

CS 31 Question Frenzy

Diogn Lei R. Mortera

Typst template from Ieuau Vinluan; Some problems from Jay Teofilo's cheatsheet

Problems from me, Patrick's "Intermediate Counting and Probability", Brualdi's "Introductory Combinatorics"

1. Grade School Math

- a. How many k -sized subarrays are there in an n -element list?
- b. How many proper divisors does 10^n have?
- c. What is the smallest number with 20 factors?
- d. How many 5-digit numbers are divisible by 3, 5, and 7?
- e. Prove that the decimal expansion of a number is either terminating or repeating.
- f. How many surjective functions are there of a function $f : A \rightarrow B$ if $|A| = m$ and $|B| = n$?

2. Crack the Codons

DNA is represented as a string over the alphabet $\{A, T, C, G\}$. RNA is represented as a string over the alphabet $\{A, U, C, G\}$. A codon is a sequence of three bases from $\{A, U, C, G\}$. (The characters A, T, C, G, U are called bases.)

Among all possible codons, three do not encode amino acids; instead, they serve as start or stop signals in the genetic sequence.

- a. How many codons encode amino acids?
- b. How many DNA strands of length 10 are there?
- c. How many DNA strands of length 10 which have at least one of each base are there?
- d. A point mutation changes one base in a DNA strand to a different base. How many distinct point mutations are possible for a DNA strand of length 10?
- e. A k -frameshift mutation replaces a contiguous substring of length k in a DNA strand with a different substring of length k . How many distinct 4-frameshift mutations are possible for a DNA strand of length 10?
- f. Prove that the number of DNA strands of length n equals the number of RNA strands of length n .

In this problem we consider a variant of alien DNA called MKNA, where data is encoded by the number of occurrences of the base.

- a. How many MKNA are there?
- b. If at least 1 of each base was required, how many possible MKNA are there?

3. Just Act Natural

We will do a bunch of things with the set below:

$$A = \{1, 2, \dots, n\}$$

- a. Calculate the number of subsets from A where no two numbers in the subset are consecutive.
- b. If a collection of subsets of A has the property that each pair of subsets has at least one element in common, show that there are at most 2^{n-1} subsets in the collection.
- c. What is the minimal n required such that there exists 2 disjoint nonempty subsets in A have the same sum? How about 3?
- d. Show that a pair in A exists such that when you sum them together, they are divisible by n .

4. Team Building

In a team of n people:

- a. How many ways can you create n pairs, if n is even? If n is odd? Each person must belong to only one pair.
- b. Prove that if each pair plays against one another, gaining a point at a win, there exists two teams with the same score. Assume no ties can occur.
- c. How many ways are there to create final standings, if ties are allowed? (Hint: express as a recurrence relation)

For the following questions, let $n = 10$:

- a. If there are 3 cabin assignments, how many ways are there to assign a cabin assignments to the teams?
- b. The team is playing “The Boat Is Sinking”, and were asked to group themselves into 3. How many possible groupings of 3 are there? Note that it only matters who is in each group; swapping the groups around doesn’t create a new arrangement.

5. Geometry Dash

- a. Show that a Venn Diagram made of circles for 4 categories is impossible.
- b. Show that the maximal number of intersection points of n lines is $\binom{n}{2}$ (hint: use recurrence relations)
- c. Show that the maximal number of regions formed by n lines is $\binom{n}{2} + n + 1$ (hint: use recurrence relations)
- d. Prove that in an equilateral triangle with

$$n^2 + 1$$

points, there exists two points whose distance is less than $\frac{1}{n}$.

- e. In a unit square, 51 points are placed arbitrarily. Prove that there is a circle of radius $1/7$ that contains 3 of the points.
- f. Find a one-to-one correspondence between the number of ways to triangulate a n -sided polygon (ways to dividing it into triangles) and ways to parenthesize an n -element list.
- g. The line segments joining 10 points pairwise are arbitrarily colored red or blue. Prove that there must exist 3 points such that the 3 line segments joining them are all red, or four points such that the 6 line segments joining them are all blue.

6. Rube Goldberg

- a. Given the linear nonhomogeneous recurrence relation

$$a_n = 9a_{n-1} - 29a_{n-2} + 39a_{n-3} - 18a_{n-4} + F(n)$$

where $F(n)$ is some function of n .

Determine the form of a particular solution if $F(n)$ is:

- i. $F(n) = 1$
 - ii. $F(n) = n2^n$
 - iii. $F(n) = (n - 5)2^n$
 - iv. $F(n) = (n^{2025} - 1)2^n$
 - v. $F(n) = 3^n + 2$
 - vi. $F(n) = 4^{n-2} + 2$
 - vii. $F(n) = n^22^n$

- b. Say I had a string

$S = \text{'I love CS31, love it so much!'}$

I then perform the following operation: Replace “love” in S with S . For example, after 2 iterations, the string should now look like:

‘I I I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! CS31, I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! it so much! CS31, I I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! CS31, I I love CS31, love it so much! CS31, I love CS31, love it so much! it so much! it so much!’

Can you count instead for n iterations how many I s are in the resulting S ?

7. Generation Failure

- a. Express the following sequences as generating functions:

$$\{1, -1, 2, -2, 3, -3, \dots\}$$

$$\{1, 2, 2, 3, 3, 3, \dots\}$$

$$\left\{ 1, 4, \binom{5}{2}, \binom{6}{3}, \binom{7}{4}, \dots \right\}$$

$$\{1^k, 2^k, 3^k, \dots\}$$

$$\{1, 1, 2, 3, 5, 8, \dots\}$$

- a. Express the following combinatorial problems using generating functions:

 - ways to assign n candies to m children where the k -th child wants to have a multiple of k amount of candies
 - ways to partition a number n using only even numbers
 - ways to put 2, 3, 5, 7, 11 peso coins in a vending machine (if order of putting in matters)

b. Derive a formula for the following using generating functions:

$$\sum_{k=0}^n k^m \binom{n}{k}$$