

# LE Reviewer

Counting, Combinatorics, and Generating Functions

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# 1 Basic Counting Principles

## 1.1 Addition Rule

*Definition.* If a task can be done in  $n_1$  ways OR  $n_2$  ways, and these ways are disjoint, then there are  $n_1 + n_2$  ways to do the task.

Generally, If there are  $k$  disjoint sets with cardinalities  $|A_1|, |A_2|, \dots, |A_k|$ , then we have:

$$|A_1 \cup A_2 \cup \dots \cup A_k| = |A_1| + |A_2| + \dots + |A_k|$$

*Example.* A student must choose one class to take next semester. They can choose to take either a math class (5 options) or a history class (3 options). Total choices:  $5 + 3 = 8$ .

Always remember that the sets must be **mutually exclusive** (disjoint)!

## 1.2 Multiplication Rule (Product Rule)

*Definition.* If a task consists of a sequence of  $k$  steps, where:

- Step 1 can be done in  $n_1$  ways
- Step 2 can be done in  $n_2$  ways (for each way of doing step 1)
- Step  $k$  can be done in  $n_k$  ways

Then the total number of ways to complete the task is given by:

$$n_1 \times n_2 \times \dots \times n_k$$

*Example.* A password consists of 2 letters followed by 3 digits.

- Letters:  $26 \times 26 = 676$  ways
- Digits:  $10 \times 10 \times 10 = 1000$  ways
- Total:  $676 \times 1000 = 676\,000$  passwords

## 1.3 Inclusion-Exclusion Principle

Given  $n$  sets  $A_1, A_2, \dots, A_n$ , the cardinality of their union is given by:

$$n \left( \bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n (-1)^{k+1} \left( \sum_{1 \leq i_1 < \dots < i_k \leq n} n(A_{i_1} \cap \dots \cap A_{i_k}) \right).$$

For two sets,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The general idea is to add individual sets, subtract pairwise intersections, add three-way intersections, subtract four-way, and so on.

*Example.* In a class of 30 students:

- 18 play basketball
- 15 play soccer
- 8 play both sports.

How many play at least one sport?

$$|B \cup S| = 18 + 15 - 8 = 25 \text{ students}$$

## 2 Pigeonhole Principle

### 2.1 Basic Pigeonhole Principle

*Definition.* If  $n$  items are placed into  $k$  containers, and  $n > k$ , then at least one container must contain more than one item.

*Example 1.* In a group of 13 people, at least 2 must share the same birth month.

- Pigeons: 13 people
- Holes: 12 months
- Since  $13 > 12$ , at least one month contains at least 2 people

*Example 2.* Among 5 points placed inside a unit square, at least 2 are within distance  $\sqrt{2}/2$  of each other.

- Divide the square into 4 quadrants (each  $1/2 \times 1/2$ )
- By pigeonhole, at least one quadrant contains at least 2 points
- Maximum distance in a quadrant:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

### 2.2 Generalized Pigeonhole Principle

*Definition.* If  $n$  objects are placed into  $k$  boxes, then at least one box contains at least  $\lceil n/k \rceil$  objects

*Example.* If 100 students take an exam graded 0-10, at least  $\lceil 100/11 \rceil = 10$  students must receive the same grade.

## 3 Permutations and Combinations

### 3.1 Permutations

*Definition.* An arrangement of objects where **order matters**.

#### 3.1.1 Permutations of n distinct objects

Number of ways to arrange  $n$  distinct objects:

$$P(n) = n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

*Example.* Arrangements of letters A, B, C:  $3! = 6$

ABC, ACB, BAC, BCA, CAB, CBA

#### 3.1.2 r-Permutations

Number of ways to arrange  $r$  objects from  $n$  distinct objects:

$$P(n, r) = \frac{n!}{(n - r)!} = n \times (n - 1) \times \dots \times (n - r + 1)$$

*Example.* Choose and arrange 2 letters from  $\{A, B, C, D\}$ :

$$P(4, 2) = \frac{4!}{(4 - 2)!} = \frac{4!}{2!} = 12$$

#### 3.1.3 Circular Permutations

Given  $n$  distinct elements, the number of ways to arrange them in a circle is  $(n - 1)!$ . Cyclical shifts of a permutation of  $n$  elements in a row are all considered the same when they are in a circle.

