2.2 Generalized Pigeonhole Principle

Definition. If n objects are placed into k boxes, then at least one box contains at least $\lceil n/k \rceil$ objects

Example. If 100 students take an exam graded 0-10, at least $\lceil 100/11 \rceil = 10$ students must receive the same grade.

3 Permutations and Combinations

3.1 Permutations

Definition. An arrangement of objects where **order matters**.

3.1.1 Permutations of n distinct objects

Number of ways to arrange n distinct objects:

$$P(n) = n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

Example. Arrangements of letters A, B, C: $3! = 6$

ABC, ACB, BAC, BCA, CAB, CBA

3.1.2 r-Permutations

Number of ways to arrange r objects from n distinct objects:

$$P(n, r) = \frac{n!}{(n-r)!} = n \times (n-1) \times ... \times (n-r+1)$$

Example. Choose and arrange 2 letters from $\{A, B, C, D\}$:

$$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$$

3.1.3 Circular Permutations

Given n distinct elements, the number of ways to arrange them in a circle is $(n-1)!$. Cyclical shifts of a permutation of n elements in a row are all considered the same when they are in a circle.

3.1.4 Permutations with Repetition

If there are n objects in total, with n_1 of type 1, n_2 of type 2, ..., n_k of type k:

$$P = \frac{n!}{n_{1!} \cdot n_{2!} \cdot ... \cdot n_{k!}}$$

Example. Arrangements of MISSISSIPPI:

- M: 1, I: 4, S: 4, P: 2
- Total:

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$$

3.2 Combinations

Definition. A selection of objects where **order does not matter**.

3.2.1 r-Combinations

Number of ways to choose r objects from n distinct objects:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example. Choose 2 letters from $\{A, B, C, D\}$:

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

Selections: AB, AC, AD, BC, BD, CD

3.2.2 Derangements

A derangement is a permutation of n elements where no element appears in its original position. This is denoted as D_n and is given by the formula

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

4 Pascal's Formulas and Binomial Theorem

Each entry is $\binom{n}{r}$ where n is the row and r is the position.

4.1 Pascal's Identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof. Consider choosing a subset of size r from a set S with n items. Fix one specific item $x \in S$:

- Case 1: we choose x. Then, choose $r-1$ more from remaining $n-1$ items:

$$\binom{n-1}{r-1}$$

- Case 2: we don't choose x. Then, choose r from remaining $n-1$ items $\to$

$$\binom{n-1}{r}$$

- Thus, in total:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

4.2 Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Proof. Consider choosing a subset of size r from two sets A, B of sizes m and n, respectively.

- Suppose k come from A and $r-k$ come from B
- There are

$$\binom{m}{k}\binom{n}{r-k}$$

  ways to choose a subset
- Sum up over all possible k:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

4.3 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

4.3.1 Special Cases

Setting $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

This gives the total number of subsets of an n-element set.

Setting $x = 1, y = -1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

This shows that the number of even-sized subsets equals odd-sized subsets.

Setting $y = 1$:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

5 Recurrence Relations

5.1 Definition

A **recurrence relation** (or recurrence) expresses a sequence term a_n in terms of previous terms.

Example. Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$

The equations $F_0 = 0, F_1 = 1$ are called **initial conditions**.

5.2 Common Recurrence Relations

5.2.1 Arithmetic Sequence

$$a_n = a_{n-1} + d$$

Closed form: $a_n = a_0 + nd$

5.2.2 Geometric Sequence

$$a_n = r \cdot a_{n-1}$$

Closed form: $a_n = a_0 \cdot r^n$

5.3 Linear Homogeneous Recurrences

A recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

is called a **linear homogeneous recurrence of order $\boldsymbol{k}$** with constant coefficients.

5.3.1 Solving Second-Order Linear Homogeneous Recurrences

For $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:

Step 1: Write the **characteristic equation**:

$$r^2 = c_1 r + c_2$$

or equivalently: $r^2 - c_1 r - c_2 = 0$

Step 2: Solve for roots r_1, r_2

Step 3: General solution depends on the roots:

Case 1: Two distinct real roots $r_1 \neq r_2$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Case 2: One repeated root $r_1 = r_2 = r$

$$a_n = \alpha_1 r^n + \alpha_2 n r^n$$

Step 4: Use initial conditions to find α_1, α_2

5.4 Linear Non-Homogeneous Recurrences

Form: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$, where $f(n)$ is the **non-homogeneous term**.

5.4.1 Solving

Step 1: Solve the associated homogeneous recurrence to get $a_n^{(h)}$

Step 2: Find a particular solution $a_n^{(p)}$ based on the form of $f(n)$:

Form of $f(n)$	**Trial Solution for $a_n^{(p)}$**
c (constant)	A
$c \cdot n$	$An + B$
$c \cdot n^2$	$An^2 + Bn + C$
$c \cdot s^n$	$A \cdot s^n$

Important: If your guess for $a_n^{(p)}$ solves the homogeneous equation (i.e., is a root of the characteristic equation), multiply by n.

