Applications of Fibonacci Numbers

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A REPORT ON THE THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

A newspaper article at Pisa, Italy, with a prominent headline: "CONVEGNO PARLANO I MATEMATICI L'INCONTRO IN OMMAGIO A FIBONACCI" heralded our Third International Conference on Fibonacci Numbers and Their Applications which was held in Pisa, Italy, July 25th-29th, 1988. A stamp: "I NUMERI DI FIBONACCI CONGRESSO INTERNAZIONALE, 26-7-1988" commemorated it.

Of course, mathematicians all across the globe, and especially those who are so fortunate as to have become interested in "Fibonacci-type mathematics," had known about it for some time. The August 1987 issue of *The Fibonacci Quarterly* had brought the glad tidings: an announcement that our third conference was to take place at the University of Pisa during the last week of July 1988.

By mid June 1988, we held the coveted program in our hands. 66 participants were listed, and they came from 22 different countries, the U.S. heading the list with a representation of 20, followed by Italy and Australia. Of course, it was to be expected that at conference time proper additional names would lengthen the count. Forty-five papers were to be presented, several of them with coauthors; there were 3 women speakers.

Theoretically sounding titles abounded. There was Andreas N. Philippou's paper, coauthored by Demetris L. Antzoulakes: "Multivariate Fibonacci Polynomials of Order K and the Multiparameter Negative Binomial Distribution of the Same Order." But, rather intriguingly, practical interests wedged themselves in also with Piero Filipponi's paper, coauthored by Emilio Montolivo: "Representation of Natural Numbers as a Sum of Fibonacci Numbers: An Application to Modern Cryptography." This again highlighted one of the joys mathematicians experience: the interplay between theoretical and applied mathematics.

What a delight it was to meet in Pisa, Italy, the birthplace of Leonardo of Pisa, son of Bonacci, "our" Fibonacci (=1170-1250). We already knew that—befittingly, and much to our pleasure—Pisa had honored its mathematical son by a statue. My friends and I were among the many (maybe it was all of them) who made a pilgrimage to Fibonacci's statue. It was a fairly long walk, eventually on Via Fibonacci(!), along the Arno River, until we finally found him in a pretty little park. He seemed thoughtful, and appeared to enjoy the sight of the nearby shrubs and flowers. I felt like thanking him for "having started it all," for having coined the sequence that now bears his name. It would have been nice to invite him to our sessions. I predict he would have been thoroughly startled. What had happened since 1202 when his *Liber Abaci* was published?!

Almost invariably, the papers were of very high caliber. The great variety of topics and the multitude of approaches to deal with a given mathematical idea was remarkable and rather appealing. And it was inspiring to coexperience the deep involvement which authors feel with their topic.

We worked hard. The sessions started at 9 a.m. and with short intermissions (coffee break and lunch) they lasted till about 5:30 p.m. As none of the papers were scheduled simultaneously, we could experience the luxury of hearing ALL presentations.

We did take out time to play. Of course, just to BE in Pisa was a treat. We stepped into the past, enwrapped into the charm of quaint, old buildings, which—could they only talk—would fascinate us with their memories of olden times. As good fortune would have it (or, was it the artistry of Roborto Dvornicich, Professor of Mathematics at the University of Pisa, who arranged housing for the conference participants) my friends and I stayed at the Villa Kinzica—across the street form the Leaning Tower of Pisa. Over a plate of spaghetti, we could see that tower, one of the "seven wonders of the world" whose very construction took 99 years. And—it REALLY leans! We were charmed by the seven bells, all chiming in different tones. But—most of all—we pictured Galileo Galilei excitedly experimenting with falling bodies . . .

I would be amiss if I did not mention the Botanical Garden of Pisa—situated adjacent to our conference room at POLO DIDATTICO DELLA FACOLTA DI SCIENZE. In the summer of 1543 (the University of Pisa itself was founded in the 12th century) this garden was opened as the first botanical garden in Western Europe. Its present location was taken up 50 years later. While we may not have been able to recognize "METASEQUOIA GLYPTOSTROBOIDES" the peace and serenity of this beautiful park struck chords in all of us.

On the third day, the Conference terminated at noon, and we took the bus to Volterra. The bus ride itself ushered in a trip long to be remembered. The incredibly luscious fields of sunflowers and sunflowers—an actual ocean of yellows—were not only joyous, but also touched our mathematical souls. Do Fibonacci numbers not play an important role in deciphering nature's handiwork in sunflowers?

Volterra, situated about 550 metres above sea-level, immediately transplanted us into enigmatic Etruscan, as well as into problematic Medieval times. While we were fascinated both by the historic memorabilia, as well as by the artifacts and master pieces, the magnificent panorama of the surrounding landscape enhanced our enjoyment still further.

As has become tradition in our conference, a banquet was held on the last night before the closing of our sessions. Lucca, the site of the meeting, provided a wonderful setting for a memorable evening, Ligurian in origin, it bespeaks of Etruscan culture, and exudes the charm of an ancient city.

The spirit at the banquet highlighted what had already become apparent during the week: that the Conference had not only been mind-streatching, but also heartwarming. Friendships which had been started, became knitted more closely. New friendships were formed. The magnetism of common interest and shared enthusiasm wove strong bonds among us. We had come from different cultural and ethnic backgrounds, and our native tongues differed. Yet, we truly understood each other. And we cared for each other.

I believe, I speak for all of us if I express by heartfelt thanks to all members of the International, as well as of the Local Committee whose dedication and industriousness gave us this unforgettable event. Our gratitude also goes to the University of Pisa whose generous hospitality we truly appreciated. I would also like to thank all participants, without whose work we could not have had this treat.

"Auf Wiedersehen" then, at Conference number Four in 1990.

Herta T. Freitag

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FOREWORD

This book contains thirty-six papers from among the forty-five papers presented at the Third International Conference on Fibonacci Numbers and Their Applications which was held in Pisa, Italy from July 25 to July 29, 1988 in honor of Leonardo de Pisa. These papers have been selected after a careful review by well known referees in the field, and they range from elementary number theory to probability and statistics. The Fibonacci numbers are their unifying bond.

It is anticipated that this book, like its two predecessors, will be useful to research workers and graduate students interested in the Fibonacci numbers and their applications.

August 1989

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INTRODUCTION

The numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$$

known as the Fibonacci numbers, have been named by the nineteenth-century French mathematician Edouard Lucas after Leonard Fibonacci of Pisa, one of the best mathematicians of the Middle Ages, who referred to them in his book *Liber Abaci* (1202) in connection with his rabbit problem.

The astronomer Johann Kepler rediscovered the Fibonacci numbers, independently, and since then several renowned mathematicians have dealt with them. We only mention a few: J. Binet, B. Lamé, and E. Catalan. Edouard Lucas studied Fibonacci numbers extensively, and the simple generalization

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots,$$

bears his name.

During the twentieth century, interest in Fibonacci numbers and their applications rose rapidly. In 1961 the Soviet mathematician N. Vorobyov published *Fibonacci Numbers*, and Verner E. Hoggatt, Jr., followed in 1969 with his *Fibonacci and Lucas Numbers.* Meanwhile, in 1963, Hoggatt and his associates founded The Fibonacci Association and began publishing *The Fibonacci Quarterly.* They also organized a Fibonacci Conference in California, U.S.A., each year for almost sixteen years until 1979. In 1984, the First International Conference on Fibonacci Numbers and Their Applications was held in Patras, Greece, and the proceedings from this conference have been published. It was anticipated at that time that this conference would set the beginning of international conferences on the subject to be held every two or three years in different countries. With this intention as a motivating force, The Second International Conference on Fibonacci Numbers and Their Applications was held in San Jose, California, U.S.A., August 13-16, 1986. The proceedings from this conference have also been published. In order to carry on this new tradition, The Third International Conference on Fibonacci Numbers and Their Applications was held in Pisa, Italy, July 25-29, 1988. This book is a result of that conference. Because of the participation at the third conference and the encouragement to hold another conference in two years, The Fourth International Conference on Fibonacci Numbers and Their Applications will take place at Winston-Salem, North Carolina, U.S.A., July 30-August 3, 1990.

It is impossible to overemphasize the importance and relevance of the Fibonacci numbers to the mathematical and physical sciences as well as other areas of study. The Fibonacci numbers appear in almost every branch of mathematics, like number theory, differential equations, probability, statistics, numerical analysis, and linear algebra. They also occur in physics, biology, chemistry, and electrical engineering.

It is believed that the contents of this book will prove useful to everyone interested in this important branch of mathematics and that this material may lead to additional results on Fibonacci numbers both in mathematics and in their applications to science and engineering.

The Editors

THE ROLE OF THE FIBONACCI SEQUENCE IN THE ISOLATION OF THE REAL ROOTS OF POLYNOMIAL EQUATIONS

A. G. Akritas and P. G. Bradford

1. INTRODUCTION

Isolation of the real roots of polynomials in $\mathbb{Z}[x]$ is the process of finding real, disjoint intervals each of which contains exactly one real root and every real root is contained in some interval. This process is of interest because, according to Fourier, it constitutes the first step involved in the computation of real roots, the second step being the approximation of these roots to any desired degree of accuracy.

Various propositions have been used to isolate the real roots of polynomial equations with integer coefficients; due to their relation to Fibonacci numbers in this paper we will only examine Vincent's theorem [10] and Wang's generalization of it as presented by Chen in her dissertation [8].

In its *original* statement Vincent's theorem of 1836 states the following [7]:

Theorem 0: If in a polynomial equation with rational coefficients and without multiple roots one makes successive substitutions of the form

$$x{:=}a_1 + 1/x',\ x'{:=}a_2 + 1/x'',\ x''{:=}a_3 + 1/x''',\ \ldots,$$

where a_1 is an arbitrary nonnegative integer and a_2, a_3, ... are any positive integers, then the resulting, transformed equation has either zero or one sign variation. In the latter, the equation has a single positive root represented by the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}},$$

whereas in the former case there is no root.

Obviously, this theorem only treats positive roots; the negative roots are investigated by replacing x by $-x$ in the original polynomial equation. The generality of the thoerem is not restricted by the fact that there should be no multiple roots, because we can first apply square-free factorization [6]. Vincent himself states that his theorem was hinted in 1827 by Fourier, who never did give any proof of it (or if he did, it was never found); moreover, Lagrange had used the basic principle of this theorem much earlier.

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Notice that Vincent's theorem does not give us a bound on the number of substitutions of the form $x{:=}a_i + 1/x$ that have to be performed; this bound was computed with the help of the Fibonacci sequence by Uspensky (with a correction by Akritas) and is described below.

In 1960, and without being aware of Vincent's theorem, Wang generalized it so that it can be applied to polynomial equations with multiple roots; more precisely, using Wang's theorem we obtain not only the isolating intervals of the roots but also their multiplicites. Like Vincent, Wang did not give us a bound on the number of substitutions of the form $x{:=}a_i + 1/x$ that have to be performed; and again, this bound was computed with the help of the Fibonacci sequence by Chen (in her Ph.D. thesis) and is also described below.

2. VINCENT'S THEOREM OF 1836 AND WANG'S THEOREM OF 1960

We begin with a formal definition of sign variations in a number sequence.

Definition: We say that a sign variation exists between two nonzero numbers c_p and c_q ($p < q$) of a finite or infinite sequence of real numbers $c_1, c_2, \ldots$, if the following holds:

for $q = p + 1$, c_p and c_q have opposite signs;

for $q \geq p + 2$, the numbers $c_{p+1}, \ldots, c_{q-1}$ are all zero and c_p and c_q have opposite signs.

We next present the extended version of Vincent's theorem of 1836 which, by the way, is based on Budan's theorem of 1807 [5]. Notice how the Fibonacci numbers are used to bound the number of partial quotients that need to be computed.

Theorem 1: Let $p(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its roots. Let m be the smallest index such that

$$F_{m-1}\Delta/2 > 1 \text{ and } F_{m-1}F_m\Delta > 1 + 1/\varepsilon_n, \tag{V}$$

where F_k is the k-th member of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... and

$$\varepsilon_n = (1 + 1/n)^{1/(n-1)} - 1.$$

Let a_1 be an arbitrary nonnegative integer and let $a_2, \ldots, a_m$ be arbitrary positive integers. Then the substitution

$$x{:=}a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_m + \cfrac{1}{y}}}} \tag{CF}$$

(which is equivalent to the series of successive substitutions of the form $x{:=}a_i + 1/y$, $i = 1, 2, \ldots, m$) transforms the equation $p(x) = 0$ into the equation $p_{ti}(y) = 0$, which has no more than one sign variation in the sequence of its coefficients.

The proof can be found in the literature [4], [6]. Since the transformed equation $p_{ti}(y) = 0$ has either 0 or 1 sign variation, the above theorem is closely related to the Cardano-Descartes rule of signs which states that the number p of positive roots of a polynomial equation $p(x) = 0$ cannot exceed the number v of sign variations in the sequence of coefficients of p(x), and if $n = v - p > 0$, then n is an even number. Notice that the Cardano-Descartes rule of signs gives the exact number of positive roots only in the following two special cases:

(i) if there is no sign variation, there is no positive root, and

(ii) if there is one sign variation, there is one positive root.

(Observe how these two special cases are used in Theorem 1 above.)

Theorem 1 can be used in the isolation of the real roots of a polynomial equation. To see how it is applied, observe the following:

i. The continued fraction substitution (CF) can also be written as

$$x := \frac{p_m y + p_{m-1}}{q_m y + q_{m-1}}, \tag{CF1}$$

where p_k/q_k is the k-th convergent to the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}},$$

and, as we know, for $k \geq 0$, $p_0 = 1$, $p_{-1} = 0$, $q_0 = 0$, and $q_{-1} = 1$ we have:

$$p_{k+1} := a_{k+1}p_k + p_{k-1},$$

$$q_{k+1} := a_{k+1}q_k + q_{k-1}.$$

ii. The distance between two consecutive convergents is

$$|p_{m-1}/q_{m-1} - p_m/q_m| = 1/q_{m-1}q_m.$$

Clearly, the smallest values of the q_i occur when $a_i = 1$ for all i. Then, $q_m = F_m$, the m-th Fibonacci number. This explains why there is a relation between the Fibonacci numbers and the distance Δ in Theorem 1.

iii. Let $p_{ti}(y) = 0$ be the equation obtained from $p(x) = 0$ after a substitution of the form (CF1), corresponding to a series of translations and inversions. Observe that (CF1) maps the interval $0 < y < \infty$ onto the x-interval whose unordered endpoints are the consecutive convergents p_{m-1}/q_{m-1} and p_m/q_m. If this x-interval has length less than Δ, then it contains at most one root of $p(x) = 0$, and the corresponding equation $p_{ti}(y) = 0$ has at most one root in $(0,\infty)$.

iv. If y' were this positive root of $p_{ti}(y) = 0$, then the corresponding root x' of $p(x) = 0$ could be easily obtained from (CF1). We only know though, that y' lies in the interval $(0,\infty)$; therefore, substituting y in (CF1) once by 0 and once by ∞ we obtain for the positive root x its isolating interval whose unordered endpoints are p_{m-1}/q_{m-1} and p_m/q_m. To each positive root there corresponds a different continued fraction; at most m partial quotients have to be computed for the isolation of any positive root. (As we mentioned before, negative roots can be isolated if we replace x by $-x$ in the original equation.)

The calculation of the partial quotients (for each positive root) constitutes the real root isolation procedure. There are two methods, Vincent's and the continued fractions method of 1978 (developed by Akritas), corresponding to the two different ways in which the computation of the a_i's may be performed. The difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue. That is, the sum 1+1+1+1+1 can be computed in the following two ways:

(a) $1+1 = 2$, $2+1 = 3$, $3+1 = 4$, $4+1 = 5$ (Riemann) and
(b) $5 \cdot 1 = 5$ (Lebesgue).

Vincent's method consists of computing a particular a_i by a series of unit incrementations $a_i := a_i + 1$, to each one of which corresponds the translation $p_{ti}(x) := p_{ti}(x+1)$ for some polynomial equation $p_{ti}(x)$. This brute force approach results in a method with exponential behavior and hence is of little practical importance.

The continued fractions method of 1978 on the contrary, consists of computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial equation; actually, we can safely conclude that $b = \lfloor a_s \rfloor$ where a_s is the smallest positive root of some equation obtained during the transformations described in Theorem 1. Implementation details can be found in the literature [1], [2]. Here we simply mention that to compute this lower bound b on the values of the positive roots we use Cauchy's rule [3] (actually presented for upper bounds).

Cauchy's Rule: Let $p(x) = x^n + c_{n-1}c^{n-1} + \ldots + c_n x + c_0 = 0$ be a monic polynomial equation with integer coefficients of degree $n>0$, with $c_{n-k}<0$ for at least one k, $1 \le k \le n$, and let λ be the number of its negative coefficients. Then

$$b = \max_{\substack{1 \le k \le n \\ c_{n-k} < 0}} \{ |\lambda c_{n-k}|^{1/k} \}$$

is an upper bound on the values of the positive roots of $p(x) = 0$.

Notice that the lower bound is obtained by applying Cauchy's rule to the polynomial $p(1/x) = 0$.

Moreover, we used Mahler's [9] bound on Δ

$$\Delta \ge \sqrt{3} \cdot n^{-(n+2)/2} \cdot |p(x)|_1^{-(n-1)}, \tag{M}$$

(where n is the degree of $p(x)$ and $|p(x)|_1$ is the sum of the absolute values of the coefficients).

According to Chen [8], and without being aware of Vincent's theorem, Wang in 1960 independently stated a more general theorem which includes the one by Vincent as a special case.

Again a bound was needed on the number m of substitutions of the form $x{:=}a_i + 1/y$ that must be performed; this bound on m was computed, again with the help of Fibonacci numbers, by Chen [8] and is described in Theorem 2 below.

<u>Theorem 2:</u> Let $p(x) = 0$ be an integral polynomial equation of degree $n \geq 3$, and assume that it has at least 2 sign variations in the sequence of its coefficients; moreover, let $\Delta > 0$ be the smallest distance between any two of its roots. Let m' be the smallest positive index such that

$$(F_{m'-1})^2 > 1/\Delta, \qquad \text{(FIB)}$$

where F_k is the k-th member of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., and let m'' be the smallest positive integer such that

$$m'' > 1 + \lceil \log_\phi n \rceil / 2.$$

If we let

$$m = m' + m'',$$

then the arbitrary continued fraction substitution

$$x{:=}a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{\ddots + \cfrac{1}{a_m + \frac{1}{y}}}}$$

with a_1 nonnegative integer and a_2, ..., a_m positive integers, transforms $p(x) = 0$ into the equation $p_{ti}(y) = 0$, which has r sign variations in the sequence of its coefficients. If $r = 0$, then there are no roots of $p(x)$ in the interval I_m with (unordered) endpoints p_m/q_m, p_{m-1}/q_{m-1} (obtained from (CF1)). If $r > 0$, then $p(x) = 0$ has a unique positive position real root of multiplicity r in I_m.

Notice how this theorem includes the one by Vincent as a special case; however, as was mentioned before, this proposition is of theoretical interest only. It has been demonstrated, both theoretically [1] and empirically [2], that, when classical arithmetic algorithms are used, Vincent's theorem together with square-free factorization is the best approach to the problem of isolating the real roots of a polynomial equation with integer coefficients.

CONCLUSION

We have illustrated the importance of the Fibonacci sequence in computing an upper bound on the number of substitutions of the form $x{:=}a_i{+}1/x$, which are required for polynomial real root isolation using Theorem 1 (Vincent) or Theorem 2 (Wang).

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A GCD PROPERTY ON PASCAL'S PYRAMID AND THE CORRESPONDING LCM PROPERTY OF THE MODIFIED PASCAL PYRAMID

Shiro Ando and Daihachiro Sato

1. INTRODUCTION

Concerning the six binomial coefficients $A_1, A_2, \ldots, A_6$		A_2		A_3	
surrounding any entry A inside Pascal's triangle, Hoggatt and	A_1		A		A_4
Hansell [1] proved the identity		A_6		A_5	

$$A_1A_3A_5 = A_2A_4A_6, \tag{1}$$

which has been generalized to the case of multinomial coefficients by Hoggatt and Alexanderson [2]. Meanwhile, Gould [3] found the remarkable property

$$\gcd(A_1, A_3, A_5) = \gcd(A_2, A_4, A_6), \tag{2}$$

which was established by Hillman and Hoggatt [4] for the generalized binomial coefficients defined by (16) for m=2. He also showed that the equality

$$\operatorname{lcm}(A_1, A_3, A_5) = \operatorname{lcm}(A_2, A_4, A_6) \tag{3}$$

does not always hold.

Later, Ando [5] proposed a modified Pascal triangle which has $(n+1)!\,/\,h!\,k!$ (where $h+k=n$) as its entries, where the situations of GCD and LCM are interchanged.

While the problem of characterizing equal product has been settled to complete satisfaction for all multinomial coefficients, the corresponding results on equal GCD and LCM properties have been less well known. In particular, the Hoggatt-Alexanderson decomposition (see the next section) of multinomial coefficients (abbreviated to "H-A decomposition" below) does not give GCD properties unless it is on binomial coefficients.

We will present here some counterexamples for these facts and give a generalization theorem concerning GCD equalities which hold for multinomial coefficients and LCM equalities for their modified number arrays. Concerning the m-nomial coefficients defined by

$$\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1\,!\,k_2\,!\ldots k_m\,!}$$

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and the modified m-nomial coefficients

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix} = \frac{(n+m-1)!}{k_1 !\, k_2 ! \ldots k_m !},$$

where $n = k_1 + k_2 + \cdots + k_m$, we have:

<u>Theorem 1:</u> In the H-A decomposition of the m(m+1) m-nomial coefficients surrounding any entry of Pascal's pyramid into m sets of m+1 m-nomial coefficients, each set consisting of the m^2-1 coefficients from any $m-1$ sets has the same GCD.

<u>Theorem 2:</u> The role of GCD in Theorem 1 can be replaced with LCM, if we replace the m-nomial coefficients in Pascal's pyramid with the modified m-nomial coefficients.

These results can be further generalized to a wide variety of similar higher dimensional number arrays, including an array of the Fibonacci-multinomial coefficients.

2. THE H-A DECOMPOSITION

Fix an entry A inside Pascal's pyramid consisting of m-nomial coefficients. Using a m-dimensional vector whose components represent offsets from $(k_1, k_2, \ldots, k_m)$, we represent it as

$$A = \binom{n}{k_1,\ k_2,\ \ldots,\ k_m} = (0, 0, \cdots, 0)$$

where $n = k_1 + k_2 + \ldots + k_m$, and the coefficients adjacent to A as

$$\binom{n \pm 1}{k_1,\ \ldots,\ k_i \pm 1,\ \ldots,\ k_m} = (0, \cdots 0, \pm 1, 0, \cdots, 0),$$

$$\binom{n}{k_1,\ \ldots,\ k_i+1,\ \ldots,\ k_j-1,\ \ldots,\ k_m} = (0, \cdots, 1, \cdots, -1, \cdots, 0),$$

where i and j run from 1 to m and $i \neq j$.

Define an $(m+1) \times m$ matrix $C = (c_{ij})$ as follows.

For m odd, let

$$c_{ij} = \begin{cases} -1 & (i = j) \\ 1 & (i + j = m + 2) \\ 0 & (\text{otherwise}). \end{cases}$$

For m even, let

$$c_{ij} = \begin{cases} -1 & (i = j) \\ 1 & (i = m-j+1 \text{ for } j \leq \frac{m}{2}, \text{ or } i = m+1 \text{ for } j = \frac{m}{2}+1, \\ & \text{or } i = m-j+2 \text{ for } j \geq \frac{m}{2}+2) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, m+1 row vectors of C represent m+1 m-nomial coefficients adjacent to A, which we denote by $A_{11}, \cdots, A_{m+1\,1}$. We put

$$S_1 = \{A_{11}, \cdots, A_{m+1\ 1}\}.$$

A cyclic permutation of the column vectors of C, which is caused by moving the last column to the first, gives a new matrix C'. The row vectors of C' represent another set of m+1 m-nomial coefficients adjacent to A denoted by

$$S_2 = \{A_{12}, \cdots, A_{m+1\ 2}\}.$$

Continuing in a similar manner, we get m sets of different m+1 m-nomial coefficients adjacent to A:

$$S_j = \{A_{1j}, \cdots, A_{m+1\ j}\}, \text{ where } j = 1, 2, \ldots, m.$$

$S_1, S_2, \ldots, S_m$ give a decomposition of the set S of m(m+1) coefficients adjacent to A into m sets of m+1 coefficients. We call it the H-A decomposition.

For simplicity, we use the notations:

$$\prod S_j = \prod_{i=1}^{m+1} A_{ij}, \qquad \gcd S_j = \gcd(A_{1j}, \ldots, A_{m+1\ j}),$$

and lcm S_j, similarly, which are being defined for $1 \le j \le m$.

Hoggatt and Alexanderson [2] proved that

$$\prod S_1 = \prod S_2 = \cdots = \prod S_m \tag{4}$$

for this decomposition. For m = 2, we have gcd S_1 = gcd S_2 as well as $\prod S_1 = \prod S_2$, which are called hexagon properties or Star of David properties. For $m \ge 3$, however, gcd S_1, gcd S_2, ... , gcd S_m are not always equal. We will give here some counter examples.

Example 1: For m=3, put k_1=2, k_2=3, k_3=4, and n=9. Then from

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

we get

$$\begin{aligned} S_1 &= \left\{\frac{k_1}{n}A, \frac{k_2}{k_3+1}A, \frac{k_3}{k_2+1}A, \frac{n+1}{k_1+1}A\right\} \\ &= \left\{\frac{k_1}{n}, \frac{k_2}{k_3+1}, \frac{k_3}{k_2+1}, \frac{n+1}{k_1+1}\right\} A = \{280, 756, 1260, 4200\}, \\ S_2 &= \left\{\frac{k_2}{n}, \frac{k_3}{k_1+1}, \frac{k_1}{k_3+1}, \frac{n+1}{k_2+1}\right\} A = \{420, 1680, 504, 3150\}, \\ S_3 &= \left\{\frac{k_3}{n}, \frac{k_1}{k_2+1}, \frac{k_2}{k_1+1}, \frac{n+1}{k_3+1}\right\} A = \{560, 630, 1260, 2520\}, \end{aligned}$$

from which we have

gcd $S_1 = 28$, gcd $S_2 = 42$, gcd $S_3 = 70$.

Notice that gcd(42, 70) = gcd(28, 70)= gcd(28, 42) = 14 as Theorem 1 asserts.

Example 2: For m=4, put $k_1=2$, $k_2=4$, $k_3=5$, $k_4=3$, and n=14. Then, from

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

we have

$$S_1 = \left\{\frac{k_1}{n}, \frac{k_2}{k_4+1}, \frac{k_3}{k_2+1}, \frac{k_4}{k_1+1}, \frac{n+1}{k_3+1}\right\} A,$$

$$S_2 = \left\{\frac{k_2}{n}, \frac{k_3}{k_1+1}, \frac{k_4}{k_3+1}, \frac{k_1}{k_2+1}, \frac{n+1}{k_4+1}\right\} A,$$

$$S_3 = \left\{\frac{k_3}{n}, \frac{k_4}{k_2+1}, \frac{k_1}{k_4+1}, \frac{k_2}{k_3+1}, \frac{n+1}{k_1+1}\right\} A,$$

$$S_4 = \left\{\frac{k_4}{n}, \frac{k_1}{k_3+1}, \frac{k_2}{k_1+1}, \frac{k_3}{k_4+1}, \frac{n+1}{k_2+1}\right\} A,$$

which give gcd S_1=30B, gcd S_2=B, gcd S_3=2B, gcd S_4=5B, where $B = 2^2 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13 = 252252$.

Example 3: For m=5, put $k_1=2$, $k_2=4$, $k_3=3$, $k_4=6$, $k_5=5$, and n=20. Then we have $A = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, and gcd S_1=2B, gcd S_2=6B, gcd S_3=B, gcd S_4=7B, and gcd S_5=5B, where $B = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

3. PROOF OF MAIN RESULTS

First we will prove Theorem 1 concerning m-dimensional Pascal's pyramid consisting of m-nomial coefficients. Choose an entry A inside the pyramid, and let

$$S = S_1 \cup S_2 \cup \cdots \cup S_m$$

be the H-A decomposition of the set S of entries surrounding A. For the complement $S_j' = S - S_j$ (j=1, 2, $\cdots$, m) of S_j in S, we will show that

$$\gcd S_1' = \gcd S_2' = \cdots = \gcd S_m', \tag{5}$$

which establishes Theorem 1.

If we represent every entry of S_j as a linear combination of the elements of S_j' with integral coefficients for all j, then we have gcd S_j'=gcd S (j=1, 2, $\cdots$, m), which implies (5). We will assume that m is odd and j=1 for simplicity since the proof is similar in every other case.

For m odd,

$$S_j = \left\{\frac{k_j}{n}, \frac{k_{j+1}}{k_{j-1}+1}, \frac{k_{j+2}}{k_{j-2}+1}, \cdots, \frac{k_{j+m-1}}{k_{j-m+1}+1}, \frac{n+1}{k_{j-m}+1}\right\} A, \tag{6}$$

where suffixes must be understood to be taken mod m. First, using S_2, A will be expressed as

$$\begin{aligned} A &= ((n+1)-k_1-k_2-\cdots-k_m)\ A \\ &= (k_2+1)A_{m\ 2}-(k_3+1)A_{m-1\ 2}-nA_{1\ 2}-\cdots-(k_4+1)A_{m-2\ 2}. \end{aligned} \tag{7}$$

From (6), we also have

$$\frac{A}{n} = \frac{A_{11}}{k_1} = \frac{A_{12}}{k_2} = \cdots = \frac{A_{1m}}{k_m} \tag{8}$$

and for $r = 1, 2, \cdots, m$

$$\frac{A}{k_{m-r}+1} = \frac{A_{2\ 1-r}}{k_{2-r}} = \frac{A_{3\ 2-r}}{k_{4-r}} = \cdots = \frac{A_{m\ m-1-r}}{k_{2m-2-r}} = \frac{A_{m+1\ m-r}}{n+1}, \tag{9}$$

where the suffixes of k and the second suffixes of A are supposed to be considered mod m.

Since m is assumed odd, m integers $m-r$, $2-r$, $4-r$, $\cdots$, $2m-2-r$ form a complete residue system modulo m, so that the denominators of (8) and (9) satisfy the equalities:

$$n=k_1+k_2+\cdots+k_m, \qquad (k_{m-r}+1)+k_{2-r}+k_{4-r}+\cdots+k_{2m-2-r}= n+1. \tag{10}$$

From the corresponding relations of numerators of (8) and (9), we have

$$\begin{aligned} A_{11} &= A-A_{12}-A_{13}-\cdots-A_{1m}, \\ A_{21} &= A_{m+1\ m}-A_{32}-A_{43}-\cdots-A_{m\ m-1}-A, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{m1} &= A_{m+1\ 2}-A_{23}-A_{34}-\cdots-A_{m-1\ m}-A, \\ A_{m+1\ 1} &= A+A_{22}+A_{33}+\cdots+A_{mm}. \end{aligned}$$

Using (7) to substitute A into these expressions, we can represent each element of S_1 as the linear combination of the elements of S_1' with integral coefficients, so that we have gcd S_1' = gcd S as desired. Similarly we have gcd S_j' = gcd S for $j = 2, 3, \cdots, m$ to complete the proof.

For m even, we replace (6) with

$$S_j = \left\{\frac{k_j}{n}, \frac{k_{j+1}}{k_{j-1}+1}, \cdots, \frac{k_{m/2+j-1}}{k_{m/2+j+1}+1}, \frac{k_{m/2+j}}{k_{m/2+j-1}+1}, \cdots, \frac{k_{j-1}}{k_j+1}, \frac{n+1}{k_{m/2+j}+1}\right\}. \tag{11}$$

Rest of the proof is similar to the case of m odd and will be omitted.

In order to prove Theorem 2, we use the same notations for the modified Pascal pyramid as for Pascal's one. Then, for odd m, we have

$$S_j = \left\{\frac{k_j}{n+m-1}, \frac{k_{j+1}}{k_{j-1}+1}, \cdots, \frac{k_{j-1}}{k_{j+1}+1}, \frac{n+m}{k_j+1}\right\} A \tag{12}$$

instead of (6). For even m, we replace n in (11) with $n+m-1$.

This time, all we have to do is to represent the reciprocal of each element of S_j as a linear combination of the reciprocals of the elements of S_j' with integral coefficients for all j. The procedure is similar to the case of Theorem 1, and will be omitted.

4. GENERALIZATION

We can find similar properties in various number arrays consisting of generalized m-nomial coefficients defined by (16). A sequence of positive integers

$$a_1, a_2, \ldots, a_n, \ldots \tag{13}$$

is called a strong divisibility sequence if it satisfies the following condition (i) (see Kimberling [6]).

(i) For any positive integers m and n,

$$a_{(m,n)} = (a_m, a_n), \tag{14}$$

where (a, b) denotes gcd (a, b).

It is not hard to see that this condition is equivalent to the condition (ii) which is used to define the generalized binomial coefficients and their modifications in [4] and [5].

(ii) For any positive integers m and n,

$$(a_m, a_n) \mid a_{m+n}, \text{ and } (a_m, a_n) \mid a_{m-n} \text{ if } m>n. \tag{15}$$

Concerning the sequence (13) which satisfies the condition (i) or (ii), we define the generalized m-nomial coefficients by

$$\frac{a_1\, a_2 \cdots a_n}{a_1 \cdots a_{k_1}\, a_1 \cdots a_{k_2} \cdots a_1 \cdots a_{k_m}}, \tag{16}$$

and the generalized modified m-nomial coefficeints by

$$\frac{a_1\, a_2 \cdots a_{n+m-1}}{a_1 \cdots a_{k_1}\, a_1 \cdots a_{k_2} \cdots a_1 \cdots a_{k_m}}, \tag{17}$$

where $k_1 + k_2 + \cdots + k_m = n$.

If we use the same notations S_j and S_j' for these generalized m-dimentional number arrays as in Pascal's pyramid, we have:

<u>Theorem 3:</u> For the m-dimensional number arrays consisting of the generalized m-nomial coefficients (16), the same equality (5) holds as in Theorem 1. For the generalized modified m-nomial coefficients, GCD in (5) is replaced with LCM as in Theorem 2.

In this generalized case, we can not apply the previous argument. Let p be a prime number. For a rational number r, we denote the p-adic valuation of r by v(r). For $r \neq 0$, it represents the integer such that $r = p^{v(r)} a/b$, where a and b are integers not divisible by p, and for $r = 0$, $v(0) = \infty$. Then it satisfies

(i) $v(1) = 0$, (ii) $v(rs) = v(r)+v(s)$, and

(iii) $v(r \pm s) \geq \min(v(r), v(s))$, where equality holds if $v(r) \neq v(s)$.

First, we prove a lemma.

Lemma: Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers that satisfies (14) or (15). If rational numbers $A_1, A_2, \ldots, A_t$ satisfy

(i) $\frac{A_1}{a_{k_1}} = \frac{A_2}{a_{k_2}} = \frac{A_3}{a_{k_3}} = \cdots = \frac{A_t}{a_{k_t}}$, where $e_1k_1+e_2k_2+\cdots+e_tk_t = 0$ for some $e_j = \pm 1$ ($j = 1, 2, \ldots, t$), and

(ii) $v(A_2) < v(A_j)$ for $j = 3, \ldots, t$,

then we have $v(A_1) = v(A_2)$.

Proof: From (i), we have

$$v(A_1)-v(a_{k_1})=v(A_2)-v(a_{k_2})=\cdots=v(A_t)-v(a_{k_t}),$$

and so $v(a_{k_2})<v(a_{k_j})$ for $j = 3, \ldots, t$. Put $h = e_3k_3+\ldots+e_tk_t$. Then,

$$v(a_{k_2}) < \min(v(a_{k_3}), \ldots, v(a_{k_t})) \leq v(a_h), \tag{18}$$

since $\gcd(a_{k_3}, \ldots, a_{k_t})$ divides a_h. Considering this inequality, we have

$$v(a_{k_1}) \geq \min(v(a_{k_2}), v(a_h))=v(a_{k_2}),$$

as $k_2=-e_2(e_1k_1+h)$ is divisible by $\gcd(k_1, h)$. If $v(a_{k_1}) > v(a_{k_2})$ here, we have

$$v(a_{k_2}) \geq \min(v(a_{k_1}), v(a_h))=v(a_h),$$

which contradicts (18) so that we can conclude $v(a_{k_1})= v(a_{k_2})$. Hence, $v(A_1) = v(A_2)$.

Proof of Theorem 3: We consider the generalized m-nomial coefficients for odd m. As in the proof of Theorem 1, we will prove that $\gcd S_1' = \gcd S$. Fixing a prime number p, put

$$M = \min_{\substack{1\leq i\leq m+1 \\ 2\leq j\leq m}} v(A_{ij}).$$

Now we will show that $v(A_{i1})\geq M$ for $i = 1, 2, \ldots, m+1$. This time, (8) and (9) are replaced with

$$\left.\begin{aligned} &\frac{A}{a_n} = \frac{A_{11}}{a_{k_1}} = \frac{A_{12}}{a_{k_2}} = \ldots = \frac{A_{1m}}{a_{km}} \\ \text{and}\quad &\frac{A}{a_{k\,m-r+1}} = \frac{A_{2\ 1-r}}{a_{k_{2-r}}} = \frac{A_{3\ 2-r}}{a_{k_{4-r}}} = \ldots = \frac{A_{m\ m-1-r}}{a_{k_{2m-2-r}}} = \frac{A_{m+1\ m-r}}{a_{n+1}}, \end{aligned}\right\} \tag{19}$$

respectively.

If $v(A_{11})<M$, then we have $v(A_{11})<v(A_{1j})$ for j=2, 3, ..., m. Using (10), we can apply above lemma to get $v(A) = v(A_{11})$. Thus we have $v(A)\leq M$. Applying the lemma to (19), we have $v(A)=v(A_{i1})$ for i=2, 3, ..., m+1 since the second inequality of (10) assure the assumption. In particular, these results imply

$$v(A_{i1})<v(A_{i2}) \text{ for } i = 1, 2, \ldots, m+1, \tag{20}$$

which contradicts the product equality (4) for H-A decomposition given in [2]. Hence, $v(A_{11})\geq M$. In a similar manner we can verify

$$v(A_{i1})\geq M \text{ for } i = 1, 2, \ldots, m+1.$$

As this relation holds for every prime p, gcd S_1'=gcd S. Similarly, we can show that gcd S_j = gcd S, establishing the first part of the theorem for odd m.

The rest of the proof of the theorem can be completed in the same way, and so we will not repeat it here.

Remark: Since the Fibonacci sequence F_1, F_2, ..., F_n, ... satisfies (14) and (15), we can apply Theorem 3 to get GCD property for Fibonacci m-nomial coefficients, which are given by replacing all a_i's in (16) and (17) with F_i's, respectively.

For m = 3 and m = 4, [2] lists other partitions of matrices (p. 356, 420). Our results apply to those partitionings as well.

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TRANSLATABLE AND ROTATABLE CONFIGURATIONS WHICH GIVE EQUAL PRODUCT, EQUAL GCD AND EQUAL LCM PROPERTIES SIMULTANEOUSLY

Shiro Ando and Daihachiro Sato

0. SALUTI A PISA

Saluti a Pisa, la città di Leonardo, dal paese del sole nascente.

1. INTRODUCTION

Let $A = \binom{n-1}{k-1}$, $B = \binom{n}{k-1}$, $C = \binom{n+1}{k}$, $D = \binom{n+1}{k+1}$, $E = \binom{n}{k+1}$, $F = \binom{n-1}{k}$ and $X = \binom{n}{k}$. The multiplicative equality

$$ACE = BDF, \tag{1}$$

was found in [1] by V. E. Hoggatt Jr. and Walter Hansell. We therefore call this configuration "Hoggatt-Hansell's perfect square hexagon" (Figure 1). The GCD counterpart of identity (1), namely

$$\mathrm{GCD}(A,C,E) = \mathrm{GCD}(B,D,F), \tag{2}$$

was found in [2] by H. W. Gould. These two identities are the first two non-trivial examples of translatable identities of binomial coefficients, which we call the "star of David theorems", [3, 4, 5, 6]. Many generalizations of (1) and (2) have been developed. In particular, S. Hitotumatu and D. Sato proved a general Star of David Theorem using the characterization theorem for translatable GCD configurations [7, 8]. The complete characterization of equal product configurations was proved by D. Sato and E. G. Straus and applied to the characterization of perfect k-th power configuration by B. Gordon, E. G. Straus and D. Sato, [9].

The LCM counterpart of identities (1) and (2), namely

$$\mathrm{LCM}[A,C,E] = \mathrm{LCM}[B,D,F], \tag{3}$$

does not hold on Pascal's triangle, and it has been a long-standing open question whether there exists any mathematically non-trivial and/or artistically interesting configurations which give a translatable LCM identity of type (3).

We salute the city and people of Pisa and all of the members of the Fibonacci Association by saying:

La risposta a questa domanda è certamente "si",

as demonstrated by the following

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 15–26.

<u>Theorem 1:</u> (Pisa triple equality theorem)

There exists a configuration which gives simultaneously equal products, equal GCD and equal LCM properties on binomial, Fibonacci-binomial and their modified coefficients.

2. TRIPLE EQUALITY CONFIGURATIONS

Our first example, named "Julia's snowflake" (Figure 2) was constructed on July 29, 1987, in Regina, Saskatchewan, Canada, exactly one year prior to the conference lecture at Pisa, Italy. Our second example, named "Tokyo bow" (Figure 3) was constructed independently in Tokyo, Japan.

A subconfiguration of "Julia's snowflake" is shown in Figure 4 and is referred to as "Saskatchewan hexagon". A subconfiguration of "Tokyo bow" is shown in Figure 5 and is referred to as "Fujiyama". In these illustrations, B, which is called the "black set" represents the set of black points, and W which is called the "white set" represents the set of white points of the same configuration. The points on Pascal's triangle which do not belong to either of these sets are indicated by small dots in the figures. The set of small dots is denoted as D and is called the "set of dots".

We now claim the following:

<u>Theorem 2:</u> (California GL - double equality theorem)

Both "Saskatchewan hexagon" and "Fujiyama" have the simultaneous equal GCD and LCM properties, but product equality does not hold for either of these configurations.

<u>Theorem 3:</u> (Pacific Glove PGL - triple equality theorem)

"Julia's snowflake" and "Tokyo bow" have the triple equality properties stated in Theorem 1.

It is to be noted that in constructing "Julia's snowflake", and "Saskatchewan hexagon", more points have been used than mathematically required, in order to achieve a high degree of symmetry and a better artistic impression. For example, in the case of "Julia's snowflake", the central hexagon which itself has the equal product and GCD properties, may be removed without violating Theorem 3. In constructing "Tokyo bow" and "Fujiyama", on the other hand, effort was made to minimize the number of points.

3. EQUAL PRODUCT PROPERTY AND PERFECT POWERS PROPERTY

As in [9], we agree that the symbol $\binom{n}{k}$ represents both the number $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and the point which is located in row n and column k of Pascal's triangle. The usual inequality $0 \leq k \leq n$ is assumed throughout. The proofs of Theorems 2 and 3 are lengthy, if done completely. To shorten the proofs, we refer to existing theorems whenever available. The following characterization theorem concerning the equal product property and the perfect m-th power property is found in [9].

Theorem 4: (Equal product characterization theorem)

Let $S = B \cup W$ be a configuration on Pascal's triangle for which B and W are sets of binomial coefficients $\binom{n}{k}$. (B and W need not be disjoint). The product of values in B is always equal to the product of values in W if and only if the number of black points equals the number of white points on each of the lines, $n = \text{constant}$, $k = \text{constant}$ and $n - k = \text{constant}$.

Theorem 5: (Perfect power characterization theorem)

The product of all values in configuration S of binomial coefficients $\binom{n}{k}$ is a perfect m-th power if and only if the number of points of S, counting their multiplicities on each line $n = \text{constant}$, $k = \text{constant}$, and $n - k = \text{constant}$, is a multiple of m.

An immediate consequence of these characterization theorems is the following

Corollary 1:

The number of black points is equal to the number of white points for any configuration for which the equal product property holds.

Corollary 2:

The number of points in any perfect m-th power configuration is always divisible by m.

By the characterization Theorem 5, we conclude that "Julia's snowflake" and "Tokyo bow" have the equal product property, but "Saskatchewan hexagon" and "Fujiyama" do not have the equal product property for the black and white sets in those configurations. A consequence of Theorem 5 and Corollary 2 is the following:

Corollary 3:

"Julia's snowflake" and "Tokyo bow" are perfect square configurations, but "Saskatchewan hexagon" and "Fujiyama" are not perfect square configurations.

4. EQUAL GREATEST COMMON DIVISOR PROPERTY

Since we are only interested in constructing configurations having the equal GCD property, the characterization theorem is not required here. The proof of the equal GCD property for all configurations given in Figures 1, 2, 3, 4, and 5 requires only repeated use of the following fundamental identities, (Figure 1).

Lemma 1: (GCD covering formulas)

$$\mathrm{GCD}(X,A,F) = \mathrm{GCD}(A,F) \tag{4}$$

$$\mathrm{GCD}(X,B,C) = \mathrm{GCD}(B,C) \tag{5}$$

$$\mathrm{GCD}(X,D,E) = \mathrm{GCD}(D,E) \tag{6}$$

$$\mathrm{GCD}(X,A,C,E) = \mathrm{GCD}(A,C,E) \tag{7}$$

$$\mathrm{GCD}(X,B,D,F) = \mathrm{GCD}(B,D,F). \tag{8}$$

<u>Proof:</u> To prove (4), (5) and (6) we only need to note the following well-known properties of binomial coefficients:

$$X = A + F = C - B = D - E. \tag{9}$$

For (7) and (8), it is sufficient to verify the following two identities:

$$X = (n - k + 1)C - nA - (k + 1)E \tag{10}$$

and

$$X = (k + 1)D - nF - (n - k + 1)B. \tag{11}$$

These combinatorial identities show that adjunction of the center point $X = \binom{n}{k}$ does not decrease the greatest common divisor of the original set. This is fundamental to our discussion.

The repetitive process used to establish the equal greatest common divisiors of two sets was demonstrated at a lecture in Pisa, using red and green transparent bingo chips (Japanese and Italian colors!) on a Chinese checker board representing Pascal's triangle.

The method involving repeated use of the covering formulas is very effective for proving the equal GCD property for many configurations. We have called it the "Pennant closure process" in [7] and have found that most (but not all) of the equal product configurations listed in [26] and [27] also possess the equal GCD property. We give here one more such example which is a simple consequence of Lemma 1.

<u>Theorem 6:</u> (Complementary equal GCD theorem)

Let D be the "set of dots" in Figure 2 or Figure 3, then for either configuration,

$$\mathrm{GCD}(D) = \mathrm{GCD}(B) = \mathrm{GCD}(W).$$

5. EQUAL LEAST COMMON MULTIPLE PROPERTY

The authors of the present paper have prepared a more detailed report on equal LCM properties of binomial and modified binomial coefficients [14]. Figures 2, 3, 4, and 5 are direct consequences of that investigation. The construction of an equal LCM configuration requires more effort than its counterpart for the GCD, but proof of the LCM equalities for these configurations, once they are constructed, requires only a finite number of applications of one of the five fundamental LCM identities called "LCM covering Formulas". In order to simplify the format of these identities, we expand the Hoggatt-Hansell perfect square hexagon (Figure 1) to a larger pattern which we call "north star". Notations for binomial coefficients, other than those already listed in the introduction are $H = \binom{n-2}{k-2}$, $I = \binom{n}{k-2}$, $J = \binom{n+2}{k}$, $L = \binom{n+2}{k+2}$, $R = \binom{n}{k+2}$, and $T = \binom{n-2}{k}$, (Figure 6A and Figure 6B).

<u>Lemma 2:</u> (LCM covering formulas)

$$\mathrm{LCM}[X,A,B,H,I] = \mathrm{LCM}[A,B,H,I] \tag{12}$$

$$\mathrm{LCM}[X,E,F,R,T] = \mathrm{LCM}[E,F,R,T] \tag{13}$$

$$\mathrm{LCM}[X,C,D,J,L] = \mathrm{LCM}[C,D,J,L] \tag{14}$$

$$\mathrm{LCM}[X,A,C,E,H,J,R] = \mathrm{LCM}[A,C,E,H,J,R] \tag{15}$$

$$\mathrm{LCM}[X,B,D,F,I,L,T] = \mathrm{LCM}[B,D,F,I,L,T]. \tag{16}$$

As in Lemma 1, these combinatorial identities show that adjunction of $X = \binom{n}{k}$ to the sets on the right hand side does not increase their least common multiple.

We will prove only (12) and (15), since the proof of (13) and (14) is similar to that of (12), and the proof of (16) is analogous to that of (15). Given an integer y and a prime p, we define the additive p-adic valuation of y denoted by $\alpha = v_p(y)$, to be the integer α such that $p^\alpha | y$ and $p^{\alpha+1} \nmid y$.

<u>Proof of (12):</u> We note that

$$A = \tfrac{k}{n}X,$$

$$B = \frac{k}{n-k+1}X,$$

$$H = \frac{k(k-1)}{n(n-1)}X,$$

and

$$I = \frac{k(k-1)}{(n-k+1)(n-k+2)}X.$$

If $L_1 = \mathrm{LCM}[X,A,B,H,I]$ and $L_2 = \mathrm{LCM}[A,B,H,I]$, then clearly $L_1 \geq L_2$. If $L_1 > L_2$ then there exists at least one prime p such that $v_p(L_1) > v_p(L_2)$. Let p be one such prime. Then

$$\max\{v_p(A), v_p(B), v_p(H), v_p(I)\} < v_p(X).$$

This p cannot divide $n - 1$, because if $p|(n-1)$, then $p \nmid n$ and hence

$$v_p(A) = v_p\left(\tfrac{k}{n}X\right) \geq v_p(X).$$

The prime p cannot divide $n - k + 2$ either, because if $p|(n-k+2)$, then $p \nmid (n-k+1)$ and

$$v_p(B) = v_p\left(\frac{k}{n-k+1}X\right) \geq v_p(X).$$

The only possibility is therefore $p|n$ and $p|(n-k+1)$. If $p^\alpha|n$ and $p^\alpha|(n-k+1)$, then $p^\alpha|(k-1)$, which means that

$$v_p(H) = v_p\left(\frac{k(k-1)}{n(n-1)}X\right)$$

and

$$v_p(I) = v_p\left(\frac{k(k-1)}{(n-k+1)(n-k+2)}X\right)$$

are $\geq v_p(X)$. This contradiction establishes the equality $L_1 = L_2$.

<u>Proof of (15):</u> Again we note that

$$A = \tfrac{k}{n}X,$$

$$C = \frac{n+1}{n-k+1}X,$$

$$E = \frac{n-k}{k+1}X,$$

$$H = \frac{k(k-1)}{n(n-1)}X,$$

$$J = \frac{(n+1)(n+2)}{(n-k+1)(n-k+2)}X,$$

and

$$R = \frac{(n-k)(n-k+1)}{(k+1)(k+2)}X.$$

If $L_1 = \mathrm{LCM}[X,A,C,E,H,J,R]$ and $L_2 = \mathrm{LCM}[A,C,E,H,J,R]$, then clearly $L_1 \geq L_2$. If $L_1 > L_2$, then there exists a prime p such that $v_p(L_1) > v_p(L_2)$.

Then

$$\max\{v_p(A), v_p(C), v_p(E), v_p(H), v_p(J), v_p(R)\} < v_p(X).$$

Now p cannot divide $n-1$, $k+2$, or $n-k+2$, because if $p|(n-1)$ then $p \not| n$, and

$$v_p(A) = v_p\left(\tfrac{k}{n}X\right) \geq v_p(X).$$

If $p|(k+2)$, then $p \not|(k+1)$ and

$$v_p(E) = v_p\left(\frac{n-k}{k+1}X\right) \geq v_p(X).$$

If $p|(n-k+2)$, then $p \not|(n-k+1)$ and

$$v_p(C) = v_p\left(\frac{n+1}{n-k+1}X\right) \geq v_p(X).$$

Therefore the only remaining possibility is that there exists $\alpha \geq 1$ such that p^α divides n, $k+1$ and $n-k+1$. But then the identity $2 = (k+1) + (n-k+1) - n$ implies that $p^\alpha = 2$. We now have that all of the numbers $k-1$, $n-k+1$ and $n+2$ are even. Therefore there exists at least one element in the set L_2 for which the p-adic exponent of 2 is $\geq v_2(X)$. This contradiction implies that $L_1 = L_2$. Since Pascal's triangle is symmetric with respect to its vertical center line, or more generally, since it is p-adically 120 degree rotatale, [25], equalities (13), (14) and (16) are also established, [28-36].

Having proved all the LCM covering formulas, it only remains to perform repeated application of these identities in order to establish the equal LCM properties of configurations in Figures 2, 3, 4, and 5.

6. FIBONACCI-BINOMIAL COEFFICIENTS AND MODIFIED BINOMIAL COEFFICIENTS

A Fibonacci-binomial coefficients or Fibonomial coefficient is a rational number defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_1F_2F_3\ldots F_n}{F_1F_2F_3\ldots F_kF_1F_2F_3\ldots F_{n-k}}$$

where F_i is the i-th Fibonacci number, i.e.;

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_n + F_{n+1} \qquad (n = 1, 2, 3, \ldots).$$

All Fibonomial coefficients are positive integers, and the triangular array of these numbers has a p-adic geometric structure similar to Pascal's triangle, [19-22]. A. P. Hillman and V. E. Hoggatt Jr. investigated these similarities and have shown that the original Star of David theorem, analogous to equalities (1) and (2), also holds on this Fibonacci version of the Pascal-like triangle [13]. Shiro Ando on the other hand, defined a modified binomial coefficient as

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{(n+1)!}{k!(n-k)!} = (n+1) \binom{n}{k}.$$

These modified binomial coefficients appear in the denominators of the numbers in Leibniz's harmonic triangle [15] and their p-adic geometric structure is algebraically dual to that of binomial coefficients. Ando proved that the translatable product and LCM equalities, similar to (1) and (3), (but not the GCD equality (2)), hold for the array of modified binomial coefficients [10]. D. Sato also gave an alternate non p-adic proof for Ando's equality in [11].

These two Pascal like number arrays can be combined further to define the modified Fibonomial coefficient, given by

$$< \begin{matrix} n \\ k \end{matrix} > = \frac{F_1F_2 \ldots F_{n+1}}{F_1F_2\ldots F_kF_1F_2\ldots F_{n-k}} = F_{n+1} \left[\begin{matrix} n \\ k \end{matrix} \right].$$

Considering the p-adic similarity between Pascal's triangle and Fibonacci Pascal's triangle, their algebraic duality to modified Pascal's triangle and modified Fibonacci Pascal's triangle, we are able to demonstrate the following.

Theorem 7: (Sakasa-Fuji quadruple equality theorem)

The configuration of Fujiyama has equal GCD and equal LCM properties on Fibonacci-Pascal's triangle, while its upside down configuration (called SAKASA-FUJI, in Japanese) has equal GCD and equal LCM properties on modified Pascal's and modified Fibonacci Pascal's triangle.

Theorem 8: (Tokyo bow sextuple equality theorem)

The configuration of Tokyo bow gives triple equality, as in Theorem 3, on Fibonacci-Pascal's triangle. The upside down configuration of Tokyo bow also gives triple equality on modified Pascal's triangle and modified Fibonacci Pascal's triangle.

Finally we have the most simultaneous equalities in:

Theorem 9: (Universal equality theorem)

The Julia's snowflake and its upside down configuration both give translatable simultaneously equal product, equal GCD and equal LCM properties on Pascal's triangle, Fibonacci Pascal's triangle, modified Pascal's triangle and modified Fibonacci Pascal's triangle. The Saskatchewan hexagon and its upside down configuration have equal GCD and equal LCM properties on all of these triangular arrays of numbers.

Thus, Julia's snowflake alone gives twelve translatable simultaneous equalities over four arrays of binomial-like coefficients. The proof of Theorems 7, 8 and 9 together with higher dimensional extensions of some of our results will be reported separately in a more general setting for similar arrays of numbers which are defined by the strong divisibility sequences, one of which is of course our well known Fibonacci number sequence [19-22].

7. EXPLANATION OF NAMING

The authors of the present paper met for the first time at the West Coast Number Theory Conference in California, U.S.A. in December, 1985. The conference was organized to commemorate the late Dr. Julia Robinson (1919-1985) of the University of California at Berkley. Dr. Robinson was a past-president of the American Mathematical Society, long-time member of the Mathematical Association of America, and a regular member of the West Coast Number Theory Conference [37, 38]. The original question and some results on equal LCM properties of binomial coefficients arose from dining room conversations at the conference. On the way back from the U.S.A., the second author was greeted by a beautiful snowfall in Canada. He saw impressive hexagonal snowflakes, sparkling and shining against a dark northern sky. The memories of Dr. Julia Robinson and the large hexagonal snowflakes were still fresh when six equal GCD-LCM hexagons were arranged to obtain the triple equality property. The configuration in Figure 2 is thus named after Dr. Julia Robinson for the friendship and support given us during many years of mathematical association. Theorem 2 and Theorem 3 are named after the place (Pacific Glove, California) where the conference was held and which incidentally includes the first letters of product, GCD and LCM.

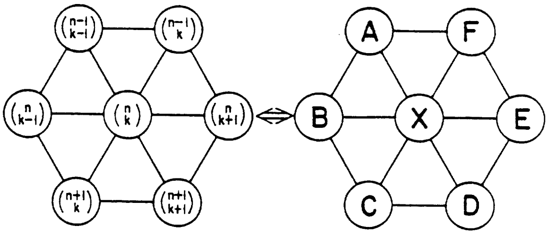
For those readers who are not familiar to the geography of Canada and Japan, we wish to mention that "Saskatchewan" is the name of a province in western Canada where, according to the best of our knowledge, the first non-trivial mutually exclusive equal GCD-LCM configuration Figure 4 was constructed. The names "Pisa" and "Tokyo" need no explanation. "Fujiyama" is a highly symmetric triangular mountain near Tokyo, after which configurations Figure 3 and Figure 5 are named.

8. ANNOUNCEMENT

While hexagons are geometrically well known and "Fujiyama" is geographically well known, they don't have much historical significance. Moreover, they are not equal product configurations. The "Star of David" is historically well known, but it is not an equal LCM configuration. "Julia's snowflake" and "Tokyo bow" may be artistically appealing, but it is difficult to relate them to any historically well known configurations. What was, then, in our minds when we prepared the title and abstract of our conference lectures? An explanation will be provided in the second part of this article, which will appear separately.

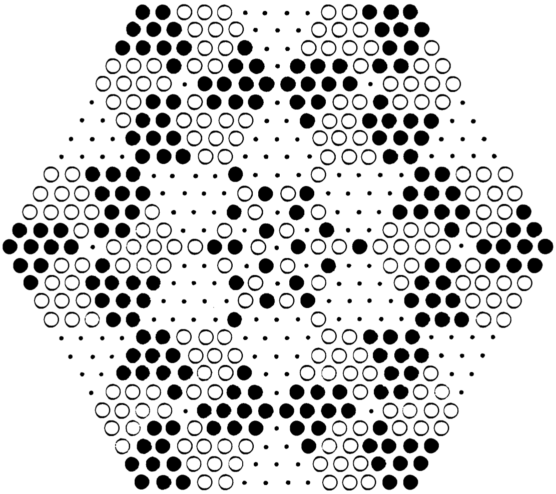
This paper was presented as the very last talk at the Third International Conference on Fibonacci Numbers and Their Applications, partially in order to entertain those participants who decided to stay until the Sayonara meeting, hoping that all of us will have good, productive years, until we see each other again in the near future.

9. CONFIGURATIONS AND THEIR NAMES



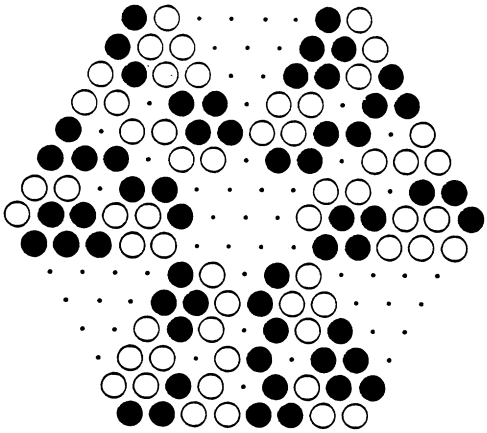
$$\boxed{(\prod, \mathrm{GCD})\{A, C, E\} = (\prod, \mathrm{GCD})\{B, D, F\}}$$

Figure 1. Hoggatt-Hansell Perfect Square Hexagon



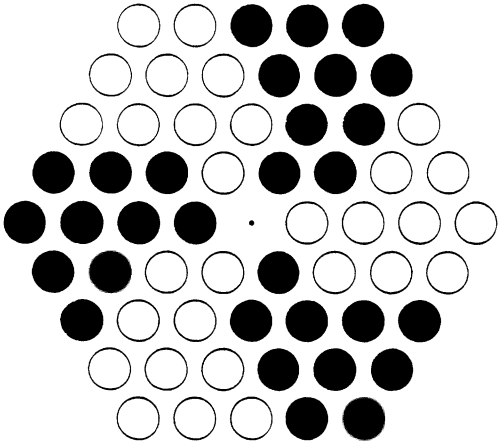
$$(\prod, \text{GCD}, \text{LCM})\ \{\bullet\} = (\prod, \text{GCD}, \text{LCM})\ \{\circ\}$$
$$\text{GCD}\ \{\cdot\} = \text{GCD}\ \{\bullet\} = \text{GCD}\ \{\circ\}$$
$$(\prod, \text{GCD}, \text{LCM})\ (\{\bullet\} \cup \{\cdot\}) = (\prod, \text{GCD}, \text{LCM})\ (\{\circ\} \cup \{\cdot\})$$

Figure 2. Julia's Snowflake



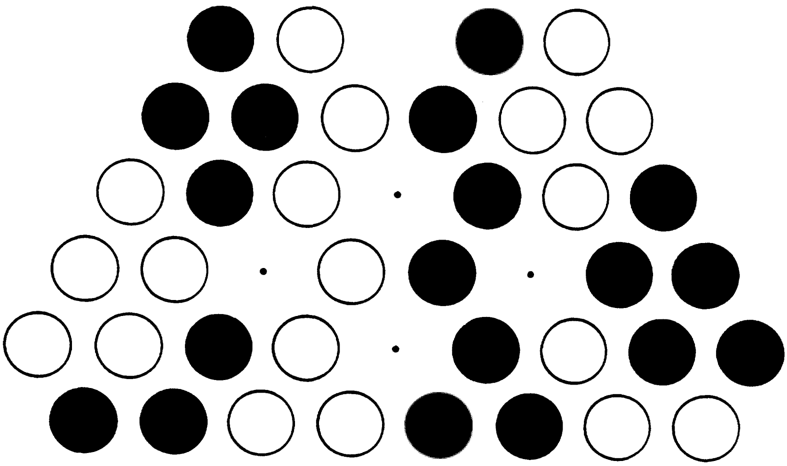
$$(\prod, \text{GCD}, \text{LCM})\ \{\bullet\} = (\prod, \text{GCD}, \text{LCM})\ \{\circ\}$$
$$\text{GCD}\ \{\cdot\} = \text{GCD}\ \{\bullet\} = \text{GCD}\ \{\circ\}$$
$$(\prod, \text{GCD}, \text{LCM})\ (\{\bullet\} \cup \{\cdot\}) = (\prod, \text{GCD}, \text{LCM})\ (\{\circ\} \cup \{\cdot\})$$

Figure 3. Tokyo Bow



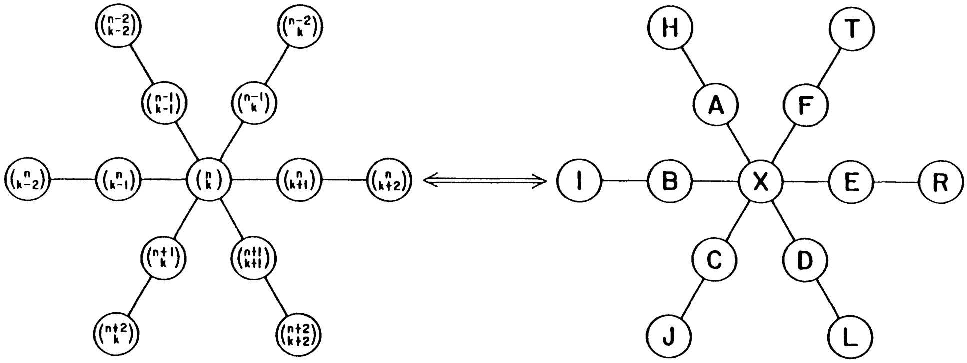
$$\boxed{(\text{GCD, LCM})\ \{\bullet\} = (\text{GCD, LCM})\ \{\circ\}}$$

Figure 4. Saskatchewan Hexagon



$$\boxed{(\text{GCD, LCM})\{\bullet\} = (\text{GCD, LCM})\ \{\circ\}}$$

Figure 5. Fujiyama



$$\boxed{\begin{array}{c}(\prod,\ \mathrm{GCD})\{A,C,E,H,J,R\}=(\prod,\ \mathrm{GCD})\{B,D,F,I,L,T\}\\ =(\prod,\ \mathrm{GCD})\{A,C,E,I,L,T\}=(\prod,\ \mathrm{GCD})\{B,D,F,H,J,R\}\end{array}}$$

Figure 6A. North Star Figure 6B. North Star

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FIBONACCI LENGTH OF GENERATING PAIRS IN GROUPS

C. M. Campbell, H. Doostie and E. F. Robertson

1. FIBONACCI LENGTH

Let G be a group and let x, y $\in$ G. If every element of G can be written as a word

$$x^{\alpha_1} y^{\alpha_2} x^{\alpha_3} \dots x^{\alpha_{n-1}} y^{\alpha_n} \tag{1}$$

where $\alpha_i \in \mathbb{Z}$, $1 \leq i \leq n$, then we say that x and y *generate* G and that G is a *2-generator group.* Although cyclic groups are 2-generator groups according to this definition we are only interested here in 2-generator groups which cannot be generated by a single element. Even among finite groups G many are not 2-generator groups; for example the abelian group of order 8 in which every element has order 2 cannot be 2-generated since given any pair of distinct non-trivial elements x, y there are only 4 words given by expressions of the form (1). However, many groups are 2-generated, in particular finite simple groups.

Let G be a finite 2-generator group and let X be the subset of G $\times$ G such that (x,y) $\in$ X if, and only if, G is generated by x and y. We call (x,y) a *generating pair* for G. We define the *Fibonacci orbit* $F_{x,y} = \{a_i\}$ of (x,y) $\in$ X by

$$a_1 = x,\ a_2 = y,\ a_{i+2} = a_i a_{i+1}, \quad i \geq 1.$$

This is a similar concept to that studied by Wilcox in [8] where only abelian groups are considered.

<u>Lemma 1:</u> For any Fibonacci orbit $F_{x,y} = \{a_i\}$ of a finite group G, there exists an integer $n \geq 2$ with

$$x = a_1 = a_{n+1} \text{ and } y = a_2 = a_{n+2}.$$

<u>Proof:</u> Since G $\times$ G is finite there exists m_1, m_2 with $m_1 \neq m_2$ and

$$a_{m_1} = a_{m_2},\ a_{m_1+1} = a_{m_2+1}.$$

Let m_1 and m_2 be minimal with respect to this property. Now $m_1 = 1$, for, if not,

$$a_{m_1-1} = a_{m_1+1}\, a_{m_1}^{-1} = a_{m_2+1}\, a_{m_2}^{-1} = a_{m_2-1}.$$

and m_1-1, m_2-1 is a smaller pair of integers with the same property.

Let n be the least positive integer with $n \geq 1$ and $x = a_1 = a_{n+1}$, $y = a_2 = a_{n+2}$. Then we consider the Fibonacci orbit $F_{x,y}$ to be finite containing n elements and suppose the suffices on the a's to be reduced mod n. We say that (x,y) $\in$ X has *Fibonacci length* n.

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 27–35.

Lemma 2: If the Fibonacci orbit $F_{x,y}$ of $(x,y) \in X$ has length n then for any i, $1 \leq i \leq n$, we have $(a_i, a_{i+1}) \in X$. Also we have $F_{x,y} = F_{a_i,a_{i+1}}$.

Proof: That $(a_i, a_{i+1}) \in X$ can be deduced by induction. The case $i = 1$ is trivially true. Suppose by way of inductive hypothesis that $(a_k, a_{k+1}) \in X$ and consider (a_{k+1}, a_{k+2}). Now $a_k = a_{k+2}a_{k+1}^{-1}$ so, since every element of G has an expression of the form (1) with $x = a_k$, $y = a_{k+1}$, we see that, on replacing a_k by $a_{k+2}\, a_{k+1}^{-1}$ every element of G is generated by a_{k+1} and a_{k+2}.

Finally suppose $F_{x,y} = \{a_i\}$ and $F_{r,s} = \{b_i\}$. Then again an inductive argument proves that if $a_1 = b_{j+1}$, $a_2 = b_{j+2}$ then $F_{x,y} = F_{r,s}$.

For suppose that $a_i = b_{i+j}$, $i < t$. Then

$$a_t = a_{t-2}\, a_{t-1} = b_{j+t-2}\, b_{j+t-1} = b_{j+t}$$

and the result is proved.

Lemma 2 gives immediately

Theorem 3: If G is a finite group, X is partitioned by the Fibonacci orbits $F_{x,y}$ for $(x,y) \in X$.

2. BASIC FIBONACCI LENGTH

To examine the concept more fully we study the action of the automorphism group Aut G of G on X and on the Fibonacci orbits $F_{x,y}$, $(x,y) \in X$. Now Aut G consists of all isomorphisms $\theta: G \rightarrow G$ and if $\theta \in$ Aut G and $(x,y) \in X$ then $(x\theta, y\theta) \in X$.

For a subset $A \subseteq G$ and $\theta \in$ Aut G the image of A under θ is

$$A\theta = \{a\theta : a \in A\}.$$

Lemma 4: Let $(x,y) \in X$ and let $\theta \in$ Aut G. Then $F_{x,y}\theta = F_{x\theta,y\theta}$.

Proof: Let $F_{x,y} = \{a_i\}$. Now $\{a_i\}\theta = \{a_i\theta\}$ and since

$$a_{i+2}\theta = (a_i\, a_{i+1})\theta = a_i\theta\, a_{i+1}\theta$$

the result follows.

Each generating pair $(x,y) \in X$ maps to |Aut G| distinct elements of X under the action of elements of Aut G. Hence there are

$$d_2(G) = |X|/|\text{Aut } G| \tag{2}$$

non-isomorphic generating pairs for G. The notation $d_2(G)$ follows Hall [4] where the number of generating pairs for a group is discussed.

Suppose k of the elements of Aut G map $F_{x,y}$ into itself. Then there are |Aut G|/k distinct Fibonacci orbits $F_{x\theta,y\theta}$ for $\theta \in$ Aut G. For a generating pair $(x,y) \in X$ we define the *basic Fibonacci orbit* $\overline{F}_{x,y}$ of *basic length* m to be the sequence $\{a_i\}$ of elements of G such that

$$a_1 = x,\ a_2 = y,\ a_{i+2} = a_i\, a_{i+1},\quad i \geq 1$$

where $m \geq 1$ is the least integer with

$$a_1 = a_{m+1}\theta,\ a_2 = a_{m+2}\theta,$$

for some $\theta \in \mathrm{Aut}\ G$. Since a_{m+1}, a_{m+2} generate G, it follows that θ is uniquely determined. Again we consider the basic Fibonacci orbit $\overline{F}_{x,y}$ to be finite containing m elements.

<u>Theorem 5:</u> Let G be a finite group and let $(x,y) \in X$. Suppose the Fibonacci orbit $F_{x,y}$ has length n and the basic Fibonacci orbit has length m. Then m divides n and there are n/m elements of Aut G which map $F_{x,y}$ into itself. Moreover

$$d_2(G) = \sum m_i \tag{3}$$

where the m_i are the basic Fibonacci lengths of the non-isomorphic orbits $F_{x,y}$.

<u>Proof:</u> Since $F_{x,y} = \overline{F}_{x,y} \cup \overline{F}_{x\theta,y\theta} \cup \overline{F}_{x\theta^2,y\theta^2} \cup \ldots$ and $|\overline{F}_{x,y}| = |\overline{F}_{x\theta,y\theta}|$ we have n = mk where

k is the order of the automorphism $\theta \in \mathrm{Aut}\ G$. Clearly $1, \theta, \theta^2, \ldots, \theta^{k-1}$ map $F_{x,y}$ into itself. Now $d_2(G) = \sum m_i$ follows using (2) and Theorem 3.

<u>3. APPLICATIONS</u>

For the rest of this article we denote the Fibonacci length of a generating pair (a,b) by LEN and we denote by BLEN the length of the basic Fibonacci orbit.

<u>3.1 DIHEDRAL GROUPS:</u>

Consider the dihedral groups D_{2n} where

$$D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle.$$

Generating pairs (x,y) of D_{2n} are of three types:

(i) $x^2 = y^2 = (xy)^n = 1$,

(ii) $x^2 = y^n = (xy)^2 = 1$,

(iii) $x^n = y^2 = (xy)^2 = 1$.

For (x,y) and (r,s) of the same type there is an automorphism $\theta \in \mathrm{Aut}\ D_{2n}$ with $x\theta = r$, $y\theta = s$. There is, up to isomorphism, only one Fibonacci orbit. If (x,y) is of type (i) then LEN = 6 and the orbit is

$$F_{x,y} = \{x,\ y,\ xy,\ yxy,\ y,\ yx\}.$$

Notice that (y, xy) is a generating pair of type (ii) and (xy, yxy) is a generating pair of type (iii). The basic Fibonacci length BLEN = 3 since $x\theta = yxy$, $y\theta = y$ where θ is the inner automorphism induced by conjugation by y. Equation (3) reduces to $d_2(D_{2n}) = 3$.

3.2 QUATERNION GROUPS:

Consider the quaternion group

$$Q_8 = < a, b \mid a^4 = 1, a^2 = b^2, b^{-1} ab = a^{-1} > .$$

There is, up to isomorphism, only one generating pair for Q_8. The Fibonacci orbit $F_{a,b}$ has LEN = 3:

$$F_{a,b} = \{a, b, ab\}$$

since bab = a and aba = b. The basic Fibonacci length BLEN = 1 since there is an outer automorphism θ of order 3 with $a\theta = b$, $b\theta = ab$. Equation (3) reduces to $d_2(Q_8) = 1$.

The generalised quaternion group Q_{2^n}, n > 3, behaves in a slightly different way from Q_8 and essentially the same as D_{2n}.

$$Q_{2^n} = < a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} > .$$

There is, up to isomorphism, only one Fibonacci orbit with LEN = 6 and BLEN = 3,

$$F_{a,b} = \{a, b, ab, a^{2^{n-2}-1}, a^{2^{n-2}+2}b, ab\}.$$

3.3 SOME COMPLETE DECOMPOSITIONS:

The two cases of dihedral groups and quaternion groups examined in 3.1 and 3.2 are not, in general, typical. The examples that we examine in detail in this subsection are chosen as particular examples by Hall [4] for computing $d_2(G)$. For each group we give the Fibonacci length LEN of each non-isomorphic Fibonacci orbit of generating pairs and its basic Fibonacci length BLEN.

$$A_5 = PSL(2,5)$$

LEN	12	14	50
BLEN	2	7	10

$$d_2(A_5) = 19 = \sum \text{BLEN}$$

A_6

LEN	10	20	10	14	80	22	128
BLEN	2	4	5	7	8	11	16

$d_2(A_6) = 53 = \sum$ BLEN

PSL(2,7)

LEN	8	14	24	10	28	42	128
BLEN	2	2	4	5	14	14	16

$d_2(\mathrm{PSL}(2,7)) = 57 = \sum$ BLEN

PSL(2,11)

LEN	20	20	24	44	48	10	18	60	14	40	80	22	12	26
BLEN	2	2	2	4	4	5	6	6	7	8	8	11	12	13
LEN	168	80	90	198	128	160	408							
BLEN	14	16	18	18	32	32	34							

$d_2(\mathrm{PSL}(2,11)) = 254 = \sum$ BLEN

PSL(2,13)

LEN	8	12	14	26	28	24	28	48	56	10	84	84
BLEN	2	2	2	2	2	4	4	4	4	5	6	6
LEN	16	56	56	18	30	22	168	196	196	64	192	
BLEN	8	8	8	9	10	11	14	14	14	16	16	
LEN	252	280	88	46	48	312	338	392	62	128	108	480
BLEN	18	20	22	23	24	24	26	28	31	32	36	40

$d_2(\mathrm{PSL}(2,13)) = 495 = \sum$ BLEN

3.4 SIMPLE GROUPS G, $|G| < 10^6$:

Minimal generating pairs of the simple groups G, $|G| < 10^6$, are given in [5] and presentations satisfied by these pairs are given in [1] and [3]. These three papers contain the definition of minimal generating pair and give a consistent numbering system for these pairs. In the tables below we list the generating pairs for a particular group in the order given by that numbering system. We list both LEN and BLEN separated by a comma for each pair and pairs are separated by a semi-colon. In the case where (a,b) and (a,b^{-1}) are both minimal generating pairs, then they must generate isomorphic Fibonacci orbits and we replace the pair LEN, BLEN for the pair (a,b^{-1}) by an asterisk since the entry is identical to the immediately preceding pair.

A_5	50, 10.
PSL(2,7)	28, 14.
A_6	128, 16.
A_7	150, 30; 288, 24.
PSL(3,3)	36, 36; 210, 70.
PSU(3,3)	114, 38; 192, 24.
M_{11}	1824, 152; *.
A_8	66, 33; 234, 78.
PSL(3,4)	250, 50.
PSU(4,2)	1120, 112; 768, 96; 456, 76; *; 64, 32; *; 450, 90; 648, 108; 324, 54.
Sz(8)	76, 38; *; 1404, 108; *; 1148, 82; *; 280, 20; *; 46, 23; *; 3328, 128; *; 404, 202; *; 270, 54; *.
PSU(3,4)	580, 58; 270, 54.
M_{12}	664, 166; 138, 46; 52, 26.
PSU(3,5)	416, 104.
J_1	2460, 246; *; 2420, 110; *; 5880, 196; *; 646, 34; 264, 22; *; 4800, 240; *; 440, 44; *; 2430, 162; 380, 20; 2520, 126; *; 1672, 88; 560, 112; 2248, 562; *; 12152, 868; *; 3268, 86; *; 2014, 106; 490, 98; 966, 138; 9040, 452; *; 3806, 346; 1050, 70.

A_9	2368, 296; *; 432, 72; 588, 84.
PSL(3,5)	240, 80; 32, 32; 2688, 336; 770, 154; 1368, 114; 2060, 206; 180, 18.
M_{22}	540, 54; 1392, 116; 1264, 316; *; 860, 172; 1656, 138.
J_2	240, 120; 304, 76; 592, 74; 60, 60; 660, 110; 1820, 182; *; 1392, 58; 246, 82; *; 924, 66; *; 496, 248; *; 670, 134; *; 308, 44; 9200, 460; *; 840, 56; 3840, 160; 520, 52; *.
PSp(4,4)	760, 76; 376, 47; 1580, 158; 352, 44; *; 2680, 268; *; 2448, 144; 23188, 682; *; 1652, 826; *; 2364, 394; *; 9030, 1806; *; 648, 216; 198, 198; 4692, 276; 1560, 156; *; 6030, 402; 4050, 270; 6800, 1360; *; 3774, 222; 390, 39; 6030, 402; 4650, 930; *; 21216, 1248; *.

There are some comments on these results which we will make in Section 4.

3.5 A PRESENTATION FOR PSL(2,p):

For p a prime we consider the generating pair (a,b) of PSL(2,p) which satisfies the presentation

$$< a,\, b \mid a^2 = b^p = (ab)^3 = (ab^4ab^{(p+1)/2})^2 = 1 >. \quad (^*)$$

As matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ maps to a and $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ maps to b under the homomorphism from SL(2,p) to PSL(2,p). We have obtained LEN and BLEN for $5 \leq p \leq 400$. Note that LEN/BLEN is the order of an element in Aut(PSL(2,p)) and so divides p or $p-1$ or $p+1$. From the numerical evidence of our results we make the following conjecture.

Conjecture: For the generating pair (a,b) of PSL(2,p) satisfying (*):

(i) if $p \equiv 0 \pmod 5$, then LEN/BLEN divides p;

(ii) if $p \equiv \pm 1 \pmod 5$, then LEN/BLEN divides $p-1$;

(iii) if $p \equiv \pm 2 \pmod 5$, then LEN/BLEN divides $p+1$.

Note that $p = 5$ is the only prime satisfying (i) and since LEN $= 50$, BLEN $= 10$ in this case, part (i) of the conjecture is proved. We also note that the conjecture can be restated as follows.

Conjecture: For the generating pair (a,b) of PSL(2,p) satisfying (*), LEN/BLEN divides $p-(5/p)$ where $(5/p)$ denotes the Legendre symbol.

This form of the conjecture exhibits a similarity with the result [7] that the restricted period of the Fibonacci sequence $\{f_n\}$ modulo p divides $p-(5/p)$. We would like to thank the referee for pointing out this second form of the conjecture and the connection with the restricted period of the Fibonacci sequence.

4. METHODS AND CONCLUSION

The results of 3.3, 3.4 and 3.5 were obtained using a specially written computer program and the CAYLEY group theory system. Groups were represented as presentations, permutation groups or matrix groups as convenient. Data concerning the minimal generating pairs for finite simple groups was taken from the CAYLEY library of finite simple groups, see [2] and [6].

The Fibonacci lengths of generating pairs were computed by generating successive terms of the Fibonacci orbit. We note that all but one of the results of Section 3 give LEN to be even. The exception is the quaternion group Q_8, see 3.2, which has a single generating pair with LEN = 3. In fact it is easy to show:

Theorem 6: (i) If G is a 2-generator group with a generating pair (a,b) with LEN = 3, then G is either the quaternion group or the Klein 4-group $C_2 \times C_2$.

(ii) There is no 2-generator group with a generating pair (a,b) with LEN = 5 or LEN = 7.

Proof: (i) If (a,b) is a generating pair with LEN = 3 then the Fibonacci orbit is

$$a,\ b,\ ab,\ bab = a,\ ab^2ab = b.$$

Hence G is a homomorphic image of

$$< ab \mid aba = b,\ bab = a >$$

which is the quaternion group.

(ii) Apply the same method as in (i). If (a,b) is a generating pair with LEN = 5 then

$$babab^2ab = a,\ ab^2ab^2abab^2ab = b$$

which defines C_{11}, the cyclic group of order 11. Similarly if (a,b) is a generating pair with LEN = 7 then G is a homomorphic image of C_{29}.

Calculating the basic Fibonacci lengths is a little harder than calculating LEN. We use the fact that BLEN divides LEN and that the orders of the elements in the cycle of length BLEN must repeat LEN/BLEN times. Finally one must check the BLEN calculations using the fact that if $\theta \in$ Aut G with $a\theta = a'$, $b\theta = b'$ then (a,b) and (a',b') satisfy the same presentation. For the minimal generating pairs we use the presentations given in [1] and [3].

Finally we observe by examining the values of LEN that for the finite simple groups examined in Section 3.4, with the exception of PSp(4,4), no two minimal generating pairs lie in the same Fibonacci orbit. However LEN and BLEN for the minimal generating pairs

PSp(4,4)V22 and PSp(4,4)V28 are identical. Examining the orbits we find that the 151st and 152nd elements of the orbit for V22 are the image under an automorphism of the generators V28.

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A GENERALIZATION OF FIBONACCI TREES

Renato M. Capocelli

1. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

In this paper we introduce *Fibonacci binary trees of degree r,* in the sequel called *R-nacci trees*, and extend to them some results on *Fibonacci binary trees* [2], [8], [9]. We also study some properties of related *tree codes.*

R-nacci trees are a natural generalization of Fibonacci trees and constitute an interesting family of trees with properties intermediate between Fibonacci trees, that correspond to the case r = 2, and complete binary trees which R-nacci trees tend to as r increases. We also introduce a variant of R-nacci trees, the *uniform* Fibonacci trees of degree r, in the sequel called *uniform R-nacci trees.* The main feature of uniform R-nacci trees is that, in the setting of Fibonacci numeration systems, they play a role analogous to that of complete binary trees in the setting of the binary numeration system:

i) The number of nodes at each level is given by a *Fibonacci number of degree r.*
ii) The path of labels to a terminal node is the *Zeckendorf representation* in terms of Fibonacci numbers of degree r.

We define the Fibonacci numbers of degree r, in the sequel called *R-nacci numbers,* as follows.

$$F_0^{(r)} = 0,\ F_1^{(r)} = 1,\ F_j^{(r)} = 2^{j-2},\ j = 2, 3, 4, \dots,\ \ r + 1;$$
$$F_j^{(r)} = F_{j-1}^{(r)} + F_{j-2}^{(r)} + F_{j-3}^{(r)} + \cdots + F_{j-r+1}^{(r)} + F_{j-r}^{(r)},\ \ j \geq r + 2. \tag{1}$$

The *Fibonacci numbers* are the case r=2, while the *Tribonacci numbers* have r=3, [5], and the *Quadranacci numbers* have r=4, [6].

In studying R-nacci numbers it is necessary to consider the *characteristic equation* $f(x) = x^r - x^{r-1} - \cdots - x - 1 = 0$. It has been shown that the roots of this equation are distinct, and in addition, one root, that will be denoted by $\Phi_1^{(r)}$, is real and lies between 1 and 2, and the remaining r−1 roots, denoted by $\Phi_i^{(r)}$ $\left(i \in [2,r]\right)$, lie within the unit circle of the complex plane, [11], [12].

From standard analysis it then follows that a Fibonacci number of degree r and order k, $F_k^{(r)}$, can be written as

$$F_k^{(r)} = \sum_{i=1}^{r} b_i^{(r)} (\Phi_i^{(r)})^k \tag{2}$$

where $b_i^{(r)}$ are constants determined by the initial conditions.

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The following theorem holds.