#### 3.1.4 Permutations with Repetition

If there are  $n$  objects in total, with  $n_1$  of type 1,  $n_2$  of type 2, ...,  $n_k$  of type  $k$ :

$$P = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

*Example.* Arrangements of MISSISSIPPI:

- M: 1, I: 4, S: 4, P: 2
- Total:

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$$

## 3.2 Combinations

*Definition.* A selection of objects where **order does not matter**.

#### 3.2.1 r-Combinations

Number of ways to choose  $r$  objects from  $n$  distinct objects:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n - r)!}$$

*Example.* Choose 2 letters from  $\{A, B, C, D\}$ :

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

Selections: AB, AC, AD, BC, BD, CD

### 3.2.2 Derangements

A derangement is a permutation of  $n$  elements where no element appears in its original position. This is denoted as  $D_n$  and is given by the formula

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

## 4 Pascal's Formulas and Binomial Theorem

Each entry is  $\binom{n}{r}$  where  $n$  is the row and  $r$  is the position.

### 4.1 Pascal's Identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

*Proof.* Consider choosing a subset of size  $r$  from a set  $S$  with  $n$  items. Fix one specific item  $x \in S$ :

- Case 1: we choose  $x$ . Then, choose  $r-1$  more from remaining  $n-1$  items:

$$\binom{n-1}{r-1}$$

- Case 2: we don't choose  $x$ . Then, choose  $r$  from remaining  $n-1$  items →

$$\binom{n-1}{r}$$

- Thus, in total:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

### 4.2 Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}$$

*Proof.* Consider choosing a subset of size  $r$  from two sets  $A, B$  of sizes  $m$  and  $n$ , respectively.

- Suppose  $k$  come from  $A$  and  $r-k$  come from  $B$
- There are

$$\binom{m}{k} \binom{n}{r-k}$$

ways to choose a subset

- Sum up over all possible  $k$ :

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}$$

### 4.3 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

#### 4.3.1 Special Cases

Setting  $x = y = 1$ :

$$2^n = \sum_{k=0}^n \binom{n}{k}$$

This gives the total number of subsets of an  $n$ -element set.

Setting  $x = 1, y = -1$ :

$$0 = \sum_{k=0}^n (-1)^k \binom{n}{k}$$

This shows that the number of even-sized subsets equals odd-sized subsets.

**Setting  $y = 1$ :**

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

# 5 Recurrence Relations

## 5.1 Definition

A **recurrence relation** (or recurrence) expresses a sequence term  $a_n$  in terms of previous terms.

*Example.* Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$

The equations  $F_0 = 0, F_1 = 1$  are called **initial conditions**.

## 5.2 Common Recurrence Relations

### 5.2.1 Arithmetic Sequence

$$a_n = a_{n-1} + d$$

Closed form:  $a_n = a_0 + nd$

### 5.2.2 Geometric Sequence

$$a_n = r \cdot a_{n-1}$$

Closed form:  $a_n = a_0 \cdot r^n$

## 5.3 Linear Homogeneous Recurrences

A recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

is called a **linear homogeneous recurrence of order  $k$**  with constant coefficients.

### 5.3.1 Solving Second-Order Linear Homogeneous Recurrences

For  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ :

**Step 1:** Write the **characteristic equation**:

$$r^2 = c_1 r + c_2$$

or equivalently:  $r^2 - c_1 r - c_2 = 0$

**Step 2:** Solve for roots  $r_1, r_2$

**Step 3:** General solution depends on the roots:

*Case 1:* Two distinct real roots  $r_1 \neq r_2$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

*Case 2:* One repeated root  $r_1 = r_2 = r$

$$a_n = \alpha_1 r^n + \alpha_2 n r^n$$

**Step 4:** Use initial conditions to find  $\alpha_1, \alpha_2$

## 5.4 Linear Non-Homogeneous Recurrences

Form:  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$ , where  $f(n)$  is the **non-homogeneous term**.

