CS 31 Question Frenzy

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Typst template from Ieuan Vinluan; Some problems from Jay Teofilo's cheatsheet

Problems from me, Patrick's "Intermediate Counting and Probability", Brualdi's "Introductory Combinatorics"

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1. Grade School Math

a. How many k-sized subarrays are there in an n-element list?
b. How many proper divisors does 10^n have?
c. What is the smallest number with 20 factors?
d. How many 5-digit numbers are divisible by 3, 5, and 7?
e. Prove that the decimal expansion of a number is either terminating or repeating.
f. How many surjective functions are there of a function $f : A \to B$ if $|A| = m$ and $|B| = n$?

2. Crack the Codons

DNA is represented as a string over the alphabet $\{A, T, C, G\}$. RNA is represented as a string over the alphabet $\{A, U, C, G\}$. A codon is a sequence of three bases from $\{A, U, C, G\}$. (The characters A, T, C, G, U are called bases.)

Among all possible codons, three do not encode amino acids; instead, they serve as start or stop signals in the genetic sequence.

a. How many codons encode amino acids?

b. How many DNA strands of length 10 are there?

c. How many DNA strands of length 10 which have at least one of each base are there?

d. A point mutation changes one base in a DNA strand to a different base. How many distinct point mutations are possible for a DNA strand of length 10?

e. A k-frameshift mutation replaces a contiguous substring of length k in a DNA strand with a different substring of length k. How many distinct 4-frameshift mutations are possible for a DNA strand of length 10?

f. Prove that the number of DNA strands of length n equals the number of RNA strands of length n.

In this problem we consider a variant of DNA called MKNA, where data is encoded by the number of occurences of the base. An example MKNA of length 10 is $\{4 : A, 3 : C, 2 : G, 1 : T\}$.

a. How many MKNA of length 10 are there?
b. If at least 1 of each base was required, how many possible MKNA are there?

3. Just Act Natural

We will do a bunch of things with the set below:

$$A = \{1, 2, ..., n\}$$

a. Calculate the number of subsets from A where no two numbers in the subset are consecutive.
b. If a collection of subsets of A has the property that each pair of subsets has at least one element in common, show that there are at most 2^{n-1} subsets in the collection.
c. What is the minimal n required such that there exists 2 disjoint nonempty subsets in A have the same sum? How about 3?
d. Show that a pair in A exists such that when you sum them together, they are divisible by n.

4. Team Building

In a team of n people:

a. How many ways can you create $\frac{n}{2}$ pairs, if n is even? If n is odd? Each person must belong to only one pair.
b. Prove that if each pair plays against one another, gaining a post at a win, there exists two teams with the same score. Assume no ties can occur.
c. How many ways are there to create final standings, if ties are allowed? (Hint: express as a recurrence relation)

For the following questions, let $n = 10$:

a. If there are 3 cabin assignments, how many ways are there to assign a cabin assignments to the teams?
b. The team is playing "The Boat Is Sinking", and were asked to group themselves into 3. How many possible groupings of 3 are there? Note that it only matters who is in each group; swapping the groups around doesn't create a new arrangement.

5. Geometry Dash

a. Show that a Venn Diagram made of circles for 4 categories is impossible.

b. Show that the maximal number of intersection points of n lines is $\binom{n}{2}$ (hint: use recurrence relations)

c. Show that the maximal number of regions formed by n lines is $\binom{n}{2} + n + 1$ (hint: use recurrence relations)

d. Prove that in an equilateral triangle with

$$n^2 + 1$$

points, there exists two points whose distance is less than $\frac{1}{n}$.

e. In a unit square, 51 points are placed arbitarily. Prove that there is a circle of radius 1/7 that contains 3 of the points.

f. Find a one-to-one correspondence between the number of ways to triangulate a n-sided polygon (ways to dividing it into triangles) and ways to parenthesize an n-element list.

g. The line segments joining 10 points pairwise are arbitrarily colored red or blue. Prove that there must exist 3 points such that the 3 line segments joining them are all red, or four points such that the 6 line segments joining them are all blue.

6. Rube Goldberg

a. Given the linear nonhomogeneous recurrence relation

$$a_n = 9a_{n-1} - 29a_{n-2} + 39a_{n-3} - 18a_{n-4} + F(n)$$

where $F(n)$ is some function of n.

Determine the form of a particular solution if $F(n)$ is:

i. $F(n) = 1$
ii. $F(n) = n2^n$
iii. $F(n) = (n-5)2^n$
iv. $F(n) = (n^{2025} - 1)2^n$
v. $F(n) = 3^n + 2$
vi. $F(n) = 4^{n-2} + 2$
vii. $F(n) = n^2 2^n$

b. Say I had a string

$$S = \text{'I love CS31, love it so much!'}$$

I then perform the following operation: Replace “love” in S with S. For example, after 2 iterations, the string should now look like:

‘I I I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! CS31, I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! it so much! CS31, I I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! CS31, I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! it so much! it so much!’

Can you count instead for n iterations how many Is are in the resulting S?

7. Generation Failure

a. Express the following sequences as generating functions:

$$\{1, -1, 2, -2, 3, -3, ...\}$$

$$\{1, 2, 2, 3, 3, 3, ...\}$$

$$\left\{1, 4, \binom{5}{2}, \binom{6}{3}, \binom{7}{4}, ...\right\}$$

$$\{1^k, 2^k, 3^k, ...\}$$

$$\{1, 1, 2, 3, 5, 8, ...\}$$

a. Express the following combinatorial problems using generating functions:

i. ways to assign n candies to m children where the k-th child wants to have a multiple of k amount of candies
ii. ways to partition a number n using only even numbers
iii. ways to put $2, 3, 5, 7, 11$ peso coins in a vending machine (if order of putting in matters)

b. Derive a formula for the following using generating functions:

$$\sum_{k=0}^{n} k^m \binom{n}{k}$$