Step 3: General solution: $a_n = a_n^{(h)} + a_n^{(p)}$

Step 4: Use initial conditions to find constants

6 Generating Functions

6.1 Definition

The **ordinary generating function** (OGF) for a sequence $\{a_n\}_{n=0}^{\infty}$ is:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

The generating function encodes the entire sequence into a single function.

6.2 Common Generating Functions

6.2.1 Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

6.2.2 Variations

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

6.3 Operations on Generating Functions

6.3.1 Addition

If $G(x) = \sum a_n x^n$ and $H(x) = \sum b_n x^n$, then:

$$G(x) + H(x) = \sum (a_n + b_n) x^n$$

6.3.2 Scalar Multiplication

$$c \cdot G(x) = \sum (c \cdot a_n) x^n$$

6.3.3 Multiplication by x^k

$$x^k G(x) = \sum a_n x^{n+k}$$

This shifts the sequence by k positions.

6.3.4 Differentiation

$$G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Equivalently: $xG'(x) = \sum_{n=1}^{\infty} n a_n x^n$

6.3.5 Integration

$$\int G(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

6.3.6 Convolution

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences, and let their OGFs be $F(x)$ and $G(x)$, respectively. Let the product $F(x)G(x)$ be the OGF of $\{c_n\}_{n=0}^{\infty}$. We have

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

The coefficient c_n counts the number of ways to achieve a total of n by:

- first choosing something worth k in a_k ways,
- then choosing something worth $n-k$ in b_{n-k} ways,
- and summing over all possible values of k.

6.4 Using Generating Functions to Solve Recurrences

6.4.1 General Method

Step 1: Let $G(x) := \sum_{n=0}^{\infty} a_n x^n$

Step 2: Multiply the recurrence by x^n and sum over appropriate values of n

Step 3: Express the result in terms of $G(x)$ and initial conditions

Step 4: Solve for $G(x)$

Step 5: Expand $G(x)$ as a power series to find a_n

Example 1. Solve the recurrence: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = 0, F_1 = 1$.

Solution.

Step 1: Let $G(x) := \sum_{n=0}^{\infty} F_n x^n$

Step 2: Multiply recurrence by x^n and sum from $n = 2$ to ∞:

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

Step 3: Express in terms of $G(x)$:

- LHS:

$$G(x) - F_0 - F_1 x = G(x) - x$$

- First sum on RHS:

$$x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} = x(G(x) - F_0) = xG(x)$$

- Second sum on RHS:

$$x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = x^2 G(x)$$

Thus, $G(x) - x = xG(x) + x^2 G(x)$.

Step 4: Solve for $G(x)$:

$$G(x)(1 - x - x^2) = x$$

$$G(x) = \frac{x}{1 - x - x^2}$$

Step 5: Use partial fractions. Factor denominator:

$$1 - x - x^2 = -(x - \varphi)(x - \hat{\varphi})$$

where $\varphi = \left(1 + \sqrt{5}\right)/2$ and $\hat{\varphi} = \left(1 - \sqrt{5}\right)/2$

$$G(x) = \frac{x}{(1 - \varphi x)(1 - \hat{\varphi} x)} = \frac{A}{1 - \varphi x} + \frac{B}{1 - \hat{\varphi} x}$$

Solving: $A = 1/\sqrt{5}$, and $B = -1/\sqrt{5}$.

Expand using geometric series:

$$G(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \varphi^n x^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \hat{\varphi}^n x^n$$

Therefore:

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n)$$

Example 2.

Solve the recurrence: $a_n = 3a_{n-1} + 2$ for $n \geq 1$, with $a_0 = 1$.

Solution. Let

$$G(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Multiply recurrence by x^n and sum from $n = 1$ to ∞:

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n$$

LHS:

$$G(x) - a_0 = G(x) - 1$$

First term of RHS:

$$3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 3xG(x)$$

Second term on right:

$$2 \sum_{n=1}^{\infty} x^n = \frac{2x}{1 - x}$$

So:

$$G(x) - 1 = 3xG(x) + \frac{2x}{1 - x}$$

Solve for $G(x)$:

$$
\begin{aligned}
G(x)(1-3x) &= 1 + \frac{2x}{1-x} \\
&= \frac{1-x+2x}{1-x} \\
&= \frac{1+x}{1-x}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
G(x) &= \frac{1+x}{(1-x)(1-3x)} \\
&= \frac{2}{1-3x} - \frac{1}{1-x}
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
G(x) &= \frac{2}{1-3x} - \frac{1}{1-x} \\
&= 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^n (2 \cdot 3^n - 1)
\end{aligned}
$$

Therefore:

$$
a_n = 2 \cdot 3^n - 1
$$

6.5 Solving Counting Problems with Generating Functions

Example. In how many ways can we make change for n cents using pennies, nickels, and dimes?

Solution. Let a_n = number of ways to make n cents.