### 5.4.1 Solving

**Step 1:** Solve the associated homogeneous recurrence to get  $a_n^{(h)}$

**Step 2:** Find a particular solution  $a_n^{(p)}$  based on the form of  $f(n)$ :

Form of $f(n)$	Trial Solution for $a_n^{(p)}$
$c$ (constant)	$A$
$c \cdot n$	$An + B$
$c \cdot n^2$	$An^2 + Bn + C$
$c \cdot s^n$	$A \cdot s^n$

**Important:** If your guess for  $a_n^{(p)}$  solves the homogeneous equation (i.e., is a root of the characteristic equation), multiply by  $n$ .

**Step 3:** General solution:  $a_n = a_n^{(h)} + a_n^{(p)}$

**Step 4:** Use initial conditions to find constants

# 6 Generating Functions

## 6.1 Definition

The **ordinary generating function** (OGF) for a sequence  $\{a_n\}_{n=0}^{\infty}$  is:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

The generating function encodes the entire sequence into a single function.

## 6.2 Common Generating Functions

### 6.2.1 Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

### 6.2.2 Variations

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

## 6.3 Operations on Generating Functions

### 6.3.1 Addition

If  $G(x) = \sum a_n x^n$  and  $H(x) = \sum b_n x^n$ , then:

$$G(x) + H(x) = \sum (a_n + b_n) x^n$$

### 6.3.2 Scalar Multiplication

$$c \cdot G(x) = \sum (c \cdot a_n) x^n$$

### 6.3.3 Multiplication by $x^k$

$$x^k G(x) = \sum a_n x^{n+k}$$

This shifts the sequence by  $k$  positions.

### 6.3.4 Differentiation

$$G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Equivalently:  $x G'(x) = \sum_{n=1}^{\infty} n a_n x^n$

### 6.3.5 Integration

$$\int G(x) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

### 6.3.6 Convolution

Let  $\{a_n\}_{n=0}^{\infty}$  and  $\{b_n\}_{n=0}^{\infty}$  be two sequences, and let their OGFs be  $F(x)$  and  $G(x)$ , respectively.

Let the product  $F(x)G(x)$  be the OGF of  $\{c_n\}_{n=0}^{\infty}$ . We have

$$c_n = \sum_{i=0}^n a_i b_{n-i}.$$

The coefficient  $c_n$  counts the number of ways to achieve a total of  $n$  by:

- first choosing something worth  $k$  in  $a_k$  ways,
- then choosing something worth  $n - k$  in  $b_{n-k}$  ways,
- and summing over all possible values of  $k$ .

## 6.4 Using Generating Functions to Solve Recurrences

### 6.4.1 General Method

**Step 1:** Let  $G(x) := \sum_{n=0}^{\infty} a_n x^n$

**Step 2:** Multiply the recurrence by  $x^n$  and sum over appropriate values of  $n$

**Step 3:** Express the result in terms of  $G(x)$  and initial conditions

**Step 4:** Solve for  $G(x)$

**Step 5:** Expand  $G(x)$  as a power series to find  $a_n$

*Example 1.* Solve the recurrence:  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ , with  $F_0 = 0, F_1 = 1$ .

*Solution.*

**Step 1:** Let  $G(x) := \sum_{n=0}^{\infty} F_n x^n$

**Step 2:** Multiply recurrence by  $x^n$  and sum from  $n = 2$  to  $\infty$ :

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

**Step 3:** Express in terms of  $G(x)$ :

- LHS:

$$G(x) - F_0 - F_1 x = G(x) - x$$

- First sum on RHS:

$$x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} = x(G(x) - F_0) = xG(x)$$

- Second sum on RHS:

$$x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = x^2 G(x)$$

Thus,  $G(x) - x = xG(x) + x^2 G(x)$ .

**Step 4:** Solve for  $G(x)$ :

$$G(x)(1 - x - x^2) = x$$

$$G(x) = \frac{x}{1 - x - x^2}$$

**Step 5:** Use partial fractions. Factor denominator:

$$1 - x - x^2 = -(x - \varphi)(x - \hat{\varphi})$$

where  $\varphi = (1 + \sqrt{5})/2$  and  $\hat{\varphi} = (1 - \sqrt{5})/2$

$$G(x) = \frac{x}{(1 - \varphi x)(1 - \hat{\varphi} x)} = \frac{A}{1 - \varphi x} + \frac{B}{1 - \hat{\varphi} x}$$

Solving:  $A = 1/\sqrt{5}$ , and  $B = -1/\sqrt{5}$ .