__Theorem 1:__ Let $F_k^{(r)}$ denote a Fibonacci number of degree r and order k; then $F_k^{(r)} = \sum_{i=1}^{r} b_i^{(r)}(\Phi_i^{(r)})^k$, where

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{-1}(\Phi_i^{(r)}-1)}{(r+1)\,\Phi_i^{(r)} - 2r}. \tag{3}$$

__Proof:__ Miles [11] has shown that the solution of recurrence (1) is $F_k^{(r)} = \sum_{i=1}^{r} b_i^{(r)}(\Phi_i^{(r)})^k$ with

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{r-2}}{(\Phi_i^{(r)} - \Phi_r^{(r)})(\Phi_i^{(r)} - \Phi_{r-1}^{(r)})\ldots(\Phi_i^{(r)} - \Phi_{i+1}^{(r)})(\Phi_i^{(r)} - \Phi_{i-1}^{(r)})\ldots(\Phi_i^{(r)} - \Phi_1^{(r)})};$$

where $\Phi_1^{(r)}$ is the dominant root and $\Phi_i^{(r)}$, $2 \le i \le r$, are the remaining roots of the characteristic equation.

We observe that $f(x) = x^r - x^{r-1} - \ldots - x - 1 = (x-\Phi_1^{(r)})(x-\Phi_2^{(r)})\ldots(x-\Phi_r^{(r)})$, and set

$$F(x) = f(x)(x-1) = (x-1)(x-\Phi_1^{(r)})(x-\Phi_2^{(r)})\ldots(x-\Phi_r^{(r)}) = x^{r+1} - 2x^r + 1.$$

Using the fact that $f(\Phi_i^{(r)}) = 0$ we find that

$$\begin{aligned} f'(\Phi_i^{(r)}) &= (\Phi_i^{(r)} - \Phi_r^{(r)})(\Phi_i^{(r)} - \Phi_{r-1}^{(r)})\ldots(\Phi_i^{(r)} - \Phi_{i+1}^{(r)})(\Phi_i^{(r)} - \Phi_{i-1}^{(r)})\ldots(\Phi_i^{(r)} - \Phi_1^{(r)}) \\ &= \frac{F'(\Phi_i^{(r)})}{(\Phi_i^{(r)}-1)} = \frac{(r+1)(\Phi_i^{(r)})^r - 2r(\Phi_i^{(r)})^{r-1}}{\Phi_i^{(r)}-1}. \end{aligned}$$

Inserting these results in Miles' formula we finally obtain

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{r-2}(\Phi_i^{(r)}-1)}{(r+1)(\Phi_i^{(r)})^r - 2r(\Phi_i^{(r)})^{r-1}} = \frac{(\Phi_i^{(r)})^{-1}(\Phi_i^{(r)}-1)}{(r+1)\Phi_i^{(r)} - 2r}. \square$$

When k becomes large, $b_1^{(r)}\Phi_1^{(r)}$ becomes the *dominant* term of the sum (2) yielding for large k

$$F_k^{(r)} \sim b_1^{(r)}(\Phi_1^{(r)})^k. \tag{4}$$

This implies $\lim_{k\to\infty} \frac{F_{k+1}^{(r)}}{F_k^{(r)}} = \Phi_1^{(r)}$ and, since $\lim_{r\to\infty} \Phi_1^{(r)} = 2$, $\lim_{k,r\to\infty} \frac{F_{k+1}^{(r)}}{F_k^{(r)}} = 2$.

A little closer analysis shows that $F_k^{(r)} = \lfloor b_1^{(r)}(\Phi_1^{(r)})^k + .5 \rfloor$, [3].

Taking the logarithm of (4) provides the formula

$$\log_2(F_k^{(r)}) \sim k \log_2 (\Phi_1^{(r)}) + \log_2 (b_1^{(r)}), \tag{5}$$

which gives account of the fact that if $log_2 F_k^{(r)}$ is plotted against k, for large k the graph is a straight line. (5) is useful for determining the extra space required in computations when Zeckendorf's representations are used rather than the usual common binary representation.

Each integer N has the following unique *Zeckendorf representation* in terms of Fibonacci numbers of degree r, [6], [16]:

$N = \alpha_2 F_2^{(r)} + \alpha_3 F_3^{(r)} + \alpha_4 F_4^{(r)} + \ldots + \alpha_k F_k^{(r)}$, where $\alpha_i \in [0,1]$ and $\alpha_i \alpha_{i-1} \alpha_{i-2} \alpha_{i-3} \ldots \alpha_{i-r+1} = 0$.

Let us write this representation as $\alpha_k \alpha_{k-1} \alpha_{k-2} \ldots \alpha_3 \alpha_2$. The Zeckendorf representation of an integer then provides a binary sequence, called an *R-nacci sequence,* that does not contain r consecutive ones.

We recall that the Zeckendorf representation is a lexicographic ordering based on $0 < 1$, [7]. For future reference we also list the following properties of the Zeckendorf representation.

Z1. The number of R-nacci sequences of length $k - 1$ is $F_{k+1}^{(r)}$.

Z2. The Zeckendorf representation of $F_{k+1}^{(r)}$ is $1\,\underbrace{00\ldots0}_{k-1}$.

Z3. The Zeckendorf representation of $\sum_{i=1}^{h} F_{k+i}^{(r)}$ is $\underbrace{11\ldots1}_{h}\,\underbrace{00\ldots0}_{k-1}$, $h < r$.

Z4. Let j be an integer such that $F_{k-t}^{(r)} \le j < F_{k-t+1}^{(r)}$, $t > 0$, $k - t \ge 2$, and let α be its Zeckendorf representation of degree r. Then the Zeckendorf representation of $j + \sum_{i=1}^{h} F_{k+i}^{(r)}$, $h < r$, is $\underbrace{11\ldots1}_{h}\,\underbrace{00\ldots0}_{t}\,\alpha$.

Labeling each branch of a binary tree with a code symbol (in the sequel we use 0 for a left branch, 1 for a right branch) and representing each terminal node with the path of labels from the root to it leads to a binary *prefix* codeword set, called a *tree code.* A code is a prefix if no codeword is the beginning of any other codeword.

Tree codes are a remarkable family of prefix codes that deserve attention for their inherent usefulness. Both encoding and decoding are natural for tree codes, but only decoding is easy for arbitrary prefix codes. Tree codes preserve the order structure of the encoded set, permitting efficient computation of the order relation. Further, for certain ordered sets, such as the integers, there is no loss of flexibility in restricting prefix codes to the set of tree codes, and for general ordered sets there is no asymptotic loss, [14].

For future reference we recall that the property of *order-preserving* (if x precedes y, the codeword for *x lexicographically* precedes the codeword for y) is a necessary condition for a code to be interpreted as a tree code.

2. FIBONACCI TREES OF DEGREE r

We define the *Fibonacci tree of degree r and order k,* in the sequel called R-nacci tree of order k and denoted by $T_k^{(r)}$, as follows:

For $k < 0$, the R-nacci tree is the empty tree Δ.

For $k = 0$ or $k = 1$, $T_k^{(r)}$ is the root only.

For $k > 1$, $T_k^{(r)} = T_{k-1}^{(r)} \wedge (T_{k-2}^{(r)} \wedge (T_{k-3}^{(r)} \wedge \ldots (T_{k-r+1}^{(r)} \wedge T_{k-r}^{(r)})))$; where $A \wedge B$ denotes the tree whose left subtree is A and whose right subtree is B. We call $T_{k-1}^{(r)}, T_{k-2}^{(r)}, \ldots, T_{k-r}^{(r)}$ *principal subtrees* of $T_k^{(r)}$.

R-nacci trees can be constructed inductively as follows:

For $1 < k \leq r$, $T_k^{(r)}$ is the complete binary tree of height $k - 1$. For $k > r$, the left subtree is $T_{k-1}^{(r)}$; the right subtree is the tree $\hat{T}_{k-1}^{(r)}$. The tree $\hat{T}_{k-1}^{(r)}$ is obtained from $T_{k-1}^{(r)}$ by replacing the subtree rooted at its rightmost internal node at level $r - 2$ with $T_{k-r}^{(r)}$. For $r = 2$, R-nacci trees reduce to the usual Fibonacci trees. By induction it is also easy to see that $T_k^{(r)}$ has height $k - 1$.

In the sequel we will find it useful to consider the tree $G^{(r)}$ of height $r - 1$ obtained from $T_k^{(r)}$ by removing all principal subtrees $T_{k-1}^{(r)}, T_{k-2}^{(r)}, \ldots, T_{k-r}^{(r)}$. We call $G^{(r)}$ the ***skeleton tree*** of $T_k^{(r)}$.

Figure 1 shows the Fibonacci tree of degree r and order k.

Figure 2 shows the Fibonacci tree of degree 3 and order 6, $T_6^{(3)}$.

The first result of this Section is the determination of the number of nodes of $T_k^{(r)}$ and the verification that $T_k^{(r)}$ is a ***balanced*** tree in the sense of Adelson-Velskii and Landis, [1]. A binary tree is called balanced if the height of the left subtree of every internal node never differs by more than 1 from the height of its right subtree. Moreover, no internal node has an empty subtree.

We also determine the number $H_k^{(r)}$ of internal nodes of $T_k^{(r)}$ that have left and right subtrees of different height. In the sequel we denote these nodes by $T_k^{(r)}(/)$.

Lemma 1: $T_k^{(r)}$ is a balanced tree with $F_{k+1}^{(r)}$ terminal nodes and $F_{k+1}^{(r)} - 1$ internal nodes. The number of nodes $T_k^{(r)}(/)$ is $\sum_{j=1}^{k-r} F_j^{(r)}$.

Proof: By induction. The lemma is trivial for $k \leq r$. Suppose that it is true for each $i < k$, $k > r$. We prove that it is true for $i \geq k$. From the inductive construction of the R-nacci tree we have that:

1) The number of terminal nodes is
$F_k^{(r)} + F_{k-1}^{(r)} + \ldots + F_{k-(r-1)}^{(r)} = F_{k+1}^{(r)}$.

2) The number of internal nodes is
$F_k^{(r)} - 1 + F_{k-1}^{(r)} - 1 + \ldots + F_{k-(r-1)}^{(r)} - 1 + r - 1 = F_{k+1}^{(r)} - 1$.

3) All internal nodes of $T_k^{(r)}$ that are ancestors of principal subtrees $T_{k-i}^{(r)}$ (i.e., all internal nodes on the rightmost path of $T_k^{(r)}$) have right and left subtrees of the same height, but one. This node is the rightmost node of $T_k^{(r)}$ at level $r - 2$. It has as left subtree $T_{k-r+1}^{(r)}$ of height $k - r$ and as right subtree $T_{k-r}^{(r)}$ of height $k - r - 1$. Therefore, by the inductive

hypothesis, $T_k^{(r)}$ is a balanced tree. Whenever a node has subtrees of different height, the height of the left subtree is the largest one.

4) Denote the number of nodes $T_k^{(r)}(/)$ by $H_k^{(r)}$. From the construction of the R-nacci tree it then results

$$H_k^{(r)} = H_{k-1}^{(r)} + H_{k-2}^{(r)} + \ldots + H_{k-r}^{(r)} + 1 = \sum_{j=1}^{k} F_j^{(r)} - F_{k+1}^{(r)} = \sum_{j=1}^{k-r} F_j^{(r)}. \square$$

We notice that the number of nodes $T_k^{(r)}(/)$ decreases as r becomes large. This means that R-nacci trees lie between Fibonacci trees, that are the balanced trees of height h with minimum number of terminal nodes, and complete binary trees that are the balanced trees of height h with maximum number of terminal nodes.

We now consider some properties of codes associated with $T_k^{(r)}$, in the sequel denoted by $C_k^{(r)}$. We assume the terminal nodes of $T_k^{(r)}$ are ordered and labeled, from left to right, by the integers 0, 1, 2, ..., $F_{k+1}^{(r)} - 1$.

The tree code $C_6^{(3)}$ associated with $T_6^{(3)}$ is

0	00000	6	0011	12	0111	18	10101
1	00001	7	01000	13	10000	19	1011
2	00010	8	01001	14	10001	20	1100
3	00011	9	01010	15	10010	21	1101
4	00100	10	01011	16	10011	22	1110
5	00101	11	0110	17	10100	23	1111

We start proving that in an R-nacci tree the left branch is taken somewhat more often than the right branch.

<u>Theorem 2:</u> Let $N(0)^{(r)}$ and $N(1)^{(r)}$ denote the asymptotic proportions of zeros and of ones in R-nacci tree codes, then

$$N(1)^{(r)} = \frac{\sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1}}{\Phi_1^{(r)} \sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1} + (r-1)};$$

$$N(0)^{(r)} = \frac{(\Phi_1^{(r)}-1) \sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1} + (r-1)}{\Phi_1^{(r)} \sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1} + (r-1)};$$

$$\frac{N(0)^{(r)}}{N(1)^{(r)}} = \frac{(\Phi_1^{(r)}-1) \sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1} + (r-1)}{\sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1}}.$$

<u>**Proof:**</u> Let $N(0)_k^{(r)}$ and $N(1)_k^{(r)}$ denote the total number of zeros and ones in $C_k^{(r)}$, respectively, and let $N_k^{(r)} = N(0)_k^{(r)} + N(1)_k^{(r)}$ denote the total number of symbols.

From the inductive construction of the R-nacci tree and from the fact that $T_k^{(r)}$ has $F_{k+1}^{(r)}$ terminal nodes, one has the following equations

$$\begin{aligned} N(1)_k^{(r)} &= N(1)_{k-1}^{(r)} + N(1)_{k-2}^{(r)} + F_{k-1}^{(r)} + N(1)_{k-3}^{(r)} + 2F_{k-2}^{(r)} + \cdots \\ &\quad + N(1)_{k-r+1}^{(r)} + (r-2)\, F_{k-r+2}^{(r)} + N(1)_{k-r}^{(r)} + (r-1)\, F_{k-r+1}^{(r)}, \end{aligned}$$

$$\begin{aligned} N(0)_k^{(r)} &= N(0)_{k-1}^{(r)} + F_k^{(r)} + N(0)_{k-2}^{(r)} + F_{k-1}^{(r)} + N(0)_{k-3}^{(r)} + F_{k-2}^{(r)} + \cdots \\ &\quad + N(0)_{k-r+1}^{(r)} + F_{k-r+2}^{(r)} + N(0)_{k-r}^{(r)}, \end{aligned}$$

$$\begin{aligned} N_k^{(r)} = {} & N_{k-1}^{(r)} + F_k^{(r)} + N_{k-2}^{(r)} + 2F_{k-1}^{(r)} + N_{k-3}^{(r)} + 3F_{k-2}^{(r)} + \cdots \\ & + N_{k-r+2}^{(r)} + (r-2)F_{k-r+3} + N_{k-r+1}^{(r)} + (r-1)F_{k-r+2}^{(r)} + N_{k-r}^{(r)} + (r-1)F_{k-r+1}^{(r)}. \end{aligned}$$

Applying these recursively, we have

$$N(1)_k^{(r)} = \sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\Big((r-1)\, F_{j+2}^{(r)} + (r-2)\, F_{j+3}^{(r)} + (r-3)\, F_{j+4}^{(r)} + \cdots + F_{j+r}^{(r)}\Big),$$

$$N(0)_k^{(r)} = \sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}(F_{j+3}^{(r)} + F_{j+4}^{(r)} + F_{j+5}^{(r)} + \cdots + F_{j+r+1}^{(r)}),$$

$$N_k^{(r)} = \sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\Big((r-1)\, F_{j+2}^{(r)} + (r-1)\, F_{j+3}^{(r)} + (r-2)\, F_{j+4}^{(r)} + \cdots + 2F_{j+r}^{(r)} + F_{j+r+1}^{(r)}\Big),$$

where for convenience we set $F_k^{(r)} = 0$ when $k \leq 0$.
We then have

$$\frac{N(1)_k^{(r)}}{N_k^{(r)}} =$$

$$\frac{\sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\Big((r-1)\, F_{j+2}^{(r)} + (r-2)\, F_{j+3}^{(r)} + (r-3)\, F_{j+4}^{(r)} + \cdots + 2F_{j+r-1}^{(r)} + F_{j+r}^{(r)}\Big)}{\sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\Big((r-1)\, F_{j+3}^{(r)} + (r-2)\, F_{j+4}^{(r)} + \cdots + 2F_{j+r}^{(r)} + F_{j+r+1}^{(r)}\Big) + (r-1) \sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\, F_{j+2}^{(r)}}.$$

Dividing the numerator and denominator by

$$\sum_{j=-r+1}^{k-r-1} F_{k-(r+j)}^{(r)}\Big((r-1)\, F_{j+2}^{(r)} + (r-2)\, F_{j+3}^{(r)} + (r-3)\, F_{j+5}^{(r)} + \cdots + 2F_{j+r-1}^{(r)} + F_{j+r}^{(r)}\Big)$$

and recalling that $\lim_{k\to\infty} \frac{F_{k+1}^{(r)}}{F_k^{(r)}} = \Phi_1^{(r)}$ we obtain, by simple calculations,

$$N(1)^{(r)} = \lim_{k\to\infty} \frac{N(1)_k^{(r)}}{N_k^{(r)}} = \frac{\sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1}}{\Phi_1^{(r)} \sum_{j=1}^{r-1} (r-j)(\Phi_1^{(r)})^{j-1} + (r-1)} . \square$$

As r becomes large $N(1)^{(r)}$ tends to $N(0)^{(r)}$ and both tend to $\frac{1}{2}$. R-nacci trees tend to complete binary trees.

Being a tree code, $C_k^{(r)}$ is a prefix and order-preserving code. In addition, $C_k^{(r)}$ is a *complete* prefix code (that is, no new codeword may be added without destroying its prefix property), *maximal* as code (that is, no new codeword may be added without destroying the property of unique decodability of it).

The occurrence that $C_k^{(r)}$ is a complete prefix code corresponds to the fact that every internal node has two sons. We prove that $C_k^{(r)}$ is maximal by induction on the order k using the Kraft's inequality, [10].

We recall that Kraft's inequality states that the codeword lengths l_i of a prefix code must satisfy the inequality $\sum D^{-l_i} \leq 1$, where D is the cardinality of the code alphabet. If *the characteristic sum of the code* $\sum D^{-l_i}$ is 1 the code is said to be maximal.

<u>Theorem 3:</u> The R-nacci tree code $C_k^{(r)}$ is a maximal code.

<u>Proof:</u> Let r be given. The theorem is trivial for $k \leq r$. Suppose that it is true for each $C_i^{(r)}$, $i < k$, $k > r$. We prove that it is true for $C_k^{(r)}$. From the construction of $T_k^{(r)}$, one has

$$\sum_{i=1}^{F_{k+1}^{(r)}} 2^{-l_i} = \sum_{i=1}^{F_k^{(r)}} 2^{-(l_{i,1}+1)} + \sum_{i=1}^{F_{k-1}^{(r)}} 2^{-(l_{i,2}+2)} + \sum_{i=1}^{F_{k-2}^{(r)}} 2^{-(l_{i,3}+3)} + \cdots +$$

$$+ \sum_{i=1}^{F_{k-(r-1)+1}^{(r)}} 2^{-(l_{i,r-1}+(r-1))} + \sum_{i=1}^{F_{k-r+1}^{(r)}} 2^{-(l_{i,r}+(r-1))}$$

where $l_{i,j}$ denotes the i^{th} codeword length of $C_{k-j}^{(r)}$.

By the inductive hypothesis, it then follows that

$$\sum_{i=1}^{F_{k+1}^{(r)}} 2^{-l_i} = \frac{1}{2}\sum_{i=1}^{F_k^{(r)}} 2^{-l_{i,1}} + \frac{1}{4}\sum_{i=1}^{F_{k-1}^{(r)}} 2^{-l_{i,2}} + \frac{1}{8}\sum_{i=1}^{F_{k-2}^{(r)}} 2^{-l_{i,3}} + \cdots +$$

$$+ \frac{1}{2^{r-1}} \sum_{i=1}^{F_{k-(r-1)+1}^{(r)}} 2^{-l_{i,r-1}} + \frac{1}{2^{r-1}} \sum_{i=1}^{F_{k-r+1}^{(r)}} 2^{-l_{i,r}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{r-1}} + \frac{1}{2^{r-1}} = 1. \square$$

By induction on the order k, it is also possible to prove the following theorem that relates R-nacci tree codes to the r^{th} degree Zeckendorf representation of integers.

Theorem 4: Consider $C_k^{(r)}$. Let w(j) denote the codeword that corresponds to terminal node j, $0 \leq j < F_{k+1}^{(r)}$. Then, w(j) and the r^{th} degree Zeckendorf representation of j contain the same number of ones.

Proof: The theorem is trivial for $k \leq r$. Suppose the theorem is true for each $C_i^{(r)}$ of order i less than k, $k > r$. We will prove that it is true also for k. From the construction of the R-nacci tree, one has that the terminal nodes of $T_k^{(r)}$ can be labeled by adding the quantity $S_i = \sum_{h=2}^{i} F_{k-(h-2)}^{(r)}$ to the labels h_j, $0 \leq h_j < F_{k-i+1}^{(r)}$, that are on the terminal nodes within principal subtrees $T_{k-i}^{(r)}$, $1 \leq i \leq r$. Moreover $C_k^{(r)}$ is obtained by prefixing each code $C_{k-i}^{(r)}$ $(i < r)$ with the sequence $\underbrace{11\ldots1}_{i-1}0$ and the code $C_{k-r}^{(r)}$ with the sequence $\underbrace{11\ldots1}_{r-1}$, respectively. By the inductive hypothesis, the theorem follows from the Property Z4 of the Zeckendorf representation. □

We now specify branch costs of an R-nacci tree. A right branch has cost 2 whenever it follows $r-2$ consecutive right branches, otherwise it has cost one. A left branch always has cost 1. The cost of a node i is the sum of costs of the branches from the root to this node. The number of branches from the root to a node gives the *level* of the node. The average cost of a tree T is defined by $s = \sum_{j=1}^{m} \frac{a_j}{m}$, where m is the number of terminal nodes in T and a_j is the cost of them. Notice that with the above cost assignment each sequence of $r-1$ consecutive right branches has cost r.

As in the case r=2 in [8], we have the following.

Lemma 2: The R-nacci tree of order k has $F_{k-(r-1)}^{(r)}$ terminal nodes of cost k and $\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ terminal nodes of cost $k-1$.

Proof: By induction. The lemma is trivially true for $k \leq r$. Suppose that it is true for each $i < k$, $k > r$. We prove that it is true for $i = k$. From the inductive construction of the R-nacci tree, one has that within each principal subtree $T_{k-i}^{(r)}$, $1 \leq i \leq r$, $T_k^{(r)}$ has $F_{k-i-(r-1)}^{(r)}$ terminal nodes of cost $(k-i)$ and $\sum_{j=1}^{r-1} F_{k-i-(j-1)}^{(r)}$ terminal nodes of cost $(k-i-1)$. These nodes within the whole tree have cost k and $k-1$, respectively. It follows that $T_k^{(r)}$ has in total

$$\sum_{i=1}^{r-1} F_{k-1-(i-1)}^{(r)} + \sum_{i=1}^{r-1} F_{k-2-(i-1)}^{(r)} + \cdots + \sum_{i=1}^{r-1} F_{k-(r-1)-(i-1)}^{(r)} + \sum_{i=1}^{r-1} F_{k-r-(i-1)}^{(r)} = \sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$$

terminal nodes of cost $k-1$ and

$$F_{k-1-(r-1)}^{(r)} + F_{k-2-(r-1)}^{(r)} + \cdots + F_{k-(r-1)-(r-1)}^{(r)} + F_{k-r-(r-1)}^{(r)} = F_{k-(r-1)}^{(r)}$$

terminal nodes of cost k. □

Lemma 3: If we split all terminal nodes of cost k−1 of $T_k^{(r)}$, the resulting tree, which then has $2\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)} + F_{k-(r-1)}^{(r)} = F_{k+2}^{(r)}$ terminal nodes, is exactly $T_{k+1}^{(r)}$.

Proof: We prove the lemma by induction on k. Suppose that the lemma is true for $T_i^{(r)}$, $i < k$, $k > r$. When $k \leq r$ the lemma is trivially true. As in Lemma 2 the principal subtrees $T_{k-i}^{(r)}$ of the tree $T_k^{(r)}$ have $\sum_{i=1}^{r-1} F_{k-i-(j-1)}^{(r)}$ terminal nodes of cost k−1. These nodes within each $T_{k-i}^{(r)}$ have cost k−i−1. Splitting these nodes changes, by the induction hypothesis, each principal subtree $T_{k-i}^{(r)}$ into $T_{k-i+1}^{(r)}$ yielding therefore a whole tree that, by definition, is exactly $T_{k+1}^{(r)}$. □

Before proving the next theorem it is convenient to recall the following growth procedure, due to Varn, that constructs a tree of minimum cost, [15].

Suppose an optimal tree with n terminal nodes has already been constructed. Split, in this tree, any one terminal node of minimum cost to produce two new terminal nodes. The resulting tree with n+1 terminal nodes will be optimal.

With the branch cost specified as above, for any fixed $k \geq 2$, the following theorem holds.

Theorem 5: The average cost of $T_k^{(r)}$ is $\dfrac{F_{k-(r-1)}^{(r)}}{F_{k+1}^{(r)}} + k - 1$. It is optimal among binary trees with $F_{k+1}^{(r)}$ terminal nodes.

Proof: By Lemma 2, $T_k^{(r)}$ has $F_{k-(r-1)}^{(r)}$ terminal nodes of cost k and $\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ terminal nodes of cost k−1. The cost of $T_k^{(r)}$ is therefore

$$\frac{1}{F_{k+1}^{(r)}}\left[(k-1)\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)} + kF_{k-(r-1)}^{(r)}\right] = \frac{1}{F_{k+1}^{(r)}}\left[(k-1)\,F_{k+1}^{(r)} + F_{k-(r-1)}^{(r)}\right] = \frac{F_{k-(r-1)}^{(r)}}{F_{k+1}^{(r)}} + k - 1.$$

By the Varn procedure, the second assertion is a straightforward consequence of Lemma 3 which states that splitting in $T_k^{(r)}$ all terminal nodes of minimum cost produces $T_{k+1}^{(r)}$. □

The next theorem generalizes a result in [9] which deals with the number of terminal nodes of $T_k^{(r)}$ at each level. Before proving it, let us look at the R-nacci trees, in particular at the process of constructing $T_{k+1}^{(r)}$ from $T_k^{(r)}$, more closely. Lemmas 2 and 3 state that $T_k^{(r)}$ has $\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ terminal nodes of cost (k−1) and $F_{k-(r-1)}^{(r)}$ terminal nodes of cost k, and that $T_{k+1}^{(r)}$ can be obtained from $T_k^{(r)}$ by splitting all terminal nodes of cost k−1. We call S-nodes (for *splitting* nodes) the terminal nodes of cost k − 1 and $\mathcal{E}$-nodes (for *ending* nodes) those of cost k.

It is convenient to classify terminal nodes of $T_k^{(r)}$ in the following r different types (Figure 3):

There are $F^{(r)}_{k-(i-1)}$ S-nodes of type i, $1 \leq i \leq r - 1$. Nodes of type i split between $T^{(r)}_k$ and $T^{(r)}_{k+1}$: the left branch produces a node that at the next stage, between $T^{(r)}_{k+1}$ and $T^{(r)}_{k+2}$, will be of type 1; the right branch produces a node that at the next stage will be of type i + 1.

The type r contains $F^{(r)}_{k-(r-1)}$ $\mathcal{S}$-nodes. These nodes do not split between $T^{(r)}_k$ and $T^{(r)}_{k+1}$. At the next stage, between $T^{(r)}_{k+1}$ and $T^{(r)}_{k+2}$, they become of type 1.

Therefore at the next stage there are $2\sum_{i=1}^{r-1} F^{(r)}_{k-(i-1)} + F^{(r)}_{k-(r-1)} = F^{(r)}_{k+2}$ terminal nodes:

$$F^{(r)}_{k+1} + F^{(r)}_{k} + \cdots + F^{(r)}_{k-(r-2)} + F^{(r)}_{k-(r-3)} \text{ S-nodes and } F^{(r)}_{k-(r-2)} \text{ } \mathcal{S}\text{-nodes.}$$

The above classification can be better understood with the aid of the following branch labeling. We label (inductively on order k) each branch with one of the four sequences $\alpha,\beta,\beta\alpha,\beta\beta$ in the following way (Figure 3):

A right branch is labeled $\beta\alpha$ if it enters a node of type 1; β if it enters a node of the type i, $2 \leq i \leq r - 1$; $\beta\beta$ if enters a node of type r.

A left branch is always labeled α.

The path of labels from a root to a node is obtained by concatenating branch labels along paths. The length of a path is the total number of α's and β's in it.

Assume that the labeling is already done for $T^{(r)}_{k-1}, T^{(r)}_{k-2}, \ldots, T^{(r)}_{k-r}$, $k - r \geq 0$. Label β all right branches of the skeleton tree $G^{(r)}$ of $T^{(r)}_k$ but the last one, the branch entering the rightmost terminal node, labeled $\beta\alpha$ ($\beta\beta$ if k = r). Then, $T^{(r)}_k$ is labeled by replacing each terminal node i of $G^{(r)}$ with $T^{(r)}_{k-i}$. For the consistency of the inductive construction, we require that if k < r, the labeling of $T^{(r)}_k$ is accomplished by labeling all right branches β. Left brańches are always labeled α.

This labeling rule implies that:

F1) Every right branch entering an S-node is labeled $\beta\alpha$ or β.
F2) A right branch labeled $\beta\beta$ is relabeled $\beta\alpha$ at the next stage.
F3) No right branch entering an internal node is labeled $\beta\beta$.
F4) No path entering an internal node contains r consecutive β's.
F5) Paths entering terminal nodes of $G^{(r)}$ are labeled respectively α, $\beta\alpha$, $\beta\beta\alpha$, ... , $\underbrace{\beta\ldots\beta\beta}_{r-2}\,\alpha$ and $\underbrace{\beta\ldots\beta\beta\beta}_{r-1}\,\alpha$ (or $\underbrace{\beta\ldots\beta\beta\beta}_{r}$ if k = r).
F6) Every path entering an $\mathcal{S}$-node ends with r consecutive β's.
F7) Every path entering a terminal node of type 1 ends with α.
F8) Every path entering a terminal node of type i, $2 \leq i \leq r - 1$, ends with i − 1 consecutive β's.

The following lemma proves that the assignment of types to terminal nodes can be made inductively on the order $k \geq r$. For k = r or r + 1, in order that the assignment satisfies the inductive construction, we make the convention that trees $T_1^{(r)}$ and $T_0^{(r)}$ that consist only of a root node are labeled, respectively, $T_1^{(r)}$ of type 1 and $T_0^{(r)}$ of type r.

For $k \leq r - 1$ the node labeling is done according to the path of labels entering the given terminal node. There are at most $r - 1$ different types of terminal nodes. Terminal nodes that are left sons are of type 1. A terminal right son is labeled according to the number of terminal consecutive β's that are on the path that enters it.

Lemma 4: The type determination within the principal subtrees $T_{k-i}^{(r)}$ gives the correct type determination for the whole tree.

Proof: Given $k \geq r$. A node of $T_k^{(r)}$ is of type 1 iff the path of labels entering it terminates with α; of type i, $i = 2 \ldots r$, iff the path of labels entering it terminates with $i - 1$ consecutive β's and of type r iff the path terminates with r consecutive β's. By construction, the path entering within a principal subtree $T_{k-j}^{(r)}$ a given terminal node is a suffix of the path entering the same terminal node in the whole tree $T_k^{(r)}$ with resulting prefix $\underbrace{\beta\ldots\beta\beta\beta}_{j-1}\ \alpha\ \underbrace{\beta\ldots\beta\beta\beta}_{r}$ if $k - j = 0$). Both paths terminate with the same number of β's. Therefore, the inductive construction does not change the node classification. □

The following theorem provides the number of nodes of each type i that $T_k^{(r)}$ has at each level.

Theorem 6: The number $\mathcal{N}_{i,k-j}^{(r)}$ of nodes of type i that $T_k^{(r)}$ has at each level $k - j$ is given by:

$$\mathcal{N}_{i,k-j}^{(r)} = \sum_{n_1,\ldots,n_{r-1}} \binom{n_1 + n_2 + n_3 + \cdots + n_{r-1} + j - 1}{n_1, n_2, n_3, \cdots, n_{r-1}, j - 1}$$

where (.) is the multinomial coefficient and the summation is over all nonnegative integers n_1, n_2, n_3, $\cdots$, n_{r-1} such that $n_1 + 2n_2 + 3n_3 + \cdots + (r - 1)n_{r-1} + r(j - 1) = k - i$.

Proof: Let $T_k^{(r)}$ be given . Consider the paths of labels from the root to leaves. As in Lemma 2, it is possible to see, by induction, that there are $\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ paths of length $k - 1$ and $F_{k-(r-1)}^{(r)}$ paths of length k. Paths of length $k - 1$ enter S-nodes, paths of length k enter $\mathcal{S}$-nodes. It has been noticed that paths entering S-nodes of type j terminate with a sequence of $(j-1)$ β's; whereas paths entering $\mathcal{S}$-nodes terminate with r consecutive β's. In both cases the beginning of a path entering a node of type i is a sequence of length $k - i$ that terminates with α and does not contain r consecutive β's. All sequences of this kind are present. Therefore the number of sequences of length $k - i$ that terminate with α and do not contain r consecutive β's (i.e., the number of different paths entering a node of type i) provides the number $F_{k-(i-1)}^{(r)}$ of nodes of type i. To complete the proof, let us notice that a path of branches contains a branch labeled $\beta\alpha$ or $\beta\beta$ iff it contains a sequence of $r - 1$ consecutive branches labeled β. This implies that if a path of labels to a given terminal node contains h sequences of $r - 1$ consecutive β's, then its length is greater by h than the number of branches that constitute it. That is, the terminal node is at level $k - 1 - h$. Therefore one sees that the number of terminal nodes of $T_k^{(r)}$ of type i at level $k - j$ coincides with the number of paths of length $k - i$ that terminate with α, do not contain r consecutive β's and contain exactly $j - 1$ sequences of $r - 1$ consecutive β's. To calculate this number, notice that a sequence of length $k - i$ terminates with α, contains exactly j $-$ 1 sequences of r $-$ 1 consecutive β's and does not contain r consecutive β's, iff n_1 of its elements are α, n_2 of its elements are $\beta\alpha$, n_3 of its elements are $\beta\beta\alpha$, ..., j $-$ 1 of its elements are $\underbrace{\beta\beta...\beta}_{r-1}\alpha$ with $n_1 + 2n_2 + 3n_3 + \ldots + r(j-1) = k - i$. By elementary combinatorics, one gets therefore

$$\mathcal{N}^{(r)}_{i,k-j} = \sum_{n_1,\ldots,n_{r-1}} \binom{n_1 + n_2 + n_3 + \cdots + n_{r-1} + j - 1}{n_1, n_2, n_3, \cdots, n_{r-1}, j-1}$$

where (.) is the multinomial coefficient and the summation is over all nonnegative integers n_1, n_2, n_3, $\cdots$, n_{r-1} such that $n_1 + 2n_2 + 3n_3 + \cdots + (r-1)n_{r-1} + r(j-1) = k - i$. □

Notice that $\sum_{j=1}^{\lfloor \frac{k-1}{r}+1 \rfloor} \mathcal{N}^{(r)}_{i,k-j}$ represents the total number $F^{(r)}_{k-i+1}$ of nodes of type i. Therefore by counting the terminal nodes of $T^{(r)}_k$ we have obtained a *proof-by-tree* of the identity [13]

$$F^{(r)}_{k-i+1} = \sum_{n_1,\ldots,n_r} \binom{n_1 + n_2 + n_3 + \cdots + n_{r-1} + n_r}{n_1, n_2, n_3, \cdots, n_{r-1}, n_r}$$

where (.) is the multinomial coefficient and the summation is over all nonnegative integers n_1, n_2, n_3, $\cdots$, n_{r-1}, n_r such that $n_1 + 2n_2 + 3n_3 + \cdots + (r-1)n_{r-1} + rn_r = k - i$.

3. UNIFORM R-NACCI TREES AND ZECKENDORF'S REPRESENTATIONS OF DEGREE r

We define the *uniform R-nacci* tree of order k, denoted in the sequel by $U^{(r)}_k$, as follows:

For $k < 0$, the uniform R-nacci tree is the empty tree Δ.

For $k = 0$ or $k = 1$, $U^{(r)}_k$ is the root only.

For $k > 1$, $U^{(r)}_k = U^{(r)}_{k-1} \wedge \left(U^{(r)}_{k-2} \wedge \left(U^{(r)}_{k-3} \wedge \ldots (U^{(r)}_{k-r} \wedge \Delta)\right)\right)$.

As before, we call the subtrees $U^{(r)}_{k-i}$, $1 \le i \le r$, principal subtrees of $U^{(r)}_k$. The name uniform was chosen because all leaves are at the same level.

Uniform R-nacci trees can be constructed inductively as follows:

For $k \le r$, $T^{(r)}_k = U^{(r)}_k$.

For $k > r$, the left subtree is $U^{(r)}_{k-1}$; the right subtree is the tree $\hat{U}^{(r)}_{k-1}$. The tree $\hat{U}^{(r)}_{k-1}$ is obtained from $U^{(r)}_{k-1}$ by deleting the right subtree of the rightmost internal node at level r $-$ 2.

Notice that, by construction, each internal node of $U^{(r)}_k$ either has subtrees of the same height or has empty right subtree.

Figure 4 shows the uniform Fibonacci tree of degree r and of order k.

Figure 5 shows the uniform Fibonacci tree of degree 3 and order 6, $U^{(3)}_6$.

By induction on the order k it is possible to determine the number of nodes that $U_k^{(r)}$ has at each level.

Theorem 7: $U_k^{(r)}$ has at each level i, $0 \le i \le k-1$, $F_{i+2}^{(r)}$ nodes.

Proof: The theorem is trivial for $k \le r$. Suppose that it is true for each $U_i^{(r)}$, $i < k$ $(k > r)$. We prove that it is true for $U_k^{(r)}$. Let us denote by $L^{(r)}(i, k)$ the number of nodes that $U_k^{(r)}$ has at level i. The construction of $U_k^{(r)}$ implies that

$$L^{(r)}(i, k) = F_{i+2}^{(r)}, \quad \text{for } 0 \le i < r$$

and

$$L^{(r)}(i, k) = L^{(r)}(i-1, k-1) + L^{(r)}(i-2, k-2) + \cdots + L^{(r)}(i-r, k-r), \quad \text{for } r \le i < k-1.$$

By the induction hypothesis, this gives

$$L^{(r)}(i, k) = F_{i+1}^{(r)} + F_i^{(r)} + \cdots + F_{i-r+1}^{(r)} = F_{i+2}^{(r)}. \ \square$$

A uniform R-nacci tree is an R-nacci tree with dummy nodes after each consecutive $(r-1)$ right branches. They force the leaves to be at the same level. The number of dummy nodes is determined by the following.

Corollary 1: $U_k^{(r)}$ is obtained by adding $\sum_{i=1}^{k-r} F_i^{(r)}$ internal nodes to $T_k^{(r)}$.

Proof: $U_k^{(r)}$ and $T_k^{(r)}$ have the same number of terminal nodes. From Theorem 7, $U_k^{(r)}$ has $\sum_{i=2}^{k} F_i^{(r)}$ internal nodes. Since $T_k^{(r)}$ has $F_{k+1}^{(r)} - 1$ internal nodes, we obtain that $U_k^{(r)}$ has $\sum_{i=2}^{k} F_i^{(r)} - F_{k+1}^{(r)} + 1 = \sum_{i=1}^{k-r} F_i^{(r)}$ additional internal nodes. $\square$

It is also possible to prove that $U_k^{(r)}$ is optimal with respect to $T_k^{(r)}$, in the sense that $U_k^{(r)}$ is the uniform tree (i.e., the tree with all terminal nodes at the same level) that is obtained from $T_k^{(r)}$ by inserting the minimum number of additional internal nodes.

Theorem 8: $U_k^{(r)}$ is optimal with respect to $T_k^{(r)}$.

Proof: By definition, every internal node of $U_k^{(r)}$ either has an empty subtree or has subtrees of equal height. $T_k^{(r)}$ is a balanced tree and has $\sum_{j=1}^{k-r} F_j^{(r)}$ internal nodes with left and right subtrees of different height. No internal node of $T_k^{(r)}$ has an empty subtree. To obtain a uniform tree from $T_k^{(r)}$ it is necessary to equalize the height of subtrees of nodes $T_k^{(r)}(/)$. This requires at least $\sum_{i=1}^{k-r} F_i^{(r)}$ internal nodes. $U_k^{(r)}$ is obtained by adding to $T_k^{(r)}$ exactly $\sum_{i=1}^{k-r} F_i^{(r)}$ internal nodes. □

As was proved for the case r = 2, [2], and as Theorem 7 suggests, $U_i^{(r)}$ can be obtained by truncating $U_k^{(r)}$ at level i − 1 (i < k). Conversely, $U_k^{(r)}$ can be constructed by properly splitting terminal nodes of $U_{k-1}^{(r)}$.

To see this let us distinguish in a uniform R-nacci tree two types of terminal nodes:

The $\mathcal{L}$-nodes, the terminal nodes that follow r − 1 consecutive right branches: they generate only the left son.

The $\mathcal{R}$-nodes, the remaining terminal nodes: they generate two sons.

The definition of $\mathcal{L}$-nodes and $\mathcal{R}$-nodes implies that the type determination within principal subtrees gives the correct type determination in the whole tree.

<u>Lemma 5:</u> $U_k^{(r)}$ has $F_{k-(r-1)}^{(r)}$ $\mathcal{L}$-nodes and $\sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ $\mathcal{R}$-nodes.

<u>Proof:</u> By induction. Trivially true for k ≤ r. Suppose the lemma is true for each $U_i^{(r)}$, i < k, k > r. By definition, $U_k^{(r)}$ has as principal subtrees $U_{k-1}^{(r)}$, 1 ≤ i ≤ r. By the induction hypothesis, each $U_{k-i}^{(r)}$ has $F_{k-i-(r-1)}^{(r)}$ $\mathcal{L}$-nodes and $\sum_{j=1}^{r-1} F_{k-i-(j-1)}^{(r)}$ $\mathcal{R}$-nodes. Therefore $U_k^{(r)}$ has $\sum_{i=1}^{r} F_{k-i-(r-1)}^{(r)} = F_{k-(r-1)}^{(r)}$ $\mathcal{L}$-nodes and $\sum_{i=1}^{r} \sum_{j=1}^{r-1} F_{k-i-(j-1)}^{(r)} = \sum_{i=1}^{r-1} F_{k-(i-1)}^{(r)}$ $\mathcal{R}$-nodes. □

The next theorem provides a way to obtain $U_{k+1}^{(r)}$ from $U_k^{(r)}$.

<u>Theorem 9:</u> If each $\mathcal{L}$-node of $U_k^{(r)}$ generates only the left son and each $\mathcal{R}$-node generates two sons, then the resulting tree that has $F_{k-(r-1)}^{(r)} + 2\sum_{j=1}^{r-1} F_{k-(i-1)}^{(r)}$ nodes is exactly $U_{k+1}^{(r)}$.

<u>Proof:</u> By induction. Suppose the theorem is true for each uniform R-nacci tree of order less than k, k > 3 (when k ≤ 3 the assertion is trivially true). By construction, principal subtrees $U_{k-i}^{(r)}$ (1 ≤ i ≤ r) of the tree $U_k^{(r)}$ have $F_{k-i-(r-1)}^{(r)}$ $\mathcal{L}$-nodes and $\sum_{j=1}^{r-1} F_{k-i-(j-1)}^{(r)}$ $\mathcal{R}$-nodes. Making all $\mathcal{L}$-nodes and all $\mathcal{R}$-nodes of each $U_{k-i}^{(r)}$ generate respectively a left son and two sons produces, by the induction hypothesis, $U_{k-i+1}^{(r)}$ yielding therefore an augmented tree that is, by definition, exactly $U_{k+1}^{(r)}$. □

We conclude by proving some properties of uniform R-nacci codes, denoted by $B_k^{(r)}$, and their relationship with the r^{th} degree Zeckendorf representation of integers.

Consider $B_6^{(3)}$

0	00000	6	00110	12	01101	18	10101
1	00001	7	01000	13	10000	19	10110
2	00010	8	01001	14	10001	20	11000
3	00011	9	01010	15	10010	21	11001
4	00100	10	01011	16	10011	22	11010
5	00101	11	01100	17	10100	23	11011

Comparing $B_6^{(3)}$ and $C_6^{(3)}$ it is seen that both have the same number of ones. Moreover, $B_6^{(3)}$ can be obtained from $C_6^{(3)}$ by inserting, from the left, a 0 after each two consecutive 1's until reaching a codeword length of 5.

These features are general features that are derived from the definition of $T_k^{(r)}$ and $U_k^{(r)}$. Indeed, by construction, $B_k^{(r)}$ has the same number of ones as $C_k^{(r)}$, does not allow r consecutive ones and is obtained form $C_k^{(r)}$ by inserting a zero after r − 1 consecutive ones. In particular the following lemma holds.

Lemma 6: The uniform R-nacci code of order k is the set of all R-nacci sequences of length k − 1.

Proof: From the construction of the uniform R-nacci tree, the uniform R-nacci code contains $F_{k+1}^{(r)}$ distinct codewords of length k − 1 and does not allow in any codeword r consecutive ones. The number of R-nacci sequences of length k − 1 is also given by $F_{k+1}^{(r)}$ (Property Z1). □

Lemma 6 is useful for determining the asymptotic proportions of ones and zeros in R-nacci sequences.

Theorem 10: The asymptotic proportion of ones, $\tilde{N}(1)^{(r)}$, in a uniform R-nacci code is

$$\tilde{N}(1)^{(r)} = \frac{(\Phi_1^{(r)})^{-1}\,(\Phi_1^{(r)} - 1)}{(r+1)\,\Phi_1^{(r)} - 2r} \sum_{i=1}^{r-1} i\,(\Phi_1^{(r)})^{-i}. \tag{6}$$

Proof: By construction, $B_k^{(r)}$ and $C_k^{(r)}$ have the same number of ones. Let $\tilde{N}(1)_k^{(r)}$ denote the total number of ones in $B_k^{(r)}$. From Theorem 2, with the same notation, it results that

$$\tilde{N}(1)_k^{(r)} = N(1)_k^{(r)} = \sum_{j=0}^{k-2} F_{j+1}^{(r)} \sum_{i=1}^{r-1} i\, F_{k-i-j}^{(r)}.$$

By Lemma 6 the total number of symbols of $B_k^{(r)}$ is $\tilde{N}_k^{(r)} = (k - 1)\, F_{k+1}^{(r)}$. Therefore

$$\tilde{N}(1)^{(r)} = \lim_{k \to \infty} \frac{\tilde{N}(1)_k^{(r)}}{\tilde{N}_k^{(r)}} = \lim_{k \to \infty} \frac{\sum_{j=0}^{k-2} F_{j+1}^{(r)} \sum_{i=1}^{r-1} i\, F_{k-i-j}^{(r)}}{(k - 1)\, F_{k+1}^{(r)}}$$

$$= \lim_{k \to \infty} \frac{\sum_{j=0}^{k-2} b_1^{(r)} (\Phi_1^{(r)})^{j+1} \sum_{i=1}^{r-1} i\, b_1^{(r)} (\Phi_1^{(r)})^{k-i-j}}{(k - 1)\, b_1^{(r)} (\Phi_1^{(r)})^{k+1}} = b_1^{(r)} \sum_{i=1}^{r-1} i (\Phi_1^{(r)})^{-i}.$$

The theorem follows from (3). □

When r becomes large $\tilde{N}(1)^{(r)}$ tends to $N(1)^{(r)}$ and both tend to $\frac{1}{2}$.

Below are shown the dominant roots and the constants $b_1^{(r)}$ for r ∈ [2, 11]

r	$\Phi_1^{(r)}$	$b_1^{(r)}$	r	$\Phi_1^{(r)}$	$b_1^{(r)}$
2	1.61803	.44721	7	1.99196	.25726
3	1.83928	.33623	8	1.99603	.25404
4	1.92756	.29381	9	1.99802	.25223
5	1.96594	.27362	10	1.99901	.25123
6	1.98358	.26304	11	1.99951	.25067

The asymptotic proportion of ones in R-nacci sequences has been determined very recently also by Chang, [4]. Our formula is simpler and seems to yield more accurate numerical values. Indeed numerical values obtained by Chang's formula do not converge increasingly to 1/2 as one would expect. This is essentially due to the fact that in calculating the constant $b_1^{(r)}$ Chang uses the identity $(2 - \Phi_2^{(r)})(2 - \Phi_3^{(r)}) \ldots (x - \Phi_r^{(r)}) = 1/(2 - \Phi_1^{(r)})$; whereas we use the identity $(2 - \Phi_2^{(r)})(2 - \Phi_3^{(r)}) \ldots (x - \Phi_r^{(r)}) = (\Phi_1^{(r)})^r$. The identity used by Chang is more sensitive to rounding errors.

The last theorem relates uniform R-nacci codes to the r^{th} degree Zeckendorf representation.

Theorem 11: In a uniform R-nacci code, the codeword that represents the terminal node i is the r^{th} degree Zeckendorf representation of the integer i.

Proof: From Lemma 6 the uniform R-nacci code of order k is the set of R-nacci sequences of length k − 1. By definition, they provide the Zeckendorf representation in terms of R-nacci numbers of nonnegative integers < $F_{k+1}^{(r)}$. Since Zeckendorf representations preserve the lexicographic ordering, [6], the assertion is a straightforward consequence of the order-preserving property of tree codes. □

Theorem 11 provides a pretty and efficient mechanism for obtaining the Zeckendorf representation of any integer in terms of Fibonacci numbers of any degree r. The procedure is the following:

Given an integer i, $0 \leq i < F_{k+1}^{(r)}$, construct the uniform R-nacci tree of order k. The Zeckendorf representation of degree r of the integer i is the path of labels from the root to terminal node i.

It is also worthwhile to note that the uniform R-nacci trees in the setting of higher-degree Fibonacci numeration systems play a role analogous to that of the complete binary trees in the setting of the binary numeration system:

The number of nodes at each level is given by an R-nacci number (power of 2, in the binary case);

The path of labels to a terminal node is the Zeckendorf representation (the binary representation, in the binary case).

Notice that the above procedure does not require any *a priori* knowledge of R-nacci numbers. It only requires the appraisal of the height k of the uniform R-nacci tree that can be assumed equal to

$$\left\lceil \frac{\log_2 i}{\log_2 \Phi_1^{(r)}} + 1 \right\rceil .$$

Once the degree r of R-nacci numbers has been chosen and the height k of the tree has been evaluated, the uniform R-nacci tree can be constructed inductively starting from the r − 1 complete trees of height 1, 2, ..., r − 1.

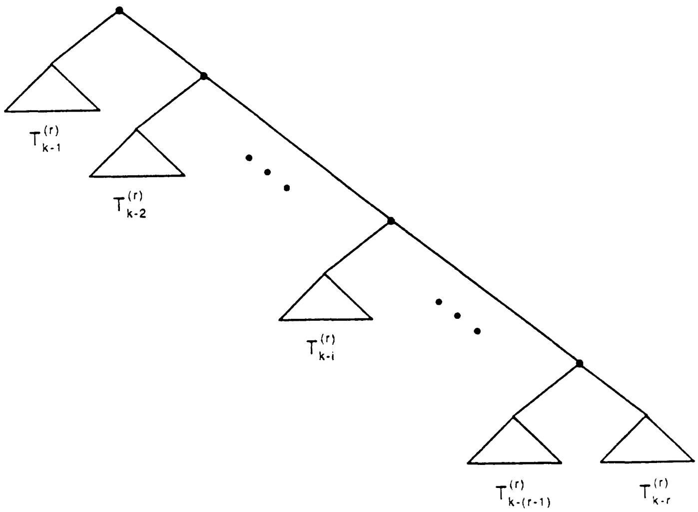

Figure 1. The Fibonacci Tree of Degree r and Order k

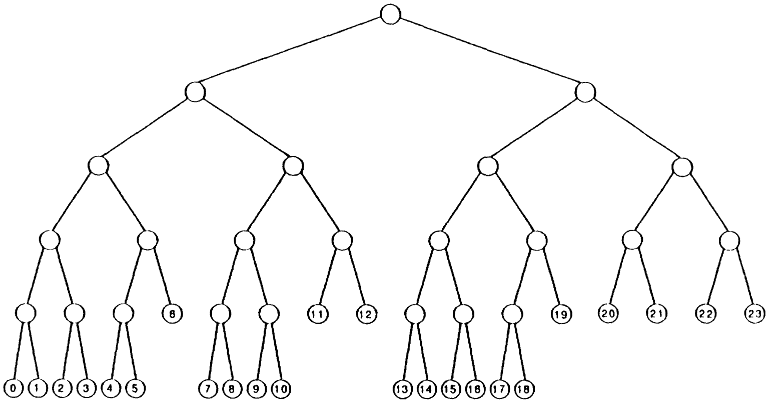

Figure 2. The Fibonacci Tree of Degree 3 and Order 6

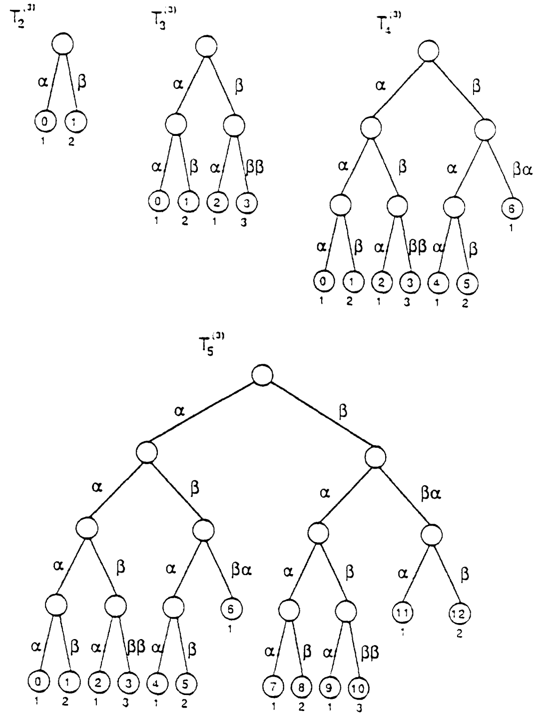

Figure 3. Fibonacci Trees of Degree 3: Branch and Node Labeling

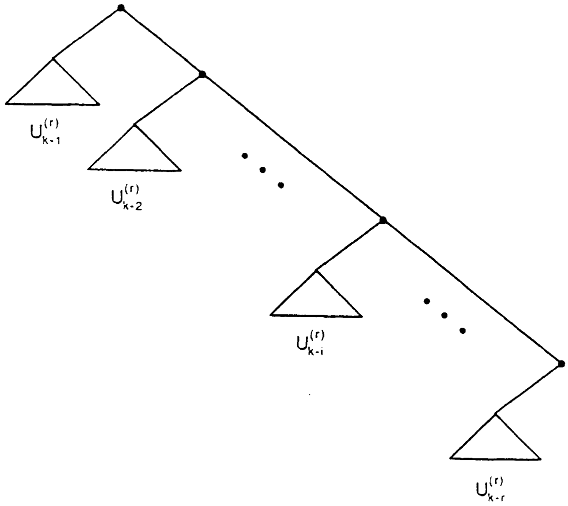

Figure 4. The Uniform Fibonacci Tree of Degree r and Order k

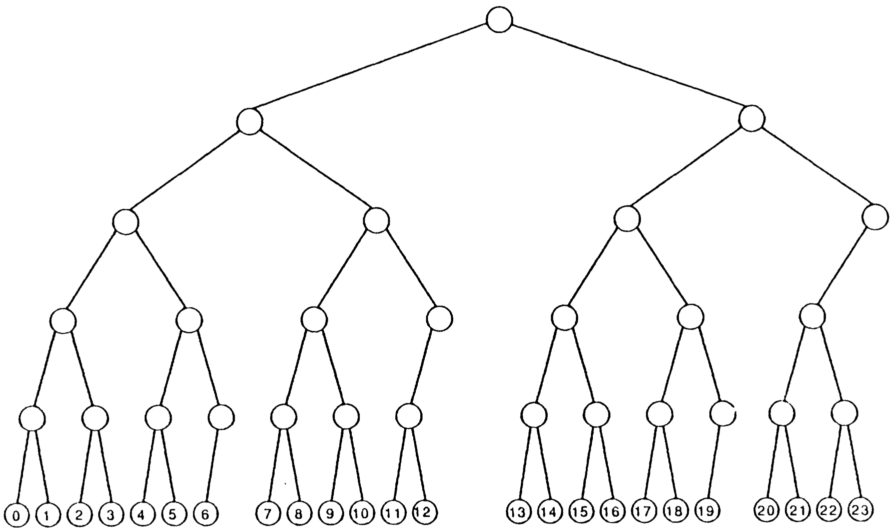

Figure 5. The Uniform Fibonacci Tree of Degree 3 and Order 6

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GENERALIZED FIBONACCI NUMBERS ARE ROUNDED POWERS

Renato M. Capocelli and Paul Cull

We show that the k^{th} degree generalized Fibonacci number of order n ≥ 0, $F_n^{(k)}$, can be computed by

$$\begin{aligned} F_n^{(k)} &= \text{ROUND}\left\{\alpha_0^{(k)}\left(\Phi_0^{(k)}\right)^n\right\} \\ &= \left\lfloor \frac{\Phi_0^{(k)} - 1}{(k+1)\Phi_0^{(k)} - 2\mathrm{k}} \left(\Phi_0^{(k)}\right)^{n-1} + .5 \right\rfloor \end{aligned}$$

where $\Phi_0^{(k)}$ is the unique positive root of the characteristic polynomial for the Fibonacci numbers of degree k and $\lfloor x \rfloor$ denotes the integral part of x.

The k^{th} degree generalized Fibonacci numbers [2] are the solution of the *difference equation*

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + F_{n-3}^{(k)} + \cdots + F_{n-k+1}^{(k)} + F_{n-k}^{(k)}, \; n \geq k; \tag{1}$$

with the initial conditions

$$F_0^{(k)} = 0,\; F_1^{(k)} = 1,\; F_j^{(k)} = 2^{j-2},\; \mathrm{j} = 2, 3, 4, \ldots, \mathrm{k} - 1.$$

We have the *Fibonacci numbers* for the case $k = 2$ and the *Tribonacci numbers* for $k = 3$ [1]. It is well known [3] that the *Fibonacci numbers*, F_n, can be calculated by

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + .5 \right\rfloor, \; n \geq 0.$$

where $\lfloor x \rfloor$ means the integer part of x. In other words, the n^{th} Fibonacci number can be calculated by raising a number to the n^{th} power, multiplying by a constant which is independent of n, and rounding the result. In this sense, Fibonacci numbers are rounded powers.

It has been also proved [5] that for *Tribonacci numbers*, $F_n^{(3)}$,

$$F_n^{(3)} = \left\lfloor \frac{p-1}{4p-6}\, p^{n-1} + .5 \right\rfloor, \; n \geq 0;$$

where $p = \frac{1}{3}\left(\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1\right)$.

This note shows that a similar formula holds for the *generalized Fibonacci numbers*, $F_n^{(k)}$. That is,

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 57–62.

$$F_n^{(k)} \quad = \quad \left\lfloor \frac{\Phi_0^{(k)} - 1}{(k+1)\Phi_0^{(k)} - 2k} \left(\Phi_0^{(k)}\right)^{n-1} + .5 \right\rfloor, \; n \geq 0;$$

where $F_n^{(k)}$ is the n^{th} generalized Fibonacci number of degree k, and $\Phi_0^{(k)}$ is the unique positive real root of the polynomial $x^k - x^{k-1} - \cdots - x - 1$, which is the *characteristic polynomial* for the Fibonacci numbers of degree k. In fact our proof shows that this rounded formula is valid for $n \geq -k + 2$, so the formula is valid even if the generalized Fibonacci numbers are defined using the above difference equation and the initial conditions $F_0^{(k)} = F_1^{(k)} = \cdots = F_{k-2}^{(k)} = 0$, $F_{k-1}^{(k)} = 1$, as for example in [4]. As is well known [4], the solution of the difference equation (1) can be written as

$$F_n^{(k)} = \sum_{i=0}^{k-1} \alpha_i^{(k)} \left(\Phi_i^{(k)}\right)^n ;$$

where the $\Phi_i^{(k)}$'s are the k roots of $x^k - x^{k-1} - \cdots - x - 1$ and the $\alpha_i^{(k)}$'s are calculated from the initial conditions. This simple form depends on the fact that the above polynomial has distinct roots. This fact can be easily be proved if one multiplies the polynomial by $x - 1$ to get $x^{k+1} - 2x^k + 1$ which has no multiple roots because it has no roots in common with its derivative.

Facts 1 and 2 will establish the approximate positions of the k roots. We have used superscripts to indicate that the roots and the coefficients in the solution of the difference equation depend on k. In the following we will suppress these superscripts. We will call the roots $\Phi_0, \Phi_1, \ldots, \Phi_{k-1}$. The unique positive real root will be called Φ_0, and when k is even the negative real root will be called Φ_{k-1}. The complex roots will occur in conjugate pairs and for $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$ we will index these roots as Φ_j, $\Phi_{j+\lfloor \frac{k-1}{2} \rfloor}$ so that Φ_j is in the upper half of the complex plane and so that $\Phi_{j+\lfloor \frac{k-1}{2} \rfloor}$ is the conjugate of Φ_j and lies in the lower half of the complex plane.

<u>Fact 1:</u> The characteristic polynomial of the generalized Fibonacci numbers has a unique positive root Φ_0 such that

$$2 - \frac{1}{2^k} > \Phi_0 > 2 - \frac{2}{2^k}.$$

Any other root Φ_i of this polynomial satisfies

$$1 > |\Phi_i| > \frac{1}{\sqrt[k]{3}} .$$

If k is even, one root, Φ_{k-1}, is negative, satisfies $-1 + \frac{2}{3k} < \Phi_{k-1} < -1 + \frac{2}{k}$ and tends monotonically to -1 as k increases.

<u>Proof:</u> From Descartes rule of signs the polynomial $x^k - x^{k-1} - \cdots - x - 1$ has exactly one positive real root Φ_0; moreover the polynomial is negative on $[0, \Phi_0)$ and positive above Φ_0. Evaluating $x^{k+1} - 2x^k + 1$ at a number greater than 1 will give the same sign as the characteristic polynomial.

Evaluating $x^{k+1} - 2x^k + 1$ at $\left(2 - \frac{1}{2^k}\right)$ we get $-\left(1 - \frac{1}{2^{k+1}}\right)^k + 1$ which is positive because

$1 - \frac{1}{2^{k+1}} < 1$. Hence $\left(2 - \frac{1}{2^k}\right) > \Phi_0$.

Evaluating $x^{k+1} - 2x^k + 1$ at $\left(2 - \frac{2}{2^k}\right)_k$ we get $-2\left(1 - \frac{1}{2^k}\right)^k + 1$ which is negative because the expression is 0 for $k = 1$ and $\left(1 - \frac{1}{2^k}\right)^k$ is increasing with k. Hence $\left(2 - \frac{2}{2^k}\right) < \Phi_0$.

From the absolute value inequality we obtain that for any root Φ_i of the characteristic polynomial $|\Phi_i|^k - |\Phi_i|^{k-1} - \cdots - |\Phi_i| - 1 \le 0$ and $|\Phi_i|^{k+1} - 2|\Phi_i|^k + 1 \ge 0$. The two inequalities imply that $|\Phi_i| = \Phi_0$ or $|\Phi_i| \le 1$. If the characteristic polynomial had a complex root $\Phi_0 w$, where w is a complex number of modulus 1, then $\Phi_0^k = \Phi_0^{k-1} w^{-1} + \Phi_0^{k-2} w^{-2} + \cdots + w^{-k}$ and the right hand side would have to be real and attain its maximum real value, but this would require all powers of w to point in the positive real direction so w would be forced to be 1. Moreover 1 is not a root of the characteristic polynomial and 1 is the only root of modulus 1 of $x^{k+1} - 2x^k + 1$. Therefore we obtain that the roots other than Φ_0 have moduli $|\Phi_i| < 1$.

To give a lower bound for these moduli consider the polynomial $x^{k+1} - 2x + 1$. Its roots are the reciprocals of roots of the polynomial $x^{k+1} - 2x^k + 1$. Let x_0 be a root of $x^{k+1} - 2x + 1$ with $|x_0| > 1$. Then, since $|x_0||x_0^k - 2| = 1$ one has $|x_0^k - 2| < 1$ which implies $|x_0^k| < 3$ and then $|x_0| < \sqrt[k]{3}$. Therefore $1 > |\Phi_i| > \frac{1}{\sqrt[k]{3}}$

When k is even, there is a unique negative root which lies in the interval $(-1,0)$ at the intersection of the curves x^k and $\frac{1}{2-x}$. Since $x^{k+2} < x^k$ on this interval, this intersection will occur further to the left as k increases. Hence the negative root monotonically decreases as k increases. The lower bound $-1 + \frac{2}{3k}$ follows from an application of Newton's method to the characteristic polynomial. The upper bound follows by showing that $x^{k+1} - 2x^k + 1$ is positive at $x = -1 + \frac{2}{k}$. □

<u>Fact 2:</u> For each j, $1 \le j \le \lfloor\frac{k-1}{2}\rfloor$, the argument of the complex root Φ_j, $arg\,[\Phi_j]$, satisfies

$$\frac{2j\pi}{k} < arg[\Phi_j] \le \frac{2j\pi}{k-1}.$$

<u>Proof:</u> Let x_0 be a nonreal root of $x^{k+1} - 2x + 1$. Write $x_0 = p(\cos\phi + i\sin\phi)$. Then we must have

$$p^{k+1}\cos(k+1)\phi - 2p\cos\phi + 1 = 0 \tag{2}$$

$$p^{k+1}\sin(k+1)\phi - 2p\sin\phi = 0. \tag{3}$$

From (3) we get $p^k = \frac{2\sin\phi}{\sin(k+1)\phi}$; that substituted in (2) gives $p = \frac{\sin(k+1)\phi}{2\sin k\phi}$. This yields

$$2^{k+1}\sin^k k\phi - \sin^{k+1}(k+1)\phi = 0 \tag{4}$$

which determines ϕ. The left hand side of (4) is negative for $\phi = \frac{2j\pi}{k}$ and nonnegative for $\phi = \frac{2j\pi}{k-1}$. By continuity, one has that $\phi \in (\frac{2j\pi}{k}, \frac{2j\pi}{k-1}]$.

Considering the conjugate of x_0, $\overline{x_0}$, and taking into account that the roots of $x^{k+1} - 2x + 1$ are the reciprocals of the roots of the polynomial $x^{k+1} - 2x^k + 1$ gives the assertion. □

Fact 3: Let L be the companion matrix of the difference equation, that is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}.$$

Then the column eigenvectors of L are $C_i = ((\Phi_i)^{k-1}, (\Phi_i)^{k-2}, \ldots, 1)$ and the row eigenvectors of L are $R_i = (1, \Phi_i - 1, (\Phi_i)^2 - \Phi_i - 1, \ldots, (\Phi_i)^{k-1} - (\Phi_i)^{k-2} - \cdots - 1)$ and $(R_i \circ C_i) = P'(\Phi_i)$ where $P(x)$ is the characteristic polynomial. If $i \neq j$, $(R_i \circ C_j) = 0$. C_0 and R_0 have only positive entries.

Proof: Direct computation and use of $\Phi_i^k = \Phi_i^{k-1} + \Phi_i^{k-2} + \cdots + 1$ shows that $L \circ C_i = \Phi_i C_i$ and $R_i \circ L = \Phi_i R_i$. consider $R_i \circ L \circ C_j$, associating to the right gives $\Phi_j(R_i \circ C_j)$ while associating to the left gives $\Phi_i(R_i \circ C_j)$. Hence if $\Phi_i \neq \Phi_j$, $(R_i \circ C_j) = 0$. Direct computation shows that $(R_i \circ C_i) = P'(\Phi_i)$. Clearly the entries of C_0 are positive. The entries of R_0 are partial sums of the characteristic polynomial evaluated at Φ_0 divided by powers of Φ_0. Since the characteristic polynomial at Φ_0 evaluates to zero, and since the missing terms in these partial sums are negative, the entries of R_0 are positive. □

Fact 4:

$$\alpha_j = \frac{\Phi_i - 1}{\Phi_i[(k+1)\Phi_i - 2k]}. \tag{5}$$

Proof:

$$\begin{aligned} R_j \circ \left(F_{k-1}^{(k)}, F_{k-2}^{(k)}, \ldots, F_0^{(k)}\right) &= R_j \circ \left(\sum_{i=0}^{k-1} \alpha_i \Phi_i^{k-1}, \sum_{i=0}^{k-1} \alpha_i \Phi_i^{k-2}, \ldots, \sum_{i=0}^{k-1} \alpha_i \Phi_i^0\right) \\ &= \sum_{i=0}^{k-1} \alpha_i (R_j \circ C_i) \\ &= \alpha_j (R_j \circ C_j). \end{aligned}$$

So

$$\alpha_j = \frac{\left(R_j \circ (F_{k-1}^{(k)}, F_{k-2}^{(k)}, \ldots, F_0^{(k)})\right)}{P'(\Phi_j)}.$$

Now

$$P(x) = x^k - x^{k-1} - \cdots - 1 = \frac{x^{k+1} - 2x^k + 1}{x - 1},$$

so

$$P'(x) = \frac{\left((k+1)x^k - 2kx^{k-1}\right)(x-1) - P(x)}{(x-1)^2};$$

and since $P(\Phi_j) = 0$

$$\frac{1}{P'(\Phi_j)} = \frac{\Phi_j - 1}{\Phi_j^{k-1}\left((k+1)\Phi_j - 2k\right)}$$

and

$$\left(R_j \circ \left(F_{k-1}^{(k)}, F_{k-2}^{(k)}, \ldots, F_0^{(k)}\right)\right) = (1, \Phi_j - 1, \Phi_j^2 - \Phi_j - 1, \ldots, \Phi_j^{k-1} - \cdots - \Phi_j - 1) \circ (2^{k-3}, \ldots, 2, 1, 1, 0)$$

$$= (2^{k-3} - 2^{k-4} - \cdots - 1 - 1) + \Phi_j(2^{k-4} - 2^{k-5} - \cdots - 1 - 1) + \cdots + \Phi_j^{k-2}(1) + \Phi_j^{k-1}(0) = \Phi_j^{k-2},$$

the fact follows. □

To show that the generalized Fibonacci numbers are rounded powers we will consider the deviations, d_n, defined by

$$d_n = F_n^{(k)} - \alpha_0 \Phi_0^n$$

and show that for all $n \geq 0$, $| d_n | < \frac{1}{2}$. That is, whatever k, the complex roots and the negative root (in the case k is even) of the characteristic polynomial combine surprisingly with each other always giving a contribution less than $\frac{1}{2}$.

Fact 5: At most $k - 1$ consecutive deviations have the same sign.

Proof: By definition

$$d_n = F_n^{(k)} - \alpha_0 \Phi_0^n = \sum_{i=1}^{k-1} \alpha_i \Phi_i^n.$$

Consider

$$\begin{aligned} &R_0 \circ (d_n, d_{n-1}, \ldots, d_{n-k+1}) \\ &= \sum_{i=1}^{k-1} \alpha_i R_0 \circ (\Phi_i^n, \Phi_i^{n-1}, \ldots, \Phi_i^{n-k+1}) \\ &= \sum_{i=1}^{k-1} \alpha_i R_0 \circ C_i = 0 \end{aligned}$$

by fact 3.

But since each entry of R_0 is positive then if k consecutive d's; $d_n, d_{n-1}, \ldots, d_{n-k+1}$ all have the same sign and at least one of these d's is nonzero, then we have the contradiction that some nonzero number equals zero. □

Fact 6: The deviations satisfy the recurrence relation

$$d_{n+1} = 2d_n - d_{n-k}.$$

Proof: Since $F_n^{(k)}$ and Φ_0^n satisfy the generalized Fibonacci difference equation of degree k and the deviations are a linear combination of $F_n^{(k)}$ and Φ_0^n, the deviations satisfy

$d_{n+1} = d_n + d_{n-1} + \cdots + d_{n-k+1} = d_n + (d_{n-1} + \cdots + d_{n-k+1} + d_{n-k}) - d_{n-k} = 2d_n - d_{n-k}$. □

Fact 7: If $| d_n | \geq \frac{1}{2}$, then $| d_{n-i} | > \frac{1}{2}$ for some i such that $k \geq i \geq 2$.

Proof: If d_n and d_{n+1} have different signs then since $d_{n+1} = 2d_n - d_{n-k}$, $d_{n+1}^2 = 2d_n d_{n+1} - d_{n-k} d_{n+1} > 0$ so d_{n+1} and d_{n-k} have different signs and $-2 | d_n | | d_{n+1} | + | d_{n-k} | | d_{n+1} | > 0$. So $| d_{n-k} | > 2 | d_n | \geq 1 > \frac{1}{2}$.

If d_n and d_{n+1} have the same sign then either $| d_{n-k} | > \frac{1}{2}$, or $| d_{n-k} | \leq \frac{1}{2}$ and $d_{n-k} = 2d_n - d_{n+1} = sgn(d_n)(2 | d_n | - | d_{n+1} |)$ and $-\frac{1}{2} \leq 2 | d_n | - | d_{n+1} | \leq \frac{1}{2}$ so $| d_{n+1} | \geq 2 | d_n | - \frac{1}{2} \geq \frac{1}{2}$.

If $| d_{n-k} | \leq \frac{1}{2}$ we can repeat our argument with d_{n+1} and d_{n+2}, and so forth, but we will not have to proceed beyond d_{n+k-2} because at most $k - 1$ consecutive deviations can have the same sign. Hence at least one among $d_{n-k}, d_{n-k+1}, d_{n-k+2}, \ldots, d_{n-2}$, will have absolute value greater than $\frac{1}{2}$. □

To complete the proof we need to show that k consecutive deviations each have absolute value less than $\frac{1}{2}$. We choose to show that d_1, d_0, ..., d_{-k+2} all have absolute value less than $\frac{1}{2}$, and then conclude from Fact 7 that $| d_n | < \frac{1}{2}$ for all $n \geq 0$.

Fact 8: $| d_i | < \frac{1}{2}$ for i such that $1 \geq i \geq -k + 2$.

Proof: Using the difference equation backwards, we find that $F_0^{(k)} = F_{-1}^{(k)} = F_{-2}^{(k)} = F_{-k+2}^{(k)} = 0$, and that $d_0 = -\alpha_0$, $d_{-1} = -\frac{\alpha_0}{\Phi_0}$, ..., $d_{-k+2} = -\frac{\alpha_0}{\Phi_0^{k-2}}$. Since $\alpha_0 > 0$ and $\Phi_0 > 1$, we have $| d_0 | > | d_{-1} | > \ldots > | d_{-k+2} |$. So we obtain the desired bound for $k - 1$ consecutive d's by showing that $\frac{1}{2} > | d_0 |$. This is equivalent to $\frac{1}{2} > \frac{\Phi_0 - 1}{\Phi_0[(k + 1)\Phi_0 - 2k]}$ or $(k + 1)\Phi_0(\Phi_0 - 2) + 2 > 0$. For this, using Fact 1, we find that $\frac{k + 1}{2^{k-1}} \Phi_0 < 2$ is sufficient and hence that $\frac{k + 1}{2^{k-1}} \leq 1$ is sufficient. This sufficient condition holds for all $k \geq 3$. Although this sufficient condition does not hold for $k = 2$, it is easy to check that $\frac{1}{2} > | d_0 |$ in this case by substituting $\Phi_0 = \frac{1 + \sqrt{5}}{2}$. Since d_0, d_{-1},..., d_{-k+2} are negative, d_1 is necessarily positive, so we only have to show that $\frac{1}{2} > 1 - \frac{\Phi_0 - 1}{(k + 1)\Phi_0 - 2k}$, but this is equivalent to $\Phi_0 < 2$ which is true. □

We summarize our result in the following theorem.

Theorem: The k^{th} degree generalized Fibonacci numbers which are the solution of the difference equation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + F_{n-3}^{(k)} + \cdots + F_{n-k+1}^{(k)} + F_{n-k}^{(k)} , \ n \geq k;$$

with the initial conditions

$$F_0^{(k)} = 0, \ F_1^{(k)} = 1, \ F_j^{(k)} = 2^{j-2}, \ \mathrm{j} = 2, 3, 4, \ldots, \mathrm{k} - 1$$

can be computed from the formula

$$F_n^{(k)} = \left\lfloor \frac{(\Phi_0^{(k)}) - 1}{(k + 1)\Phi_0^{(k)} - 2k} \left(\Phi_0^{(k)}\right)^{n-1} + .5 \right\rfloor, \ n \geq -k + 2;$$

where $\lfloor x \rfloor$ denotes the integral part of x and $\Phi_0^{(k)}$ is the unique positive real root of $x^k - x^{k-1} - \cdots - x - 1$ which is the characteristic polynomial for the k^{th} degree Fibonacci numbers.

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ON GENERALIZED FIBONACCI NUMBERS OF GRAPHS

Michael Drmota

1. INTRODUCTION

A subset I of vertices of a graph G is called independent if no two vertices of I are joined by an edge of G. It is of some interest to determine the number g(G) of independent vertex-sets. For example, consider the graphs G_n with n vertices $x_1, \ldots, x_n$ such that only the pairs (x_i, x_{i+1}), $i = 1, \ldots, n-1$, are joined by an edge. It is easy to see that the numbers $g_n = g(G_n)$ satisfy the relation

$$g_{n+1} = g_n + g_{n-1}$$

with the initial conditions $g_1 = 2$, $g_2 = 3$. Therefore the numbers g_n are essentially the Fibonacci numbers.

This is the motivation to call the number g(G) of a graph G the Fibonacci number of G. This notation has been introduced by H. Prodinger and R. F. Tichy [8], (see also [3], [4]). P. Kirschenhofer, H. Prodinger and R. F. Tichy derived some explicit and asymptotic formulas for the Fibonacci numbers of complete t-ary trees and for the average number of some special families of trees. Later on A. Meir and J. W. Moon [7] determined asymptotic formulas for the average number of maximal independent subsets of simply generated families of trees. The intention of this paper is to generalize this concept.

Let T denote a rooted tree with root x_{root}. A subset I is called i-independent ($i \geq 1$), if I is independent and if every vertex $x \in T$ not contained in I can be joined to a vertex $y \in I$ by a path $x_0 = x, x_1, \ldots, x_j = y$ of length $j \leq i$ such that the sequence of distances to the root $d(x_i, x_{root})$ is monotone. Obviously a subset I is maximal independent if and only if it is 1-independent. Furthermore every i-ind. subset is j-ind. if $i \leq j$.

The last part of this definition seems to be unnatural due to the monotonicity condition. The first and obvious reason for this restriction is that it is much easier to handle with this notion in simply generated families of trees as it is done in this paper. But there is another reason. If such a tree is interpreted as a recursive data structure it is apparently easier to store the information of the path to the next vertex of an i-ind. subset with this montonicity restriction than without it.

Now let F denote a family of rooted trees and F_n the subset of F of trees T with $|T| = n$. Such a family of trees is called simply generated family of trees if the generating function

$$Y(x) = \sum_{n\geq 1} y_n x^n \tag{1}$$

of the sequence $y_n = |F_n|$ satisfies a functional equation of the type

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 63–76.

$$Y = x\varphi(Y), \tag{2}$$

where

$$\varphi(t) = 1 + \varphi_1 t + \varphi_2 t^2 + \cdots \tag{3}$$

is a power series with non-negative coefficients. The coefficients of $\varphi(t)$ can be used to define weights $\omega(T)$ of rooted trees T such that $\omega(T) > 0$ if and only if $T \in F$. Let $D_j(T)$ denote the number of vertices of T incident with j edges leading away from the root, then the weight $\omega(T)$ can be defined by ($\varphi_0 = 1$)

$$\omega(T) = \prod_{j\geq 0} \varphi_j^{D_j(T)}. \tag{4}$$

Now it is easy to see that

$$\sum_{n\geq 1} \left(\sum_{|T|=n} \omega(T) \right) x^n$$

is the solution of (2). Thus

$$y_n = \sum_{|T|=n} \omega(T). \tag{5}$$

For example, we have $\varphi(t) = 1 + t^2$ for binary trees, $\varphi(t) = 1/(1-t)$ for plane trees, and $\varphi(t) = e^t$ for labelled trees.

In the next section we derive functional equations for the generating functions of the average number of i-ind. subsets. In the third section a singularity analysis will give asymptotic formulas for the average number and size, and in the last section we present some numerical results.

2. FUNCTIONAL EQUATIONS

The aim of this section is to derive functional equations for the generating functions related to the average number

$$e^{(i)}(n) = \frac{1}{y_n} \sum_{|T|=n} \omega(T)\, g^{(i)}(T) = \frac{g_n}{y_n} \tag{6}$$

and the average size

$$\mu^{(i)}(n) = \frac{1}{g_n} \sum_{|T|=n} \omega(T) \sum_{k=1}^{n} k g_k^{(i)}(T) \tag{7}$$

of i-ind. subsets. ($g_k^{(i)}(T)$ and $g^{(i)}(T)$ are defined in the sequel.)

Let T denote a rooted tree. Then let $a_k^{(i)}(T)$ denote the number of i-ind. subsets of size k of vertices of T that contain the root, $b_k^{(i)}$ the corresponding number that do not contain the root, and $g_k^{(i)}(T) = a_k^{(i)}(T) + b_k^{(i)}(T)$. Now set

$$a_k^{(i)} = \sum_{|T|=n} \omega(T)a_k^{(i)}(T), \quad b_k^{(i)} = \sum_{|T|=n} \omega(T)b_k^{(i)}(T), \tag{8}$$

and $g_{nk}^{(i)} = a_{nk}^{(i)} + b_{nk}^{(i)}$, furthermore

$$a_{nk}^{(i)} = \sum_{k\geq 0} a_{nk}^{(i)}, \quad b_{nk}^{(i)} = \sum_{k\geq 0} b_{nk}^{(i)}, \tag{9}$$

and $g_{nk}^{(i)} = a_{nk}^{(i)} + b_{nk}^{(i)}$. Thus $g_n^{(i)}$ is the total number of i-ind. subsets in trees in F_n. If we introduce the formal generating functions

$$A^{(i)}(x,z) = \sum_{n,k} a_{nk}^{(i)} z^k x^n, \quad B^{(i)}(x,z) = \sum_{n,k} b_{nk}^{(i)} z^k x^n, \tag{10}$$

and

$$G^{(i)}(x,z) = A^{(i)}(x,z) + B^{(i)}(x,z) = \sum_{n,k} g_{nk}^{(i)} z^k x^n \tag{11}$$

we get immediately (compare with [7]).

<u>Lemma 1:</u>

$$G^{(i)}(x,1) = \sum_{n\geq 1} g_n^{(i)} x^n = \sum_{n\geq 1} e^{(i)}(n) y_n x^n, \tag{12}$$

$$G_z^{(i)}(x,1) = \sum_{n\geq 1} \mu^{(i)}(n) g_n^{(i)} x^n. \tag{13}$$

Now we define two sequences of functions recursively by

$$\begin{aligned} &I^{(1)}(x,z,u,v) = xz\varphi(x\varphi(v)) \\ &I^{(i+1)}(x,z,u,v) = I^{(i)}(x,z,u,u + x\varphi(v)) \quad (i \geq 1) \end{aligned} \tag{14}$$

and

$$\begin{aligned} &J^{(1)}(x,u,v,w) = x\varphi(v) - x\varphi(w) \\ &J^{(i+1)}(x,u,v,w) = J^{(i)}(x,u,v,u - x\varphi(v) + x\varphi(w)) \quad (i \geq 1). \end{aligned} \tag{15}$$

Then we can formulate

<u>Theorem 1:</u> If $Y(x) = x\varphi(Y(x))$ we have

$$A^{(i)}(x,z) = I^{(i)}(x,z,A^{(i)}(x,z), G^{(i)}(x,z)) \tag{16}$$

and

$$B^{(i)}(x,z) = J^{(i)}(x,B^{(i)}(x,z),\ G^{(i)}(x,z),\ B^{(i)}(x,z)). \tag{17}$$

For i = 1 we have

$$A^{(1)} = xz\varphi(x\varphi(G^{(1)})) \text{ and } B^{(1)} = x\varphi(G^{(1)}) - x\varphi(B^{(1)})$$

or after some simple calculation

$$B^{(1)} = x\varphi(B^{(1)} + xz\varphi(B^{(1)} + x\varphi(B^{(1)}))) - x\varphi(B^{(1)})$$

and

$$G^{(1)} = xz\varphi(x\varphi(G^{(1)})) + x\varphi(G^{(1)} - x\varphi(G^{(1)} - xz\varphi(x\varphi(G^{(1)})))).$$

This result is due to A. Meir and J. W. Moon [7]. The difference between i = 1 and i > 1 is that the functional equations (15) and (16) cannot be reduced to one equation for i > 1. For example, we get for i = 2

$$\begin{aligned} A^{(2)} &= xz\varphi(x\varphi(A^{(2)} + x\varphi(G^{(2)}))) \\ B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(B^{(2)} - x\varphi(G^{(2)}) + x\varphi(B^{(2)})) \end{aligned} \tag{18}$$

and for i = 3

$$\begin{aligned} A^{(3)} &= xz\varphi(x\varphi(A^{(3)} + x\varphi(A^{(3)} + x\varphi(G^{(3)})))) \\ B^{(3)} &= x\varphi(G^{(3)}) - x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)}))). \end{aligned} \tag{19}$$

Proof: We will only prove the cases i = 2,3 in order to demonstrate the methods and ideas. The proof of the general case runs along the same lines.

In order to formulate the arguments more clearly we will use following additional notations. If u is any vertex in a tree T with root-vertex x_{root}, let T_u denote the subtree (rooted at u) that contains all vertices v such that the path from x_{root} to v passes through u. And if I is any subset of the vertex-set V(T) of T, let $I_u = I \cap V(T_u)$ for any vertex u.

i=2: We first observe that if $C^{(2)}(x,z) = A^{(2)}(x,z) + x\varphi(G^{(2)}(x,z))$, then $C^{(2)}$ enumerates those subsets I of the vertex-sets of trees T in F such that

(1.1) $x_{root} \in I$ and I is a 2-ind. subset of T, or
(1.2) $x_{root} \notin I$ and I_u is a 2-ind. subset for every vertex u joined to x_{root} in T.