Generating function:

- Pennies (1¢): can use 0, 1, 2, 3, …

$$
(1 + x + x^2 + x^3 + \ldots) = \frac{1}{1-x}
$$

- Nickels (5¢): can use 0, 1, 2, 3, …

$$
(1 + x^5 + x^{10} + x^{15} + \ldots) = \frac{1}{1-x^5}
$$

- Dimes (10¢): can use 0, 1, 2, 3, …

$$
(1 + x^{10} + x^{20} + \ldots) = \frac{1}{1-x^{10}}
$$

Total generating function:

$$
G(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}
$$

The coefficient of x^n in the expansion of $G(x)$ gives a_n.

Example. How many ways are there to distribute 10 identical candies to 3 children such that each child gets at least 1 candy?

Solution. Give 1 candy to each child first. Now distribute remaining 7 candies with no restrictions.

Generating function for each child:

$$1 + x + x^2 + ... = \frac{1}{1 - x}$$

For 3 children:

$$G(x) = \frac{1}{(1 - x)^3} = \sum_{n=0}^{\infty} \binom{n + 2}{2} x^n$$

Coefficient of x^7:

$$\binom{7 + 2}{2} = \binom{9}{2} = 36$$

So there are **36 ways**.

7 Key Formulas

7.1 Counting

- Addition Rule:

$$|A \cup B| = |A| + |B|$$

- Multiplication Rule:

$$n_1 \times n_2 \times ... \times n_k$$

- Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Pigeonhole: n objects in k boxes $\Rightarrow$ at least $\lceil n/k \rceil$ in one box

7.2 Permutations & Combinations

- Permutations:

$$P(n, r) = \frac{n!}{(n - r)!}$$

- Permutations with repetition:

$$n^r$$

- Permutations with repeated elements:

$$\frac{n!}{n_1! n_2! ... n_k!}$$

- Combinations:

$$\binom{n}{r} = \frac{n!}{r!(n - r)!}$$

- Combinations with repetition:

$$\binom{n + r - 1}{r}$$

7.3 Pascal & Binomial

- Pascal's Identity:

$$\binom{n}{r} = \binom{n - 1}{r - 1} + \binom{n - 1}{r}$$

- Binomial Theorem:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

7.4 Recurrences

- Characteristic equation for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$: $r^2 - c_1 r - c_2 = 0$
- Distinct roots: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- Repeated root: $a_n = \alpha_1 r^n + \alpha_2 n r^n$

7.5 Generating Functions

- Geometric:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- Power of $(1-x)$ as denominator:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

8 Exercises

Instructions. Answer the following correctly, completely, and precisely.

1. Let $n \coloneqq 2^{12} \cdot 3^5 \cdot 4^3$. How many positive factors does n have?
2. Let $S(x)$ denote the sum of the digits of x. Let $A \coloneqq \{x \mid 1 \leq x < 1000, S(x) = 12\}$. Give $|A|$.
3. A palindrome is a number that reads the same when it is read backwards. For instance, 121, 1001, and 1 are palindromes, but 31 and 23 are not.

   Let $S(x)$ denote the sum of the digits of x. A positive 5-digit palindrome y is chosen at random. If the probability that $S(y) = 10$ is given by a/b where a and b are relatively prime positive integers, find $a + b$.
4. Suppose $|A| = 23$ and $|B| = 34$. Give the lower and upper bounds of $|A \cup B|$.
5. Let S be the set containing 6-letter words made from the first 11 letters of the alphabet without repeated letters. Let T be the set containing all elements of S that do not contain the sub-word `bed`. Give $|T|$.
6. Prove or disprove the following statement: for all integers $n > 1$, in a group of n people, there exist two people who are friends with the same number of people in the group. Note that friendship is a symmetrical relation (i.e., if person X is friends with person Y, then person Y is friends with person X).
7. There is a row of 35 chairs. Prove or disprove the following statement: if 28 people are to occupy the chairs, there will always exist 4 consecutive occupied chairs.
8. Prove or disprove the following statement: in any sequence of $n^2 + 1$ distinct real numbers, there exists an increasing or decreasing subsequence of length $n + 1$.
9. Given an integer $n \geq 2$ and a set S consisting of n distinct odd integers in the range $[3, 2^X]$, prove or disprove the following statement: if $n \geq X$, then there always exists two distinct integers $x, y \in S$ such that $y > x$ and $y \bmod x$ is even.
10. Let S be a set of n integers, with $n \geq 2$. Prove or disprove the following statement: there exist a pair of integers $x, y \in S$ such that $x - y$ is divisible by $n - 1$.
11. How many permutations of the string `COOKIEZI` are there such that there exists some `I` that comes before some `E`?
12. Let A be the multiset $\{1, 1, 3, 4, 5\}$. How many permutations of A are lexicographically greater than the permutation $(3, 1, 4, 1, 5)$?
13. How many ways are there to seat 6 men and 6 women in a row so that no two women sit next to each other?
14. Let n be a positive integer. How many permutations p of the integers from 1 to n are there such that the following hold:
    - $p_i \neq i$ for all $1 \leq i \leq n$, and
    - $p_1 \neq 2$.