Expand using geometric series:

$$G(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \varphi^n x^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \hat{\varphi}^n x^n$$

Therefore:

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n)$$

*Example 2.*

Solve the recurrence:  $a_n = 3a_{n-1} + 2$  for  $n \geq 1$ , with  $a_0 = 1$ .

*Solution.* Let

$$G(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Multiply recurrence by  $x^n$  and sum from  $n = 1$  to  $\infty$ :

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n$$

LHS:

$$G(x) - a_0 = G(x) - 1$$

First term of RHS:

$$3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 3xG(x)$$

Second term on right:

$$2 \sum_{n=1}^{\infty} x^n = \frac{2x}{1-x}$$

So:

$$G(x) - 1 = 3xG(x) + \frac{2x}{1-x}$$

Solve for  $G(x)$ :

$$\begin{aligned}
G(x)(1 - 3x) &= 1 + \frac{2x}{1-x} \\
&= \frac{1-x+2x}{1-x} \\
&= \frac{1+x}{1-x}
\end{aligned}$$

Thus:

$$\begin{aligned}
G(x) &= \frac{1+x}{(1-x)(1-3x)} \\
&= \frac{2}{1-3x} - \frac{1}{1-x}
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
G(x) &= \frac{2}{1-3x} - \frac{1}{1-x} \\
&= 2 \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^n (2 \cdot 3^n - 1)
\end{aligned}$$

Therefore:

$$a_n = 2 \cdot 3^n - 1$$

## 6.5 Solving Counting Problems with Generating Functions

*Example.* In how many ways can we make change for  $n$  cents using pennies, nickels, and dimes?

*Solution.* Let  $a_n$  = number of ways to make  $n$  cents.

### Generating function:

- Pennies (1¢): can use 0, 1, 2, 3, ...

$$(1 + x + x^2 + x^3 + \dots) = \frac{1}{1-x}$$

- Nickels (5¢): can use 0, 1, 2, 3, ...

$$(1 + x^5 + x^{10} + x^{15} + \dots) = \frac{1}{1-x^5}$$

- Dimes (10¢): can use 0, 1, 2, 3, ...

$$(1 + x^{10} + x^{20} + \dots) = \frac{1}{1-x^{10}}$$

Total generating function:

$$G(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}$$

The coefficient of  $x^n$  in the expansion of  $G(x)$  gives  $a_n$ .

*Example.* How many ways are there to distribute 10 identical candies to 3 children such that each child gets at least 1 candy?

*Solution.* Give 1 candy to each child first. Now distribute remaining 7 candies with no restrictions.

Generating function for each child:

$$1 + x + x^2 + \dots = \frac{1}{1-x}$$

For 3 children:

$$G(x) = \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$$

Coefficient of  $x^7$ :

$$\binom{7+2}{2} = \binom{9}{2} = 36$$

So there are **36 ways**.

## 7 Key Formulas

### 7.1 Counting

- Addition Rule:

$$|A \cup B| = |A| + |B|$$

- Multiplication Rule:

$$n_1 \times n_2 \times \dots \times n_k$$

- Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Pigeonhole:  $n$  objects in  $k$  boxes  $\Rightarrow$  at least  $\lceil n/k \rceil$  in one box

### 7.2 Permutations & Combinations

- Permutations:

$$P(n, r) = \frac{n!}{(n-r)!}$$

- Permutations with repetition:

$$n^r$$

- Permutations with repeated elements:

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

- Combinations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Combinations with repetition:

$$\binom{n+r-1}{r}$$

### 7.3 Pascal & Binomial

- Pascal's Identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

- Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

### 7.4 Recurrences

- Characteristic equation for  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ :  $r^2 - c_1 r - c_2 = 0$
- Distinct roots:  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- Repeated root:  $a_n = \alpha_1 r^n + \alpha_2 n r^n$

### 7.5 Generating Functions

- Geometric:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- Power of  $(1 - x)$  as denominator:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