We now consider the generating function $A^{(2)}$ that enumerates 2-ind. subsets I of T such that $x_{root} \in I$. Since $x_{root} \in I$ no vertex joined to the root of T can belong to any such subset I. Now let a denote any vertex in T at distance two from the root x_{root} of T. Then it follows readily from the definition of a 2-ind. set that a subset I of the vertices of T is counted in $A^{(2)}$ if and only if $x_{root} \in I$ and either

(2.1) $a \in I_a$ and I_a is a 2-ind. subset of T_a, or
(2.2) $a \notin I_a$ and I_b is a 2-ind. subset of T_b for every vertex b joined to a in T_a.

If we compare (2.1) and (2.2) with (1.1) and (1.2) and observe that

$$|I| = 1 + \sum_{d(a,x_{root})=2} |I_a| ,$$

it is not difficult to see that

$$\begin{aligned} A^{(2)} &= xz\varphi(x\varphi(C^{(2)})) \\ &= xz\varphi(x\varphi(A^{(2)} + x\varphi(G^{(2)}))) \\ &= I^{(2)}(x,z,A^{(2)}, G^{(2)}), \end{aligned}$$

as required.

Figure 1 tries to give a graphical interpretation for the two equations

$$A^{(2)} = xz\varphi(x\varphi(C^{(2)}))$$

$$C^{(2)} = x\varphi(G^{(2)}) + A^{(2)}$$

for binary trees.

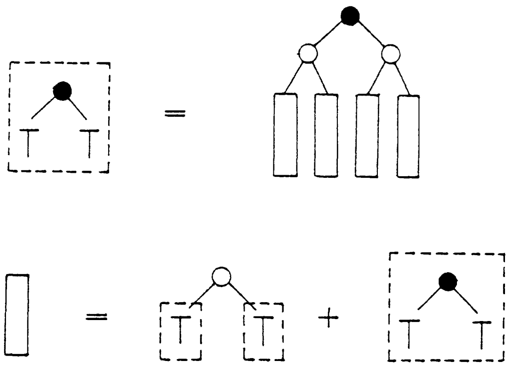

Figure 1

Now let $D^{(2)}(x,z)$ enumerate those subsets I of the vertex-sets of trees T in F such that

(3.1) $x_{root} \notin I$ and I is a 2-ind. subset of T and

(3.2) $u \notin I$ for every vertex u of T joined to x_{root}.

First, observe that $x\varphi(G^{(2)}(x,z))$ enumerates those subsets I of the vertex-sets of trees T in F such that

(4.1) $x_{root} \notin I$ and

(4.2) I_u is a 2-ind. subset of T_u for every vertex u of T joined to x_{root}.

It is easy to verify that I satisfies (4.1) and (4.2) if and only if

(5.1) $x_{root} \notin I$ and I is a 2-ind. subset of T, or

(5.2) $x_{root} \notin I$ and for every vertex u joined to x_{root}

(5.2.1) $u \notin I_u$ an I_u is a 2-ind. subset of T_u, and
(5.2.2) $v \notin I_u$ for every vertex v of T_u joined to u.

Note that (5.1) and (5.2) cannot occur at once. Comparing (3.1) and (3.2) with (5.2.1) and (5.2.2) we get $x\varphi(G^{(2)}) = B^{(2)} = B^{(2)} + x\varphi(D^{(2)})$. Similarly it can be shown that $x\varphi(B^{(2)}) = D^{(2)} + x\varphi(D^{(2)})$.

Thus we get $B^{(2)} - D^{(2)} = x\varphi(G^{(2)}) - x\varphi(B^{(2)})$ and after substituting $D^{(2)}$ in the first equation

$$\begin{aligned} B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(B^{(2)} - x\varphi(G^{(2)}) + x\varphi(B^{(2)})) \\ &= J^{(2)}(x, B^{(2)}, G^{(2)}, B^{(2)}) . \end{aligned}$$

Figure 2 gives a graphical interpretation of the two equations

$$\begin{aligned} B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(D^{(2)}) \\ D^{(2)} &= x\varphi(B^{(2)}) - x\varphi(D^{(2)}) \end{aligned}$$

for binary trees.

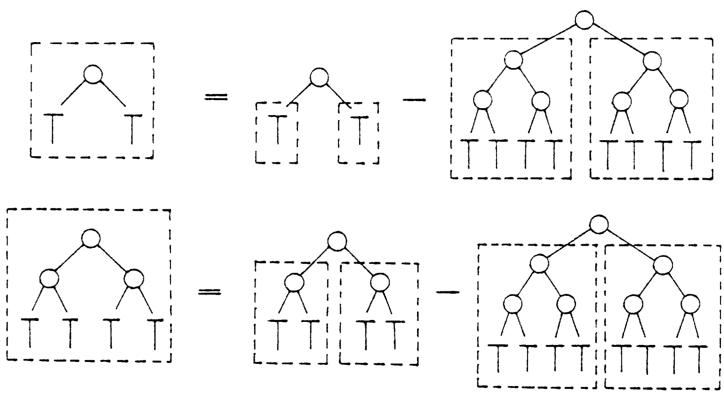

Figure 2

<u>i=3:</u> First we deal with 3-ind. subsets that contain the root. We omit the verbal description. It can be easily deduced from the graphical interpretation

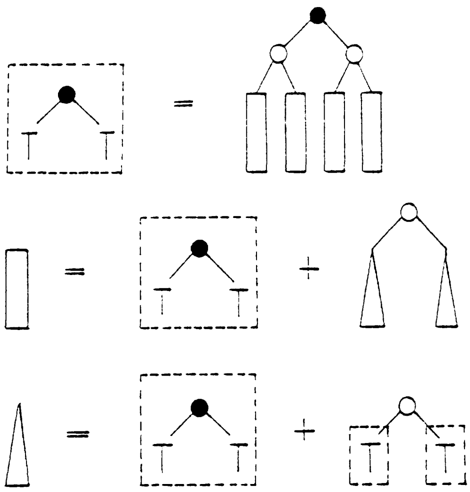

Figure 3

This directly yields the functional equation

$$A^{(3)} = xz\varphi(x\varphi(A^{(3)} + x\varphi(A^{(3)} + x\varphi(G^{(3)})))).$$

For the 3-ind. subsets that do not contain the root we have

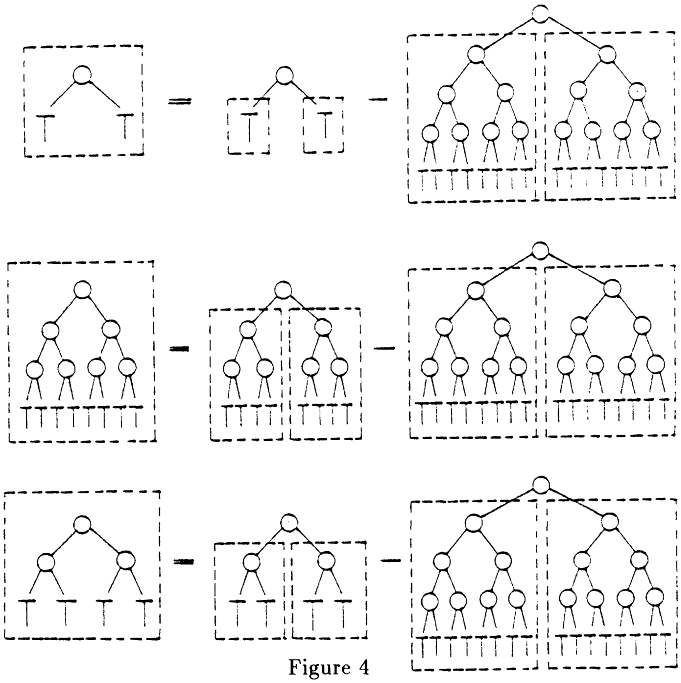

Figure 4

This gives

$$\begin{aligned}B^{(3)} &= x\varphi(G^{(3)}) \\ &\quad - x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)}))) \\ &= J^{(3)}(x, B^{(3)}, G^{(3)}, B^{(3)}).\end{aligned}$$

3. ASYMPTOTIC RESULTS

If we set

$$\begin{aligned}F_1^{(i)}(x,z,a,b) &= a - I^{(i)}(x,z,a,a + b) \\ F_2^{(i)}(x,z,a,b) &= b - J^{(i)}(x,b,a + b,b)\end{aligned} \tag{20}$$

we formally have

$$F_1^{(i)}(x,z,A^{(i)}(x,z), B^{(i)}(x,z)) = F_2^{(i)}(x,z,A^{(i)}(x,z), B^{(i)}(x,z)) \equiv 0.$$

Now suppose that $\varphi(t)$ has a positive radius of convergence which implies that $F_1^{(i)}(x,1,a,b)$ and $F_2^{(i)}(x,1,a,b)$ are analytic around $(0,1,0,0)$. (We have $A^{(i)}(0,1) = B^{(i)}(0,1) = 0$.) If

$$\frac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(a,b)}(0,1,0,0) = (F_{1a}^{(i)}F_{2a}^{(i)} - F_{1b}^{(i)}F_{2a}^{(i)})(0,1,0,0) \neq 0$$

the implicit function theorem says that there is a unique analytic solution $A(x) = A^{(i)}(x,1)$, $B(x) = B^{(i)}(x,1)$ around 0 of the implicit equations $F_1^{(i)}(x,1,A(x),B(x)) = F_2^{(i)}(x,1,A(x),B(x)) \equiv 0$. Since we know that the Taylor coefficients of $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are non-negative the first singularity on the positive real axis is the radius of convergence. As long as

$$\frac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(a,b)}(x_0,1,A^{(i)}(x_0,1),B^{(i)}(x_0,1)) \neq 0 \text{ with } x_0 > 0$$

$A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are analytic in the circle $|x| < x_0$.

Let $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ be the first solution of

$$\begin{aligned}F_1^{(i)}(x,1,a,b) &= 0 \\ F_2^{(i)}(x,1,a,b) &= 0 \\ \frac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(a,b)}(x,1,a,b) &= 0\end{aligned} \tag{21}$$

with $r^{(i)} > 0$ and suppose that

$$\frac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(a,x)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0 \tag{22'}$$

and

$$\frac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(x,b)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0 \tag{22''}$$

Then $r^{(i)}$ is the common radius of convergence of $A^{(i)}(x,1)$, $B^{(i)}(x,1)$. This can be seen by following observation. If $A^{(i)}(x,1)$ is regular at $x = r^{(i)}$ then $A_x^{(i)}(x,1)$ is regular, too. But this is a contradiction to the fact that the limit $x \to r^{(i)}-$ of

$$A_x^{(i)}(x,1) = -\frac{\dfrac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(x,b)}(x,1,A^{(i)}(x,1), B^{(i)}(x,1))}{\dfrac{\partial(F_1^{(i)}, F_2^{(i)})}{\partial(a,b)}(x,1,A^{(i)}(x,1), B^{(i)}(x,1))}$$

does not exist. The same is true for $B^{(i)}(x,1)$.

Similarly to A. Meir and J.W. Moon [7] it can be shown that $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ converge at $x = r^{(i)}$ if there is a $k \geq 2$ with $\varphi_k > 0$. Thus

$$\lim_{x \to r^{(i)}} A^{(i)}(x,1) = \alpha_0^{(i)} \text{ and } \lim_{x \to r^{(i)}} B^{(i)}(x,1) = \beta_0^{(i)}$$

If

$$\left[\left(F_{1bb}^{(i)}\frac{(F_{1a}^{(i)})^2}{(F_{1b}^{(i)})^2} - 2F_{1ab}^{(i)}\frac{F_{1a}^{(i)}}{F_{1b}^{(i)}} + F_{1aa}^{(i)}\right)F_{2b}^{(i)} - \right. \tag{23}$$

$$\left. - \left(F_{2bb}^{(i)}\frac{(F_{2a}^{(i)})^2}{(F_{2b}^{(i)})^2} - 2F_{2ab}^{(i)}\frac{F_{2a}^{(i)}}{F_{2b}^{(i)}} + F_{2aa}^{(i)}\right)F_{1b}^{(i)}\right](r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0$$

and

$$\left[\left(F_{1aa}^{(i)}\frac{(F_{1b}^{(i)})^2}{(F_{1a}^{(i)})^2} - 2F_{1ab}^{(i)}\frac{F_{1b}^{(i)}}{F_{1a}^{(i)}} + F_{1bb}^{(i)}\right)F_{2a}^{(i)} - \right. \tag{24}$$

$$\left. - \left(F_{2aa}^{(i)}\frac{(F_{2b}^{(i)})^2}{(F_{2a}^{(i)})^2} - 2F_{2ab}^{(i)}\frac{F_{2b}^{(i)}}{F_{2a}^{(i)}} + F_{2bb}^{(i)}\right)F_{1a}^{(i)}\right](r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0$$

the inverse functions $X_1(a)$ of $A^{(i)}(x,1)$ and $X_2(b)$ of $B^{(i)}(x,1)$ satisfy $X_1'(\alpha_0^{(i)}) = X_2'(\beta_0^{(i)}) = 0$ and $X_1''(\alpha_0^{(i)}) \neq 0$, $X_2''(\beta_0^{(i)}) \neq 0$. Hence we have a local expansion of the form

$$\begin{aligned} A^{(i)}(x,1) &= \alpha_0^{(i)} - \alpha_1^{(i)}(r^{(i)} - x)^{1/2} - \alpha_2^{(i)}(r^{(i)} - x) + \cdots \\ B^{(i)}(x,1) &= \beta_0^{(i)} - \beta_1^{(i)}(r^{(i)} - x)^{1/2} - \beta_2^{(i)}(r^{(i)} - x) + \cdots , \end{aligned} \tag{25}$$

where

$$(\alpha_1^{(i)})^2 = \frac{(F_{1x}^{(i)}F_{2b}^{(i)} - F_{2x}^{(i)}F_{1b}^{(i)})(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})}{\left[\left(F_{1bb}^{(i)}\frac{(F_{1a}^{(i)})^2}{(F_{1b}^{(i)})^2} - 2F_{1ab}^{(i)}\frac{F_{1a}^{(i)}}{F_{1b}^{(i)}} + F_{1aa}^{(i)}\right)F_{2b}^{(i)} - \left(F_{2bb}^{(i)}\frac{(F_{2a}^{(i)})^2}{(F_{2b}^{(i)})^2} - 2F_{2ab}^{(i)}\frac{F_{2a}^{(i)}}{F_{2b}^{(i)}} + F_{2aa}^{(i)}\right)F_{1b}^{(i)}\right]}$$

$$(\beta_1^{(i)})^2 = \frac{(F_{1x}^{(i)}F_{2a}^{(i)} - F_{2x}^{(i)}F_{1a}^{(i)})(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})}{\left[\left(F_{1bb}^{(i)}\frac{(F_{1a}^{(i)})^2}{(F_{1b}^{(i)})^2} - 2F_{1ab}^{(i)}\frac{F_{1a}^{(i)}}{F_{1b}^{(i)}} + F_{1aa}^{(i)}\right)F_{2b}^{(i)} - \left(F_{2bb}^{(i)}\frac{(F_{2a}^{(i)})^2}{(F_{2b}^{(i)})^2} - 2F_{2ab}^{(i)}\frac{F_{2a}^{(i)}}{F_{2b}^{(i)}} + F_{2aa}^{(i)}\right)F_{1a}^{(i)}\right]} \tag{26}$$

If there are in addition two distinct j,k with gcd(j,k) = 1, $\varphi_j > 0$, and $\varphi_k > 0$ then it can be shown similarly to A. Meir and J. W. Moon [7] but in a more complicated way that $F_{1a}^{(i)}F_{2b}^{(i)} - F_{1b}^{(i)}F_{2a}^{(i)} \neq 0$ on the circle of convergence except at $x = r^{(i)}$. Thus $r^{(i)}$ is the only singularity on the circle of convergence. (Compare also with A. Bender [1, Theorem 4].) Therefore we get by Darboux's theorem [2] that

$$g_n^{(i)} \sim \frac{\alpha_1^{(i)} + \beta_1^{(i)}}{2\sqrt{\pi}} (r^{(i)})^{-n+1/2} n^{-3/2}. \tag{27}$$

If τ is the smallest positive solution of $\tau\varphi'(\tau) = \varphi(\tau)$ then (see [6])

$$y_n \sim \sqrt{\frac{\varphi(\tau)}{2\pi\varphi''(\tau)}} \left(\frac{\tau}{\varphi(\tau)}\right)^{-n} n^{-3/2} \tag{28}$$

and therefore

$$e^{(i)}(n) = \frac{g_n^{(i)}}{y_n} \sim (\alpha_1^{(i)} + \beta_1^{(i)}) \sqrt{\frac{r^{(i)}\varphi''(\tau)}{2\varphi(\tau)}} \left(\frac{\tau/\varphi(\tau)}{r^{(i)}}\right)^n \tag{29'}$$

or

$$\log e^{(i)}(n) \sim E^{(i)}n \tag{29''}$$

with $E^{(i)} = \log(\tau/(r^{(i)}\varphi(\tau)))$.

If $\varphi(t) = 1+t^2$ (binary trees) $y_{2k} = a_{2k}^{(i)} = b_{2k}^{(i)} = 0$. Thus $A^{(i)}(-x,1) = -A^{(i)}(x,1)$ and $B^{(i)}(-x,1) = -B^{(i)}(x,1)$. Therefore there are at least two singularities $x = \pm r^{(i)}$. Since f(x) with $f(x^2) = \frac{1}{x}A^{(i)}(x^2,1)$ has only one singularity $A^{(i)}(x,1)$ has only these two singularities. (The same is true for $B^{(i)}(x,1)$.) Therefore the above considerations can easily be adapted for this case and we get an analoguous asymptotic formula

$$\log e^{(i)}(2n+1) \sim (2n+1) \log \frac{\tau/\varphi(\tau)}{r^{(i)}}. \tag{30}$$

We have proved

Theorem 2: Let F be a simply generated family of trees where $\varphi(t)$ has a radius of convergence large enough that $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$, the solution of (21) where $r^{(i)} > 0$ is minimal, is a regular point of $F_1^{(i)}(x,z,a,b)$ and $F_2^{(i)}(x,z,a,b)$. Suppose that (22), (23), and (24) are satisfied and that there are two distinct positive integers j,k with $\gcd(j,k) = 1$, $\varphi_j > 0$, and $\varphi_k > 0$. Then the average number $e^{(i)}(n)$ of i-independent subsets in trees $T \in F_n$ can be determined asymptotically by

$$e^{(i)}(n) \sim C^{(i)}(D^{(i)})^n$$

where

$$C^{(i)} = (\alpha_1^{(i)} + \beta_1^{(i)}) \sqrt{\frac{r^{(i)}\varphi''(\tau)}{2\varphi(\tau)}} \quad \text{and} \quad D^{(i)} = \frac{\tau/\varphi(\tau)}{r^{(i)}}.$$

($\alpha_1^{(i)}$ and $\beta_1^{(i)}$ are defined in (25) and (26) and τ is the smallest positive solution of $t\varphi'(t) = \varphi(t)$.)

Since $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are absolutely convergent for $|x| \le r^{(i)}$, $A^{(i)}(x,z)$ and

$B^{(i)}(x,z)$ are analytic functions for $|x| < r^{(i)}$ and $|z| < 1$ and satisfy

$$F_1^{(i)}(x,z,A^{(i)}(x,z),\ B^{(i)}(x,z)) = F_2^{(i)}(x,z,A^{(i)}(x,z),\ B^{(i)}(x,z)) \equiv 0.$$

Thus we get

$$G_x^{(i)}(x,z) = A_x^{(i)}(x,z) + B_x^{(i)}(x,z)$$

$$= -\frac{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)} + \dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,x)}}{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}}\ (x,z,A^{(i)}(x,z),\ B^{(i)}(x,z))$$

$$G_z^{(i)}(x,z) = A_z^{(i)}(x,z) + B_z^{(i)}(x,z)$$

$$= -\frac{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(z,b)} + \dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,z)}}{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}}\ (x,z,A^{(i)}(x,z),\ B^{(i)}(x,z))$$

and therefore

$$G_z^{(i)}(x,1) =$$

$$= \frac{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(z,b)} + \dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,z)}}{\dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)} + \dfrac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,x)}}\ (x,1,A^{(i)}(x,1),\ B^{(i)}(x,1))\ G_x^{(i)}(x,1) \tag{31}$$

$$= H^{(i)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))\ G_x^{(i)}(x,1).$$

Suppose that $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ is a regular point of $H^{(i)}(x,1,a,b)$ then $f(x) = H^{(i)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))$ has an expansion of the form

$$f(x) = \delta_0^{(i)} - \delta_1^{(i)}(r^{(i)} - x)^{1/2} - \delta_2^{(i)}(r^{(i)} - x) - \cdots$$

with $\delta_0^{(i)} = f(r^{(i)})$. Since

$$G_x^{(i)}(x,1) = \frac{\alpha_1^{(i)} + \beta_1^{(i)}}{2}(r^{(i)} - x)^{-1/2} + (\alpha_2^{(i)} + \beta_2^{(i)}) + \cdots$$

we have

$$G_z^{(i)}(x,1) = \frac{\delta_0^{(i)}(\alpha_1^{(i)} + \beta_1^{(i)})}{2}(r^{(i)} - x)^{-1/2} + \cdots . \tag{32}$$

Using again Darboux's theorem [2] and comparing this with Lemma 1 and Theorem 2 we conclude

Theorem 3: If we assume in addition to the assumptions of Theorem 2 that $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ is a regular point of $H^{(i)}(x,z,a,b)$ (defined in (31)), the average size $\mu^{(i)}(n)$ of i-independent subsets of trees $T \in F_n$ can be evaluated asymptotically by

$$\mu^{(i)}(n) \sim S^{(i)}n \tag{33}$$

where

$$S^{(i)} = \frac{\delta_0^{(i)}}{r^{(i)}} = \frac{H^{(i)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})}{r^{(i)}} .$$

Again we have the same asymtotic formula

$$\mu\,(2n+1) \sim S^{(i)}(2n+1)$$

for binary trees.

4. NUMERICAL RESULTS

The author has approximately evaluated the numerical constants $D^{(i)}$ and $S^{(i)}$ (see (29) and (33)) for binary trees ($\varphi(t) = 1 + t^2$), plane trees ($\varphi(t) = 1/(1 - t)$), and for labelled trees ($\varphi(t) = e^t$) for i = 2,3. $r^{(i)}$, $\alpha_0^{(i)}$, and $\beta_0^{(i)}$ can be determined by solving the equations (21) approximately by a Newton iteration method. Then it is easy to examine the conditions (22), (23), (24) and to determine $D^{(i)}$ and $S^{(i)}$. In the following two tables these constants are compared with $D^{(1)}$, $S^{(1)}$ [7], with the corresponding numbers $D^{(\infty)}$, $S^{(\infty)}$ for all ind. subsets [3], and with $S^{(max)}$, the average size of the largest ind. subset of a tree $T \in F_n$ [5]. (All constants are rounded.)

	$D^{(\infty)}$	$D^{(3)}$	$D^{(2)}$	$D^{(1)}$
binary trees	1.657	1.531	1.433	1.250
plane trees	1.687	1.564	1.455	1.240
labelled trees	1.655	1.562	1.474	1.274

Table 1

	$S^{(\infty)}$	$S^{(3)}$	$S^{(2)}$	$S^{(1)}$	$S^{(max)}$
binary trees	0.309	0.401	0.438	0.482	0.585
plane trees	0.333	0.410	0.416	0.5	0.618
labelled trees	0.307	0.380	0.393	0.463	0.567

Table 2

It would be interesting to determine the asymptotic expansion of $D^{(i)}$ and $S^{(i)}$ if i tends to infinity. The author conjectures that $\lim_{i\to\infty} D^{(i)} = D^{(\infty)}$ and $\lim_{i\to\infty} S^{(i)} = S^{(\infty)}$ and that the order of convergence depends on $\varphi(t)$ but he could not find a way to prove this.

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AN INVESTIGATION OF SEQUENCES DERIVED FROM HOGGATT SUMS AND HOGGATT TRIANGLES

Daniel C. Fielder and Cecil O. Alford

1. INTRODUCTION

In a recent note [1], the authors discuss derivations of integer sequences called *Hoggatt Sums* and associated triangular arrays called *Hoggatt Triangles*. The nomenclature was proposed as a tribute to the late Verner Hoggatt, Jr. since the investigation and extension of an unpublicized conjecture of Hoggatt ultimately resulted in the above sums and triangles. In personal correspondence [2], Hoggatt conjectured that the third (counting as 0, 1, 2, 3, ...) right diagonal of Pascal's triangle could be used to determine the sequence of integers, S_0, S_1, S_2, ... S_m, ... , which are identically the Baxter permutation counts [3] of indices 0, 1, 2, ... , m, ... Hoggatt based his calculation algorithm for S_m on sums of products between third diagonal terms from Pascal's triangle and appropriately corresponding terms from a completed S_{m-1}. The authors' note [1] supplied the missing proof of Hoggatt's conjecture. Hoggatt's conjecture was then extended to include all right Pascal triangle diagonals indexed as 0, 1, 2, 3, ... , d, ... For each d, the set of S_m's became Hoggatt sums of order d, and the individual integers which sum to a particular S_m became row members of a triangular array called a Hoggatt triangle of order d. With the inclusion of d as a variable parameter, the numerical results of [1] can be interpreted as sequences of $(S_d)_m$'s with fixed index d and variable index m. For example, the Baxter permutation count values are Hoggatt sums of order three whose general sequence term is $(S_3)_m$. Sequences of Hoggatt sums follow a linear recursion which is *index-variant* in m, i.e., the calculation of $(S_d)_m$ for d fixed depends not only on previous members of the sequence but also depends on the value of m. Difference equations for this type of recursion are known to be difficult, if not impossible, to obtain by operational methods [4].

In the present investigation, Hoggatt sums and triangles are used to develop two new types of sequences. The first type (defined as *Hoggatt Second Kind Sequences*) consists of sequences of Hoggatt sums, $(S_d)_m$, with m fixed and d variable. The second type (defined as *Hoggatt Position Sequences*) consists of sequences of terms from the p^{th} positions of the m^{th} rows of d^{th} order Hoggatt triangles. Here p and m are fixed while d is the variable index. Both new sequences are *index-invariant* in d and belong to a class of recursions we have taken the liberty of calling "elementary". Of all the formally documented recursive sequences (see [5]), these stand out as the simplest. The index-invariant feature permits the use of operational methods to determine general forms for properties of the sequences once the degree, n, of recursion is established.

Since many of the results of this paper rely on numerical calculations, a word about calculation methods is in order. The calculations often involve extremely large but exact integers. There can be no rounding approximations! Except for lowest order calculations, hand methods or pocket calculators are out of the question. No matter how sophisticated they may be, conventional high-level computer languages eventually produce floating point and/or round-off

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errors. Fortunately, in recent years computer algebra systems such as MACSYMA, muMath, etc., [6] have been introduced. As well as performing complicated symbolic algebraic operations, the systems admit exact rational arithmetic with integers limited in size only by a well-managed available memory. We used a version of muMath on an AT clone personal computer. Thus, with size no obstacle, we were able to extend our integer use to meet the demands of almost any test situation.

2. BACKGROUND DETAILS

Prior to presenting our development, a very brief review of pertinent features of Hoggatt sums and triangles is essintial. The general form of Hoggatt's conjecture from [1], recast and extended for use herein, follows:

"Select the zeroth and d^{th} right diagonals of Pascal's triangle and let them be the zeroth and first right diagonals of a new triangle with as yet undetermined values for the remaining right diagonals. For m=2, 3, 4, ... in succession, compute the m^{th} row sum and m^{th} row individual entries as

$$(S_d)_m = 1 + \left(\tfrac{m+d-1}{d}\right)\left\{\frac{(R_{m-1})_0}{D_0} + \frac{(R_{m-1})_1}{D_1} + \ldots. + \frac{(R_{m-1})_{m-1}}{D_{m-1}}\right\} \tag{1}$$

where the $(R_{m-1})_q$'s are the row integers from the previously completed $(m-1)$st row starting with q=0 on the left. The D_q's are the first diagonal integers starting with q=0 on the top right.

THEN, the m_{th} row sum, $(S_d)_m$, given by (1) is identically Hoggatt's sum with indices d and m. Morover,

$$1, \left(\tfrac{m+d-1}{d}\right)\left\{\frac{(R_{m-1})_0}{D_0}\right\}, \left(\tfrac{m+d-1}{d}\right)\left\{\frac{(R_{m-1})_1}{D_1}\right\}, \ldots, \left(\tfrac{m+d-1}{d}\right)\left\{\frac{(R_{m-1})_{m-1}}{D_{m-1}}\right\} \tag{2}$$

are the values of the row integers, P(m, p, d), of row m as assigned to positions 0, 1, ... , m."

An example taken from [1] illustrates for d=3, m=5 how the algorithm generates the fifth row terms from completed rows zero through four. The imcomplete Hoggatt triangle is shown as

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 4 && 1 &&& \\ && 1 && 10 && 10 && 1 && \\ & 1 && 20 && 50 && 20 && 1 & \\ 1 && 35 && x && x && x && x \end{array} \tag{3}$$

The values chosen according to the conjecture result in the following generation of terms to complete the fifth row.

$$1,\ 35\left(\tfrac{1}{1}\right)= 35,\ 35\left(\tfrac{20}{4}\right)= 175,\ 35\left(\tfrac{50}{10}\right)= 175,\ 35\left(\tfrac{20}{20}\right)= 35,\ 35\left(\tfrac{1}{35}\right)= 1. \tag{4}$$

(Note that the zeroth and first terms of any row are always present.) The row sum $(S_3)_5$ = 1+35+175+175+35+1 = 422. A typical position term, say the fourth, is P(5,4,3) = 35.

A general form readily adaptable for computing Hoggatt sums is

$$1+\frac{\binom{m+d-1}{d}}{\binom{d}{d}}+\frac{\binom{m+d-1}{d}\binom{m+d-2}{d}}{\binom{d}{d}\binom{d+1}{d}}+\cdots+\frac{\binom{m+d-1}{d}\binom{m+d-2}{d}\cdots\binom{d+1}{d}\binom{d}{d}}{\binom{d}{d}\binom{d+1}{d}\cdots\binom{m+d-2}{d}\binom{m+d-1}{d}}, \tag{5}$$

which can be condensed to

$$(\mathrm{S}_d)_m = 1 + \sum_{h=0}^{m-1}\prod_{k=0}^{h}\frac{\binom{m+d-1-k}{d}}{\binom{d+k}{d}}. \tag{6}$$

The value for P(m, p, d), the member in the p^{th} position of the m^{th} row of a d^{th} order Hoggatt triangle, is given by

$$\mathrm{P(m, p, d)} = \prod_{h=0}^{p-1}\frac{\binom{m+d-1-h}{d}}{\binom{d+h}{d}}, \tag{7}$$

where P(m, 0, d) = 1 and p≤m. For example, P(5, 3, 4) is calculated as

$$\frac{\binom{8}{4}\binom{7}{4}\binom{6}{4}}{\binom{4}{4}\binom{5}{4}\binom{6}{4}} = 490. \tag{8}$$

Hoggatt triangles of orders 0 through 5, rows of index 0 through 6, are shown in Figure 1. The rectangular and circular enclosures are reserved for later use to illustrate properties of sets of P(m, p, d) entries for fixed m and p values.

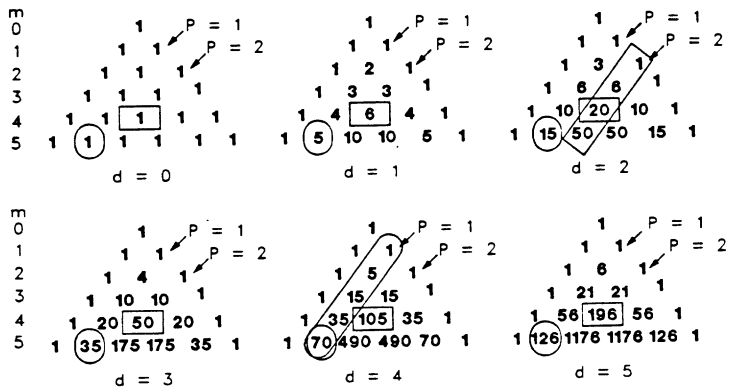

Figure 1. Partial Hoggatt Triangles, d = 0 to 5.

3. PROPERTIES OF ELEMENTARY RECURSIONS

In this section we define "elementary" recursions and develop general forms for their generating functions, homogeneous difference equations, and general sequence terms. If the order of recursion is n, the first n terms of the sequence can serve as initial conditions in a homogeneous equation or elsewhere as needed. A numerical algorithm by which the order, n, of any elementary recursion can be found is developed in the next section.

In determining the type and order of recursion in a sequence of integers, a convenient first trial is to calculate sucessive forward differences and hope to observe some unifying relation among the differences of various orders. (See Sloane [5], pp. 5-10.) The simplest of such observations occurs when the n^{th} differences are all zero. Regardless of order, these recursions are defined herein as "elementary". Such recursions are linear and index-invariant. Once the order of an elementary recursion is determined, its generating function, homogeneous difference equation, and general sequence term can be found from regular and predictable procedures.

As an illustration of the procedures, consider the sequence for $(S_d)_3$ d = 0, 1, 2, . . . (as calculated from (6)). For difference representation, Δ_d^0 replaces $(S_d)_3$. The partial triangle of differences becomes

$$\begin{array}{ccccc}
d & \Delta_d^0 & \Delta_d^1 & \Delta_d^2 & \Delta_d^3 \\
0 & 4 & & & \\
 & & 4 & & \\
1 & 8 & & 2 & \\
 & & 6 & & 0 \\
2 & 14 & & 2 & \\
 & & 8 & & 0 \\
3 & 22 & & 2 & \\
 & & 10 & & 0 \\
4 & 32 & & 2 & \\
 & & 12 & & \\
5 & 44 & & &
\end{array} \tag{9}$$

For local computations to follow, $(S_d)_3 = \Delta_d^0 = a_d$. We have $\Delta_d^3 = 0$, $\Delta_{d+1}^2 - \Delta_d^2 = 0$, $\Delta_{d+2}^1 - 2\Delta_{d+1}^1 + \Delta_d^1 = 0$, and $\Delta_{d+3}^0 - 3\Delta_{d+2}^0 + 3\Delta_{d+1}^0 - \Delta_d^0 = 0$. This leads to the homogeneous difference equation

$$a_{d+3} - 3a_{d+2} + 3a_{d+1} - a_d = 0. \tag{10}$$

The z-transform [4] of both sides of (10) yields

$$\left\{z^3\left[(a_d) - a_0\right] - z^2 a_1 - z a_2\right\} - 3\left\{z^2\left[Z(a_d) - a_0\right] - z a_1\right\} + 3\left\{z\left[Z(a_d) - a_0\right]\right\} - Z(a_d) = 0, \tag{11}$$

where a_0, a_1, a_2 are the initial conditions for (10). After factoring out $Z(a_d)$ and rearranging terms, we have the closed form of $Z(a_d)$ which is also the generating function in powers of $1/z$ of the sequence. $Z(a_d)$ becomes

$$Z(a_d) = \frac{z^3a_0+z^2(a_1\text{-}3a_0)+z(a_2\text{-}3a_1+3a_0)}{(z\text{-}1)^3} = \frac{4z^3\text{-}4z^2+2z}{(z\text{-}1)^3}. \tag{12}$$

(For those who wish a more conventional generating function form, 1/z can be replaced by x.) One way of getting a general sequence term is to find the inverse z-transform of (12) using an inversion integral [4] whose integrand is $Z(a_d)$ multiplies by z^{d-1}. The inversion integral is a contour integral in the z-plane where the contour can be taken as any counter-clockwise encirclement of the origin in the finite z-plane which completely encloses the unit circle without touching it.

$$a_d = \frac{1}{2\pi i}\oint \frac{\{z^3a_0+z^2(a_1\text{-}3a_0)+z(a_2\text{-}3a_1+3a_0)\}z^{d-1}\,dz}{(z\text{-}1)^3} \tag{13}$$

The general term, a_d, is equal to the residue of the integrand in the third order pole at $z = 1$. The integrand can be expanded in a Laurent expansion about the third order pole at $z = 1$. By dividing out he pole factor $(z\text{-}1)^3$, the Laurent expansion assumes the form shown below where the braced ({ }) portion is the numerator of the integrand with no singularities at $z = 1$. Hence the braced portion can be expanded in a Taylor's series about $z = 1$.

$$\frac{1}{(z\text{-}1)^3}\left\{\text{xxxx} + \text{xxxx}(z-1) + \text{Residue}(z-1)^2 + \text{xxxx}(z-1)^3 + \ldots\right\} \tag{14}$$

The only coefficient of interest to us in the Taylor expansion is the coefficient of $(z-1)^2$ because it is the coefficient of $1/(z-1)$ in the Laurent expansion of the integrand and, hence, the desired residue. A convenient way to avoid the differentiations of Taylor expansions is to shift plane origins by letting $(z-1) = W$, replace z by $(1+W)$, and after binomial expansion, pick the total coefficient of W^2 as the residue. The numerator of the integrand in W is

$$(1+W)^{d+2}a_0 + (1+W)^{d+1}(a_1-3a_0) + (1+W)^d(a_2-3a_1+3a_0). \tag{15}$$

The coefficient of W^2 in (15) becomes

$$a_d = \binom{d+2}{2}a_0 + \binom{d+1}{2}(a_1-3a_0) + \binom{d}{2}(a_2-3a_1+3a_0), \tag{16}$$

or, alternately,

$$a_d = \frac{(d+2)^{(2)}a_0 + (d+1)^{(2)}(a_1\text{-}3a_0) + (d)^{(2)}(a_2\text{-}3a_1+3a_0)}{2!}. \tag{17}$$

where $(d+2)^{(2)}$, for example, is the partial factorial $(d+2)(d+1)$, etc. For the numerical example, substitution leads to

$$a_d = (S_d)_3 = d^2 + 3d + 2. \tag{18}$$

Although the numerical results of (10), (12), and (18) serve as satisfying computational examples, (10) through (17) suggest what similar expressions for other specific orders and even the general order, n, might look like. For the third order case presented above, a guiding "threeness" is very evident in the expressions. As difference triangles are extended to higher orders, it becomes evident that for elementary recurisons of order n, it is Δ_d^n which first becomes zero and that n sequence tersm, a_0 through a_{n-1} are necessary to complete the triangle. The general homogeneous difference equation becomes

$$\sum_{r=1}^{n}(-1)^r\binom{n}{r}a^{d+n-r} = a^{d+n}-\binom{n}{1}a^{d+n-1} + \binom{n}{2}a^{d+n-2}+ \cdots + (-1)^n\binom{n}{n}a^d = 0. \tag{19}$$

Application of the z-transform to (19) yields the general generating function

$$Z(a_d) = \frac{\begin{array}{c} z^n a_0+z^{n-1}(a_1-\binom{n}{1}a_0)+z^{n-2}(a_2-\binom{n}{1}a_1+\binom{n}{2}a_0) \\ + \cdots +z\left\{a_{n-1}-\binom{n}{1}a_{n-2}+ \cdots +(-1)^{n-1}\binom{n}{n-1}a_0\right\} \end{array}}{(z-1)^n} \tag{20}$$

Applying the inversion integral to (20), there results two equivalent forms of the general sequence term for elementary recursions of order n,

$$a_d=\left[\begin{array}{c} \binom{d+n-1}{n-1}a_0 + \binom{d+n-2}{n-1}\left\{a_1-\binom{n}{1}a_0\right\} + \binom{d+n-3}{n-1}\left\{a_2-\binom{n}{1}a_1 + \binom{n}{2}a_0\right\} \\ +\cdots+ \binom{d}{n-1}\left\{a_{n-1} - \binom{n}{1}a_{n-2} + \cdots + (-1)^{n-1}\binom{n}{n-1}a_0\right\} \end{array}\right], \tag{21}$$

$$a_d = \frac{\left[\begin{array}{c} (d+n-1)^{(n-1)}a_0 + (d+n-2)^{(n-1)}\left\{a_1-\binom{n}{1}a_0\right\} + \binom{d+n-3}{n-1}\left\{a_2 - \binom{n}{1}a_1 + \binom{n}{2}a_0\right\} \\ +\ldots+(d)^{(n-1)}\left\{a_{n-1} - \binom{n}{1}a_{n-2} + \ldots +(-1)^{n-1}\binom{n}{n-1}a_0\right\} \end{array}\right]}{(n-1)!} \tag{22}$$

Note that the example solutions are merely special cases of the above for n = 3. If n and a_0 through a_{n-1} of an elementary recursion are known, it is now possible to write the homogeneous difference equation, the generating function, and the general sequence term almost by inspection.

4. ALGORITHM FOR FINDING ORDER, n

The expressions for homogeneous difference equations, generating functions, and general sequence terms derived above, are worthless unless n and the first n sequence terms, a_0 through a_{n-1} are known. However, the primary purpose of this paper is to investigate the above properties in two types of elementary recursive sequences from Hoggatt sums and triangles in which n is not known. On the other hand, because of (6) and (7), all sequence terms, a_0 through a_{n-1} and beyond, are known for any n whatever. All that is needed is the m specification in (6) and the m, p specification in (7). If we hope to generalize n as a function of m (or m and p) through computer simulation, we must obtain n for a sufficiently large sampling involving m (or m and p).

Even though a recursion is known to be elementary, the difference triangle approach can become overwhelmingly tedious for large n and is not particularly suitable for machine computation. As a practical algorithm for finding n, consider the generating function (20) for any order n. When expanded, (20) is the open form of the z-transform, and infinite sequence in powers of 1/z. This sequence can be found by dividing $(z-1)^n$ into the numerator of (20). Since the first n sequence coefficients, a_0 through a_{n-1} are presumed known for any chosen n, the quotient can be expressed as

$$a_0 + a_1/z + a_2/z^2 + \cdots + a_{n-1}/z^{n-1} + (-1)^{n-1} + \left\{a_0 - \binom{n}{1}a_1 + \cdots + (-1)^{n-1}\binom{n}{n-1}a_{n-1}\right\}/z^n + \cdots \tag{23}$$

The coefficient, a_n, of $1/z^n$ (in braces { }) is the first coefficient to depend on all n coefficients, a_0 through a_{n-1}. Because of recursion, all coefficients beyond and including a_n thereby depend on a_0 through a_{n-1}. For a_n in particular, we have

$$a_n = (-1)^{n-1}\left\{a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 + \ldots + (-1)^{n-1}\binom{n}{n-1}a_{n-1}\right\}. \tag{24}$$

If a_n is subtracted from both sides of (24), the result, <u>regardless of $(-1)^{n-1}$</u>, is

$$\left\{a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 + \cdots + (-1)^n a_n\right\} = 0. \tag{25}$$

For our two types of sequences, the value of n is not known but as many of a_0, a_1, a_2, $\cdots$ as are needed are available. Start with n=1 and test (25) for { } = 0. If { } $\neq$ 0, then try successively larger trial n's. The smallest n which satisfies (25) is the order, n, of the elementary recursion. It can be shown that any larger trial n also satisfies (25), so that a wide variety of bracketing steps could be employed to find n. In a simple illustration of the algorithm as used herein, sequence terms with the sign of every odd-indexed term reversed are multiplied by binomial coefficeints corresponding to a test n. The products are added and the sum tested for zero. If the sum is not zero, n is incremented and the process repeated as required using the next order of binomial coefficeints. An example of a test which results in the recursion order seven for the sequence for $(S_d)_5$ is shown in (26).

	n=5		n=6		n=7
$+a_0$	6x 1= 6	$+a_0$	6x 1= 6	$+a_0$	6x 1= 6
$-a_1$	−32x 5= −160	$-a_1$	−32x 6= −192	$-a_1$	−32x 7= −224
$+a_2$	132x10= 1320	$+a_2$	132x15= 1980	$+a_2$	132x21= 2772
$-a_3$	−422x10=−4220	$-a_3$	−422x20= −8440	$-a_3$	−422x35=−14770
$+a_4$	1122x 5= 5610	$+a_4$	1122x15= 16830	$+a_4$	1122x35= 39270
$-a_5$	−2606x 1=−2606	$-a_5$	−2606x 6=−15636	$-a_5$	−2606x21=−54726
	TOTAL= −50	$+a_6$	5462x 1= 5462	$+a_6$	5462x 7= 38234
			TOTAL= 10	$-a_7$	−10562x 1=−10562
					TOTAL= 0

(26)

5. HOGGATT SEQUENCES OF THE SECOND KIND

Since the original Hoggatt sums were sequences of $(S_d)_m$ with m as the variable index and d fixed, it seemed logical to name sequences of $(S_d)_m$ in which d is the variable index and m is fixed, Hoggatt sequences of the second kind. The second kind sequences are much easier to work with since the recursion is elementary. Examples of several second kind sequences calculated using (5) or (6), are listed in (27). Shown also are the values of recursion order, n, versus m. The values of n were found using members of the second kind sequences and the order algorithm discussed in the previous section.

d	$(S_d)_0$	$(S_d)_1$	$(S_d)_2$	$(S_d)_3$	$(S_d)_4$	$(S_d)_5$	$(S_d)_6$	$(S_d)_7$
0	1	2	3	4	5	6	7	8
1	1	2	4	8	16	32	64	128
2	1	2	5	14	42	132	429	1430
3	1	2	6	22	92	422	2074	10754 (27)
4	1	2	7	32	177	1122	7898	60398
5	1	2	8	44	310	4606	25202	272582
6	1	2	9	58	506	5462	70226	1038578
7	1	2	10	74	782	10562	175826	3457742
8	1	2	11	92	1157	19142	403691	10312304
9	1	2	12	112	1652	32892	862864	28066040
10	1	2	13	134	2290	54056	1736737	70702634
m	0	1	2	3	4	5	6	7
n	1	1	2	3	5	7	10	13

It is suggested by (27) that n is a function of m and, except for m = 0, the order is never less than m. This makes m a good first candidate for a trial n in the order algorithm. However, before any explicit relationships can be found, more comparisons must be observed. Through extensive use of muMath, the order algorithm, and values generated by (5) or (6), the following tabulation was found:

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n	1	1	2	3	5	7	10	13	17	21	26	31	37	43	50	57	65	73	82	91

(28)

The repeating differences observed between n values in the "order" sequence (28) suggest that two sequences exist, one for m odd and one for m even.

For m odd, let h = (m+1)/2 so that h varies as 1, 2, 3, ... as m takes on 1, 3, 5, ... The order calculation is so simple that the difference triangle method quickly shows $\Delta_h^3 = 0$ and thereby indicates an elementary third order recursion for the sequences of n for m odd. By observing the difference behavior, the missing "a_0" can be found and included. Now, by using $a_0 = 1$, $a_1 = 1$, $a_2 = 3$, and in (22) replacing "n" by three and d by h, the general sequence term for the odd m sequence appears as

$$\frac{(h+2)^{(2)} - 2(h+1)^{(2)} + 3(h)^{(2)}}{2!} = h^2 - h + 1. \tag{29}$$

When h is replaced by $(m+1)/2$, the general order value, n, for second kind sequences for odd m becomes

$$n = (m^2+3)/4. \tag{30}$$

A repeat of the process using m even and h $-$ m/2 yields

$$n = (m^2+4)/4. \tag{31}$$

By combining (30) and (31), the recursion order, n, for any m for second kind sequences of $(S_d)_m$ becomes

$$n = \frac{m^2 + 3.5 + (-1)^m(0.5)}{}. \tag{32}$$

Now, for any specified m, the exact number, n, of sequence terms, $(S_0)_m$, $(S_1)_m$, ... , $(S_{n-1})_m$, can be calculated from (5) or (6) to serve as initial constants for an almost at sight development of homogeneous difference equations [see (19)], generating functions in 1/z [see (20)], and general sequence term values [see (21) or (22)].

6. HOGGATT POSITION SEQUENCES

Expression (7) determines the value of the integer in the pth position of the mth row of a dth order Hoggatt triangle and assigns the ordered functional designation, P(m, p, d). The sequences formed by P(m, p, d) values with m and p fixed and variable d = 0, 1, 2, 3, ... are called Hoggatt position sequences. Fortunately, they exhibit elementary recursion so that the methods of previous sections can be used to again obtain general expressions for recursion order, n, homogeneous difference equations, generating functions, and sequence terms. As before, the ultimate goal is the ability to write the expressions almost by inspection once n and the first n sequence terms are available.

As illustrations, portions of position sequences can be observed in Figure 1. If m = 2, p = 2, the integers of sequence 1, 6, 20, 50, 105, 196, ... are set off by rectangular enclosures, while for m = 5, p = 1 oval enclosures are used to enclose the sequence 1, 5, 15, 20, 70, 126, ... While the definition of position sequences is global with respect to d, an interesting feature is that any particular position sequence has an equal, matching diagonal sequence completely contained in some Hoggatt triangle. This property is identified in Figure 1 for the above position sequences for the diagonal sequence in triangle two by a rectangular enclosure and for triangle four by an oval enclosure. These phenomena have the practical aspect of making two observational and calculation approaches available.

A position sequence term can be distinguished from a diagonal sequence term by choosing $P(m_n, P_h, d)$ for the position term and $P(m, p_d, d_d)$ for the diagonal term where m_h, p_h, p_d and d_d are constants. Because of symmetry of Hoggatt triangles about a central, "vertical" axis [1], we restrict our position sequence computations to $0 \le p \le m/2$ for m even and to $0 \le p \le (m-1)/2$ for m odd. The matching diagonal sequences proceed from upper right to lower left.

At the integer of intersection of a position sequence with its matching equal diagonal sequence, $m = m_h$, $d = d_d$, and $p_h = p_d$. Since p_h and p_d are constants, they can be replaced by a general constant, p. Also the integer of intersection must be the d_d term of each sequence since the intersection takes place in the d_dth triangle. Counting back along the diagonal sequence from the intersection integer determines the starting m of the diagonal sequence as $m_h - d_d$. Since the diagonal sequence must start on the extreme right of the d_dth triangle, the m value for that row of the d_dth triangle fixes p as

$$p = m_h - d_d, \tag{33}$$

with p also the starting value of m of the diagonal sequence. The order of the triangle in which intersection takes place is, thereby,

$$d_d = m_h - p, \tag{34}$$

where m_h and p are established position sequence constants. Correspondingly, if a diagonal sequence is already specified, the m_h of its matching position sequence is given as

$$m_h = d_d + p. \tag{35}$$

With release of the position restriction and with p taken as $m/2 \leq p \leq m$ for m even, or $(m-1)/2 \leq p \leq m$ for m odd, it is seen that the matching diagonal sequences start on the upper left and proceed to the lower right.

As an illustration of (35), the diagonal shown on Figure 1 for d = 4 starting with 1, 35, 490, ... has p = 3, d_d = 4 which results in m_h = 7. The matching position sequence representation is P(7, 3, d) which is beyond the limits of Figure 1. However, calculations made using (7) verify the first few terms of the position sequence as 1, 35, 490, 4116, 24696, 116424, 457380, ...

The remaining study of position sequences is devoted to finding the general expression for recursion order, n, as a function of m and p. Through use of the order algorithm, the orders of position sequences P(m, p, d) were calculated in m increments from zero through nine with p subincrements (for each m) from zero through m/2 for m even and through (m−1)/2 for m odd. The full range of p was not needed because of Hoggatt triangle symmetry. The values of recursion order, n, for the ranges of m and p are given in (36).

p \ m	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2			1	3	5	7	9	11	13	15
3				1	4	7	10	13	16	19
4					1	5	9	13	17	21
5						1	6	11	16	21
6							1	7	13	19
7								1	8	15

← n values (36)

Each of the sequences of n values with m assuming the rôle of index is seen to have an elementary recursion of order two, with the first difference identically p regardless of m. The "a_0, a_1" first two terms are always 1 and 1 + p. However, the starting index for each sequence changes with m which is not consistent with our methods. To force a starting index of zero for each sequence, the index must be adjusted to m − p. When this is done, replacement of d by m − p, replacement of "n" by 2, substitution of 1 and 1 + p for a_0 and a_1 in (22) yields the general n as

$$n = (m-p+1)a_0 + (m-p)(a_1-2a_0) = (m-p+1) + (m-p)(p-1) = (m-p)p + 1. \tag{37}$$

Thus the order of any Hoggatt position sequence P(m, p, d) can be specified directly in terms of its fixed m and p parameters. To further aid in visual identification, (37) is arranged in (38) as an "order" triangle, allowing the order, n, of a position sequence, P(m, p, d), to be selected at sight from the mth row, pth position integer.

$$\begin{array}{l|ccccccccccccccccccc}
m & & & & & & & & & & & & & & & & & & & \\
0 & & & & & & & & & & 1 & & & & & & & & & \\
1 & & & & & & & & & 1 & & 1 & & & & & & & & \\
2 & & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
3 & & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
4 & & & & & & 1 & & 4 & & 5 & & 4 & & 1 & & & & & \\
5 & & & & & 1 & & 5 & & 7 & & 7 & & 5 & & 1 & & & & \\
6 & & & & 1 & & 6 & & 9 & & 10 & & 9 & & 6 & & 1 & & & \\
7 & & & 1 & & 7 & & 11 & & 13 & & 13 & & 11 & & 7 & & 1 & & \\
8 & & 1 & & 8 & & 13 & & 16 & & 17 & & 16 & & 13 & & 8 & & 1 & \\
9 & 1 & & 9 & & 15 & & 19 & & 21 & & 21 & & 19 & & 15 & & 9 & & 1 \; .
\end{array} \tag{38}$$

Now that the recursion order for Hoggatt position sequences is established as (m−p)p + 1, it is possible to state a set of initial conditions, the homogeneous difference equation, the generating function in 1/z, and the general sequence term for any Hoggatt position sequence.

The Hoggatt position sequences seem to have some value in counting specialized distributions of objects. As a parting, proverbial "exercise for the reader," deduce the P(m, p, d) position sequence for which the a_r [or P(m, d, r)] term answers the following:

In how many ways can r indistinct objects be distributed in six distinct cells with five of the cells permitted any occupancy (including none) but with the sixth cell permitted either no occupancy or single occupancy?

7. CONCLUSIONS

We have introduced two new classes of sequences, Hoggatt second kind sequences and Hoggatt position sequences. Even though we have classified the sequences as having the most elementary form of recursion, each sequence has exhibited an interesting behavior. An order algorithm developed for use with the sequences provided data for completing the specifications of homogeneous difference equations, generating functions, and formulas for general terms for the sequences.

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REPRESENTATION OF NATURAL NUMBERS AS SUMS OF FIBONACCI NUMBERS: AN APPLICATION TO MODERN CRYPTOGRAPHY

Piero Filipponi and Emilio Montolivo

1. INTRODUCTION

After giving some basic concepts of a cryptographic technique referred to as *Stream Ciphering* [7] (Sec. 2) we have a look at the translation of natural numbers into Fibonacci binary sequences (Sec. 3) and show that no two of the 1's occurring in these sequences are adjacent.

The probability of occurrence of the symbol 1 in a predetermined position of a NAO-*sequence* (Non Adjacent Ones sequence) plays an important role in the procedure for generating ciphering sequences proposed in this paper. An explicit formula for such a probability is established in Sec. 4.

In Sec. 5 we show in detail how a fairly satisfactory ciphering sequence $\mathfrak{C}$ can be generated by combining Fibonacci binary sequences obtained by the F-*addend representation* [2] [3] of random positive integers. In Sec. 5.1 the cryptographic robustness of the sequence $\mathfrak{C}$ is evaluated partly theoretically and partly with the aid of a computer experiment.

2. STREAM CIPHERING: BASIC CONCEPTS

Stream ciphering [7] is a cryptographic technique used for encryption and decryption of a *numerical* message P (plaintext). If the sender S wants to send the alphabetical message A to the receiver R, he first translates A into the binary message $P = \{p_1, p_2, \dots p_L\}$ using a conversion table (common knowledge) an example of which is given in table 1.

	= 00000	P	= 10000
A	= 00001	Q	= 10001
B	= 00010	R	= 10010
C	= 00011	S	= 10011
D	= 00100	T	= 10100
E	= 00101	U	= 10101
F	= 00110	V	= 10110
G	= 00111	W	= 10111
H	= 01000	X	= 11000
I	= 01001	Y	= 11001
J	= 01010	Z	= 11010
K	= 01011	.	= 11011
L	= 01100	,	= 11100
M	= 01101	:	= 11101
N	= 01110	?	= 11110
O	= 01111	!	= 11111

Table 1 - An example of an alphabetical - numerical conversion table.

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 89–99.

Then he generates a binary sequence $C = \{c_1, c_2, \ldots, c_L\}$ (ciphering sequence) using a certain ***algorithm*** (which can be common knowledge) governed by a ***secret key*** K (known only to S and R), performs modulo two the bitwise addition $P \oplus C = X = \{x_1. x_2, \ldots, x_L\}$ (i.e., $x_i = 0$ (1) if $p_i = c_i$ ($p_i \neq c_i$)) and sends the ciphertext X to R.

The receiver R generates C using K, performs $C \oplus X = P$ and recovers A from P using the conversion table.

According to table 1, a small example of stream ciphering technique is given in the following, where the ciphering sequence C has been obtained by tossing a coin:

$A = \ldots$	F	I	B	O	N	A	C	C	I ...
$P = \ldots$	00110	01001	00010	01111	01110	00001	00011	00011	01001...
$C = \ldots$	00101	01110	10110	01100	00011	10001	01001	01000	11100...
$X = \ldots$	00011	00111	10100	00011	01101	10000	01010	01011	10101...
	C	G	T	C	M	P	J	K	U

It should be noted that any unauthorized intruder who intercepts the transmitted ciphertext X along the transmission channel and uses the above conversion table obtains nothing but the nonsense CGTCMPJKU. We point out that, in general, the transformations $A \to P$ and $P \to A$ are performed automatically by the transmitting and receiving devices, respectively.

Some other words must be put in for the ciphering sequence C. This sequence must be ***cryptographically strong***, that is it must at least *look* random, even if upon closer inspection it can be shown to have certain regularities. Since C must be generated again by the receiver R, it must be deterministic. So, since it must be generated by a finite state sequential machine, it is periodic (possibly furnished with an anti-period). No periodic sequence is ever truly random ***stricto sensu.*** The best that can be done is to single out certain properties as being associated with randomness, and to accept C as a random sequence if it has these properties.

The following properties are associated with randomness (Golomb's randomness postulates [5]):

(*i*) 1. The number of 1's approximately equal to the number of 0's.

2. The number of any possible (0,1) n-tuple is approximately equal to $\Lambda/2^n$, Λ being the length of C.

3. The auto-correlation function $\Gamma(\tau)$ is two-valued. Explicitly,

$$\Gamma(\tau) = \sum_{i=1}^{\Lambda} c_i c_{i+\tau} = \begin{cases} \Lambda/2 \text{ if } \tau = 0 \\ \Lambda/4 \text{ if } 0 < \tau < \Lambda , \end{cases}$$

where the subscript $i + \tau$ must be considered as reduced modulo Λ whenever it is greater than Λ.

Besides (*i*), *C* must have the following further properties:

(*ii*) The period of repetition Λ of *C* must be at least equal to the length *L* of the message *P*, even though a period $\Lambda \gg L$ is usually required.

(*iii*) Recovering the secret key *K* from the knowledge of a portion C' of *C* must be either impossible or not computationally feasible. It must be noted that $C' = P' \oplus X'$ can be immediately obtained by an opponent cryptanalyst who enters into possession of portions of the plaintext (P') and the ciphertext (X') (*plaintext attack* [7]).

3. TRANSFORMING AN INTEGER INTO A FIBONACCI BINARY SEQUENCE

It is known [2] that a natural number *M* can be represented uniquely as a sum of F-*addends* (distinct nonconsecutive Fibonacci numbers)

$$M = \sum_{j=1}^{r} F_{m_j}, \begin{cases} m_1 \geq 2 \\ m_1 \leq m_2 - 2 \leq \cdots \leq m_r - 2 \end{cases} \tag{3.1}$$

where $r = f(M)$ ($f(x)$ being defined [3] as the number of *F*-addends in the Zeckendorf decomposition of any natural number x) and m_r is the subscript of the greatest Fibonacci number not exceeding *M*. Considering an arbitrary natural number $\lambda \geq m_r - 1$, (3.1) can be rewritten as

$$M = \sum_{i=2}^{\lambda} a_i F_i \tag{3.2}$$

where

$$a_i = \begin{cases} 1, \text{ if } i \in \{m_1, m_2, \ldots, m_r\} \\ 0, \text{ otherwise.} \end{cases} \tag{3.3}$$

By (3.2) and (3.3) it is evident (e.g., see [1]) that *M* can be represented uniquely by the binary sequence

$$S_L(M) = \{a_\lambda, a_{\lambda-1}, \ldots, a_3, a_2\} = \{s_1, s_2, \ldots, s_L\} \tag{3.4}$$

of length $L = \lambda - 1$. For example, letting $\lambda = 14$, the integer $M = 138 = F_3 + F_7 + F_9 + F_{11}$ is represented by the binary sequence

$$S_{13}(138) = \{ \underset{F_{14}}{0}, \underset{F_{13}}{0}, \underset{F_{12}}{0}, \underset{F_{11}}{1}, \underset{F_{10}}{0}, \underset{F_9}{1}, \underset{F_8}{0}, \underset{F_7}{1}, \underset{F_6}{0}, \underset{F_5}{0}, \underset{F_4}{0}, \underset{F_3}{1}, \underset{F_2}{0} \}.$$

By (3.2) and (3.3) it follows immediately (e.g., see [9, Ch. 1.1]) that the sequences $S_L(M)$ never contain adjacent 1's. In the following such sequences will be referred to as NAO-sequences (Non Adjacent Ones sequences).

4. EVALUATION OF THE PROBABILITY OF THE OCCURRENCE OF THE SYMBOL 1 IN THE k^{th} POSITION OF A NAO-SEQUENCE.

Let $S_{L,m}$ be a NAO-sequence of given length L containing exactly m 1's ($L-m$ 0's). It is known [3] that the number $N_{L,m}$ of distinct $S_{L,m}$ is given by

$$N_{L,m} = \begin{cases} \binom{L-m+1}{m}, \text{ if } 0 \le m \le [(L+1)/2] \\ 0, \text{ if } m > [(L+1)/2] \end{cases} \tag{4.1}$$

where $[x]$ denotes the greatest integer not exceeding x.

Let $N_{L,m}(k)$ be the number of sequences $S_{L,m}$ having 1 in the k^{th} position. In [4] we proved that

$$N_{L,m}(k) = N_{L,m}(L-k+1) \tag{4.2}$$

and

$$N_{L,m}(k) = \sum_{i=0}^{k-1} (-1)^i \binom{L-m-i}{m-1-i}, \ (1 \le k \le [(L+1)/2]) \tag{4.3}$$

or, under the same restriction,

$$N_{L,m}(k) = \frac{1-(-1)^k}{2} \binom{L-m-k+1}{m-k} + \sum_{i=0}^{[\frac{k-2}{2}]} \binom{L-m-2i-1}{m-2i-1}. \tag{4.4}$$

On the basis of the assumption

$$\binom{a}{-|b|} = 0 \tag{4.5}$$

[8], it appears clearly that the addends of the sum (4.3) vanish for $i > m-1$. If $k > m$ such a condition is verified for $m \le i < k-1$. This fact, together with the symmetry relation (4.2) allows us to state the following.

<u>Theorem 1:</u> $N_{L,m}(k) = \text{const.} = N_{L,m}(m)$ for $m < k < L-m+1$.

The quantity $N_{L,m}(k)$ can be obtained using *ad libitum* either (4.3) or (4.4). As particular cases we have

$$N_{L,0}(k) = 0, \ \forall\ k \tag{4.6}$$

$$N_{L,1}(k) = 1, \ \forall\ k \tag{4.7}$$

$$N_{L,2}(k) = \begin{cases} L-2, \text{ if } k = 1 \text{ or } L \\ L-3, \text{ otherwise} \end{cases} \tag{4.8}$$

and (cf. (4.4))

$$\mathrm{N}_{\mathrm{L},m}(\mathrm{m}) = \frac{1-(-1)^m}{2} + \sum_{i=0}^{[\frac{m-2}{2}]} \binom{\mathrm{L-m-2i-1}}{\mathrm{L-2m}} \tag{4.9}$$

whence

$$\mathrm{N}_{2m\text{-}1,m}(\mathrm{m}) = \begin{cases} 1, \text{ if m is odd} \\ 0, \text{ if m is even} \end{cases} \tag{4.10}$$

$$\mathrm{N}_{2m,m}(\mathrm{m}) = [(\mathrm{m}+1)/2] \tag{4.11}$$

$$\mathrm{N}_{2m+1,m}(\mathrm{m}) = \begin{cases} (\mathrm{m}+1)^2/4, \text{ if m is odd} \\ \mathrm{m}(\mathrm{m}+2)/4, \text{ if m is even.} \end{cases} \tag{4.12}$$

The probability $p_{\mathrm{L},m}(k)$ of the occurrence of the symbol 1 in the k^{th} position of a NAO-sequence of length L with m 1's is given clearly by (cf. (4.3), (4.1) and (4.2))

$$p_{\mathrm{L},m}(k) = \mathrm{N}_{\mathrm{L},m}(k)/\mathrm{N}_{\mathrm{L},m}\ . \tag{4.13}$$

As particular cases we have

$$p_{\mathrm{L},m}(1) = p_{\mathrm{L},m}(\mathrm{L}) = m/(\mathrm{L}-m-1) \tag{4.14}$$

$$p_{\mathrm{L},m}(2) = p_{\mathrm{L},m}(\mathrm{L}-1) = m(\mathrm{L}-2m+1)/((\mathrm{L}-m+1)(\mathrm{L}-m)) \tag{4.15}$$

$$\mathrm{P}_{2m\text{-}1,m}(\mathrm{m}) = \begin{cases} 1, \text{ if m is odd} \\ 0, \text{ if m is even} \end{cases} \tag{4.16}$$

$$\mathrm{P}_{2m,m}(\mathrm{m}) = \begin{cases} 1/2, \text{ if m is odd} \\ \mathrm{m}/(2\mathrm{m}+2), \text{ if m is even} \end{cases} \tag{4.17}$$

$$\mathrm{P}_{2m+1,m}(\mathrm{m}) = \begin{cases} (\mathrm{m}+1)/(2\mathrm{m}+4), \text{ if m is odd} \\ \mathrm{m}/(2\mathrm{m}+2), \text{ if m is even.} \end{cases} \tag{4.18}$$

The characteristic behavior of $p_{\mathrm{L},m}(k)$ against k is shown in figure 1 (L=30,m=9) where the *central portion* (broken line) of the curve shows the interval where $p_{\mathrm{L},m}(k)$ is constant (independent of k).

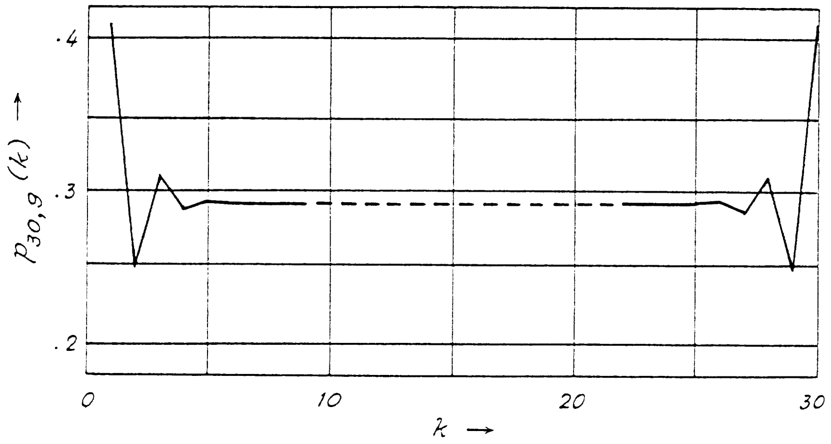

Figure 1 - Behavior of $p_{L,m}(k)$ against k for L = 30 and $m = 9$.

5. CIPHERING SEQUENCE GENERATION VIA THE F-ADDEND DECOMPOSITION OF RANDOM INTEGERS

We propose the following procedure to generate a fairly satisfactory ciphering sequence C. On the basis of the pair (K_1,K_2) of triples of integers (the secret key)

$$\begin{cases} K_1 = (a_n,\ c_n,\ \mu_n) \\ K_2 = (a_m,\ c_m,\ \mu_m)\ , \end{cases} \tag{5.1}$$

and the pair of integers $(N_0,\ M_0)$(starting values) satisfying

$$\begin{cases} \mu_n,\mu_m \text{ primes of the same order of magnitude} \\ 0 < N_0,\ a_n,\ c_n < \mu_n \\ 0 < M_0,\ a_m,\ c_m < \mu_m\ , \end{cases} \tag{5.2}$$

the *pseudorandom* integral sequences

$$\begin{cases} \mathcal{N} = \{\ N_1,N_2,\ldots,N_H\ \} \\ \mathcal{M} = \{\ M_1,M_2,\ldots,M_H\ \} \end{cases} \tag{5.3}$$

are obtained by means of the algorithm (see [6])

$$\begin{cases} N_{h+1} = (a_n N_h + c_n) \bmod \mu_n \ , \ 0 \le h \le H-1 \\ M_{h+1} = (a_m M_h + c_m) \bmod \mu_m \ , \ 0 \le h \le H-1 \ , \end{cases} \tag{5.4}$$

where H is a positive integer depending on the length of the plaintext P. The integers N_i and M_i ($i = 1,2,\ldots,H$) are transformed into the Fibonacci binary sequences (see Sec. 3)

$$\begin{cases} S_L(N_i) = \{ n_1, n_2, \ldots, n_L \} \\ S_L(M_i) = \{ m_1, m_2, \ldots, m_L \} \end{cases}$$

of length L (see (5.8)). Then, we perform the bitwise *logical sum* (OR or +) of their central portions (where the probability of occurrence of the symbol 1 is constant) thus obtaining the binary subsequence C_i of length t

$$C_i = \{ c_1, c_2, \ldots, c_t \} \ , \tag{5.5}$$

where $t = L - 2U_L + 2$ (see (5.9) for the definition of U_L) and

$$c_j = n_k + m_k \ (j = 1, 2, \ldots, t \, ; \ k = U_L + j - 1). \tag{5.6}$$

The ciphering sequence $\mathfrak{C}$ of length Ht is obtained by juxtaposing the various subsequences C_i

$$\mathfrak{C} = \{ C_1, C_2, \ldots, C_H \} \ . \tag{5.7}$$

5.1. EVALUATION OF THE CRYPTOGRAPHIC ROBUSTNESS OF THE CIPHERING SEQUENCE $\mathfrak{C}$

The order of magnitude of μ_n, μ_m and H must warrant that $\mathfrak{C}$ has the property (*ii*) (see Sec. 2) (we recall that, if μ_n and μ_m are primes, then the period of repetition of $\mathcal{N}$ and $\mathcal{M}$ is μ_n and μ_m, respectively [6]).

The nonlinear operation (5.6) ensures that the property (*iii*) (see Sec. 2) is adequately satisfied.

5.1.1. EVALUATION OF THE RANDOMNESS PROPERTIES OF $\mathfrak{C}$: THEORETICAL CONSIDERATIONS

Let us show how the property (*i*1) (1^{st} Golomb's postulate) is satisfied. If n is the subscript of the greatest Fibonacci number not exceeding N_i, then the length L of the shortest binary sequence $S_L(N_i)$ which can represent N_i is $L = n - 1$ (cf. (3.4)). It can be readily proved that

$$L = [\log_\alpha ((N_i + 1/2) \sqrt{5})] - 1 \tag{5.8}$$

where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio. In [3] it has been proved that the most probable number U_L of 1's in $S_L(N_i)$ is given by

$$U_L = \left[\frac{5(L + 2) - 8 - (5(L + 2)^2 + 4)^{1/2}}{10} \right] + 1 \,. \tag{5.9}$$

The probability $p(1) = p_{L,U_L}(k)$ of the occurrence of the symbol 1 in the central positions of $S_L(N_i)$ is given by (cf. Theorem 1, (4.13), (4.9) and (4.1))

$$p(1) = p_{L,U_L}(U_L) = N_{L,U_L}(U_L)/\, N_{L,U_L} \,. \tag{5.10}$$

The same argument holds for $S_L(M_i)$.

The behavior of $p(1)$ against L is shown in figure 2. As expected, [3], such a quantity tends to $1/(\alpha + 2)$ as L tends to infinity. Moreover, $p(1)$ appears to be sufficiently close to this limit for comparatively small values of L (say, $L \geq 25$). Therefore, for $L \geq 25$, we can consider

$$p(1) \approx 1/(\alpha + 2) \approx 0.276. \tag{5.11}$$

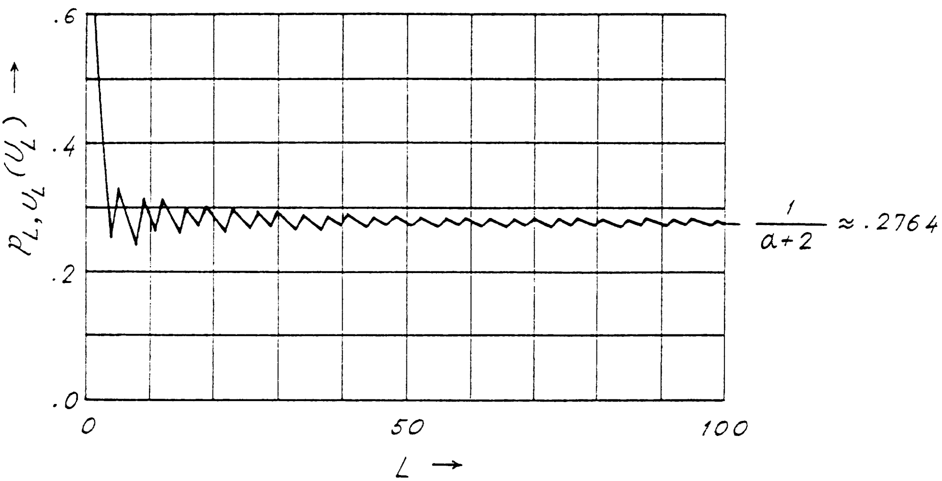

Figure 2 - Behavior of $p_{L,U_L}(U_L)$ against L.

The probability $P(0)$ of the occurrence of the symbol 0 in the ciphering sequence $\mathfrak{C}$ (simultaneous occurrence of 0 in the same position of both $S_L(N_i)$ and $S_L(M_i)$) is clearly given by

$$P(0) = p^2(0) \approx \alpha^2/5 \approx 0.524 \,, \tag{5.12}$$

where

$$p(0) = 1 - p(1) \approx (\alpha+1)/(\alpha+2) \approx 0.724 \tag{5.13}$$

is the probability of the occurrence of the symbol 0 in the central portion of both $S_L(N_i)$ and $S_L(M_i)$. Therefore, the probability $P(1)$ of the occurrence of the symbol 1 in $\mathfrak{C}$ is given by

$$P(1) = 1 - P(0) = 1 - p^2(0) \approx (5 - \alpha^2)/5 \approx 0.476. \tag{5.14}$$

From (5.12) and (5.14) it follows that, for $L \geq 25$, the property (*i*1) is adequately satisfied.

The property (*i*2) (2^{nd} Golomb's postulate) does not appear to be so well satisfied. In this context, we shall evaluate only the probabilities $P(00)$, $P(01)$, $P(10)$ and $P(11)$ of the occurrence of any possible binary *pair* in $\mathfrak{C}$.

First, we consider the probabilities $p(00)$, $p(01)$, and $p(10)$ of the occurrence of any possible pair in the central portion of both $S_L(N_i)$ and $S_L(M_i)$ and observe that, since the pair (11) does not exist, the symbol 1 is necessarily preceded and followed by the symbol 0. Therefore, we have

$$p(01) = p(10) = p(1) \approx 1/(\alpha + 2) \tag{5.15}$$

and

$$p(00) = 1 - (p(10) + p(01)) = 1 - 2p(1) \approx \alpha/(\alpha + 2). \tag{5.16}$$

Then, considering the pairs in $\mathfrak{C}$, we note that

(a) (00) can be obtained by the bitwise logical sum (00)+(00) (simultaneous occurrence of (00) in both $S_L(N_i)$ and $S_L(M_i)$) ,
(b) (01) can be obtained by either (01)+(00) or (00)+(01) or (01)+(01),
(c) (10) can be obtained by either (10)+(00) or (00)+(10) or (10)+(10),
(d) (11) can be obtained by either (10)+(01) or (01)+(10).

Finally, from (a)-(d), (5.15) and (5.16) we get

$$P(00) = p^2(00) = 1/5 = 0.2 \tag{5.17}$$

$$P(01) = P(10) = 2\,p(00)\,p(01) + p^2(01) \approx \alpha/5 \approx 0.324 \tag{5.18}$$

$$P(11) = 2\,p^2(01) \approx 2/(5\alpha^2) \approx 0.152\,. \tag{5.19}$$

5.1.2 COMPUTER SIMULATION

A computer experiment has been carried out to obtain a statistical evaluation of the properties (*i*2) and (*i*3) (2^{nd} and 3^{rd} Golomb's postulates, respectively) and to check the exactness of the theoretical results presented in Sec. 5.1.1.

More precisely, on the basis of the following parameter values

$$\begin{cases} N_0 = 5{,}123{,}123{,}007 \\ M_0 = 4{,}901{,}976{,}445\,, \end{cases}$$

$$K_1 = \begin{cases} a_n = 29 \\ c_n = 4{,}713{,}321{,}103 \\ \mu_n = 4{,}500{,}000{,}013 \ , \end{cases}$$

$$K_2 = \begin{cases} a_m = 31 \\ c_m = 5{,}666{,}778{,}007 \\ \mu_m = 5{,}127{,}312{,}451 \ , \end{cases}$$

$$H = 1000 \ ,$$

a binary sequence $\mathfrak{C}$ has been generated using the algorithm developed in Sec. 5. From the order of magnitude of μ_n and μ_m, it follows that

$$L = 47, \ U_L = 13, \ t = 23.$$

So, the length of $\mathfrak{C}$ is $\Lambda = Ht = 23000$.

1^{st} Experiment (2^{nd} Golomb's postulate)

The *a posteriori* probability of the occurence of the n-tuples has been found for $1 \leq n \leq 15$ by counting the number of times they occur in $\mathfrak{C}$ and dividing this number by Λ. The results are presented in table 2 for $1 \leq n \leq 4$.

n = 1	prob.
(0)	0.525
(1)	0.475

n = 2	prob.
(00)	0.201
(01)	0.322
(10)	0.322
(11)	0.155

n = 3	prob.
(000)	0.078
(001)	0.124
(010)	0.225
(011)	0.096
(100)	0.124
(101)	0.197
(110)	0.096
(111)	0.060

n = 4	prob.	n = 4	prob.
(0000)	0.030	(1000)	0.047
(0001)	0.047	(1001)	0.076
(0010)	0.089	(1010)	0.136
(0011)	0.036	(1011)	0.060
(0100)	0.086	(1100)	0.038
(0101)	0.138	(1101)	0.058
(0110)	0.059	(1110)	0.037
(0111)	0.037	(1111)	0.023

Table 2.- Probability of the occurrence of the binary n-tuples in $\mathfrak{C}$ ($1 \leq n \leq 4$).

The experimental values obtained for $n=1$ and $n=2$ show that (cf. (5.12), (5.14) and (5.17)-(5.19)) the values of $P(0)$, $P(1)$, $P(00)$, $P(01)$, $P(10)$ and $P(11)$ for comparatively small L ($L=47$) are very close to the corresponding theoretical values for $L \to \infty$. Moreover, from the result of this experiment we can infer that the longer the burst of 1's (or 0's) contained in an n-tuple the lower the occurrence probability of this n-tuple.

2^{nd} *Experiment (3^{rd} Golomb's postulate)*

We chose randomly a portion of $\mathcal{C}$ of length $\Lambda' = 1000$ and calculated the values of the auto-correlation function (see Sec. 2)

$$\Gamma(\tau) = \sum_{i=1}^{\Lambda'} c_i \, c_{i+\tau}$$

for $\tau = 0, 1, 2, \ldots \Lambda' - 1 = 999$, obtaining the following results

$$\Gamma(0) = 0.47 \; \Lambda'$$

$$\Gamma(1) = \Gamma(\Lambda' - 1) = 0.163 \; \Lambda'$$

$$0.2 \; \Lambda' \le \Gamma(\tau) \le 0.247 \; \Lambda' \quad (\overline{\Gamma}(\tau) = 0.223 \; \Lambda') \text{ for } 2 \le \tau \le \Lambda' - 2,$$

where $\overline{\Gamma}(\tau)$ denotes the average value of $\Gamma(\tau)$.

Several further results obtained by varying the parameters N_0, M_0, K_1 and K_2 showed nothing but negligible differences with respect to the result presented in this section.

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A NOTE ON RAMIFICATIONS CONCERNING THE CONSTRUCTION OF PYTHAGOREAN TRIPLES FROM RECURSIVE SEQUENCES

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An interesting method of generating Pythagorean triples is based on recursive sequences [1], [2]. We define a Pythagorean triple (a,b,c) as a triple of integers such that $a^2 + b^2 = c^2$. If the numbers are mutually relatively prime to each other, it is a primitive Pythagorean triple. Let $\{A_n\}$ be a second order linear sequence such that

$$\left.\begin{array}{l} A_1 \text{ and } A_2 \text{ are arbitrary positive integers} \\ \text{and} \\ A_{n+2} = A_{n+1} + A_n \text{ for all } n \geq 1 \end{array}\right\} \quad . \tag{1}$$

Then, choosing any four consecutive terms

$$(A_k, A_{k+1}, A_{k+2}, A_{k+3})$$

from the sequence, the triple defined by

$$\left.\begin{array}{l} a = A_k\ A_{k+3} \\ b = 2A_{k+1}\ A_{k+2} \\ c = {A_{k+1}}^2 + {A_{k+2}}^2 \end{array}\right\} \tag{2}$$

is Pythagorean for k being any natural number. To see this, note, by (1), that

$$a = (A_{k+2} - A_{k+1})(A_{k+2} + A_{k+1}) = {A_{k+2}}^2 - {A_{k+1}}^2 .$$

Hence, by (2),

$$a^2 + b^2 = ({A_{k+2}}^2 - {A_{k+1}}^2)^2 + (2A_{k+1}\ A_{k+2})^2 = ({A_{k+2}}^2 + {A_{k+1}}^2)^2 = c^2 .$$

The triple is primitive if A_k is odd and $(A_k, A_{k+1}) = 1$.

We now give some geometric applications of this construction. Suppose the sides of a right triangle constitute a Pythagorean triple (a,b,c), as defined in (2). First, we note that

$$\tan B = \frac{2A_{k+1}\ A_{k+2}}{A_k\ A_{k+3}} .$$

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Hence, taking successive quadruples in our sequence, we investigate

$$L = \lim_{k \to \infty} \tan B.$$

In early presentation of A_m we conjecture that

$$A_m = F_{m-2} A_1 + F_{m-1} A_2$$

Where F_n represents the n-th Fibonacci number. This is readily proved by mathematical induction. Hence,

$$L = 2 \lim_{k \to \infty} \frac{(F_{k-1} A_1 + F_k A_2)(F_k A_1 + F_{k+1} A_2)}{(F_{k-2} A_1 + F_{k-1} A_2)(F_{k+1} A_1 + F_{k+2} A_2)}$$

$$= 2 \lim_{k \to \infty} \frac{(A_1 + (F_k/F_{k-1}) A_2)(A_1 + (F_{k+1}/F_k) A_2)}{((F_{k-2}/F_{k-1}) A_1 + A_2)((F_{k+1}/F_k) A_1 + (F_{k+2}/F_k) A_2)} .$$

Here we invoke the fact that $\lim_{k \to \infty} F_{n+a}/F_n = G^a$, where $G = \left(\sqrt{5+1}\right)/2$, the golden ratio. Then

$$L = \lim_{k \to \infty} \tan B = 2 . \tag{3}$$

Hence, our sequence of triangles, formed by "sweeping over" any of our sequences of numbers, reveals triangles which become more and more similar to each other with tan B tending to 2.

Now, we derive some further, rather peculiar relationships. Let K denote the area of our triangle, s the semiperimeter, r its inradius, and r_a, r_b, and r_c its exradii. Then, as can readily be established from (1) and (2),

$$\left.\begin{aligned} K &= A_k A_{k+1} A_{k+2} A_{k+3} \\ \text{and} \quad s &= A_{k+2} A_{k+3} \end{aligned}\right\} \tag{4}$$

which means that

$$r = A_k A_{k+1} . \tag{5}$$

This may also serve to establish that the inradius of a right triangle of integral side measures is integral. Furthermore,

$$\left.\begin{aligned} s - a &= A_{k+1} A_{k+3} \\ s - b &= A_k A_{k+2} \\ s - c &= A_k A_{k+1} = r \end{aligned}\right\} \tag{6}$$

,

from which:

$$\left.\begin{aligned}
r_a &= K/(s-a) = A_k\, A_{k+2} &&= s-b \\
r_b &= K/(s-b) = A_{k+1}\, A_{k+3} &&= s-a \\
r_c &= K/(s-c) = A_{k+2}\, A_{k+3} &&= s
\end{aligned}\right\} \quad . \tag{7}$$

Thus, the exradii too are integral.

It may also be of interest to point out that

$$\left.\begin{aligned}
&\left.\begin{aligned}
c + a &= 2\, A_{k+2}^{\;2} \\
c - a &= 2\, A_{k+1}^{\;2} \\
c + b &= A_{k+3}^{\;2} \\
c - b &= A_k^{\;2}
\end{aligned}\right\} \\
&\text{and, thus, also} \\
&3c = A_k^{\;2} + A_{k+1}^{\;2} + A_{k+2}^{\;2} + A_{k+3}^{\;2}
\end{aligned}\right\} \quad . \tag{8}$$

Finally, a few more surprises. By similar analyzations as above,

$$\left.\begin{aligned}
&\lim_{k\to\infty} r_a/r = \lim_{k\to\infty} r_c/r_b = G \\
&\qquad \text{where } G = \left(\sqrt{5+1}\right)/2, \text{ the golden ratio:} \\
&\lim_{k\to\infty} r_b/r_a = G^2 \\
&\lim_{k\to\infty} r_b/r = G^3 \\
&\lim_{k\to\infty} r_c/r = G^4
\end{aligned}\right\} \quad . \tag{9}$$

This also means that, in the order given, these limits constitute a geometric sequence whose first term, as well as its ratio, equal G.