    You may express your answer in terms of D_n, the number of derangements for a sequence of n elements.
15. For a permutation p of the integers from 1 to n, a fixed point is defined as an index i such that $p_i = i$. How many permutations of $(1, 2, ..., n)$ contain exactly one fixed point?

16. Find the coefficient of x^7 in $\left(1+2x+3x^2\right)^{10}$. There is no need to simplify your answer.
17. Simplify the expression
$$\sum_{k=0}^{n} \binom{n}{k} 99^k.$$
18. Chu Shih-chieh's identity is
$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$
for $n, r \in \mathbb{N}$ and $n \geq r$. Use a combinatorial proof to verify this.
19. Use a combinatorial proof to show that
$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$
for all $n \in \mathbb{N}$.
20. Use a combinatorial proof to show that
$$\sum_{k=0}^{n} k^3 \binom{n}{k} = 2^{n-1}n + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}$$
for all $n \in \mathbb{N}$.
21. Consider an $n \times n$ grid. We denote by (i, j) the cell on the ith row from the bottom and the jth column from the left. An ant starts on $(1, 1)$ and makes its way to (n, n) by only moving up or right. Let $a_{i,j}$ be the number of ways to reach the (i, j) by only moving up or right. Give a recurrence relation for $a_{i,j}$ and give the closed form of $a_{n,n}$.
22. Consider the sequence
$$(1, 0, 1, 0, 1, 0, ...).$$
Let a_i denote the ith term of the sequence, starting from $i = 0$. Give a recurrence relation for a_i and its closed form.
23. Consider a staircase. You may climb 1, 2, or 3 steps at a time, but you cannot take consecutive 2-steps. Let a_n be the number of ways to reach step n. Give the recurrence.
24. Give the closed form: $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$, where $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$.
25. Give the closed form: $a_n = 13a_{n-1} - 40a_{n-2} + 2^n$, with $a_0 = 0, a_1 = 0$.
26. We know that
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$
Use generating functions to prove this identity.
27. Find the generating function for $a_n = n$. Hint: start from the generating function of the sequence $b_n = 1$ for all $n \geq 0$.
28. The sequence of triangular numbers T_n is given by

$$T_n = \frac{n(n+1)}{2}.$$

Give the generating function for T_n.

29. Four fair six-sided dice are rolled. Use generating functions to count how many ways there are to achieve a total of 12.

30. We know that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Use generating functions to prove this identity. Hint: start from the identity

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

and differentiate r times.

9 Solutions

1. We note that $n = 2^{12} \cdot 3^5 \cdot 4^3 = 2^{18} \cdot 3^5$. All factors of n are of the form $2^a \cdot 3^b$, where a and b are integers and $0 \le a \le 18$ and $0 \le b \le 5$ hold. Since there are 19 possible values for a and 6 possible values for b, there are $19 \cdot 6 = 114$ total factors.

2. Let the digits of some $x \in A$ be d_1, d_2, and d_3. We must have $d_1 + d_2 + d_3 = 12$.

   Without restrictions, there are

   $$\binom{12+3-1}{12} = \binom{14}{12}$$

   ways to assign values to d_1, d_2, and d_3. However, we must remove the cases where at least one of them is greater than 9.

   Note that since $d_1 + d_2 + d_3 = 12$, at most one of them can be greater than 9. Thus, we can fix which of the digits is greater than 9.

   Suppose $d_1 > 9$. Then, let $d_1' := d_1 - 10$. Thus, we have:

   $$\begin{aligned} d_1 + d_2 + d_3 &= d_1' + 10 + d_2 + d_3 = 12 \\ d_1' + d_2 + d_3 &= 2 \end{aligned}$$

   There are

   $$\binom{2+3-1}{2} = 6$$

   solutions to this. Thus, we have 6 solutions where $d_1 > 9$. Note that by symmetry, we also have 6 solutions where $d_2 > 9$ and 6 solutions where $d_3 > 9$.

   Thus, there are

   $$\binom{14}{12} - 6 - 6 - 6 = 73$$

   valid numbers.

3. We first note that there are $9 \cdot 10 \cdot 10 = 900$ positive 5-digit palindromes.

   Consider some positive 5-digit palindrome whose digit sum is 10. Let x be the first and fifth digit, y be the second and fourth digit, and z be the third digit. Then, we have

   $$2x + 2y + z = 10,$$

   which implies z is even. Then, we have

   $$x + y = 5 - \frac{z}{2}.$$

   Note that $x > 0$, $y \ge 0$, and $z \in \{0, 2, 4, 6, 8\}$. For each z, there are $5 - z/2$ valid pairs (x, y). Thus, over all z, we have $5 + 4 + 3 + 2 + 1 = 15$ positive 5-digit palindromes whose digit sum is 10.

   The final probability is

   $$\frac{15}{900} = \frac{1}{60},$$

and the final answer is $1 + 60 = 61$.

4. We note that $|A \cup B| = |A| + |B| - |A \cap B|$. We have

$$0 \leq |A \cap B| \leq 23.$$

Thus, $34 \leq |A \cup B| \leq 57$.