## 8 Exercises

**Instructions.** Answer the following correctly, completely, and precisely.

1. Let  $n := 2^{12} \cdot 3^5 \cdot 4^3$ . How many positive factors does  $n$  have?
2. Let  $S(x)$  denote the sum of the digits of  $x$ . Let  $A := \{x \mid 1 \leq x < 1000, S(x) = 12\}$ . Give  $|A|$ .
3. A palindrome is a number that reads the same when it is read backwards. For instance, 121, 1001, and 1 are palindromes, but 31 and 23 are not.  
Let  $S(x)$  denote the sum of the digits of  $x$ . A positive 5-digit palindrome  $y$  is chosen at random. If the probability that  $S(y) = 10$  is given by  $a/b$  where  $a$  and  $b$  are relatively prime positive integers, find  $a + b$ .
4. Suppose  $|A| = 23$  and  $|B| = 34$ . Give the lower and upper bounds of  $|A \cup B|$ .
5. Let  $S$  be the set containing 6-letter words made from the first 11 letters of the alphabet without repeated letters. Let  $T$  be the set containing all elements of  $S$  that do not contain the sub-word **bed**. Give  $|T|$ .
6. Prove or disprove the following statement: for all integers  $n > 1$ , in a group of  $n$  people, there exist two people who are friends with the same number of people in the group. Note that friendship is a symmetrical relation (i.e., if person  $X$  is friends with person  $Y$ , then person  $Y$  is friends with person  $X$ ).
7. There is a row of 35 chairs. Prove or disprove the following statement: if 28 people are to occupy the chairs, there will always exist 4 consecutive occupied chairs.
8. Prove or disprove the following statement: in any sequence of  $n^2 + 1$  distinct real numbers, there exists an increasing or decreasing subsequence of length  $n + 1$ .
9. Given an integer  $n \geq 2$  and a set  $S$  consisting of  $n$  distinct odd integers in the range  $[3, 2^X]$ , prove or disprove the following statement: if  $n \geq X$ , then there always exists two distinct integers  $x, y \in S$  such that  $y > x$  and  $y \bmod x$  is even.
10. Let  $S$  be a set of  $n$  integers, with  $n \geq 2$ . Prove or disprove the following statement: there exist a pair of integers  $x, y \in S$  such that  $x - y$  is divisible by  $n - 1$ .
11. How many permutations of the string **C00KIEZI** are there such that there exists some **I** that comes before some **E**?
12. Let  $A$  be the multiset  $\{1, 1, 3, 4, 5\}$ . How many permutations of  $A$  are lexicographically greater than the permutation  $(3, 1, 4, 1, 5)$ ?
13. How many ways are there to seat 6 men and 6 women in a row so that no two women sit next to each other?
14. Let  $n$  be a positive integer. How many permutations  $p$  of the integers from 1 to  $n$  are there such that the following hold:

- $p_i \neq i$  for all  $1 \leq i \leq n$ , and
- $p_1 \neq 2$ .

You may express your answer in terms of  $D_n$ , the number of derangements for a sequence of  $n$  elements.

15. For a permutation  $p$  of the integers from 1 to  $n$ , a fixed point is defined as an index  $i$  such that  $p_i = i$ . How many permutations of  $(1, 2, \dots, n)$  contain exactly one fixed point?
16. Find the coefficient of  $x^7$  in  $(1 + 2x + 3x^2)^{10}$ . There is no need to simplify your answer.
17. Simplify the expression

$$\sum_{k=0}^n \binom{n}{k} 99^k.$$

18. Chu Shih-chieh's identity is

$$\sum_{i=r}^n \binom{i}{r} = \binom{n+1}{r+1}$$

for  $n, r \in \mathbb{N}$  and  $n \geq r$ . Use a combinatorial proof to verify this.

19. Use a combinatorial proof to show that

$$\sum_{k=0}^n k^3 = \left( \sum_{k=0}^n k \right)^2$$

for all  $n \in \mathbb{N}$ .

20. Use a combinatorial proof to show that

$$\sum_{k=0}^n k^3 \binom{n}{k} = 2^{n-1} n + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}$$

for all  $n \in \mathbb{N}$ .

21. Consider an  $n \times n$  grid. We denote by  $(i, j)$  the cell on the  $i$ th row from the bottom and the  $j$ th column from the left. An ant starts on  $(1, 1)$  and makes its way to  $(n, n)$  by only moving up or right. Let  $a_{i,j}$  be the number of ways to reach the  $(i, j)$  by only moving up or right. Give a recurrence relation for  $a_{i,j}$  and give the closed form of  $a_{n,n}$ .

