

# CS 31 Question Frenzy

Diogn Lei R. Mortera

Typst template from Ieuan Vinluan; Some problems from Jay Teofilo's cheatsheet

Problems from me, Patrick's "Intermediate Counting and Probability", Brualdi's "Introductory Combinatorics"

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## 1. Grade School Math

- How many  $k$ -sized subarrays are there in an  $n$ -element list?
- How many proper divisors does  $10^n$  have?
- What is the smallest number with 20 factors?
- How many 5-digit numbers are divisible by 3, 5, and 7?
- Prove that the decimal expansion of a number is either terminating or repeating.
- How many surjective functions are there of a function  $f : A \rightarrow B$  if  $|A| = m$  and  $|B| = n$ ?

## 2. Crack the Codons

DNA is represented as a string over the alphabet  $\{A, T, C, G\}$ . RNA is represented as a string over the alphabet  $\{A, U, C, G\}$ . A codon is a sequence of three bases from  $\{A, U, C, G\}$ . (The characters A, T, C, G, U are called bases.)

Among all possible codons, three do not encode amino acids; instead, they serve as start or stop signals in the genetic sequence.

- How many codons encode amino acids?
- How many DNA strands of length 10 are there?
- How many DNA strands of length 10 which have at least one of each base are there?
- A point mutation changes one base in a DNA strand to a different base. How many distinct point mutations are possible for a DNA strand of length 10?
- A  $k$ -frameshift mutation replaces a contiguous substring of length  $k$  in a DNA strand with a different substring of length  $k$ . How many distinct 4-frameshift mutations are possible for a DNA strand of length 10?
- Prove that the number of DNA strands of length  $n$  equals the number of RNA strands of length  $n$ .

In this problem we consider a variant of alien DNA called MKNA, where data is encoded by the number of occurrences of the base.

- How many MKNA are there?
- If at least 1 of each base was required, how many possible MKNA are there?

### 3. Just Act Natural

We will do a bunch of things with the set below:

$$A = \{1, 2, \dots, n\}$$

- Calculate the number of subsets from  $A$  where no two numbers in the subset are consecutive.
- If a collection of subsets of  $A$  has the property that each pair of subsets has at least one element in common, show that there are at most  $2^{n-1}$  subsets in the collection.
- What is the minimal  $n$  required such that there exists 2 disjoint nonempty subsets in  $A$  have the same sum? How about 3?
- Show that a pair in  $A$  exists such that when you sum them together, they are divisible by  $n$ .

### 4. Team Building

In a team of  $n$  people:

- How many ways can you create  $n$  pairs, if  $n$  is even? If  $n$  is odd? Each person must belong to only one pair.
- Prove that if each pair plays against one another, gaining a point at a win, there exists two teams with the same score. Assume no ties can occur.
- How many ways are there to create final standings, if ties are allowed? (Hint: express as a recurrence relation)

For the following questions, let  $n = 10$ :

- If there are 3 cabin assignments, how many ways are there to assign a cabin assignments to the teams?
- The team is playing “The Boat Is Sinking”, and were asked to group themselves into 3. How many possible groupings of 3 are there? Note that it only matters who is in each group; swapping the groups around doesn’t create a new arrangement.

### 5. Geometry Dash

- Show that a Venn Diagram made of circles for 4 categories is impossible.
- Show that the maximal number of intersection points of  $n$  lines is  $\binom{n}{2}$  (hint: use recurrence relations)
- Show that the maximal number of regions formed by  $n$  lines is  $\binom{n}{2} + n + 1$  (hint: use recurrence relations)
- Prove that in an equilateral triangle with

$$n^2 + 1$$

points, there exists two points whose distance is less than  $\frac{1}{n}$ .

- In a unit square, 51 points are placed arbitrarily. Prove that there is a circle of radius  $1/7$  that contains 3 of the points.
- Find a one-to-one correspondence between the number of ways to triangulate a  $n$ -sided polygon (ways to dividing it into triangles) and ways to parenthesize an  $n$ -element list.
- The line segments joining 10 points pairwise are arbitrarily colored red or blue. Prove that there must exist 3 points such that the 3 line segments joining them are all red, or four points such that the 6 line segments joining them are all blue.