5. We proceed with complementary counting. First, we have $|S| = P(11, 6)$. Then, we count how many elements of S contain `bed`. Note that we have 8 choices for the rest of the letters, and then we must permute the 4 elements (treating `bed` as a single unit). Thus, there are

$$\binom{8}{3} \cdot 4!$$

elements of S that contain `bed`. Thus, the final answer is

$$\frac{11!}{5!} - \binom{8}{3} \cdot 4!.$$

6. The statement is true.

Proof. Let $f(x)$ be the number of friends of some person x in the group. Thus, for all x, we have $f(x) \in [0, n-1]$. Consider two cases.

Case 1. There is a person with no friends in the group.

Then, it is impossible for someone else to have $n-1$ friends in the group; thus, for all x, $f(x) \in [0, n-2]$. We have n people and $n-1$ possible values for f.

Case 2. There is a person who is friends with everyone in the group.

Then, it is impossible for someone else to have no friends in the group; thus, for all x, $f(x) \in [1, n-1]$. We have n people and $n-1$ possible values for f.

Since in all cases, there are n people and $n-1$ possible values for $f(x)$, there exists two people a and b such that $f(a) = f(b)$. Thus, two people must have the same number of friends in the group. ■

7. The statement is true.

Proof. There are 7 empty chairs and, therefore, 8 segments of consecutive occupied seats. Since there are 28 people, one segment must have at least $\lceil 28/8 \rceil = 4$ people. Thus, there always exists 4 consecutive occupied chairs. ■

8. The statement is true.

Proof. Let the sequence be $a_1, a_2, ..., a_{n^2+1}$. We proceed with a proof by contradiction.

Suppose there exists a sequence of $n^2 + 1$ distinct real numbers where length of the longest monotone subsequence is at most n. Let x_i be the length of the longest increasing subsequence ending at a_i, and let y_i be the length of the longest decreasing subsequence ending at a_i. Since $1 \leq x_i, y_i \leq n$, there are $n \cdot n = n^2$ distinct pairs (x_i, y_i).

There are $n^2 + 1$ elements, so by the pigeonhole principle, there exist two elements a_i and a_j such that $(x_i, y_i) = (x_j, y_j)$.

Without loss of generality, suppose $i < j$. Since $a_i \neq a_j$, we can consider two cases.

If $a_i < a_j$, then we can extend the longest increasing subsequence ending at a_i by appending a_j. Thus, we cannot have $x_i = x_j$.

Similarly, if $a_i > a_j$, then we can extend the longest decreasing subsequence ending at a_i by appending a_j. Thus, we cannot have $y_i = y_j$.

Since both cases lead to contradictions, the initial assumption was false. Thus, the original statement is true. ■

9. The statement is true.

   Proof. Consider two odd integers x and y, where $x < y$ and $y \bmod x$ is even. Note that we have

$$y = \left\lfloor \frac{y}{x} \right\rfloor \cdot x + (y \bmod x).$$

   Since y and x are odd and $y \bmod x$ is even, $\lfloor y/x \rfloor$ should be odd.

   Let $s := \min_{x \in S} x$. Partition the range $[3, 2^X]$ into at most $X - 1$ subranges: $[s, 2s)$, $[2s, 4s)$, and so on, until $[2^{X-2}s, 2^{X-1}s)$. Note that $2^X < 2^{X-1}s$ because $s > 2$; thus, the union of these ranges, $[s, 2^{X-1}s)$, fully covers $[3, 2^X]$.

   Claim. If two odd integers a, b with $a < b$, are in the same subrange, then $b \bmod a$ is even.

   Proof of claim. Note that because a and b are in the same subrange, $b < 2a$. Since $a < b < 2a$, we must have

$$1 < \frac{b}{a} < 2,$$

   which implies

$$\left\lfloor \frac{b}{a} \right\rfloor = 1.$$

   Since a and b are both odd and $\lfloor b/a \rfloor$ is odd, $b \bmod a$ is even. ■

   Now, since $n \geq X$ and there are $X - 1$ subranges, by the pigeonhole principle, at least two elements of S will be in the same subrange. Let these two elements be x and y, and without loss of generality, suppose $x < y$. Since they are in the same subrange, $y \bmod x$ is even. Thus, we can always find two distinct integers $x, y \in S$ such that $y > x$ and $y \bmod x$ is even. ■

10. The statement is true.

    Proof. Note that there are $n - 1$ possible remainders on division by $n - 1$. Since there are n integers in S, at least two must have the same remainder on division by $n - 1$. Let these two integers be a and b. Then, we must have

$$a \equiv b \pmod{n-1} \Rightarrow a - b \equiv 0 \pmod{n-1},$$

    as desired. ■

11. Observe that if we permute all other characters besides `I` and `E`, only 2 out of the 3 ways to arrange the 2 `I`'s and the 1 `E` satisfy the condition (namely, `IIE` and `IEI`). Thus, 2/3 of all permutations of the string `COOKIEZI` satisfy the condition. The answer is

$$\frac{2}{3} \cdot \frac{8!}{2!2!} = 6720.$$

12. We can count how many permutations are lexicographically smaller than $(3, 1, 4, 1, 5)$.

    We first count how many begin with 1. There are $4! = 24$ permutations.