22. Consider the sequence

$$(1, -1, 1, -1, 1, -1, \dots).$$

Let  $a_i$  denote the  $i$ th term of the sequence, starting from  $i = 0$ . Give a recurrence relation for  $a_i$  and its closed form.

23. Consider a staircase. You may climb 1, 2, or 3 steps at a time, but you cannot take consecutive 2-steps. Let  $a_n$  be the number of ways to reach step  $n$ . Give the recurrence.
24. Give the closed form:  $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$ , where  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = 2$ , and  $a_3 = 3$ .
25. Give the closed form:  $a_n = 13a_{n-1} - 40a_{n-2} + 2^n$ , with  $a_0 = 0$ ,  $a_1 = 0$ .
26. We know that

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}.$$

Use generating functions to prove this identity.

27. Find the generating function for  $a_n = n$ . Hint: start from the generating function of the sequence  $b_n = 1$  for all  $n \geq 0$ .
28. The sequence of triangular numbers  $T_n$  is given by

$$T_n = \frac{n(n+1)}{2}.$$

Give the generating function for  $T_n$ .

29. Four fair six-sided dice are rolled. Use generating functions to count how many ways there are to achieve a total of 12.
30. We know that

$$\sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}.$$

Use generating functions to prove this identity. Hint: start from the identity

$$\frac{1}{1-x} = \sum_{k \geq 0} x^k$$

and differentiate  $r$  times.

## 9 Solutions

1. We note that  $n = 2^{12} \cdot 3^5 \cdot 4^3 = 2^{18} \cdot 3^5$ . All factors of  $n$  are of the form  $2^a \cdot 3^b$ , where  $a$  and  $b$  are integers and  $0 \leq a \leq 18$  and  $0 \leq b \leq 5$  hold. Since there are 19 possible values for  $a$  and 6 possible values for  $b$ , there are  $19 \cdot 6 = 114$  total factors.
2. Let the digits of some  $x \in A$  be  $d_1$ ,  $d_2$ , and  $d_3$ . We must have  $d_1 + d_2 + d_3 = 12$ .

Without restrictions, there are

$$\binom{12+3-1}{12} = \binom{14}{12}$$

ways to assign values to  $d_1$ ,  $d_2$ , and  $d_3$ . However, we must remove the cases where at least one of them is greater than 9.

Note that since  $d_1 + d_2 + d_3 = 12$ , at most one of them can be greater than 9. Thus, we can fix which of the digits is greater than 9.

Suppose  $d_1 > 9$ . Then, let  $d'_1 := d_1 - 10$ . Thus, we have:

$$\begin{aligned} d_1 + d_2 + d_3 &= d'_1 + 10 + d_2 + d_3 = 12 \\ d'_1 + d_2 + d_3 &= 2 \end{aligned}$$

There are

$$\binom{2+3-1}{2} = 6$$

solutions to this. Thus, we have 6 solutions where  $d_1 > 9$ . Note that by symmetry, we also have 6 solutions where  $d_2 > 9$  and 6 solutions where  $d_3 > 9$ .

Thus, there are

$$\binom{14}{12} - 6 - 6 - 6 = 73$$

valid numbers.

3. We first note that there are  $9 \cdot 10 \cdot 10 = 900$  positive 5-digit palindromes.

Consider some positive 5-digit palindrome whose digit sum is 10. Let  $x$  be the first and fifth digit,  $y$  be the second and fourth digit, and  $z$  be the third digit. Then, we have

$$2x + 2y + z = 10,$$

which implies  $z$  is even. Then, we have

$$x + y = 5 - \frac{z}{2}.$$

Note that  $x > 0$ ,  $y \geq 0$ , and  $z \in \{0, 2, 4, 6, 8\}$ . For each  $z$ , there are  $5 - z/2$  valid pairs  $(x, y)$ . Thus, over all  $z$ , we have  $5 + 4 + 3 + 2 + 1 = 15$  positive 5-digit palindromes whose digit sum is 10.