These results make it tempting to investigate special cases involving the golden ratio G. Thus, we delete the restriction that $\{A_n\}$ be a set of integers, and define a Golden Pythagorean Triangle by choosing the quadruple

$$\left(G^{(n-3)/2}\,,\ G^{(n-1)/2}\,,\ G^{(n+1)/2}\,,\ G^{(n+3)/2}\right)$$

from the sequence $\{G^m\}$ where $G = \left(\sqrt{5+1}\right)/2$, is the golden ratio and n an arbitrarily fixed odd positive integer. Then, with the same construction as given in (2), we have a "pseudo-Pythagorean triple" (a,b,c) such that

$$\left.\begin{aligned} a &= G^n \\ b &= 2G^n \\ c &= G^{n+1} + G^{n-1} \end{aligned}\right\} \quad . \tag{10}$$

The ensuing relationships pertaining to geometric parts of the golden triangle surpassed expectations. Notable among those is the surprisingly frequent occurrence of powers of G. Arithmetic, geometric, and harmonic means also enter the picture. None of the relationships involving means pertain to the general right triangle.

We will highlight just a brief account of some of these relationships. All of these can be etablished readily by simplifications involving $G^n = F_n\, G + F_{n-1}$ where F_m denotes the m-th Fibonacci number. Then, employing the same symbolism as before,

$$\left.\begin{aligned} K &= G^{2n} \\ s &= G^{n+2} \\ r &= G^{n-2} \end{aligned}\right\} \quad . \tag{11}$$

As, clearly, tan B = 2 (compare with (3)), examining various triangular relationships suggests itself. Among these, we may list:

$$\left.\begin{aligned} s - a &= G^{n+1} \\ s - b &= G^{n-1} \\ s - c &= G^{n-2} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \cos A/2 \cos B/2 \cos C/2 &= G^4 \sin A/2 \sin B/2 \sin C/2 \\ \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2 &= G^2(\sin^2 A/2 + \sin^2 B/2 + \sin^2 C/2) \end{aligned}\right\}. \tag{13}$$

Also,

$$\left.\begin{aligned} \cot A/2 &= G^3 \\ \cot B/2 &= G \end{aligned}\right\} \tag{14}$$

and letting A. M. (a,b), G. M. (a,b), and H. M. (a,b) represent the arithmetic, geometric, and harmonic mean, respectively.

$$\tan B/2 = \text{A. M. } (\tan A/2, \tan C/2) \ . \tag{15}$$

Similarly,

$$\left.\begin{aligned} \sin^2 B/2 &= \text{A. M. } (\sin^2 A/2, \sin^2 C/2) \\ \cos^2 B/2 &= \text{A. M. } (\cos^2 A/2, \cos^2 C/2) \end{aligned}\right\} \ . \tag{16}$$

Furthermore,

$$\left.\begin{aligned} r_a &= G^{n-1} \\ r_b &= G^{n+1} \\ r_c &= G^{n+2} = s \end{aligned}\right\} \quad . \tag{17}$$

and, thus, another arithmetic mean emerges:

$$r_b = \text{A. M. } (r_a,\ r_c)\ . \tag{18}$$

Also,

$$\left.\begin{aligned} a &= \text{A. M. } (r_b,r) = \text{H. M. } (r_a,r_c) = \text{G. M. } (r_a,r_b) = \text{G. M. } (r_c,r) \\ b &= \text{H. M. } (r_b,r_c) \end{aligned}\right\} \quad . \tag{19}$$

Now to some area considerations. The triangles determined by the contactpoints of the incircle as related to those of the excircles, deserve attention. Let $\overline{K}$, K_a, K_b, and K_c denote, respectively, the areas of the triangles whose vertices are the contactpoints of the incircle, and the excircles, respectively. Then,

$$\overline{K} = G^{3n-2}/c\ . \tag{20}$$

$$\left.\begin{aligned} K_a &= G^{2n-3}M \\ K_b &= G^{2n-1}M \\ K_c &= G^{2n}M \end{aligned}\right\} \quad , \text{ where } M = 2\cos A/2\cos B/2\cos C/2 \tag{21}$$

That is,

$$1/K_a + 1/K_b + 1/K_c = 1/\overline{K}\ . \tag{22}$$

Furthermore,

$$\begin{aligned} K_b &= \text{A. M. } (K_a,\ K_c) = \text{G. M. } (K_c, K_b - K_a) \\ K_b/K_a &= \text{G. M. } (K_c/K_a,\ K_c/K_b)\ . \end{aligned} \tag{23}$$

Finally, we let M be the incenter, and P_a, P_b, and P_c be the contactpoints of the incircle with sides a, b, and c, respectively. We consider the area T_a of kite AP_cMP_b, and the corresponding ones by cyclic permutation. As

$$\left.\begin{aligned} T_a &= G^{2n-1} \\ T_b &= G^{2n-3} \\ T_c &= G^{2n-4} \end{aligned}\right\} \quad , \tag{24}$$

again an harmonic mean emerges:

$$T_b = \text{H. M.}\ (T_a, T_c)\ . \tag{25}$$

The golden Pythagorean triangle does, indeed, exhibit a wealth of rather striking relationships, of which the given ones are but a sample.

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ON THE REPRESENTATION OF $\{F_{kn}/F_n\}$, $\{F_{kn}/L_n\}$, $\{L_{kn}/L_n\}$, and $\{L_{kn}/F_n\}$ as ZECKENDORF SUMS

Herta T. Freitag

1. INTRODUCTION

Zeckendorf's Theorem guarantees that every positive integer can be uniquely expressed as a sum of Fibonacci numbers, provided no two consecutive numbers are taken. The same holds for Lucas numbers, with the one additional condition that L_2 and L_0 not occur in the same representation (or else, $5 = L_3 + L_1 = L_2 + L_0$). This note deals with sets of integers $\{F_{kn}/F_n\}$, $\{F_{kn}/L_n\}$, $\{L_{kn}/L_n\}$, and $\{L_{kn}/F_n\}$ where n and k are positive integers obeying appropriate conditions to assure that all elements in our sequence are integral. Functions ϕ and λ are displayed where ϕ denotes the NUMBER OF addends in the FIBONACCI representation and λ the NUMBER OF LUCAS terms.

These functions are arrived at by first forming a lemma on the basis of the structure revealed by individual cases. Invariably, a RECURSIVE behavior of the ϕ - and λ -functions becomes apparent. Upon PROVING the lemma, the functions sought can be stated, as the recursiveness assures the correctness of the pattern throughout.

This procedure will be exhibited in detail in the first two theorems presented. Since similar methods are employed in the proofs of the remaining theorems, results only will be given for these. The most frequently used relationships follow.

2. RELATIONSHIPS USED IN THE PROOFS OF THE THEOREMS

$$L_{2n} + 2(-1)^{n-1} = 5F_n^2 \tag{1}$$

$$F_{m+n} + F_{m-n} = \begin{cases} L_mF_n & \text{if n is odd} \\ F_mL_n & \text{if n is even} \end{cases} \tag{2}$$

$$F_{m+n} - F_{m-n} = \begin{cases} F_mL_n & \text{if n is odd} \\ L_mF_n & \text{if n is even} \end{cases} \tag{3}$$

$$L_{m+n} + L_{m-n} = \begin{cases} 5F_mF_n & \text{if n is odd} \\ L_mL_n & \text{if n is even} \end{cases} \tag{4}$$

$$L_{m+n} - L_{m-n} = \begin{cases} L_mL_n & \text{if n is odd} \\ 5F_mF_n & \text{if n is even} \end{cases} \tag{5}$$

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$$\sum_{i=1}^{n} F_{an+(2i-1)} = F_{(a+2)n} - F_{an} = \begin{cases} F_{(a+1)n}L_n & \text{if n is odd} \\ L_{(a+1)n}F_n & \text{if n is even} \end{cases} \tag{6}$$

$$\sum_{i=1}^{n} F_{an+2(i+b)} = F_{(a+2)n+(2b+1)} - F_{an+2b+1} = \begin{cases} F_{(a+1)n+2b+1}L_n & \text{if n is odd} \\ L_{(a+1)n+2b+1}F_n & \text{if n is even} \end{cases} \tag{7}$$

$$\sum_{i=1}^{n} L_{ai} = \frac{(L_{an+a} - L_{an}) + (2(-)^a - L_a)}{L_a - ((-1)^a + 1)} \tag{8}$$

$$\text{for a} = 4: \quad \sum_{i=1}^{n} L_{4i} = (L_{4n+4} - L_{4n})/5 - 1 = F_{4n+2} - 1$$

$$\text{for a} = 8: \quad \sum_{i=1}^{n} L_{8i} = (L_{8n+8} - L_{8n})/45 - 1 = F_{8n+4}/3 - 1$$

$$\sum_{i=1}^{n} L_{8i-4} = F_{8n}/3 \tag{9}$$

$$\sum_{i=1}^{n} L_{a(2i-1)} = \frac{L_{2an+a} - L_{2an-a}}{L_{2a} - 2} \tag{10}$$

$$\sum_{i=1}^{n} L_{an+(2i+b)} = L_{(a+2)n+(b+1)} - L_{an+(b+1)} \tag{11}$$

$$= \begin{cases} L_{(a+1)n+(b+1)}L_n & \text{if n is odd} \\ 5F_{(a+1)n+(b+1)}F_n & \text{if n is even} \end{cases}$$

$$\sum_{i=1}^{n} L_{2an+(4i+a)} = F_{(a+2)(2n+1)} - F_{a(2n+1)+2} = L_{2(a+1)n+(a+2)} F_{2n} \tag{12}$$

$$\sum_{i=1}^{n} L_{2an+2(2i-1)} = F_{2(a+2)n} - F_{2an} = L_{2an+2n}F_{2n}. \tag{13}$$

3. THEOREMS

A. F_{kn}/F_n

Theorem 1: Let n be an even number and k any natural number.

Then $$\phi(F_{kn}/F_n) = \begin{cases} 1 & \text{if } n = 2 \\ k & \text{if } n \geq 4. \end{cases}$$

$\lambda(F_{kn}/F_n = \lfloor (k+1)/2 \rfloor$ where $[\ \ ]$ denotes the greatest integer function.

Proof: Part 1: $\phi(F_{kn}/F_n)$

For n = 2, trivially, $\phi(F_{kn}/F_n) = 1$.

Let $n \geq 4$: Here we consider the parity of k. If k is odd, we note, by using (8), that

$$S_1 = \sum_{i=1}^{(k-1)/2} L_{2in} = \frac{L_{(k+1)n} - L_{(k-1)n}}{L_{2n} - 2} - 1.$$

Invoking (5) and (1), $S_1 = F_{kn}/F_n - 1$. Thus, $F_{kn}/F_n = S_1 + 1$, and $\phi(F_{kn}/F_n) = (k-1) + 1 = k$. For an even k, we consider $S_2 = \sum_{i=1}^{k/2} L_{(2i-1)n}$ and, by using (10), (5), and (1), $S_2 = F_{kn}/F_n$, from which $\phi(F_{kn}/F_n) = k$ for all k-values.

Part 2: $\lambda(F_{kn}/F_n)$

Case 1: $n = 2$

For odd k-values, as $F_{kn}/F_n = F_{2k}$ for $n = 2$, we use $S_3 = \sum_{i=1}^{(k-1)/2} L_{4i}$ and, by (8) and (5), $S_3 = F_{2k} - 1$. Thus, $\lambda(F_{2k}) = (k-1)/2 + 1 = (k+1)/2$. If k is even, we take $S_4 = \sum_{i=1}^{k/2} L_{4i-2}$ where, by (10) and (5), $S_4 = F_{2k}$. Hence, $\lambda(F_{2k}) = k/2$. Therefore, for all k-values, $\lambda(F_{2k}) = [(k+1)/2]$.

Case 2: $n \geq 4$

From Part 1,

$$\lambda(F_{kn}/F_n) = \begin{cases} (k-1)/2 + 1 = (k+1)/2 & \text{for odd k's} \\ k/2 & \text{for even k's,} \end{cases}$$

and, thus, for all k-values,

$$\lambda(F_{kn}/F_n) = [(k+1)/2].$$

Theorem 2: Let n be an odd number and k any natural number. Then

$$\phi(F_{kn}/F_n) = \begin{cases} 1 & \text{if } n = 1 \\ \left.\begin{array}{ll} k\,(n+1)/4 & \text{if } k \equiv 0 \pmod 4 \\ ((k-1)\,n + (k+3))\,/4 & \text{if } k \equiv 1 \pmod 4 \\ (k-2)\,n + (k+6)\,/4 & \text{if } k \equiv 2 \pmod 4 \\ (k+1)\,n + (k-3)\,/4 & \text{if } k \equiv 3 \pmod 4 \end{array}\right\} & \text{and } n \geq 3. \end{cases}$$

$$\lambda(F_{kn}/F_n) = \begin{cases} kn/4 & \text{if } k \equiv 0 \pmod 4 \\ ((k-1)n + 4))\,/4 & \text{if } k \equiv 1 \pmod 4 \\ ((k-2)n + 4)\,/4 & \text{if } k \equiv 2 \pmod 4 \\ (k+1)n/4 & \text{if } k \equiv 3 \pmod 4). \end{cases}$$

<u>Proof:</u> <u>Part 1: $\phi(F_{kn}/F_n)$</u>

As, trivially, $\phi(F_{kn}/F_n) = 1$ for n = 1, we let n $\geq$ 3. By observing individual cases, we state

<u>Lemma 1:</u> $F_{kn}/F_n = F_{(k-2)n+1} + S_1 + F_{(k-3)n+1} + F_{(k-3)n-2} + F_{(k-4)n}/F_n$, where $S_1 = \sum_{i=1}^{n-2} F_{(k-3)n+2i+2}$. Using (7), $S_1 = F_{(k-1)n-1} - F_{(k-3)n+3}$. But, $S_1 + F_{(k-3)n+1} + F_{(k-3)n-2} + F_{(k-1)n+1} = (F_{(k-1)n+1} + F_{(k-1)n-1}) - (F_{(k-3)n+2} - F_{(k-3)n-2})$, which, by (2) and (3) becomes $L_{(k-1)n} - L_{(k-3)n} = L_{(k-2)n}L_n$. Now by (3) and (5), $F_{kn} - F_{(k-4)n} = L_{(k-2)n}F_n$, and, as $F_{2n} = F_nL_n$, Lemma 1 holds.

$$\begin{aligned} \text{Therefore}\quad \phi(F_{kn}/F_n) &= ((n-2) + 3)) + \phi(F_{(k-4)n}/F_n) \\ &= (n+1) + \phi(F_{(k-4)n}/F_n). \end{aligned}$$

Before displaying this recursive behavior of our ϕ-function, we need to determine F_{kn}/F_n for k ϵ {1, 2, 3}. Trivially, for k = 1, $\phi(F_{kn}/F_n) = 1$ and $\phi(F_{kn}/F_n) = \phi(L_n) = 2$ for k = 2. For k = 3, we form

<u>Lemma 2:</u> $F_{3n}/F_n = F_{2n+1} + S_2 + F_1$ where $S_2 = \sum_{i=1}^{n-2} F_{2i+2}$ and, by (7), $S_2 = F_{2n-1} - 2$. Substituting, noting that $F_{2n+1} + F_{2n-1} = L_{2n}$, and using relationship (2), the lemma holds and $\phi(F_{kn}/F_n) = n$ for k = 3. Thus we have

Chart 1 n $\geq$ 3

k	$\phi(F_{kn}/F_n)$			
	k = 0 (mod 4)	k = 1 (mod 4)	k = 2 (mod 4)	k = 3 (mod 4)
1		1		
2			2	
3				n
4	n+1			
5		(n+1)+1=n+2		
6			(n+1)+2=n+3	
7				(n+1)+n=2n+1
8	(n+1)+n+1=2(n+1)			
9		(n+1)+n+2=2n+3		
10			(n+1)+n+3=2n+4	
11				(n+1)+2n+1=3n+2
12	(n+1)+2(n+1)=3(n+1)			
13		(n+1)+2n+3=3n+4		
14			(n+1)+2n+4=3n+5	
15				(n+1)+3n+2=4n+3
⋮	⋮	⋮	⋮	⋮

Therefore,

$$\phi(F_{kn}/F_n) = \begin{cases} 1 & \text{if } n = 1 \\ k(n+1)/4 & \text{if } k \equiv 0 \pmod 4 \\ ((k-1)n+(k+3))/4 & \text{if } k \equiv 1 \pmod 4 \\ ((k-2)n+(k+6))/4 & \text{if } k \equiv 2 \pmod 4 \\ ((k+1)n+(k-3))/4 & \text{if } k \equiv 3 \pmod 4 \end{cases} \quad \left.\right\} \text{and } n \geq 3.$$

Part 2: $\lambda(F_{kn}/F_n)$

Proceeding similarly, we coin

Lemma 3: $F_{kn}/F_n = S_3 + F_{(k-4)n}/F_n$ where $S_3 = \sum_{k=1}^{n} L_{(k-3)n+2i-1}$

which can readily be established. Thus, for all odd n-values, $\lambda(F_{kn}/F_n) = n + \lambda(F_{(k-4)n}/F_n)$, again a recursive function. Here, $\lambda(F_{kn}/F_n)$ for k = 1 equals $\lambda(F_{kn}/F_n)$ for k = 2. For $n \geq 3$ and k = 3, we employ

Lemma 4: $F_{3n}/F_n = \sum_{i=1}^{n-1} L_{2i+1} + L_0$ which can easily be established.

Thus, for k = 3, $\lambda(F_{kn}/F_n) = \lambda(F_{3n}/F_n) = n$ and we have CHART 2.

Chart 2 $n \geq 3$

k	$\lambda(F_{kn}/F_n)$			
	$k \equiv 0 \pmod 4$	$k \equiv 1 \pmod 4$	$k \equiv 2 \pmod 4$	$k \equiv 3 \pmod 4$
1		1		
2			1	
3				n
4	n			
5		(n)+1 = n+1		
6			(n)+1 = n+1	
7				(n)+n = 2n
8	(n)+n = 2n			
9		(n)+n+1 = 2n+1		
10			(n)+n+1 = 2n+1	
11				(n)+2n = 3n
12	(n)+2n = 3n			
13		(n)+2n+1 = 3n+1		
14			(n)+2n+1 = 3n+1	
15				(n)+3n = 4n
⋮	⋮	⋮	⋮	⋮

Thus, for $n \geq 3$,

$$\lambda(F_{kn}/F_n) = \begin{cases} kn/4 & \text{if } k \equiv 0 \pmod 4 \\ ((k-1)n+4)/4 & \text{if } k \equiv 1 \pmod 4 \\ ((k-2)n+4)/4 & \text{if } k \equiv 2 \pmod 4 \\ ((k+1)n/4 & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

Now, for $n = 1$, $F_{kn}/F_n = F_k$ and, using $F_k = L_{k-2} + F_{k-4}$, $\lambda(F_k) = 1 + \lambda(F_{k-4})$.

But, $\lambda(F_k) = 1$ for $k = 1, 2, 3$, and 4.

Therefore, for $n = 1$,

$$\lambda(F_{kn}/F_n) = \begin{cases} k/4 & \text{if } k \equiv 0 \pmod 4 \\ (k+3)/4 & \text{if } k \equiv 1 \pmod 4 \\ (k+2)/4 & \text{if } k \equiv 2 \pmod 4 \\ (k+1)/4 & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

However, these relationships for $n = 1$ are just special cases of $\lambda(F_{kn}/F_n)$ for $n \geq 3$ and, thus, the above-stated results for $\lambda(F_{kn}/F_n)$ hold for all odd n-values.

B. F_{kn}/L_n

Theorem 3: Let n be an odd, and k an even, integer. Then

$$\phi(F_{kn}/L_n) = \begin{cases} 1 & \text{if } n = 1 \\ k/2 & \text{if } n \geq 3 \end{cases}$$

$$\lambda(F_{kn}/L_n) = \begin{cases} \left.\begin{array}{ll} k(n+1)/8 & \text{if } k \equiv 0 \pmod 4 \\ (kn + (k+4))/8 & \text{if } k \equiv 2 \pmod 4 \end{array}\right\} & \text{and } n \equiv 1 \pmod 4 \\ k(n+1)/8 \quad \text{for all } k & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 4: If n and k are even,

$$\phi(F_{kn}/L_n) = \begin{cases} kn/4 & \text{if } k \equiv 0 \pmod 4 \\ ((k-2)n+4)/4 & \text{if } k \equiv 2 \pmod 4 \end{cases}$$

$$\lambda(F_{kn}/L_n) = \begin{cases} kn/8 & \text{if } n \equiv 0 \pmod 4 \\ kn/8 & \text{if } k \equiv 0 \pmod 4 \\ (kn+4)/8 & \text{if } k \equiv 2 \pmod 4 \end{cases} \quad \text{(last two cases: and } n \equiv 2 \pmod 4).$$

C. L_{kn}/L_n

Theorem 5: If n and k are both odd numbers,

$$\phi(L_{kn}/L_n) = \begin{cases} 2 & \text{if } n = 1 \\ k & \text{if } n \geq 3 \end{cases}$$

$$\lambda(L_{kn}/L_n) = \begin{cases} 1 & \text{if } n = 1 \\ (k+1)/2 & \text{if } n \geq 3. \end{cases}$$

Theorem 6: Let n be an even number and k an odd one. Then

$$\phi(L_{kn}/L_n) = \begin{cases} (k+1)/2 & \text{if } n = 2 \\ ((k-1)n + (k+3))/4 & \text{if } k \equiv 1 \pmod 4 \\ ((k+1)n + (k-3))/4 & \text{if } k \equiv 3 \pmod 4 \end{cases} \quad \text{(last two cases: and } n \geq 4).$$

$$\lambda(L_{kn}/L_n) = \begin{cases} ((k-2)n + 4)/4 & \text{if } k \equiv 1 \pmod 4 \\ (k+1)n/4 & \text{if } k \equiv 3 \pmod 4). \end{cases}$$

D. L_{kn}/F_n

Theorem 7: Let $n \leq 3$ and k any integer. (For $n > 3$, L_{kn}/F_n is not integral.)

Then:

$$\phi(L_{kn}/F_n) = \begin{cases} 1 & \text{if } k = 1 \text{ or } k = 2 \\ 2 & \text{if } k \geq 3 \end{cases} \text{ and } n = 1; \quad \begin{cases} 1 & \text{if } k = 1 \\ 2 & \text{if } k \geq 2 \end{cases} \text{ and } n = 2; \quad k \text{ if } n = 3.$$

$$\lambda(L_{kn}/F_n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ k & \text{if } n = 3. \end{cases}$$

4. CONCLUSION

This completes our discussion of ϕ- and λ-functions for A_{kn}/B_n with A_m and B_m being either Fibonacci- or Lucas-numbers. Noteworthy factors, in the eyes of the author, are the frequency of the occurrence of recursive functions, and the fact that modulus four plays such a big role. This did not seem evident from the start.

While this study was undertaken for theoretical interests, it is hoped that the results will also be of help in some applications. Cryptanalysis suggests itself as a possible candidate.

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FUNCTIONAL RECURRENCES

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1. INTRODUCTION

In the theory of recurrences many papers are devoted to the examination of number - theoretic polynomials, in particular Pell polynomials $P_n(x)$, Pell - Lucas polynomials $Q_n(x)$, and others (see [1], [2]).

The purpose of our paper is to give some connections between solutions of differential equation of the second order, power matrices, and some functional recurrences.

2. RESULTS

Theorem 1: Let

$$A = A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

be the matrix with $a = a(x)$, $b = b(x)$, $c = c(x)$, $d = d(x)$ which are non-zero and real-valued functions defined on $J = (x_1, x_2) \subset R$ and let $\det A(x) \neq 0$ on J. Moreover, let

$$r = r(x) = a(x) + d(x) = \mathrm{Tr}\, A(x)$$

$$s = s(x) = -\det A(x)$$

and

$$u_0 = r,\ u_1 = ru_0 + s$$

$$u_n(x) = ru_{n-1}(x) + su_{n-2}(x) \quad \text{for } n \geq 2.$$

Then for every natural number $n \geq 2$ we have

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^n = \begin{pmatrix} a(x)u_{n-2}(x)+v_{n-2}(x) & b(x)u_{n-2}(x) \\ c(x)u_{n-2}(x) & d(x)u_{n-2}(x)+v_{n-2}(x) \end{pmatrix}$$

where

$$v_{n-2}(x) = s(x)u_{n-3}(x)\ ,\quad u_{-1}(x) = 1 \quad \text{for } x \in J.$$

Theorem 2: Let the functions s_0, t_0, u, v of x and the constant λ satisfy the following conditions

$$s_0,\ t_0,\ u,\ v \in C^2(J), \quad J = (x_1, x_2) \subset R,$$

$$u \neq 0,\ v \neq 0 \text{ on } J,\ \lambda \in R_+ .$$

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 115–121.

Then the functions

$$y_1 = s_0 u^\lambda \text{ and } y_2 = t_0 v^\lambda \tag{2.1}$$

are the solutions of the differential equation

$$D_0 y'' + D_1 y' + D_2 y = 0 \tag{2.2}$$

where

$$D_0 = \det\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, D_1 = \det\begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, D_2 = \det\begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}, \tag{2.3}$$

$$\begin{aligned} s_1 &= s'_0 + \lambda\, s_0 \tfrac{u'}{u}, \qquad s_2 = s'_1 + \lambda\, s_1 \tfrac{u'}{u} \\ t_1 &= t'_0 + \lambda\, t_0 \tfrac{v'}{v}, \qquad t_2 = t'_1 + \lambda\, t_1 \tfrac{v'}{v} \end{aligned} \tag{2.4}$$

and the dashes ($'$, $''$) denote differentiation with respect to x.

3. PROOF OF THEOREMS.

Proof of Theorem 1: The proof is by induction with respect to $n \geq 2$. Since

$$\det(A(x) - \lambda I) = \lambda^2 - \text{Tr } A(x)\, \lambda + \det A(x),$$

we have, by the Cayley - Hamilton theorem,

$$A^2(x) = (\text{Tr } A(x))A(x) - \det A(x)I.$$

Hence

$$A^2 = rA + sI = r\begin{pmatrix} a & b \\ c & d \end{pmatrix} + s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ra+s & br \\ cr & rd+s \end{pmatrix} =$$

$$= \begin{pmatrix} au_0 + v_0 & bu_0 \\ cu_0 & du_0 + v_0 \end{pmatrix}$$

and our theorem is true for $n = 2$. For convenience, we temporarily drop the functional notation. Suppose that the theorem is true for $n = m$, i.e.

$$A^m = \begin{pmatrix} au_{m-2} + v_{m-2} & bu_{m-2} \\ cu_{m-2} & du_{m-2} + v_{m-2} \end{pmatrix}. \tag{3.1}$$

From (3.1) we get

$$A^m = u_{m-2}A + v_{m-2}I \tag{3.2}$$

From (3.2) it follows that

$$A^{m+1} = A^m A = (u_{m-2}A + v_{m-2}I)A = u_{m-2}A^2 + v_{m-2}A. \tag{3.3}$$

Since $A^2 = rA + sI$, by (3.3) we get

$$A^{m+1} = (ru_{m-2} + v_{m-2})A + su_{m-2}I. \tag{3.4}$$

Using the conditions in the statement of the theorem, we have

$$u_{m-1} = ru_{m-2} + v_{m-2}, \qquad v_{m-1} = su_{m-2}$$

so we get

$$A^{m+1} = \begin{pmatrix} au_{m-1} + v_{m-1} & bu_{m-1} \\ cu_{m-1} & du_{m-1} + v_{m-1} \end{pmatrix}.$$

Hence, the theorem also holds for $n = m+1$. Therefore, the theorem is proved.

Proof of Theorem 2: Let

$$L(y) = D_0y'' + D_1y' + D_2y.$$

Then for $y_1 = s_0u^\lambda$ we have

$$L(s_0u^\lambda) = D_0(s_0u^\lambda)'' + D_1(s_0u^\lambda)' + D_2(s_0u^\lambda). \tag{3.5}$$

It is easy to see that

$$(s_0u^\lambda)' = s'_0u^\lambda + \lambda\, s_0u^{\lambda-1}u' = (s'_0 + \lambda\, s_0\, \tfrac{u'}{u})u^\lambda$$

and because $s'_0 + \lambda\, s_0\, \frac{u'}{u} = s_1$ from (2.4), we have

$$(s_0u^\lambda)' = s_1u^\lambda\ . \tag{3.6}$$

Similarly we get

$$(s_0u^\lambda)'' = (s_1u^\lambda)' = s_2u^\lambda\ . \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get

$$L(s_0u^\lambda) = (D_0s_2 + D_1s_1 + D_2s_0)u^\lambda. \tag{3.8}$$

We remark that

$$D_0s_2 + D_1s_1 + D_2s_0 = \det \begin{pmatrix} s_2 & s_0 & s_1 \\ s_2 & s_0 & s_1 \\ t_2 & t_0 & t_1 \end{pmatrix} = 0$$

and therefore from (3.8) and (3.9) it follows that

$$L(s_0u^{\lambda}) = 0, \text{ since } u \neq 0 \text{ on } J.$$

Analogously we obtain that $L(t_0u^{\lambda}) = 0$ and the proof is complete.

4. APPLICATIONS

Corollary 1: Let the assumptions of Theorem 1 be satisfied. Then for every natural number $n \geq 2$ we have

$$u_{n-2}^2 - (\mathrm{Tr}\ A)u_{n-2}u_{n-3} + (\det A)u_{n-3}^2 = (\det A)^{n-1} \tag{4.1}$$

The proof follows from Theorem 1 and Cauchy's theorem on the product of determinants.

Formula (4.1) is a general form of Simson's formula (see for example [1], p. 291),

$$u_{m-1}^2 - u_mu_{m-2} = (\det A)^m, \tag{4.2}$$

Indeed, since

$$u_m = ru_{m-1} + su_{m-2}$$

then from (4.1) we have

$$(ru_{m-1} + su_{m-2})^2 - (\mathrm{Tr}\ A)u_mu_{m-1} + (\det A)u_{m-1}^2 = (\det A)^{m+1} \tag{4.3}$$

Because $r = \mathrm{Tr}\ A$, $s = -\det A$, from (4.3), we get

$$su_{m-2}(ru_{m-1} + su_{m-2}) - su_{m-1}^2 = (-s)^{m+1}.$$

Thus

$$u_{m-1}^2 - u_mu_{m-2} = (\det A)^m.$$

Corollary 2: Let the assumptions of Theorem 1 be satisfied. Then for every natural number $n \geq 2$ we have

$$\begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix}^n = \begin{pmatrix} R_n(x) & S_n(x) \\ S_n(x) & R_n(x) \end{pmatrix}$$

where

$$R_n(x) = \tfrac{1}{2}\left[(a(x) + b(x))^n + (a(x) - b(x))^n\right]$$

$$S_n(x) = \tfrac{1}{2}\left[(a(x) + b(x))^n - (a(x) - b(x))^n\right].$$

The proof is by induction.

Corollary 3: Let the assumptions of Theorem 2 be satisfied and let u, v be linearly independent over R. Let $s_0 = t_0 = 1$. Then the general solution of the differential equation

$$\det \begin{pmatrix} 1 & \lambda\frac{u'}{u} \\ 1 & \lambda\frac{v'}{v} \end{pmatrix} y'' + \det \begin{pmatrix} s_2 & 1 \\ t_2 & 1 \end{pmatrix} y' + \det \begin{pmatrix} \lambda\frac{u'}{u} & s_2 \\ \lambda\frac{v'}{v} & t_2 \end{pmatrix} y = 0 \tag{4.4}$$

is of the form

$$y = c_1 u^\lambda + c_2 v^\lambda \tag{4.5}$$

where c_1, c_2 are arbitrary constants and

$$s_2 = \lambda\left[\frac{u''}{u} - (1 - \lambda)\frac{(u')^2}{u^2}\right]$$

$$t_2 = \lambda\left[\frac{v''}{v} - (1 - \lambda)\frac{(v')^2}{v^2}\right].$$

Proof: First we remark that the functions u, v are linearly independent over R iff the functions u^λ and v^λ are also independent over R. Since

$$D_0 = \begin{vmatrix} 1 & \lambda\frac{u'}{u} \\ 1 & \lambda\frac{v'}{v} \end{vmatrix} = \lambda\,(uv)^{-1} \begin{vmatrix} u & v \\ u' & v' \end{vmatrix},$$

then the Wronskian

$$W(u^\lambda, v^\lambda) = (uv)^\lambda D_0\,.$$

Now, from Theorem 1 it follows that the general solution of (4.4) has the form (4.5) and the proof is finished. We remark that the equation (4.4) determines a new class of equations effectively integrable. In particular this equation includes the classes of the differential equations connected with Tshebysheff, Pell, Pell-Lucas, and other polynomials.

Corollary 4: Let $a(x), b(x) \in C^2(J)$, and

$$A = \begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix}, \; b(x) \neq 0 \text{ on } J.$$

Then the characteristic roots of the matrix A, namely,

$$u = a(x) + b(x),\ v = a(x) - b(x)$$

are generators of the general solution of the equation

$$\begin{vmatrix} 1 & n\frac{u'}{u} \\ 1 & n\frac{v'}{v} \end{vmatrix} y'' + \begin{vmatrix} s_2 & 1 \\ t_2 & 1 \end{vmatrix} y' + \begin{vmatrix} n\frac{u'}{u} & s_2 \\ n\frac{v'}{v} & t_2 \end{vmatrix} y = 0 \tag{4.6}$$

where $n \in N$ and

$$s_2 = n\left[\frac{u''}{u} - (1 - n)\left(\frac{u'}{u}\right)^2\right]$$

$$t_2 = n\left[\frac{v''}{v} - (1 - n)\left(\frac{v'}{v}\right)^2\right]$$

and then the general solution of (4.6) is of the form

$$y = C_1(a(x) + b(x))^n + C_2(a(x) - b(x))^n \tag{4.7}$$

where C_1, C_2 are arbitrary constants.

Corollary 5: Let $a(x) = x$, $b(x) = \sqrt{x^2 - 1}$, where $|x| > 1$. Then the characteristic roots of the matrix

$$A = \begin{pmatrix} x & \sqrt{x^2 - 1} \\ \sqrt{x^2 - 1} & x \end{pmatrix}$$

are generators of the general solution of the Tshebysheff equation

$$(x^2 - 1)y'' + xy' - n^2y = 0 \tag{4.8}$$

so that the function

$$y = C_1(x + \sqrt{x^2 - 1})^n + C_2(x - \sqrt{x^2 - 1})^n$$

is the general solution of the equation (4.8).

Analogously as in Corollary 5 we can use the results for other number-theoretic polynomials. We consider only Pell-Lucas polynomials and Fibonacci Polynomials. We have

Corollary 6: Let $a(x) = x$, $b(x) = \sqrt{x^2 + 1}$. Then the characteristic roots of the matrix

$$A = \begin{pmatrix} x & \sqrt{x^2 + 1} \\ \sqrt{x^2 + 1} & x \end{pmatrix}$$

are generators of the general solution of the differential equation

$$(x^2 + 1)y'' + xy' - n^2y = 0 \tag{4.9}$$

so that the function

$$y = C_1(x + \sqrt{x^2 + 1})^n + C_2(x - \sqrt{x^2 + 1})^n$$

is the general solution of (4.9).

Corollary 7: Let $a(x) = x$, $b(x) = \sqrt{x^2 + 4}$. Then the characteristic roots of the matrix

$$A = \begin{pmatrix} x & \sqrt{x^2+4} \\ \sqrt{x^2+4} & x \end{pmatrix}$$

are generators of the general solution of the differential equation

$$(x^2 + 4)y'' + xy' - n^2y = 0, \tag{4.10}$$

and the function

$$y = C_1(x + \sqrt{x^2 + 4})^n + C_2(x - \sqrt{x^2 + 4})^n$$

is the general solution of (4.10).

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CONCENTRIC CYCLES IN MOSAIC GRAPHS

Heiko Harborth

Mosaic graphs, such as the plane representations of the five platonic solids or the three regular and the hyperbolic tesselations of the plane, often are connected with problems in group theory, geometry and especially hyperbolic geometry (see [1], [2]). However, simple combinatorial enumeration problems for mosaic graphs do not seem to have been treated very often in the mathematical literature. In this paper formulas for the numbers of vertices and cells on concentric cycles of (p,q)-tesselations shall be developed.

The (p,q)-mosaic graph $M_n=M_n(p,q)$, where $t=(p-2)(q-2) \geq 4$, is defined as follows: M_1 consists of a vertex V with degree q, and of all q cells, which are p-gons, and which have V as its common vertex. Then M_n is obtained from M_{n-1}, if p-gons are added to every vertex V_i of the boundary of M_{n-1} such that every V_i becomes the central vertex of a subgraph M_1. Figure 1 shows $M_n(4,5)$ for n=1, 2, 3, and Figures 4 to 6 are examples for M_3. The dual of M_n is constructed in the familiar way: If one new vertex is

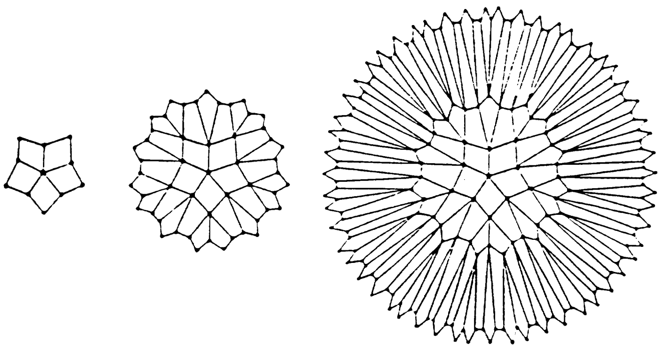

Figure 1.

drawn in every cell of $M_n(p,q)$, and two new vertices are connected, if their cells have a common edge, then the dual (q,p)-mosaic graph $T_n(q,p)$ is constructed, which starts now with a q-gon instead of a vertex (Figure 2 depicts $T_n(5,4)$ for n=1, 2, 3). Note that the dual graph of $T_n(p,q)$ is $M_{n-1}(q,p)$, $n \geq 2$. The cells of M_n and T_n are arranged in concentric cycles around the starting vertex or cell.

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 123–128.

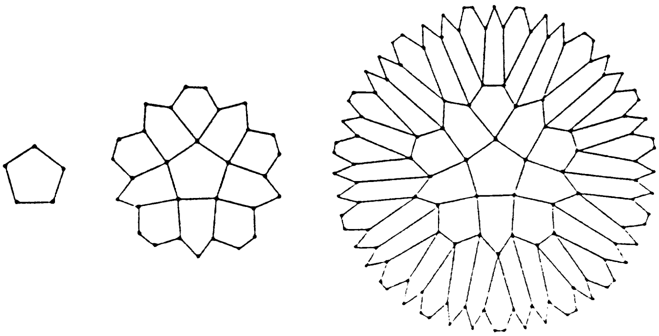

Figure 2.

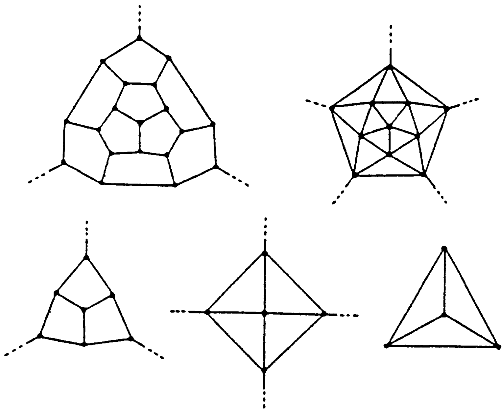

Figure 3.

The sequences of M_n and T_n are infinite for $t=(p-2)(q-2) \geq 4$. For t=4 they converge to the regular and for t>4 to the hyperbolic (p,q)-mosaic tesselations of the plane (see [1], [2]). For t<4 the sequences of M_n and T_n are finite and they end with the graphs of the five platonic solids (see Figure 3 for M_n, where the dotted lines are incident to a common vertex). This follows also from equation (4) below, where $a_2=-3$, $a_2=0$, $a_3=0$, $a_2=0$, $a_3=0$ for (p,q) = (3,3), (3,4), (3,5), (4,3), (5,3), respectively.

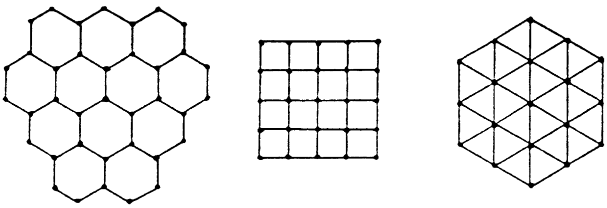

Figure 4.

It is the purpose of this note to determine the numbers $a_n=a_n(p,q)$ and $b_n=b_n(p,q)$ of vertices (or edges) on the boundary of $M_n(p,q)$ and $T_n(p,q)$, also the numbers $c_n=c_n(p,q)$ and $d_n=d_n(p,q)$ of p-gons which have at least one point in common with the boundary of $M_n(p,q)$ and $T_n(p,q)$, respectively.

Note that the vertices on the boundary all have degree 2 or degree 3. Let $a_{n,2}$ and $a_{n,3}$ denote the numbers of vertices of degrees 2 and 3 of the boundary of M_n. It is easily checked, that

$$a_{1,2} = q(p-3), \qquad a_{1,3} = q, \tag{1}$$

$$a_{n,3} = (q-3)\, a_{n-1,3} + (q-2)\, a_{n-1,2}, \tag{2}$$

$$a_{n,2} = (p-3)\,(q-4)\, a_{n-1,3} + (p-3)\,(q-3)\, a_{n-1,2} + (p-4)\, a_{n-1}. \tag{3}$$

Hence, with $a_n=a_{n,2} + a_{n,3}$ it follows

$$\begin{aligned} a_n &= (p-4)\, a_{n-1} + (pq-2q-3p+6)\,(a_{n-1,2} + a_{n-1,3}) + a_{n-1,2} - (p-3)\, a_{n-1,3} \\ &= ((p-2)\,(q-2)-2)\, a_{n-1} - (p-3)\, a_{n-2,3} - (p-3)\, a_{n-2,2} + (p-4)\, a_{n-2} \\ &= ((p-2)\,(q-2)-2)\, a_{n-1} - a_{n-2}, \end{aligned}$$

and thus the following recurrence is obtained,

$$a_n = (t-2)\, a_{n-1} - a_{n-2}, \qquad t = (p-2)\,(q-2). \tag{4}$$

The first two values,

$$a_1 = q(p-2) \text{ and } a_2 = (t-2)\, a_1, \tag{5}$$

are deduced for (1) to (3). The characteristic roots of (4) are

$$\alpha = \frac{1}{2}\,(t - 2 + \sqrt{t(t-4)}\,) \text{ and } \beta = \frac{1}{2}\,(t - 2 - \sqrt{t(t-4)}\,), \tag{6}$$

so that for $t \geq 5$ together with (5) the numbers a_n can be determined to be

$$a_n = a_n(p,q) = \frac{q(p-2)}{\sqrt{t(t-4)}}\,(\alpha^n - \beta^n). \tag{7}$$

For $t=4$, equation (4) becomes $a_n=2a_{n-1}-a_{n-2}$. Repeated application leads to the well known results for the three regular tesselations

$$a_n(6,3) = 12n, \quad a_n(4,4) = 8n, \quad a_n(3,6) = 6n. \tag{8}$$

Repeated application of (4) yields

$$a_n = a_1 \sum_{i=1}^{n} (-1)^{i-1}\binom{2n-i}{i-1} t^{n-i}, \tag{9}$$

which also can be verified by (4).

The number $A_n=A_n(p,q)$ of vertices of $M_n(p,q)$ is obtained by

$$A_n = A_n(p,q) = 1 + \sum_{i=1}^{n} a_i. \tag{10}$$

If (4) for 3, 4, ..., n is added up, then the recurrence

$$A_n = (t-2)A_{n-1} - A_{n-2} + 2p \tag{11}$$

is obtained for A_n. Equations (10) and (8) yield the well known results for regular tesselations,

$$A_n(6,3) = 6n^2 + 6n + 1,\ \ A_n(4,4) = (2n+1)^2,\ \ A_n(3,6) = 3n^2+3n+1. \tag{12}$$

If $q-2$ cells are counted around every vertex of the boundary of $M_{n-1}(p,q)$, then every cell of the n-th and $(n-1)$st cycle of cells of $M_n(p,q)$ is counted exactly once, that is,

$$c_n + c_{n-1} = (q-2)\, a_{n-1}, \tag{13}$$

and with (4) the recurrence

$$c_n = (t-3)\, c_{n-1} + (t-3)\, c_{n-2} - c_{n-3}, \tag{14}$$

follows for $c_n=c_n(p,q)$. Another counting argument, however, leads directly to c_n. If it is observed, that $p-2$ edges for every of the c_n cells are counted in a_n+a_{n-1}, then

$$c_n = c_n(p,q) = \frac{a_n(p,q)+a_{n-1}(p,q)}{p-2}\ ,\ c_1 = q. \tag{15}$$

The number $C_n=C_n(p,q)$ of p-gons of $M_n(p,q)$ is then

$$C_n = C_n(p,q) = \sum_{i=1}^{n} c_i = \frac{A_{n-1}+A_n-2}{p-2}\ . \tag{16}$$

Concerning the (p,q)-mosaic graph $T_n(p,q)$, all corresponding numbers can be obtained by duality from the above formulas, for example,

$$b_n = b_n(p,q) = c_n(q,p), \tag{17}$$

$$d_n = d_n(p,q) = a_{n-1}(q,p),\ d_1 = 1. \tag{18}$$

The following remark may be of interest. Let v be the ratio of the number of cells versus the number of edges of the boundary of the above mosaic graphs. Then, by the deduced formulas, for $t=4$ this ratio v is of order n, whereas for $t\geq 5$ this ratio v is a constant, that means, the number of cells and the number of edges of the boundary increase in the same order of magnitude for hyperbolic mosaic graphs.

Now consider $t=5$, the smallest parameter for hyperbolic mosaic graphs, where $(p,q) =$ (7,3) or (3,7) (see Figures 5 and 6 for $M_3(7,3)$ and $M_3(3,7)$). With Fibonacci numbers F_n, $F_{n+2}=F_{n+1}+F_n$, $F_0=0$, $F_1=1$, and Lucas numbers L_n, $L_{n+2}=L_{n+1}+L_n$, $L_0=2$, $L_1=1$, the following formulas are easily checked by equations (4), (5), (10), (15), and (16). For $M_n(7,3)$ holds

$$a_n(7,3) = 15F_{2n}, \quad c_n(7,3) = 3L_{2n-1},$$

$$A_n(7,3) = 15F_{2n+1} - 14, \quad C_n(7,3) = 3L_{2n} - 6,$$

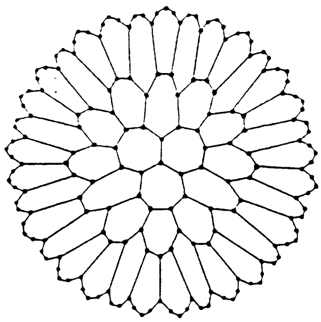

Figure 5.

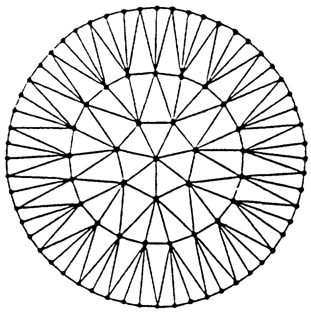

Figure 6.

and for $M_n(3,7)$

$$a_n(3,7) = 7F_{2n}, \qquad c_n(3,7) = 7L_{2n-1},$$

$$A_n(3,7) = 7F_{2n+1} - 6, \qquad C_n(3,7) = 7L_{2n} - 14.$$

An identity for F_{2n} is thus obtained from (9) for a_n/a_1,

$$F_{2n} = \sum_{i=1}^{n} (-1)^{i-1} \binom{2n-i}{i-1} 5^{n-i}.$$

So for t=5 the numbers a_n/a_1 and c_n/c_1 are always Fibonacci and Lucas numbers for all (p,q)-mosaic graphs $M_n(7,3)$, $M_n(3,7)$, $T_n(7,3)$ and $T_n(3,7)$. Therefore these graphs could be qualified for the notation "Fibonacci mosaic graphs" or "Fibonacci tesselations" in the infinite case. However, do there exist other values t with Fibonacci or Lucas numbers for all a_n/a_1? This is not the case, since $a_3/a_1 = (t-2)^2-1$ follows from (4) and (5), and $F_m=(t-2)^2-1$ so as $L_m=(t-2)^2-1$ both are impossible for t>5 (see [4]). Thus the notation "Fibonacci mosaic graph" seems to be justified for t=5.

There remain many combinatorial problems to be worked on. Which games, played on a checker board, are possible on a (p,q)-mosaic graph? All problems concerning polyominoes, that are simple connected sets of cells of (p,q)-mosaic tesselations, are unknown so far. A special problem is the determination of the smallest number $E_m(p,q)$ of edges of a (p,q)-mosaic tesselation ($t\geq5$) such that these edges are the edges of m cells? For the three regular tesselations (t=4) these numbers were determined in [3] to be

$$E_m(3,6) = m + \{\tfrac{1}{2}(m + \sqrt{6m})\},$$

$$E_m(4,4) = 2m + \{2\sqrt{m}\},$$

$$E_m(6,3) = 3m + \{\sqrt{12m-3}\},$$

where $\{x\}$ denotes the smallest integer greater than or equal to x. What are the corresponding numbers $E_m(3,7)$ and $E_m(7,3)$ for the Fibonacci tesselations?

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FIBONACCI TRIANGLES

Heiko Harborth and Arnfried Kemnitz

Let F_n denote the n^{th} Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let L_n denote the n^{th} Lucas number, that is, $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Since ancient times, mathematicians have discussed triangles with integral sides. There are $[(d+1)(d+3)(2d+1)/24]$ incongruent integral triangles with sides of length at most d (see [3]). If in addition the area is an integer, then such a triangle is sometimes called Heronian. It is the purpose of this note to investigate Heronian triangles such that the length of each side is a Fibonacci number. Such a triangle, if it exists, is called a Fibonacci triangle.

The triangle inequality implies that a Fibonacci triangle cannot be scalene. Furthermore, a Fibonacci triangle cannot be equilateral, since an equilateral triangle with integer sides has irrational area. Therefore a Fibonacci triangle must be one of two types of isosceles triangle: I: (F_k, F_k, F_n) or II: (F_{n-k}, F_n, F_n), with $1 \leq k < n$. Again, the triangle inequality implies $k = n - 1 \geq 3$ for Fibonacci triangles of type I. Let us consider triangles (F_{n-1}, F_{n-1}, F_n) with $n \geq 4$.

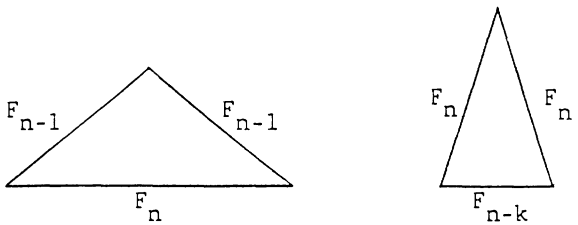

Figure 1. Candidates for Fibonacci Triangles

We will need the following identities, in which m, n denote non-negative integers:

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m, \tag{1}$$

$$F_m^2 + F_{m+1}^2 = F_{2m+1}, \tag{2}$$

$$F_{m-1} + F_{m+1} = L_m, \tag{3}$$

$$(F_m, L_{m\pm n}) \mid L_n, \tag{4}$$

$$F_m = a^2 \text{ iff } m = 0, 1, 2, 12, \tag{5}$$

$$F_m = 2a^2 \text{ iff } m = 0, 3, 6, \tag{6}$$

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 129–132.

$$L_m = a^2 \text{ iff } m = 1, 3, \tag{7}$$

$$L_m = 2a^2 \text{ iff } m = 0, 6. \tag{8}$$

Remarks: (1), (2), and (3) are (I_{13}), (I_{11}) and (I_8) in [2]; (4) is Theorem 1 in [4]; (5) through (8) are Theorems 3, 4, 1, 2 in [1].

Lemma 1: (a) $F_{n-3}L_n = 4F_{n-1}^2 - F_n^2 = F_{2n-3} + 2(-1)^n$,

(b) $F_{n+3}L_n = 4F_{n+1}^2 - F_n^2 = F_{2n+3} + 2(-1)^n$.

Proof of (a): $4F_{n-1}^2 - F_n^2 = (2F_{n-1} - F_n)(2F_{n-1} + F_n) = (F_{n-1} - F_{n-2})(F_{n-1} + F_{n+1}) =$

$F_{n-3}L_n$, using (3). $F_{2n-3} = F_{n-1}^2 + F_{n-2}^2 = 2F_{n-1}^2 - (F_n - F_{n-2})^2 + F_{n-2}^2 = 2F_{n-1}^2 - F_n^2 + 2F_nF_{n-2}$

$= 4F_{n-1}^2 - F_n^2 - 2(-1)^n$, using (2) and (1).

(The proof of (b) is similar.)

Lemma 2: (a) $F_{n-3}L_n = s^2$ iff $n = 3, 6$,

(b) $F_{n+3}L_n = s^2$ iff $n = 0$.

Proof of (a): $F_0L_3 = 0\cdot 4 = 0^2$; $F_3L_6 = 2\cdot 18 = 6^2$. Now suppose $F_{n-3}L_n = s^2$. Let

$d = (F_{n-3}, L_n)$. Then $F_{n-3} = dy^2$, $L_n = dz^2$. (4) implies $d \mid L_3$, that is, $d \mid 4$. If $d = 1$ or 4,

then (5) and (7) imply $n = 3$. If $d = 2$, then (6) and (8) imply $n = 6$.

Proof of (b): $F_3L_0 = 2\cdot 2 = 2^2$. Now suppose $F_{n+3}L_n = s^2$. Let $g = (F_{n+3}, L_n)$. Then $F_{n+3} =$

gv^2, $L_n = gw^2$. (4) implies $g \mid L_3$, that is, $g \mid 4$. If $g = 1$ or 4, then (5) and (7) yield an

inconsistency. If $g = 2$, then (6) and (8) imply $n = 0$.

Theorem 1: (F_{n-1}, F_{n-1}, F_n) is a Fibonacci triangle iff $n = 6$.

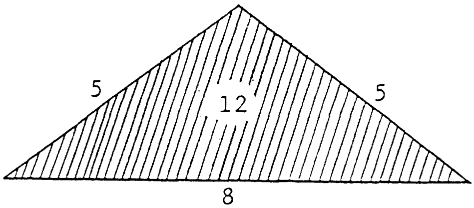

Figure 2. The Unique Fibonacci Triangle of Type I

Proof: If $n = 6$, then $(F_{n-1}, F_{n-1}, F_n) = (5, 5, 8)$ and has area 12. Now suppose (F_{n-1}, F_{n-1}, F_n)

is a Fibonacci triangle with $n \geq 4$. Let h be the altitude to the base. Since our triangle is

Heronian, h must be an integer. Furthermore, $4h^2 = 4F_{n-1}^2 - F_n^2$. Lemma 1a implies $4h^2 = F_{n-3}L_n$; Lemma 2a implies n = 3 or 6. Since $n \geq 4$, we must have n = 6.

<u>Remark:</u> The unique Fibonacci triangle of type I may be formed by the juxtaposition of two copies of the smallest primitive Pythagorean triangle: (3, 4, 5).

For Fibonacci triangles of type II, only the case k = 1 has been settled so far.

<u>Theorem 2:</u> Fibonacci triangles (F_{n-1}, F_n, F_n) do not exist.

<u>Proof:</u> Assuming the contrary, let h be the altitude to the base. Again, h is an integer and $4h^2 = 4F_n^2 - F_{n-1}^2$. Lemma 1b implies $4h^2 = F_{n+2}L_{n-1}$. Lemma 2b implies n = 1, an impossibility, since $n \geq 4$.

As a byproduct of Lemmas 1 and 2, we obtain all Fibonacci numbers F_m of the forms $x^2 \pm 2$, where m is odd.

<u>Theorem 3:</u> Let m be odd and positive. Then (a) $F_m = x^2-2$ iff m = 3, 9; (b) $F_m = x^2+2$ iff m = 3.

<u>Proof of (a):</u> $F_3 = 2 = 2^2-2$; $F_9 = 34 = 6^2-2$. Now suppose $F_m = x^2-2$. If $m \equiv 1 \pmod 4$, then $m = 2n-3$, $2 \mid n$. Now $x^2 = F_m + 2 = F_{2n-3} + 2(-1)^n$. Lemmas 1a and 2a imply n = 6, so m = 9. If $m \equiv 3 \pmod 4$, then $m = 2n+1$, $2 \nmid n$. Now $x^2 = F_m + 2 = F_{2n+1} - 2(-1)^n$. Lemmas 1b and 2b imply n = 1, so m = 3.

<u>Proof of (b):</u> $F_3 = 2 = 0^2 + 2$. Now suppose $F_m = x^2 + 2$. If $m \equiv 1 \pmod 4$, then $m = 2n+1$, $2 \mid n$. Now $x^2 = F_m - 2 = F_{2n+1} - 2(-1)^n$. Lemmas 1b and 2b imply n=1, a contradiction. If $m \equiv 3 \pmod 4$, then $m = 2n-3$, $2 \nmid n$. Now $x^2 = F_m - 2 = F_{2n-3} + 2(-1)^n$. Lemmas 1a and 2a imply n = 3, so m = 3.

It may be conjectured that $F_{2n} = x^2 \pm 2$ is impossible for $n \geq 3$.

The existence of Fibonacci triangles (F_{n-k}, F_n, F_n) with $k \geq 2$ remains an open question. We have verified that no such examples exist for $n \leq 25$. Perhaps (5, 5, 8) is the unique Fibonacci triangle?

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MOSAIC NUMBERS OF FIBONACCI TREES

Heiko Harborth and Sabine Lohmann

Different meanings of Fibonacci trees are used in the mathematical literature. Here we will consider those drawings of trees which represent the old rabbit story as in V. E. Hoggatt's book [3], p. 2. These Fibonacci trees T_n will be realized as polyominoes in the square grid such that vertices correspond to unit squares and edges to certain strings of edge-to-edge unit squares. Because of their patterns we will call these realizations mosaics of T_n. Subsequently we define the mosaic number $M(n)$ of T_n to be the smallest number of unit squares which are necessary for realizations of T_n. It is the purpose of this note to determine general bounds of $M(n)$ and exact values for small n.

A Fibonacci tree T_n has vertices of two different types, A and B, on levels 2 to n. Starting with a vertex A on level 2, each vertex of type A on level h is joined by two edges to vertices on level $h+1$, right to a vertex of type A and left to a vertex of type B, and each vertex of type B on level h is joined by an edge to a vertex of type A on level $h+1$ (see Figure 1). Thus

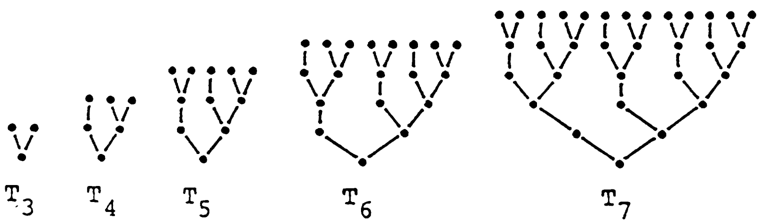

Figure 1. Fibonacci Trees.

we have F_h vertices on level h, F_{h-1} of type A and F_{h-2} of type B, where F_n denotes the Fibonacci number with $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Altogether a Fibonacci tree T_n has $F_2 + F_3 + \ldots + F_n = F_{n+2} - 2$ vertices and $F_{n+2} - 3$ edges.

We now define mosaics of T_n, that are tree polyominoes P_n, corresponding to the Fibonacci tree T_n. A polyomino is a simple connected part of the square grid [2]. Every vertex of T_n corresponds to a square of P_n. For two adjacent vertices of T_n the corresponding squares of P_n either have an edge in common, or they are connected by a sequence of additional squares each having two edges in common, one with its predecessor and one with its successor. No other common edges of squares and no common vertex points of exactly two squares are allowed. That square of P_n corresponding to the root vertex of the start level 2 of T_n is marked by a cross. This root square is allowed to have its common edge with the square of type B only one or two edges to left of its common edge with the square of type A.

Two mosaics of T_n are isomorphic if congruence is possible by translation and rotation. Reflection would change the directions right and left and thus is not allowed. All nonisomorphic mosaics of T_n without additional squares are constructed for $2 \leq n \leq 6$ in Figures 2 to 5.

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Figure 2. All Mosaics of T_2 and T_3.

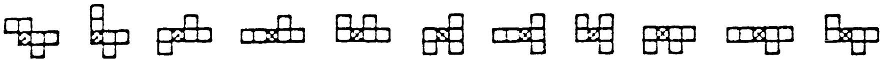

Figure 3. All Mosaics of T_4.

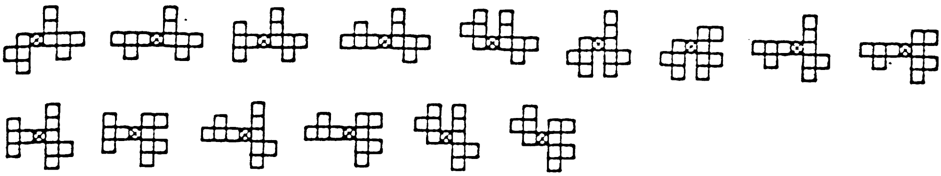

Figure 4. All Mosaics of T_5.

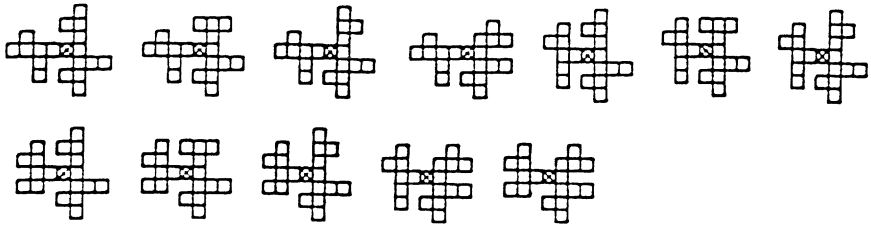

Figure 5. All Mosaics of T_6.

The number of vertices of T_n with degree 3 equals the number of squares with 3 neighbors in any mosaic of T_n, and so do the number of vertices with degree 1 and the number of squares with a single neighbor.

The existence of mosaics of T_n in general is evident if enough additional squares are used. Consequently the smallest number $M(n)$ of squares of all possible mosaics of T_n is of interest. This number is called the mosaic number $M(n)$, and

$$M(n) = F_{n+2} - 2 \quad \text{for } 2 \leq n \leq 6 \tag{1}$$

is proved by Figures 2 to 5. For $n \geq 7$ additional squares are necessary, which will be hatched in the figures.

We will prove the following upper bound for $M(n)$:

$$M(n) \leq \frac{n+4}{5} F_n + \frac{2n-10}{5} F_{n-1} + 2. \tag{2}$$

First special tree polyominoes P_n are constructed for T_n. The right path of T_n with vertices only of type A is represented by a horizontal sequence of squares, interrupted by $F_{n-i+1} - 1$ squares between those squares corresponding to vertices of levels i and $i+1$, $2 \leq i \leq n-1$. The left path of T_n with vertices alternating of types A and B is represented by a vertical uninterrupted sequence of $n-1$ squares. If now the $(F_{n-1}+1)$-st square horizontally is the root square of P_{n-1}, and the third square vertically is the root square of P_{n-2}, then we have a recurrence definition of P_n (see Figure 6), which has a horizontal length of $F_{n+1} - 1$ squares. The number $Q(n)$ of squares of this P_n fulfills

$$Q(n) = Q(n-1) + Q(n-2) + F_{n-1} + 1 \quad \text{with } Q(2) = 1,\ Q(3) = 3. \tag{3}$$

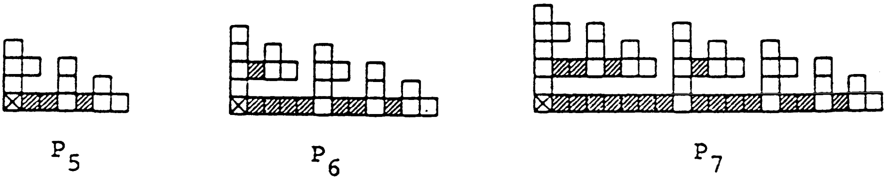

Figure 6.

By verification we receive an upper bound for the mosaic number

$$M(n) \leq Q(n) = \frac{n+4}{5} F_n + \frac{2n}{5} F_{n-1} - 1. \tag{4}$$

If a direct solution of (3) is desired, then repeated application together with the identity

$$F_i F_{n-i} = \frac{1}{5}\left(L_n + (-1)^{i+1} L_{n-2i}\right),\quad L_n = F_{n+1} + F_{n-1}, \tag{5}$$

from [1] leads straightforward to (4).

If P_{n-2}, P_{n-3} rotated by $\frac{\pi}{2}$, P_{n-3} rotated by π, and P_{n-4} rotated by $\frac{3\pi}{2}$ are fitted together for $n \geq 6$ as in Figure 7, then we obtain a slight improvement of the upper bound $Q(n)$. We can

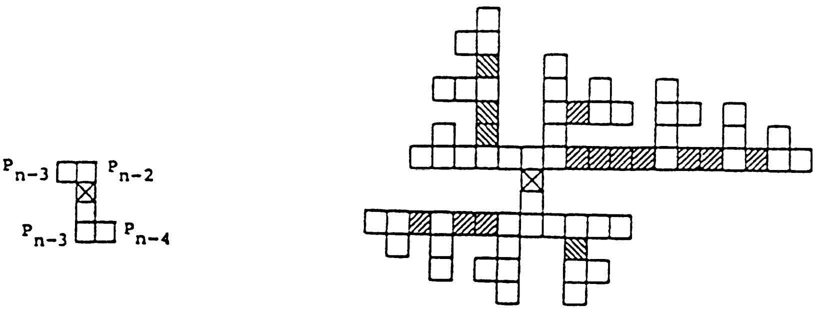

Figure 7.

subtract $F_{n-1} - 1 + F_{n-2} - 1 + F_{n-3} - 1 = 2F_{n-1} - 3$ from $Q(n)$, and get the asserted bound in (2) for $n \geq 6$, which is valid also for $n \geq 2$.

Further small improvements for large values of n are possible, however, at this moment we do not see how to get a smaller order of magnitude.

Since $M(n) \geq F_{n+2} - 2$ is trivial, we have proved in general

$$c_1 F_n < M(n) < c_2 n F_n, \tag{6}$$

or

$$c_3 \alpha^n < M(n) < c_4 n \alpha^n, \quad \alpha = \frac{1+\sqrt{5}}{2}, \tag{7}$$

with constant values c_1, c_2, c_3, c_4.

To improve the lower bound $F_{n+2} - 2$ at least a little we first assert

$$M(7) = 34 = F_9 - 2 + 2. \tag{8}$$

Figure 9 proves $M(7) \leq 34$. Assume that at most one additional square suffices for a mosaic of T_7. We check that the subgraph of T_7 drawn in Figure 8 allows mosaics with at most one additional square only in those six possibilities shown in Figure 8.

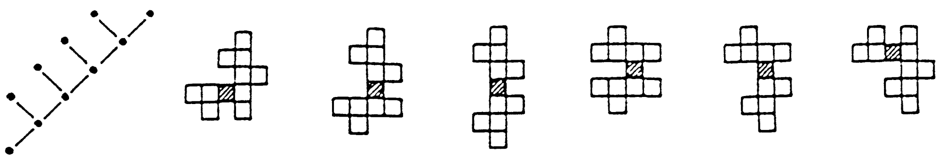

Figure 8.

Now the outside right square in the second row from bottom has to have a neighbor square which then is the neighbor of the root square of one of the mosaics of T_5 in Figure 4. Furthermore the outside right square in the bottom row has to be the neighbor of the root square of one of the mosaics of T_4 in Figure 3. It can be checked that both cases cannot occur simultaneously. Thus at least two additional squares are necessary, and $M(7) \geq 34$.

For $i = 1, 2, \ldots, \lfloor \frac{n-2}{5} \rfloor$ we find in T_n from level $n - 5i$ up to level $n - 5i + 5$ exactly F_{n-5i-1} disjunct subgraphs T_7 and F_{n-5i-2} disjunct subgraphs as in Figure 8. Therefore at least

$$2F_{n-5i-1} + F_{n-5i-2} = F_{n-5i+1} \tag{9}$$

additional squares are needed for every i, and we obtain the lower bound

$$M(n) \geq F_{n+2} - 2 + \sum_{i=1}^{\lfloor \frac{n-2}{5} \rfloor} F_{n-5i+1} \,. \tag{10}$$

What is about the exact values of $M(8)$, $M(9)$, ... ? Figure 10 and 11 give our best upper bounds. For small n it is a challenging problem to search for mosaics of T_n with few additional squares. For the general estimation we conjecture that neither the lower nor the upper bound in (6) will be the right order of magnitude.

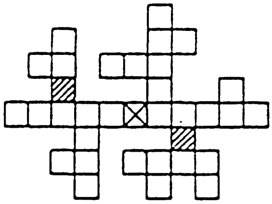

Figure 9. Mosaic of T_7, $M(7) = 34$.

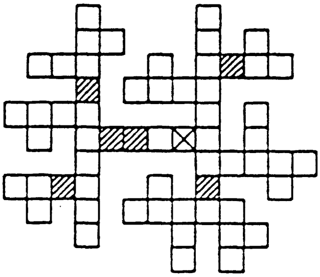

Figure 10. Mosaic of T_8, $M(8) \leq 59 = F_{10} - 2 + 6$

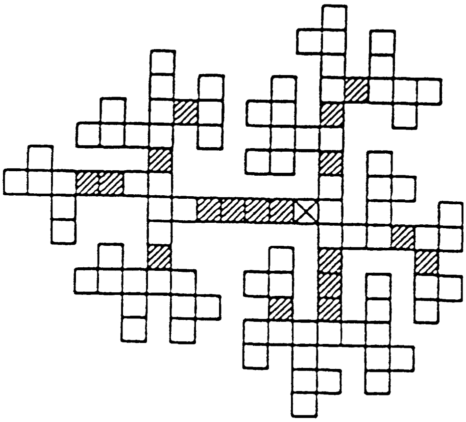

Figure 11. Mosaic of T_9, $M(9) \leq 105 = F_{11} - 2 + 18$.

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FALLING FACTORIAL POLYNOMIALS OF GENERALIZED FIBONACCI TYPE

A. F. Horadam

1. INTRODUCTION

In [4], the author extended the work of Asveld [2] for his Fibonacci-type sequence of polynomials G_n by means of the polynomials H_n (hereafter relabelled $\mathcal{H}_n$ to avoid confusion with Asveld's later symbolism which he used in [3]), and Pell numbers. The nature of the polynomials G_n, $\mathcal{H}_n$, and H_n is detailed in (2.4), (2.5), and (2.6) respectively.

Among the main concerns of [4] was an investigation of the properties of

$$a_{ij} = -\sum_{m=i+1}^{j} \beta_{im} a_{mj} \qquad (i<j) \qquad 1(\alpha)$$

with

$$a_{ii} = 1 \qquad 1(\alpha)'$$

where

$$\beta_{im} = \binom{m}{i}(-1)^{m-i}(p-2^{m-i}q) \qquad (i\leq m) \qquad 1(\beta)$$

so that

$$\beta_{ii} = p-q. \qquad 1(\beta)'$$

For future reference, we here record that, from $1(\alpha)$ and $1(\beta)$,

$$a_{25} = 10\{6(p-2q)^3 - 6(p-2q)(p-2^2q) + (p-2^3q)\}. \qquad 1(\gamma)$$

Another main thrust of [4] was an analysis of the structure of polynomials

$$P_j(n) = \sum_{i=0}^{j} a_{ij} n^i. \qquad 1(\delta)$$

Some particular results for the a_{ij} included

$$\frac{a_{n,n+1}}{a_{n-1,n}} = \frac{\beta_{n,n+1}}{\beta_{n-1,n}} = \frac{n+1}{n} \qquad 1(\epsilon)$$

$$a_{nm} = \binom{m}{n} a_{0,m-n} \qquad 1(\zeta)$$

$$\lim_{m\to\infty} \left(\frac{a_{0m}}{a_{nm}}\right) = n!k^n \qquad 1(\eta)$$

where

$$k = -1/\log[(p \pm \sqrt{p^2-4q(p-1-q)}/2q], \qquad 1(\theta)$$

with the positive root only being of interest to us.

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Before proceeding, I have to acknowledge an error in [4] which was detected by me too late for an alteration to be made in the published article. Specifically, equation (3.8) in [4], i.e. equation $1(\alpha)$ above, does not generally satisfy equation (3.6) of [4], though equation $1(\alpha)'$ does. Consequently, by the method used, the complete generalization — as claimed in [4] — does not extend beyond Fibonacci numbers. However, all the results in the bulk of [4] which follow, involving a_{ij} and β_{ij}, are valid.

In [3] Asveld generalized his work in [2] by replacing his ordinary power polynomials G_n by factorial polynomials H_n.

Here, we extend the work in [3]. Furthermore, we include some results for Jacobsthal numbers which were not considered in [4].

To facilitate comparison among the new results in this paper and those in [3] and [4], we shall use format in [4] as a model, and base our symbolism on that employed in [2], [3], and [4].

We define the sequence $\{U_n\}$ recursively by

$$U_n = pU_{n-1} - q\,U_{n-2}, \qquad U_0=1,\ U_1=p\ (U_{-1}=0) \tag{1.1}$$

so that when p=1 and q=−1, U_n becomes the Fibonacci number F_n. For U_n, the Binet form is

$$U_n = (\alpha^{n+1} - \beta^{n+1})/\Delta \tag{1.2}$$

where

$$\alpha = (p+\sqrt{p^2-4q})/2, \qquad \beta = (p-\sqrt{p^2-4q})/2, \tag{1.3}$$

whence

$$\alpha+\beta = p,\ \alpha\beta = q,\ \alpha-\beta = \sqrt{p^2-4q} = \Delta. \tag{1.4}$$

Let us designate by α' the value of α in (1.3) for the Fibonacci numbers F_n so that

$$\alpha' = (1+\sqrt{5})/2. \tag*{(1.3)$'$}$$

Numbers other than the Fibonacci numbers which will interest us are the *Pell numbers* P_n occuring when p=2, q=−1 in (1.1), and the *Jacobsthal numbers* J_n [5] which arise when p=1, q=−2. Thus

$$P_n = 2P_{n-1} + P_{n-2} \qquad P_0=1,\ P_1=2, \qquad P_{-1}=0 \tag{1.5}$$

and

$$J_n = J_{n-1} + 2J_{n-2} \qquad J_0=1,\ J_1=1, \qquad J_{-1}=0 \tag{1.6}$$

in which the initial conditions have been chosen to conform to Asveld's choice of notation for F_n.

Values of α in (1.3) for Pell and Jacobsthal numbers are

$$\alpha'' = 1+\sqrt{2} \qquad \text{for } P_n \tag*{(1.3)$''$}$$

and

$$\alpha''' = 2 \qquad \text{for } J_n. \tag*{(1.3)$'''$}$$

2. FALLING FACTORIAL POLYNOMIALS

Consider the set of polynomials $\{W_n\}$ defined recursively by

$$W_n = pW_{n-1} - qW_{n-2} + (p-q-1)\sum_{j=0}^{k} \gamma_j n^{(j)} \tag{2.1}$$

for which

$$W_0 = b,\ W_1 = pb-qa \qquad (W_{-1}=a,\ p-q-1 \neq 0) \tag{2.2}$$

and

$$n^{(j)} = n(n-1)(n-2)\ldots(n-j+1) = j!\binom{n}{j}, \quad \text{for } j \geq 1,\ n^{(0)}=1\ (0^{(0)}=1) \tag{2.3}$$

whence

$$n^{(n)} = n^{(n-1)} = n!. \tag{2.3}$$

In our nomenclature, $n^{(j)}$ is the *falling factorial* function of n. Consequently, we may refer to W_n in (2.1) as a *falling factorial polynomial of generalized Fibonacci type.*

Thus for Asveld's G_n in [2] we have

$$W_n = G_n \text{ when } a=0,\ b=1,\ p=1,\ q=-1 \text{ and } n^{(j)} \text{ is replaced by } n^j, \tag{2.4}$$

whereas for our $\mathcal{H}_n$ in [4],

$$W_n = \mathcal{H}_n \text{ when } n^{(j)} \text{ is replaced by } n^j, \tag{2.5}$$

while for Asveld's H_n in [3],

$$W_n = H_n \text{ when } a=0,\ b=1,\ p=1,\ q=-1. \tag{2.6}$$

In establishing some properties of falling factorials which we need, it is desirable to exploit the "Binomial Theorem for Factorial Polynomials", the proof of which Asveld [3] attributed to A. A. Jagers. This theorem is designated by Asveld as Lemma 1:

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}. \tag{2.7}$$

Hence,

$$\begin{cases} (n-1)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} n^{(k)} \\ (n-2)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-2)^{m-k} n^{(k)} \end{cases} \tag{2.8}$$

where, by (2.3),

$$(-x)^{(n)} = (-1)^n (x+n-1)^{(n)} \tag{2.9}$$

with, in particular,

$$\begin{cases} (-1)^{(m-i)} = (-1)^{m-i}(m-i)! \\ (-2)^{(m-i)} = (-1)^{m-i}(m-i+1)! \end{cases} \tag{2.9$'$}$$

by $(2.3)'$.

For subsequent availability, we now introduce the symbol δ_{im} which corresponds to β_{im} in $1(\beta)$ and is defined by

$$\delta_{im} = \binom{m}{i}\{p(-1)^{(m-i)} - q(-2)^{(m-i)}\} \qquad (i \leq m) \tag{2.10}$$

$$= \frac{m!}{i!}(-1)^{m-i}\{p-(m-i+1)q\} \qquad \text{by } (2.9)'$$

so that

$$\delta_{ii} = p-q. \tag*{(2.10)$'$}$$

In particular,

$$\delta_{j-i,j} = (-1)^i j^{(i)}\{p-(i+1)q\}. \tag*{(2.10)$''$}$$

Other definitions we need include

$$b_{ij} = -\sum_{m=i+1}^{j} \delta_{im} b_{mj}, \qquad i<j \quad (b_{ij}=0,\ i>j) \tag{2.11}$$

with

$$b_{ii} = 1, \tag*{(2.11)$'$}$$

which are analogues of a_{ij} and a_{ii} in $1(\alpha)$ and $1(\alpha)'$, and

$$\prod\nolimits_j(n) = \sum_{i=0}^{j} b_{ij} n^{(i)}, \tag{2.12}$$

the analogue of $P_j(n)$ in $1(\delta)$.

Thus, for instance,

$$b_{14} = -\sum_{m=2}^{4} \delta_{1m} b_{m4} = -\ \{\delta_{12}b_{24} + \delta_{13}b_{34} + \delta_{14}b_{44}\} \tag{2.13}$$

$$= -\{-2(p-2q)[12(p-2q)^2 - 12(p-3q)] + 6(p-3q).4(p-2q) - 24(p-4q)\}$$

$$= 24\{(p-2q)^3 - 2(p-2q)(p-3q) + (p-4q)\},$$

and

$$b_{03} = -\sum_{m=1}^{3} \delta_{0m} b_{m3} = -\ \{\delta_{01}b_{13} + \delta_{02}b_{23} + \delta_{03}\delta_{33}\} \tag*{(2.13)$'$}$$

$$= -\{-(p-2q)[6(p-2q)^2 - 6(p-3q)] + 2(p-3q).3(p-2q) - 6(p-4q)\}$$

$$= 6\{(p-2q)^3 - 2(p-2q)(p-3q) + (p-4q)\}$$

whence

$$\frac{b_{14}}{b_{03}} = 4\left[= \binom{4}{1}\right]. \tag*{(2.13)$''$}$$

Because of the falling factorials in (2.1), we might reasonably expect the appearance of Stirling numbers as the theory progresses. Furthermore, the existence in (2.9) of the falling factorial of a negative number suggests the possible appearance of Lah numbers.

We revert now to (2.1). Substituting $W_n = \sum_{i=0}^{k} B_i n^{(i)}$ in (2.1), we have, with the aid of (2.9)$'$,

$$\sum_{i=0}^{k} B_i n^{(i)} - \sum_{i=0}^{k}\left[\sum_{\ell=0}^{i} B_i\binom{i}{\ell}\{p(-1)^{(i-\ell)} - q(-2)^{(i-\ell)}\}n^{(\ell)}\right] - (p-q-1)\sum_{i=0}^{k}\gamma_i n^{(i)} = 0 \tag{2.14}$$

so, by (2.10),

$$B_i - \sum_{m=i}^{k} \delta_{im} B_m - (p-q-1)\gamma_i = 0. \tag{2.15}$$

Thus, we see how δ_{im} appears in this treatment in a natural way.

Subsequently, we will be largely concerned with an investigation of the properties of b_{ij}, δ_{im}, and $\prod_j(n)$.

Apropos of his pleasing Proposition 3 in [3], Asveld commented that it allows him to connect his main coefficients in [2] and [3] by means of Stirling numbers. This connection can be generalized to our a_{ij} and b_{ij} in $1(\alpha)$ and (2.11) respectively.

In terms of falling factorials, *Stirling numbers of the first kind* s(k,n) and *Stirling numbers of the second kind S(k,n)* are defined by

$$x^{(n)} = \sum_{k=0}^{n} s(k,n)x^k \tag{2.16}$$

and

$$x^n = \sum_{k=0}^{n} S(k,n)x^{(k)}. \tag{2.17}$$

Tables of values of s(k,n) and S(k,n) may be found in [1], [6], and [8]. Leading properties of the Stirling numbers may also be located in [1] and [6].

Our generalization of Asveld's expression alluded to above is

$$a_{ij} = \sum_{m=i}^{j} s(i,m)\left(\sum_{\ell=m}^{j} b_{m\ell}\, S(\ell,j)\right) \qquad (i \le j). \tag{2.18}$$

We may use (2.18) to show, for example, that

$$a_{25} = 60(p-2q)^3 - 60(p-2q)(p-2^2q) + 10(p-2^3q), \quad \text{which is } (1\gamma).$$

In like manner, (2.18) may be used to show that for Fibonacci, Pell, and Jacobsthal numbers

$$a_{25} = 810,\ 2500,\ 4970.$$

From (2.10) and $1(\beta)$, and $(2.10)'$ and $1(\beta)'$, we observe that

$$\frac{\delta_{im}}{\beta_{im}} = \frac{(m-i)!\,\{p-(m-i+1)q\}}{p-2^{m-i}q} \tag{2.19}$$

with

$$\delta_{ii} = \beta_{ii} = p-q \qquad \left(\text{i.e., } \frac{\delta_{ii}}{\beta_{ii}} = 1\right). \tag{2.19$'$}$$

3. THE POLYNOMIALS $\prod_j(n)$

A detailed tabulation of the generalized polynomials $P_j(n)$, which are a special case of $\prod_j(n)$, was given in [4]. To avoid a repetition of this complicated table (call it now Table A for convenience), we merely state how our corresponding table for polynomials $\prod_j(n)$ - call it Table B for reference purposes - may be quite simply obtained from Table A by the following rule (cf. (2.20)):

$$\text{Rule:} \quad \begin{cases} \text{replace} & p-2^k q & \text{in Table A} \\ \text{by} & k!\{p-(k+1)q\} & \text{in Table B.} \end{cases} \tag{3.1}$$

When k=1, this rule implies that $p-2q$ is the same in both tables.

Table B may now be constructed by the reader (though details can be provided on request). For example,

$$\begin{aligned}\prod_3(n) &= \sum_{i=0}^{3} b_{i3}n^{(i)} = b_{03} + b_{13}\, n^{(1)} + b_{23}n^{(2)} + b_{33}n^{(3)} \\ &= 6(p-2q)^3 - 6(p-2q).2(p-3q) + 6(p-4q) + \\ &+ [6(p-2q)^2 - 6(p-3q)]\, n^{(1)} + 3(p-2q)n^{(2)} + n^{(3)}.\end{aligned}$$

Putting p=1, q=−1 in this expression, we find that it reduces to the form given in Asveld's Table 1 [3, p. 363] for Fibonacci numbers (i.e., our $\prod_3(n)$ = Asveld's $\pi_3(n)$):

$$\prod_3(n) = 48 + 30n^{(1)} + 9n^{(2)} + n^{(3)}.$$

As an illustration, we have, with n=4,

$$\prod_3(4) = 48 + 30.4 + 9.12 + 24 = 300$$

for Fibonacci numbers.

On the other hand, for Pell numbers (p=2, q=−1), we get

$$\prod_3(n) = 180 + 66n^{(1)} + 12n^{(2)} + n^{(3)}$$

which yields, for n=4,

$$\prod_3(4) = 180 + 66.4 + 12.12 + 24 = 612.$$

Moreover, for Jacobsthal numbers (p=1, q=−2), we find

$$\prod_3(n) = 384 + 108n^{(1)} + 15n^{(2)} + n^{(3)}$$

producing

$$\prod_3(4) = 384 + 108.4 + 15.12 + 24 = 1020.$$

Appended are Tables 1, 2 and 3 which give some detailed information about $\prod_j(n)$ for the Fibonacci, Pell, and Jacobsthal numbers respectively. Table 1 is, in effect, a truncated version of Asveld's Table 1 [3]. It has been abbreviated for comparison with Tables 2 and 3.