    Then, we count how many begin with $3, 1, 1$. There are $2! = 2$ permutations.

    We observe that $(3, 1, 4, 1, 5)$ is the first permutation starting with $3, 1, 4$, so we can conclude that there are $24 + 2 = 26$ permutations lexicographically smaller than $(3, 1, 4, 1, 5)$. Since there are

    $$\frac{5!}{2!} = 60$$

    total permutations, there are $60 - 1 - 26 = 33$ permutations lexicographically larger than $(3, 1, 4, 1, 5)$.

13. We can seat the 6 men first and then seat the 6 women in the 7 spots between the men and at the ends. The answer is $6! \cdot 7!$.

14. We proceed with complementary counting; count how many permutations there are such that $a_i \neq i$ for all i and $a_1 = 2$.

    There are two cases.

    Case 1. $a_2 = 1$

    Then, we must count how many permutations of the remaining $n - 2$ there are such that $a_i = i$. This is simply D_{n-2}.

    Case 2. $a_2 \neq 1$

    We must count how many permutations of the $n - 1$ elements there are such that $a_i = i$ for $2 < i \leq n$ and $a_2 \neq 1$. This is D_{n-1}.

    Thus, the final answer is $D_n - (D_{n-1} + D_{n-2})$.

15. We can first choose the index of the fixed point and then permute everything else to satisfy the condition. Thus, there are nD_{n-1} permutations.

16. Let the exponent of $2x$ be i and the exponent of $3x^2$ be j. We must find

    $$\sum_{\substack{i+2j=7 \\ i,j\geq 0}} \binom{10}{i, j, 10-i-j} 2^i 3^j.$$

    We note that the only valid pairs of (i, j) are $(1, 3)$, $(3, 2)$, $(5, 1)$, and $(7, 0)$. Thus, the answer is

    $$\binom{10}{1,3,6} 2^1 3^3 + \binom{10}{3,2,5} 2^3 3^2 + \binom{10}{5,1,4} 2^5 3^1 + \binom{10}{7,0,3} 2^7 3^0.$$

17. Recall that

    $$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

    Substituting $x = 99$ yields

$$\sum_{k=0}^{n}\binom{n}{k}99^k = (1+99)^n = 100^n.$$

18. *Proof.* Consider a set S of $n+1$ people, each ranked from 0 to n.

    Suppose we want to choose a subset of size $r+1$ from S. We can do this in two ways.

    The first way is simply directly taking a subset of size $r+1$ from S. There are

    $$\binom{n+1}{r+1}$$

    ways to do this.

    The second way is by first fixing the highest rank of the $r+1$ people in our subset. Suppose the person with the highest rank is i. Then, we need to choose r people with ranks $[0, i)$ because the person with rank i is already in our subset. There are

    $$\binom{i}{r}$$

    ways to do this.

    Then, we must sum this quantity over all possible values of i. We have $i \in [r, n]$. Thus, in total, there are

    $$\sum_{i=r}^{n}\binom{i}{r}$$

    ways to do this task.

    Since both quantities count the same set, they must be equal. Thus,

    $$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1},$$

    as desired. ■

19. *Proof.* For some integer $n \in \mathbb{N}$, consider the problem of counting how many triples (i, j, k) there are such that $1 \leq i, j < k \leq n+1$.

    Fix k. Then, there are $(k-1)^2$ valid pairs (i, j). Summing over all valid values of k gives the answer:

    $$\sum_{k=2}^{n+1}(k-1)^2 = \sum_{k=1}^{n}k^2.$$

    Then, consider choosing the values of i, j, and k directly. There are two cases.

    Case 1. $i = j$

    Thus, there are

    $$\binom{n+1}{2}$$

    ways to pick valid values for i, j, and k.

    Case 2. $i \neq j$

There are

$$\binom{n+1}{3}$$

ways to pick the set of values for i, j, and k. Then, because swapping the values of i and j yields a valid triple, this must be multiplied by 2. Thus, there are

$$2\binom{n+1}{3}$$

ways to pick valid values for i, j, and k.

Since both ways count the same set, they must be equal. Thus,

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3},$$

as desired. ■

20. *Proof.* Consider a set S consisting of n people. We are to pick a subset of S, and we must assign the roles of president, secretary, and treasurer to some people in this subset, allowing the possibility of some person having more than 1 role. Let us do this in two ways.

Consider first fixing the size of the subset to be k and then assigning the roles to those in the subset. For a subset $A \subseteq S$, where $|A| = k$, there are

$$\binom{n}{k}$$

ways to choose A and k^3 ways to assign the three roles. Thus, for some k, there are

$$k^3\binom{n}{k}$$

ways to do this task, and summing up for all values of k gives the desired result. This is

$$\sum_{k=0}^{n} k^3\binom{n}{k}.$$

Next, consider assigning people to the three roles first and then choosing the remaining people who will be part of the subset. There are three cases.