The final probability is

$$\frac{15}{900} = \frac{1}{60},$$

and the final answer is  $1 + 60 = 61$ .

4. We note that  $|A \cup B| = |A| + |B| - |A \cap B|$ . We have

$$0 \leq |A \cap B| \leq 23.$$

Thus,  $34 \leq |A \cup B| \leq 57$ .

5. We proceed with complementary counting. First, we have  $|S| = P(11, 6)$ . Then, we count how many elements of  $S$  contain **bed**. Note that we have 8 choices for the rest of the letters, and then we must permute the 4 elements (treating **bed** as a single unit). Thus, there are

$$\binom{8}{3} \cdot 4!$$

elements of  $S$  that contain **bed**. Thus, the final answer is

$$\frac{11!}{5!} - \binom{8}{3} \cdot 4!.$$

6. The statement is true.

*Proof.* Let  $f(x)$  be the number of friends of some person  $x$  in the group. Thus, for all  $x$ , we have  $f(x) \in [0, n - 1]$ . Consider two cases.

*Case 1.* There is a person with no friends in the group.

Then, it is impossible for someone else to have  $n - 1$  friends in the group; thus, for all  $x$ ,  $f(x) \in [0, n - 2]$ . We have  $n$  people and  $n - 1$  possible values for  $f$ .

*Case 2.* There is a person who is friends with everyone in the group.

Then, it is impossible for someone else to have no friends in the group; thus, for all  $x$ ,  $f(x) \in [1, n - 1]$ . We have  $n$  people and  $n - 1$  possible values for  $f$ .

Since in all cases, there are  $n$  people and  $n - 1$  possible values for  $f(x)$ , there exists two people  $a$  and  $b$  such that  $f(a) = f(b)$ . Thus, two people must have the same number of friends in the group. ■

7. The statement is true.

*Proof.* There are 7 empty chairs and, therefore, 8 segments of consecutive occupied seats. Since there are 28 people, one segment must have at least  $\lceil 28/8 \rceil = 4$  people. Thus, there always exists 4 consecutive occupied chairs. ■

8. The statement is true.

*Proof.* Let the sequence be  $a_1, a_2, \dots, a_{n^2+1}$ . We proceed with a proof by contradiction.

Suppose there exists a sequence of  $n^2 + 1$  distinct real numbers where length of the longest monotone subsequence is at most  $n$ . Let  $x_i$  be the length of the longest increasing subsequence ending at  $a_i$ , and let  $y_i$  be the length of the longest decreasing subsequence ending at  $a_i$ . Since  $1 \leq x_i, y_i \leq n$ , there are  $n \cdot n = n^2$  distinct pairs  $(x_i, y_i)$ .

There are  $n^2 + 1$  elements, so by the pigeonhole principle, there exist two elements  $a_i$  and  $a_j$  such that  $(x_i, y_i) = (x_j, y_j)$ .

Without loss of generality, suppose  $i < j$ . Since  $a_i \neq a_j$ , we can consider two cases.

If  $a_i < a_j$ , then we can extend the longest increasing subsequence ending at  $a_i$  by appending  $a_j$ . Thus, we cannot have  $x_i = x_j$ .

Similarly, if  $a_i > a_j$ , then we can extend the longest decreasing subsequence ending at  $a_i$  by appending  $a_j$ . Thus, we cannot have  $y_i = y_j$ .

Since both cases lead to contradictions, the initial assumption was false. Thus, the original statement is true. ■

9. The statement is true.

*Proof.* Consider two odd integers  $x$  and  $y$ , where  $x < y$  and  $y \bmod x$  is even. Note that we have

$$y = \left\lfloor \frac{y}{x} \right\rfloor \cdot x + (y \bmod x).$$

Since  $y$  and  $x$  are odd and  $y \bmod x$  is even,  $\lfloor y/x \rfloor$  should be odd.

Let  $s := \min_{x \in S} x$ . Partition the range  $[3, 2^X]$  into at most  $X - 1$  subranges:  $[s, 2s], [2s, 4s],$  and so on, until  $[2^{X-2}s, 2^{X-1}s]$ . Note that  $2^X < 2^{X-1}s$  because  $s > 2$ ; thus, the union of these ranges fully covers  $[3, 2^X]$ .