From the information in these tables, we may construct tabulations for the values of b_{nm} as n,m take the values 0,1,2,... for Fibonacci, Pell, and Jacobsthal polynomials similar to those for a_{nm} in [4].

j	$\prod_j(n)$					
0						1
1					$n^{(1)}$	+ 3
2				$n^{(2)}$ +	$6n^{(1)}$	+ 10
3			$n^{(3)}$ +	$9n^{(2)}$ +	$30n^{(1)}$	+ 48
4		$n^{(4)}$ +	$12n^{(3)}$ +	$60n^{(2)}$ +	$192n^{(1)}$	+ 312
5	$n^{(5)}$ +	$15n^{(4)}$ +	$100n^{(3)}$ +	$480n^{(2)}$ +	$1560n^{(1)}$	+ 2520

Table 1. Factorial Polynomials $\prod_j(n)$ for Fibonacci Numbers (j=0,1,...5).

j	$\prod_j(n)$					
0						1
1					$n^{(1)}$	+ 4
2				$n^{(2)}$ +	$8n^{(1)}$	+ 22
3			$n^{(3)}$ +	$12n^{(2)}$ +	$66n^{(1)}$	+ 180
4		$n^{(4)}$ +	$16n^{(3)}$ +	$132n^{(2)}$ +	$720n^{(1)}$	+ 1968
5	$n^{(5)}$ +	$20n^{(4)}$ +	$220n^{(3)}$ +	$1800n^{(2)}$ +	$9840n^{(1)}$	+ 26880

Table 2. Factorial Polynomials $\prod_j(n)$ for Pell Numbers (j=0,1,...5).

j	$\prod_j(n)$					
0						1
1					$n^{(1)}$	+ 5
2				$n^{(2)}$ +	$10n^{(1)}$	+ 36
3			$n^{(3)}$ +	$15n^{(2)}$ +	$108n^{(1)}$	+ 384
4		$n^{(4)}$ +	$20n^{(3)}$ +	$216n^{(2)}$ +	$1536n^{(1)}$	+ 5472
5	$n^{(5)}$ +	$25n^{(4)}$ +	$360n^{(3)}$ +	$3840n^{(2)}$ +	$27360n^{(1)}$	+ 97440

Table 3. Factorial Polynomials $\prod_j(n)$ for Jacobsthal Numbers (j=0,1,...5).

Next, we introduce two infinite sequences $M_{1,-1}$ and $\mu_{1,-1}$, the ordered subscripts indicating that p=1, q=−1 respectively, where $M_{1,-1}$ are the constants in $\prod_0(n)$, $\prod_1(n)$, $\prod_2(n)$, ... and $\mu_{1,-1}$ are the sums of the coefficients of the $n^{(i)}$ in $\prod_1(n)$, $\prod_2(n)$, $\prod_3(n)$, ... More generally, for unspecified p and q, we emply the symbols $M_{p,q}$ and $\mu_{p,q}$.

From Table 1 (for Fibonacci numbers), we derive

$$M_{1,-1}: \quad 1, \quad 3, \quad 10, \quad 48, \quad 312, \quad 2520, \ldots \tag{3.2}$$

$$\mu_{1,-1}: \quad 0, \quad 1, \quad 7, \quad 40, \quad 265, \quad 2156, \ldots \tag{3.3}$$

while from Table 2 (Pell numbers)

$$M_{2,-1}: \quad 1, \quad 4, \quad 22, \quad 180, \quad 1968, \quad 26880, \ldots \tag{3.4}$$

$$\mu_{2,-1}: \quad 0, \quad 1, \quad 9, \quad 79, \quad 869, \quad 11881, \ldots \tag{3.5}$$

and from Table 3 (for Jacobsthal numbers)

$$M_{1,-2}: \quad 1, \quad 5, \quad 36, \quad 384, \quad 5472, \quad 97440, \ldots \tag{3.6}$$

$$\mu_{1,-2}: \quad 0, \quad 1, \quad 11, \quad 124, \quad 1773, \quad 31586, \ldots . \tag{3.7}$$

Of the sequences (3.2)-(3.7), only (3.2) is listed in [3]. Though (3.3) is a direct analogue of equation (4.3) in [3], which generalises the corresponding sequence λ in [2], Asveld [3] does not consider this interpretation, but restricts his μ-sequence to the coefficients of $n^{(1)}$. If we now consider this interpretation and label our sequences

$\mu'_{1,-1}$ $\mu'_{2,-1}$ and $\mu'_{1,-2}$, we obtain

$$\mu'_{1,-1}: \quad 0, \quad 1, \quad 6, \quad 30, \quad 192, \quad 1560, \quad 15120, \ldots \tag{3.3$'$}$$

$$\mu'_{2,-1}: \quad 0, \quad 1, \quad 8, \quad 66, \quad 720, \quad 9840, \ldots \tag{3.5$'$}$$

$$\mu'_{1,-2}: \quad 0, \quad 1, \quad 10, \quad 108, \quad 1536, \quad 27360, \ldots . \tag{3.7$'$}$$

None of the sequences (3.2)-(3.7) and $(3.3)'$, $(3.5)'$, and $(3.7)'$ is listed in [9].

Asveld's restriction in [3] of his μ-sequence to the coefficient of $n^{(1)}$ arises by strict analogy with his work in [2]. In our generalization, for the μ'-sequence this becomes

$$\begin{aligned} \prod_k(1)-\prod_k(0) &= \left(b_{0k}1^{(0)} + b_{1k}1^{(1)}\right) - b_{0k}0^{(0)} \\ &= b_{1k} \text{ since } 1^{(0)}=1^{(1)}=1,\ 1^{(2)}=1^{(3)}=\ldots=1^{(k)}=0. \end{aligned} \tag{3.8}$$

On the other hand, strict analogy with our work in [4] [namely, $P_k(1)-P_k(0) = a_{1k}+a_{2k}+\ldots+a_{kk}$, since $1^1=1^2=\ldots=1^k=1$] produces for the generalized μ-sequence

$$b_{1k}+b_{2k}+\ldots+b_{kk}.$$

4. THE NUMBERS b_{ij}

Properties of the coefficients b_{ij} occurring in (2.11) in the expression for $\prod_j(n)$ will now be investigated. Further attention will also be accorded to the elements of the sequences $M_{p,-q}$ and $\mu_{p,-q}$.

The results in this section are new, but parallel similar results obtained in [4].

Consequently we will not reproduce the proofs in [4] which serve as our models. Analogues of some of the results below are given earlier in $1(\epsilon)$, $1(\zeta)$, $1(\eta)$, and $1(\theta)$.

Note that by $(2.10)''$

$$\delta_{n,n+1} = -(n+1)(p-2q). \tag{4.1}$$

We deduce:

<u>Lemma 1:</u> $$\frac{b_{n,n+1}}{b_{n-1,n}} = \frac{\delta_{n,n+1}}{\delta_{n-1,n}} = \frac{n+1}{n}. \tag{4.2}$$

<u>Lemma 2:</u> $$\frac{b_{n,m}}{b_{n-1,m-1}} = \frac{m}{n}. \tag{4.3}$$

<u>Theorem 1:</u> $$b_{nm} = \binom{m}{n} b_{0,m-n}. \tag{4.4}$$

<u>Theorem 2:</u> $$\lim_{m\to\infty} \left\{ \frac{b_{0m}}{b_{0,m-1}} - mk \right\} = 0 \text{ where } k = \lim_{m\to\infty} \left(\frac{b_{0m}}{b_{1m}}\right). \tag{4.5}$$

<u>Theorem 3:</u> $$\lim_{m\to\infty} \left(\frac{b_{0m}}{b_{nm}}\right) = n!k^n. \tag{4.6}$$

<u>Theorem 4:</u> $$\lim_{m\to\infty} \left(\frac{b_{0m}}{b_{1m}}\right) = \left(\frac{1}{t}-1\right)^{-1} \text{ [=k by Theorem 2]} \tag{4.7}$$

where

$$t = (p \pm \sqrt{p^2-4q(p-q-1)}/2q. \tag{4.8}$$

It should be particularly noted that t here, and in [4], satisfies the quadratic equation

$$qt^2 - pt + p-q-1 = 0 \tag{4.9}$$

though the value of k is different in the two cases, namely, $-(\log t)^{-1}$ in [4] and $(\frac{1}{t}-1)^{-1}$ here, all of which indicates the formal similarity of the work in this paper to that in [4].

Values of k (Theorem 4) are

$$\begin{aligned} k &= (1+\sqrt{5})/2 = \alpha' && \text{for Fibonacci numbers} \\ &= 1+\sqrt{3} && \text{for Pell numbers} \\ &= (\sqrt{17}+3)/2 && \text{for Jacobsthal numbers.} \end{aligned} \tag{4.10}$$

We denote by $M_i^{(p,q)}$ and $\mu_i^{(p,q)}$ the elements in the ith positions in $M_{p,q}$ and $\mu_{p,q}$ respectively. Proceeding as in [4], we may demonstrate that

<u>Theorem 5:</u> $$\lim_{i\to\infty}\left(\frac{M_i^{(p,q)}}{\mu_i^{(p,q)}}\right) = \left(e^{\frac{1}{k}}-1\right)^{-1}. \tag{4.11}$$

Suppose, on the other hand, we examine limits corresponding to the sequences $\mu'_{1,-1}$, $\mu'_{2,-1}$ and $\mu'_{1,-2}$ in $(3.3)'$, $(3.5)'$, and $(3.7)'$ respectively. Using the symbol $\mu'_{p,-q}$ in the obvious way, we find, by Theorem 4,

$$\lim_{i\to\infty}\left(\frac{M_i^{(p,-q)}}{\mu_i'^{(p,-q)}}\right) = \lim_{i\to\infty}\left(\frac{b_{0i}}{b_{1i}}\right) = k \tag{4.12}$$

$$= \frac{1+\sqrt{5}}{2} = \alpha' \qquad \text{from (4.10)}$$

$$= \lim_{i\to\infty}\left(\frac{F_{i+1}}{F_i}\right)$$

which is in accordance with Asveld's Proposition [3] for Fibonacci numbers.

Before discussing the nature of b_{ij} in more depth, we need some basic information on the number p_n of unrestricted partitions of an integer n, for which the enumerating generating function p(t) is (Riordan [8])

$$p(t) = \sum_{n=0}^{\infty} p_n t^n = (1-t)^{-1}(1-t^2)^{-1}\ldots(1-t^k)^{-1}\ldots \quad (p_0=1). \tag{4.13}$$

Values of p_n, obtained by Gupta, are given in Riordan [8], as are other relevant details about partitions, e.g., Ferrers graphs. The first few values of p_n are: $p_i=i(i=1,2,3)$, $p_4=5$, $p_5=7$, $p_6=11$, $p_7=15$, $p_8=22$. Following Riordan [8], we adopt the exponent convention for representing the repeated part of a partition, namely, for example, the partition of 8 as 221111 is written 2^21^4, for any order (composition) of the digits 1 and 2 in the partition, the number of such compositions being $\frac{6!}{2!4!} = 15$.

Table 4 for partitions of n when n=1,2,...,8, (Riordan [8]), will be sufficient for our purposes. (The integers collected to form a partition, or composition, are called its "parts".)

Number of Parts

n	1	2	3	4	5	6	7	8
1	1							
2	2	1^2						
3	3	21	1^3					
4	4	31	21^2	1^4				
		2^2						
5	5	41	31^2	21^3	1^5			
		32	2^21					
6	6	51	41^2	31^3	21^4	1^6		
		42	321	2^21^2				
		3^2	2^3					
7	7	61	51^2	41^3	31^4	21^5	1^7	
		52	421	321^2	2^21^3			
		43	3^21	2^31				
			32^2					
8	8	71	61^2	51^3	41^4	31^5	21^6	1^8
		62	521	421^2	321^3	2^21^4		
		53	431	3^21^2	2^31^2			
		4^2	42^2	32^21				
			3^22	2^4				

Table 4. Partitions of n by Number of Parts (n=1,2,...,8).

It should be noted that the total number of compositions of n in column k of Table 4 is $\binom{n-1}{n-k}$ so that the total number of compositions of n is

$$\binom{n-1}{n-1}+\binom{n-1}{n-2}+\ldots+\binom{n-1}{n-k}+\ldots+\binom{n-1}{1}+\binom{n-1}{0}=(1+1)^{n-1}=2^{n-1}. \tag{4.14}$$

When $n=j-i$, the critical number in our work, this number of compositions is 2^{j-i-1}.

We revert now to the b_{ij}, which we will describe in terms of sums and products of δ's. To illustrate the principles involved, choose b_{16}. Now

$$\begin{aligned}
b_{16} &= \binom{6}{1} b_{05} && \text{by Theorem 1}\\
&= \binom{6}{1}\left\{-\sum_{m=1}^{5} \delta_{om} b_{m5}\right\} && \text{by (2.11)}\\
&= -\binom{6}{1}\left\{\delta_{01}b_{15}+\delta_{02}b_{25}+\delta_{03}b_{35}+\delta_{04}b_{45}+\delta_{05}b_{55}\right\}\\
&= -\binom{6}{1}\left\{\delta_{01}\binom{5}{1}b_{04}+\delta_{02}\binom{5}{2}b_{03}+\delta_{03}\binom{5}{3}b_{02}+\delta_{04}\binom{5}{4}b_{01}+\delta_{05}\right\} && \text{by Theorem 1}
\end{aligned}$$

$$\begin{aligned}
&= 6^{(5)} \Big\{\delta_{01}\delta_{12}\delta_{23}\delta_{34}\delta_{45} - \Big(\delta_{02}\delta_{23}\delta_{34}\delta_{45}+\delta_{01}\delta_{13}\delta_{34}\delta_{45}+\\
&\qquad \delta_{01}\delta_{12}\delta_{24}\delta_{45}+\delta_{01}\delta_{12}\delta_{23}\delta_{35}\Big)+\Big([\delta_{01}\delta_{12}\delta_{25}+\delta_{01}\delta_{14}\delta_{45}+\\
&\qquad \delta_{03}\delta_{34}\delta_{45}]+[\delta_{01}\delta_{13}\delta_{35}+\delta_{02}\delta_{23}\delta_{35}+\delta_{02}\delta_{24}\delta_{45}]\Big)-\\
&\qquad \Big([\delta_{02}\delta_{25}+\delta_{03}\delta_{35}]+[\delta_{01}\delta_{15}+\delta_{04}\delta_{45}]\Big)+\delta_{05}\Big\} \ldots\ldots\ldots\ldots\, \text{(I)}
\end{aligned}$$

by repeated use of Theorem 1 and (2.11)

$$\begin{aligned}
&= 6^{(5)} \Big\{(p-2q)^5 - 4(p-2q)^3(p-3q)+\Big(3(p-2q)(p-3q)^2+3(p-2q)^2(p-4q)\Big)-\\
&\qquad \Big(2(p-3q)(p-4q)+2(p-2q)(p-5q)\Big)+(p-6q)\Big\} \ldots\ldots\, \text{(II) by (2.10)}''.
\end{aligned}$$

Notice that, in (II), $6^{(5)}(=720)$ is of the form $j^{(j-i)}$ for b_{16} so that the falling factorial factor required by Asveld [3] in his Proposition 3 exists here.

A close analysis of (I) reveals a clearly discernible pattern of mathematical behaviour which relates to the theory of partitions of integers when $n=5(=j-i)$.

Suppose we consider the differences $\ell-k$ of the two subscripts of $\delta_{k\ell}$, where $0\leq k<\ell\leq 5$, i.e., $1\leq\ell-k\leq 5$, as the numbers of a set of 5 to be investigated. Then (I) shows us that the following tabular representation (Table 5) of the products (and sums of products) of δ's in terms of subscript differences exists:

Term	Compositions	Partitions	Number of Compositions
$\delta_{01}\delta_{12}\delta_{23}\delta_{34}\delta_{45}$	11111	1^5	$\binom{4}{0}=1$
$\delta_{02}\delta_{23}\delta_{34}\delta_{45}+\cdots+\delta_{01}\delta_{12}\delta_{23}\delta_{35}$	2111, 1211, 1121, 1112	21^3	$\binom{4}{1}=4$
$\delta_{01}\delta_{12}\delta_{25}+\cdots+\delta_{03}\delta_{34}\delta_{45}$	113, 131, 311	31^2	$\binom{4}{2}=6$
$\delta_{01}\delta_{13}\delta_{35}+\cdots+\delta_{02}\delta_{24}\delta_{45}$	122, 212, 221	2^21	
$\delta_{02}\delta_{25}+\delta_{03}\delta_{35}$	23, 32	32	$\binom{4}{3}=4$
$\delta_{01}\delta_{15}+\delta_{04}\delta_{45}$	14, 41	41	
δ_{05}	5	5	$\binom{4}{4}=1$

Table 5. Subscript Differences as Partitions.

Observe that the subscripts in each product of δ's form a chain beginning with 0 and ending with 5:

$$0 \to a \to b \to \ldots \to 5 \qquad (0 \le a < b < \ldots \le 5)$$

with, in general, some of the digits 1,2,3,4 missing. The number m of digits missing in each row of Table 5 appears in the binomial coefficient $\binom{4}{m}$ for that row (m=0,1,2,3,4).

These $2^4 = 16$ partition representations of the subscript differences in Table 5 are precisely those given for $n(=j-i) = 5$ in Table 4.

It is not difficult to see why the δ-subscript differences are necessarily related to partitions. This connection must be so because <u>each difference corresponds to one of the integers 1,2,3,4,5, and the sum of the differences in any product of δ's is 5.</u> A similar pattern of representations exists for all b_{ij} for which $j-i=5$, e.g., $b_{05}, b_{27}, b_{38}, \ldots, b_{i,i+5}, \ldots$.

In passing, note that the $\binom{4}{2} = 6$ compositions occurring in Table 5 when 2 digits are missing are made up of $\frac{3!}{2!} + \frac{3!}{2!}$ when we consider the repetitions.

Any partition appearing in Table 4 may now be interpreted in terms of δ's. Thus, for $n(=j-i)=7$, e.g. for b_{07}, and the number of parts equal to 5, i.e., products of 5 δ's, we know that there are $\binom{6}{2} = 15$ terms (compositions) with 2 digits of the set 1,2,3,4,5,6 missing. These 15 terms are made up of:

$$\begin{cases} \text{type } 31^4 : & \frac{5!}{4!} = 5 \\ \text{type } 2^21^3 : & \frac{5!}{2!3!} = 10. \end{cases}$$

Similarly, for $n=8(=j-i)$ and the number of parts 4, the $\binom{7}{4} = 35$ compositions with 4 digits of the set $\{1,2,3,4,5,6,7\}$ missing are made up of:

$$\begin{cases} \text{type } 51^3 & : & \frac{4!}{3!} & = 4 \\ \text{type } 421^2 & : & \frac{4!}{2!} & = 12 \\ \text{type } 3^21^2 & : & \frac{4!}{2!2!} & = 6 \\ \text{type } 32^21 & : & \frac{4!}{2!} & = 12 \\ \text{type } 2^4 & : & \frac{4!}{4!} & = 1. \end{cases}$$

From this combinatorial structure of the δ's, the expression for any b_{ij} in terms of p and q may be, with perseverance, written down by means of $(2.10)''$.

Working more generally and following Asveld, we may now write

$$b_{ij} = j^{(j-i)} f(p,q) ,$$

where f(p,q) contains 2^{j-i-1} terms.

Then, as j = 1,2,3,... for b_{oj}, we have the following sequences of values for f(p,q):

$f(1,-1)$	:	3,	5,	8,	13,	21,...	(truncated Fibonacci sequence)
$f(2,-1)$	:	4,	11,	30,	82,	224,...	
$f(1,-2)$	:	5,	18,	64,	228,	812,... .	

Neither of the two latter (truncated) sequences appears in [9]. Repetitions of these three sequences recur when we consider b_{1j}(j=2,3,4,...), b_{2j} (j=3,4,5,...), b_{3j} (j=4,5,6,...),..., b_{ij} (j=i+1, i+2, i+3,...),... .

That we have previously recognised a relationship of the b_{ij} to Stirling numbers may now be not so surprising if we remind ourselves of some aspects of the nature of Stirling numbers:

(i) $(-1)^{n-k}$ s(k,n) = the number of permutations of n symbols which have exactly k cycles,
(ii) S(k,n) = the number of ways of partitioning a set of n elements into k non-empty sets.

In the truncated Fibonacci sequence $f(1,-1)$ we may notice the numbers F_{j-1+2} (j>i), in accordance with Asveld's Proposition 3 [3]:

$$b_{ij} = j^{(j-i)} F_{j-i+2} \qquad (j>i).$$

5. CONCLUDING REMARKS

Before concluding this exposition, we mention a few further interesting facets of factorials which are peripherally relevant.

5.1 Lah Numbers

Earlier, in (2.9), it was necessary to use the relationship between the falling factorials for a positive quantity and a negative quantity. We explore this concept with Lah numbers.

Lah numbers L_{nk}, which have a tenuous and indirect connection with our work, are defined [7], [8] as the coefficients in the expression

$$(-x)^n = \sum_{k=0}^{n} L_{nk}\, x^{(k)} \qquad (k \leq n) \tag{5.1}$$

with inverse expression

$$(x)^n = \sum_{k=0}^{n} L_{nk}\, (-x)^{(k)} \tag{5.2}$$

so that

$$\delta^*_{nm} = \sum_{j=k}^{m} L_{nj} L_{jm} \tag{5.3}$$

in which δ^*_{nm} is the Kronecker delta ($\delta^*_{nn}=1$; $\delta^*_{nm}=0$, $n \neq m$).

It may be shown [8] that

$$L_{nk} = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1} \qquad (L_{n0} = \delta^*_{n0},\ L_{n1} = (-1)^n n!), \tag{5.4}$$

whence the numbers $(-1)^n\, L_{nk}$ are positive. Moreover [8],

$$L_{nk} = \sum (-1)^j\, s(n,j) S(j,k) \tag{5.5}$$

where s(n,j) and S(j,k) are Stirling numbers of the first and second kinds respectively ((2.16), (2.17)), and where the summation extends over values of j for which s(n,j) and S(j,k) are non-zero.

From (4.2) and (5.4), we may derive

$$\frac{\delta_{n,n+1}}{\delta_{n-2,n-1}} = \frac{b_{n,n+1}}{b_{n-2,n-1}} = -\frac{L_{n+1,n}}{L_{n,n-1}} = \frac{n+1}{n-1}\,. \tag{5.6}$$

Other formulas connecting the ratios of Lah numbers and δ_{ij} are obtainable.

5.2 Rising Factorials

One might reasonably ask whether there is a theory for rising factorials corresponding to that for falling factorials.

The *rising factorial* $n_*^{(j)}$ is clearly defined as:

$$\begin{aligned} n_*^{(j)} &= n(n+1)(n+2)\ldots(n+j-1) \\ &= (-1)^j\,(-n)^{(j)}. \end{aligned} \tag{5.7}$$

Thus, rising factorials are directly related to falling factorials of negative numbers which, by (2.9), are connected with ordinary falling factorials, so it would appear that no fresh development flows from this consideration.

5.3 Fermat Numbers

It is of minor interest to note that while *Fermat numbers*, for which p=3, q=2 in (1.1), do not fit into our scheme (since for them $p-q-1 = 3-2-1 = 0$, which does not satisfy the requirement of (2.2)) yet it is possible to obtain a table for them similar to Tables 1,2, and 3.

Patterns of numbers which we would expect to occur do exist, though the numbers in alternate sloping lines are all positive or all negative. The reason for the existence of this pattern is that $\prod_j(n)$, on which the tables depend, is a function of p and q, but does not involve the factor $p-q-1$.

Of course, the Fermat numbers (0), 1,3,7,15,31,63,... are merely the numbers of the form $2^n - 1$ for n=(0), 1,2,3,4,5,6,... respectively.

5.4 Finale

Omitted from [4] was any reference to, or application of, Jacobsthal numbers. Interested readers might, on the basis of material in this paper, obtain relevant results for themselves.

In [4], and in this article, we have progressed in our extensions of [2] and [3] through the Λ,λ and M,μ notations. Will there yet be in this mathematical saga an additional development involving an N,ν notation? ("God forbid!" I hear you thinking.)

Postscript: After the above material had been submitted, I was informed that another article by P.R.J. Asveld is to appear in *The Fibonacci Quarterly*, a copy of which was kindly sent to me by the author. Much, if not all, of this article could probably be extended to involve Pell, Pell-Lucas, and Jacobsthal polynomials. However, at the moment I have no plans for offering a generalization.

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SOME NOTES ON FIBONACCI BINARY SEQUENCES

Yasuichi Horibe

The Fibonacci binary sequence is a 0,1-sequence with no two 1's adjacent. In this paper, we consider the probabilistic such sequences generated by a modified Bernoulli trial, and some basic properties on these sequences will be given.

1. FIBONACCI TREE

The (branch-labeled) Fibonacci tree of order k [1], denoted by T_k, is a binary tree having three types of branches 0, 1, 10, and grown as follows: T_3 has only two terminal branches 0 and 1. If we let, in $T_k (k \geq 3)$, every terminal 0 or 10 be branched into 0 and 1, and every terminal 1 be changed ("matured") into 10, then we have T_{k+1} (see Fig. 1). (We are using 0 and 1 for convenience. In [1], α and β were used instead.) It was shown in [1] that T_k has F_k terminal branches in all, F_{k-1} of them being 0 or 10 and F_{k-2} being 1. Here F_k is the k-th Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$.

A sequence of 0 and 1 with no two 1's adjacent is sometimes called Fibonacci sequence or simply F-sequence. It was also observed in [1] that T_{n+2} displays all the possible F_{n+2} distinct F-sequences of length n, if we concatenate the branch labels along a path from the root (see Fig. 1) to a terminal branch.

Remark 1: The number of F-sequences $\alpha_1 \ldots \alpha_n$ of length n such that $\alpha_k = 1$ is given by $F_k F_{n-k+1}$, $k = 1, \ldots, n$. For we then have $\alpha_{k-1} = \alpha_{k+1} = 0$, and $\alpha_1 \ldots \alpha_{k-2}$ $(k \geq 3)$ and $\alpha_{k+2} \ldots \alpha_n (k \leq n-2)$ can be arbitrary F-sequences of lengths $k - 2$ and $n - k - 1$, respectively.

Consider now the F-sequences of length n that have r 1's. A branch is said to be at level l if it is the l^{th} branch of the path from the root to it.

Lemma 1: The number of F-sequences of length n with r 1's and ending with 0 [resp. 1] is equal to the number of terminal branches 0 or 10 [branches 1] at level $n - r$ $[n + 1 - r]$ in T_{n+2}.

This number is given by $\binom{n-r}{r}\left[\binom{n-r}{r-1}\right]$.

Proof: The lemma is essentially a corollary of Theorem 1 in [1]. An F-sequence of length n with r 1's and ending with 0 contains r 10's. Hence, the sequence corresponds to a path terminating with branch 0 or 10 and having r 10-branches and $n - 2r$ 0-branches, hence $n - r$ branches in all. On the other hand, an F-sequence with r 1's and ending with 1 has $r - 1$ 10's. Hence, the sequence corresponds to a path terminating with branch 1 and having $r - 1$ 10-branches, one 1-branch, and $n - 2(r - 1) - 1$ 0-branches, hence $n - r + 1$ branches in all. This proves the former part of the lemma. For the latter part: The number of F-sequences with r 1's and ending with 0 is the number of ways to choose r positions to receive 1 from the $n - r$ starred positions in the alternating sequence $* 0 * 0 \ldots * 0$. This is $\binom{n-r}{r}$. Similarly, the number of F-sequences with r 1's and ending with 1 is the number of ways to choose $r - 1$ positions to receive 1 from the $n - r$ starred positions in the sequence $* 0 * 0 \ldots * 01$. This is $\binom{n-r}{r-1}$. □

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 155–160.

Since $\binom{n-r}{r-1} + \binom{n-r}{r} = \binom{n+1-r}{r}$ and, as mentioned earlier, T_{n+2} has F_n terminal branches 1, F_{n+1} terminal branches 0 or 10, and F_{n+2} terminal branches in all, we have $F_n = \sum_r \binom{n-r}{r-1}$, $F_{n+1} = \sum_r \binom{n-r}{r}$, and $F_{n+2} = \sum_r \binom{n+1-r}{r}$, the successive three diagonal sums in the Pascal triangle (see also [1]).

2. FIBONACCI TRIAL

Consider the Bernoulli trial of "success (=1)" probability $1 - p$ and "failure (=0)" probability p ($0 < p < 1$). If the trial is continued until a binary sequence $\alpha_1 \ldots \alpha_n$ is obtained, the probability that r 1's will appear in this sequence, i.e., $\alpha_1 + \ldots + \alpha_n = r$, is given by $\binom{n}{r}(1 - p)^r p^{n-r}$ (binomial distribution).

We now modify this Bernoulli trial only in this: When 1 appears, it is followed by a single 0. The trial, therefore, if continued until we can have n bits, will give an F-sequence $\alpha_1 \ldots \alpha_n$. Furthermore, the trial corresponds to the choice of a branch at each branching node in T_{n+2}, which chooses 10-branch (or 1-branch) with probability $1 - p$ and 0-branch with p. It was shown in [2] that, when $p = \psi = (\sqrt{5} - 1)/2$ and n is large, every F-sequence of length n will occur almost uniformly and as uniformly as possible (in the sense of maximizing entropy) (see also the next section). This corresponds to the situation for the Bernoulli trial with $p = 1/2$.

We first show the next lemma. Put $p_n = P\{\alpha_n = 0\}$ and $q_n = P\{\alpha_n = 1\}$, then:

<u>Lemma 2:</u>

$$p_n = \sum_r \binom{n-r}{r}(1 - p)^r p^{n-2r} = \frac{1}{2-p}\left\{1 - (-(1-p))^{n+1}\right\}.$$

$$q_n = \sum_r \binom{n-r}{r-1}(1 - p)^r p^{n+1-2r} = \frac{1-p}{2-p}\left\{1 - (-(1-p))^{n}\right\}.$$

Before proving the lemma, we note the following. The p_n [q_n] is the probability to reach a terminal branch 0 or 10 [branch 1] in T_{n+2}. (It is convenient to put $p_0 = 1$, $q_0 = 0$.) When $p = \psi$, the lemma implies $p_n = \sum_r \binom{n-r}{r}\psi^n = F_{n+1}\psi^n$, $q_n = \sum_r \binom{n-r}{r-1}\psi^{n+1} = F_n\psi^{n+1}$, since $1 - \psi = \psi^2$, and hence $F_{n+1}\psi^n + F_n\psi^{n+1} = 1$.

<u>Proof:</u> From Lemma 1 and its proof, we readily see the left side equalities. For the right side, consider the two cases $\alpha_1 = 0$ and $\alpha_1 = 1$ (the latter implies $\alpha_2 = 0$), then $p_1 = p$ and $p_n = pp_{n-1} + (1 - p)p_{n-2} (n \geq 2)$. Alternatively: $\alpha_n = 1$ implies $\alpha_{n-1} = 0$, so we have $q_n = (1 - p)p_{n-1}$, hence $p_n = 1 - (1 - p)p_{n-1} (n \geq 1)$, since $p_n + q_n = 1$. We deduce at any rate that

$$p_n - p_{n-1} = -(1 - p)(p_{n-1} - p_{n-2}) \quad (n \geq 2)$$

$$p_n - p_{n-1} = (-(1 - p))^n \quad (n \geq 1)$$

$$p_n = \frac{1}{2 - p}\left\{1 - (-(1 - p))^{n+1}\right\}. \square$$

<u>Remark 2:</u> It is easy to see

$$P\{\alpha_i = 1, \alpha_j = 1\} = q_i q_{j-i-1} \quad (i < j),$$

$$P\{\alpha_i = 1, \alpha_j = 1, \alpha_k = 1\} = q_i q_{j-i-1} q_{k-j-1} \quad (i < j < k), \text{ etc.}$$

Now consider the distribution obeyed by the number $\alpha_1 + \cdots + \alpha_n$ of 1's. Denoting $P\{\alpha_1 + \cdots + \alpha_n = r\}$ by $P_n(r)$, we have, in place of the binomial distribution,

$$P_n(r) = \binom{n-r}{r}(1-p)^r p^{n-2r} + \binom{n-r}{r-1}(1-p)^r p^{n+1-2r}.$$

The generating function $G_n(x) = \sum_r P_n(r)x^r$ of this distribution can be expressed as follows in terms of the Fibonacci polynomial $f_n(x) = \sum_r \binom{n-1-r}{r} x^r$, $(f_n(1) = F_n)$.

$$\begin{aligned} G_n(x) &= p^n \sum_r \binom{n-r}{r}\left(\frac{1-p}{p^2}x\right)^r + p^{n-1}(1-p)x \sum_r \binom{n-1-(r-1)}{r-1}\left(\frac{1-p}{p^2}x\right)^{r-1} \\ &= p^n f_{n+1}\left(\frac{1-p}{p^2}x\right) + p^{n-1}(1-p)x f_n\left(\frac{1-p}{p^2}x\right). \end{aligned}$$

Note that $f_n(x)$ has the following closed form expression (see [4]):

$$f_n(x) = \frac{1}{\sqrt{1+4x}}\left\{\left(\frac{1+\sqrt{1+4x}}{2}\right)^n - \left(\frac{1-\sqrt{1+4x}}{2}\right)^n\right\}.$$

The expected number of 1's $\mu_n = E[\alpha_1 + \cdots + \alpha_n] = \sum_r rP_n(r)$, though can be found by computing $G'_n(1)$, is derived simply, using Lemma 2, as follows:

$$\begin{aligned} \mu_n &= E[\alpha_1] + \ldots + E[\alpha_n] \\ &= q_1 + \ldots + q_n \\ &= \left(\frac{1-p}{2-p}\right)\left\{n - \sum_{j=1}^{n}(-(1-p))^j\right\} \\ &= \left(\frac{1-p}{2-p}\right)n + \left(\frac{1-p}{2-p}\right)^2\left\{1 - (-(1-p))^n\right\}. \end{aligned}$$

(Note that, for the binomial distribution $\binom{n}{r}(1-p)^r p^{n-r}$, the corresponding expectation is $(1-p)n$.) We have, therefore, that

$$\frac{\mu_n}{n} \to \frac{1-\psi}{2-\psi} = \frac{\psi}{\sqrt{5}}\ (n \to \infty), \text{ when } p = \psi. \tag{1}$$

In [3], the following was shown: Let N_n be the total number of 1's of all the F_{n+2} F-sequences of length n, then

$$\frac{1}{n}\left(\frac{N_n}{F_{n+2}}\right) \to \frac{\psi}{\sqrt{5}}\ (n \to \infty). \tag{2}$$

This (2) can be seen by noting that $N_n = \sum_{k=1}^{n} F_k F_{n-k+1}$ from Remark 1 given in Section 1 and this sum is, in turn, equal to $(nF_{n+2} + (n+2)F_n)/5$ (see [4]).

The limits in (1) and (2) coincide. This is because, when $p = \psi$, every F-sequence occurs almost uniformly [2].

3. ENTROPY PER BIT

In our Fibonacci trial, 10 occurs with probability $1 - p$ and 0 (not preceded by 1) with p. The average number of bits is, therefore, $2(1 - p) + 1 \cdot p = 2 - p$. Hence, the entropy (see [2]) per bit is given by $H(p, 1 - p)/(2 - p)$, where $H(p, 1 - p) = -p \log p - (1 - p) \log(1 - p)$ is the binary entropy function. If the trial is continued until 10 occurs for the first time, the probability that $j + 1$ bits are obtained is $p^{j-1}(1 - p)$, $j = 1, 2, \ldots$. The entropy of this geometric distribution is given by $H = -\sum_{j\geq1} p^{j-1}(1 - p) \log p^{j-1}(1 - p) = H(p, 1 - p)/(2 - p)$. On the other hand, the average number of bits (called also average cost) is $C = \sum_{j\geq1}(j + 1)p^{j-1}(1 - p) = (2 - p)/(1 - p)$. Hence, the entropy per bit H/C becomes again $H(p, 1 - p)/(2 - p)$.

This is not a mere coincidence. We will show this fact in general in the following. Consider an arbitrary finite or infinite binary tree T. Each left branch corresponds to the failure 0 (cost 1) and has probability p, and each right branch corresponds to the success 10 (cost 2) and has probability $1 - p$. (The trees corresponding to the above mentioned two cases are shown in Fig. 2.) The cost of a node of T is defined as: The cost of the root is 0, and the cost of the left [right] son of a node of cost b is $b + 1$ [$b + 2$]. On the other hand, the probability of a node is defined as: The probability of the root is 1, and the probability of the left [right] son of a node of probability q is pq [$(1 - p)q$].

Suppose T is finite and has $n - 1$ internal nodes of probabilities $q_1, \ldots, q_{n-1}$ and n terminal nodes of costs $c_1, \ldots, c_n$ and probabilities $p_1, \ldots, p_n$. The average terminal cost (average number of bits) $C = \sum_{i=1}^{n} p_i c_i$ can also be expressed as:

$$C = (2 - p) \sum_{j=1}^{n-1} q_j. \tag{3}$$

<u>Proof of (3)</u>: The T can be viewed as grown by $n - 1$ successive branchings, starting with the branching of the root node. The average cost is initially 0. The average cost increased by the branching of a node of cost b and probability q is seen to be

$$pq(b + 1) + (1 - p)q(b + 2) - qb = (2 - p)q,$$

hence finishing the proof by induction. □

On the other hand, the entropy H of terminal distribution $p_1, \ldots, p_n$ was shown to be expressed as [2]:

$$H = H(p, 1 - p) \sum_{j=1}^{n-1} q_j. \tag{4}$$

We therefore have, from (3) and (4), that the entropy per bit is given by

$$\frac{H}{C} = \frac{H(p, 1-p)}{2-p}.$$

If T is infinite, we define its entropy per bit as the limiting value of H_l/C_l (as $l \to \infty$), which is surely given by $H(p, 1 - p)/(2 - p)$, where H_l and C_l are the entropy of terminal distribution and the average terminal cost of the finite tree obtained by truncating or pruning T at level l. □

The F-sequence generated by the Fibonacci trial may be said to be a rhythmic pattern of long (=10) and short (=0 not preceded by 1), a "free" rhythm not in a regular, metrical sense [5]. And the entropy measures the spread of a distribution. Thus, the $p = \psi$, which maximizes $H(p, 1 - p)/(2 - p)$, is the proportion that produces the long-and-short rhythmic patterns most abundantly.

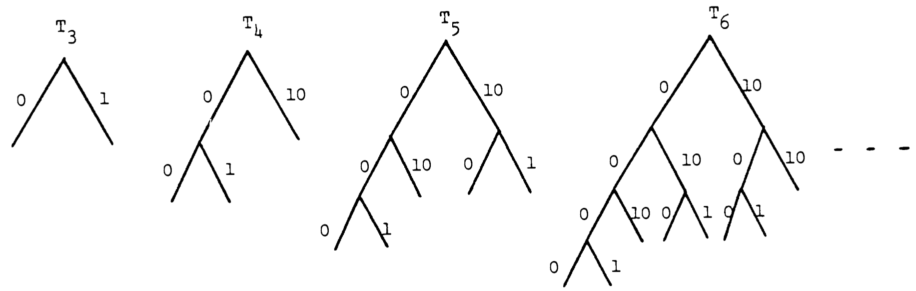

Figure 1.

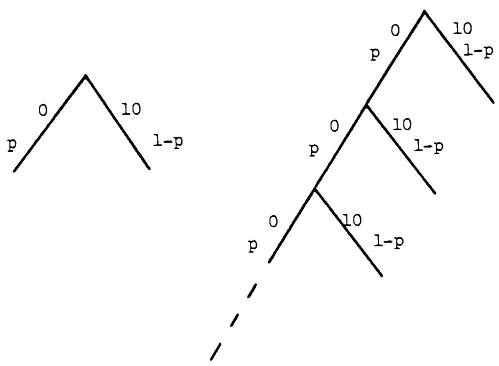

Figure 2.

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CONGRUENCES FOR WEIGHTED AND DEGENERATE STIRLING NUMBERS

F. T. Howard

1. INTRODUCTION

Let s(n,k) and S(n,k) be the (unsigned) Stirling numbers of the first and second kinds respectively [8, pp. 204-219]. In [1] and [10] congruence formulas (mod p), p prime, are proved for s(n,k) and S(n,k). As applications, the residues (mod p) for p = 2, 3, and 5 are worked out for both kinds of Stirling numbers. In [10] similar formulas are proved for the associated Stirling numbers d(n,k) and b(n,k).

The purpose of the present paper is to extend the results of [1] and [10] to the weighted Stirling numbers [2], [3], the weighted associated Stirling numbers [11], the degenerate Stirling numbers [4] and the degenerate weighted Stirling numbers [9].

Throughout this paper we assume λ and θ are rational numbers. When a particular prime p is being considered, we always assume λ and θ are integral (mod p).

2. WEIGHTED STIRLING NUMBERS OF THE FIRST KIND

The weighted Stirling number of the first kind $R_1(n,k,\lambda)$ can be defined by the recurrence

$$R_1(n,k,\lambda) = (\lambda+n-1)R_1(n-1,k,\lambda) + R_1(n-1,k-1,\lambda), \tag{2.1}$$

with

$$R_1(n,0,\lambda) = \lambda(\lambda+1)\cdots(\lambda+n-1), \text{ if } n > 0, \tag{2.2}$$
$$R_1(n,n,\lambda) = 1 \text{ if } n \geq 0, \tag{2.3}$$
$$R_1(n,k,\lambda) = 0 \text{ if } k > n \text{ or } k < 0. \tag{2.4}$$

It follows that

$$R_1(n,k,0) = s(n,k),$$

the ordinary Stirling number of the first kind. Many other properties of $R_1(n,k,\lambda)$, including generating functions and combinatorial interpretations, can be found in [2] and [3]. In particular, we will make use of

$$R_1(n,k,\lambda) = \sum_{j=0}^{n-k} \begin{bmatrix} j+k \\ j \end{bmatrix} \lambda^j \, s(n,j+k). \tag{2.5}$$

If p is a prime number, it follows from (2.2)-(2.5) and properties of s(p,k) [8, p. 218], that

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$$R_1(p,0,\lambda) \equiv 0 \pmod{p}, \tag{2.6}$$
$$R_1(p,1,\lambda) \equiv -1 \pmod{p}, \tag{2.7}$$
$$R_1(p,k,\lambda) \equiv 0 \pmod{p} \ (k=2, \ldots, p-1). \tag{2.8}$$

Theorem 2.1: If p is a prime number, $h > 0$ and $0 \le m < p$, then

$$R_1(hp+m,k,\lambda) \equiv \sum_{i=0}^{h} \begin{bmatrix} h \\ i \end{bmatrix} (-1)^{h-i} R_1(m,k-h-i(p-1),\lambda) \pmod{p}.$$

Proof: We first prove Theorem 2.1 for $h = 1$; that is,

$$R_1(p+m,k,\lambda) \equiv -R_1(m,k-1,\lambda) + R_1(m,k-p,\lambda) \pmod{p}. \tag{2.9}$$

By (2.2)-(2.4) and (2.6)-(2.8), congruence (2.9) is true for $m = 0$. A simple induction argument on m, making use of (2.1), now proves (2.9) for $m = 0,1,\ldots$. Now let

$$A_m(t) = \sum_{r=0}^{m} R_1(m,r,\lambda)t^r,$$

so that

$$A_{p+m}(t) \equiv (t^p - t) \ A_m(t) \pmod{p}.$$

It follows immediately that

$$A_{hp+m}(t) \equiv (t^p - t)^h A_m(t) \pmod{p}. \tag{2.10}$$

Comparing coefficients of t^k in (2.10), we complete the proof of Theorem 2.1.

It follws from Theorem 2.1 that if $m = 0$ or 1 and $h > 0$,

$$R_1(hp+m,k,\lambda) \equiv 0 \pmod{p},$$

except for the following: For $i = 0,1,\ldots,h$,,

$$R_1(hp,h+(p-1)i,\lambda) \equiv \begin{bmatrix} h \\ i \end{bmatrix} (-1)^{h-i} \pmod{p},$$

$$R_1(hp+1,h+(p-1)i,\lambda) \equiv \lambda \begin{bmatrix} h \\ i \end{bmatrix} (-1)^{h-i} \pmod{p} \ (p > 2),$$

$$R_1(hp+1,h+1+(p-1)i,\lambda) \equiv \begin{bmatrix} h \\ i \end{bmatrix} (-1)^{h-i} \pmod{p} \ (p > 2).$$

For $p = 2$ we have for $i = 0,1,\ldots,h+1$,

$$R_1(2h,h+i,\lambda) \equiv \begin{bmatrix} h \\ i \end{bmatrix} \pmod{2},$$

$$R_1(2h+1,h+i,\lambda) \equiv \lambda \begin{bmatrix} h \\ i \end{bmatrix} + \begin{bmatrix} h \\ i-1 \end{bmatrix} \pmod{2}.$$

3. WEIGHTED STIRLING NUMBERS OF THE SECOND KIND

We now turn to the numbers $R(n,k,\lambda)$, which can be defined by means of the recurrence

$$R(n,k,\lambda) = (\lambda+k)R(n-1,k,\lambda) + R(n-1,k-1,\lambda), \tag{3.1}$$

with

$$R(n,0,\lambda) = \lambda^n \text{ if } n \geq 0, \tag{3.2}$$
$$R(n,n,\lambda) = 1 \text{ if } n \geq 0, \tag{3.3}$$
$$R(n,k,\lambda) = 0 \text{ if } n < k \text{ or } k < 0. \tag{3.4}$$

It follows that

$$R(n,k,0) = S(n,k),$$

the ordinary Stirling number of the second kind. In [2] and [3], Carlitz proved many other properties of $R(n,k,\lambda)$, including

$$R(n,k,\lambda) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \lambda^r S(n-r,k), \tag{3.5}$$

$$\frac{x^k}{(1-\lambda x)(1-(\lambda+1)x)\cdots(1-(\lambda+k)x)} = \sum_{n=0}^{\infty} R(n,k,\lambda)x^n. \tag{3.6}$$

Let p be a prime number. It follows from (3.2)-(3.5) and properties of $S(p,k)$ [8, p. 219] that

$$R(p,0,\lambda) \equiv \lambda \pmod p, \tag{3.7}$$
$$R(p,1,\lambda) \equiv 1 \pmod p, \tag{3.8}$$
$$R(p,k,\lambda) \equiv 0 \pmod p (k=2,\ldots,p-1). \tag{3.9}$$

<u>Theorem 3.1:</u> If p is a prime number, $h > 0$ and $0 \leq m < p$, then

$$R(hp+m,k,\lambda) \equiv \sum_{i=0}^{h} \begin{bmatrix} h \\ i \end{bmatrix} R(m+h-i,k-ip,\lambda) \pmod p.$$

<u>Proof:</u> For $h = 1$, we want to prove for $m = 0,1,\ldots,$

$$R(p+m,k,\lambda) \equiv R(m+1,k,\lambda) + R(m,k-p,\lambda) \pmod p. \tag{3.10}$$

Congruence (3.10) is true for $m = 0$ by (3.2)-(3.4) and (3.7)-(3.9). A simple induction argument on m establishes (3.10), and then an induction argument on h establishes Theorem 3.1. The proof is identical to the proof for $\lambda = 0$ in [10].

To get a more useful result, we let

$$r_n(t) = \sum_{k=0}^{n} R(n,k,\lambda)t^k.$$

It follows from a theorem in [10], which is based on results in [1], that any set of numbers satisfying (3.10), with $R(0,0,\lambda) = 1$, has the property:

$$r_n(t) \equiv f_n(t) - \sum_{j=0}^{p-1} r_{p-1-j}(t)\, f_{n+j}(t) \pmod p, \tag{3.11}$$

where

$$f_n(t) = -\sum_{i} \begin{bmatrix} i \\ n-(p-1)(i+1) \end{bmatrix} t^{p(n-(p-1)(i+1))}. \tag{3.12}$$

The summation on the right of (3.12) is over all i such that

$$(p-1)(i+1) \leq n \leq p(i+1)-1.$$

A careful examination of (3.11) gives us the next theorem.

Theorem 3.2: Suppose p is a prime number, $h \geq 0$ and $0 \leq m < p-1$. If $h = n-(p-1)(i+1)$, then

$$R(n,hp,\lambda) \equiv \begin{bmatrix} i \\ h-1 \end{bmatrix} + \lambda^{p-1} \begin{bmatrix} i \\ h \end{bmatrix} \pmod{p}.$$

If $h = n-(p-1)i-t$ with $1 \leq t \leq p-2$, then

$$R(n,hp,\lambda) \equiv \lambda^t \begin{bmatrix} i \\ h \end{bmatrix} \pmod{p}.$$

If $h = n-(p-1)i-j$ with $1 \leq m \leq j \leq p-2$, then

$$R(n,hp+m,\lambda) \equiv R(j,m,\lambda) \begin{bmatrix} i \\ h \end{bmatrix} \pmod{p}.$$

In all other cases, $R(n,k,\lambda) \equiv 0 \pmod{p}$.

For example, if $h \neq n-(p-1)(i+1)$ then

$$R(n,hp+(p-1),\lambda) \equiv 0 \pmod{p}.$$

If $h = n-(p-1)(i+1)$, then

$$R(n,hp+(p-1),\lambda) \equiv \begin{bmatrix} i \\ h \end{bmatrix} \pmod{p}.$$

If $h \neq n-(p-1)i-j$ with $1 \leq j \leq p-1$, then

$$R(n,hp+1,\lambda) \equiv 0 \pmod{p}.$$

If $h = n-(p-1)i-j$ with $1 \leq j \leq p-1$, then

$$R(n,hp+1,\lambda) \equiv \left[(1+\lambda)^j - \lambda^j\right] \begin{bmatrix} i \\ h \end{bmatrix} \pmod{p}.$$

If $p = 2$, we have

$$R(n,2h,\lambda) \equiv \begin{bmatrix} n-h-1 \\ h-1 \end{bmatrix} + \lambda \begin{bmatrix} n-h-1 \\ h \end{bmatrix} \pmod{2},$$

$$R(n,2h+1,\lambda) \equiv \begin{bmatrix} n-h-1 \\ h \end{bmatrix} \pmod{2}.$$

Using Theorem 3.2, we can easily write down the congruences for $R(n,k,\lambda)$ for $p = 3,5,\ldots$. For $\lambda = 0$ this has been done for $p = 2,3,5$ [1].

Let p be prime and let $m \geq 1$. It is known that the Stirling numbers of the second kind are periodic (mod p^m); that is, there exists $N \geq k$ and $\pi \geq 1$ such that

$$S(n+\pi,k) \equiv S(n,k) \pmod{p^m} \quad (n > N).$$

See, for example, [5], [12] and [13]. Let $g(\lambda,k,p^m)$ be the minimum period (mod p^m) for the numbers $R(n,k,\lambda)$. By using (3.6) and the methods of [12], we can show that if $p > 2$ and $p^{b-1} \leq k < p^b$ with $b \geq 2$, then for $\lambda \not\equiv 0 \pmod{p}$,

$$g(\lambda,k,p^m) = (p-1)p^{m+b-2}.$$

If $\lambda \equiv 0$ (mod p) the boundaries for k are changed to $p^{b-1} < k \le p^b$. The details of the proof are similar to the details of [12] and will not be given here.

4. WEIGHTED ASSOCIATED STIRLING NUMBERS

It is known that $R_1(n,k,\lambda)$ and $R(n,k,\lambda)$ are both polynomials in n of degree 2k. In fact,

$$R_1(n,n-k,\lambda) = \sum_{j=0}^{k} Q_1(2k-j,k-j,\lambda) \begin{bmatrix} n \\ 2k-j \end{bmatrix},$$

$$R(n,n-k,\lambda) = \sum_{j=0}^{k} Q(2k-j,k-j,\lambda) \begin{bmatrix} n \\ 2k-j \end{bmatrix},$$

where $Q_1(n,k,\lambda)$ and $Q(n,k,\lambda)$ are weighted associated Stirling numbers of the first and second kinds respectively. In [11] many properties, including generating functions and combinatorial interpretations, are given for $Q_1(h,k,\lambda)$ and $Q(n,k,\lambda)$. Note that when $\lambda = 0$ we have

$$Q_1(n,k,0) = d(n,k), \quad Q(n,k,0) = b(n,k),$$

where d(n,k) and b(n,k) are the associated Stirling numbers [8, pp. 256, 222].

For $Q_1(n,k,\lambda)$ the recurrence is

$$Q_1(n,k,\lambda) = (\lambda+n-1)Q_1(n-1,k,\lambda) + (n-1)Q_1(n-2,k-1,\lambda) \tag{4.1}$$

with

$$Q_1(0,0,\lambda) = 1, \tag{4.2}$$
$$Q_1(n,0,\lambda) = \lambda(\lambda+1)\cdots(\lambda+n-1) \text{ if } n > 0, \tag{4.3}$$
$$Q_1(n,k,\lambda) = 0 \text{ if } n < 2k \text{ or } k < 0, \tag{4.4}$$

$$Q_1(n,k,\lambda) = \sum_{i=0}^{n-2k} \begin{bmatrix} n \\ i \end{bmatrix} \lambda(\lambda+1)\cdots(\lambda+i-1)\, d(n-i,k). \tag{4.5}$$

If p is prime, we see by (4.2)-(4.5) that

$$Q_1(p,0,\lambda) = \lambda(\lambda+1)\cdots(\lambda+p-1) \equiv 0 \pmod p, \tag{4.6}$$
$$Q_1(p,1,\lambda) \equiv -1 \pmod p, \tag{4.7}$$
$$Q_1(p,k,\lambda) \equiv 0 \pmod p \text{ for } k = 2,3,\ldots \tag{4.8}$$

Theorem 4.1: If p is a prime number, $h > 0$ and $0 \le m < p$, then

$$Q_1(hp+m,k,\lambda) \equiv (-1)^h\, Q_1(m,k-h,\lambda) \pmod p.$$

Proof: A simple induction argument on m, making use of (4.6)-(4.8), establishes for $m = 0,1,\ldots$

$$Q_1(p+m,k,\lambda) \equiv -Q_1(m,k-1,\lambda) \pmod p.$$

Now an induction argument on h, exactly like the argument in [10] for the case $\lambda = 0$, proves Theorem 4.1.

We see from Theorem 4.1 that

$$Q_1(hp+m,k,\lambda) \equiv 0 \pmod p \text{ if } 2(k-h) > m,$$
$$Q_1(hp+m,k,\lambda) \equiv 0 \pmod p \text{ if } k \le h \ (m \ne 0).$$

Also, for m = 0 and m = 1, and h > 0,

$$Q_1(hp+m,k,\lambda) \equiv 0 \pmod p$$

except for the following cases:

$$Q_1(hp,h,\lambda) \equiv (-1)^h \pmod p,$$
$$Q_1(hp+1,h,\lambda) \equiv (-1)^h\, \lambda \pmod p.$$

We note that it would be easy to write down congruence formulas (mod p) for p = 2,3,5. This has been done for the case $\lambda = 0$ [10].

We turn now to $Q(n,k,\lambda)$. The recurrence is

$$Q(n,k,\lambda) = (\lambda+k)\, Q(n-1,k,\lambda) + (n-1)\, Q(n-2,k-1,\lambda), \tag{4.9}$$

with

$$Q(n,0,\lambda) = \lambda^n \text{ if } n \geq 0, \tag{4.10}$$
$$Q(n,k,\lambda) = 0 \text{ if } n < 2k \text{ or } k < 0, \tag{4.11}$$

$$Q(n,k,\lambda) = \sum_{i=0}^{n-2k} \begin{bmatrix} n \\ i \end{bmatrix} b(n-i,k)\lambda^i. \tag{4.12}$$

See [11] for many other properties of $Q(n,k,\lambda)$.

If p is prime, we see by (4.10)-(4.12) that

$$Q(p,0,\lambda) \equiv \lambda \pmod p, \tag{4.13}$$
$$Q(p,1,\lambda) \equiv 1 \pmod p, \tag{4.14}$$
$$Q(p,k,\lambda) \equiv 0 \pmod p \text{ for } k = 2,3,\ldots . \tag{4.15}$$

<u>Theorem 4.2:</u> If p is a prime number, h > 0 and $0 \leq m < p$, then

$$Q(hp+m,k,\lambda) \equiv \sum_{i=0}^{h} \begin{bmatrix} h \\ i \end{bmatrix} Q(m+h-i,k-i,\lambda) \pmod p.$$

<u>Proof:</u> The proof is omitted since it is almost identical to the proof for $\lambda = 0$ in [10]. The case h = 1 should be noted: For m = 0,1,...,

$$Q(p+m,k,\lambda) \equiv Q(m+1,k,\lambda) + Q(m,k-1,\lambda) \pmod p. \tag{4.16}$$

It follows from Theorem 4.2 that if either h + m < k or h + m $\leq$ k with m > 0, then

$$Q(hp+m,k,\lambda) \equiv 0 \pmod p.$$

Now let $\quad q_n(t) = \sum_{k=0}^{n} Q(n,k,\lambda)t^k.$

It follows from a theorem in [10] that any set of numbers satisfying (4.16), with $Q(0,0,\lambda) = 1$, has the property

$$q_n(t) \equiv f_n(t) - \sum_{j=0}^{p-1} q_{p-1-j}(t)\, f_{n+j}(t) \pmod p \tag{4.17}$$

where

$$f_n(t) = -\sum_i \begin{bmatrix} i \\ n-(p-1)(i+1) \end{bmatrix} t^{n-(p-1)(i+1)}. \tag{4.18}$$

The summation on the right of (4.17) is over all i such that

$$(p-1)(i+1) \leq n \leq p(i+1)-1.$$

A careful examination of both sides of (4.17) gives us the next theorem. The proof is identical to the proof for $\lambda = 0$ in [10].

<u>Theorem 4.3:</u> Suppose p is a prime number, $h \geq 0$ and $0 \leq m < p-1$. If $n-k = (p-1)h$, then

$$Q(n,k,\lambda) \equiv \begin{bmatrix} h-1 \\ k-1 \end{bmatrix} + \lambda^{p-1} \begin{bmatrix} h-1 \\ k \end{bmatrix} \pmod p.$$

If $n-k = (p-1)h+m$ with $1 \leq m \leq p-2$, then

$$Q(n,k,\lambda) \equiv \sum_{i=0}^{p-1-m} \begin{bmatrix} h \\ k-i \end{bmatrix} Q(m+i,i,\lambda) \pmod p.$$

If $m \leq (p-1)/2$, the upper limit of summation can be replaced by m.

For example, if $n-k = (p-1)h+1$,

$$Q(n,k,\lambda) \equiv \lambda \begin{bmatrix} h \\ k \end{bmatrix} + \begin{bmatrix} h \\ k-1 \end{bmatrix} \pmod p.$$

If $n-k = (p-1)h+2$, then

$$Q(n,k,\lambda) \equiv \lambda^2 \begin{bmatrix} h \\ k \end{bmatrix} + (1+3\lambda) \begin{bmatrix} h \\ k-1 \end{bmatrix} + 3 \begin{bmatrix} h \\ k-2 \end{bmatrix} \pmod p.$$

More examples for $\lambda = 0$ are given in [10].

5. DEGENERATE STIRLING NUMBERS

The degenerate Stirling number of the second kind, $S(n,k \mid \theta)$ can be defined by means of

$$S(n,k \mid \theta) = (k-(n-1)\theta)\, S(n-1,k \mid \theta) + S(n-1,k-1 \mid \theta), \tag{5.1}$$

with

$$S(n,n \mid \theta) = 1 \text{ if } n \geq 0, \tag{5.2}$$
$$S(n,0 \mid \theta) = 0 \text{ if } n > 0, \tag{5.3}$$
$$S(n,k \mid \theta) = 0 \text{ if } n < k \text{ or } k < 0, \tag{5.4}$$

$$k!S(n,k \mid \theta) = \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} j(j-\theta)\cdots(j-(n-1)\theta). \tag{5.5}$$

Many more properties of $S(n,k \mid \theta)$ can be found in [4]. Note that

$$S(n,k \mid 0) = S(n,k),$$

the ordinary Stirling number of the second kind.

If p is prime and $\theta \not\equiv 0 \pmod p$, it follows from (5.3) and (5.5) that

$$S(p,k \mid \theta) \equiv 0 \pmod p \quad (k = 0,1,\ldots,p-1). \tag{5.6}$$

Theorem 5.1: If p is a prime number, $h > 0$, $0 \le m < p$ and $\theta \not\equiv 0 \pmod p$, then

$$S(hp+m,k \mid \theta) \equiv S(m,k-hp \mid \theta) \pmod p.$$

Proof: For $h = 1$ we want to prove that if $\theta \not\equiv 0 \pmod p$, then for $m = 0,1,\ldots,$

$$S(p+m,k \mid \theta) \equiv S(m,k-p \mid \theta) \pmod p. \tag{5.7}$$

For $m = 0$, (5.7) is true by (5.2)-(5.6). Now a simple induction argument on m, using (5.1), establishes (5.7). The proof of Theorem 5.1 is now easily completed by using induction on h.

For example, if $\theta \not\equiv 0 \pmod p$ we have for $m = 0,1,2$, and $h > 0$,

$$S(hp+m,k \mid \theta) \equiv 0 \pmod p,$$

except for the following cases:

$$S(hp,hp \mid \theta) \equiv S(hp+1,hp+1 \mid \theta) \equiv S(hp+2,hp+2 \mid \theta) \equiv 1 \pmod p,$$
$$S(hp+2,hp+1 \mid \theta) \equiv 1 - \theta \pmod p.$$

In general, if $\theta \not\equiv 0 \pmod p$ and $h > 0$,

$$S(hp+m,k \mid \theta) \equiv 0 \pmod p \text{ for } k = 0,\ldots,hp-1, \tag{5.8}$$
$$S(hp+m,hp+t \mid \theta) \equiv S(m,t \mid \theta) \pmod p \text{ for } t = 0,1,\ldots,m. \tag{5.9}$$

Thus $S(n,k \mid \theta)$ is even unless $k = n$ (provided $\theta \not\equiv 0 \pmod p$). If $\theta \equiv 0 \pmod p$, we have

$$S(n,k \mid \theta) \equiv S(n,k) \pmod p.$$

The degenerate Stirling number of the first kind $S_1(n,k \mid \theta)$ can be defined by the recurrence

$$S_1(n,k \mid \theta) = (n-1-k\theta)S_1(n-1,k \mid \theta) + S_1(n-1,k-1 \mid \theta),$$

with

$$S_1(n,n \mid \theta) = 1 \text{ if } n \ge 0,$$
$$S_1(n,0 \mid \theta) = 0 \text{ if } n > 0,$$
$$S_1(n,k \mid \theta) = 0 \text{ if } n < k \text{ or } k < 0.$$

It follows that

$$S_1(n,k \mid 0) = s(n,k).$$

Carlitz [4] has shown that if $\theta \ne 0$,

$$S_1(n,k \mid \theta) = (-\theta)^{n-k}S(n,k \mid \theta^{-1}). \tag{5.10}$$

From (5.10) and Theorem 5.1 we can easily deduce the next theorem.

Theorem 5.2: Suppose p is prime, $h > 0$ and $0 \le m < p$. If $\theta \not\equiv 0 \pmod p$, then

$$S_1(hp+m,k \mid \theta) \equiv S_1(m,k-hp \mid \theta) \pmod p.$$

It follows immediately from Theorem 5.2 that $S_1(n,k \mid \theta)$ also satisfies (5.8) and (5.9). It also follows that if $\theta \not\equiv 0 \pmod 2$,

$$S_1(n,k \mid \theta) \equiv 0 \pmod 2 \quad (k=0,\ldots,n-1).$$

6. DEGENERATE WEIGHTED STIRLING NUMBERS

In [9] the degenerate weighted Stirling numbers of the first and second kinds, $S_1(n,k,\lambda \mid \theta)$ and $S(n,k,\lambda \mid \theta)$ respectively, are defined, and their properties, including generating functions and combinatorial interpretations, are discussed. The following special cases are of interest:

$$\begin{array}{ll} S_1(n,k,0 \mid 0) = s(n,k), & S(n,k,0 \mid 0) = S(n,k), \\ S_1(n,k,\lambda \mid 0) = R_1(n,k,\lambda), & S(n,k,\lambda \mid 0) = R(n,k,\lambda), \\ S_1(n,k,\theta \mid \theta) = S_1(n,k \mid \theta), & S(n,k,0 \mid \theta) = S(n,k \mid \theta). \end{array}$$

The recurrence for $S_1(n,k,\lambda \mid \theta)$ is

$$S_1(n,k,\lambda \mid \theta) = S_1(n-1,k-1,\lambda \mid \theta) + (\lambda+n-\theta-\theta k-1)\, S_1(n-1,k,\lambda \mid \theta), \tag{6.1}$$

with

$$S_1(n,0,\lambda \mid \theta) = (\lambda-\theta)(\lambda-\theta+1)\cdots(\lambda-\theta+n-1) \text{ if } n > 0, \tag{6.2}$$

$$S_1(n,n,\lambda \mid \theta) = 1 \text{ if } n \geq 0, \tag{6.3}$$

$$S_1(n,k,\lambda \mid \theta) = 0 \text{ if } k > n \text{ or } k < 0, \tag{6.4}$$

$$S_1(n,k,\lambda \mid \theta) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (\lambda-\theta)(\lambda-\theta+1)\cdots(\lambda-\theta+i-1)\, S_1(n-i,k \mid \theta). \tag{6.5}$$

If p is prime and $\theta \not\equiv 0 \pmod p$, it follows from (6.2)-(6.5) that

$$S_1(p,k,\lambda \mid \theta) \equiv 0 \pmod p \text{ for } k = 0,\ldots,p-1.$$

An easy argument gives us, like Theorem 5.1:

<u>Theorem 6.1:</u> Suppose p is a prime number, $h > 0$ and $0 \leq m < p$. If $\theta \not\equiv 0 \pmod p$ then

$$S_1(hp+m,k,\lambda \mid \theta) \equiv S_1(m,k-hp,\lambda \mid \theta) \pmod p.$$

It is clear that if $\theta \not\equiv 0 \pmod p$, $S(n,k,\lambda \mid \theta)$ satisfies (5.8) and (5.9), that is, we can replace $S(n,k \mid \theta)$ by $S_1(n,k,\lambda \mid \theta)$ in (5.8) and (5.9).

We turn now to $S(n,k,\lambda \mid \theta)$. The recurrence is

$$S(n,k,\lambda \mid \theta) = (k+\lambda-\theta(n-1))\, S(n-1,k,\lambda \mid \theta) + S(n-1,k-1,\lambda \mid \theta), \tag{6.6}$$

with

$$S(n,0,\lambda \mid \theta) = \lambda(\lambda-\theta)\cdots(\lambda-n\theta+\theta) \text{ if } n > 0, \tag{6.7}$$

$$S(n,n,\lambda \mid \theta) = 1 \text{ if } n \geq 0), \tag{6.8}$$

$$S(n,k,\lambda \mid \theta) = 0 \text{ if } k > n \text{ or } k < 0, \tag{6.9}$$

$$S(n,k,\lambda \mid \theta) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \lambda(\lambda-\theta)\cdots(\lambda-i\theta+\theta)\, S(n-i,k \mid \theta). \tag{6.10}$$

If p is prime, it follows from (6.7)-(6.10) that if $\theta \not\equiv 0 \pmod p$, then

$$S(p,k,\lambda \mid \theta) \equiv 0 \pmod p \text{ for } k = 0,1,\ldots,p-1. \tag{6.11}$$

Induction gives us, like Theorems 5.1 and 6.1:

Theorem 6.2: Suppose p is a prime number, $h > 0$ and $0 \leq m < p$. If $\theta \not\equiv 0 \pmod p$ then

$$S(hp+m,k,\lambda \mid \theta) \equiv S(m,k-hp,\lambda \mid \theta) \pmod p.$$

It is clear that $S(n,k,\lambda \mid \theta)$ satisfies (5.8) and (5.9) if $\theta \not\equiv 0 \pmod p$. We note that if $\theta \equiv 0 \pmod p$, then

$$S_1(n,k,\lambda \mid \theta) \equiv R_1(n,k,\lambda) \pmod p,$$
$$S(n,k,\lambda \mid \theta) \equiv R(n,k,\lambda) \pmod p.$$

For $p = 2$ and θ odd, we see that $S_1(n,k,\lambda \mid \theta)$ and $S(n,k,\lambda \mid \theta)$ are both even except for the following cases:

$$S_1(n,n,\lambda \mid \theta) \equiv S(n,n,\lambda \mid \theta) \equiv 1 \pmod 2,$$
$$S_1(2h+1,2h,\lambda \mid \theta) \equiv \lambda-1 \pmod 2,$$
$$S(2h+1,2h,\lambda \mid \theta) \equiv \lambda \pmod 2.$$

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AN INVERSE THEOREM ON FIBONACCI NUMBERS

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1. INTRODUCTION.

Let us begin with the particular case. Fibonacci Sequence $\{F_n\}$ (n=0, 1, 2, . . .) is defined by the three-term linear recurrence formula

$$F_n = F_{n-1} + F_{n-2}\ (n \geq 2),\ \ F_0 = 0,\ F_1 = 1. \tag{1.1}$$

Whence we have the following well-known relation

$$F_{n-1}(F_n + F_{n+1}) - F_nF_{n+1} = (-1)^n\ (n \geq 1). \tag{1.2}$$

Conversely, under the additional condition

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_3 = 2$$

we can derive (1.1) from (1.2) by induction.

In [2, Theorem 4], [1], [4], and [7] we find an interesting relation

$$\tan^{-1}\frac{1}{F_{2n}} = \tan^{-1}\frac{1}{F_{2n+1}} + \tan^{-1}\frac{1}{F_{2n+2}}\ (n \geq 1). \tag{1.3}$$

So it is natural to ask the question whether

$$\tan^{-1}\frac{1}{F_{2n+1}} = \tan^{-1}\frac{1}{F_{2n+2}} + \tan^{-1}\frac{1}{F_{2n+3}}$$

holds or not. By simple calculation we see that the answer is negative. However, if we replace the arctangent function by the hyperbolic arctangent function, we can prove that it is affirmative, i.e., we get

$$\tanh^{-1}\frac{1}{F_{2n+1}} = \tanh^{-1}\frac{1}{F_{2n+2}} + \tanh^{-1}\frac{1}{F_{2n+3}}\ (n \geq 1), \tag{1.4}$$

in other words ([4])

$$\frac{1}{2}\log\frac{1+1/F_{2n+1}}{1-1/F_{2n+1}} = \frac{1}{2}\log\frac{1+1/F_{2n+2}}{1-1/F_{2n+2}} = \frac{1}{2}\log\frac{1+1/F_{2n+3}}{1-1/F_{2n+3}}\ .$$

Putting

$$\theta_n = \tan^{-1}\frac{1}{F_n},\ \xi_n = \sin^{-1}\frac{1}{F_n},\ u_n = \tanh^{-1}\frac{1}{F_n},$$

we have some geometrical interpretations. In [1], [4], and [7], the angles θ_n are introduced as

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"hogan-kaku ($\theta_n = \tan^{-1} M/N$, where M and N are positive integers)." In Lobachevskii's non-Euclidean space we may interpret the angles ξ_n as those between two non-Euclidean lines 1_n and 1_∞ which pass through the point (0.1) and have radius F_n and ∞ respectively. Using ξ_n, we can write ([5, p. 217])

$$u_n = \tanh^{-1}(\sin \xi_n) = \tfrac{1}{2}\log \frac{1+\sin \xi_n}{1-\sin \xi_n}$$
$$= \int_0^{\xi_n} \frac{d\xi}{\cos \xi} = \log|\tan(\tfrac{\xi_n}{2} + \tfrac{\pi}{4})| = gd^{-1}\, \xi_n,$$

that is, the numbers u_n are regarded as the inverse Gudermannian function of ξ_n. Namely u_n are regarded as the distances from the equator to the latitude ξ_n on Mercator's map, or the area of the region lying below the curve $\eta = 1/\cos \xi$ and above the ξ-axis, from $\xi=0$ to $\xi=\xi_n$. u_n are also regarded as the angles such that

$$u_n = \tan^{-1} \frac{1}{W_n},\ W_n = \{\tan(\tanh^{-1} \frac{1}{F_n})\}^{-1}.$$

But u_n are not hogan-kaku with M=1.

In this paper we will prove that the relations such as (1.3) and (1.4) are derived from a linear recurrence formula such as (1.1) under some additional conditions, and the converse is also true under those conditions.

2. GENERALIZED SEQUENCE.

For the given sequence of complex numbers $\{H_n\}$ (n=0, 1, 2, . . .) with two non-zero complex numbers a and b such that

$$H_n = a\, H_{n-1} + b\, H_{n-2}\ (n \geq 2), \tag{2.1}$$

let K and I be the numbers defined by

$$K = \frac{a}{\sqrt{-b}},$$
$$I = \frac{1}{b} H_1^2 - H_0^2 - \frac{a}{b} H_0 H_1$$

respectively.

In this paper we consider only the case $I \neq 0$.

For $I \neq 0$, let G_n be the numbers defined by

$$G_n = \frac{H_n}{\sqrt{I}\left(\sqrt{-b}\right)^n}\ (\text{for } I \neq 0).$$

For $G_n \neq 0$, $\neq \pm i$, and $\neq \pm Ki$, let Z_n, Z_n^* be the numbers defined by

$$Z_n = \tan^{-1} \frac{1}{G_n} = \tan^{-1} \frac{\sqrt{I}\left(\sqrt{-b}\right)^n}{H_n}\ (\text{for } G_n \neq 0, \pm i),$$

$$Z_n^* = \tan^{-1} \frac{K}{G_n} = \tan^{-1} \frac{(a/\sqrt{-b})\sqrt{I}\left(\sqrt{-b}\right)^n}{H_n}\ (\text{for } G_n \neq 0, \pm iK),$$

respectively, where for the complex number z, we mean that

$$\tan^{-1} z = \tfrac{1}{i} \tanh^{-1} iz = \tfrac{i}{2} \operatorname{Log} \tfrac{1-iz}{1+iz}.$$

For the sake of simplicity, let S be the set of complex numbers $\pm i$, 0, i.e.,

$$S = \{-i, 0, i\},$$

and let N_o and N_s be the sets of the suffixes defined by

$$N_o = \{n \mid G_n \notin S,\ G_{n-1}/K \notin S, \text{ and } G_{n-2} \notin S\},$$

$$N_s = \{n \mid G_n \in S \text{ or } G_{n-1}/K \in S, \text{ or } G_{n-2} \in S\},$$

respectively, i.e., let us decompose the set of the suffixes $\{n\}$ into three parts $\{0, 1\}$, N_o, and N_s.

For example, in the case $a=b=1$ and the sequence

$$\{H_n\} = \{2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots\},$$

we have $K=-i$, $I=-1$. Hence we get

$$\{G_n\} = \{H_n/i^{n+1}\} = \{-2i, 1, i, 0, -i, -1, 2i, 3, -5i, \ldots\},$$

$$\{G_n/K\} = \{2, i, -1, 0, 1, -i, -2, 3i, 5, \ldots\}.$$

Thus we have

$$N_s = \{2, 3, 4, 5, 6\}, \qquad N_o = \{7, 8, \ldots\}.$$

Then we can derive the following

<u>Lemma:</u> Assume that $\{H_n\}$ fulfills the condition such that

$$I = \tfrac{1}{b} H_1^2 - H_0^2 - \tfrac{a}{b} H_0 H_1 \neq 0,$$

$$n \in N_o. \tag{2.2}$$

Then the relation

$$Z_n = Z^*_{n-1} - Z_{n-2} \tag{2.3}$$

is equivalent to the relation

$$G_{n-2}(G_n - \tfrac{1}{K} G_{n-1}) - \tfrac{1}{K} G_{n-1} G_n = 1. \tag{2.4}$$

<u>Proof:</u> Since Z_n, Z^*_{n-1}, Z_{n-2} are not equal to $\pm\pi/2$ and (2.3) holds, by the addition theorem of the tangent function we see that

$$\frac{1}{G_{n-2}} = \tan(Z^*_{n-1} - Z_n) = \frac{\dfrac{K}{G_{n-1}} - \dfrac{1}{G_n}}{1 + \dfrac{K}{G_{n-1}G_n}} = \frac{G_n - \frac{1}{K} G_{n-1}}{\frac{1}{K} G_n G_{n-1} + 1}.$$

Therefore we get (2.4). The converse is obviously true. Q.E.D.

Next, we can easily verify the following

<u>Theorem 1:</u> If $\{H_n\}$ is satisfied by the condition (2.2), then from the linear recurrence formula such that

$$H_n = aH_{n-1} + bH_{n-2} \ (n \in N_o) \tag{2.5}$$

the relation (2.3) holds. That is,

$$\tan^{-1} \frac{\sqrt{I}\left(\sqrt{-b}\right)^n}{H_n} = \tan^{-1} \frac{a\sqrt{I}\left(\sqrt{-b}\right)^{n-1}}{\sqrt{-b}\ H_{n-1}} - \tan^{-1} \frac{\sqrt{I}\left(\sqrt{-b}\right)^{n-2}}{H_{n-2}} .$$

<u>Proof:</u> Dividing both sides in (2.5) by $\sqrt{I}\left(\sqrt{-b}\right)^n$, we have

$$G_n = KG_{n-1} - G_{n-2} \ \ (n \in N_o). \tag{2.6}$$

In order to prove the theorem, it is sufficient to derive (2.4) instead of (2.3) under the assumption $N_s = \phi$, by virtue of the lemma. For all $n \geq 2$, (2.4) will be proved by induction. When $n = 2$, the left side in (2.4) is equal to 1. Indeed,

$$G_0\,(G_2 - \tfrac{1}{K}G_1) - \tfrac{1}{K}G_1G_2 = KG_0G_1 - G_0{}^2 - G_1{}^2 = 1.$$

Hence the case $n = 2$ is proved.

Assume next that (2.4) holds for $n = k$, i.e.,

$$G_{k-2}\,(G_k - \tfrac{1}{K}G_{k-1}) - \tfrac{1}{K}G_{k-1}G_k = 1. \tag{2.7}$$

Then in case $n = k+1$ we shall proceed as follows:

$$\begin{aligned} &G_{k-1}\,(G_{k+1} - \tfrac{1}{K}G_k) - \tfrac{1}{K}G_kG_{k+1} \\ &= KG_{k-1}G_k - G_{k-1}^2 - G_k^2 \\ &= G_{k-2}\,(G_k - \tfrac{1}{K}G_{k-1}) - \tfrac{1}{K}G_kG_{k-1} = 1. \end{aligned}$$

This proves the theorem. Q.E.D.

Here as an application of the theorem some examples are given.

<u>Examples:</u> Ex. 1. In the case a=b=1, $H_0=0$, $H_1=1$ ($H_n=F_n$: Fibonacci numbers), we have $K=-i$ and $I=1$. Hence from (2.3) we have

$$\tan^{-1} \frac{i^{n+1}}{F_{n-1}} = \tan^{-1} \frac{i^{n+1}}{F_n} + \tan^{-1} \frac{i^{n+1}}{F_{n+1}} ,$$

i.e.

$$\theta_{2n} = \theta_{2n+1} + \theta_{2n+2}, \quad u_{2n+1} = u_{2n+2} + u_{2n+3} \ (n \geq 1).$$

Therefore we have Lehmer's result

$$\pi = 4 \sum_{k=1}^{\infty} \tan^{-1} \frac{1}{F_{2k+1}} \quad ([2, \text{Theorem } 5]).$$

Ex. 2. In the case a=2x, b=1, $H_0=0$, $H_1=1$($H_n=P_n(x)$: Pell polynomials ([3, p. 55])), we have $K=-2xi$ and I=1. Hence from (2.3) we have

$$\tan^{-1} \frac{1}{P_{2n}(x)} = \tan^{-1} \frac{2x}{P_{2n+1}(x)} + \tan^{-1} \frac{1}{P_{2n+2}(x)} \ (n \geq 1),$$

$$\tanh^{-1} \frac{1}{P_{2n+1}(x)} = \tanh^{-1} \frac{2x}{P_{2n+2}(x)} + \tanh^{-1} \frac{1}{P_{2n+3}(x)} \ (n \geq 1).$$

Hence

$$\pi = 4 \tan^{-1} \{\exp (\sum_{k=2}^{\infty} \tanh^{-1} \frac{2x}{P_{2k}(x)})\} - 2 \sin^{-1} \frac{1}{4x^2+1} .$$

Ex. 3. In the case a=2x, b=1, $H_0=Q_0(x)=2$, $H_1=Q_1(x)=2x$($H_n=Q_n(x)$: Pell-Lucas polynomials ([3, p. 55])), we have $K=-2xi$ and $I=-4(x^2+1)$. Hence from (2.3) we have

$$\tanh^{-1} \frac{2\sqrt{x^2+1}}{Q_{2n}(x)} = \tanh^{-1} \frac{4x\sqrt{x^2+1}}{Q_{2n+1}(x)} + \tanh^{-1} \frac{2\sqrt{x^2+1}}{Q_{2n+2}(x)},$$

$$\tan^{-1} \frac{2\sqrt{x^2+1}}{Q_{2n+1}(x)} = \tan^{-1} \frac{4x\sqrt{x^2+1}}{Q_{2n+2}(x)} + \tan^{-1} \frac{2\sqrt{x^2+1}}{Q_{2n+3}(x)} .$$

Hence

$$\pi = 4 \tan^{-1} \{\exp (\sum_{k=1}^{\infty} \tanh^{-1} \frac{4x\sqrt{x^2+1}}{Q_{2k+1}(x)})\} - 2 \sin^{-1} \frac{\sqrt{x^2+1}}{2x^2+1} .$$

Putting x=1/2, we have

$$\pi = 4 \tan^{-1} \{\exp (\sum_{k=1}^{\infty} \tanh^{-1} \frac{\sqrt{5}}{L_{2k+1}})\} - 2 \sin^{-1} \frac{\sqrt{5}}{3},$$

where L_n are Lucas numbers.

Ex. 4. The Chebyshev polynomials $T_n(x)$ satisfy (2.1) for a=2x, b=−1, $H_0=T_0(x)=1$, $H_1=T_1(x)=x$, $H_n=T_n(x)$ ($n\geq 2$) ([6, p. 35]). Thus from (2.3) we get an identity equation

$$\tanh^{-1} \frac{\sqrt{1-x^2}}{T_{n-2}(x)} = \tanh^{-1} \frac{2x\sqrt{1-x^2}}{T_{n-1}(x)} - \tanh^{-1} \frac{\sqrt{1-x^2}}{T_n(x)} .$$

Ex. 5 For the sequence $\{V_n\}$ defined by

$$V_n = V_{n-1} + (\tfrac{d-1}{4})V_{n-2} \ (n\geq 2), \ V_0=0, \ V_1=1,$$

we have I=1. If d=5, then we have Ex. 1. From (2.3) we have

$$\tan^{-1} \frac{i^{n+1}\left(\sqrt{\frac{d-1}{4}}\right)^{n-1}}{V_{n-1}} = \tan^{-1} \frac{i^{n+1}\left(\sqrt{\frac{d-1}{4}}\right)^{n-1}}{V_n} + \tan^{-1} \frac{i^{n+1}\left(\sqrt{\frac{d-1}{4}}\right)^{n+1}}{V_{n+1}} .$$

Hence we obtain a generalized Lehmer's result

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{\left(\frac{d-1}{4}\right)^k}{V_{2k+1}} = \tan^{-1} \left(\frac{d-1}{4}\right).$$

3. INVERSE PROBLEMS

In this section we shall discuss the conditions which are imposed on $\{H_n\}$ $(n = 0, 1, 2, \ldots)$ in order to derive the linear recurrence formula (2.1). In this section we use the same symbols as used previously. The inverse of Theorem 1 is the following

Theorem 2: Under the conditions such that

(A) $\quad H_2 = aH_1 + bH_0,$ (3.1)

(B) $\quad H_n = aH_{n-1} + bH_{n-2} \ (n \in N_s),$

(C) $\quad Z_n = Z^*_{n-1} - Z_{n-2}, \ (n \in N_o)$, i.e.

$$\tan^{-1} \frac{\sqrt{I}\left(\sqrt{-b}\right)^n}{H_n} = \tan^{-1} \frac{a\sqrt{I}\left(\sqrt{-b}\right)^{n-1}}{\sqrt{-b}\ H_{n-1}} - \tan^{-1} \frac{\sqrt{I}\left(\sqrt{-b}\right)^{n-2}}{H_{n-2}}.$$

$$(n \in N_o)$$

are fulfilled, then

$$H_n = aH_{n-1} + bH_{n-2} \ (n \geq 2).$$

Remark: Condition (B) is included for the convenience of applications.

Proof of Theorem 2: It is sufficient to prove the theorem in the case $N_s = \phi$. And we have only to derive

$$G_n = KG_{n-1} - G_{n-2} \ (n \geq 2)$$

from (3.1) and (2.4).

Dividing both sides of (3.1) by $-b$, we have

$$G_2 = KG_1 - G_0. \tag{3.2}$$

Assume now that

$$G_k = KG_{k-1} - G_{k-2} \ (k \geq 2). \tag{3.3}$$

In (2.4) putting n=k, n=k+1, we have

$$G_{k-2}\left(G_k - \tfrac{1}{K}G_{k-1}\right) - \tfrac{1}{K}G_kG_{k-1} = 1,$$

$$G_{k-1}\left(G_{k+1} - \tfrac{1}{K}G_k\right) - \tfrac{1}{K}G_{k+1}G_k = 1.$$

Subtracting these equations, we get

$$G_{k+1}\,(G_{k-1} - \tfrac{1}{K}G_k) - KG_k\,(G_{k-1} - \tfrac{1}{K}G_k) + G_{k-1}\,(G_{k-1} - \tfrac{1}{K}G_k) = 0. \tag{3.4}$$

Here we notice that

$$G_{k-1} - \tfrac{1}{K}G_k \neq 0 \quad (k \geq 2),$$

because, if $G_{k-1} - (1/K)G_k = 0 (k \geq 2)$, then from (3.3) we see that $G_{k-2} = 0$ which is a contradiction to $G_n \notin S (n \geq 0)$.

Dividing both sides of (3.4) by $G_{k-1} - (1/K)G_k$, we get

$$G_{k+1} = KG_k - G_{k-1}.$$

Hence the theorem is proved. Q.E.D.

When a=b=1, we get the following corollary of Theorem 2. Its investigation brought about the starting point of this study.

Putting $\theta_n = \tan^{-1}(1/H_n)$, $u_n = \tanh^{-1}(1/H_n)$ we obtain

Corollary: Under the conditions $H_0=0$, $H_1=H_2=1$, $H_3=2$, and

$$\theta_{2n-2} = \theta_{2n-1} + \theta_{2n}, \quad u_{2n-1} = u_{2n} + u_{2n+1} \quad (n \geq 2),$$

it holds that

$$H_n = H_{n-1} + H_{n-2} \quad (n \geq 2),$$

i.e., $H_n = F_n$ (Fibonacci numbers).