Case 1. All three roles are given to the same person.

In this case, there are n ways to choose this one person with all three roles and 2^{n-1} ways to build the rest of the subset. Thus, there are $2^{n-1}n$ ways to do the task in this case.

Case 2. Two of the roles are given to one person; the last role is given to another person.

In this case, there are n ways to pick the person who will take two roles, $n-1$ ways to pick the person who will take the final role, 2^{n-2} ways to build the rest of the subset, and

$$\binom{3}{2} = 3$$

ways to pick which two roles will be assigned to a single person. Thus, there are $3n(n-1)2^{n-2}$ ways to do the task in this case.

Case 3. All the roles are given to different people.

In this case, there are n ways to pick the president, $n-1$ ways to pick the secretary, $n-2$ ways to pick the treasurer, and 2^{n-3} ways to build the rest of the subset. Thus, there are $n(n-1)(n-2)2^{n-3}$ ways to do the task in this case.

Thus, there are

$$2^{n-1}n + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}$$

ways in total to do the task in this manner.

Since both ways count the number of ways to do the same task, they must be equal. Thus,

$$\sum_{k=0}^{n} k^3 \binom{n}{k} = 2^{n-1}n + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}. \blacksquare$$

21. Consider a cell (i,j). A path to (i,j) can end in either an up move or a right move. Thus, we must have $a_{i,j} = a_{i,j-1} + a_{i-1,j}$, with the base cases $a_{1,1} = 1$ and $a_{i,j} = 0$ if $i \leq 0$ or $j \leq 0$. That is:

$$a_{i,j} = \begin{cases} a_{i-1,j} + a_{i,j-1} \text{ if } i > 1 \text{ and } j > 1, \\ 1 \text{ if } (i,j) = (1,1), \\ 0 \text{ otherwise} \end{cases}$$

Moving to the cell (n,n) consists of n up moves and n right moves. Thus, there are

$$\frac{(2n)!}{n!n!} = \binom{2n}{n}$$

ways to move to (n,n). Thus,

$$a_{n,n} = \binom{2n}{n}.$$

22. We have $a_n = a_{n-2}$, with the base cases $a_0 = 1$ and $a_1 = 0$.

    The characteristic equation is $r^2 = 1$, which means $r_1 = 1$ and $r_2 = -1$. Thus,

$$a_n = A(1)^n + B(-1)^n$$

    for some constants A and B. Solving yields $A = B = 1/2$. Thus,

$$a_n = \frac{(-1)^n + 1}{2}.$$

    Alternatively, we have $a_n = 1 - a_{n-1}$, with the base case $a_0 = 1$. Solving this gives the same closed form.

23. Let x_n be the number of ways to reach step n without ending on a 2-step, and let y_n be the number of ways to reach step n by ending with a 2-step. Thus, $a_n = x_n + y_n$.

    We have $x_n = x_{n-1} + y_{n-1} + x_{n-3} + y_{n-3} = a_{n-1} + a_{n-3}$ and $y_n = x_{n-2}$. Thus:

$$\begin{aligned} a_n &= x_n + y_n \\ &= a_{n-1} + a_{n-3} + x_{n-2} \\ &= a_{n-1} + a_{n-3} + a_{n-3} + a_{n-5} \\ &= a_{n-1} + 2a_{n-3} + a_{n-5}. \end{aligned}$$

The recurrence is $a_n = a_{n-1} + 2a_{n-3} + a_{n-5}$, with base cases $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 4$.

24. Observe that the characteristic equation is simply $(r-1)^4 = 0$. Thus,

$$\begin{aligned} a_n &= A(1)^n + Bn(1)^n + Cn^2 + (1)^n + Dn^3(1)^n \\ &= A + Bn + Cn^2 + Dn^3. \end{aligned}$$

Thus, $A = C = D = 0$ and $B = 1$. Then, $a_n = n$.

25. The homogeneous part is $a_n = 13a_{n-1} - 40a_{n-2}$. We have $r^2 - 13r + 40 = 0$, and so $(r_1, r_2) = (5, 8)$. The trial solution for the non-homogeneous part is $C(2)^n$ for some constant C. Thus, $C(2)^n = 13C(2)^{n-1} - 40C(2)^{n-2} + 2^n$. Solving:

$$\begin{aligned} 13 \cdot 2 - 40 + \frac{4}{C} &= 4 \\ C &= \frac{2}{9} \end{aligned}$$

Thus,

$$a_n = A(5)^n + B(8)^n + \frac{2}{9}2^n.$$

Solving for A and B:

$$\begin{aligned} a_0 = 0 &= A + B + \frac{2}{9} \\ a_1 = 0 &= 5A + 8B + \frac{4}{9} \\ 3A &= -\frac{12}{9} \Rightarrow A = -\frac{4}{9} \\ B &= -\frac{2}{9} - A = \frac{2}{9} \end{aligned}$$

