Computability Theory

May 6, 2014

Exercises (group I):

(1) Define a program for the Turing machine TM_1^1 which computes the function $\{(n, 2 * n) \mid n \in \mathbb{N}\}$.

(2) Define a program for the register machine RM_1^1 which computes the function f for $f(0) = 1,\ f(sy) = 2 * f(y)$.

(3) Let $P = (1, A)$ be a program for the register machine RM_1^2, where $A =$

$$\begin{aligned}&\{1\colon\ \underline{\text{if}}\ t_1\ \underline{\text{then}}\ 4\ \underline{\text{else}}\ 2,\ 2\colon A_3\ \underline{\text{then}}\ 3,\ 3\colon S_1\ \underline{\text{then}}\ 1,\\ &4\colon\ \underline{\text{if}}\ t_3\ \underline{\text{then}}\ 0\ \underline{\text{else}}\ 5,\ 5\colon\ \underline{\text{if}}\ t_2\ \underline{\text{then}}\ 9\ \underline{\text{else}}\ 6,\ 6\colon A_4\ \underline{\text{then}}\ 7,\\ &7\colon A_1\ \underline{\text{then}}\ 8,\ 8\colon S_2\ \underline{\text{then}}\ 5,\ 9\colon S_3\ \underline{\text{then}}\ 10,\\ &10\colon\ \underline{\text{if}}\ t_4\ \underline{\text{then}}\ 4\ \underline{\text{else}}\ 11,\ 11\colon\ S_4\ \underline{\text{then}}\ 12,\ 12\colon A_2\ \underline{\text{then}}\ 10\}\end{aligned}$$

Determine $\mathit{Func}(P)$.

(4) Let Σ and Π be the following operators on functions $\mathbb{N}^{n+1} \to \mathbb{N}$:

$$\Sigma(f)(x, y) = \sum_{i=0}^{y} f(x, i),\ \Pi(f)(x, y) = \prod_{i=0}^{y} f(x, i).$$

Show that **PR** and **R** are closed under Σ and Π.

(5) Show that **PR** and **R** are closed under bounded quantification.

(6) Prove that **PR** is closed under bounded maximization, i.e.: Let Q be an $n + 1$-place primitive recursive predicate and let

$$\begin{aligned}(\max Q)(x, y) &= \max\{k \mid k \le y \wedge Q(x, k)\} \text{ if } (\exists i \le y) Q(x, i),\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then $(\max Q)$ is primitive recursive.

(7) Let $p(0) = 1$ and for all $n \ge 1$ $p(n) =$ the n-th prime. Prove that p is primitive recursive.