The following proposition related to the above corollary does not need the condition $H_3=2$. In a similar theorem of [7] it is assumed that H_n, except H_0, are positive integers instead of positive numbers. Our proof is different from that of [7].

Proposition 1: Assume that $H_0=0$, $H_1=1$, $H_n>0$ $(n \geq 2)$, and

$$\theta_{2n-2} = \theta_{2n-1} + \theta_{2n} \quad (n \geq 2), \tag{3.5}$$

$$H_{2n} = H_{2n-1} + H_{2n-2} \quad (n \geq 1). \tag{3.6}$$

Then $$H_n = H_{n-1} + H_{n-2} \quad (n \geq 2),$$

i.e., $$H_n = F_n \quad (n \geq 0).$$

Proof: From (3.5) we get

$$H_{2n-2}(H_{2n} + H_{2n-1}) - H_{2n}H_{2n-1} = -1 \quad (n \geq 2). \tag{3.7}$$

Replacing n by n+1

$$H_{2n}(H_{2n+2} + H_{2n+1}) - H_{2n+2}H_{2n+1} = -1. \tag{3.8}$$

From (3.7) and (3.6) we get

$$H_{2n}^2 - H_{2n-1}^2 - H_{2n}H_{2n-1} = -1. \tag{3.9}$$

Equation (3.6) replaced n by n+1 is written as

$$H_{2n+2} = H_{2n+1} + H_{2n}.$$

So from (3.8)

$$H_{2n}H_{2n+1} + H_{2n}^2 - H_{2n+1}^2 = -1. \tag{3.10}$$

From (3.9) and (3.10)

$$(H_{2n+1} + H_{2n-1})\,(H_{2n} + H_{2n-1} - H_{2n+1}) = 0.$$

By virtue of the condition $H_n > 0$ $(n \geq 1)$ we have

$$H_{2n+1} = H_{2n} + H_{2n-1} \quad (n \geq 2). \tag{3.11}$$

Combining (3.11) with (3.6), we produce

$$H_{n+1} = H_n + H_{n-1} \quad (n \geq 3). \tag{3.12}$$

The starting relation $H_3 = H_2 + H_1 = 2$ is given from (3.6), (3.7), and $H_n > 0$ $(n \geq 2)$. This completes the proof. Q.E.D.

By a similar proof we can derive the following proposition, so its proof is omitted.

Proposition 2: Assume that $H_0=0$, $H_1=H_2=1$, $H_n > 0$ $(n \geq 3)$, and

$$u_{2n-1} = u_{2n} + u_{2n+1} \ (n \geq 2), \ H_{2n+1} = H_{2n} + H_{2n-1} \ (n \geq 1).$$

Then $$H_n = H_{n-1} + H_{n-2} \ (n \geq 2),$$

i.e., $$H_n = F_n \quad (n \geq 0).$$

Remark: In the proof of the Theorem 2 we see that the condition $G_n \neq KG_{n-1}$ plays an important role. Noting this fact, we can easily obtain a generalization of Theorem 2:

Let M_o and M_s be the sets of the suffixes defined by

$$M_o = \{n \mid G_n \neq 0,\ G_{n-1} \neq 0, \text{ and } G_{n-2} \neq 0\},$$

$$M_s = \{n \mid G_n = 0 \text{ or } G_{n-1} = 0, \text{ or } G_{n-2} = 0\},$$

respectively, i.e., let us decompose

$$\{n\} = \{0, 1\} + M_o + M_s.$$

Then we have the following

Theorem 3: Assume that

(A′) $H_2 = aH_1 + bH_0$,

(B′) $H_n = aH_{n-1} + bH_{n-2}$ $(n \in M_s)$,

(C′) $H_{n-2}(aH_n + bH_{n-1}) - H_nH_{n-1} = aI(-b)^{n-1}$ $(n \in M_o)$.

Then the formula

$$H_n = aH_{n-1} + bH_{n-2} \quad (n \geq 2)$$

is valid.

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SOME RESULTS ON DIVISIBILITY SEQUENCES

Norbert Jensen

1. BASIC CONCEPTS

Let g be a sequence of rational integers. g is called a *divisibility sequence* (DS) iff

$$n \mid m \quad \rightarrow \quad g(n) \mid g(m) \tag{1.1}$$

holds for all positive integers n, m. (This concept has been introduced by Ward in [8], [9]. Ward generalized a concept used by Hall in [2] where he considered DS satisfying a linear recurrence relation of order k with the coefficients being rational integers. Hall's work has been inspired by works of other authors for special cases, e.g.: E. Lucas, D. H. Lehmer, T. A. Pierce.)

g is called a *strong divisiblity sequence* (SDS), iff even

$$(g(n), g(m)) = g((n,m)) \tag{1.2}$$

holds for all positive integers n,m. (This term has first been used by Kimberling in [4]. Here g satisfies a linear recurrence relation of arbitrary order. The general definition is given by Kimberling in [5]. However, such sequences have already been considered by Ward in [8], [9], [12], 1936-39.) The Fibonacci- Sequence defined by $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ on the set of positive integers $\mathbb{N}$ is a special SDS, as can be seen from Theorem 6.1, see Carmichael [1].

Many examples for DS or SDS considered in literature satisfy a homogenous linear recurrence relation of order two. We say that g satisfies a homogenous linear recurrence relation of order k iff there are rational integers $a_1, a_2, \ldots a_k$, such that for each argument n

$$g(n+k) = a_1\, g(n+k-1) + \cdots + a_k\, g(n) \tag{1.3}$$

holds. Sequences satisfying (1.1) and (1.3) Ward calls Lucasian Sequences. The polynomial

$$\chi(x) = x^k - a_1\, x^{k-1} - \ldots - a_k \tag{1.4}$$

is said to be a characteristic polynomial of g.

2. RESULTS

First we define two positive-valued SDS without using a linear recursion (1.3). The first will be a minimal univalent SDS for arguments greater or equal than 2 in the sense that all positive-valued SDS pointwise below this sequence can't be univalent or are equal to the considered SDS. The second is a minimal increasing SDS where minimal should be understood in the analogue way. Both sequences have the same initial values as the sequence of Fibonacci-

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numbers. We consider their growth-properties and their divisibility by primes. It turns out that the minimal univalent sequence is polynomially bounded whereas the second is not. It will be seen that Ward's concept of the Dedekind generator [13] is a useful tool for handling such sequences. Second we shall determine all DS satisfying a recursion (1.3) where

$$\chi(x) = \prod_{j=0}^{k-1} (x - a^j b^{k-1-j}) \tag{2.1}$$

and a, b are the roots of a non-degenerate polynomial $x^2 - u x + v$ with u, v being coprime rational integers. This is a generalization of a former result obtained by several authors ([3], [4], and [2] in connection with [14]). It gives a proof of a special case of a conjecture for Lucasian Sequences of M. Ward [11], which is about fifty years old. It also gives a partial answer to Kimberling's question for a classification of all Lucasian Sequences [5].

The proof we shall give can be generalized such that it is applicable when instead of rational integers we take an integral domain where each element has just a finite number of divisors. So it is also applicable to sequences of polynomials with the coefficients being rational integers.

3. A THEOREM OF WARD.

In [13] Ward established

Theorem 3.1: For each positive valued SDS g there is exactly one mapping $g' : \mathbb{N} \to \mathbb{N}$ which has the property

$$g(n) = \prod_{d \mid n} g'(d) \quad \text{for each } n \in \mathbb{N}. \tag{3.1}$$

g' will be called the derivative of g.

A mapping $f : \mathbb{N} \to \mathbb{N}$ is the derivative of a SDS, iff for all n, m ε $\mathbb{N}$

$$(f(n), f(m)) = 1 \tag{3.2}$$

if none of the two arguments divides the other.

With a slight modification this theorem is also true for arbitrary semigroups with unique prime factorization. Then the factorization of $x^n - 1$ into its cyclotomic factors in $\mathbb{Z}[x]$ is an example for (3.1). The proof of theorem 3.1 is based upon an application of the Moebius inversion formula and Dedekind's formula for the least common multiple of a finite number of integers.

4. TWO SPECIAL SDS DEFINED WITHOUT USING A LINEAR RECURSION FORMULA.

Using theorem 3.1 we define two SDS g_1, g_2 by constructing their derivatives.

$$g_1'(1) := 1, \; g_1'(2) := 1, \tag{4.1}$$

$$g_1'(n) := \min \{ m \in \mathbb{N} \mid \text{(i) } g_1'(k), m \text{ are coprime for each } k \le n-1 \text{ not dividing } n,$$
$$\text{(ii) } m \prod_{\substack{d \mid n \\ \neq}} g_1'(d) \text{ is different from } g_1(k) \text{ for each } k \le n-1\},$$

where $n \ge 3$ and $g_1'(1), \ldots g_1'(n-1)$ have been defined previously.

$$g_2'(1) := 1,\ g_2'(2) := 1, \tag{4.2}$$

$$g_2'(n) := \min \{m \in \mathbb{N} \mid \text{(i) } g_2'(k), m \text{ are coprime for each } k \leq n-1 \text{ not dividing } n,$$

$$\text{(ii) } m \prod_{\substack{d \mid n \\ \neq}} g_2'(d) \geq g_2(n-1)\},$$

where $n \geq 3$ and $g_2'(1), \ldots g_2'(n-1)$ have been defined previously.

<u>Remark:</u> If $g_1'(2) = 2$ instead of 1 both sequences would obviously be the identity mapping on $\mathbb{N}$. In this case, however, they are different from each other. g_1 is univalent on the positive rational integers ≥ 2, g_2 is even strictly increasing on the same set. It will be shown that g_1 is polynomially bounded (it is not linearly bounded) and that for each polynomial h $g_2(n)$ is greater than $h(n)$, provided n is sufficiently large.

5. PROPERTIES OF THE SDS'S g_1, g_2.

5.1 g_1 : It is easily seen that $g_1'(n)$ is always a prime or unity and that it is different from unity iff there are $k \in \mathbb{N}$ and an odd prime p such that $n = 2^{k+1}$ or $n = 2$ p or $n = p^k$. By evaluation of (4.1) one gets:

$$g_1(2^{k+1}) = 3^k,\ g_1(2p) = g_1(p)^2,\ g_1(p^k) = g_1(p)\, g_1'(p^2)^{k-1} \tag{5.1.1}$$

for each $k \in \mathbb{N}$ and each odd prime p.

From (3.2) it is evident that $g_1'(4)$, $g_1'(2p) = g_1'(p)$, $g_1'(p^2)$, p an odd prime, are pairwise coprime. So $g_1(n)$ contains a characteristic factor iff n is an odd prime or the square of a prime. Since by the formula (4.1) it is obvious that the primes occur in g_1 in their natural order, one sees: If p is an odd prime then $g_1'(p^2)$ must be the $\pi(p^2) + \pi(p) - 1$ th prime number. Now we can prove:

There are constants $c, k \in \mathbb{N}$ such that $g_1(n) \leq c\, n^k$ for each $n \in \mathbb{N}$. (5.1.2)

<u>Proof:</u> By Chebychev's theorem and the characterization of $g_1'(p^2)$ given above there is an $\iota \in \mathbb{N}$ such that $g_1'(p^2) \leq p^\iota$. So we get for $a \in \mathbb{N}$:

$$g_1(p^a) = g_1(p)\, g_1'(p^2)^{a-1} \leq g_1'(p^2)^a \leq p^{\iota a} = (p^a)^\iota.$$

So for $n = 2^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ it follows by (5.1.1) that

$$g_1(n) \leq g_1'(2p_2) \ldots g_1'(2p_r)\, g_1(2^{a_1}) \ldots g(p_r{}^{a_r}) \leq$$

$$\leq g_1'(p_2^2) \ldots g_1'(p_r^2)\, g_1(2^{a_1}) \ldots g_1(p_r{}^{a_r}) \leq n^{2\iota}.$$

The assertion follows for $k = 2\,\iota$.

5.2 g_2 : g_2 is not polynomially bounded. In fact, for each $k \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the inequality

$$g_2(n) \geq n^k \tag{5.2.1}$$

holds. This is a consequence of

<u>Theorem 5.1:</u> Let g be a positive-valued SDS satisfying $g(1) = 1$. Let P be the set of all rational integers greater than an arbitrarily chosen positive constant m_0 and divisible by at most 3 different primes. Assume g is increasing on P and not always $= 1$ on P. Then g is univalent or it is not polynomially bounded.

To prove this theorem we need

<u>Lemma 5.2:</u> Let g be a positive-valued SDS. Then g is univalent iff for each prime-power p^n greater than 1 one has $g'(p^n)$ is greater than 1.

The proof is a simple application of Theorem 3.1 and will be left to the reader.

<u>Proof of 5.1:</u> W l. o. g. we assume $g(m) > 1$ for all $m \varepsilon P$. Suppose, g is not univalent. By Lemma 5.2, there is a prime power $q^n > 1$, such that

$$g'(q^n) = 1 \tag{5.1.1}$$

It will be shown that for $k \in \mathbb{N}$ there is an infinite number of $m \in \mathbb{N}$ such that $g(m) > m^k$. For almost all primes $p_1 \in P$ one has

$$1 < g'(q^n\ p_1). \tag{5.1.2}$$

Choose a prime $p_2 \in P$ different from q. Let $p_1 \in P$ be a prime such that $p_1 > \max\{q, p_2\}$. Choose $m \in \mathbb{N}_0$ such that

$$q^m\ p_2 < q^n\ p_1 < q^{m+1}\ p_2\ .$$

So $q^{n-1}\ p_1 < q^m\ p_2 < q^n\ p_1$.

Since g is increasing on P the inequalities

$$g(q^{n-1}\ p_1) \leq g(q^m\ p_2) \leq g(q^n\ p_1)$$

hold. If one of them were an equation we had $g(p_2) = 1$, a contradiction, since $p_2 \in P$. $(5.1.2)'$ follows from (5.1.1) and (3.1).

By (3.2) the sequence $g'(q^n q_1)$ where q_1 runs through all primes tends to infinity. Now let p be a prime of P, $p > q^n$. For all $m \in \mathbb{N}$ the inequality

$$g'(p^m\ p_1)\ g'(p^m) \geq g'(q^n\ p_1) \tag{5.1.3}$$

holds. (5.1.3) is a consequence of $g(p^m\ p_1) \geq g(q^n p^{m-1}\ p_1)$ and (3.1). (5.1.3) implies:

$$g(p^m\ p_1) \geq (g'(q^n\ p_1))^m = (p^m\ p_1)^{(\log g'(q^n\ p_1))/(\log p\ +\ (\log p_1)/m)}\ .$$

By first choosing p_1 sufficiently large we get $\log g'(q^n\ p_1) \geq k\ (\log p + 1)$. Choosing $m > \log p_1$ the assertion follows.

Now by the previous considerations for $j \in \mathbb{N}$, sufficiently large p_1 we get the following inequalities:
For $n \in \mathbb{N}$ sufficiently large let $m \in \mathbb{N}$ be such that

$$3^{m+1}\ p_1 > n \geq 3^m\ p_1.$$

Then it follows that:

$$g_2(n) \geq g_2(3^m\ p_1) \geq (3^m\ p_1)^j = (3^{2m}\ {p_1}^2)^{(j/2)} \geq n^{(j/2)}.$$

6. CHARACTERIZATION OF DS AND SDS SATISFYING A CERTAIN LINEAR RECURSION FORMULA

Let u, v ϵ $\mathbb{Z}$ be coprime, v different from 0. Assume $p(x) := x^2 - u\,x + v$ is different from $Q_1\, Q_2, Q_3, Q_4, Q_6$ where Q_m denotes the m-th cyclotomic polynomial. Let a, b ϵ $\mathbb{C}$ be such that $p(x) = (x - a)(x - b)$. Let $K := Q[a,b]$ and let R denote the ring of integers of K. Let f ϵ K[x] be of positive degree φ.

Carmichael proved in [1]

Theorem 6.1: Every sequence h in $\mathbb{Z}$ for which p is a characteristic polynomial is a SDS if the initial values satisfy $h(2) = u\,h(1)$.

This also holds for artibrary domains R with a unity, if u, v coprime is understood as meaning $uR + vR = R$, see Penzin [6].

According to a theorem of Ward [10] a sequence g with values in a finite extension field L of a domain D of characteristic 0 satisfies a homogenous linear recurrence relation with coefficients in D iff there is a polynomial F ϵ $L[x_1, \dots x_r, y]$ and there are integral elements $w_1, \dots w_r$ ϵ L, such that $g(n) = F(w_1^n, \dots w_r^n, n)$ for each n ϵ $\mathbb{N}$, where the x and y are indeterminates and r ϵ $\mathbb{N}$ depends on g. So in the following we will essentially characterize the polynomial F which arises in our special case. We will be interested in $F(x_1, x_2) = x_2^{\varphi} f(x_1/x_2)$. The main result proved here is given by

Theorem 6.2: Let g be defined by $g(n) = b^{n\varphi} f((a/b)^n)$, $a \neq \pm 1$, that is let g satisfy a recursion (1.3) with a characteristic polynomial χ as given in (2.1), g(n) not always zero. g is a DS iff there is a c ϵ K and there are $w, d_1, \dots d_k, \alpha_1, \dots \alpha_k$ ϵ $\mathbb{N}$, such that

$$f(x) = c\, x^w\, \mu(x), \tag{6.2.1}$$

where μ denotes the least common multiple of $(x^{d_1} - 1)^{\alpha_1}, \dots (x^{d_k} - 1)^{\alpha_k}$ with the highest coefficient being unity.

If g is a SDS then $w = 0$ and $d_1, \dots d_k, \alpha_1, \dots \alpha_k$ can be chosen such that

$$d_1 \mid d_2 \mid \dots \mid d_k,\ d_1 < d_2 < \dots < d_k$$

and

$$f(x) = c \prod_{j=1}^{k} (x^{d_j} - 1)^{\alpha_j}. \tag{6.2.2}$$

Corollary 6.3: If the sequence h in $\mathbb{Z}$ has p as a characteristic polynomial and is not of first order then the following assertions are pairwise equivalent:

(i) h is a DS,
(ii) h is a SDS,
(iii) $h(2) = u\,h(1)$.

Corollary 6.3 follows also from the results of previous authors. Ward proved in [14] (1954) that a recurrent sequence having p as a characteristic polynomial and not of first order has infinitely many prime divisors. Hence 6.3 (i) $\rightarrow$ (iii) follows from Hall's Theorem 1 in [2] (1936). (ii) $\rightarrow$ (iii) is a consequence of a theorem by Kimberling in [4] (1978). (ii) $\longleftrightarrow$ (iii) has also been proved by Horak & Skula in [3] (1985).

A Schinzel extended (ii) $\longleftrightarrow$ (iii) to the ring of integers of an algebraic number field in [7].

Our proof: (iii) $\rightarrow$ (ii) is 6.1. Since every SDS is a DS we have (ii) $\rightarrow$ (i). (i) $\rightarrow$ (iii) is a consequence of 6.2. From the theory of difference equations it is clear that there are c_1, $c_2 \in K\setminus\{0\}$ such that $h(n) = c_1 a^n + c_2 b^n$. Define f by $f(x) = c_1 x + c_2$. So the premises of 6.2 are fulfilled. Hence $c_1 = - c_2$. But then $h(2) = (a + b) c_1 (a - b) = u h (1)$.

7. PROOF OF 6.2

Here we will just consider the only-if-part of the proof of (6.2.1) and leave the rest of (6.2.1) to the reader.

If p is irreducible K has a uniquely determined automorphism σ of order two which leaves $\mathbb{Q}$ fixed. σ induces an automorphism ψ of the multiplicative semigroup of K[x] as follows:

If $h_0, \ldots h_l \in K$, $h_o h_l \neq 0$, $h(x) = \sum_{j=0}^{l} h_j x^j$, let $h\psi = \sum_{j=0}^{l} h_{l-j}\sigma x^j$, $(x^w h)\psi = x^w (h\psi)$, $0\psi = 0$. Since (a/b) is not a root of unity the condition that g is a sequence in $\mathbb{Z}$ is equivalent to $f = f\psi$. Let the sequence $\hat{g} : \mathbb{N} \rightarrow R[x]$ be defined by $\hat{g}(n) = f(x^n)$. In the sequel we shall also use the extended definitions of DS and SDS for K[x]-valued sequences. Since there is a $z \in \mathbb{Z}\setminus\{0\}$, such that $zf \in R[x]$, and since zg is a DS or a SDS if g is, we will assume $f \in R[x]$ in the sequel.

Lemma 7.1: Let $r \in R[x]\setminus\{0\}$, $\rho := \deg r$, $m_1, m_2 \in \mathbb{N}$, $c_1, c_2 \in R[x]$. Assume further

$$\sum_{i=1}^{2} c_i \hat{g}(m_i) = r. \tag{7.1}$$

If g is a DS then $g(j) \mid f_\varphi b^{\rho j} r((a/b)^j)$ in R where f_φ is the highest coefficient of f.

Proof: Since $aR + bR = R$ one has for arbitrary $u \in \mathbb{N}$ that

$$f_\varphi R \subset f_\varphi R + b^u R \subset g(ju)R + b^u R \subset g(j)R + b^u R.$$

$$(7.1) \text{ yields } g(j)\, R \supset b^{(N-\rho)j}\, b^{\rho j}\, r((a/b)^j),$$

$N := \rho + \max\{\deg c_i + m_i \varphi \mid i \in \{1, 2\}\}$, since g is a DS. By choosing $u = (N - \rho) j$ one finds that there are $\gamma \in g(j) R$, $s \in R$ such that $\gamma + b^{(N-\rho)j} s = f_\varphi$. The assertion follows from

$$f_\varphi b^{\rho j} r((a/b)^j) = (\gamma + b^{(N-\rho)j} s)\, b^{\rho j} r((a/b)^j) =$$
$$= \gamma\, b^{\rho j} r((a/b)^j) + s b^{(N-\rho)j} (b^{\rho j} r((a/b)^j)) \in g(j)R,$$

since both summands belong to this ideal.

Lemma 7.2: Let $m, n \in \mathbb{N}$, $d \in R[x]$ be such that

$$d \in (\hat{g}(m), \hat{g}(n))_{K[x]}. \tag{7.2.1}$$

Assume that for each $j \in \mathbb{N}$ the relation

$$g((m,n)j) = (g(mj), g(nj)) \tag{7.2.2}$$

holds. Then there is a $k \in K$ such that $d^2 = k\, \hat{g}((m,n))^2$.

<u>Proof:</u> Let $\delta := \deg d$. By (7.2.1) and (7.2.2) there is a $w_1 \in \mathbb{N}$ such that (*) $b^{\delta j}\, d((a/b)^j) \mid w_1\, g((m,n)j)$ for each $j \in \mathbb{N}$. By (7.2.1) and since K[x] is euclidian there is a $w_2 \in \mathbb{N}$ and there are q_1, $q_2 \in R[x]$, such that $q_1\, \hat{g}(m) + q_2\, \hat{g}(n) = w_2\, d$. This yields for each $j \in \mathbb{N}$:

$$b^{(N-\varphi m)j}\, q_1\, ((a/b)^j)\, g(mj) + b^{(N-\varphi n)j}\, q_2((a/b)^j)\, g(nj) = b^{(N-\delta)j}\, w_2\, b^{\delta j}\, d((a/b)^j).$$

Hence: (**) $g((m,n)j) \mid f_\varphi\, w_2\, b^{\delta j}\, d((a/b)^j)$, by lemma 7.1. (*) and (**) yield

$$(***) \qquad ((w_1\, g((m,n)j)) \,/\, (b^{\delta j}\, d((a/b)^j))) \mid w_1\, w_2\, f_\varphi$$

in R if j is sufficiently large.

<u>Case 1:</u> p is irreducible: Since $f = f\psi$, there is a $k \in K\backslash\{0\}$ such that $d\psi = kd$.

Using $(b^{\delta j}\, d((a/b)^j))\sigma = b^{\delta j}\, (d\psi)\, ((a/b)^j) = b^{\delta j}\, k\, d((a/b)^j)$ and taking the norms on both sides in (***) one gets:

$$((w_1^2\, g((m,n)j)^2)/(k(b^{\delta j}\, d((a/b)^j))^2)) \mid w_1^2\, w_2^2\, (f_\varphi\sigma\, f_\varphi) \text{ in } \mathbb{Z}.$$

Since the right side has only a finite number of divisors there is a $z \in \mathbb{Z}\backslash\{0\}$, such that

$$w_1^2\, g((m,n)j)^2 = z\, k\, (b^{\delta j}\, d((a/b)^j))^2$$

holds for an infinite number of $j \in \mathbb{N}$. Since (a/b) is not a root of unity the assertion follows by a little computation. Use $f(0), d(0) \neq 0$; u,v coprime.

<u>Case 2:</u> p is reducible: This is similar to case 1, but simpler.

<u>Corollary 7.3:</u> If g is a DS then so is $\hat{g}$ in K[x]. If g is a SDS then so is $\hat{g}$ in K[x].

<u>Proof:</u> Suppose g is a DS. Choose $m = 1$, n arbitrarily in lemma 7.2. It follows that $f(x)^2 \mid f(x^n)^2$ in K[x]. Since K[x] admits a unique prime factorization $f(x) \mid f(x^n)$ whence the assertion follows by replacing x by an arbitrary power x^j. If g is a SDS the proof is similar.

<u>Lemma 7.4:</u> $\hat{g}$ is a DS in K[x] iff there are $w \in \mathbb{N}_0$, $c \in K\backslash\{0\}$, $l \in \mathbb{N}$, $d_1, \dots d_l$, $c_1, \dots c_l \in \mathbb{N}$ such that

$$f(x) = c\, x^w\, \mu(x), \tag{7.4.1}$$

where $\mu \in R[x]$ denotes the least common multiple of $(x^{d_1} - 1)^{e_1}, \dots, (x^{d_l} - 1)^{e_l}$ in K[x] with the highest coefficient being unity.

$\hat{g}$ is a SDS in K[x] iff $d_1, \dots d_l$, $e_1, \dots e_l$ can be chosen such that

d_j is a proper divisor of d_{j-1} for $j = l, l-1, \dots 2$, and (7.4.2)

$$f(x) = c \prod_{j=1}^{l} (x^{d_j} - 1)^{e_j}$$

<u>Proof :</u> Here we will just consider the only-if-implications and leave the rest to the reader.

<u>Proof of (7.4.1):</u> For each root $x_0 \in \mathbb{C}$ of f and each $n \in \mathbb{N}$ we have $f(x_0{}^n) = 0$, since $f(x) \mid f(x^n)$ in K[x]. Hence $x_0 = 0$ or x_0 is a root of unity. Let $w := \max \{w \in \mathbb{N}_0 \mid x^w \mid f\}$. Let $\tilde{f}$ be defined by $f = x^w \tilde{f}$. Let $c \in R\backslash\{0\}$, $x_1, \ldots x_m \in \mathbb{C}$ be such that $\tilde{f} = c \prod_{j=1}^{m} (x - x_j)$. For each $n \in \mathbb{N}$ one has $\tilde{f}(x) \mid \tilde{f}(x^n)$. Let $l \in \{1, \ldots m\}$ such that $(x - x_j) \mid (x^n - x_l)$. Since $(x - x_j) \mid (x^n - x_j{}^n)$ it follows that $(x - x_j)$ divides the difference $x_l - x_j{}^n$. Hence this difference must be 0. But each factor $x^n - x_l$ is squarefree, so for each zero x_j of $\tilde{f}$ all its powers must also be zeroes of $\tilde{f}$ and must have at least the same multiplicity. Let x_j be a d_j-th primitive root of unity, let

$$e_j := \max \{e \in \mathbb{N} \mid (x^{d_j} - 1)^e \mid \tilde{f}\}.$$

From the definition of f and the premise that $\hat{g}$ is a DS it follows that $e \geq 1$, since all d_j-th roots of unity are powers of x_j. From the previous consideration it is evident that e_j is the multiplicity of x_j in $\tilde{f}$. So μ and $\tilde{f}$ have the same roots including multiplicities. The assertion follows.

<u>Proof of 7.4.2:</u> It is easily seen that w must be 0.

We shall prove the assertion by induction on deg f. It is a consequence of the first part of the theorem, if deg f = 1. Otherwise let $n_1 := \max \{n \in \mathbb{N} \mid (x^n - 1) \text{ divides } f\}$. Now for each $n \in \mathbb{N}$ the implication $(x^n - 1) \mid f \rightarrow n \mid n_1$ holds. Let $d := (n, n_1)$. Then we have

$$(x^{(n\ n1)/d} - 1) \mid (f(x^{n/d}), f(x^{n1/d})) = f(x)$$

and $[n, n_1] = (n\ n_1)/d \leq n_1$ whence $n \mid n_1$ follows. Define f_1 by $f = (x^{n_1} - 1) f_1$ and $g_1{}^* : \mathbb{N} \rightarrow K[x]$ by $g_1{}^*(n) = f_1(x^n)$. We shall show that $g_1{}^*$ is a SDS. Then our assertion will follow from the induction hypothesis.

Let q be a prime of K[x] such that $q^\gamma \parallel f_1$, $\gamma \in \mathbb{N}$. Then $q \mid (x^{n1} - 1)$. So $q^{\gamma+1} \mid f(x) \mid f(x^m)$. Since $x^{n_1 m} - 1$ is squarefree $q^\gamma \mid f_1(x^m)$. So g_1^* is a DS. If $n, m \in \mathbb{N}$, $d \in K[x]$ such that $d \mid (f_1(x^n), f_1(x^m))$, then $(x^{(n,m)n_1} - 1)\ d \mid (f(x^n), f(x^m)) = f(x^{(n,m)})$. By cancellation we get $d \mid f_1(x^{(n,m)})$.

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ON MENTAL CALCULATION OF REPEATING DECIMALS, FINDING FIBONACCI NUMBERS AND A CONNECTION TO PASCAL'S TRIANGLE

Marjorie Bicknell-Johnson

1. ON MENTAL CALCULATION OF REPEATING DECIMALS

There is a well-known expression for 1/89,

$$
\begin{array}{rl}
1/89 = & .\,0\,1\,1\,2\,3\,5\,8 \\
 & 1\,3 \\
 & 2\,1 \\
 & 3\,4 \\
 & 5\,5 \\
 + & \ldots \\
\hline
= & .\,0\,1\,1\,2\,3\,5\,9\,5\,5\,0\,5\ldots
\end{array}
$$

where successive terms of the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$, appear and are summed. Of course, the decimal expansion of 1/89 can be found by long division and will have a period length of 44 digits, but there is a short-cut for obtaining those digits which leads to other entertainments. If we calculate $(89 + 1)/10 = 9$, we can generate 1/89 by "special division" of 0.1 by 9, where "special division" means that each successive digit in the quotient gives the next digit in the dividend, as

```
        .0 1 1 2 3 5 9 ...
9 )     .1 0 1 1 2 3 5 9...
          9
          1 1
            9
            2 1
            1 8
              3 2
              2 7
                5 3
                4 5
                  8 5
                  8 1
                    4 9
                      ...
```


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where the quotient will be 1/89 and the dividend will be 9/89. (Of course, one soon does this by short division.) Since 1/89 has a period of 44 digits, $10^{44} \equiv 1 \pmod{89}$ is preceded by $10^{43} \equiv 9 \pmod{89}$ so that 9 is the next to last remainder in the long division process to find 1/89, and 1/89 and 9/89 have the digits in their repetends displaced by one digit as well as the trivial but significant relationship, 9/89 = (9)(1/89). Thus, we can write the digits in the repetend of 1/89 in reverse order by multiplication by 9. We write the last digit of 1/89, which has to be a 1 since its product with 89 must end in a 9, and use "special multiplication" of 1 by 9, where we use successive digits in the product to write successive digist in the multiplicand, as

$$\begin{array}{r} \ldots 0\ 2\ 2\ 4\ 7\ 1\ 9\ 1 \\ \text{x}\ 9 \\ \hline \ldots 2\ 0\ 2\ 2\ 4\ 7\ 1\ 9 \end{array}$$

Further, it can be shown that 1/89 ends in

$$\begin{array}{r} 1 \\ 9 \\ 81 \\ 729 \\ 6561 \\ 59049 \\ +\quad \ldots\qquad . \\ \hline \ldots\quad 247191 \end{array}$$

where the powers of 9 appear successively and are summed.

This generalizes to any fraction 1/n, where n is an integer ending in the digit 9. The "special division" will divide 0.1 by (n + 1)/10, and the proof is the same as for the special case, since

$$10^{L(n)} \equiv 1 \pmod{n}$$

$$10^{L(n)-1} \equiv \frac{n+1}{10} \pmod{n}$$

where L(n) is the period length of n. Also, the repetend of 1/n, where n = 10k − 1, contains powers of k as seen from the right, since the sum after L(n) terms

$$S = \frac{1}{10^{L(n)}} + \frac{k}{10^{L(n)-1}} + \frac{k^2}{10^{L(n)-2}} + \ldots$$

is given by summing the geometric progression for L(n) terms as

$$S = \frac{1}{10^{L(n)}} * \frac{10^{L(n)}k^{L(n)} - 1}{10k - 1}$$

$$= \frac{1}{10^{L(n)}} * \frac{10^{L(n)}k^{L(n)} - 10^{L(n)} + 10^{L(n)} - 1}{10k - 1}$$

$$= \frac{k^{L(n)} - 1}{n} + \frac{10^{L(n)} - 1}{10^{L(n)} * n}$$

where the lefthand term is an integer since (k,n) = 1 and $k^{L(n)} \equiv 1 \pmod{n}$ is known [1], [2], and the righthand term represents the L(n) digits of the repetend of 1/n, following the decimal point.

2. ON FIBONACCI NUMBERS AND PASCAL'S TRIANGLE

Next we turn to a second Fibonacci generating fraction 1/109, which is of the same form. Thus, (109 + 1)/10 = 11 is its special number, and "special division" of 0.1 by 11 gives us

$$\begin{array}{rl} & .0\;0\;9\;1\;7\;4\;3\;1\;\ldots \\ \hline 11\;) & .1\;0\;0\;9\;1\;7\;4\;3\;\ldots \end{array}$$

while "special multiplication" of 1 by 11 gives us

$$\begin{array}{r} \ldots 3\;8\;5\;3\;2\;1\;1 \\ x\;1\;1 \\ \hline \\ \ldots 2\;3\;8\;5\;3\;2\;1 \end{array}$$

where either process will continue for 108 digits. But, 1/109 ends in powers of 11, as

$$\begin{array}{lr} & 1 \\ & 11 \\ & 121 \\ & 1331 \\ & 14641 \\ & 161051 \\ & 1771561 \\ + & \ldots \\ \hline \\ & \ldots\;3853211 \end{array}$$

The powers of 11 = (10 + 1) are related to Pascal's triangle,

$$\begin{array}{rrrrrr} & & & & & 1 \\ & & & & 1 & 1 \\ & & & 1 & 2 & 1 \\ & & 1 & 3 & 3 & 1 \\ & 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

where the sums of the diagonals rising leftward are the Fibonacci numbers. Notice that the sums of powers of 11 use the same scheme but shifted left n places to make sums in columns. Until carrying is involved, the Fibonacci numbers appear. The carrying in the multiplication, as in 11^5 as compared to the 5^{th} row of Pascal's triangle, gives the same final sum whether we add to form 11^5 first and then add down, or just add all of the columns down. So, we have a Pascal connection, and the repetend of 1/109 is given from the right as a reverse diagonalized sum of Fibonacci numbers,

$$
\begin{array}{r}
13853211 \\
21 \\
34 \\
55 \\
\underline{+\ldots.} \\
\ldots\ 8623853211
\end{array}
$$

This result can be proved by other methods [3]. In a similar way, 1/10099 can be expressed in terms of powers of 101 from the far right, where 101^k generates rows of Pascal's triangle with the columns interspersed by zeroes, so that 1/10099 ends in ...13080503020101. Also, 1/1000999 is related to powers of 1001 and ends in ...013008005003002001001.

It is also entertaining to notice that 1/89 can be expressed as the sum of successive powers of 11 (with an adjustment of the decimal point) as

$$
\begin{array}{rl}
1/89 = & .01 \\
& .0011 \\
& .000121 \\
& .00001331 \\
+ & .0000014641\ \ldots,
\end{array}
$$

which is proved by summing the infinite geometric series

$$1/89 = 1/10^2 + 11/10^4 + 11^2/10^6 + \ldots$$

and which Pascal connection is one way to explain the curiosity given at the beginning. In a similar manner [3], 1/9899 is related to 101^k and has successive Fibonacci numbers in every second place, as

$$1/9899 = .0001010203050813\ldots$$

while a Pascal connection with 1001^k leads to

$$1/998999 = .000001001002003005008013\ldots$$

3. ON MORE GENERALIZED MENTAL CALCULATION

To finish this discussion, we give a process for rapid listing of the digits of repeating decimals for 1/n where (n,10) = 1 but n does not end in the digit 9. If n ends in the digit k = 3 or 7, then kn will end in 9, and use "special division" of k/10 by (kn + 1)/10. For example, when k = 3, 1/13 = 3/39, and

$$\begin{array}{r@{\;}l} & .0\;7\;6\;9\;2\;3\ldots \\ \hline 4\,) & .3\;0\;7\;6\;9\;2\ldots \end{array}$$

If n ends in 7, then 7n ends in 9, but the "special division" of 0.7 by (7n + 1)/10 might not be a short-cut. For example, 1/17 = 7/119 can be done with "special division" of 0.7 by 12, but it is easier to use 1/17 = 47/799, where we can use "special division" of 0.47 by 8 but displace the digits two places, as

$$\begin{array}{r@{\;}l} & .0\;5\;8\;8\;2\ldots \\ \hline 8\,) & .4\;7\;0\;5\;8\;8\ldots \end{array}$$

Our most generalized statement, then, is that if 1/n ends in k last digits which make up the integer K, then nK ends in k nines and we use a displacement of k places. Thus, 1/97 = 7/679 requires "division" by 68, not particularly a short-cut, but the first one-digit divisor possible occurs for 1/97 = 6185567/599999999, where "special division" of 0.06185567 by 6 needs an eight digit displacement, and the numerator just happens to be the last eight digits of the repetend of 1/97. The numerators are always the last k digits of the repetend of 1/n, and the divisors are always the k^{th} remainders from the last if 1/n is found by long division. Note that "special multiplication" can also be done with these numbers to get the digits of 1/n appearing from right to left.

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DIOPHANTINE REPRESENTATION OF FIBONACCI NUMBERS OVER NATURAL NUMBERS

James P. Jones

The sequence of Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, ..., defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}$, played an important role in the solution of one of the Hilbert Problems. The Fibonacci sequence was used in 1970 by the Russian mathematician Y.V. Matijasevič to solve the Tenth Problem of Hilbert. The Tenth Problem of Hilbert was the problem of existence of an algorithm for deciding solvability of Diophantine equations. Matijasevič [8] [9] made use of divisibility properties of the Fibonacci sequence to prove that every recursively enumerable set is Diophantine. This solved Hilbert's Tenth Problem in the negative.

1. DIOPHANTINE SETS.

A *set* A is said to be *Diophantine* if there exists a polynomial $P(a_1, a_2, ..., a_k, x_1, x_2, ..., x_n)$, with integer coefficients, such that for all values of the parameters $a_1, a_2, ..., a_k$, we have

$$A(a_1, a_2, ..., a_k) \Longleftrightarrow (\exists x_1, x_2, ..., x_n)[P(a_1, a_2, ..., a_k, x_1, x_2, ..., x_n) = 0]. \tag{1}$$

If for each $(a_1, a_2, ..., a_k)$ the equation $P(a_1, a_2, ..., a_k, x_1, x_2, ..., x_n) = 0$ has only a single solution, i.e. the unknowns $(x_1, x_2, ..., x_n)$ are unique, then the set A is said to be *singlefold* Diophantine. If for each $(a_1, a_2, ..., a_k)$, the solutions $x_1, x_2, ... x_n$ can be chosen to be elements of a fixed set B, then the set A is said to be *Diophantine over B.* When the set A is Diophantine over itself, we say that A is *self Diophantine.*

The theorem of Matijasevič [8], that every r.e. set is Diophantine, was surprising when it was proved because it implies that every r.e. set is equal to the set of positive values of a polynomial. To see why this is so, suppose A is Diophantine. Then, (following H. Putnam [10]),

$$\begin{aligned} y \in A &\Longleftrightarrow (\exists\, x_1, x_2, ..., x_n)\, [P(y, x_1, x_2, ..., x_n) = 0], \\ &\Longleftrightarrow (\exists\, x_1, x_2, ..., x_n)\, [\, 1 \le 1 - P(y, x_1, x_2, ..., x_n)^2\,], \\ &\Longleftrightarrow (\exists\, x_0, x_1, ..., x_n)\, [\, y = x_0(\, 1 - P(\, x_0, x_1, ..., x_n)^2)\,]. \end{aligned} \tag{2}$$

Now put $Q(x_0, x_1, ..., x_n) = x_0(1 - P(x_0, x_1, ..., x_n)^2)$. Then Q is a polynomial and it is easy to see that A is the positive part of the range of Q,

$$y \in A \Longleftrightarrow (\exists\, x_0, x_1, ..., x_n)\, [\, Q\, (\, x_0, x_1, ..., x_n) = y\,]. \tag{3}$$

So every r.e. set is the positive part of the range of a polynomial. Since the set of Fibonacci numbers is r.e., the Fibonacci numbers can also be represented as the set of positive values of some polynomial Q.

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If we use the original [8] proof of unsolvability of Hilbert's Tenth Problem to construct a polynomial Q representing the set of Fibonacci numbers, then Q will be a polynomial in hundreds of variables. It will also have very large degree. However, it is possible to construct a relatively small polynomial in only two variables and of very low degree, 5. In 1975 this was done in the author's paper [2],

$$2y^4x + y^3x^2 - 2y^2x^3 - y^5 - yx^4 + 2y. \tag{4}$$

It was proved in the paper [2] that, as the variables x and y run over the positive integers, the positive values of the polynomial (4) coincide with the set of positive Fibonacci numbers. (4) is a polynomial of degree 5.

In the present paper, we consider the problem of constructing a Fibonacci number representing polynomial over the natural numbers. That is, we want to construct a polynomial whose nonnegative values represent the set of nonnegative Fibonacci numbers, as the variables x and y run over the nonnegative integers. At first glance, this appears to be a simple problem and in fact this was already done. The polynomial (4) already has this property. (The construction in [2] clearly shows this.) However the polynomial (4) had, over the positive integers, some additional attractive properties which we would also like to preserve over the natural numbers.

The 1975 polynomial (4) took on each Fibonacci number exactly once, because it was constructed from a singlefold Diophantine definition of the Fibonacci numbers. In fact, this Diophantine definition was simultaneously singlefold and self Diophantine. So the earlier polynomial (4) which took on each Fibonacci number exactly once, did so only at Fibonacci lattice points, that is, for values of x and y which were Fibonacci.

Over the natural numbers, (nonnegative integers) the polynomial (4) losses some of these special properties. For example the Fibonacci value 1 is taken on twice. It is taken on at the point $x = 0$, $y = 1$, and again at the point $x = 1$ and $y = 1$. Somewhat worse, the Fibonacci value 0 is taken on infinitely many times by (4). The polynomial equals 0 when $y = 0$ and $x = 0, 1, 2, \ldots$.

The problem with the construction in [2], and the motivation for the present paper, is that over the natural numbers, the Diophantine definition used in [2] is not singlefold. This raises the questions: Is the set of Fibonacci numbers singlefold Diophantine over the natural numbers? Is the set of Fibonacci numbers self Diophantine over the natural numbers in one unknown? Is the set of Fibonacci numbers singlefold self Diophantine over the natural numbers in one unknown? It turns out that the answers to the first two questions are "yes", (and this is proved below). We don't know the answer to the third question. It is left as an open problem at the end of the paper. Nevertheless, we were able to prove the theorem which would have followed if we had a positive answer to it. There exists a polynomial in two variables, which has, over the natural numbers, the same properties as (4). This polynomial takes on each Fibonacci number exactly once, and does so only at a Fibonacci point. It is a polynomial not much more complicated than (4). It has degree 6 instead of 5, see (16).

2. FIBONACCI NUMBERS AS VALUES OF A POLYNOMIAL.

The proof of existence of this polynomial is a refinement of the construction in [2]. As in [2], we use again the polynomial

$$L(x,y) = y^2 - xy - x^2. \tag{5}$$

This polynomial L may be considered to have its origin in the familiar identity,

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \tag{6}$$

The identity (6) is due to Lucas [7]. Twenty five years later, J. Wasteels [12] proved a form of converse to (6). He showed that

$$L(x,y) = +1 \iff x = F_{2n} \text{ and } y = F_{2n+1}, \tag{7}$$

$$L(x,y) = -1 \iff x = F_{2n-1} \text{ and } y = F_{2n}. \tag{8}$$

Thus the polynomial L(x,y) gives a Diophantine definition of the Fibonacci numbers. It is a Diophantine definition in the sense of definition (1). That is, L defines the set of Fibonacci numbers by

$$y \in F \iff (\exists x)[L(x,y) = \pm 1]. \tag{9}$$

The equation $L(x, y) = \pm 1$ represents two hyperbolas in the x,y plane. On these two hyperbolas one finds pairs of adjacent Fibonacci numbers $x = F_n$ and $y = F_{n+1}$ and only pairs of adjacent Fibonacci numbers among points with integer coordinates. (One can find a proof of this in [2].) Another pair of curves with the same property is $x^2 - 5y^2 = \pm 4$. They are also due to Wasteels. We will now use (5) to answer one of the questions raised earlier, about singlefold representation.

<u>Theorem 1:</u> The set of nonnegative Fibonacci numbers is singlefold Diophantine definable over the nonnegative integers in one unknown. There exists a polynomial P(x, y) with parameter y and one unknown x, such that for every natural y, we have $y \in F \iff (\exists! x)\ [P(x, y) = 0]$. P(x,y) is a polynomial of degree 4.

<u>Proof:</u> Here the use of $\exists! x$ is intended to assert both that $y \in F \iff (\exists x)[P(x, y) = 0]$ and also that when $y \in F$, there is a unique x satisfying $P(x, y) = 0$. The variables x and y range over nonnegative integers. The proof uses property (9). We have

$$y \in F \iff (\exists x)\ [\ L(x, y) = -1 \quad \text{or} \quad L(x + 1, y) = +1], \tag{10}$$

$$\iff (\exists x)\ [\ L(x, y) + 1)(\ L(x + 1, y)\ - 1) = 0]. \tag{11}$$

So one can take P(x,y) to be the polynomial given in (11). It will have degree 4. By (7) and (8) this is a singlefold Diophantine definition of F. Theorem 1 is proved.

In general, in condition (10) in the proof of Theorem 1, when $y \geq 13$ and y has odd index, the number x will not be Fibonacci. E.g. $P(7, 13) = 0$ and 7 is not Fibonacci. We don't know whether there is a singlefold self Diophantine definition of the set of Fibonacci numbers in one unknown. Nevertheless, we can still derive the theorem which would follow from that. We can prove

<u>Theorem 2:</u> There exists a polynomial $Q(x_1, x_2)$, in two variables and degree 6, such that for each nonnegative integer y,

$$y \in F \iff (\exists x_1, x_2)\ [y = Q(x_1, x_2)\ ]. \tag{12}$$

Furthermore, the numbers x_1, x_2 when they exist, are unique and are Fibonacci numbers.

Proof: The idea of the construction is that each pair of adjacent nonnegative Fibonacci numbers (x, y), with $L(x, y) = \pm 1$ will produce a unique Fibonacci number $x + 2y$, two units to the right. As the sequence of adjacent nonnegative Fibonacci numbers started at $F_{-1} = 1$ begins 1, 0, 1, 1, 2, 3, 5, ..., we see that the Fibonacci number $x + y$ is not unique, but that the next one, $x + 2y$, is unique. That is, there is only one pair of nonnegative Fibonacci numbers (x, y) producing $x + 2y$. Hence, we have the following lemma.

Lemma 1: For each nonnegative number z, in order that $z \in F$, it is necessary and sufficient that there exist unique nonnegative numbers x, y such that $z = x + 2y$, and either both $x = y = 0$ or else $L(x,y) = \pm 1$.

Observe that we cannot replace $z = x + 2y$ by $z = x + y$ in Lemma 1 because then for example both (0,1) and (1,0) would produce $z = 1$. The definition would no longer be singlefold. Note also that when $z \in F$, both x and y must be Fibonacci numbers (by [9]). So the Lemma provides us with a Diophantine definition of the set of F over F. It is also a singlefold definition. In two unknowns,

$$z \in F \iff (\exists x,y)[z = x + 2y \text{ and } ((x = 0 \text{ and } y = 0) \text{ or } L(x,y) = \pm 1)]. \tag{13}$$

For the proof of Theorem 2, we need only write condition (13) as an equation. Use $A = 0$ or $B = 0 \iff AB = 0$, and proceed approximately as in (2). First the conjunction $x = 0$ and $y = 0$ is rendered as $x^2 + y^2 = 0$, or more simply as $x + y = 0$, because $0 \leq x$ and $0 \leq y$. Hence we have from (13)

$$z \in F \iff (\exists x,y)[z = x + 2y \text{ and } (x + y)(L(x,y)^2 - 1) = 0\,]. \tag{14}$$

Using the fact that $0 < x$ or $0 < y$ implies $L(x, y) \neq 0$, (see [2] Lemma 4 p. 85), we have for all x and y that $0 \leq (x + y)(L(x + y)^2 - 1)$. From this and (14) it follows that for every nonnegative number z,

$$z \in F \iff (\exists! x,y)\ [z = (x + 2y)(1 - (x + y)(L(x, y)^2 - 1))\,]. \tag{15}$$

Theorem 2 is proved. (15) is the promised polynomial of degree 6. Below it is written out as a sum of monomials.

Corollary 1: The set of nonnegative Fibonacci numbers is equal to the set of the nonnegative values of the polynomial,

$$7y^4x^2 - 7y^2x^4 - 5y\,x^5 + y^3x^3 + y^5\,x - 2y^6 + 3yx + 2y^2 + 2y - x^6 + x^2 + x \tag{16}$$

as the variables x and y range over the nonnegative integers. The polynomial takes on each Fibonacci number value once, and then only at Fibonacci values of x an y.

In conclusion one might say that the polynomial L(x, y) is very useful. Besides its use in constructing (16), the polynomial L also makes it possible to give a Diophantine definition of the integer Fibonacci numbers among integers. Here, by *integer Fibonacci numbers* is meant both the positive and negative Fibonacci numbers, including $F_{-n} = -(-1)^n F_n$. The Diophantine definition of the integer Fibonacci numbers is

$$\text{Y is a Fibonacci integer} \iff (\exists X)[\, 0 \leq X \text{ and } L(X, Y) = \pm 1 \,]. \tag{17}$$

Here in (17) the variables X and Y range over integers. It would be nice if from this one could proceed to construct a polynomial whose integer values are exactly the integer Fibonacci numbers. Unfortunately this is not possible. In fact it can be proved that there exists no polynomial of this type (see [2] Theorem 3, p. 85). For this type of theorem generally one must work in natural numbers or positive integers.

Open Probelm. *Is the set of nonnegative Fibonacci numbers singlefold Diophantine over itself in one unknown?*

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ON PRIME DIVISORS OF THE TERMS OF SECOND ORDER LINEAR RECURRENCE SEQUENCES

Peter Kiss*

Let (R_n), n=0,1,2,..., be a non-degenerate second order linear recurrence sequence defined by $R_n=AR_{n-1}+BR_{n-2}$ (n>1), where A and B are fixed non-zero coprime integers and $R_0=0$, $R_1=1$.

It is known that if p is a prime and $p\nmid B$, then there are terms of the sequence (R_n) divisible by p. The least positive index of these terms is called the rank of apparition of p in the sequence and we denote it by r(p). Thus r(p)=n if $p|R_n$ but $p\nmid R_m$ for 0<m<n. We say p is a primitive prime divisor of R_n if r(p)=n. It is also known that there is no term of (R_n) divisible by the prime p if p|B and (A, B)=1, furthermore if $p\nmid B$, $D=A^2+4B$ and (D/p) denotes the Legendre symbol with (D/p)=0 in the case p|D, then

(i) $\qquad r(p)|(p-(D/p))$

and

(ii) $\qquad p|R_n$ if and only if $r(p)|n$

(see e.g. D. H. Lehmer [3]).

The prime and primitive prime divisors of the terms of (R_n) have been investigated by several authors. A. Schinzel [5] and C. L. Stewart [6] proved that there is an absolute constant $n_0>0$ such that the terms R_n have at least one primitive prime divisor for any $n>n_0$. We know several results on the greatest prime factors of the terms of linear recurrence sequences (see e.g. C. L. Stewart [7] and its references) and some papers dealt with the reciprocal sum of the divisors which also characterises the magnitude of them. In the special case (A;B)=(3;−2), that is if $(R_n)=(2^n-1)$ is the sequence of Mersenne numbers, P. Erdös [1] proved that the reciprocal sum of the prime divisors of 2^n-1 is less than log log log n + c_1, where c_1 is a constant. In the case of Mersenne numbers C. Pomerance [4] also obrained results for the recirpocal sum of the primitive divisors (not necessarily prime ones) and gave an estimation for the average order of them.

Let

$$p(n) = \sum_{r(p)=n} \frac{1}{p}$$

be the reciprocal sum of the primitive prime divisors of R_n (n>0). We write p(n)=0 if there is no prime for which r(p)=n. Further let

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$$f(n) = \sum_{p|R_n} \frac{1}{p}$$

be the reciprocal sum of all prime divisors of R_n ($n>0$). We write $f(n)=0$ if R_n has no prime divisor.

For the function $p(n)$ we prove:

<u>Theorem 1:</u> There is a positive absolute constant c, which does not depend on the sequence (R_n), such that

$$p(n) < c \cdot \frac{(\log\log n)^2}{n}$$

for any sufficiently large n. Furthermore, for the average order of $p(n)$ we have

$$\sum_{n\leq x} p(n) = \log\log x + 0(1)$$

if x is a sufficiently large real number.

This theorem shows that $f(n)$ can be arbitrarily "small", e.g. if n is a large prime, since by (ii) all prime divisors of R_n are primitive ones if n is a prime and so $f(n)=p(n)$. On the other hand $f(n)$ can be also "large". To see it let $n=m!$, where m is a sufficiently large integer. Then, by (i), $p|R_n$ for any prime p for which $p \nmid B$ and $p<m$. So we have

$$f(n) \geq \sum_{\substack{p<m \\ p\nmid B}} \frac{1}{p} > \log\log m + c_2 > \log\log\log n + c_3$$

with some absolute constants c_2 and c_3. However in general $f(n)$ is not "small" and not "large", from a joint paper with B.M. Phong [2] it follows that the average order of $f(n)$ is less than a constant even if we consider the interval $(x, x + \log\log x)$ with sufficiently large x, that is

$$\sum_{\substack{n>x \\ n\leq x+\log\log x}} f(n) = 0(\log\log x) .$$

Now we give an asymptotic formula for the average order of the function $f(n)$.

<u>Theorem 2:</u> There is a constant c_0, depending only on the sequence (R_n), such that

$$\sum_{n\leq x} f(n) = c_0 x + 0(\log\log x)$$

for any sufficiently large x.

We note that in a joint work with P. Erdös we have obtained a similar result, without error term but in a shorter interval, for the sequence of Mersenne numbers.

<u>Proof of Theorem 1:</u> Let $n > \max(n_0, D)$, where n_0 and D are defined above. In this case, as we have seen, R_n has at least one primitive prime divisor and $(D/p)\neq 0$ for these primes. If p is a primitive prime divisor of R_n then, by (i), p is of the form $p=kn+1$ or $p=kn-1$ and so we can write

$$p(n) = \sum_{r(p)=n} \frac{1}{p} = \sum\nolimits_1 + \sum\nolimits_{-1} + \sum\nolimits_0 ,$$

where

$$\sum_1 = \sum_{\substack{r(p)=n \\ p\equiv 1 (mod\ n) \\ p<n^3}} \frac{1}{p}, \quad \sum_{-1} = \sum_{\substack{r(p)=n \\ p\equiv -1 (mod\ n) \\ p<n^3}} \frac{1}{p} \text{ and } \sum_0 = \sum_{\substack{r(p)=n \\ p\geq n^3}} \frac{1}{p}.$$

R_n has less than n distinct primitive prime divisors if n is sufficiently large (since $|R_n|<e^{\partial n}$, where $\partial>0$ is a constant depending on the sequence) and so

$$\sum_0 < \frac{1}{n^3}\cdot n = \frac{1}{n^2}. \tag{1}$$

Now we give an upper bound for $\sum_1$. Let $\omega(n)$ be the number of distinct primitive prime divisors of R_n for which $p\equiv 1 \pmod n$ and $p<n^3$, that is

$$\omega(n) = \sum_{\substack{r(p)=n \\ p\equiv 1 (mod\ n) \\ p<n^3}} 1.$$

If $\omega(n)=0$, then $\sum_1=0$. If $\omega(n)\neq 0$, then let $y>0$ be a real number for which

$$\sum_{\substack{p\leq y \\ p\equiv 1 (mod\ n)}} 1 = \omega(n). \tag{2}$$

Then naturally we can suppose that $y\leq n^3$ and we have

$$\sum_1 \leq \sum_{\substack{p\leq y \\ p\equiv 1 (mod\ n)}} \frac{1}{p}. \tag{3}$$

It is clear that we can choose y such that $y>2n$ since $\omega(n)\geq 1$. It is known that there are positive absolute constants c_4 and c_5 such that

$$\sum_{\substack{p\leq y \\ p\equiv 1 (mod\ n)}} 1 < c_4 \cdot \frac{y}{\varphi(n)\cdot \log (y/n)}, \tag{4}$$

where φ is the Euler's function (Brun-Titchmarsh theorem), and

$$\varphi(n) > c_5 \cdot \frac{n}{\log\log n}. \tag{5}$$

From (2), (4) and (5)

$$\frac{\omega(n)}{y} < c_4 \cdot \frac{1}{\varphi(n)\cdot \log(y/n)} < c_6 \cdot \frac{\log\log n}{n} \tag{6}$$

follows with some positive c_6 since $y>2n$.

Let a(m) be an arithmetical function defined by $a(m)=1$ if m is a prime with condition $m\equiv 1 \pmod n$ and $a(m)=0$ otherwise. Then, using partial summation, by (3), (4), (5) and (6) we obtain the estimation

$$\sum_1 \leq \sum_{\substack{m\leq y\\ m>n}} a(m)\cdot\frac{1}{m} = \sum_{\substack{m\leq y\\ m>2n}} a(m)\cdot\frac{1}{m} + 0\left(\frac{1}{n}\right) \leq$$

$$\leq \omega(n)\cdot\frac{1}{y} + \int_{2n}^{y} \frac{c_4 t}{\varphi(n)\cdot\log(t/n)}\cdot\frac{1}{t^2}\,dt + 0\left(\frac{1}{n}\right) <$$

$$< c_6\cdot\frac{\log\log n}{n} + \frac{c_4}{c_5}\cdot\frac{\log\log n}{n}\cdot\int_{2n}^{y} \frac{dt}{t\cdot\log(t/n)} + 0\left(\frac{1}{n}\right).$$

But $y\leq n^3$, by the definition of y, which implies that

$$\int_{2n}^{y} \frac{dt}{t\cdot\log(t/n)} = 0(\log\log n)$$

and

$$\sum_1 < c_7\cdot\frac{(\log\log n)^2}{n}, \tag{7}$$

where c_7 is an absolute constant.

We can prove similarly that

$$\sum_{-1} = 0\left(\frac{(\log\log n)^2}{n}\right) \tag{8}$$

thus by (1), (7) and (8) the first part of the theorem is proved.

The proof of the second part of the theorem is more simple. If x is sufficiently large then, using that $r(p) \leq p+1$ by (i),

$$\sum_{n\leq x} p(n) = \sum_{r(p)\leq x} \frac{1}{n} > \sum_{p\leq x} \frac{1}{p} + 0(1) = \log\log x + 0(1)$$

follows. On the other hand R_n has at most n distinct prime divisors for any sufficiently large n, and so

$$\sum_{n\leq x} p(n) \leq \sum_{p<x^3} \frac{1}{p} = \log\log x + 0(1),$$

which completes the proof of the theorem.

<u>Proof of Theorem 2:</u> By (ii) and Theorem 1 we obtain the estimation

$$\begin{aligned}\sum_{n\leq x} f(n) &= \sum_{n\leq x}\left(\left[\frac{x}{n}\right]\cdot\sum_{r(p)=n}\frac{1}{p}\right) = \\ &= \sum_{n\leq x}\frac{x}{n}\cdot p(n) + 0\left(\sum_{n\leq x} p(n)\right) = \\ &= x\cdot\sum_{n\leq x}\frac{1}{n}\cdot p(n) + 0(\log\log x).\end{aligned} \tag{9}$$

But by Theorem 1 there is a real valued function S(t) such that S(t)=0(1) and

$$\sum_{n\leq t} p(n) = \log\log t + S(t)$$

for any $t \geq 3$, so by partial summation

$$\sum_{n\leq x} \frac{1}{n}\cdot p(n) = \sum_{3<n\leq x} \frac{1}{n}\cdot p(n) + c_8 = \tag{10}$$

$$= \frac{\log\log x + S(x)}{x} + c_9 + \int_3^x \frac{\log\log t + S(t)}{t^2}\,dt$$

follows, where c_8 and c_9 are constants depending on the sequence (R_n).

From (9) and (10) the theorem follows with

$$c_0 = c_9 + \int_3^\infty \frac{\log\log t + S(t)}{t^2}\,dt\ .$$

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AN ALTERNATING PRODUCT REPRESENTATION FOR REAL NUMBERS

Arnold Knopfmacher and John Knopfmacher

INTRODUCTION

In 1770 Lambert introduced two positive series expansions for the real numbers in terms of rationals. These were subsequently rediscovered by Sylvester (1880) and Engel (1913) after whom they are respectively named. A further positive series expansion for the real numbers was discovered by Lüroth (1883). Also of particular interest to us is the product expansion of Cantor (1869). More recently, Oppenheim [3] defined a general algorithm for expressing real numbers in terms of a positive series of rational numbers. All of the previously mentioned expansions were shown to be special cases of the Oppenheim algorithm.

In [1] we introduced a new algorithm, analogous to that of Oppenheim, that gives rise to an expansion for real numbers in terms of a general alternating series of rationals. In particular, the algorithm was used to derive a new alternating series representation for real numbers corresponding to the expansion of Lüroth. In addition the metrical properties of this representation were investigated. Alternating series corresponding to the positive expansions of Sylvester and Engel had previously been considered, e.g. in [2], [5].

In this paper we use this algorithm to show that (corresponding to the well-known Cantor product): *every real number $A > 1$ can be uniquely expressed as an alternating product*

$$A = 2^k \prod_{i=1}^{\infty} \left(1 + \frac{(-1)^{i-1}}{a_i}\right)$$

where $a_{i+1} \geq (a_i + 1)(a_i + (-1)^{i-1})$, $k \geq 0$ in $\mathbb{Z}$. *Furthermore, A is rational if and only if A can be expressed as a finite product.*

As an interesting special case of this product, it will be shown below that

$$\sqrt{\frac{a+1}{a-1}} = \prod_{i=1}^{\infty} \left(1 + \frac{(-1)^{i-1}}{a_i}\right)$$

where $a_1 = a-1$, $a_{2i} = 2(a_{2i-1} + 1)^2 - 1$, $a_{2i+1} = 2a_{2i}^2 - 2$. In particular this yields (very rapidly converging) explicit products for concrete examples like $\sqrt{2}$, $\sqrt{6}$ and so on.

In addition, the algorithm below can be used to obtain rapidly converging representations for transcendental numbers. For example

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 209–216.

$$1 + \tfrac{1}{e} = (1 + \tfrac{1}{2})\,(1 - \tfrac{1}{11})\,(1 + \tfrac{1}{321})\,(1 - \tfrac{1}{272985}) \cdots .$$

If we take the first four terms only, the corresponding error does not exceed 1.6×10^{-11} . Similarly,

$$\pi - 2 = (1 + \tfrac{1}{7})\,(1 - \tfrac{1}{903})\,(1 + \tfrac{1}{1007341}) \cdots .$$

The paper is divided up as follows. In section 1 we show the existence of the product expansion. In section 2 we consider expansions for rational numbers and in section 3 we show the uniqueness and order properties of the representation.

1. ALTERNATING PRODUCT REPRESENTATIONS FOR REAL NUMBERS

We define the general alternating series algorithm as follows (see [1]): Given any real number A, let $a_0 = [A]$, $A_1 = A - a_0$. Then we recursively define

$$a_n = \left[\frac{1}{A_n} \right] \geq 1 \text{ for } n \geq 1,\ A_n > 0,$$

where $A_{n+1} = (\frac{1}{a_n} - A_n)\,(c_n/b_n)$, for $a_n > 0$.

Herein

$$b_i = b_i(a_1, a_2, \ldots, a_i),\ c_i = c_i(a_1, a_2, \ldots, a_i)$$

are positive numbers (usually integers), chosen so that $A_n \leq 1$ for $n \geq 1$. Note that $A_{n+1} \geq 0$,

since $a_n \leq \dfrac{1}{A_n}$ for $A_n > 0$. Using this algorithm we now show:

<u>Theorem 1:</u> Every real number A with $1 \leq A < 2$ has a unique product expansion of the form

$$A = (1 + \tfrac{1}{a_1})\,(1 - \tfrac{1}{a_2}) \cdots (1 + \frac{(-1)^{n-1}}{a_n}) \cdots ,$$

where

$$a_{i+1} \geq \begin{cases} (a_i + 1)^2, & i \text{ odd}, \\ a_i^2 - 1\,, & i \text{ even}. \end{cases}$$

(We denote the product by $A = [a_1, a_2, \ldots, a_n, \ldots]$.)

<u>Proof:</u> We will deduce this result from the general alternating algorithm by a suitable choice of the parameters b_n and c_n for $n \geq 1$.

Let $\quad B_k = \prod_{i=1}^{k} \frac{a_i + (-1)^{i-1}}{a_i}$, $k \geq 1$, $B_0 = 1$. Then

$$B_k - B_{k-1} = \frac{(-1)^{k-1}}{a_k} B_{k-1} \ ;$$

so $\quad B_n = 1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_k} B_{k-1}.$

We have therefore shown that the alternating product can be written as an equivalent alternating series which can be derived from the algorithm by setting $a_0 = 1$, $b_n = a_n + (-1)^{n-1}$, $c_n = a_n$, $n \geq 1$. In addition,

$$A - B_n = (-1)^n B_n A_{n+1}.$$

Now $a_n = \left[\frac{1}{A_n} \right]$ implies

$$\frac{1}{a_n + 1} < A_n \leq \frac{1}{a_n}, \quad \text{for } 0 < A_n \leq 1.$$

Thus $0 \leq A_{n+1} = (\frac{1}{a_n} - A_n)(c_n/b_n) = (\frac{1}{a_n} - A_n)\, a_n/(a_n + (-1)^{n-1})$

$$< (\frac{1}{a_n} - \frac{1}{a_n + 1})\, a_n/(a_n + (-1)^{n-1})$$

$$= 1/((a_n + 1)(a_n + (-1)^{n-1})), \quad \text{if } 0 < A_n \leq 1.$$

Therefore, if $A_{n+1} > 0$,

$$a_{n+1} = \left[\frac{1}{A_{n+1}}\right] \geq \begin{cases} (a_n + 1)^2, & n \text{ odd,} \\ a_n^2 - 1, & n \text{ even.} \end{cases}$$

So we have $a_1 \geq 1$, $a_2 \geq 4$, $a_3 \geq 15$, ..., or in general, $a_n \geq 2^{2^{n-1}}$ if n is even, and $a_n \geq 2^{2^{n-1}} - 1$, if n is odd. Also, we find $B_{n-1} \leq 2^{n/2}$ and $B_n \leq 2^{n/2}$ for even n. Thus,

$$A_{n+1} B_n \leq \frac{B_n}{(a_n + 1)(a_n + (-1)^{n-1})} \to 0, \ n \to \infty .$$

We note that we derived the product from the series algorithm

$$A_1 = A - 1.$$

$$a_n = \left[\frac{1}{A_n}\right] \geq 1 \text{ for } n \geq 1, \text{ if } A_n > 0,$$

where $$A_{n+1} = (\tfrac{1}{a_n} - A_n)/(1 + \frac{(-1)^{n-1}}{a_n}) .$$

We can however derive the alternating product using a product algorithm:

Corollary: Every real $A > 1$ has a unique product representation of the form

$$A = 2^k \prod_{i=1}^{\infty} (1 + \frac{(-1)^{i-1}}{a_i}),$$

where $a_{i+1} \geq (a_i + 1)\,(a_i + (-1)^{i-1})$, $k \geq 0$ in $\mathbb{Z}$, by means of the algorithm $A = 2^k A_1^*$ for $1 \leq A_1^* < 2$, and

$$a_n = \left\lceil \frac{(-1)^{n-1}}{A_n^* - 1} \right\rceil \quad \textit{for } n \geq 1,\ A_n^* \neq 1,$$

$$A_{n+1}^* = \left(1 + \frac{(-1)^{n-1}}{a_n}\right)^{-1} A_n^*, \quad \textit{for } n \geq 1.$$

Here $A_n^* = 1 + (-1)^{n-1} A_n \to 1$ as $n \to \infty$, and

$$A_1^* = A_{n+1}^* \prod_{i=1}^{n} (1 + \frac{(-1)^{i-1}}{a_i}) .$$

Proof: By the series expansion for an alternating product,

$$\begin{aligned} A_1^* - 1 &= \prod_{i=1}^{n}(1 + \frac{(-1)^{i-1}}{a_i}) - 1 + (-1)^n A_{n+1} (1 + \tfrac{1}{a_1}) (1 - \tfrac{1}{a_2})\ldots(1 + \frac{(-1)^{n-1}}{a_n}) \\ &= (1 + (-1)^n A_{n+1}) \prod_{i=1}^{n}(1 + \frac{(-1)^{i-1}}{a_i}) - 1 . \end{aligned}$$

Then $$a_n = \left\lceil \frac{1}{A_n} \right\rceil = \left\lceil \frac{(-1)^{n-1}}{A_n^* - 1} \right\rceil$$ for $A_n^* \neq 1$, and the rest follows.

Remark: A particular case of the alternating product is given by the following identity:

$$\sqrt{\frac{a + 1}{a - 1}} = \prod_{n=1}^{\infty} (1 + \frac{(-1)^{n-1}}{a_n}),$$

where $a_1 = a-1$, $a_{2n} = 2(a_{2n-1} + 1)^2 - 1$, and

$$a_{2n+1} = 2a_{2n}^2 - 2,\ n \geq 1 .$$

Proof: We use the known formula of Engel (see e.g. Perron [4]):

$$\sqrt{\frac{a+1}{a-1}} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{q_n}\right),$$

where $q_1 = a$, $a_{n+1} = 2q_n^2 - 1$, $n \geq 1$, and the fact that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{2x^2-1}\right) = \left(1 + \frac{1}{x-1}\right)\left(1 - \frac{1}{2x^2-1}\right).$$

Thus for each $n = 1, 2, 3\ldots$ we can replace the pairs of terms in the Engel product of type

$$\left(1 + \frac{1}{q_{2n-1}}\right)\left(1 + \frac{1}{q_{2n}}\right), \text{ where } q_{2n} = 2q_{2n-1}^2 - 1,$$

by the alternating terms

$$\left(1 + \frac{1}{q_{2n-1} - 1}\right)\left(1 - \frac{1}{q_{2n}}\right) = \left(1 + \frac{1}{a_{2n-1}}\right)\left(1 - \frac{1}{a_{2n}}\right)$$

where $a_{2n} = q_{2n} = 2q_{2n-1}^2 - 1 = 2(a_{2n-1} + 1)^2 - 1$.

Also $1 + a_{2n+1} = q_{2n+1} = 2q_{2n}^2 - 1 = 2a_{2n}^2 - 1$.

Hence the result is proved.

As concrete illustrations of the above identity, we mention:

$$\sqrt{2} = \left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{17}\right)\left(1 + \frac{1}{576}\right)\left(1 - \frac{1}{665857}\right)\cdots,$$

and

$$\sqrt{6} = 2\left(1 + \frac{1}{4}\right)\left(1 - \frac{1}{49}\right)\left(1 + \frac{1}{4800}\right)\left(1 - \frac{1}{46099201}\right)\cdots.$$

These examples illustrate the very rapid convergence behaviour of the above products - the first four brackets alone yield $\sqrt{2}$ to within an error of at most 1.6×10^{-12} and $\sqrt{6}$ to within an error of at most 5.8×10^{-16}.

Uniqueness of the digits a_n in the alternating product expansion is dealt with in section 3.

2. EXPANSIONS FOR RATIONAL NUMBERS

As in the case of the ordinary (positive) Cantor product, rational numbers have special types of alternating expansions.

Proposition 2: The alternating product terminates if and only if A is rational.

Proof: Clearly any number represented by a finite product is rational. Conversely, since A_i , $i \geq 1$, is rational, let $A_i = p_i/q_i$, $(p_i, q_i) = 1$.

Now since $a_i = \left[\frac{1}{A_i}\right] \geq \frac{1}{A_i} - 1$ it follows that $q_i - a_i p_i < p_i$.

Using the series recurrence relation we now obtain

$$\frac{p_{i+1}}{q_{i+1}} = \left(\frac{1}{a_i} - \frac{p_i}{q_i}\right) \frac{a_i}{(a_i+(-1)^{i-1})} = \frac{q_i - a_i p_i}{q_i(a_i+(-1)^{i-1})} .$$

Thus $0 \leq p_{i+1} \leq q_i - a_i p_i < p_i$. Since $\{p_i\}$ is a strictly decreasing sequence of non-negative integers, we must eventually reach a stage at which $p_{n+1} = 0$, whence

$$A = \prod_{i=1}^{n} \left(1 + \frac{(-1)^{i-1}}{a_i}\right).$$

There is the possibility of an ambiguity in the alternating product representation for rational numbers, analogous to that for continued fractions, which we eliminate as follows:

Convention 3: If n is even, we replace the finite sequence $[a_1, a_2, \ldots, a_n]$ by the sequence $[a_1, a_2, \ldots, a_{n-2}, a_{n-1} + 1]$ in the case $a_n = (a_{n-1} + 1)^2$. If n is odd, we replace the finite sequence $\{a_1, a_2, \ldots, a_n]$ by $[a_1, a_2, \ldots, a_{n-2}, a_{n-1} - 1]$ in the case $a_n = a_{n-1}^2 - 1$.

3. UNIQUENESS AND ORDER PROPERTIES

In order to be able to compare finite expansions of different lengths in size we introduce the symbol ω with the property $n < \omega$, for any $n \in \mathbb{N}$. We can now represent finite sequences by infinite sequences as follows:

For every $A = [a_0, a_1, \ldots, a_n]$ let $a_j = \omega$ for $j > n$ and hence $A = [a_0, a_1, \ldots, a_n, \omega, \omega, \ldots]$.

Proposition 4: (Uniqueness of Product). Let $A = \prod_{i=1}^{\infty} \left(1 + \frac{(-1)^{i-1}}{a_i}\right) \neq B = \prod_{i=1}^{\infty} \left(1 + \frac{(-1)^{i-1}}{b_i}\right)$.

The condition $A < B$ is equivalent to

(i) $a_{2n} < b_{2n}$, or
(ii) $a_{2n+1} > b_{2n+1}$, where $i = 2n$ or $i = 2n+1$ is the first index $i \geq 0$ such that $a_i \neq b_i$.

Proof: We shall use the notation

$$A'_n = \frac{1}{a_n} - \frac{(a_n + (-1)^{n-1})}{a_n} \cdot \frac{1}{a_{n+1}} + \frac{(a_n+(-1)^{n-1})(a_{n+1}+(-1)^{n})}{a_n \qquad a_{n+1}} \cdot \frac{1}{a_{n+2}} - \cdots .$$

Now the growth condition $a_{n+1} \geq (a_n+1)(a_n+(-1)^{n-1})$, $n \geq 1$, implies that

$$A'_{2n} = \frac{1}{a_{2n}} - \frac{(a_{2n}-1)}{a_{2n}} \cdot \frac{1}{a_{2n+1}} + \frac{(a_{2n}-1)(a_{2n+1}+1)}{a_{2n} \qquad a_{2n+1}} \cdot \frac{1}{a_{2n+2}} - \cdots$$

$$\geq \frac{1}{a_{2n}}\left(1 - \frac{1}{a_{2n}+1}\right) + \frac{(a_{2n}-1)(a_{2n+1}+1)}{a_{2n}\ a_{2n+1}\ a_{2n+2}}\left(1 - \frac{1}{a_{2n+2}+1}\right) + \cdots$$

$$> \frac{1}{a_{2n}+1},$$

Convention 3 eliminates the case $A = [a_1, a_2, \ldots, a_{2n}, a_{2n+1}]$ with $a_{2n+1} = (a_{2n}+1)(a_{2n}-1)$, for which equality would hold in the above. Similarly,

$$A'_{2n} \leq \frac{1}{a_{2n}} - \frac{(a_{2n}-1)}{a_{2n}\ a_{2n+1}}\left(1 - \frac{1}{a_{2n+1}+1}\right) - \cdots$$

$$\leq \frac{1}{a_{2n}}.$$

Thus $$A'_{2n} > \frac{1}{a_{2n}+1} \geq \frac{1}{b_{2n}} \geq B'_{2n},$$

and the result $A < B$ now follows from

$$A = 1 + \frac{1}{a_1} - \frac{(a_1+1)}{a_1}\frac{1}{a_2} + \cdots - \frac{(a_1+1)\ldots(a_{2n-1}+(-1)^{2n-2})}{a_1 \cdots a_{2n-1}} A'_{2n},$$

$$B = 1 + \frac{1}{a_1} - \frac{(a_1+1)}{a_1}\frac{1}{a_2} + \cdots - \frac{(a_1+1)\ldots(a_{2n-1}+(-1)^{2n-2})}{a_1 \cdots a_{2n-1}} B'_{2n}.$$

Note that if $b_{2n} = \omega$ then $B'_{2n} = 0$ and the result remains valid in this case. In a similar fashion we show that $A < B$ if (ii) holds.

To conclude, we remark that as in the case of the alternating Lüroth expansion (see [1]), a study of the metrical properties of the alternating product expansion may prove to be interesting.

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MAXIMUM LENGTH OF THE EUCLIDEAN ALGORITHM AND CONTINUED FRACTIONS IN $\mathbb{F}(X)$

Arnold Knopfmacher and John Knopfmacher

1. INTRODUCTION

In this paper, a bound is derived for the maximum length of the Euclidean algorithm for pairs of polynomials h, k in $\mathbb{F}[x]$, where $\mathbb{F}$ is an arbitrary field, and various explicit examples of maximum length are considered. In particular we show that pairs of consecutive Fibonacci polynomials and, more generally, many types of orthogonal polynomials lead to examples of maximum length.

Let $\mathbb{F}[x]$ denote a polynomial ring in an indeterminate X over a field $\mathbb{F}$ and let $\partial = \deg$ denote the degree function on $\mathbb{F}[x] - \{0\}$.

It is known that, with uniqueness up to non-zero (scalar) factors in $\mathbb{F}$, the g.c.d. (h, k) of two non-zero polynomials $h, k \in \mathbb{F}[x]$ is obtainable by the *Euclidean algorithm:*

$$\left\{\begin{array}{ll} h = a_0k + k_1 & (0 \leq \partial k_1 < \partial k)\ , \\ k = a_1k_1 + k_2 & (0 \leq \partial k_2 < \partial k_1,\ \partial a_1 = \partial k - \partial k_1 \geq 1), \\ \cdots & \\ k_{i-1} = a_ik_i + k_{i+1} & (0 \leq \partial k_{i+1} < \partial k_i,\ \partial a_i = \partial k_{i-1} - \partial k_i \geq 1), \\ \cdots & \\ k_{N-1} = a_Nk_N + 0 & (0 \leq \partial k_N < \partial k_{N-1},\ \partial a_N = \partial k_{N-1} - \partial k_N \geq 1). \end{array}\right. \tag{1}$$

Then

$$(h, k) = (k_1, k) = \cdots = (k_{N-1}, k_N) = k_N\ ,$$

and we denote the *length N* of the algorithm by *L(h, k)*. The equivalent representation of *h/k* as a finite *continued fraction* in the field of fractions $\mathbb{F}(x)$ yields an interesting alternative interpretation of $N = L(h,k)$:

$$\frac{h}{k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_N}}} \qquad (\partial a_i \geq 1 \text{ for } i \geq 1).$$

For these continued fractions in $\mathbb{F}(x)$ there exist analogues of standard results about ordinary finite continued fractions with integer digits. Where such results follow by essentially the same argument as in a standard reference like Chapter X of Hardy & Wright [3], we shall merely state them below, without proof; otherwise, slight modifications are mentioned:

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 217–222.

Firstly, if

$$(a_0; a_1 \ldots, a_N) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \frac{1}{a_N}}}$$

we have

$$(a_0; a_1, \ldots, a_N) = \frac{p_N}{q_N}\text{, where inductively} \tag{2}$$

$$\begin{array}{llll} p_0 = a_0, & p_1 = a_1a_0 + 1, & p_n = a_np_{n-1} + p_{n-2} & (2 \le n \le N), \\ q_0 = 1, & q_1 = a_1, & q_n = a_nq_{n-1} + q_{n-2} & (2 \le n \le N). \end{array}$$

Also

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}, \qquad p_nq_{n-2} - p_{n-2}q_n = (-1)^n a_n. \tag{3}$$

In particular, when all the "digits" a_i lie in some *unique factorization domain D*, (3) implies that p_n, q_n are *coprime* elements of D.

Further, if D is a polynomial ring $\mathbb{F}[x]$ in an indeterminate x over a field $\mathbb{F}$, the equation (2) for q_n, together with the condition $\partial a_i \ge 1$ for $i \ge 1$, implies inductively that

$$\partial q_n = \partial q_{n-1} + \partial a_n = \partial a_1 + \cdots + \partial a_n \ge n \ (n \ge 1). \tag{4}$$

In addition, some theorems of Magnus [6] on "P-fractions" of formal Laurent series imply the *uniqueness* condition for finite continued fractions in $\mathbb{F}(x)$:

$$M = N \text{ and } b_i = a_i \quad (i \ge 0) \text{ whenever} \tag{5}$$

$$(b_0; b_1, \ldots, b_M) = (a_0; a_1, \ldots, a_N) \ \ (b_i, a_i \in \mathbb{F}[x], \partial b_i \ge 1, \partial a_i \ge 1 \text{ for } i \ge 1).$$

(Actually Magnus [6] considers Laurent series in $z = x^{-1}$, with coefficients implicitly in $\mathbb{C}$, but his relevant arguments do not seem to need the restriction $\mathbb{F} = \mathbb{C}$).

2. CASES OF MAXIMUM LENGTH

Our main aim here is to investigate cases of maximum length for the Euclidean algorithm, or equivalently the maximum number of terms in the continued fraction expansion of h/k for polynomials $h, k \in \mathbb{F}[x]$ with $0 \le \partial h < \partial k$.

In the case of the ordinary Euclidean algorithm for natural numbers the corresponding result is due to G. Lamé (1844) (see e.g. Thoro [8]):

LAMÉ'S THEOREM - The number of divisions required to find the g.c.d. of two numbers is never greater than five times the number of digits in the smaller number.

In the case of $\mathbb{F}(x)$, the equations (1) show that

$$\partial k = \partial a_1 + \cdots + \partial a_N + \partial k_N , \text{ and} \tag{6}$$

$$\partial k \geq N, \quad \partial h \geq N - 1. \tag{7}$$

Thus as a simple analogue to Lamé's Theorem for polynomials $h, k \in \mathbb{F}[x]$ with $0 \leq \partial h < \partial k$ we have

$$L(h, k) \leq \partial k.$$

We are interested below in cases of maximum length, for which $L(h, k) = \partial k$ is true. It is well known that for natural numbers the smallest pair for which the Euclidean algorithm requires n steps is (F_{n+2}, F_{n+1}) where $\{F_n\}_{n=0}^{\infty}$ is the sequence of Fibonacci numbers. Analogously, the Fibonacci polynomials (see e.g. Bicknell [1]) defined by

$$F_1(x) = 1, \quad F_2(x) = x \text{ and } F_{n+1}(x) = x\, F_n(x) + F_{n-1}(x)$$

give rise to the simplest example of maximum length

$$\frac{F_n(x)}{F_{n+1}(x)} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{}{\ddots + \cfrac{1}{x}}}} \qquad \text{(n terms)}, \tag{8}$$

However, unlike the case of natural numbers there exists many other interesting examples of maximum length. Suppose now that $\mathbb{F}$ is a field of characteristic $\neq 2$. The Lucas polynomials $\{L_n(x)\}$ (see [1]) are defined by

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x).$$

Thus we have

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x)$$

$$\vdots$$

$$L_2(x) = xL_1(x) + L_0(x)$$

$$L_1(x) = (x/2)L_0(x) .$$

By a comparison with the general equations (1) we see that

$$\frac{L_{n-1}(x)}{L_n(x)} = \frac{1}{x} + \frac{1}{x} + \cdots + \frac{1}{x} + \frac{1}{x/2} \quad \text{(n terms)}, \tag{9}$$

Similarly in the case of the Pell polynomials, $P_n(x)$, studied by Horadam and Mahon [4], using the fact that $P_n(x) = F_n(2x)$,

$$\frac{P_n(x)}{P_{n+1}(x)} = \frac{1}{2x} + \frac{1}{2x} + \cdots + \frac{1}{2x} \quad \text{(n terms)}, \tag{10}$$

and in the case of the Pell-Lucas polynomials $Q_n(x)$ for which $Q_n(x) = L_n(2x)$,

$$\frac{Q_{n-1}(x)}{Q_n(x)} = \frac{1}{2x} + \frac{1}{2x} + \cdots + \frac{1}{2x} + \frac{1}{x} \quad \text{(n terms)}, \tag{11}$$

In addition the above equations show that pairs of consecutive polynomials $(F_n(x), F_{n-1}(x))$ etc. are coprime.