*Claim.* If two odd integers  $a, b$  with  $a < b$  are in the same subrange, then  $b \bmod a$  is even.

*Proof of claim.* Note that because  $a$  and  $b$  are in the same subrange,  $b < 2a$ . Since  $a < b < 2a$ , we must have

$$1 < \frac{b}{a} < 2,$$

which implies

$$\left\lfloor \frac{b}{a} \right\rfloor = 1.$$

Since  $a$  and  $b$  are both odd and  $\lfloor b/a \rfloor$  is odd,  $b \bmod a$  is even. ■

Now, since  $n \geq X$ , by the pigeonhole principle, at least two elements of  $S$  will be in the same subrange. Let these two elements be  $x$  and  $y$ , and without loss of generality, suppose  $x < y$ . Since they are in the same subrange,  $y \bmod x$  is even. Thus, we can always find two distinct integers  $x, y \in S$  such that  $y > x$  and  $y \bmod x$  is even. ■

10. The statement is true.

*Proof.* Note that there are  $n - 1$  possible remainders on division by  $n - 1$ . Since there are  $n$  integers in  $S$ , at least two must have the same remainder on division by  $n - 1$ . Let these two integers be  $a$  and  $b$ . Then, we must have

$$a \equiv b \pmod{n-1} \Rightarrow a - b \equiv 0 \pmod{n-1}.$$

As desired. ■

11. Observe that if we permute all other characters besides I and E, only 2 out of the 3 ways to arrange the 2 I's and the 1 E satisfy the condition (namely, II E and IEI). Thus,  $2/3$  of all permutations of the string COOKIEZI satisfy the condition. The answer is

$$\frac{2}{3} \cdot \frac{8!}{2!2!} = 6720.$$

12. We can count how many permutations are lexicographically smaller than (3, 1, 4, 1, 5).

We first count how many begin with 1. There are  $4! = 24$  permutations.

Then, we count how many begin with 3, 1, 1. There are  $2! = 2$  permutations.

We observe that (3, 1, 4, 1, 5) is the first permutation starting with 3, 1, 4, so we can conclude that there are  $24 + 2 = 26$  permutations lexicographically smaller than (3, 1, 4, 1, 5). Since there are

$$\frac{5!}{2!} = 60$$

total permutations, there are  $60 - 1 - 26 = 33$  permutations lexicographically larger than (3, 1, 4, 1, 5).

13. We can seat the 6 men first and then seat the 6 women in the 7 spots between the men and at the ends. The answer is  $6! \cdot 7!$ .

14. We proceed with complementary counting; count how many permutations there are such that  $a_i \neq i$  for all  $i$  and  $a_1 = 2$ .

There are two cases.

*Case 1.*  $a_2 = 1$

Then, we must count how many permutations of the remaining  $n - 2$  there are such that  $a_i = i$ . This is simply  $D_{n-2}$ .

*Case 2.*  $a_2 \neq 1$

We must count how many permutations of the  $n - 1$  elements there are such that  $a_i = i$  for  $2 < i \leq n$  and  $a_2 \neq 1$ . This is  $D_{n-1}$ .

Thus, the final answer is  $D_n - (D_{n-1} + D_{n-2})$ .

15. We can first choose the index of the fixed point and then permute everything else to satisfy the condition. Thus, there are  $nD_{n-1}$  permutations.
16. Let the exponent of  $2x$  be  $i$  and the exponent of  $3x^2$  be  $j$ . We must find

$$\sum_{\substack{i+2j=7 \\ i,j \geq 0}} \binom{10}{i,j,10-i-j} 2^i 3^j.$$

We note that the only valid pairs of  $(i, j)$  are  $(1, 3)$ ,  $(3, 2)$ ,  $(5, 1)$ , and  $(7, 0)$ . Thus, the answer is

$$\binom{10}{1,3,6} 2^1 3^3 + \binom{10}{3,2,5} 2^3 3^2 + \binom{10}{5,1,4} 2^5 3^1 + \binom{10}{7,0,3} 2^7 3^0.$$

17. Recall that

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

Substituting  $x = 99$  yields

$$\sum_{k=0}^n \binom{n}{k} 99^k = (1+99)^n = 100^n.$$

18. *Proof.* The

*End*