Thus,

$$a_n = -\frac{4}{9}(5)^n + \frac{2}{9}(8)^n + \frac{2^{n+1}}{9}.$$

26. *Proof.* Consider an integer $n \geq 0$. Define the sequence

$$a_k = \binom{n}{k}.$$

The generating function for a_n is

$$G(x) := \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Note that $a_k = 0$ for $k > n$. We know that

$$G(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Taking the derivative of both sides with respect to x:

$$\sum_{k=0}^{n} k\binom{n}{k}x^{k-1} = n(1+x)^{n-1}.$$

Substitute $x = 1$:

$$\sum_{k=0}^{n} k\binom{n}{k}1^{k-1} = n(1+1)^{n-1}$$

Therefore,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1},$$

as desired. ■

27. The generating function of a_n is

$$G(x) := 0x^0 + 1x^1 + 2x^2 + ... = \sum_{k=0}^{n} kx^k.$$

We know that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Taking the derivative of both sides:

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1}$$
$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$$

Thus, the generating function of a_n is

$$G(x) = \frac{x}{(1-x)^2}.$$

28. The generating function of T_n is

$$G(x) := \sum_{k=0}^{\infty} \frac{k(k+1)}{2}x^k.$$

We proceed as follows:

$$
\begin{aligned}
G(x) &= \sum_{k=0}^{\infty} \frac{k(k+1)}{2} x^k \\
&= \sum_{k=0}^{\infty} \left(\binom{k+2}{2} - k - 1 \right) x^k \\
&= \sum_{k=0}^{\infty} \binom{k+2}{2} x^k - \sum_{k=0}^{\infty} k x^k - \sum_{k=0}^{\infty} x^k \\
&= \frac{1}{(1-x)^3} - \frac{x}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1 - x(1-x) - (1-x)^2}{(1-x)^3} \\
&= \frac{1 - x + x^2 - 1 + 2x - x^2}{(1-x)^3} \\
&= \frac{x}{(1-x)^3}
\end{aligned}
$$

29. The generating function for each dice is $\sum_{k=1}^{6} x^k$. Thus, the generating function for four dice is

$$
\left(\sum_{k=1}^{6} x^k \right)^4 .
$$

We need to find the coefficient of x^{12}.

$$
\begin{aligned}
\left(\sum_{k=1}^{6} x^k \right)^4 &= \left(x + x^2 + x^3 + x^4 + x^5 + x^6 \right)^4 \\
&= x^4 \left(1 + x + x^2 + x^3 + x^4 + x^5 \right)^4
\end{aligned}
$$

We can find the coefficient of x^8 in $\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^4$ instead.

This is given by

$$
\begin{aligned}
&\binom{4}{4} + \binom{4}{3,1} + \binom{4}{2,2} \cdot 2 + \binom{4}{2,1,1} \cdot 5 + \binom{4}{1,1,1,1} \cdot 2 \\
&= 1 + 4 + 6 \cdot 2 + 12 \cdot 5 + 24 \cdot 2 \\
&= 1 + 4 + 12 + 60 + 48 \\
&= 125.
\end{aligned}
$$

Thus, the coefficient of x^{12} in the original expression is 125. This means that there are 125 ways to achieve a total of 12.

30. *Proof.* We have

$$
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.
$$

Differentiating both sides with respect to x r times gives

$$
\frac{r!}{(1-x)^{r+1}} = \sum_{k=0}^{\infty} \frac{k!}{(k-r)!} x^{k-r},
$$

which implies

$$
\begin{aligned}
\frac{x^r}{(1-x)^{r+1}} &= \sum_{k=0}^{\infty} \frac{k!}{(k-r)!r!} x^k \\
&= \sum_{k=0}^{\infty} \binom{k}{r} x^k.
\end{aligned}
$$

Since for all $k < r$, we have

$$\binom{k}{r} = 0,$$

we can rewrite

$$
\begin{aligned}
\frac{x^r}{(1-x)^{r+1}} &= \sum_{k=0}^{\infty} \binom{k}{r} x^k \\
&= \sum_{k=r}^{\infty} \binom{k}{r} x^k.
\end{aligned}
$$

Define a new sequence b as follows:

$$b_n := \sum_{k=r}^{n} \binom{k}{r} x^k.$$

Note that $b_n = 0$ for $n < r$.

We can use the fact that if $A(x)$ is the ordinary generating function of a sequence $\{x_n\}$, then the ordinary generating function of the sequence of partial sums $\left\{\sum_{k=0}^{n} a_k\right\}$ is

$$\frac{A(x)}{1-x}.$$

The generating function for b is

$$
\begin{aligned}
\sum_{k=r}^{\infty} \left(\sum_{i=r}^{k} \binom{i}{r} \right) x^k &= \frac{x^r}{(1-x)^{r+1} \cdot (1-x)} \\
&= \frac{x^r}{(1-x)^{r+2}} \\
&= x^r \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} x^k \\
&= \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} x^{k+r} \\
&= \sum_{k=r}^{\infty} \binom{k+1}{r+1} x^k.
\end{aligned}
$$

Equating coefficients, we have

$$\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1},$$

which is equivalent to

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1},$$

as desired. ■

End