3. APPLICATIONS TO ORTHOGONAL POLYNOMIALS

A further wide class of interesting examples of maximum length in $\mathbb{R}(x)$ is obtained by considering the class of real orthogonal polynomials:

Let [a, b] be a finite or infinite interval and let w(x) be a positive weight function defined there. We assume that the integrals

$$\int_a^b w(x)x^n dx, \quad n = 0, 1, 2, \ldots$$

all exist. Let $\{p_n(x)\}_{n=0}^{\infty}$ denote the set of orthonormal polynomials with respect to w(x), that is

$$\int_a^b p_m(x)p_n(x)w(x)dx = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases}$$

Then it is known (see e.g. Davis [2, Chapter X]) that the orthonormal polynomials satisfy a three term recurrence relationship $p_{-1} = 0$, $p_0 = 1$,

$$p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x), \quad n = 2, 3, \dots . \tag{12}$$

By a repeated application of this recurrence we are led to the following *n* term continued fraction expansion

$$\frac{p_{n-1}(x)}{p_n(x)} = \cfrac{1}{a_n x + b_n + \cfrac{1}{e_2(a_{n-1}x + b_{n-1}) + \ddots + \cfrac{1}{e_n(a_1 x + b_1)}}}, \tag{13}$$

where, for $2 \le i \le n$,

$$e_i = \begin{cases} -\left(\frac{c_{n-1}}{c_n}\right)\left(\frac{c_{n-3}}{c_{n-2}}\right)\cdots\left(\frac{c_{n-i+1}}{c_{n-i+4}}\right)\left(\frac{1}{c_{n-i+2}}\right), & i \text{ even}, \\ \left(\frac{c_n}{c_{n-1}}\right)\left(\frac{c_{n-2}}{c_{n-3}}\right)\cdots\left(\frac{c_{n-i+3}}{c_{n-i+2}}\right), & i \text{ odd} . \end{cases}$$

If we consider for example the Tchebychev polynomials of the first kind, $\{T_n(x)\}$ for which

$$T_0(x) = 1, \; T_1(x) = x, \; T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

then

$$\frac{T_{n-1}(x)}{T_n(x)} = \cfrac{1}{2x - \cfrac{1}{2x - \ddots - \cfrac{1}{2x - \cfrac{1}{x}}}} . \tag{14}$$

Further explicit expressions of maximum length can be derived in the case of the Legendre, Jacobi, Hermite and Tchebychev (2nd kind) polynomials.

4. CONCLUSION

In conclusion we note that any expression $(0; a_1, a_2, \dots, a_n)$ with $\partial a_i = 1$ for $i = 1, 2, \dots, n$ leads by (2) to a case of maximum length. Mesirov and Sweet [7] consider many examples for polynomials over the finite field $\mathbb{F}_2$.

A further question of interest is that of determining the *average length* of the Euclidean algorithm for polynomials $h, k \in \mathbb{F}[X]$ with $0 \le \partial h < \partial k$. For the case of a finite field $\mathbb{F}_q$ this problem is treated in [5].

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RECURRENCE RELATIONS IN SINUSOIDS AND THEIR APPLICATIONS TO SPECTRAL ANALYSIS AND TO THE RESOLUTION OF ALGEBRAIC EQUATIONS

Joseph Lahr

INTRODUCTION

We consider time-discrete sine signals, as they appear typically in digital electronics, when signals are sampled at discrete time instants which are equally spaced along the independent variable.

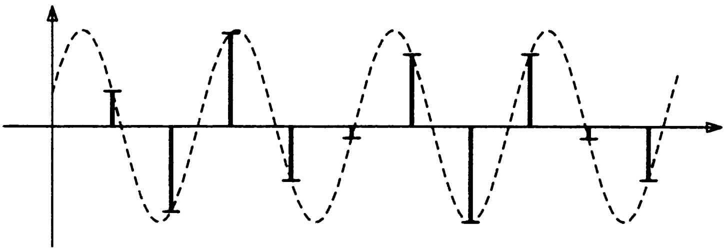

Figure 1. A Sampled Sine Signal.

In a first step we treat the case of one sine signal and the relations existing between the sampled values. Then we extend our examination to the general case of, in principle, an unlimited number of superposed sinusoids.

1. RECURRENCE RELATIONS IN A SAMPLED SINUSOID

1.1. Definitions and Notations

Suppose that we have a sine signal s(t) of frequency f,

$$s(t) = A \sin(2\pi ft + c). \tag{1}$$

A is the peak amplitude and c the phase angle at the time $t = 0$. By sampling this sine every T_S units we generate the following time-discrete signal:

$$S_n = A \sin(2\pi f T_S n + c) \quad \text{with } n = 0, 1, 2, 3, \ldots . \tag{2}$$

The sampling rate or sampling frequency f_S is determined by the sampling period T_S.

$$f_S = \frac{1}{T_S}. \tag{3}$$

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 223–238.

For practical purpose the notation of normalized frequency f_N is introduced. It is expressed in cycles per sample and is given by the formula*

$$f_N = fT_S = \frac{f}{f_S}. \tag{4}$$

Using f_N relation (2) may be written in the following manner:

$$S_n = A \sin (2\pi f_N n + c). \tag{5}$$

In this context it is useful to mention an important theorem about sampled signals:

$$\boxed{\text{A faithful representation of a signal requires a sampling rate greater than twice the highest frequency of the signal.}} \tag{6}$$

This theorem, also called the sampling theorem or theorem of C. Shannon, signifies in the case of one sine signal, that f_S must be greater than twice the frequency f. In other words, the sine wave must be sampled at least twice a period. If it is not, the digital sequence will be interpreted as coming from a lower frequency.

Considered in this way the normalized frequency f_N might only have values less than 0.5.

$$f_N < 0.5. \tag{7}$$

1.2. The main recurrence relation and the parameter of a sampled sinusoid.

Let us consider three values of a sinusoid which are equally spaced along the time axis and denoted by S_n, S_{n-1} and S_{n-2}. The angle between two consecutive values is constant and is denoted by a.

We have immediately:

$$a = 2\pi fT_S \tag{8}$$

and

$$S_n = A \sin(a\, n + c) \tag{9}$$

or

$$S_n = A \sin(2\pi fT_S n + c). \tag{10}$$

We shall show the more important recurrence relation:

$$\boxed{\begin{array}{c} S_n - P\, S_{n-1} + S_{n-2} = 0 \quad (11) \\ \text{with } P = 2 \cos a. \quad (12) \end{array}}$$

Equation (11) is a second-order linear recursion relation and represents the main relation of a sampled sinusoid. In the further development it plays a dominant part and is named "the fundamental recurrence relation of a sampled sinusoid".

The parameter P also is of central importance in the characterization of a sinusoid and is called in this paper "the parameter of a sampled sinusoid". Limiting ourselves to real values of the angle a and considering the theorem of Shannon, we have:

* Some authors denote by normalized frequency the product $2\pi fT_S$.

$$-2 < P < 2. \tag{13}$$

In addition the following relations are useful:

$$P = 2 \cos (2\pi f T_S) \tag{14}$$

and

$$a = \cos^{-1}(P/2). \tag{15}$$

The demonstration of the fundamental recurrence relation of a sinusoid may be given by means of Simson's formulae [1].

$$\sin(na) = 2 \cos(a) \sin [(n-1)a] - \sin [(n-2)a]. \tag{16}$$

$$\cos(na) = 2 \cos(a) \cos [(n-1)a] - \cos [(n-2)a]. \tag{17}$$

For the purpose of these considerations a constant phase angle c must be added to obtain the more general relations:

$$\sin(na+c) = 2 \cos(a) \sin [(n-1)a+c] - \sin [(n-2)a+c]. \tag{18}$$

$$\cos(na+c) = 2 \cos(a) \cos [(n-1)a+c] - \cos [(n-2)a+c]. \tag{19}$$

The demonstration of these relations may be made by trivial trigonometric identities.

With the relations (18), (12) and (9) the fundamental relation (11) may be verified. The relationships established in this chapter permit the following remarks:

a) Independently of having a sampled sine or cosine signal, the fundamental recurrence relation and the parameter are the same. (20)
b) The fundamental recurrence relation and the parameter are independent of the peak amplitude. (21)
c) The phase angle c which may admit any positive or negative value gives the fundamental recurrence relation the property of "shift invariance". This property signifies that the sampling positions may be shifted linearly without modifying the fundamental recurrence relation. (22)

1.3. The determination of the frequency of a sampled sinusoid.

The parameter P may also be determined by three consecutive samples.

$$P = \frac{S_n + S_{n-2}}{S_{n-1}}. \tag{23}$$

If the sampling period T_S is known, the frequency may be determined using relation (14).

$$f = \frac{\cos^{-1}(P/2)}{2\pi T_S}. \tag{24}$$

It is obvious that any formula for the determination of the frequency should not be dependent on the peak amplitude. This condition holds in our case.

1.4. The determination of the peak amplitude and the phase angle

Let us again consider the fundamental recurrence relation

$$S_n - P\,S_{n-1} + S_{n-2} = 0.$$

The initial terms of this difference equation may be S_1 and S_2. Since the value of P lies between -2 and 2, the general term of S_n may be given by:

$$S_n=2\sqrt{\frac{S_1^2+S_2^2-PS_1S_2}{4-P^2}}\ \cos\left[n\cos^{-1}\left(\frac{P}{2}\right)+\tan^{-1}\frac{2S_1+PS_2-P^2S_1}{(S_2-S_1P)\sqrt{4-P^2}}\right]. \tag{25}$$

See [3] page 72.

The general term yields directly the formula for the peak amplitude:

$$A = 2\sqrt{\frac{S_1^2+S_2^2-PS_1S_2}{4-P^2}}\ . \tag{26}$$

The point is now to find out if this formula possesses the property of shift invariance. By mathematical induction and using relation (23) it is possible to demonstrate the following relationship:

$$A = 2\sqrt{\frac{S_n^2+S_{n-1}^2-PS_nS_{n-1}}{4-P^2}}\ . \tag{27}$$

Since $S_n = A\sin(na + c)$ and the values of a and c are arbitrary, the property of shift invariance is true.

For the determination of the phase angle it is useful to transform the general term in an expression containing the sine function. This may be done with the reduction formula:

$$\cos x = \sin(x + \tfrac{\pi}{2}). \tag{28}$$

Hence we obtain a relation for the phase angle if we set $n = 0$.

$$c = \tan^{-1}\left(\frac{2S_1 + PS_2 - P^2S_1}{(S_2 - PS_1)\sqrt{4 - P^2}}\right) + \frac{\pi}{2}. \tag{29}$$

The identity

$$\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{-a}\right) = \frac{\pi}{2} \tag{30}$$

permits a further transformation.

$$c = \tan^{-1}\left(\frac{(PS_1 - S_2)\sqrt{4 - P^2}}{2S_1 + PS_2 - P^2S_1}\right). \tag{31}$$

If we allow the use of sample S_0 then:

$$c = \tan^{-1}\left(\frac{S_0\sqrt{4 - P^2}}{2S_1 - PS_0}\right). \tag{32}$$

Now using the relation (23) we may express the phase angle c by means of three consecutive samples.

It is easy to demonstrate that the shift invariance is not true in the case of the relation for the phase angle c. If we substitute in (32) S_0 by S_1 and S_1 by S_2, that signifies an incrementation of the subscripts by one, then we obtain a phase angle c_1, which value equals $c + a$.

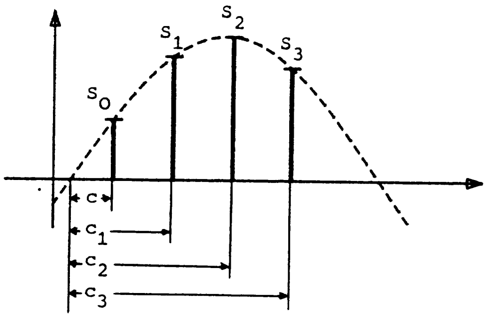

Figure 2. The Phase Angles c, c_1, c_2 and c_3

By mathematical induction we may verify that the following relation holds:

$$c_n = an + c = \tan^{-1}\left(\frac{S_n\sqrt{4 - P^2}}{2S_{n+1} - PS_n}\right). \tag{33}$$

Let us now summarize the preliminary results.

Given are the samples of a sinusoid, the frequency, the amplitude and the phase angle being unknown. The following expressions permit their determination:

$$P = \frac{S_n + S_{n-2}}{S_{n-1}} \quad \text{for any value of n.}$$

$$f = \frac{\cos^{-1}(P/2)}{2\pi T_S}.$$

$$A = 2\sqrt{\frac{S_n^2 + S_{n-1}^2 - PS_nS_{n-1}}{4 - P^2}}. \tag{34}$$

$$c_n = \tan^{-1}\left(\frac{S_n\sqrt{4 - P^2}}{2S_{n+1} - PS_n}\right).$$

The main characteristic of this development is that

$$\boxed{\text{A discrete sinusoid is completely determined by three samples.}} \tag{35}$$

2. RECURRENCE RELATIONS IN SUPERPOSED SINUSOIDS

2.1. The superposition of two sinusoids

Let us denote the samples of a first sinusoid by A_n and those of a second sinusoid by B_n. The corresponding parameters may be called P and Q. So we have:

$$A_n = P\,A_{n-1} - A_{n-2}. \tag{36}$$
$$B_n = Q\,B_{n-1} - B_{n-2}. \tag{37}$$

The samples of the superposed sinusoids are denoted by S_n.

$$S_n = A_n + B_n. \tag{38}$$

To determine the fundamental recurrence relation of two superposed sinusoids we use some theorems established by Maurice d'Ocagne [2]. If A_n and B_n are linear recurrence sequences of order N_A and N_B with characteristic polynomials $f_A(x)$ and $f_B(x)$ respectively, then $S_n = A_n + B_n$ is also a linear recurrence sequence of order $N_A + N_B$ with the characteristic polynomial $f_A(x)\cdot f_B(x)$. In our case the characteristic polynomials are $x^2 - Px + 1$ and $x^2 - Qx + 1$; so the characteristic polynomial of the superposed sinusoids is:

$$f_A(x)\cdot f_B(x) = x^4 - (P+Q)x^3 + (2+PQ)x^2 - (P+Q)x + 1. \tag{39}$$

The coefficients of the fundamental recurrence relation of two superposed sinusoids can now be obtained as coefficients of the powers of x.

$$\boxed{S_n - (P + Q)S_{n-1} + (2 + PQ)S_{n-2} - (P + Q)S_{n-3} + S_{n-4} = 0.} \tag{40}$$

2.2. The superposition of n sinusoids

The fundamental recurrence relation of n superposed sinusoids may be found via the product of the characteristic polynomials of the n sinusoids. In the case of three sinusoids we obtain, if we denote by R the parameter of the third sinusoid, the following fundamental recurrence relation:

$$\boxed{\begin{array}{c} S_n - (P+Q+R)S_{n-1} + (3+PQ+PR+QR)S_{n-2} - [2(P+Q+R)+PQR]S_{n-3} \\ + (3+PQ+PR+QR)S_{n-4} - (P+Q+R)S_{n-5} + S_{n-6} = 0. \end{array}} \tag{41}$$

In the case of four superposed sinusoids and introducing the parameter T related to the fourth sinusoid we have:

$$\boxed{\begin{aligned} S_n &- (P+Q+R+T)S_{n-1} + (4+PQ+PR+PT+QR+QT+RT)S_{n-2} \\ &- [3(P+Q+R+T)+PQR+PQT+PRT+QRT]S_{n-3} \\ &+ [6+2(PQ+PR+PT+QR+QT+RT)+PQRT]S_{n-4} \\ &- [3(P+Q+R+T)+PQR+PQT+PRT+QRT]S_{n-5} \\ &+ (4+PQ+PR+PT+QR+QT+RT)S_{n-6} - (P+Q+R+T)S_{n-7} + S_{n-8} = 0 \end{aligned}} \tag{42}$$

It is possible to determine the coefficients of the fundamental recurrence relation of n sinusoids in a recurrent manner in terms of the coefficients of the fundamental recurrence relation of $n-1$ sinusoids.

For this proposal let us denote by P_i the parameters of the sinusoid number i. A general coefficient is denoted by $T_{i,j}$, where the first subscript i designates the number of the superposed sinusoids and where the second subscript designates the position of the coefficient in the fundamental recurrence relation.

In Figure 3 the coefficients are written in tabular form, and the following recurrent relation may be established.

$$\boxed{\begin{aligned} T_{i,j} &= T_{i-1,j-2} - P_i T_{i-1,j-1} + T_{i-1,j}, \\ &\text{with } T_{i,j} = 0 \text{ if } j \le 0 \text{ or } j > 2i+1. \end{aligned}} \tag{43}$$

We see that the coefficients of the fundamental recurrence relations are disposed in a symmetrical way and that they appear as sums and products of the parameters P_i.

Further we may establish the following statement:

$$\boxed{\text{The fundamental recurrence relation of n superposed sinusoids is of the order 2n.}} \tag{44}$$

i \ j	1	2	3	4	5	6	7	8	9
1	1	−P	1						
2	1	−(P+Q)	2+PQ	−(P+Q)	1				
3	1	−(P+Q +R)	3+PQ+PR +QR	−(P+Q+R) −PQR	3+PQ+PR+QR	−(P+Q+R)	1		
4	1	−(P+Q +R+T)	4+PQ+PR +PT+QR +QT+RT	−3(P+Q+R+T) −PQR−PQT −PRT−QRT	6+2(PQ+PR+PT +QR+QT+RT) +PQRT	−3(P+Q+R+T) −PQR−PQT −PRT−QRT	4+PQ+PR +PT+QR +QT+RT	−(P+Q +R+T)	1

$T_{i,j} = T_{i-1,j-2} - P_i T_{i-1,j-1} + T_{i-1,j}$

with $T_{i,j} = 0$ if $j \le 0$ or $j > 2i+1$.

Figure 3. The Triangle of the Coefficients of the Fundamental Recurrence Relations of Superposed Sinusoids.

2.3 The modified fundamental recurrence relations

It may be useful to express the fundamental recurrence relations in terms of the different sums of the parameters.

We find the following modified relations:

In the case of one sinusoid:

$$S_n + S_{n-2} - PS_{n-1} = 0. \tag{45}$$

In the case of two sinusoids:

$$S_n+2S_{n-2}+S_{n-4} - (P+Q)(S_{n-1}+S_{n-3}) + PQS_{n-2} = 0. \tag{46}$$

In the case of three sinusoids:

$$\begin{aligned} &S_n+3S_{n-2}+3S_{n-4}+S_{n-6} - (P+Q+R)(S_{n-1}+2S_{n-3}+S_{n-5}) \\ &\quad + (PQ+PR+QR)(S_{n-2}+S_{n-4}) - PQRS_{n-3} = 0. \end{aligned} \tag{47}$$

And in the case of four sinusoids:

$$\begin{aligned} &S_n+4S_{n-2}+6S_{n-4}+4S_{n-6}+S_{n-8} \\ &\quad - (P+Q+R+T)(S_{n-1}+3S_{n-3}+3S_{n-5}+S_{n-7}) \\ &\quad + (PQ+PR+PT+QR+QT+RT)(S_{n-2}+2S_{n-4}+S_{n-6}) \\ &\quad - (PQR+PQT+PRT+QRT)(S_{n-3}+S_{n-5}) + PQRTS_{n-4} = 0. \end{aligned} \tag{48}$$

In this development the coefficients in the different parentheses containing the samples S_{n-i} are given by Pascal's triangle. Further we see that in these parentheses the subscripts are always decremented in steps of two.

Using the relation (43) or the generalization, which may be established on the basis of the modified fundamental recurrence relations using the elements of Pascal's triangle, it is not difficult to proceed to the determination of the fundamental recurrence relation of a superposition of n sinusoids.

3. A FIRST APPLICATION: SPECTRAL ANALYSIS

3.1 A special case: three superposed sinusoids

Suppose we have at least 9 samples of a signal composed by three superposed sinusoids A_n, B_n and C_n, and we want to compute the corresponding frequencies, peak amplitudes and phase angles.

Let us, for this purpose, introduce the following notations:

$$U_1 = P + Q + R, \tag{49}$$

$$U_2 = PQ + PR + QR, \tag{50}$$

$$U_3 = PQR, \tag{51}$$

$$K_{n-3} = \frac{S_{n-2} + S_{n-4}}{S_{n-3}}, \tag{52}$$

$$L_{n-3} = \frac{S_{n-1} + 2S_{n-3} + S_{n-5}}{S_{n-3}}, \tag{53}$$

$$J_{n-3} = \frac{S_n + 3S_{n-2} + 3S_{n-4} + S_{n-6}}{S_{n-3}}. \tag{54}$$

With the modified fundamental recurrence relation (47) we may establish a system of 3 linear equations to determine the unknown variables U_1, U_2 and U_3.

$$\begin{aligned} U_3 - K_{n-3}U_2 + L_{n-3}U_1 - J_{n-3} = 0.\\ U_3 - K_{n-4}U_2 + L_{n-4}U_1 - J_{n-4} = 0.\\ U_3 - K_{n-5}U_2 + L_{n-5}U_1 - J_{n-5} = 0.\end{aligned} \tag{55}$$

Using a theorem of Vieta we then solve the following cubic equation to determine the values of P, Q and R.

$$x^3 - U_1x^2 + U_2x - U_3 = 0. \tag{56}$$

The three roots of this equation are identical with the parameters P, Q and R, and permit the determination of the frequencies of the three sinusoids.

For the computation of the peak amplitudes and the phase angles we need the values of the six samples A_1, A_2, B_1, B_2, C_1 and C_2 which may be determined by the system:

A_1	A_2	B_1	B_2	C_1	C_2	
1	0	1	0	1	0	$-S_1$
0	1	0	1	0	1	$-S_2$
-1	P	-1	Q	-1	R	$-S_3$
$-P$	P^2-1	$-Q$	Q^2-1	$-R$	R^2-1	$-S_4$
$-P^2+1$	P^3-2P	$-Q^2+1$	Q^3-2Q	$-R^2+1$	R^3-2R	$-S_5$
$-P^3+2P$	P^4-3P^2+1	$-Q^3+2Q$	Q^4-3Q^2+1	$-R^3+2R$	R^4-3R^2+1	$-S_6$

(57)

It may be mentioned that the Morgan-Voyce polynomials of first kind appear in the matrix [3], [4].

The algebraic resolution gives for instance the following value for the variable C_2:

$$
\begin{aligned}
C_2 \; = \; & \frac{RS_1 - [R(P+Q)+1]\,S_2 + [R(2+PQ)+P+Q]\,S_3}{(R-P)(R-Q)} \\
+ \; & \frac{-\,[R(P+Q)+2+PQ]\,S_4 + (P+Q+R)\,S_5 - S_6}{(R-P)(R-Q)}.
\end{aligned}
\tag{58}
$$

With the fundamental recurrence relation for three superposed sinusoids (41) we obtain:

$$
\begin{aligned}
&-S_6 + (P+Q+R)S_5 - (2+PQ+PR+QR)S_4 + (2R+P+Q+PQR)S_3 \\
&- (1+PR+QR)S_2 + RS_1 = S_0 - (P+Q)S_1 + (2+PQ)S_2 - (P+Q)S_3 + S_4,
\end{aligned}
$$

and finally:

$$C_2 = \frac{S_0 - (P+Q)S_1 + (2+PQ)S_2 - (P+Q)S_3 + S_4}{(R-P)(R-Q)}. \tag{59}$$

For C_1 we find:

$$C_1 = \frac{S_{-1} - (P+Q)S_0 + (2+PQ)S_1 - (P+Q)S_2 + S_3}{(R-P)(R-Q)}. \tag{60}$$

By these two last results we recognize the general expression for C_n, which may be demonstrated by mathematical induction.

$$C_n = \frac{S_{n-2} - (P+Q)S_{n-1} + (2+PQ)S_n - (P+Q)S_{n+1} + S_{n+2}}{(R-P)(R-Q)}. \tag{61}$$

Similarly we have:

$$B_n = \frac{S_{n-2} - (P+R)S_{n-1} + (2+PR)S_n - (P+R)S_{n+1} + S_{n+2}}{(Q-P)(Q-R)}, \tag{62}$$

$$A_n = \frac{S_{n-2} - (Q+R)S_{n-1} + (2+QR)S_n - (Q+R)S_{n+1} + S_{n+2}}{(P-Q)(P-R)}. \tag{63}$$

Two remarks may be made:

The numerator is identical with the fundamental recurrence relation of two superposed sinusoids.

(64)

In the formula for a specific sinusoid defined by a parameter P_S the numerator contains only the two other parameters denoted for this purpose by P_{o1} and P_{o2}. The denominator has the form: $(P_S - P_{o1})(P_S - P_{o2})$.

(65)

3.2. The general case of n superposed sinusoids

Suppose that the samples of n superposed sinusoids are given, and that the frequency, the peak amplitude and the phase angle of each particular sinusoid is wanted:

The parameters are denoted by $P_1, P_2, P_3, P_4, \ldots, P_n$.

The different sums of products of the parameters are designated by the following auxiliary variables:

$$\begin{aligned} U_1 &= P_1 + P_2 + P_3 + \ldots + P_n, \\ U_2 &= P_1P_2 + P_1P_3 + P_1P_4 + \ldots + P_{n-1}P_n, \\ U_3 &= P_1P_2P_3 + P_1P_2P_4 + P_1P_2P_5 + \ldots + P_{n-2}P_{n-1}P_n, \\ &\vdots \\ U_n &= P_1P_2P_3 \ldots P_n. \end{aligned} \tag{66}$$

The coefficients of the system of linear equations that must be solved are established by using auxiliary polynomials in which appear the elements of Pascal's triangle.

$$\begin{aligned} L_{1,i} &= S_i. \\ L_{2,i} &= S_{i+1} + S_{i-1}. \\ L_{3,i} &= S_{i+2} + 2S_i + S_{i-2}. \\ L_{4,i} &= S_{i+3} + 3S_{i+1} + 3S_{i-1} + S_{i-3}. \\ L_{5,i} &= S_{i+4} + 4S_{i+2} + 6S_i + 4S_{i-2} + S_{i-4}. \\ &\vdots \\ L_{n+1,i} &= S_{i+n} + nS_{i+n-1} + \cdots + S_{i-n}. \end{aligned} \tag{67}$$

The scheme of the system to determine the unknowns $U_1 \ldots U_n$ has then the following form:

U_n	U_{n-1}	U_{n-2}	$\cdots$	
$L_{1,2n}$	$-L_{2,2n}$	$L_{3,2n}$	$\cdots$	$(-1)^n L_{n+1,2n}$
$L_{1,2n-1}$	$-L_{2,2n-1}$	$L_{3,2n-1}$	$\cdots$	$(-1)^n L_{n+1,2n-1}$
$L_{1,2n-2}$	$-L_{2,2n-2}$	$L_{3,2n-2}$	$\cdots$	$(-1)^n L_{n+1,2n-2}$
$\vdots$				
$L_{1,n+1}$	$-L_{2,n+1}$	$L_{3,n+1}$	$\cdots$	$(-1)^n L_{n+1,n+1}$

(68)

The determination of the parameters is now possible; we only need to solve the following general algebraic equation:

$$x^n - U_1x^{n-1} + U_2x^{n-2} - U_3x^{n-3} + \cdots + (-1)^n U_n = 0. \tag{69}$$

The roots of this equation are the parameters $P_1 \ldots P_n$. And the frequencies may now be computed using (24):

$$f_i = \frac{\cos^{-1}(P_i/2)}{2\pi T_S} \qquad i = 1, 2, 3, \ldots, n. \tag{70}$$

The determination of at least two samples of each particular sinusoid may be executed according to the following flowchart.

Step 1 Begin with the sinusoid number n; the corresponding parameter is P_n.

Step 2 Establish the coefficients $T_{n-1,j}$ of the fundamental recurrence relation of $n-1$ superposed sinusoids with the parameters $P_1, P_2, \ldots, P_{n-1}$.

$T_{i,j} = T_{i-1,j-2} - P_i T_{i-1,j-1} + T_{i-1,j}$.

Initial conditions:

$T_{i,j} = 0$ for $j \leq 0$,

$T_{0,j} = 0$ for $j \geq 2$,

$T_{0,1} = 1$. (See Figure 3)

Step 3 The samples N_j of the nth sinusoid are given by:

$$N_j = \frac{\sum_{k=1}^{2n-1} T_{n-1,k}\, S_{j+k-n}}{\prod_{k=1}^{n-1} (P_n - P_k)}. \tag{71}$$

Compute the two samples N_n and N_{n+1} (to avoid the use of samples like $S_0, S_{-1}, \ldots$) If a third sample is needed, then compute: $N_{n+2} = P_n N_{n+1} - N_n$.

Step 4 Exchange the values of the parameters according to the following prescription:

$H = P_n$,

$P_n = P_{n-1}$,

$P_{n-1} = P_{n-2}$,

. . . .

$P_2 = P_1$,

$P_1 = H$.

Step 5 Go back to step 2.

The steps 2 and 3 are executed n times, and every time the samples of a sinusoid are determined.

After these computations the peak amplitudes and the phase angles may be obtained using the relations (27) and (31).

3.3 Some considerations and remarks

The main characteristics of this new approach for spectral analysis may be enumerated as follows:

a) $\boxed{\text{n discrete sinusoids are completely determined by 3n samples. By completely determined we mean the determination of frequencies, peak amplitudes and phase angles.}}$ (72)

b) $\boxed{\text{The main mathematical operations consist of the solution of a system of n linear equations and the determination of the n roots of an algebraic equation.}}$ (73)

From the practical point of view, this method contains two difficulties:

a) If two frequencies are very close together, then the corresponding parameters are almost identical and we are confronted with the problem of numerical instability resulting from the loss of significant digits in formula (71).
b) The resolution of an algebraic equation of degree n could be difficult.

The theory of this new method is not closed by far. Two main extensions are possible.

a) The spectral analysis of complex sinusoids.
b) The adaptation of the theory in the case of noisy signals, perhaps in combination with the method of least squares. That is, of course, a separate research project.

4. A SECOND APPLICATION: THE USE OF THE FUNDAMENTAL RECURRENCE RELATIONS OF SUPERPOSED SINUSOIDS FOR THE RESOLUTION OF ALGEBRAIC EQUATIONS

Let us consider a general algebraic equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

Let x_1, x_2, ..., x_n be the corresponding roots. Then, applying the well-known theorem of Vieta, we have:

$$\boxed{\begin{array}{lcl} x_1 + x_2 + x_3 + \dots\dots\dots\dots\dots + x_n & = & -a_1, \\ x_1x_2 + x_1x_3 + \dots\dots\dots\dots + x_{n-1}x_n & = & a_2, \\ x_1x_2x_3 + x_1x_2x_4 + \dots\dots + x_{n-2}x_{n-1}x_n & = & -a_3, \\ \dots & & \\ x_1x_2x_3 \dots x_n & = & (-1)^n a_n. \end{array}} \tag{74}$$

If the coefficients of an algebraic equation are known then, according to this theorem, the sum of the unknowns and the sum of the different products are determined. Considering the fundamental recurrence relations, for instance (46), (47) and (48), we find that it is possible to establish using the coefficients of an algebraic equation a difference equation describing superposed sinusoids, whose parameters are the unknowns of the algebraic equation.

With this difference equation we may in a first step compute some hundred samples and then use a classical method to ascertain the frequencies f_i of the spectrum of the samples, for instance the Fourier Transform or the Maximum Entropy Spectral Analysis. It must be observed that the fixing of the initial conditions of the difference equation is an open problem.

Finally the unknowns may be computed by the relation:

$$x_i = 2\cos(2\pi f_i T_S) \qquad i = 1, 2, 3, \ldots, n. \tag{75}$$

Example

Let us consider the following quintic equation:

$$x^5 + 1.225147x^4 - 3.379541x^3 - 3.411415x^2 + 1.927772x + 0.640502 = 0.$$

The coefficients may be composed by the following parameters:

$$\begin{aligned} P_1 &= P = 1.618034, \\ P_2 &= Q = 0.618034, \\ P_3 &= R = -0.250666, \\ P_4 &= S = -1.457937, \\ P_5 &= T = -1.752613. \end{aligned}$$

The sums may be computed by the following relations:

$$\begin{aligned} U_1 &= P+Q+R+S+T = -a_1, \\ U_2 &= PQ+PR+PS+PT+QR+QS+QT+RS+RT+ST = a_2, \\ U_3 &= PQR+PQS+PQT+PRS+PRT+PST+QRS+QRT+QST+RST = -a_3, \\ U_4 &= PQRS+PQRT+PQST+PRST+QRST = a_4, \\ U_5 &= PQRST = -a_5. \end{aligned}$$

The modified fundamental recurrence relation in question has the form:

$$\begin{aligned} S_n &+5S_{n-2} + 10S_{n-4} + 10S_{n-6} + 5S_{n-8} + S_{n-10} \\ &-U_1(S_{n-1} + 4S_{n-3} + 6S_{n-5} + 4S_{n-7} + S_{n-9}) \\ &+U_2(S_{n-2} + 3S_{n-4} + 3S_{n-6} + S_{n-8}) \\ &-U_3(S_{n-3} + 2S_{n-5} + S_{n-7}) + U_4(S_{n-4} + S_{n-6}) - U_5S_{n-5} = 0. \end{aligned}$$

For practical reasons we write:

$$\begin{aligned} S_n = {} & U_1S_{n-1} - (5 + U_2)S_{n-2} + (4U_1 + U_3)S_{n-3} \\ & - (10 + 3U_2 + U_4)S_{n-4} + (6U_1 + 2U_3 + U_5)S_{n-5} \\ & - (10 + 3U_2 + U_4)S_{n-6} + (4U_1 + U_3)S_{n-7} \\ & - (5 + U_2)S_{n-8} + U_1S_{n-9} - S_{n-10}. \end{aligned}$$

The initial conditions may have arbitrary values:

$$S_1 = S_2 = \ldots = S_{10} = 1.$$

Then we compute 300 samples for instance, and use the discrete Fourier Transform to determine the spectrum of this time discrete signal.

$$\boxed{X(f) = \sum_{n=1}^{300} S_n e^{-i2\pi fn}.} \tag{76}$$

Since the Fourier Transform is periodic and since our signal is real, only the domain $0 \leq f \leq 0.5$ is significant.

In figure 4 the 300 samples are displayed and the absolute value of the Fourier Transform is represented.

We find peaks for the following frequencies:

$$f_1 = 0.10,$$
$$f_2 = 0.20,$$
$$f_3 = 0.27,$$
$$f_4 = 0.38,$$
$$f_5 = 0.42.$$

This result gives with relation (75) the parameters P_i and so the roots of the algebraic equation. ($T_S = 1$.)

Remarks

a) The method can be used only if the roots are in the interval $(-2,2)$ since in other cases there are no corresponding superposed sinusoids.

b) The magnitude of the different peaks may vary on a large scale. These variations depend on the initial conditions of the recurrence relation. It is an open problem to determine these initial conditions in a way that the peaks have the same magnitude (without solving an algebraic equation of degree n).

We think that this method gives good initial approximations. In conjunction with other known algorithms we can obtain excellent values for the roots.

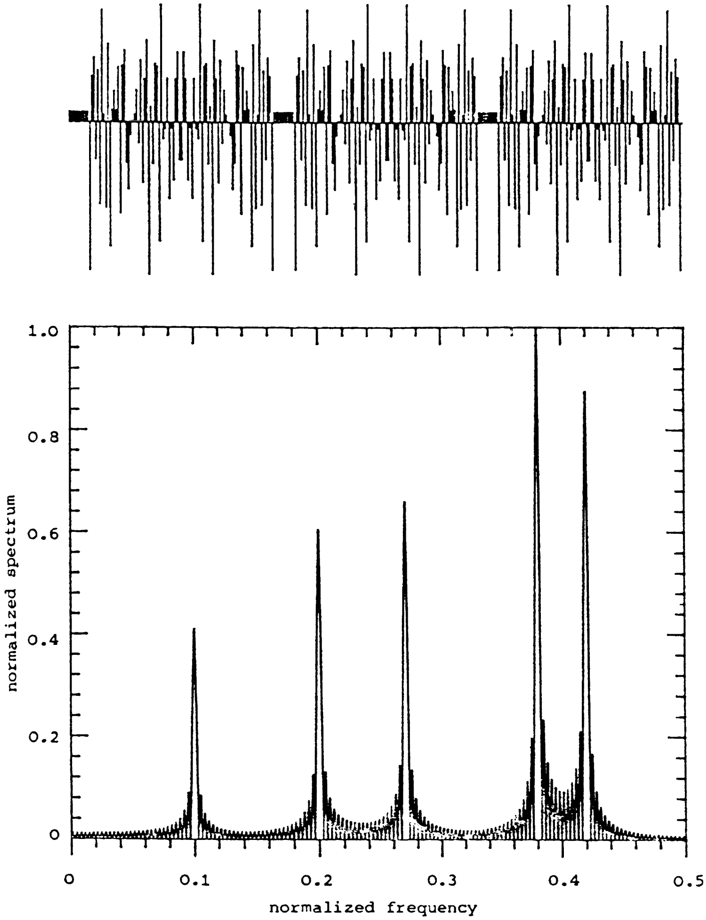

Figure 4. The 300 Samples (above) and the Fourier Transform With the Peaks: f = 0.1; 0.2; 0.27; 0.38; 0.42.

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A RECURRENCE RELATION FOR GAUSSIAN MULTINOMIAL COEFFICIENTS

S. L. Lee and G. M. Phillips

Recently we have obtained some new results concerning interpolation on the simplex (Lee and Phillips [2]). As a by-product of these researches we discovered the recurrence relation (9), which appears to be new and which we find interesting in its own right. We did not discuss this recurrence relation explicitly in [2], where it would have appeared as an isolated curiosity, nor will we retrace here the tortuous path through the work on interpolation which led us to it. In this note we present the recurrence relation in an appropriate combinatorial context.

Andrews [1] defines

$$G(n,m;q) = \frac{(1-q^{n+m})(1-q^{n+m-1})\ldots(1-q^{m+1})}{(1-q^{n})(1-q^{n-1})\ldots(1-q)} \tag{1}$$

and shows that

$$G(n,m;q) = \sum_{r=0}^{nm} p(n,m,r)q^{r}, \tag{2}$$

where p(n,m,r) denotes the number of partitions of r into at most m parts, each not greater than n. Let us define

$$[r] = (1 - q^{r})/(1 - q) \tag{3}$$

for r = 0, 1, ... We note that

$$\lim_{q \to 1} [r] = r$$

and we refer to [r] as a q-integer. We then define

$$[r]! = \begin{cases} 1, & r = 0 \\ [r]\cdot[r-1]\cdots[1], & r = 1, 2\ldots \end{cases}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]\,!}{[m]\,!\,[n-m]\,!} \tag{4}$$

for integers $n \geq m \geq 0$. On comparing (1) and (4) it is easily verified that

$$G(n,m;q) = \begin{bmatrix} n+m \\ m \end{bmatrix} \tag{5}$$

The expressions in (4) and (5) are called Gaussian polynomials or Gaussian binomial coefficients. This terminology is used by Andrews [1] and Pólya and Szegö [3] respectively, and in both [1] and [3] we find the recurrence relation

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$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{6}$$

for $n \geq m \geq 1$. We may verify (6) directly from (4). Alternatively, we may deduce (6) from (2), (5) and the recurrence relation

$$p(n,m,r) = p(n-1,m,r) + p(n,m-1,r-n) \tag{7}$$

for $r \geq n$ (see Andrews [1]).

We now generalize the Gaussian binomial coefficient (4). Let $\bar{\alpha}$ denote the k-tuple $(\alpha_1, \ldots, \alpha_k)$, where each α_j is a non-negative integer, and let us write $|\bar{\alpha}| = \alpha_1 + \ldots + \alpha_k$. Then, for $n \geq |\bar{\alpha}|$, we define

$$\begin{bmatrix} n \\ \bar{\alpha} \end{bmatrix} = \frac{[n]!}{[\alpha_1]! \ldots [\alpha_k]! \, [n-|\bar{\alpha}|]!} \tag{8}$$

which we will call a Gaussian multinomial coefficient, following Andrews [1]. (Our notation, which we prefer because it is consistent with that used in (4) for the Gaussian binomial coefficient, differs slightly from that used by Andrews.)

Let $\bar{e}_i$ denote the k-tuple $(\delta_{i1}, \delta_{i2}, \ldots, \delta_{ik})$ for $1 \leq i \leq k$, where δ_{ij} denotes the Kronecker delta. Then we can now state the recurrence relation which generalizes (6) and is the subject of this note.

Theorem: The Gaussian multinomial coefficients satisfy the recurrence relation

$$\begin{bmatrix} n \\ \bar{\alpha} \end{bmatrix} = \begin{bmatrix} n-1 \\ \bar{\alpha} \end{bmatrix} + q^{n-|\bar{\alpha}|} \sum_{i=1}^{k} \left(\prod_{v=1}^{i-1} q^{\alpha_v} \right) \begin{bmatrix} n-1 \\ \bar{\alpha}-\bar{e}_i \end{bmatrix} \tag{9}$$

for $n > |\bar{\alpha}|$ and each $\alpha_i > 0$.

Proof: First we remark that, for $i = 1$ in the summation on the right of (9), the empty product is assigned the value 1, following the usual convention. We now write

$$C = \frac{[n-1]!}{[\alpha_1]! \ldots [\alpha_k]! \, [n-|\bar{\alpha}|]!}$$

Then the right side of (9) may be expressed as

$$C\left([n - |\bar{\alpha}|] + q^{n-|\bar{\alpha}|} \sum_{i=1}^{k} \left(\prod_{v=1}^{i-1} q^{\alpha_v} \right) [\alpha_i] \right) = C\left(1 - q^{n-|\bar{\alpha}|} + q^{n-|\bar{\alpha}|} \sum_{i=1}^{k} (1 - q^{\alpha_i}) \prod_{v=1}^{i-1} q^{\alpha_v} \right)/(1-q)$$

$$= C[n] = \begin{bmatrix} n \\ \bar{\alpha} \end{bmatrix},$$

which completes the proof.

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SOME BINOMIAL FIBONACCI IDENTITIES

Calvin T. Long

1. INTRODUCTION

Interest in binomial Fibonacci identities goes back at least to E. Lucas [8] who obtained formulas like

$$\sum_{i=0}^{r} \binom{r}{i} F_{n+i} = F_{n+2r} \text{ and } \sum_{i=0}^{r} \binom{r}{i} L_{n+i} = L_{n+2r} \tag{1}$$

where F_n and L_n represent, respectively, the nth Fibonacci and Lucas numbers. Indeed, Lucas used the Binet formulas and the characteristic equation $x^2 = x + 1$ to argue that equations (1) can be written in the form

$$\begin{aligned} F^n(F + 1)^r &= F^n(F^2)^r = F^{n+2r} \\ L^n(L + 1)^r &= L^n(L^2)^r = L^{n+2r} \end{aligned} \tag{1$'$}$$

where, after simplification, the powers of F and L are replaced by the appropriately subscripted F's and L's. This same approach for other identities was investigated further by Hoggatt and Lind in [4] and Ruggles [9]. Using a combination of symbolic and matrix methods, Hoggatt and Bicknell [2], obtained the following quadratic binomial Fibonacci results

$$\sum_{i=0}^{2s+1} \binom{2s+1}{i} F_{n+i}^2 = 5^s F_{2(s+n)+1} \tag{2}$$

$$\sum_{i=0}^{2s+1} \binom{2s+1}{i} L_{n+i}^2 = 5^{s+1} F_{2(s+n)+1} \tag{3}$$

$$\sum_{i=0}^{2s+2} \binom{2s+2}{i} F_{n+i}^2 = 5^s L_{2(s+n+1)} \tag{4*}$$

$$\sum_{i=0}^{2s+2} \binom{2s+2}{i} L_{n+i}^2 = 5^{s+1} L_{2(s+n+1)} \tag{5*}$$

$$\sum_{i=0}^{2s+1} \binom{2s+1}{i} F_{n-1+i} F_{n+i} = 5^s F_{2(s+n)} \tag{6}$$

$$\sum_{i=0}^{2s+2} \binom{2s+2}{i} F_{n-1+i} F_{n+i} = 5^s L_{2(s+n)+1} \tag{7}$$

and, in [3], certain biquadratic binomial Fibonacci results. In the present paper we derive a number of linear and quadratic binomial identities, some of which generalize (2)-(7) above, but which cannot be obtained by the symbolic method of Lucas.

*There were typos in the original paper. The 1's in the subscripts on the right sides of these two equations were missing.

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2. PRELIMINARY CONSIDERATIONS

We consider the two general second order recurrence sequences $\{H_n\}$ and $\{K_n\}$ defined by

$$\begin{aligned} &H_0 = c,\ H_1 = d,\ H_{n+2} = aH_{n+1} + bH_n && \forall n, \\ &K_0 = 2d - ca,\ \ K_1 = da + 2cb,\ K_{n+2} = aK_{n+1} + bK_n && \forall n. \end{aligned} \tag{8}$$

Since the recurrences are the same except for initial conditions, we may set

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \text{ and } \beta = \frac{a - \sqrt{a^2 + 4b}}{2} \tag{9}$$

whence it follows that

$$\alpha + \beta = a,\ \alpha - \beta = \sqrt{a^2 + 4b},\ \text{and } \alpha\beta = -b. \tag{10}$$

For brevity we also set $\Delta = \sqrt{a^2 + 4b}$.

We also consider the sequences $\{U_n\}$ and $\{V_n\}$ of Lucas defined by

$$\begin{aligned} &U_0 = 0,\ U_1 = 1,\ U_{n+2} = aU_{n+1} + bU_n && \forall n, \\ &V_0 = 2,\ V_1 = a,\ V_{n+2} = aV_{n+1} + bV_n && \forall n. \end{aligned} \tag{11}$$

It is well known (see Lucas [8, p. 185]) that

$$U_n = \frac{\alpha^n - \beta^n}{\Delta} \text{ and } V_n = \alpha^n + \beta^n \qquad \forall n \tag{12}$$

and it is trivial to show that

$$b^n U_{-n} = (-1)^{n-1} U_n \text{ and } b^n V_{-n} = (-1)^{n-1} V_n. \tag{13}$$

It is also known (see Horadam [5, p. 161]) that

$$H_n = \frac{d - c\beta}{\Delta} \cdot \alpha^n - \frac{d - c\alpha}{\Delta} \cdot \beta^n \qquad \forall n, \tag{14}$$

and it is not difficult to show that

$$K_n = \frac{4cb + ca^2 + (2d - ca)\Delta}{2\Delta} \cdot \alpha^n - \frac{4cb + ca^2 - (2d - ca)\Delta}{2\Delta} \cdot \beta^n. \tag{15}$$

More importantly, however, from (10) and (14) we obtain

$$\begin{aligned} H_n &= \frac{d - c(-b/\alpha)}{\Delta} \cdot \alpha^n - \frac{d - c(-b/\beta)}{\Delta} \cdot \beta^n \\ &= d \cdot \frac{\alpha^n - \beta^n}{\Delta} + cb \cdot \frac{\alpha^{n-1} - \beta^{n-1}}{\Delta} \\ &= dU_n + cbU_{n-1} \qquad \forall n, \end{aligned} \tag{16}$$

and, from (10) and (15), we obtain

$$K_n = dV_n + cbV_{n-1} \qquad \forall n. \qquad (17)$$

Also, though we never use it, it is not difficult to show that

$$H_{n+2} + bH_n = K_{n+1} \qquad \forall n. \qquad (18)$$

We now note a sequence of identities entirely analogous to well known identities for the Fibonacci and Lucas numbers (see, for example [1], [6], and [7]).

$$U_{r+2s} - b^s U_r = \begin{cases} U_s V_{r+s} & \text{s even} \\ V_s U_{r+s} & \text{s odd} \end{cases} \qquad (19)$$

$$V_{r+2s} - b^s V_r = \begin{cases} \Delta^2 U_s U_{r+s} & \text{s even} \\ V_s V_{r+s} & \text{s odd} \end{cases} \qquad (20)$$

$$U_{r+2s} + b^s U_r = \begin{cases} V_s U_{r+s} & \text{s even} \\ U_s V_{r+s} & \text{s odd} \end{cases} \qquad (21)$$

$$V_{r+2s} + b^s V_r = \begin{cases} V_s V_{r+s} & \text{s even} \\ \Delta^2 U_s U_{r+s} & \text{s odd .} \end{cases} \qquad (22)$$

Note, in particular, that the right sides of the (19), (21) pair and the (20), (22) pair are the same except that the cases for s even and s odd are just reversed. Since the proofs of (19)-(22) are all the same, we only give the details for (19) with s even. Using (10), we have

$$\begin{aligned}
\frac{U_{r+2s}}{b^s} - U_r &= \frac{1}{\Delta}\left[\frac{\alpha^{r+2s} - \beta^{r+2s}}{b^s} - (\alpha^r - \beta^r)\right] \\
&= \frac{1}{\Delta}\left[\frac{\alpha^{r+2s} - \beta^{r+2s}}{b^s} - \left(\frac{\alpha\beta}{b}\right)^s(\alpha^r - \beta^r)\right] \\
&= \frac{1}{b^s\Delta}\left[(\alpha^{r+s} + \beta^{r+s})(\alpha^s - \beta^s)\right] \\
&= \frac{U_s V_{r+s}}{b^s}
\end{aligned}$$

and this implies the desired result.

3. SOME LINEAR BINOMIAL FIBONACCI IDENTITIES

Equations (19)-(22) could be used to obtain linear binomial identies for the sequences $\{U_n\}$ and $\{V_n\}$. However, we can obtain even more general results for the sequences $\{H_n\}$ and $\{K_n\}$ which contain as special cases results for the sequences $\{U_n\}$, $\{V_n\}$, $\{F_n\}$, $\{L_n\}$, the Pell sequence and others. We must first obtain more general analogs of (19)-(22). They are as follows:

$$H_{r+2s} - b^s H_r = \begin{cases} U_s K_{r+s} & \text{s even} \\ V_s H_{r+s} & \text{s odd} \end{cases} \tag{23}$$

$$K_{r+2s} - b^s K_r = \begin{cases} \Delta^2 U_s H_{r+s} & \text{s even} \\ V_s K_{r+s} & \text{s odd} \end{cases} \tag{24}$$

$$H_{r+2s} + b^s H_r = \begin{cases} V_s H_{r+s} & \text{s even} \\ U_s K_{r+s} & \text{s odd} \end{cases} \tag{25}$$

$$K_{r+2s} + b^s K_r = \begin{cases} V_s K_{r+s} & \text{s even} \\ \Delta^2 U_s H_{r+s} & \text{s odd .} \end{cases} \tag{26}$$

Of course, if $c = 0$ and $d = 1$, these reduce to equations (19)-(22) above, and if it is also true that $a = b = 1$, they reduce the well-known identities for the Fibonacci and Lucas sequences. Again, we note in particular that the right sides of the (23), (25) pair and the (24), (26) pair are the same except that the cases for s even and s odd are reversed. Also, as before, since all the proofs are similar, we derive only (23) for s even. Using (19) with s even and (16) and (17), we have

$$\begin{aligned} H_{r+2s} - b^s H_r &= dU_{r+2s} + bcU_{r+2s-1} - b^s(dU_r + bcU_{r-1}) \\ &= d(U_{r+2s} - b^s U_r) + bc(U_{r+2s-1} - b^s U_{r-1}) \\ &= dU_s V_{r+s} + bcU_s V_{r+s-1} \\ &= U_s(dV_{r+s} + bcV_{r+s-1}) \\ &= U_s K_{r+s} \end{aligned}$$

as claimed.

We are now in position to obtain eight binomial sums, one each for each of the eight possibilities in equations (23)-(26). They are

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} b^{(m-i)s} H_{r+2si} = \begin{cases} \Delta^m U_s^m H_{r+ms} & \text{s even, m even} \\ -\Delta^{m-1} U_s^m K_{r+ms} & \text{s even, m odd} \end{cases} \tag{27}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} b^{(m-i)s} H_{r+2si} = (-1)^m V_s^m H_{r+ms} \qquad \text{s odd} \tag{28}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} b^{(m-i)s} K_{r+2si} = \begin{cases} \Delta^m U_s^m K_{r+ms} & \text{s even, m even} \\ -\Delta^{m+1} U_s^m H_{r+ms} & \text{s even, m odd} \end{cases} \tag{29}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} b^{(m-i)s} K_{r+2si} = (-1)^m V_s^m K_{r+ms} \qquad \text{s odd} \tag{30}$$

$$\sum_{i=0}^{m} \binom{m}{i} b^{(m-i)s} H_{r+2si} = V_s^m H_{r+ms} \qquad \text{s even} \tag{31}$$

$$\sum_{i=0}^{m} \binom{m}{i} b^{(m-i)s} H_{r+2si} = \begin{cases} \Delta^m U_s^m H_{r+ms} & \text{s odd, m even} \\ \Delta^{m-1} U_s^m K_{r+ms} & \text{s odd, m odd} \end{cases} \tag{32}$$

$$\sum_{i=0}^{m}\binom{m}{i}b^{(m-i)s}K_{r+2si} = V_s^m K_{r+ms} \qquad \text{s even} \qquad (33)$$

$$\sum_{i=0}^{m}\binom{m}{i}b^{(m-i)s}K_{r+2si} = \begin{cases} \Delta^m U_s^m K_{r+ms} & \text{s odd, m even} \\ \Delta^{m+1} U_s^m H_{r+ms} & \text{s odd, m odd .} \end{cases} \qquad (34)$$

Again, since the proofs are all similar, we prove only (27) for s even. The argument is by induction on m. For m = 1, the assertion is that

$$b^s H_r - H_{r+2s} = -U_s K_{r+s}$$

and this is immediate from (23). Assume that

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i)s}H_{r+2si} = -\Delta^{k-1}U_s^k K_{r+ks} \qquad (35)$$

where k is odd. Multiplying (35) through by b^s we obtain

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i+1)s}H_{r+2si} = -b^s\Delta^{k-1}U_s^k K_{r+ks}, \qquad (36)$$

and replacing r by r + 2s in (35) we obtain

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i)s}H_{r+2si+2s} = -\Delta^{k-1}U_s^k K_{r+(k+2)s} \qquad (37)$$

or equivalently

$$\sum_{j=1}^{k+1}(-1)^{j-1}\binom{k}{j-1}b^{(k-j+1)s}H_{r+2sj} = -\Delta^{k-1}U_s^k K_{r+(k+2)s}. \qquad (37')$$

Now, by subtracting (37′) from (36), we obtain on the left

$$b^{(k+1)s}H_r + \sum_{i=1}^{k}(-1)^i\left[\binom{k}{i}+\binom{k}{i-1}\right]b^{(k-i+1)s}H_{r+2si} + (-1)^{k+1}H_{r+2sk+2s}$$
$$= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}b^{(k+1-i)s}H_{r+2si} \qquad (38)$$

and on the right, using (24),

$$\begin{aligned} &-b^s\Delta^{k-1}U_s^k U_{r+ks} + \Delta^{k-1}U_s^k K_{r+(k+2)s} \\ &= \Delta^{k-1}U_s^k(K_{r+(k+2)s} - b^s K_{r+ks}) \qquad (39) \\ &= \Delta^{k+1}U_s^{k+1}H_{r+(k+1)s}. \end{aligned}$$

Equating the right sides of (38) and (39), we see that the desired result is true for m = k + 1. Because of the two possibilities for (27) for s even, we must also show that the result is true for m = k + 2. The work on the left-hand side of the equation proceeds as before and, on the right-hand side, we again first multiply the right-hand side of (39) by b^s, then replace r by r + 2s and then subtract to obtain

$$b^s\Delta^{k+1}U_s^{k+1}H_{r+(k+1)s} - \Delta^{k+1}U_s^{k+1}H_{r+(k+3)s}$$
$$= -\Delta^{k+1}U_s^{k+1}(H_{r+(k+3)s} - b_sH_{r+(k+1)s})$$
$$= -\Delta^{k+1}U_s^{k+2}K_{r+(k+2)s}$$

by (23) and the desired result follows.

Of course, appropriate choices for a, b, c, and d give linear binomial sums for the Fibonacci and Lucas sequences, the Pell sequence, and others as desired. We list here only the results for the Fibonacci and Lucas sequences obtained by setting c = 0 and a = b = d = 1.

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}F_{r+2si} = \begin{cases} 5^{m/2}F_s^mF_{r+ms} & \text{s even, m even} \\ -5^{(m-1)/2}F_s^mL_{r+ms} & \text{s even, m odd} \end{cases} \tag{40}$$

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}F_{r+2si} = (-1)^mL_s^mF_{r+ms} \qquad \text{s odd} \tag{41}$$

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}L_{r+2si} = \begin{cases} 5^{m/2}F_s^mL_{r+ms} & \text{s even, m even} \\ -5^{(m+1)/2}F_s^mF_{r+ms} & \text{s even, m odd} \end{cases} \tag{42}$$

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}L_{r+2si} = (-1)^mL_s^mL_{r+ms} \qquad \text{s odd} \tag{43}$$

$$\sum_{i=0}^{m}\binom{m}{i}F_{r+2si} = L_s^mF_{r+ms} \qquad \text{s even} \tag{44}$$

$$\sum_{i=0}^{m}\binom{m}{i}F_{r+2si} = \begin{cases} 5^{m/2}F_s^mF_{r+ms} & \text{s odd, m even} \\ 5^{(m-1)/2}F_s^mL_{r+ms} & \text{s odd, m odd} \end{cases} \tag{45}$$

$$\sum_{i=0}^{m}\binom{m}{i}L_{r+2si} = L_s^mL_{r+ms} \qquad \text{s even} \tag{46}$$

$$\sum_{i=0}^{m}\binom{m}{i}L_{r+2si} = \begin{cases} 5^{m/2}F_s^mL_{r+ms} & \text{s odd, m even} \\ 5^{(m+1)/2}F_s^mF_{r+ms} & \text{s odd, m odd}\,. \end{cases} \tag{47}$$

It is to be noticed that the subscripts in all of these linear binomial identities are in arithmetic progression with even difference. Of course, this stems directly from identities (23)-(26) where the subscripts of the terms of the left-hand side of the equalities have an even difference. At present there seem to be no analogous general identities where the differences of the subscripts are odd. Of course, results are easily found for any given odd difference. For example, it is easily seen that

$$F_n + F_{n+3} = 2F_{n+2} \text{ and } L_n + L_{n+3} = 2L_{n+2} \tag{48}$$

and these lead to the binomial identities

$$\sum_{i=0}^{m}\binom{m}{i}F_{n+3i} = 2^mF_{n+2m} \text{ and } \sum_{i=0}^{m}\binom{m}{i}L_{n+3i} = 2^mL_{n+2m}, \tag{49}$$

but general results, valid for all odd differences seem not to be known.

4. SOME QUADRATIC BINOMIAL FIBONACCI IDENTITIES

We now turn our attention to quadratic identities analogous to the linear identities of the preceding section. As before we first develop a list of identities each with two terms on the left-hand side and with the subscripts in the terms differing by a constant amount. The identities are these:

$$b^sH_nH_m - H_{n+s}H_{m+s}$$
$$= \begin{cases} b^sU_{-s}[dH_{m+n+s} + bcH_{m+n+s-1}] & \text{s even} \\ \dfrac{b^s}{\Delta^2}[V_{-s}(dK_{m+n+s} + bcK_{m+n+s-1}) & \text{s odd} \\ \qquad -2(-b)^n(dK_{m-n} - cK_{m-n+1})] & \end{cases} \tag{50}$$

$$b^sH_nH_m + H_{n+s}H_{m+s}$$
$$= \begin{cases} b^sU_{-s}[dH_{m+n+s} + bcH_{m+n+s-1}] & \text{s odd} \\ \dfrac{b^s}{\Delta^2}[V_{-s}(dK_{m+n+s} + bcK_{m+n+s-1}) & \text{s even} \\ \qquad -2(-b)^n(dK_{m-n} - cK_{m-n+1})] & \end{cases} \tag{50'}$$

where the cases for s even and s odd are just reversed for the difference and the sum of the two terms,

$$b^sK_nK_m - K_{n+s}K_{m+s}$$
$$= \begin{cases} \Delta^2b^sU_{-s}[dH_{m+n+s} + bcH_{m+n+s-1}] & \text{s even} \\ b^s[V_{-s}(dK_{m+n+s} + bcK_{m+n+s-1}) & \text{s odd} \\ \qquad +2(-b)^n(dK_{m-n} - cK_{m-n+1})] & \end{cases} \tag{51}$$

$$b^sK_nK_m + K_{n+s}K_{m+s}$$
$$= \begin{cases} \Delta^2b^sU_{-s}[dH_{m+n+s} + bcH_{m+n+s-1}] & \text{s odd} \\ b^s[V_{-s}(dK_{m+n+s} + bcK_{m+n+s-1}) & \text{s even} \\ \qquad +2(-b)^n(dK_{m-n} - cK_{m-n+1})] & \end{cases} \tag{51'}$$

$$b^sK_nH_m - K_{n+s}H_{m+s}$$
$$= \begin{cases} b^sU_{-s}[dK_{m+n+s} + bcK_{m+n+s-1}] & \text{s even} \\ b^s[V_{-s}(dH_{m+n+s} + bcH_{m+n+s-1}) & \text{s odd} \\ \qquad +2(-b)^n(dH_{m-n} - cH_{m-n+1})] & \end{cases} \tag{52}$$

$$b^sK_nH_m + K_{n+s}H_{m+s}$$
$$= \begin{cases} b^sU_{-s}[dK_{m+n+s} + bcK_{m+n+s-1}] & \text{s odd} \\ b^s[V_{-s}(dH_{m+n+s} + bcH_{m+n+s-1}) & \text{s even} \\ \qquad +2(-b)^n(dH_{m-n} - cH_{m-n+1})] & \end{cases} \tag{52'}$$

$$H_nH_m - H_{n-s}H_{m+s} = -(-b)^nU_{-s}[dH_{m-n+s} - cH_{m-n+s+1}] \tag{53}$$

$$H_nH_m + H_{n-s}H_{m+s} = \frac{-(-b)^n}{\Delta^2}V_{-s}[dK_{m-n+s} - cK_{m-n+s+1}] + \frac{2}{\Delta^2}[dK_{m+n} + bcK_{m+n-1}] \tag{54}$$

$$K_nK_m - K_{n-s}K_{m+s} = \Delta^2(-b)^nU_{-s}[dH_{m-n+s} - cH_{m-n+s+1}] \tag{55}$$

$$K_nK_m + K_{n-s}K_{m+s} = (-b)^nV_{-s}[dK_{m-n+s} - cK_{m-n+s+1}] + 2[dK_{m+n} + bcK_{m+n-1}] \tag{56}$$

$$K_nH_m - K_{n-s}H_{m+s} = (-b)^nU_{-s}[dK_{m-n+s} - cK_{m-n+s+1}] \tag{57}$$

$$K_nH_m + K_{n-s}H_{m+s} = (-b)^nV_{-s}[dH_{m-n+s} - cH_{m-n+s+1}] + 2[dH_{m+n} + bcH_{m+n-1}] \tag{58}$$

$$K_nH_m - K_{n+s}H_{m-s} = (-b)^nU_s[dK_{m-n-s} - cH_{m-n-s+1}] \tag{59}$$

$$K_nH_m + K_{n+s}H_{m-s} = (-b)^nV_s[dH_{m-n-s} - cH_{m-n-s+1}] + 2[dH_{m+n} + bcH_{m+n-1}]\,. \tag{60}$$

Note that the U_s and V_s in (59) and (60) are not typos; unlike in the other identities they should not be U_{-s} and V_{-s}.

These identities are interesting in their own right and, of course, for special choices of a, b, c, and d, they specialize to identities for the Fibonacci and Lucas sequences, the Pell sequence, the sequences $\{U_n\}$ and $\{V_n\}$, and others. Indeed, the proofs of (50)-(60) depend on the corresponding results for $\{U_n\}$ and $\{V_n\}$ which we must treat first. They are as follows:

$$b^sU_nU_m - U_{n+s}U_{m+s} = \begin{cases} b^sU_{-s}U_{m+n+s} & \text{s even} \\ \dfrac{b^s}{\Delta^2}[V_{-s}V_{m+n+s} - 2(-b)^nV_{m-n}] & \text{s odd} \end{cases} \tag{61}$$

$$b^sU_nU_m + U_{n+s}U_{m+s} = \begin{cases} b^sU_{-s}U_{m+n+s} & \text{s odd} \\ \dfrac{b^s}{\Delta^2}[V_{-s}V_{m+n+s} - 2(-b)^nV_{m-n}] & \text{s even} \end{cases} \tag{61'}$$

$$b^sV_nV_m - V_{n+s}V_{m+s} = \begin{cases} \Delta^2b^sU_{-s}U_{m+n+s} & \text{s even} \\ b^s[V_{-s}V_{m+n+s} + 2(-b)^nV_{m-n}] & \text{s odd} \end{cases} \tag{62}$$

$$b^sV_nV_m + V_{n+s}V_{m+s} = \begin{cases} \Delta^2b^sU_{-s}U_{m+n+s} & \text{s odd} \\ b^s[V_{-s}V_{m+n+s} + 2(-b)^nV_{m-n}] & \text{s even} \end{cases} \tag{62'}$$

$$b^sV_nU_m - V_{n+s}U_{m+s} = \begin{cases} b^sU_{-s}V_{m+n+s} & \text{s even} \\ b^s[V_{-s}U_{m+n+s} + 2(-b)^nU_{m-n}] & \text{s odd} \end{cases} \tag{63}$$

$$b^sV_nU_m + V_{n+s}U_{m+s} = \begin{cases} b^sU_{-s}V_{m+n+s} & \text{s odd} \\ b^s[V_{-s}U_{m+n+s} + 2(-b)^nU_{m-n}] & \text{s even} \end{cases} \tag{63'}$$

$$U_nU_m - U_{n-s}U_{m+s} = -(-b)^nU_{-s}U_{m-n+s} \tag{64}$$

$$U_nU_m + U_{n-s}U_{m+s} = \frac{1}{\Delta^2}[2V_{m+n} - (-b)^nV_{-s}V_{m-n+s}] \tag{65}$$

$$V_nV_m - V_{n-s}V_{m+s} = (-b)^n\Delta^2U_{-s}U_{m-n+s} \tag{66}$$

$$V_nV_m + V_{n-s}V_{m+s} = 2V_{m+n} + (-b)^nV_{-s}V_{m-n+s} \tag{67}$$

$$V_nU_m - V_{n-s}U_{m+s} = (-b)^nU_{-s}V_{m-n+s} \tag{68}$$

$$V_nU_m + V_{n-s}U_{m+s} = 2U_{m+n} + (-b)^nV_{-s}U_{m-n+s} \tag{69}$$

$$V_nU_m - V_{n+s}U_{m-s} = (-b)^nU_sV_{m-n-s} \tag{70}$$

$$V_nU_m + V_{n+s}U_{m-s} = 2U_{m+n} + (-b)^nV_sU_{m-n+s}\,. \tag{71}$$

Since the proofs of (61)-(71) are similar, we only give the details for (61) with s even. Using (12) and (10) we have that

$$\frac{U_nU_m}{b^n} - \frac{U_{n+s}U_{m+s}}{b^{n+s}}$$

$$= \frac{1}{\Delta^2} \cdot \left\{\frac{(\alpha^n - \beta^n)(\alpha^m - \beta^m)}{b^n} - \frac{(\alpha^{n+s} - \beta^{n+s})(\alpha^{m+s} - \beta^{m+s})}{b^{n+s}}\right\}$$

$$= \frac{1}{\Delta^2} \cdot \left\{\frac{(\alpha^{m+n} + \beta^{m+n})}{b^n} - \left(\frac{\alpha\beta}{b}\right)^n (\alpha^{m-n} + \beta^{m-n})\right.$$

$$\left. - \left[\frac{\alpha^{m+n+2s} + \beta^{m+n+2s}}{b^{n+s}} - \left(\frac{\alpha\beta}{b}\right)^{n+s} (\alpha^{m-n} + \beta^{m-n})\right]\right\}$$

$$= \frac{1}{\Delta^2} \cdot \left(\frac{V_{m+n}}{b^n} - \frac{V_{m+n+2s}}{b^{n+s}}\right)$$

$$= \frac{-1}{\Delta^2 b^{n+s}} (V_{m+n+2s} - b^sV_{m+n}) = \frac{-U_sU_{m+n+s}}{b^{n+s}}$$

by (20) since s is even. Since $-U_s = b^sU_{-s}$ by (13), we only need to make this replacement and multiply through by b^{n+s} to obtain the desired result.

We now turn to the proofs of (50)-(60). Again the proofs are essentially all alike, so we prove only (50) for the difference and for s even. Using (16) and (61) we have, since s is even,

$$
\begin{aligned}
b^s H_n H_m &- H_{n+s}H_{m+s} \\
&= b^s\,(dU_n + bcU_{n-1})(dU_m + bcU_{m-1}) - (dU_{n+s} + bcU_{n+s-1})(dU_{m+s} + bcU_{m+s-1}) \\
&= d^2(b^sU_nU_m - U_{n+s}U_{m+s}) + bcd(b^sU_nU_{m-1} - U_{n+s}\,U_{m+s-1}) \\
&\quad +bcd(b^sU_{n-1}U_m - U_{n-1+s}U_{m+s} + b^2c^2(b^sU_{n-1}U_{m-1} - U_{n-1+s}U_{m-1+s}) \\
&= d^2b^sU_{-s}U_{m+n+s} + bcdb^sU_{-s}U_{m+n+s-1} + bcdb^sU_{-s}U_{m+n+s-1} + b^2c^2b^sU_{-s}U_{m+n+s-2} \\
&= b^sU_{-d}[d(dU_{m+n+s} + bcU_{m+n+s-1}) + bc(dU_{m+n+s-1} + bcU_{m+n+s-2})] \\
&= b^sU_{-s}[dH_{m+n+s} + bcH_{m+n+s-1}]
\end{aligned}
$$

as claimed.

We finally are in position to state and prove certain quadratic binomial Fibonacci identities. In particular we have the following, each of which derives from the corresponding identity from among (50)-(60).

$$
\sum_{i=0}^{r}(-1)^i\binom{r}{i}b^{(r-i)s}H_{n+si}H_{m+si}
$$

$$
= \begin{cases}
\Delta^{r-2}b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s even, r even} \\
\Delta^{r-1}b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s even, r odd} \\
\dfrac{b^{rs}}{\Delta^2}[V^r_{-s}(dK_{m+n+rs} + bcK_{m+n+rs-1}) & \\
\qquad - 2^r(-b)^n(dK_{m-n} - cK_{m-n+1})] & \text{s odd}
\end{cases} \tag{72}
$$

$$
\sum_{i=0}^{r}\binom{r}{i}b^{(r-i)s}H_{n+si}H_{m+si}
$$

$$
= \begin{cases}
\Delta^{r-2}b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s odd, r even} \\
\Delta^{r-1}b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s odd, r odd} \\
\dfrac{b^{rs}}{\Delta^2}[V^r_{-s}(dK_{m+n+rs} + bcK_{m+n+rs-1}) & \\
\qquad - 2^r(-b)^n(dK_{m-n} - cK_{m-n+1})] & \text{s even}
\end{cases} \tag{72'}
$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}b^{(r-i)s}K_{n+si}K_{m+si}$$

$$= \begin{cases} \Delta^r b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s even, r even} \\ \Delta^{r+1} b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s even, r odd} \\ b^{rs}[V^r_{-s}(dK_{m+n+rs} + bcK_{m+n+rs-1}) & \\ \qquad + 2^r(-b)^n(dK_{m-n} - cK_{m-n+1})] & \text{s odd} \end{cases} \tag{73}$$

$$\sum_{i=0}^{r}\binom{r}{i}b^{(r-i)s}K_{n+si}K_{m+si}$$

$$= \begin{cases} \Delta^r b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s odd, r even} \\ \Delta^{r+1} b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s odd, r odd} \\ b^{rs}[V^r_{-s}(dK_{m+n+rs} + bcK_{m+n+rs-1}) & \\ \qquad + 2^r(-b)^n(dK_{m-n} - cK_{m-n+1})] & \text{s even} \end{cases} \tag{73'}$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}b^{(r-i)s}K_{n+si}H_{m+si}$$

$$= \begin{cases} \Delta^r b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s even, r even} \\ \Delta^{r-1} b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s even, r odd} \\ b^{rs}[V^r_{-s}(dH_{m+n+rs} + bcH_{m+n+rs-1}) & \\ \qquad + 2^r(-b)^n(dH_{m-n} - cH_{m-n+1})] & \text{s odd} \end{cases} \tag{74}$$

$$\sum_{i=0}^{r}\binom{r}{i}K_{n+si}H_{m+si}$$

$$= \begin{cases} \Delta^r b^{rs}U^r_{-s}[dH_{m+n+rs} + bcH_{m+n+rs-1}] & \text{s odd, r even} \\ \Delta^{r-1} b^{rs}U^r_{-s}[dK_{m+n+rs} + bcK_{m+n+rs-1}] & \text{s odd, r odd} \\ b^{rs}[V^r_{-s}(dH_{m+n+rs} + bcH_{m+n+rs-1}) & \\ \qquad + 2^r(-b)^n(dH_{m-n} - cH_{m-n+1})] & \text{s even} \end{cases} \tag{74'}$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}H_{n-si}H_{m+si}$$

$$= \begin{cases} -\Delta^{r-2}(-b)^n U^r_{-s}[dK_{m-n+rs} - cK_{m-n+rs+1}] & \text{r even} \\ -\Delta^{r-1}(-b)^n U^r_{-s}[dH_{m-n+rs} - cH_{m-n+rs+1}] & \text{r odd} \end{cases} \tag{75}$$

$$\sum_{i=0}^{r}\binom{r}{i}H_{n-si}H_{m+si} = \frac{-(-b)^n V_{-s}^r}{\Delta^2}[dK_{m-n+rs} - cK_{m-n+rs+1}] + \frac{2^r}{\Delta^2}[dK_{m+n} + bcK_{m+n-1}] \tag{76}$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}K_{n-si}K_{m+si} = \begin{cases} \Delta^r(-b)^n U_{-s}^r[dK_{m-n+rs} - cK_{m-n+rs+1}] & \text{r even} \\ \Delta^{r+1}(-b)^n U_{-s}^r[dH_{m-n+rs} - cK_{m-n+rs+1}] & \text{r odd} \end{cases} \tag{77}$$

$$\sum_{i=0}^{r}\binom{r}{i}K_{n-si}K_{m+si} = (-b)^{n-1}V_{-s}^r[dK_{m-n+rs} - cK_{m-n+rs+1}] + 2^r[dK_{m+n} + bcK_{m+n-1}] \tag{78}$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}K_{n-si}H_{m+si} = \begin{cases} \Delta^r(-b)^n U_{-s}^r[dH_{m-n+rs} + cH_{m-n+rs+1}] & \text{r even} \\ \Delta^{r-1}(-b)^n U_{-s}^r[dK_{m-n+rs} - cK_{m-n+rs+1}] & \text{r odd} \end{cases} \tag{79}$$

$$\sum_{i=0}^{r}\binom{r}{i}K_{n-si}H_{m+si} = (-b)^n V_{-s}^r[dH_{m-n+rs} - cH_{m-n+rs+1}] + 2^r[dH_{m+n} + bcH_{m+n-1}] \tag{80}$$

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}K_{n+si}H_{m-si} = \begin{cases} \Delta^r(-b)^n U_s^r[dH_{m-n-rs} - cH_{m-n-rs+1}] & \text{r even} \\ \Delta^{r-1}(-b)^n U_s^r[dK_{m-n-rs} - cK_{m-n-rs+1}] & \text{r odd} \end{cases} \tag{81}$$

$$\sum_{i=0}^{r}\binom{r}{i}K_{n+si}H_{m-si} = (-b)^n V_s^r[dH_{m-n-rs} - cH_{m-n-rs+1}] + 2^r[dH_{m+n} + bcH_{m+n-1}] \,. \tag{82}$$

Once more the proofs are all similar and we prove only the result for the sum in (72) with s odd.

For $r = 1$, the assertion is that

$$b^s H_n H_m - H_{n+s}H_{m+s} = \frac{b^s}{\Delta^2}[V_{-s}(dK_{m+n+rs} + bcK_{m+n+rs-1}) - 2(-b)^n(dK_{m-n} - cK_{m-n+1})]$$

and this follows from (50) for s odd. Assume that

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i)s}H_{n+si}H_{m+si}\\&\quad=\frac{b^{ks}}{\Delta^2}[V_{-s}^k(dK_{m+n+ks}+bcK_{m+n+ks-1})-2^k(-b)^n(dK_{m-n}-cK_{m-n+1})]\ .\end{aligned}\tag{83}$$

Multiplying through by b^s, we obtain

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i+1)s}H_{n+si}H_{m+si}\\&\quad=\frac{b^{(k+1)s}}{\Delta^2}[V_{-s}^k(dK_{m+n+ks}+bcK_{m+n+ks-1})-2^k(-b)^n(dK_{m-n}-cK_{m-n+1})]\end{aligned}\tag{84}$$

and, replacing n and m by n + s and m + s in (83), we obtain

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k}{i}b^{(k-i)s}H_{n+si+s}H_{m+si+s}\\&\quad=\frac{b^{ks}}{\Delta^2}[V_{-s}^k(dK_{m+n+(k+2)s}+bcK_{m+n+(k+2)s-1})-2^k(-b)^{n+s}(dK_{m-n}-cK_{m-n+1})]\end{aligned}\tag{85}$$

or equivalently,

$$\begin{aligned}&\sum_{j=0}^{k+1}(-1)^{j-1}\binom{k}{j-1}b^{(k-j+1)s}H_{n+sj}H_{m+sj}\\&\quad=\frac{b^{ks}}{\Delta^2}[V_{-s}^k(dK_{m+n+(k+2)s}+bcK_{m+n+(k+2)s-1})-2^k(-b)^{n+s}(dK_{m-n}-cK_{m-n+1})].\end{aligned}\tag{85'}$$

Subtracting (85′) from (84), we now obtain on the left

$$\begin{aligned}&b^{(k+1)s}H_nH_m+\sum_{i=1}^{k}(-1)^i\left[\binom{k}{i}+\binom{k}{i-1}\right]b^{(k+1-i)s}H_{n+si}H_{m+si}\\&\quad+(-1)^{k+1}H_{n+s(k+1)}H_{m+s(k+1)}\\&=\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}b^{(k+1-i)s}H_{n+si}H_{m+si}\end{aligned}\tag{86}$$

and on the right

$$\frac{b^{(k+1)s}}{\Delta^2}[V_{-s}^k(dK_{m+n+ks}+bcK_{m+n+ks-1})-2^k(-b)^n(dK_{m-n}-cK_{m-n+1})]$$

$$-\frac{b^{ks}}{\Delta^2}[V_{-s}^k(dK_{m+n+(k+2)s}+bcK_{m+n+(k+2)s-1})-2^k(-b)^{n+s}(dK_{m-n}-cK_{m-n+1})]$$

$$= \frac{-b^{ks}}{\Delta^2}\{V_{-s}^{k}[d(K_{m+n+(k+2)s} - b^sK_{m+n+ks}) + bc(K_{m+n+(k+2)s-1} - b^sK_{m+n+ks-1})]$$
$$+ 2^{k+1}(-b)^n b^s(dK_{m-n} - cK_{m-n+1})\}$$
$$= \frac{-b^{ks}}{\Delta^2}\{V_{-s}^{k}V_d[dK_{m+n+(k+1)s} + bcK_{m+n+(k+1)s-1}] + 2^{k+1}(-b)^n b^s(dK_{m-n} - cK_{m-n+1})\}$$
$$= \frac{b^{(k+1)s}}{\Delta^2}\{V_{-s}^{k+1}[dK_{m+n+(k+1)s} + bcK_{m+n+(k+1)s-1}] - 2^{k+1}(-b)^n(dK_{m-n} - cK_{m-n+1})\}$$

by (24) and, since $-V_s = b^sV_{-s}$ for s odd by (13), this completes the proof.

Of course, for special choices of a, b, c, and d, the preceding give results about the sequences $\{F_n\}$, $\{L_n\}$, $\{U_n\}$, $\{V_n\}$, the Pell sequence and others. In particular, while we have proved much more, equations (2)-(7) of Hoggatt and Bicknell all derive from either (72) or (73) on setting a = b = d = 1, c = 0, m = n, and r = 2s + 1 or r = 2s + 2 as appropriate.

As we noted earlier, Hoggatt and Bicknell have obtained some biquadratic binomial identities but results in all generality are not known and could use some study.

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A SURVEY OF PROPERTIES OF THIRD ORDER PELL DIAGONAL FUNCTIONS

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1. INTRODUCTION

This paper is concerned with the simpler properties of some third order sequences of polynomials. These sequences are $\{r_n(x)\}$, $\{s_n(x)\}$ and $\{t_n(x)\}$, defined thus:

$$\left.\begin{array}{l} r_o(x) = 0,\ r_1(x) = 1,\ r_2(x) = 2x \\ r_{n+1}(x) = 2xr_n(x) + r_{n-2}(x) \quad (n \geq 2) \end{array}\right\}, \tag{1.1}$$

$$\left.\begin{array}{l} s_o(x) = 0,\ s_1(x) = 2,\ s_2(x) = 2x \\ s_{n+1}(x) = 2xs_n(x) + s_{n-2}(x) \quad (n \geq 2) \end{array}\right\}, \tag{1.2}$$

$$\left.\begin{array}{l} t_o(x) = 3,\ t_1(x) = 2x,\ t_2(x) = 4x^2 \\ t_{n+1}(x) = 2xt_n(x) + t_{n-2}(x) \quad (n \geq 2) \end{array}\right\}. \tag{1.3}$$

These are called third order diagonal functions of Pell polynomials or simply Pell diagonal functions because the first two coincide with sequences obtained by considering the "diagonals" of gradient 1 derived from arrays produced by Pell and Pell-Lucas polynomials [5]. The three sequences can be constructed from the "diagonals" of gradient 2 produced by arrays arising from the expansions of

$$(2x + 1)^n,\ (2x + 2)(2x + 1)^{n-1},\ (2x + 3)(2x + 1)^{n-1},$$

where $n \geq 1$.

By adapting Lucas' ideas [9], $\{r_n(x)\}$ and $\{t_n(x)\}$ may be regarded as the fundamental and primordial sequences respectively, for those polynomials obeying the recurrence formulas in (1.1) - (1.3).

Simple relations exist between $\{r_n(x)\}$ and $\{s_n(x)\}$, and the diagonal functions of Chebyshev polynomials studied by Jaiswal [6] and Horadam [4], as would be expected because the Pell and Pell-Lucas polynomials are complex Chebyshev polynomials [5].

One reason for presenting Pell diagonal functions here is that they generate new identities for the Fibonacci numbers.

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 255–271.

2. ROOTS OF THE AUXILIARY EQUATION OF THE PELL DIAGONAL FUNCTIONS

The auxiliary equation of the Pell diagonal functions is

$$f(y) \equiv y^3 - 2xy^2 - 1 = 0. \tag{2.1}$$

If we let the roots of this equation be α, β and γ, then we have by elementary equation theory

$$\left.\begin{aligned} &\alpha + \beta + \gamma = 2x, \\ &\alpha\beta + \beta\gamma + \gamma\alpha = 0, \\ \text{and}\quad &\alpha\beta\gamma = 1. \end{aligned}\right\} \tag{2.2}$$

By applying Cardano's procedure [3], it is found that the roots of (2.1) are given by

$$\left.\begin{aligned} \alpha &= \frac{2x}{3} + \frac{u}{3\sqrt[3]{2}} + \frac{v}{3\sqrt[3]{2}}, \\ \beta &= \frac{2x}{3} + \frac{u\omega}{3\sqrt[3]{2}} + \frac{v\omega^2}{3\sqrt[3]{2}}, \\ \gamma &= \frac{2x}{3} + \frac{u\omega^2}{3\sqrt[3]{2}} + \frac{v\omega}{3\sqrt[3]{2}}, \end{aligned}\right\} \tag{2.3}$$

where

$$u = \sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}} \tag{2.4}$$

and

$$v = \sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}, \tag{2.5}$$

and where ω and ω^2 are complex cube roots of unity.

Let

$$d = -3/(2\sqrt[3]{4}). \tag{2.6}$$

For $x > d$, u and v are real and so β and γ are conjugate complex numbers. For $x = d$, $\beta = \gamma = -\sqrt[3]{2}$, and for $x < d$, β and γ are negative numbers; α is the positive root of (2.1) required by Descartes' rule.

From (2.3) - (2.5), we can prove that

$$\alpha^2 - |\beta|^2 = 2x\alpha \tag{2.7}$$

for $x > d$. Hence

$$\left.\begin{array}{ll} & \alpha > |\beta| = |\gamma| \text{ for } x > 0, \\ & \alpha = |\beta| = |\gamma| = 1 \text{ for } x = 0 \\ \text{and} & \alpha < |\beta| = |\gamma| \text{ for } d < x < 0. \end{array}\right\} \tag{2.8}$$

Further, it can be shown that, for $x < d$,

$$\left.\begin{array}{ll} & |\beta| > |\gamma| > \alpha, \\ & |\beta| > 1, \\ & |\gamma| > 1 \text{ for } -1 < x < d \\ \text{and} & |\gamma| < 1 \text{ for } x < -1. \end{array}\right\} \tag{2.9}$$

When $x = -1$,

$$\alpha = (\sqrt{5}-1)/2,\ \beta = -(\sqrt{5}+1)/2 \text{ and } \gamma = -1. \tag{2.10}$$

From (2.2) and (2.9),

$$0 < \alpha < 1, \text{ for } x < d. \tag{2.11}$$

3. BINET FORMULAS FOR THE PELL DIAGONAL FUNCTIONS

Standard procedures give formulas for the Pell diagonal functions in terms of α, β, and γ for $\beta \neq \gamma$, or $x \neq d$. Firstly,

$$r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \tag{3.1}$$

and

$$r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^{n+3} & \beta^{n+3} & \gamma^{n+3} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \end{vmatrix}. \tag{3.2}$$

Next,

$$r_n(x) = A\alpha^n + B\beta^n + C\gamma^n, \tag{3.3}$$

where

$$A = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)},\ B = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)} \text{ and } C = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\ . \tag{3.4}$$

Also,

$$r_n(x) = \frac{\alpha^{n+1}}{f'(\alpha)} + \frac{\beta^{n+1}}{f'(\beta)} + \frac{\gamma^{n+1}}{f'(\gamma)}, \tag{3.5}$$

where $f'(y)$ is the derivative of $f(y)$ given by (2.1). A Binet formula for $s_n(x)$ is

$$s_n(x) = A'\alpha^n + B'\beta^n + C'\gamma^n, \tag{3.6}$$

where

$$A' = \frac{\alpha-\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)},\ B' = \frac{\beta-\gamma-\alpha}{(\beta-\gamma)(\beta-\alpha)} \text{ and } C' = \frac{\gamma-\alpha-\beta}{(\gamma-\alpha)(\gamma-\beta)}\ . \tag{3.7}$$

The next formula holds for all values of x:

$$t_n(x) = \alpha^n + \beta^n + \gamma^n. \tag{3.8}$$

The formula (3.1) may be considered as the third order analogue of the Binet formula for the Fibonacci numbers expressed as the quotient of two determinants. The formulas (3.3) and (3.4) occur in Jarden [7] and Spickerman [15] for third order sequences of numbers; (3.5) may be compared with a formula of Levesque [8].

It can be shown that

$$r_n(d) = (-1)^{n-1}2^{(2-2n)/3}\{(3n+1)2^n-(-1)^n\}/9. \tag{3.9}$$

The Binet formulas may be used to establish simple relations between the diagonal functions, such as

$$\left.\begin{aligned} s_n(x) &= r_n(x) + r_{n-3}(x) \\ &= 2xr_{n-1}(x) + 2r_{n-3}(x) \end{aligned}\right\} \tag{3.10}$$

and

$$\left.\begin{aligned} t_n(x) &= r_{n+1}(x) + 2r_{n-2}(x) \\ &= 2xr_n(x) + 3r_{n-2}(x) \\ &= s_{n+1}(x) + r_{n-2}(x) \end{aligned}\right\}\ . \tag{3.11}$$

It is noted from (2.10) that when $x = -1$, α and β are negatives of the roots of the auxiliary equation of the Fibonacci numbers. This suggests that there should be simple relations between $r_n(-1)$, $s_n(-1)$ and $t_n(-1)$, and the Fibonacci and Lucas numbers. In fact,

$$\left.\begin{aligned} r_n(-1) &= (-1)^{n-1}\,(F_{n+2}-1), \\ s_n(-1) &= (-1)^{n-1}\,2F_n \\ \text{and}\qquad t_n(-1) &= (-1)^n\,(L_n+1), \end{aligned}\right\} \tag{3.12}$$

where F_n and L_n are Fibonacci and Lucas numbers, respectively. The work of Jaiswal [6] and Horadam [4] pointed to the existence of such relations.

It can be shown that (1.1) and (3.1) are equivalent for $n \geq 0$. If we define $r_n(x)$ by (3.1), we can introduce negative subscripts with Pell diagonal functions. A way to establish the rather complex relation between polynomials with positive and negative subscripts is indicated in §7. However, we may readily show that

$$\left.\begin{aligned} r_{-n}(-1) &= F_{n-2} + (-1)^n, \\ s_{-n}(-1) &= 2F_n \\ \text{and}\qquad t_{-n}(-1) &= L_n + (-1)^n\ . \end{aligned}\right\} \tag{3.13}$$

4. EXPLICIT SUMMATION EXPRESSIONS FOR PELL DIAGONAL FUNCTIONS

A variety of well known approaches may be employed to prove that, for $n > 0$,

$$r_n(x) = \sum_{i=0}^{[(n-1)/3]} \binom{n-1-2i}{i}(2x)^{n-1-3i}, \tag{4.1}$$

$$s_n(x) = (2x)^{n-1} + \sum_{i=1}^{[(n-1)/3]} \frac{n-1-i}{i}\binom{n-2-2i}{i-1}(2x)^{n-1-3i}, \tag{4.2}$$

$$t_n(x) = \sum_{i=0}^{[n/3]} \frac{n}{n-2i}\binom{n-2i}{i}(2x)^{n-3i}. \tag{4.3}$$

The explicit summation expressions for polynomials with negative subscripts are

$$r_{-2n-1}(x) = \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1}(-2x)^{n-2-3i}, \tag{4.4}$$

$$r_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \binom{n-1-i}{2i}(-2x)^{n-1-3i}, \tag{4.5}$$

$$s_{-2n-1}(x) = (-2x)^{n+1} + \sum_{i=1}^{[(n+1)/3]} \frac{n+1+i}{2i}\binom{n-i}{2i-1}(-2x)^{n+1-3i}, \tag{4.6}$$

$$s_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+1}\binom{n-1-i}{2i}(-2x)^{n-1-3i}, \tag{4.7}$$

$$t_{-2n-1}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{2n+1}{n-i}\binom{n-i}{2i+1}(-2x)^{n-1-3i} \tag{4.8}$$

and

$$t_{-2n}(x) \quad = \quad \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i}(-2x)^{n-3}. \tag{4.9}$$

These formulas, (3.12) and (3.13) give rise to some new explicit expressions for the Fibonacci and Lucas numbers, namely,

$$F_n \quad = \quad 1 + \sum_{i=0}^{[(n-3)/3]} (-1)^i\binom{n-3-2i}{i}2^{n-3-3i}, \tag{4.10}$$

$$F_n \quad = \quad 2^{n-2} + \sum_{i=1}^{[(n-1)/3]} (-1)^i \frac{n-1-i}{i}\binom{n-2-2i}{i-1}2^{n-2-3i}, \tag{4.11}$$

$$L_n \quad = \quad -1 + \sum_{i=0}^{[n/3]} (-1)^i \frac{n}{n-2i}\binom{n-2i}{i} 2^{n-3i}, \tag{4.12}$$

$$F_{2n-1} \quad = \quad 1 + \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1} 2^{n-2-3i}, \tag{4.13}$$

$$F_{2n} \quad = \quad -1 + \sum_{i=0}^{[n/3]} \binom{n-i}{2i} 2^{n-3i}, \tag{4.14}$$

$$F_{2n-1} \quad = \quad 2^{n-1} + \sum_{i=1}^{[n/3]} \frac{n+i}{2i}\binom{n-1-i}{2i-1} 2^{n-1-3i}, \tag{4.15}$$

$$F_{2n} \quad = \quad \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+i}\binom{n-1-i}{2i}2^{n-2-3i}, \tag{4.16}$$

$$L_{2n-1} \quad = \quad 1 + \sum_{i=0}^{[(n-2)/3]} \frac{2n-i}{n-1-i}\binom{n-1-i}{2i+1} 2^{n-2-3i}, \tag{4.17}$$

$$L_{2n} \quad = \quad -1 + \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i} 2^{n-3i}. \tag{4.18}$$

Divisibility properties are suggested by some of these explicit expressions. Thus, from (4.17), for p being an odd prime, we obtain the well known result:

$$p \mid (L_p - 1). \tag{4.19}$$

5. ROOTS OF PELL DIAGONAL FUNCTIONS

By Descartes' Rule, $r_n(x)$ can have no positive roots and at most $[(n-1)/3]$ negative roots. It is believed that this maximum number is, in fact, the actual number of negative roots. Plausible reasoning based on patterns emerging from the application of Sturm's Theorem [2] lead us to make the following conjectures (11):

Conjecture 5.1: $r_n(x)$ has $[(n-1)/3]$ roots in $(d,0)$ for $n > 0$, where d was defined in (2.6).

Similarly,

Conjecture 5.2: $s_n(x)$ has $[(n-1)/3]$ roots in $(d,0)$ for $n > 0$

and

Conjecture 5.3: $t_n(x)$ has $[n/3]$ roots in $(d,0)$ for $n > 0$.

Where the subscripts are negative, we have

Conjecture 5.4: (a) $r_{-2n}(x)$ has $[(n-1)/3]$ positive roots for $n > 0$,
(b) $r_{-2n-1}(x)$ has $[(n-2)/3]$ positive roots for $n > 0$,

Conjecture 5.5: (a) $s_{-2n}(x)$ has $[(n-1)/3]$ positive roots for $n > 0$,
(b) $s_{-2n-1}(x)$ has $[(n+1)/3]$ positive roots for $n > 0$

and

Conjecture 5.6: (a) $t_{-2n}(x)$ has $[n/3]$ positive roots for $n > 0$,
(b) $t_{-2n-1}(x)$ has $[(n-1)/3]$ positive roots for $n > 0$.

Using standard procedures, Conjectures 5.1-5.3, the results (2.8) and (2.9), and the Binet formulas of §3, we can show that

$$\lim_{n\to\infty} \frac{r_n(x)}{r_{n+1}(x)} = \lim_{n\to\infty} \frac{s_n(x)}{s_{n+1}(x)} = \lim_{n\to\infty} \frac{t_n(x)}{t_{n+1}(x)} = \begin{cases} 1/\alpha \text{ for } x > 0 \\ 1/\beta \text{ for } x \le d \end{cases}. \tag{5.1}$$

Similarly,

$$\lim_{n\to\infty} \frac{r_{-n}(x)}{r_{-n-1}(x)} = \lim_{n\to\infty} \frac{s_{-n}(x)}{s_{-n-1}(x)} = \lim_{n\to\infty} \frac{t_{-n}(x)}{t_{-n-1}(x)} = \alpha \text{ for } x < 0 . \tag{5.2}$$

6. A MATRIX GENERATOR FOR PELL DIAGONAL FUNCTIONS

If

$$A = \begin{pmatrix} 2x & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.1}$$

it can be shown by induction that

$$A^n = \begin{pmatrix} r_{n+1}(x) & r_{n-1}(x) & r_n(x) \\ r_n(x) & r_{n-2}(x) & r_{n-1}(x) \\ r_{n-1}(x) & r_{n-3}(x) & r_{n-2}(x) \end{pmatrix}, \tag{6.2}$$

where n is an integer. It is noted that

$$|A| = 1. \tag{6.2'}$$

Elementary matrix identities involving the matrix A include:

$$A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} r_{n+1}(x) \\ r_n(x) \\ r_{n-1}(x) \end{pmatrix}, \tag{6.3}$$

$$(1 \;\; 0 \;\; 0)A^n = (r_{n+1}(x) \;\; r_{n-1}(x) \;\; r_n(x)) \tag{6.4}$$

and

$$(1 \;\; 0 \;\; 0)A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = r_{n+1}(x). \tag{6.5}$$

It is simple to derive many other matrix identities including ones involving the other diagonal functions. Here we note only a new matrix generator of the Fibonacci numbers:

$$\begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = (-1)^n \begin{pmatrix} f_{n+3}-1 & f_{n+1}-1 & -(f_{n+2}-1) \\ -(f_{n+2}-1) & -(f_n-1) & f_{n+1}-1 \\ f_{n+1}-1 & f_{n-1}-1 & -(f_n-1) \end{pmatrix}. \tag{6.6}$$

Among the identities that may be proved with the aid of the matrix A are:

$$r_{m+n}(x) = r_{m+1}(x)\, r_n(x) + r_m(x)\, r_{n-2}(x) + r_{m-1}(x)\, r_{n-1}(x) \tag{6.7}$$

and

$$t_{m+n}(x) = r_{m+1}(x)\, t_n(x) + r_m(x)\, t_{n-2}(x) + r_{m-1}(x)\, t_{n-1}(x). \tag{6.8}$$

Formulas similar to (6.7) occur in Agronomoff [1] and Jarden [7] for third order number sequences.

If

$$B_k = \begin{pmatrix} r_{2k}(x) & r_{2k-2}(x) & r_{2k-1}(x) \\ r_k(x) & r_{k-2}(x) & r_{k-1}(x) \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.9}$$

and if we apply (6.7), we can prove that

$$B_k A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} r_{n+2k}(x) \\ r_{n+k}(x) \\ r_n(x) \end{pmatrix}. \tag{6.10}$$

7. SIMSON'S FORMULAS FOR PELL DIAGONAL FUNCTIONS

We attend to one form of Simson's formula here. Consider

$$r_{n+1}(x)\, r_{n-1}(x) - r_n^2(x)$$

$$= \begin{vmatrix} r_{n+2}(x) & r_{n+1}(x) & 1 \\ r_{n+1}(x) & r_n(x) & 0 \\ r_n(x) & r_{n-1}(x) & 0 \end{vmatrix}$$

$$= \left| A^{n+1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right| \quad \text{by (6.3)}$$

$$= |A^n| \begin{vmatrix} 2x & 1 & r_{-n+1}(x) \\ 1 & 0 & r_{-n}(x) \\ 0 & 0 & r_{-n-1}(x) \end{vmatrix} \quad \text{by (6.3)}\,.$$

It follows then from (6.2′) that

$$r_{n+1}(x)\, r_{n-1}(x) - r_n^2(x) = -r_{-n-1}(x). \tag{7.1}$$

Simson's formulas for the other Pell diagonal functions are quite awkward and seem to be of little use [10]. The same comment may be made concerning the generalisations of (7.1). However, two extensions of (7.1) are worthy of note, namely,

$$r_{n+2}(x)\, r_{n-2}(x) - r_n^2(x) = -r_{-n-4}(x) \tag{7.2}$$

and

$$r_{n+3}(x)\, r_{n-3}(x) - r_n^2(x) = -4x^2 r_{-n-3}(x). \tag{7.3}$$

These can be proved with the aid of A, B_2 and B_3, where B_k was defined in (6.9). Two further observations are made. Firstly, Simson's formula for the general third order fundamental sequence is similar to (7.1). Secondly, the right side of (7.1) is dependent on x, but is simpler in form than the left side.

While it appears to be impossible to generalise on (7.1) or to devise useful Simson's formulas for the other Pell diagonal functions, several other highly specific but applicable formulas can be obtained in the same way as (7.1) was derived. These include:

$$r_{n+1}(x)\, t_n(x) - r_n(x)\, t_{n+1}(x) = 3r_{-n-2}(x) \tag{7.4}$$

and

$$s_{n+2}(x)\, r_n(x) - s_n(x)\, r_{n+2}(x) = 2xr_{-n-3}(x). \tag{7.5}$$

8. CONSTELLATIONS OF GENERALISED PELL DIAGONAL FUNCTIONS

We commence by defining two generalisations of $r_n(x)$, namely,

$$\left.\begin{array}{l} r_{o,m}(x) = 0,\ r_{1,m}(x) = 1,\ r_{2,m}(x) = t_m(x) \\ r_{n+1,m}(x) = t_m(x)\, r_{n,m}(x) - t_{-m}(x)\, r_{n-1,m}(x) + r_{n-2,m}(x) \end{array}\right\} \tag{8.1}$$

and

$$\left.\begin{array}{l} v_{o,m}(x) = 0,\ v_{1,m}(x) = -t_{-m}(x),\ v_{2,m}(x) = -t_{-m}(x)\, t_m(x) + 1 \\ v_{n+1,m}(x) = t_m(x)\, v_{n,m}(x) - t_{-m}(x)\, v_{n-1,m}(x) + v_{n-2,m}(x) \end{array}\right\}. \tag{8.2}$$

A third constellation is noted

$$\left.\begin{array}{l} s_{o,m}(x) = 0,\ s_{1,m}(x) = 2,\ s_{2,m}(x) = t_m(x) \\ s_{n+1,m}(x) = t_m(x)\, s_{n,m}(x) - t_{-m}(x)\, s_{n-1,m}(x) + s_{n-2,m}(x) \end{array}\right\}. \tag{8.3}$$

By putting m = 1, it is simple to prove that

$$\left.\begin{array}{l} r_{n,1}(x) = r_n(x), \\ v_{n,1}(x) = r_{n-1}(x) \\ s_{n,1}(x) = s_n(x). \end{array}\right\} \tag{8.4}$$

and (the last line)

We have been unable to prove anything concerning the roots of the generalised Pell diagonal functions. However, a number of conjectures have been made on the basis of a limited number of observations [10]. For example, for $n,m > 0$, $r_{n,m}(x)$ has $[(n-1)m/3]$ roots in $(d,0)$.

The auxiliary equation of these sequences is

$$f_m(y) \equiv y^3 - t_m(x)\, y^2 + t_{-m}(x)\, y - 1 = 0, \tag{8.4$'$}$$

and its roots are α^m, β^m and γ^m, where α, β and γ are the roots of (2.1).

Binet formulas are readily derived, for $\beta \neq \gamma$ or $x \neq d$. Firstly,

$$r_{n,m}(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m \\ \alpha^{(n+1)m} & \beta^{(n+1)m} & \gamma^{(n+1)m} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m \\ \alpha^{2m} & \beta^{2m} & \gamma^{2m} \end{vmatrix}. \tag{8.5}$$

Secondly,

$$r_{n,m}(x) = A_m\alpha^{nm} + B_m\beta^{nm} + C_m\gamma^{nm}, \tag{8.6}$$

where

$$A_m = \frac{\alpha^m}{(\alpha^m-\beta^m)(\alpha^m-\gamma^m)}\,,\ B_m = \frac{\beta^m}{(\beta^m-\gamma^m)(\beta^m-\alpha^m)}\,,\ C_m = \frac{\gamma^m}{(\gamma^m-\alpha^m)(\gamma^m-\beta^m)}\,. \tag{8.7}$$

Thirdly,

$$r_{n,m}(x) = \frac{\alpha^{(n+1)m}}{f'_m(\alpha^m)} + \frac{\beta^{(n+1)m}}{f'_m(\beta^m)} + \frac{\gamma^{(n+1)m}}{f'_m(\gamma^m)}\,, \tag{8.8}$$

where $f'_m(y)$ is the derivative of $f_m(y)$ defined in (8.4′).

Binet formulas for $v_{n,m}(x)$ include

$$v_{n,m}(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^{2m} & \beta^{2m} & \gamma^{2m} \\ \alpha^{(n+2)m} & \beta^{(n+2)m} & \gamma^{(n+2)m} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha^{2m} & \beta^{2m} & \gamma^{2m} \\ \alpha^{m} & \beta^{m} & \gamma^{m} \end{vmatrix} \tag{8.9}$$

and

$$v_{n,m}(x) = E\alpha^{nm} + F\beta^{nm} + G\gamma^{nm}, \tag{8.10}$$

where

$$E = \frac{-\alpha^{2m}(\beta^m+\gamma^m)}{(\alpha^m-\beta^m)(\alpha^m-\gamma^m)}\,,\ F = \frac{-\beta^{2m}(\gamma^m+\alpha^m)}{(\beta^m-\gamma^m)(\beta^m-\alpha^m)}\,,\ G = \frac{-\gamma^{2m}(\alpha^m+\beta^m)}{(\gamma^m-\alpha^m)(\gamma^m-\beta^m)}\,. \tag{8.11}$$

Finally, we have

$$s_{n,m}(x) = A'_m\alpha^{nm} + B'_m\beta^{nm} + C'_m\gamma^{nm}, \tag{8.12}$$

where

$$A'_m = \frac{\alpha^m-\beta^m-\gamma^m}{(\alpha^m-\beta^m)(\alpha^m-\gamma^m)}\,,\ B'_m = \frac{\beta^m-\gamma^m-\alpha^m}{(\beta^m-\gamma^m)(\beta^m-\alpha^m)}\,,\ C'_m = \frac{\gamma^m-\alpha^m-\beta^m}{(\gamma^m-\alpha^m)(\gamma^m-\beta^m)}\,. \tag{8.13}$$

Two relations between the generalised diagonal functions and the Pell diagonal functions are noted, namely,

$$r_{n,m}(x) = \frac{r_{m\text{-}3}(x)\ r_{(n+1)m\text{-}1}(x)\ -\ r_{m\text{-}1}(x)\ r_{(n+1)m\text{-}3}(x)}{r_{m\text{-}3}(x)\ r_{2m\text{-}1}(x)\ -\ r_{m\text{-}1}(x)\ r_{2m\text{-}3}(x)} \tag{8.14}$$

and

$$v_{n,m}(x) = \frac{r_{2m\text{-}1}(x)\ r_{(n+2)m\text{-}3}(x)\ -\ r_{2m\text{-}3}(x)\ r_{(n+2)m\text{-}1}(x)}{r_{m\text{-}3}(x)\ r_{2m\text{-}1}(x)\ -\ r_{m\text{-}1}(x)\ r_{2m\text{-}3}(x)}\,. \tag{8.15}$$

These can be proved from the Binet formulas.

Some special cases are recorded below

$$\left.\begin{aligned}
r_{n,2}(x) &= r_{2n-1}(x),\\
r_{n,3}(x) &= r_{3n}(x)/r_3(x),\\
v_{n,2}(x) &= r_{2n+3}(x) - 4x^2 r_{2n+1}(x),\\
r_{n,m}(-1) &= \frac{(-1)^{(n-1)m}\{F_m - F_{(n+1)m} + (-1)^m F_{nm}\}}{F_m\{1-L_m+(-1)^m\}},\\
v_{n,m}(-1) &= \frac{(-1)^{(n-1)m}\{F_{(n+2)m} - F_{2m} - F_{nm}\}}{F_m\{1-L_m+(-1)^m\}},\\
r_{n,m}(d) &= \frac{-ng^{(n+1)m} + (n+1)\,g^{(n-2)m} - g^{-2(n+1)m}}{-g^{2m}+2g^{-m}-g^{-4m}},\\
v_{n,m}(d) &= \frac{-ng^{(n+3)m} + (n+2)\,g^{(n-3)m} - 2g^{-(2n+3)m}}{g^{2m}+g^{-4m}-2g^{-m}},
\end{aligned}\right\} \tag{8.16}$$

where $g = -\sqrt[3]{2} = \beta(d) = \gamma(d)$.

Consider

$$\begin{aligned}
r_{-n-1,-m}(x) &= A_{-m}\alpha^{(n+1)m} + B_{-m}\beta^{(n+1)m} + C_{-m}\gamma^{(n+1)m} \quad \text{from (8.6)}\\
&= A_m\alpha^{nm} + B_m\beta^{nm} + C_m\gamma^{nm} \text{ from (8.7)}\ .
\end{aligned}$$

Hence

$$r_{-n-1,-m}(x) = r_{n,m}(x)$$

or

$$r_{-n,m}(x) = r_{n-1,-m}(x)\ . \tag{8.17}$$

It follows that

$$r_{-n}(x) = r_{-n,1}(x) = r_{n-1,-1}(x). \tag{8.18}$$

Thus a relation has been established between polynomials with positive and negative subscripts. In the same way as (8.17) was obtained, it can be shown that

$$v_{-n,m}(x) = v_{n-2,-m}(x)\ . \tag{8.19}$$

An explicit summation for $r_{n,m}(x)$ is noted, namely,

$$r_{n,m}(x) = \sum_{a=0}^{[(n-1)/2]} \sum_{b=0}^{[(n-1)/3]} \binom{n-1-a-2b}{a+b}\binom{a+b}{b} t_m^{n-1-2a-3b}(x)(-t_{-m}(x))^a. \tag{8.20}$$

This was found by adapting a procedure found in Shannon [14].

If

$$A_m = \begin{pmatrix} t_m(x) & -t_{-m}(x) & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{8.21}$$

then, by induction,

$$A_m^n = \begin{pmatrix} r_{n+1,m}(x) & v_{n,m}(x) & r_{n,m}(x) \\ r_{n,m}(x) & v_{n-1,m}(x) & r_{n-1,m}(x) \\ r_{n-1,m}(x) & v_{n-2,m}(x) & r_{n-2,m}(x) \end{pmatrix}. \tag{8.22}$$

Thus A_m generates $\{r_{n,m}(x)\}$ and $\{v_{n,m}(x)\}$. The matrix may be applied to derive formulas, such as

$$r_{n+p,m}(x) = r_{n+1,m}(x)\, r_{p,m}(x) + r_{n,m}(x)\, v_{p,m}(x) + r_{n-1,m}(x)\, r_{p-1,m}(x), \tag{8.23}$$

$$s_{n,m}(x) = r_{n,m}(x) + v_{n-2,m}(x), \tag{8.24}$$

$$r_{n+1,m}(x)\, r_{n-1,m}(x) - r_{n,m}^2(x) = -r_{-n-1,m}(x) \tag{8.25}$$

and

$$r_{n+1,m}(x)\, v_{n-1,m}(x) - r_{n-1,m}(x)\, v_{n+1,m}(x) = v_{-n-1,m}(x). \tag{8.26}$$

A fuller treatment of these generalisations of the Pell diagonal functions may be found in [10] and [12].

9. SOME SERIES OF FRACTIONS WITH PELL DIAGONAL FUNCTIONS IN THE DENOMINATORS

The formulas derived in §7 may be employed to sum series of fractions with Pell diagonal functions in the denominators. Thus, from (7.1), it can be shown that, for $x \le d$ or $x > 0$,

$$\sum_{i=1}^{n} \frac{r_{-i-2}(x)}{r_i(x)\, r_{i+1}(x)} = 2x - \frac{r_{n+2}(x)}{r_{n+1}(x)}. \tag{9.1}$$

Then, from (5.1),

$$\sum_{i=1}^{\infty} \frac{r_{-i-2}(x)}{r_i(x)\, r_{i+1}(x)} = \begin{cases} 2x-\alpha = \beta+\gamma & \text{for } x > 0 \\ 2x-\beta = \gamma+\alpha & \text{for } x \le d \end{cases}. \tag{9.2}$$

Similarly, from (8.25) and the conjecture of §8,

$$\sum_{i=1}^{n} \frac{r_{-i-2,m}(x)}{r_{i,m}(x)\, r_{i+1,m}(x)} = t_m(x) - \frac{r_{n+2,m}(x)}{r_{n+1,m}(x)} \tag{9.3}$$

for x > 0 or x $\le$ d, and

$$\sum_{i=1}^{\infty} \frac{r_{-i-2,m}(x)}{r_{i,m}(x)\, r_{i+1,m}(x)} = \begin{cases} \beta^m+\gamma^m & \text{for } x > 0 \\ \gamma^m+\alpha^m & \text{for } x \le d \end{cases} . \tag{9.4}$$

It can be shown from (7.2), (7.4) and (7.5) that

$$\sum_{i=1}^{n} \frac{r_{-2i-3}(x)}{r_{2i-1}(x)\, r_{2i+1}(x)} = \frac{r_{2n-1}(x)}{r_{2n+1}(x)} \text{ for } x > 0 \text{ or } x \le d, \tag{9.5}$$

$$\sum_{i=1}^{\infty} \frac{r_{-2i-3}(x)}{r_{2i-1}(x)\, r_{2i+1}(x)} = \begin{cases} 1/\alpha^2 & \text{for } x > 0 \\ 1/\beta^2 & \text{for } x \le d \end{cases} , \tag{9.6}$$

$$\sum_{i=1}^{n} \frac{r_{-i-2}(x)}{t_i(x)\, t_{i+1}(x)} = \frac{1}{3}\left(\frac{r_{n+1}(x)}{t_{n+1}(x)} - \frac{1}{2x}\right) \text{ for } x > 0 \text{ or } x \le d, \tag{9.7}$$

$$\sum_{i=1}^{\infty} \frac{r_{-i-2}(x)}{t_i(x)\, t_{i+1}(x)} = \begin{cases} \frac{1}{3}\left(\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)} - \frac{1}{2x}\right) & \text{for } x > 0 \\ \frac{1}{3}\left(\frac{\beta}{(\beta-\gamma)(\beta-\alpha)} - \frac{1}{2x}\right) & \text{for } x \le d \end{cases} , \tag{9.8}$$

$$\sum_{i=1}^{n} \frac{r_{-2i-3}(x)}{s_{2i}(x)\, s_{2i+2}(x)} = \frac{1}{2x}\left(1 - \frac{r_{2n+2}(x)}{s_{2n+2}(x)}\right) \text{ for } x > 0 \text{ or } x \le d, \tag{9.9}$$

$$\sum_{i=1}^{\infty} \frac{r_{-2i-3}(x)}{s_{2i}(x)\, s_{2i+2}(x)} = \begin{cases} \frac{1}{2x}\left(1 - \frac{\alpha}{\alpha-\beta-\gamma}\right) & \text{for } x > 0 \\ \frac{1}{2x}\left(1 - \frac{\beta}{\beta-\gamma-\alpha}\right) & \text{for } x \le d \end{cases} . \tag{9.10}$$

One can also use formulas such as (1.1) and (3.11) to sum series of fractions. Thus, for x > 0 or x $\le$ d,

$$\sum_{i=1}^{n} \frac{r_{3i}(x)}{r_{3i-2}(x)\, r_{3i+1}(x)} = \frac{1}{2x}\left(1 - \frac{1}{r_{3n+1}(x)}\right), \tag{9.11}$$

$$\sum_{i=1}^{\infty} \frac{r_{3i}(x)}{r_{3i-2}(x)\, r_{3i+1}(x)} = \frac{1}{2x}, \tag{9.12}$$

$$\sum_{i=1}^{n} \frac{(-1)^{i-1} r_{2i+2}(x)}{(2x)^{i-1} r_{2i-1}(x)\, r_{2i+1}(x)} = 2x + \frac{(-1)^{n-1}}{(2x)^{n-1} r_{2n+1}(x)}, \tag{9.13}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1} r_{2i+2}(x)}{(2x)^{i-1} r_{2i-1}(x)\, r_{2i+1}(x)} = 2x, \tag{9.14}$$

$$\sum_{i=1}^{n} \frac{(-1)^{i-1} 2^{i-1} t_{3i}(x)}{r_{3i-2}(x)\, r_{3i+1}(x)} = 1 + \frac{(-1)^{n-1} 2^n}{r_{3n+1}(x)}, \tag{9.15}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1} 2^{i-1} t_{3i}(x)}{r_{3i-2}(x)\, r_{3i+1}(x)} = 1 . \tag{9.16}$$

In the case of the last summation, x must be chosen so that $\alpha^3 > 2$ or $|\beta^3| > 2$. The latter condition certainly holds if $x \leq d$.

Again all these formulas lead to summations involving Fibonacci and Lucas numbers. Thus, from (9.1) and (9.2),

$$\sum_{i=1}^{n} \frac{F_i + (-1)^i}{(F_{i+2}-1)(F_{i+3}-1)} = 2 - \frac{F_{n+4}-1}{F_{n+3}-1}, \tag{9.17}$$

$$\sum_{i=1}^{\infty} \frac{F_i + (-1)^i}{(F_{i+2}-1)(F_{i+3}-1)} = 2 - a_1, \tag{9.18}$$

where $a_1 = -\beta(-1) = (\sqrt{5}+1)/2$. We also obtain from (9.7), (9.8) and (9.15), (9.16)

$$\sum_{i=1}^{n} \frac{F_i + (-1)^i}{(L_i+1)(L_{i+1}+1)} = \frac{1}{3}\left(\frac{F_{n+3}-1}{L_{n+1}+1} - \frac{1}{2}\right), \tag{9.19}$$

$$\sum_{i=1}^{\infty} \frac{F_i + (-1)^i}{(L_i+1)(L_{i+1}+1)} = \frac{1}{3}\left(\frac{a_1^2}{\sqrt{5}} - \frac{1}{2}\right), \tag{9.20}$$

$$\sum_{i=1}^{n} \frac{2^{i-1}(L_{3i}+1)}{(F_{3i}-1)(F_{3i+3}-1)} = 1 - \frac{2^n}{F_{3n+3}-1}, \tag{9.21}$$

$$\sum_{i=1}^{\infty} \frac{2^{i-1}(L_{3i}+1)}{(F_{3i}-1)(F_{3i+3}-1)} = 1 . \tag{9.22}$$

10. CONVOLUTIONS OF PELL DIAGONAL FUNCTIONS

Using (1.1) as a difference equation, we readily obtain the generating function for Pell diagonal functions:

$$\sum_{n=0}^{\infty} r_{n+1}(x)\, y^n = 1/(1-2xy-y^3). \tag{10.1}$$

If we define the k^{th} convolution sequence derived from the Pell Diagonal functions thus

$$r_n^{(k)}(x) = \sum_{i=1}^{n} r_i(x)\, r_{n+1-i}^{(k-1)}(x), \tag{10.2}$$

it is relatively simple to establish the next generating function:

$$\sum_{i=0}^{\infty} r_{n+1}^{(k)}(x)\, y^n = 1/(1-2xy-y^3)^{k+1} . \tag{10.3}$$

From this it follows that

$$r_n^{(k)}(x) = \sum_{i=1}^{n} r_i^{(j)}(x)\, r_{n+1-i}^{(k-1-j)}(x) \quad (0 \leq j \leq k-1) . \tag{10.4}$$

We record some of the elementary identities for the sequence $\{r_n^{(k)}(x)\}$ which are readily obtained by adapting procedures used with Gegenbauer, Humbert and Generalised Humbert polynomials [10]:

$$r_{n+1}^{(k)}(x) - 2xr_n^{(k)}(x) - r_{n-2}^{(k)}(x) = r_{n+1}^{(k-1)}(x), \tag{10.5}$$

$$\left.\begin{aligned} nr_{n+1}^{(k-1)}(x) &= k\{2xr_n^{(k)}(x) + 3r_{n-2}^{(k)}(x)\} \\ &= k\sum_{i=1}^{n} t_i(x)\, r_{n+1-i}^{(k-1)}(x), \end{aligned}\right\} \tag{10.6}$$

$$nr_{n+1}^{(k)}(x) - 2x(n+k)\, r_n^{(k)}(x) - (n+3k)\, r_{n-2}^{(k)}(x) = 0, \tag{10.7}$$

$$(n-1+k)\, r_n^{(k-1)}(x) = k\{r_n^{(k)}(x) + 2r_{n-3}^{(k)}(x)\}. \tag{10.8}$$

Among the results which can be established by getting the derivative of the generating function (10.3) are

$$D_x\,(r_{n+1}^{(k)}(x)) = 2(k+1)\, r_n^{(k+1)}(x), \tag{10.9}$$

$$D_x\,(t_n(x)) = 2nr_n(x), \tag{10.10}$$

$$D_x^a\,(r_{n+a}^{(k)}(x)) = 2^a(k+a)\,!\, r_n^{(k+a)}(x), \tag{10.11}$$

$$D_x^k\,(r_{n+k}^{(k)}(x)) = 2(k+1)\,!\, K_k r_n^{(2k)}(x), \tag{10.12}$$

$$r_n^{(a)}(x) = D_x^a\,(r_{n+a}(x))/(2^a a\,!\,), \tag{10.13}$$

where K_k is the k^{th} <u>Catalan</u> <u>number</u>, given by

$$K_k = \binom{2k}{k}/(k+1). \tag{10.14}$$

We conclude this section by quoting an explicit summation formula for $r_n^{(k)}(x)$ found from (10.3)

$$r_n^{(k)}(x) = \sum_{i=0}^{[(n-1)/3]} \binom{k+n-1-2i}{k}\binom{n-1-2i}{i}(2x)^{n-1-3i} \tag{10.15}$$

and an inversion formula

$$(2x)^n\binom{k+n}{n} = \sum_{i=0}^{[n/3]} (-1)^i\, \frac{k+1+n-3i}{k+1+n-i}\binom{k+n}{i}\, r_{n+1-3i}^{(k)}(x). \tag{10.16}$$

11. CONCLUSION

Formulas obtained here are, for the most part, new though the procedures are well known. They are often lacking in aesthetic appeal and are highly specific. The absence of a high level of generalisation may be a feature of third order sequences. On the other hand, the development of further concepts and techniques may point the way to more comprehensive generalisations.

The results obtained in §§1-7 and 10 are not at all typical of third order sequences. By studying a generalisation of the Pell diagonal function in §8 and more fully in [10] and [12], formulas more characteristic of third order sequences have been found.

Despite the shortcomings of our study, it still seems to us that the Pell diagonal functions provide a rich field awaiting further exploitation. Moreover, it does enable us to obtain easily a large number of new identities for the Fibonacci and Lucas numbers, and points the way to a method of obtaining many more for second order sequences of numbers generally.

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MULTIVARIATE FIBONACCI POLYNOMIALS OF ORDER K AND THE MULTIPARAMETER NEGATIVE BINOMIAL DISTRIBUTION OF THE SAME ORDER

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1. INTRODUCTION AND SUMMARY

Unless otherwise explicitly stated, in this paper k and r are fixed positive integers, n and n_i $(1\leq i\leq k)$ are non-negative integers as specified, p and q_i $(1\leq i\leq k)$ are real numbers in the interval (0,1) which satisfy the relation $p+q_1+\cdots+q_k=1$, and x and x_i $(1\leq i\leq k)$ are real numbers in the interval $(0,\infty)$. Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k, i.e. $F_0^{(k)}(x)=0$, $F_1^{(k)}(x)=1$, and

$$F_n^{(k)}(x) = \begin{cases} x\sum_{i=1}^{n} F_{n-i}^{(k)}(x) & \text{if } 2\leq n\leq k+1, \\ x\sum_{i=1}^{k} F_{n-i}^{(k)}(x) & \text{if } n\geq k+2. \end{cases} \tag{1.1}$$

This definition is due to Philippou, Georghiou and Philippou [8] (see also [6]), who obtained the following results:

$$\sum_{n=0}^{\infty} s^n F_{n+1}^{(k)}(x) = \frac{1}{1-xs-\ldots-xs^k}, \quad |s| < 1/(1+x), \tag{1.2}$$

and

$$F_{n+1}^{(k)}(x) = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} x^{n_1+\ldots+n_k}, \quad n\geq 0. \tag{1.3}$$

Let now $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself. Then

$$\sum_{n=0}^{\infty} s^n F_{n+1,r}^{(k)}(x) = \frac{1}{(1-xs-\ldots-xs^k)^r}, \quad |s| < 1/(1+x), \tag{1.4}$$

$$F_{n+1,r}^{(k)}(x) = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k+r-1}{n_1,\ldots,n_k,r-1} x^{n_1+\ldots+n_k}, \quad n\geq 0, \tag{1.5}$$

and

$$F_{n+1,r}^{(k)}(x) = \frac{x}{n}\sum_{i=1}^{k} [n+i(r-1)]\, F_{n+1-i,r}^{(k)}(x), \quad n\geq 1. \tag{1.6}$$

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The above results are due to Philippou and Georghiou [5], who have also derived simple recurrences for computing the probabilities of the two types of negative binomial distributions of order k.

In the present paper we introduce the multivariate Fibonacci polynomials of order k, to be denoted by $\{H_n^k(\underline{x})\}_{n=0}^{\infty}$ (see Definition 2.1), and derive appropriate analogs of (1.2)-(1.6) for $H_n^{(k)}(\underline{x})$ (see Theorems 2.1-2.3, Lemma 2.1, and relation (2.2)). Furthermore, we derive a recurrence for computing the probabilities of the multiparameter negative binomial distribution of order k of Philippou [3] (see Theorem 3.1), and we provide a new proof of a recurrence of Aki [1] along with a new result for his shifted extended negative binomial distribution of order k.

2. MULTIVARIATE FIBONACCI POLYNOMIALS OF ORDER K MULTINOMIAL EXPANSIONS AND CONVOLUTIONS

In this section we introduce the sequence of multivariate Fibonacci polynomials of order k, to be denoted by $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$, along with the (r−1)-fold convolution of $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ with itself, to be denoted by $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ and derive expansions for $H_n^{(k)}(\underline{x})$ and $H_{n,r}^{(k)}(\underline{x})$, in terms of the multinomial coefficients. We also derive a linear recurrence for $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ with variable coefficients.

Definition 2.1: The sequence of polynomials $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ is said to be the sequence of multivariate Fibonacci polynomials of order k, if

$$H_0^{(k)}(\underline{x}) \equiv H_0^{(k)}(x_1, \ldots, x_k)=0,\ H_1^{(k)}(\underline{x})=1, \text{ and}$$

$$H_n^{(k)}(\underline{x}) = \begin{cases} \sum_{i=1}^{n} x_i H_{n-i}^{(k)}(\underline{x}) & \text{if } 2 \le n \le k+1, \\ \sum_{i=1}^{k} x_i H_{n-i}^{(k)}(\underline{x}) & \text{if } n \ge k+2. \end{cases}$$

It follows from the definition of $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ that

$$H_n^{(k)}(x,\ldots,x) = F_n^{(k)}(x), \qquad n \ge 0,$$

and

$$H_n^{(k)}(x^{k-1},x^{k-2},\ldots,1) = f_n^{(k)}(x), \qquad n \ge 0,$$

where $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ is given by (1.1) and $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ is the sequence of Fibonacci polynomials of order k.

Theorem 2.1: Let $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the sequence of multivariate Fibonacci polynomials of order k. Then

$$H_{n+1}^{(k)}(\underline{x}) = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} x_1^{n_1}\cdots x_k^{n_k}, \quad n \ge 0.$$

We shall first establish the following lemma:

Lemma 2.1: Let $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the sequence of multivariate Fibonacci polynomials of order k, and denote its generating function by $G_k(s;x)$. Then, for $|s| < 1/(1+M)$, where $M=\max\{x_1,\ldots,x_k\}$,

$$G_k(s;\underline{x}) = \frac{s}{1-\sum_{i=1}^{k} x_i s^i}.$$

Proof: We see from Definition 2.1 that if $M=\max\{x_1,\ldots,x_k\}$ then

$$\begin{aligned} H_n^{(k)}(x_1,\ldots,x_k) &\leq H_n^{(k)}(M,\ldots,M) \\ &= F_n^{(k)}(M) \\ &= \begin{cases} M(1+M)^{n-2}, & 2\leq n\leq k+1, \\ (1+M)F_{n-1}^{(k)}(M)-MF_{n-1-k}^{(k)}(M), & n\geq k+2, \end{cases} \\ &\leq M(1+M)^{n-2},\ n\geq 2, \text{ by induction on n,} \end{aligned}$$

which shows the convergence of $G_k(s;\underline{x})$ for $|s| < 1/(1+M)$.

Next, be means of Definition 2.1, we have

$$\begin{aligned} G_k(s;\underline{x}) &= \sum_{n=0}^{\infty} s^n H_n^{(k)}(\underline{x}) = s + \sum_{n=2}^{k+1} s^n H_n^{(k)}(\underline{x}) + \sum_{n=k+2}^{\infty} s^n H_n^{(k)}(\underline{x}) \\ &= s + \sum_{n=2}^{k+1} s^n \sum_{i=1}^{n} x_i H_{n-i}^{(k)}(\underline{x}) + \sum_{n=k+2}^{\infty} s^n \sum_{i=1}^{k} x_i H_{n-i}^{(k)}(\underline{x}) \\ &= s + \sum_{i=1}^{k} x_i s^i \sum_{n=i+1}^{k+1} s^{n-i} H_{n-i}^{(k)}(\underline{x}) + \sum_{i=1}^{k} x_i s^i \sum_{n=k+2}^{\infty} s^{n-i} H_{n-i}^{(k)}(\underline{x}) \\ &= s + \sum_{i=1}^{k} x_i s^i \left[\sum_{n=i+1}^{k+1} s^{n-i} H_{n-i}^{(k)}(\underline{x}) + \sum_{n=k+2}^{\infty} s^{n-i} H_{n-i}^{(k)}(\underline{x}) \right] \\ &= s + \sum_{i=1}^{k} x_i s^i \left[\sum_{n=1}^{k+1-i} s^{n} H_{n}^{(k)}(\underline{x}) + \sum_{n=k+2-i}^{\infty} s^{n} H_{n}^{(k)}(\underline{x}) \right] \\ &= s + \left(\sum_{i=1}^{k} x_i s^i \right) G_k(s;\underline{x}), \end{aligned}$$

from which the lemma follows.

Proof of Theorem 2.1: It follows by expanding the generating function $G_k(s;\underline{x})$ in a Taylor series about s=0 as in the proof of Theorem 2.1 of [8].

Now let $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ with itself, i.e. $H_{0,r}^{(k)}(\underline{x})=0$, and for $n\geq 1$

$$H_{n,r}^{(k)}(\underline{x}) = \begin{cases} H_n^{(k)}(\underline{x}), & \text{if } r=1, \\ \sum_{i=1}^{n} H_{i,r-1}^{(k)}(\underline{x}) H_{n+1-i}^{(k)}(\underline{x}), & \text{if } r\geq 2. \end{cases} \tag{2.1}$$

As a consequence of (2.1) and in view of Lemma 2.1, we get

$$\sum_{n=0}^{\infty} s^n H_{n+1,r}^{(k)}(\underline{x}) = \frac{1}{(1-\sum_{i=1}^{k} x_i s^i)^r}, \quad |s| < 1/(1+M). \tag{2.2}$$

Expanding now (2.2) as a Taylor series about s=0, we readily find the following closed formula for $\{H_{n,r}^{(k)}(\underline{x})\}_{n=1}^{\infty}$, in terms of the multinomial coefficients.

<u>Theorem 2.2:</u> Let $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ with itself. Then

$$H_{n+1,r}^{(k)}(\underline{x}) = \sum_{\substack{n_1,\dots,n_k \ni \\ n_1+2n_2+\dots+kn_k=n}} \binom{n_1+\dots+n_k+r-1}{n_1,\dots,n_k,r-1} x_1^{n_1}\dots x_k^{n_k}, \quad n\geq 0.$$

We note next that $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ satisfies the following linear recurrence with variable coefficients.

<u>Theorem 2.3:</u> Let $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{H_n^{(k)}(\underline{x})\}_{n=0}^{\infty}$ with itself, and set $H_{n,r}^{(k)}(\underline{x})=0$ for $-k+1\leq n\leq -1$. Then, $H_{0,r}^{(k)}(\underline{x})=0$, $H_{1,r}^{(k)}(\underline{x})=1$, and

$$H_{n+1,r}^{(k)}(\underline{x}) = \sum_{i=1}^{k} \frac{x_i}{n}[n+i(r-1)]\, H_{n+1-i,r}^{(k)}(\underline{x}), \quad n\geq 1.$$

<u>Proof:</u> It follows along the same lines as Theorem 2.2 of [5]. In fact, letting $|s| < 1/(1+M)$, where $M=\max\{x_1,\dots,x_k\}$, and noting that

$$(1+x_1s-\dots-x_ks^k)^{-r} = (1-x_1s-\dots-x_ks^k)^{-r-1}(1-\sum_{i=1}^{k}s^i) \tag{2.3}$$

and

$$s^i(1-x_1s-\dots-x_ks^k)^{-r-1} = \sum_{n=0}^{\infty} s^n H_{n+1-i,r+1}^{(k)}(\underline{x}), \quad 1\leq i\leq n, \text{ by (2.2)}, \tag{2.4}$$

we get

$$\sum_{n=0}^{\infty} s^n H_{n+1,r}^{(k)}(\underline{x}) = \sum_{n=0}^{\infty} s^n H_{n+1,r+1}^{(k)}(\underline{x}) - \sum_{i=1}^{k} x_i \sum_{n=0}^{\infty} s^n H_{n+1-i,r+1}^{(k)}(\underline{x}),$$

by (2.2), (2.3) and (2.4).

Therefore,

$$H_{n+1,r}^{(k)}(\underline{x}) = H_{n+1,r+1}^{(k)}(\underline{x}) - \sum_{i=1}^{k} x_i H_{n+1-i,r+1}(\underline{x}), \quad n\geq 0.$$

Next, differentiating both sides of (2.2) with respect to s, we get

$$\sum_{n=0}^{\infty} n s^{n-1} H_{n+1,r}^{(k)}(\underline{x}) = r\sum_{i=1}^{k} i x_i \sum_{n=0}^{k} s^n H_{n+2-i,r+1}^{(k)}(\underline{x}), \text{ by (2.2) and (2.4)},$$

which implies

$$nH_{n+1,r}^{(k)}(\underline{x}) = r\sum_{i=1}^{k} i x_i H_{n+1-i,r+1}^{(k)}(\underline{x}), \quad n\geq 1.$$

Combining (2.5) and (2.6), we obtain

$$H_{n+1,r+1}^{(k)}(\underline{x}) = \sum_{i=1}^{k} \frac{x_i}{n}(n+ir)\, H_{n+1-i,r+1}^{(k)}(\underline{x}), \quad n\geq 1,$$

which, along with the definition of $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$, establishes the theorem.

3. MULTIBARIATE FIBONACCI POLYNOMIALS OF ORDER K AND PROBABILITY APPLICATIONS

In the present section we employ Theorems 2.2 and 2.3 to derive a recurrence for the following multiparameter negative binomial distribution of order k [3].

Definition 3.1: A random variable X is said to have the negative binomial distribution of order k with paramettoers $r,q_1,\ldots,q_k$ ($r>0$, $0<q_i<1$ for $1\le i\le k$, and $0<q_1+\ldots+q_k<1$), to be denoted by $NB_k(r;q_1,\ldots,q_k)$, if

$$P(X=n)=p^r \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k+r-1}{n_1,\ldots,n_k,r-1} q_1^{n_1}\ldots q_k^{n_k}, \quad n=0,1,\ldots,$$

where $p=1-q_1\ldots-q_k$.

It was noted by Philippou [3] that if $q_i=P^{i-1}Q$ $(1\le i\le k)$, so that $p=P^k$, then $NB_k(r:q_1,\ldots,q_k)=\overline{NB}_{k,I}(r,P)$, where $\overline{NB}_{k,I}(r,P)$ is the shifted negative binomial distribution of order k [2,7]. If $q_i=q/k$ $(1\le i\le k)$, then $NB_k(r;q_1,\ldots,q_k)=NB_{k,II}(r,p)$, where $NB_{k,II}(r,p)$ denotes the compound Poisson (or negative binomial) distribution of order k with $p=\alpha/(\alpha+k)$, where α is a parameter for the compound Poisson distribution of order k (as in [4]). Finally, if $q_1=Q_1$ and $q_i=P_1\ldots P_{i-1}Q_i$ $(2\le i\le k)$, which imply $p=P_1\ldots P_k$, then $NB_k(r;q_1,\ldots,q_k)=\overline{ENB}_k(r;P_1,\ldots,P_k)$, where $\overline{ENB}_k(r;P_1,\ldots,P_k)$ is the shifted extended negative binomial distribution of order k of Aki [1].

As a consequence of Theorem 2.2 and Definition 3.1 we have the following relationship.

Proposition 3.1: Let X be a random variable distributed as $NB_k(r,q_1,\ldots,q_k)$ and let $\{H_{n,r}^{(k)}(\underline{x})\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of $\{H_n^{(k)}(\underline{x})\}$ with itself. Then

$$P(X=n)=p^r H_{n+1,r}^{(k)}(q_1,\ldots,q_k), \quad n\ge 0.$$

Using the first two of the above transformations, respectively, Proposition 3.1 reduces to the following corollaries.

Corollary 3.1: Let X be a random variable distributed as $\overline{NB}_{k,I}(r,P)$. Then

$$P(X=n)=P^{n+kr}F_{n+1,r}^{(k)}(Q/P), \quad n\ge 0.$$

Corollary 3.2: Let X be a random variable distributed as $\overline{NB}_{k,II}(r,p)$. Then

$$P(X=n)=p^r F_{n+1,r}^{(k)}(q/k), \quad n\ge 0.$$

Theorem 2.3 and Proposition 3.1 imply

Theorem 3.1: Let X be a random variable distributed as $NB_k(r;q_1,\ldots,q_k)$, and set $\tilde{P}_n=P(X=n)$. Then

$$\tilde{P}_n=\begin{cases} 0, & n\leq -1,\\ p^r, & n=0,\\ \sum_{i=1}^{k}\frac{q_i}{n}[n+i(r-1)]\tilde{P}_{n-i}, & n\geq 1.\end{cases}$$

Proof: For $n\leq -1$, $(X=n)=\emptyset$, which implies $\tilde{P}_n=P(\emptyset)=0$. For n=0, Definition 3.1 gives $\tilde{P}_n=p^r$. For $n\geq 1$, we have

$$\begin{aligned}\tilde{P}_n&=p^rH^{(k)}_{n+1,r}(q_1,\ldots,q_k), \text{ by Proposition 3.1},\\ &=\sum_{i=1}^{k}\frac{q_i}{n}[n+i(r-1)]\ p^rH^{(k)}_{n+1-i,r}(q_1,\ldots,q_k), \text{ by Theorem 2.3},\\ &=\sum_{i=1}^{k}\frac{q_i}{n}[n+i(r-1)]\ \tilde{P}_{n-i}, \text{ by Proposition 3.1}.\end{aligned}$$

Theorem 3.1 has the following two corollaries.

Corollary 3.3: Let X be a random variable distributed as $\overline{NB}_{k,I}(r,P)$, and set $\tilde{P}_n=P(X=n)$. Then

$$\tilde{P}_n=\begin{cases} 0, & n\leq -1,\\ P^{kr}, & n=0,\\ \sum_{i=1}^{k}\frac{P^{i-1}Q}{n}[n+i(r-1)]\tilde{P}_{n-i}, & n\geq 1.\end{cases}$$

Corollary 3.4: Let X be a random variable distributed as $NB_{k,II}(r,p)$, and set $\tilde{P}_n=P(X=n)$. Then

$$\tilde{P}_n=\begin{cases} 0, & n\leq -1,\\ p^r, & n=0,\\ \frac{q}{kn}\sum_{i=1}^{k}[n+i(r-1)]\ \tilde{P}_{n-i}, & n\geq 1.\end{cases}$$

Remark 3.1: Corollaries 3.1 and 3.3 are the shifted versions of Proposition 3.1 and Theorem 3.1 of [5], and Corollaries 3.2 and 3.4 are the same as Proposition 3.2 and Theorem 3.2 of [5].

We finally note that, if we use the third tranformation mentioned above, we get the following corollary.

Corollary 3.5: Let X be a random variable distributed as $\overline{ENB}_k(r;P_1,P_2,\ldots,P_k)$, and set $\tilde{P}_n=P(X=n)$. Then

$$\tilde{P}_n=\begin{cases} 0, & n\leq -1,\\ (P_1P_2\ldots P_k)^r, & n=0,\\ \sum_{i=1}^{k}\frac{P_1\ldots P_{i-1}Q_i}{n}[n+i(r-1)]\ \tilde{P}_{n-i}, & n\geq 1.\end{cases}$$

The specialization of Corollary 3.5 for r=1, is the shifted version of Proposition 3.1 of Aki [1].

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LONGEST CIRCULAR RUNS WITH AN APPLICATION IN RELIABILITY VIA THE FIBONACCI-TYPE POLYNOMIALS OF ORDER K

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1. INTRODUCTION AND SUMMARY

Unless otherwise stated, in this paper k and n are possitive integers, n_j $(1 \le j \le k+1)$ are non-negative integers as specified, p and x are real numbers in the intervals (0,1) and $(0,\infty)$, respectively, and $q=1-p$.

In a recent paper, Philippou and Makri (1985) considered the length of the longest success run in n Bernoulli trials, and they obtained its distribution function, probability generating function and factorial moments, all in terms of the multinomial coefficients, as well as in terms of the Fibonacci-type polynomials of order k of Philippou et al (1985), properly extended to include k=0 and k=1. This paper was subsequently employed by Philippou (1986) to derive three new formulas for the reliability of the so-called linear-consecutive-k-out-of-n:F system, and to provide a new proof of a recursive formula of Shanthikumar (1982).

In the present paper, we assume that the outcomes of n Bernoulli trials are ordered circularly, so that the first outcome is adjacent to and follows the nth, and we consider the length L_n of the longest circular run of successes in these n trials. Modifying a combinatorial argument of Philippou (1987), we derive the distribution function $F_n(k)$ of L_n in terms of the Fibonacci-type polynomials of order k+1 (see Theorem 3.1). We also obtain the mean of L_n and a simple recurrence on n for $F_n(k)$ (see Proposition 3.1 and Theorem 3.2).

Since our present model provides a theoretical description of what reliability engineers call circular-consecutive-k-out-of-n:F system, we get directly a useful recurrence for the reliability of such a system, as a simple corollary of Theorem 3.2 (see Corollary 3.2).

2. AUXILIARY RESULTS

For easy reference we gather together in Lemma 2.1 below a few results of Philippou et al (1985) and Philippou (1986) on Fibonacci-type polynomials of order k, which will be instrumental in the derivations of Section 3.

We recall first the following definition of Philippou et. al. (1985), as modified by Philippou and Makri (1985).

Definition 2.1: The sequence of polynomials $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequence of Fibonacci-type polynomials of order k if $F_0^{(k)}(x) = 0$, $F_1^{(k)}(x) = 1$, and

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$$F_n^{(k)}(x) = \begin{cases} x[F_{n-1}^{(k)}(x) + \ldots + F_1^{(k)}(x)], & 2 \leq n \leq k+1, \\ x[F_{n-1}^{(k)}(x) + \ldots + F_{n-k}^{(k)}(x)], & n \geq k+2. \end{cases}$$

Lemma 2.1: Let $\{F_n^{(k)}(.)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k. Then

(a) $F_{n+1}^{(k)}(q/p) \sum_{\sum_{i=1}^{k} in_i=n} \binom{n_1+\ldots+n_k}{n_1, \ldots, n_k}(q/p)^{n_1+\ldots+n_k}$, $n \geq 0$;

(b) $F_n^{(k)}(q/p) = qp^{1-n}$, $2 \leq n \leq k+1$;

(c) $F_n^{(k)}(q/p) = \frac{1}{p}F_{n-1}^{(k)}(q/p) - \frac{q}{p}F_{n-1-k}^{(k)}(q/p)$, $n \geq k+2$;

(d) $F_n^{(k)}(q/p) = qp^{1-n}\{1-p^k[1+(n-2-k)q]\}$, $k+2 \leq n \leq 2k+2$.

3. LONGEST CIRCULAR RUNS WITH AN APPLICATION IN RELIABILITY

We shall first derive the distribution function $F_n(.)$ of L_n, by modifying a combinatorial argument of Philippou (1987). See, also, Philippou and Muwafi (1982), and Philippou and Makri (1985), (1986) and (1987).

Theorem 3.1: Let L_n be a random variable denoting the length of the longest circular run of successes in n Bernoulli trials and denote its distribution function by $F_n(.)$. Also let $F_n^{(k)}(.)$ be the Fibonacci-type polynomials of order k. Then

$$F_n(k) = \begin{cases} 0, & k \leq -1, \\ qp^{n-1} \sum_{i=0}^{k} (i+1)F_{n-i}^{(k+1)}(q/p), & 0 \leq k \leq n-1, \\ 1, & k \geq n. \end{cases}$$

Proof: The theorem is trivially true for both $k \leq -1$ and $k \geq n$, since $(L_n \leq k)$ reduces to an empty set for $k \leq -1$, while it becomes the sample space for $k \geq n$. Consider then the case $0 \leq k \leq n-1$, and dentoe by S and F, respectively, success and failure. A typical element of the event $(L_n \leq k)$ is a circular arrangement

$$\underbrace{SS\ldots S}_{\alpha}FO_1\ldots O_{n_1+\ldots+n_{k+1}}\underbrace{SS\ldots S}_{\beta}, \tag{3.1}$$

which means that the first outcome is adjacent to and follows the last one, such that n_j of the O's are $e_j = \underbrace{SS\ldots S}_{j-1}F$ $(1 \leq j \leq k+1)$, and

$$\textstyle\sum_{j=1}^{k+1} jn_j = n-1-\alpha-\beta; \quad 0 \leq \alpha \leq k, \quad 0 \leq \beta \leq k \quad \text{and } \alpha+\beta \leq k. \tag{3.2}$$

If $n_1, \ldots, n_{k+1}$, α and β are kept fixed, then the number of the above arrangements (3.1) is

$$\binom{n_1 + \ldots + n_{k+1}}{n_1, \ldots, n_{k+1}},$$

and each one has probability

$$qq^{n_1 + \ldots + n_{k+1}} p^{n-1-(n_1 + \ldots + n_{k+1})} = qp^{n-1}\left(\frac{q}{p}\right)^{n_1 + \ldots + n_{k+1}},$$

by the independence of the trials, the definition of e_j $(1 \leq j \leq k+1)$, $P\{S\} = p$ and $P\{F\} = q$. But the non-negative integers $n_1, \ldots, n_{k+1}$, α and β may vary subject to (3.2). Therefore,

$$\begin{aligned} F_n(k) &= P(L_n \leq k) \\ &= qp^{n-1} \sum_{\substack{\alpha=0 \\ \alpha+\beta \leq k}}^{k} \sum_{\beta=0}^{k} \sum_{\sum_{j=1}^{k+1} jn_j = n-1-\alpha-\beta} \binom{n_1 + \ldots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \ldots + n_{k+1}} \\ &= qp^{n-1} \sum_{i=0}^{k} (i+1) \sum_{\sum_{j=1}^{k+1} jn_j = n-1-i} \binom{n_1 + \ldots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \ldots + n_{k+1}} \\ &= qp^{n-1} \sum_{i=0}^{k} (i+1) F_{n-i}^{(k+1)}(q/p), \text{ by Lemma 2.1(a),} \end{aligned}$$

and this completes the proof of the theorem.

We now note

<u>Proposition 3.1:</u> Let L_n and $F_n^{(k)}(.)$ be as in Theorem 3.1. Then

$$E(L_n) = n - qp^{n-1} \sum_{k=0}^{n-1} \sum_{i=0}^{k} (i+1) F_{n-i}^{(k+1)}(q/p).$$

<u>Proof:</u> In fact, Theorem 3.1 gives

$$\begin{aligned} E(L_n) &= \sum_{k=0}^{n} [1 - F_n(k)] = \sum_{k=0}^{n-1} [1 - F_n(k)] \\ &= n - qp^{n-1} \sum_{k=0}^{n-1} \sum_{i=0}^{k} (i+1) F_{n-i}^{(k+1)}(q/p), \text{ QED.} \end{aligned}$$

The determination of the distribution function of L_n and its mean by means of Theorem 3.1 and Proposition 3.1, respectively, both involve the calculation of the Fibonacci type polynomials $F_{n-i}^{(k+1)}(.)$ for $0 \leq i \leq k$ and $0 \leq k \leq n-1$. This task may be simplified considerably by means of Lemma 2.1. The same lemma is instrumental in the derivation of the following recurrence.

Theorem 3.2: Let $F_n(.)$ be as in Theorem 3.1. Then,

(a) $F_n(k) = 1$, $1 \le n \le k$;

(b) $F_n(k) = 1-p^{k+1}$, $n=k+1$;

(c) $F_n(k) = 1-p^n-nqp^{k+1}$, $k+2 \le n \le 2k+3$;

(d) $F_n(k) = F_{n-1}(k)-qp^{k+1}F_{n-2-k}(k)$, $n \ge 2k+4$;

(e) $F_n(k) = 0$, if $k \le -1$, and $F_n(0) = q^n$.

Proof: Both (a) and (e) are trivially true.

(b) It is also true, since

$$F_{k+1}(k) = P(L_{k+1} \le k) = 1-P(L_{k+1} \ge k+1) = 1-P(L_{k+1}= k+1) = 1-p^{k+1}.$$

(c) We shall establish (c) by induction on n. We first observe that Theorem 3.1 and Definition 2.1 imply

$$\begin{aligned} &F_{n+1}(k)-pF_n(k) \\ &= qp^n[F_{n+1}^{(k+1)}(q/p) + \ldots + F_{n+1-k}^{(k+1)}(q/p)-(k+1)F_{n-k}^{(k+1)}(q/p)] \\ &= qp^n[(p/q)\, F_{n+2}^{(k+1)}\, (q/p)-(k+1)F_{n-k}^{(k+1)}(q/p)]\ , \quad n \ge k+1. \end{aligned} \tag{3.3}$$

Next, setting $n=k+1$ in (3.3), we get

$$\begin{aligned} F_{k+2}(k) &= pF_{k+1}(k)+qp^{k+1}[(p/q)F_{k+3}^{(k+1)}\, (q/p)-(k+1)F_1^{(k+1)}\, (q/p)] \\ &= p-p^{k+2} + qp^{k+1}[p^{-k-1}(1-p^{k+1})-(k+1)] = 1-p^{k+2}-(k+2)qp^{k+1}, \end{aligned}$$

by Theorem 3.2(b), Lemma 2.1(d) and Definition 2.1, which establishes (c) for $n=k+2$. Assume now that (c) is true for $n=m$ $(k+2 \le n \le 2k+2)$. We shall show that it is also true for $n=m+1$. In fact, relation (3.3), the assumption, and Lemma 2.1(b) and (d) give

$$\begin{aligned} F_{m+1}(k) &= pF_m(k) + qp^m[(p/q)F_{m+2}^{(k+1)}(q/p)-(k+1)F_{m-k}^{(k+1)}(q/p)] \\ &= p-p^{m+1}-mqp^{k+2} + qp^m\{p^{-m}\{1-p^{k+1}[1+(m-k-1)q]\}-(k+1)qp^{k+1-m}\} \\ &= 1-p^{m+1}- (m+1)qp^{k+1}, \text{ QED.} \end{aligned}$$

(d) It follows from Theorem 3.1, by means of Lemma 2.1(b) and (c). In fact, for $n \ge 2k+3$,

$$\begin{aligned} F_n(k) - F_{n-1}(k) &= qp^{n-1}\sum_{i=0}^{k}(i+1)[F_{n-i}^{(k+1)}(q/p)- \tfrac{1}{p}F_{n-1-i}^{(k+1)}(q/p)] \\ &= -q^2p^{n-2}\sum_{i=0}^{k}(i+1)F_{n-2-k-i}^{(k+1)}(q/p) \\ &= -qp^{k+1}F_{n-2-k}(k), \end{aligned}$$

which completes the proof of the theorem.

To Theorem 3.2, we have the following obvious corollary.

Corollary 3.1: Let $F_n(.)$ be as in Theorem 3.1. Then

$$F_n(k) = \begin{cases} 0, & k \leq -1, \\ q^n, & k=0 \\ F_{n-1}(k) - qp^{k+1} F_{n-2-k}(k), & 1 \leq k \leq [(n-4)/2], \\ 1-p^n - nqp^{k+1}, & [(n-2)/2] \leq k \leq n-2, \\ 1-p^{k+1}, & k=n-1, \\ 1, & k \geq n, \end{cases}$$

where [x] is the greatest integer in x.

Note that Corollary 3.1 immediately gives

$$F_n(k) = \begin{cases} 0, & k \leq -1, \\ q^n, & k=0 \\ 1-p^n - nqp^{k+1}, & 1 \leq k \leq n-2, \\ 1-p^n, & k=n-1, \\ 1, & k \geq n, \end{cases} \tag{3.4}$$

for $1 \leq n \leq 5$. For $n \geq 6$, it provides $F_n(.)$ recursively. For example,

$$F_6(k) = \begin{cases} 0, & k \leq -1, \\ q^6, & k=0, \\ 1-p^6-6qp^2+3q^2p^4, & k=1, \\ 1-p^6-6qp^{k+1}, & 2 \leq k \leq 4, \\ 1-p^6, & k=5, \\ 1, & k \geq 6, \end{cases} \tag{3.5}$$

since

$$F_6(1) = F_5(1) - qp^2 F_3(1) = 1-p^6-6qp^2+3q^2p^4 \tag{3.6}$$

by means of the corollary and relation (3.4).

We finish this paper by showing the following result, as a simple application of Theorem 3.2.

Corollary 3.2: Assume that the n components of a consecutive-k-out-of-n:F system are ordered circularly and function independently with probability p, and denote the reliability of the system by $R_n(k) = R(p;k,n)$. Then for any fixed integer $k \geq 2$, we have

(a) $R_n(k) = 1$, $\quad 1 \leq n \leq k-1$;

(b) $R_n(k) = 1-q^k$, $\quad n=k$;

(c) $R_n(k) = 1-q^n-npq^k$, $\quad k+1 \leq n \leq 2k+1$;

(d) $R_n(k) = R_{n-1}(k)-pq^kR_{n-1-k}(k)$, $\quad n \geq 2k+2$.

Proof: Denoting by $\widetilde{L}_n$ the length of the longest failure run of the n components of the system, we observe that

$$R_n(k) = P(\widetilde{L}_n \leq k-1).$$

The corollary then follows immediately from Theorem 3.2, upon replacing k with $k-1$ and interchanging p and q.

A different proof of the above recurrence was given by Lambiris and Papastavridis (1985).

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FIBONACCI NUMBERS AND AN ALGORITHM OF LEMOINE AND KÁTAI

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Dedicated to Professor Ernst S. Selmer on the occasion of his 70th birthday on February 11, 1990.

1. INTRODUCTION

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers. Let n be a positive integer. We write

$$n = a_{(1)} + a_{(2)} + \cdots + a_{(s)}, \tag{1.1}$$

where $a_{(1)}$ is the greatest element of the sequence $\leq n$, $a_{(2)}$ is the greatest element $\leq n - a_{(1)}$, and, generally, $a_{(i)}$ is the greatest element $\leq n - a_{(1)} - a_{(2)} - \cdots - a_{(i-1)}$. This algorithm for additive representation of positive integers was introduced in 1969 by Kátai ([2], [3], [4]). Lemoine had earlier considered the special cases $a_i = i^k$, $k \geq 2$ ([5], [6]) and $a_i = i(i+1)/2$ ([7]). (See [10] for further information.) The above algorithm is, in turn, a special case of a more general algorithm introduced by Nathanson [9] in 1975. The algorithm of Lemoine and Kátai is closely related to the Postage stamp problem, especially to the so-called regular representations, introduced by Hofmeister in 1963. (See [12] for more information).

In this paper we discuss the connections of the algorithm of Lemoine and Kátai with the Fibonacci numbers, defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n = 1, 2, \ldots$, and Lucas numbers, defined by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, $n = 0, 1, \ldots$.

The following basic definitions and results are taken from [8] and [10]. We denote here the set of positive integers by N.

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers with the first element equal to 1. We call it an A-*sequence* and denote by A the sequence itself or sometimes the set consisting of the elements of the sequence. We denote the number s of terms in (1.1) by h(n). If the set $\{n \in N \mid h(n) = m\}$ is nonempty for some $m \in N$, we say that y_m *exists* and define y_m to be the *smallest* element of this set. If y_m exists for every $m \in N$, we say that the Y-*sequence exists* and we denote the sequence $1 = y_1 < y_2 < \cdots$ by Y. The elements y_m are also called *minimal elements.*

Theorem 1.1: (Lord). Let y_k be given ($k \in N$). Then y_{k+1} exists if and only if there exists a number $n \in N$ such that $a_{n+1} - a_n - 1 \geq y_k$. Furthermore, if y_{k+1} exists, then $y_{k+1} = y_k + a_m$, where m is the smallest number in the set $\{n \in N \mid a_{n+1} - a_n - 1 \geq y_k\}$.

Proof: [8], [10, p. 9].□

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It follows that the Y-sequence exists if and only if the set $\{a_{n+1}-a_n \mid n \in N\}$ is not bounded.

Suppose that we have an A-sequence for which the Y-sequence exists. Then the set $\{y_{n+1}-y_n \mid n \in N\}$ is not bounded [10, p. 11], so that we have a sequence A_1, A_2, ... of A-sequences, where $A_1 = A$, $A_2 = Y$ and, generally, A_{n+1} is the Y-sequence for the A-sequence A_n ([10, p. 12]).

In this paper we are mainly concerned with the periodicity properties of the sequence A_1, A_2, ... This is a problem not discussed in [10].

Definition 1.2: We say that a sequence x_1, x_2, ... is *periodic* if there exist integers $n \geq 0$ and $h \geq 1$ such that

$$x_{n+i} = x_{n+h+i},\ i = 1, 2, \ldots . \tag{1.2}$$

Let n and h be the smallest numbers such that (1.2) holds. If $n = 0$ we say that the sequence is *purely periodic,* otherwise *ultimately periodic.* In both cases h will be called *the length of the period.*

The main new result of this paper is a necessary and sufficient condition for the sequence A_1, A_2,... to be purely periodic (Theorem 4.1). The most useful tools are Theorem 1.1 and mathematical induction.

2. SOME OLD RESULTS

The most natural connection between the Fibonacci numbers and the algorithm of Lemoine and Kátai is to consider the A-sequence $1 = a_1 < a_2 < \cdots$ where $a_i = F_{i+1}$, $i = 1, 2, \ldots$. Similarly, we can take $a_i = L_i$, $i = 1,2,\ldots$. In these cases the corresponding Y-sequences exist and are easy to determine using Theorem 1.1. Lord [8] gives the results

$$y_n = F_{2n+1}-1,\ n = 1, 2,\ldots \tag{2.1}$$

for the Y-sequence of the A-sequence $a_i = F_{i+1}$, $i = 1, 2,\ldots$ and

$$y_n = L_{2(n-1)}-1,\ n = 1,2,\ldots \tag{2.2}$$

for the Y-sequence of the A-sequence $a_i = L_i$, $i = 1, 2, \ldots$.

Suppose that $a_1 = 1$, $a_2 \in N$, $a_2 > 1$, and $a_{n+2} = a_{n+1} + a_n$, $n = 1, 2, \ldots$. This clearly defines an A-sequence, which we call a *recursive A-sequence.* It is completely determined by the value of a_2.

It follows from our definition that for a recursive A-sequence with $a_2 = 2$ we have $a_n = F_{n+1}$, $n = 1, 2,\ldots$ and for a recursive A-sequence with $a_2 = 3$ we have $a_n = L_n$, $n = 1, 2,\ldots$. In [10, p. 40] we proved the following generalization of (2.1) and (2.2):

Theorem 2.1: The Y-sequence for a recursive A-sequence with $a_2 = k$ is given by

(a) $y_i = i$ for $i < a_2 = k$,

(b) $y_{(k-1)+n} = a_{2(n+1)} - 1$, $n = 0, 1,\ldots$.

We note that (a) is trivial.

During the ICM 83 in Warsaw Helge Tverberg (Bergen, Norway) asked me the following question: *"Can the sequence $A_1, A_2 \dots$ be periodic?"* (See Definition 1.2.)

In 1984 I showed ([11]) that if the sequence A_1, $A_2, \dots$ is periodic, with a period of length h, then h is *even*. Further, there *exists* a periodic sequence A_1, $A_2, \dots$ with period length 2:

Theorem 2.2: Suppose that the integer $a_2 > 1$ is given. Let A_1 be the A-sequence $1 = a_1 < a_2 < \dots$ defined by

$$a_n = F_{2n-3} \cdot a_2 - F_{2n-4},\ n = 3, 4, \dots . \tag{2.3}$$

Then $A_3 = A_1$.

We also mention that with $a_2 = k$, the Y-sequence $Y = A_2$ corresponding to $A = A_1$ is given by

$$\begin{aligned} &y_i = i,\ i<k, \\ &y_{k+i} = F_{2i+4} \cdot k - F_{2i+3},\ i = 0, 1, \dots . \end{aligned}$$

While Theorem 2.2 indeed gives an answer to Tverberg's original question, it suggests the following problem: *Are there any periodic sequences A_1, A_2, ... with period length $h > 2$?* We shall make a deeper study of the question of Tverberg and also give an answer to the question above (in the negative, see Theorem 4.1).

3. A SECOND SOLUTION

We continue directly. Compare the following result with Theorem 2.2:

Theorem 3.1: Suppose that the integer $a_2 > 1$ is given. Let A_1 be the A-sequence $1 = a_1 < a_2 < \dots$ defined by

$$a_n = F_{2n-2} \cdot a_2 - F_{2n-4},\ n = 3, 4 \dots . \tag{3.1}$$

Then $A_3 = A_1$.

With $a_2 = k$, the Y-sequence $Y = A_2$ corresponding to $A = A_1$ is given by

$$\begin{cases} y_i = i,\ i < k, \\ y_{k+i} = F_{2i+3} \cdot k - F_{2i+1},\ i = 0, 1, \dots . \end{cases} \tag{3.2}$$

Theorem 3.1 can be proved in the same manner as the corresponding Theorem 2.2 was proved in [11]. Moreover, it will follow as a special case from results proved later in this paper (see (3.4) and Lemma 4.7).

I found the *second solution* (3.1) in 1987. Independently, and about the same time, it was also discovered by Ernst S. Selmer (Bergen, Norway). He also noted that the first solution (2.3) and the second solution (3.1) can be given in the alternative forms

$$a_n = F_{2n-3} \cdot a_2 - F_{2n-4} = \begin{cases} 2a_{n-1} - a_{n-2},\ n = 3 \\ 3a_{n-1} - a_{n-2},\ n > 3 \end{cases} \tag{3.3}$$

and

$$a_n = F_{2n-2} \cdot a_2 - F_{2n-4} = 3a_{n-1} - a_{n-2},\ n \geq 3, \tag{3.4}$$

respectively. Both (3.3) and (3.4) follow easily from the formula

$$F_{t+4} = 3F_{t+2} - F_t,\ t \geq 1.$$

Substituting the three smallest possible values of a_2 into (3.1) we easily get the following result:

a) If $a_2 = 2$, then $a_n = F_{2n-1}$, $n = 1, 2, \ldots$.

b) If $a_2 = 3$, then $a_n = F_{2n}$, $n = 1, 2, \ldots$.

c) If $a_2 = 4$, then $a_n = L_{2n-1}$, $n = 1, 2, \ldots$.

We see from this and Theorem 3.1 that both the even-indexed Fibonacci numbers F_{2n}, $n = 1, 2 \ldots$ and the odd-indexed Fibonacci numbers F_{2n-1}, $n = 1, 2 \ldots$ start a periodic sequence A_1, A_2, ... with period length 2. These purely periodic sequences are, however, essentially the same:

Theorem 3.2: The Y-sequence of the A-sequence $a_n = F_{2n-1}$, $n = 1, 2, \ldots$ is $y_n = F_{2n}$, $n = 1, 2, \ldots$. The Y-sequence of the A-sequence $a_n = F_{2n}$, $n = 1,2,\ldots$ is $y_n = F_{2n-1}$ $n = 1, 2, \ldots$.

Proof: Both results follow easily from (3.2) (or directly, using Theorem 1.1). □

4. A NECESSARY AND SUFFICIENT CONDITION FOR PURE PERIODICITY

Our purpose is to prove the following result which generalizes Theorems 2.2 and 3.1 (see (3.3) and (3.4)):

Theorem 4.1: The sequence A_1, A_2, ... is purely periodic if and only if A_1 satisfies the following conditions:

a) For every $n \in N$, either

(i) $a_{n+1} - a_n = a_n - a_{n-1}$ (we define $a_0 = 0$), or
(ii) $a_{n+1} = s_n a_n - a_{n-1}$ with $s_n \in N$, $s_n \geq 3$.

b) The condition (ii) in a) holds for infinitely many $n \in N$.

Moreover, if a) and b) hold, then $A_3 = A_1$.

Before starting with the proof of Theorem 4.1 we discuss the following *finite version of the algorithm of Lemoine and Kátai:* Let an integer $L > 1$ be given. Consider any finite sequence $\tilde{A}$: $1 = a_1 < a_2 < \cdots < a_m \leq L$. We will call it a *finite A-sequence with respect to the bound L.* Consider now a positive integer $n \leq L$. We can find the representation (1.1) and define $s = h(n)$ exactly as before, and consequently we can define the corresponding *finite Y-sequence with respect to the bound L,* $\tilde{Y}$: $1 = y_1 < y_2 < \cdots < y_k \leq L$. The following result is obvious:

Lemma 4.2: If we continue $\widetilde{A}$ in any way to an infinite A-sequence such that $L < a_{m+1} < a_{m+2} < \cdots$ and such that the Y-sequence exists, then $Y \cap [1, L] = \widetilde{Y}$.

Note that if a finite A-sequence $\widetilde{A}$: $1 = a_1 < a_2 < \cdots < a_m \leq L$ is given, then the corresponding finite Y-sequence $\widetilde{Y}$: $1 = y_1 < y_2 < \cdots < y_k \leq L$ exists *always* (in contrast to the infinite case). Therefore, given any finite A-sequence $\widetilde{A}$, we can form an infinite sequence of finite A-sequences $\widetilde{A}_1, \widetilde{A}_2, \ldots$ where $\widetilde{A}_1 = \widetilde{A}$, $\widetilde{A}_2 = \widetilde{Y}$, and, generally, $\widetilde{A}_{n+1}$ is the finite Y-sequence corresponding to the sequence $\widetilde{A}_n$ (all with respect to the bound L).

Remark 4.3: It was suggested to me by Kalevi Suominen (Helsinki, Finland) that I should study the above finite version of the algorithm of Lemoine and Kátai. In addition to its usefulness in theoretical considerations, it has the advantage of making available (with the help of a computer) numerical evidence on the behaviour of the sequence $\widetilde{A}_1, \widetilde{A}_2, \ldots$ (which clearly must be periodic).

Using Lemma 4.2 together with Theorem 1.1 we easily get the following result:

Theorem 4.4: Suppose that $\widetilde{A}$: $1 = a_1 < a_2 < \cdots < a_m \leq L$ is a finite A-sequence with respect to the bound L. Suppose that we are given a member $y_t \leq L$ of the corresponding finite Y-sequence $\widetilde{Y}$. Let $S = \{i \in N \mid i < m, a_{i+1} - a_i - 1 \geq y_t\}$.

a) If $S \neq \emptyset$ and $j = \min S$, then $y_{t+1} = y_t + a_j$.

b) If $S = \emptyset$ then, $y_{t+1} = y_t + a_m$, if $y_t + a_m \leq L$,

$$\widetilde{Y}:\ 1 = y_1 < y_2 < \cdots < y_t \leq L, \text{ if } y_t + a_m > L.$$

It follows that we can get $\widetilde{Y}$, starting with $y_1 = 1$, by using first a) as many times as possible and then b) as many times as possible.

We are now ready to begin the proof of Theorem 4.1. The "if" and the "only if" parts are given as the two separate Lemmas 4.5 and 4.9.

Lemma 4.5: Suppose that A: $1 = a_1 < a_2 < \cdots$ is an infinite A-sequence which satisfies the conditions a) and b) of Theorem 4.1. Then $A_3 = A_1$.

Proof: We start by introducing some notation and terminology. Remember that we defined $a_0 = 0$. We give names for the two conditions (i) and (ii) in a): we call (i) *a step* and (ii) *a jump*. Let $n_1^*, n_2^*, \ldots$ be the sequence of the integers n which correspond to a jump. Let $s_i^* = s_{n_i^*}$ and $a_i^* = a_{n_i^*}$. We define $d_n = y_{n+1} - y_n$. Let $t_i \geq 0$ be the number of steps between the $(i-1)^{th}$ and the i^{th} jump. We can express our A-sequence uniquely as the sequence of pairs (t_1, s_1^*), (t_2, s_2^*),....

Lemma 4.6: The "D-sequence" $d_1, d_2, \ldots$ of the Y-sequence corresponding to the A-sequence (t_1, s_1^*), (t_2, s_2^*), ... has the form

$$\underbrace{a_1^*, a_1^*, \ldots, a_1^*}_{s_1^* - 2},\ \underbrace{a_2^*, a_2^*, \ldots, a_2^*}_{s_2^* - 2},\ \ldots . \tag{4.1}$$

Proof: We consider the finite A-sequence $\widetilde{A} = A \cap [1, L]$ with respect to the bound $L = a_{n_i^*+1}$, determine the corresponding finite Y-sequence $\widetilde{Y}$ and use induction on i.

1° $i = 1$. We consider two cases A) $t_1 = 0$ and B) $t_1 > 0$.

A) $t_1 = 0$. In this case the A-sequence starts with a jump: $a_1 = 1$, $a_2 = s_1^* a_1^* - a_0 = s_1^* \cdot 1 - 0 = s_1^* \geq 3$. It follows that the corresponding finite Y-sequence is

$$y_1 = 1,\ y_2 = 2,\ \ldots,\ y_{s_1^*-1} = s_1^* - 1 = a_{n_1^*+1} - a_1^*,$$

and so

$$d_j = 1 = a_1^*,\ j = 1, 2, \ldots, s_1^* - 2.$$

B) $t_1 > 0$. In this case the A-sequence starts with t_1 steps: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, ..., $a_{t_1+1} = t_1 + 1$. Therefore $n_1^* = t_1 + 1$, $a_1^* = a_{t_1+1} = t_1 + 1$ and $a_{n_1^*+1} = s_1^* a_1^* - a_{t_1} = s_1^*(t_1 + 1) - t_1$.

We use now Theorem 4.4, starting with $y_1 = 1$ and observing that

$$a_{j+1} - a_j - 1 = 0 \text{ for } j = 1, 2, \ldots, t_1,$$

$$a_{j+1} - a_j - 1 = s_1^*(t_1 + 1) - t_1 - (t_1 + 1) - 1 = (s_1^* - 2)a_1^* \text{ for } j = t_1 + 1.$$

Note that we get new elements y_{h+1} only from the case a) of Theorem 4.4, that is, only by adding the number $a_{t_1+1} = a_1^*$ to the old element y_h. Therefore, $y_h = y_1 + (h-1)a_1^* = 1 + (h-1)a_1^*$ for all h such that

$$1 + (h-2)a_1^* \leq (s_1^* - 2)a_1^*. \tag{4.2}$$

The greatest h which satisfies (4.2) is obviously $h = s_1^* - 1$ and so

$$y_{s_1^*-1} = 1 + (s_1^* - 2)a_1^* = a_{n_1^*+1} - a_1^*,$$

and the D-sequence of the finite Y-sequence $\widetilde{Y}$ is

$$d_j = a_1^*,\ j = 1, 2, \ldots, s_1^* - 2.$$

2° Suppose now that for the finite A-sequence $A \cap [1, L]$ with respect to the bound $L = a_{n_i^*+1}$ the corresponding finite Y-sequence is $1 = y_1 < y_2 < \cdots < y_k \leq L$ with the D-sequence

$$\underbrace{a_1^*, a_1^*, \ldots, a_1^*}_{s_1^*-2},\ \underbrace{a_2^*, a_2^*, \ldots, a_2^*}_{s_2^*-2},\ \ldots,\ \underbrace{a_i^*, a_i^*, \ldots, a_i^*}_{s_i^*-2}$$

and

$$y_k = a_{n_i^*+1} - a_i^*. \tag{4.3}$$

We take a new bound $L' = a_{n_{i+1}^*+1}$ and consider the finite A-sequence $A \cap [1, L']$. Let $1 = y_1 < y_2 < \cdots < y_{k'} \leq L'$ be the corresponding finite Y-sequence. We calculate the new elements $y_{k+1}, \ldots, y_{k'}$ starting from the old element (4.3) and using Theorem 4.4. The situation is analogous to the case 1° B) considered above. Because all steps (if there are any) between two consecutive jumps have the same length as the first of these jumps, we see that we get the new elements $y_{k+1}, \ldots, y_{k'}$ only by adding elements a_{i+1}^* to y_k.

From (4.3) and the equations

$$a_{n^*_{i+1}+1} = s^*_{i+1}a^*_{i+1} - a_{n^*_{i+1}-1},\ a_{n^*_i+1} - a^*_i = a^*_{i+1} - a_{n^*_{i+1}-1},$$

it follows that

$$y_k + (s^*_{i+1}-2)a^*_{i+1} = a_{n^*_{i+1}+1} - a^*_{i+1}. \tag{4.4}$$

From (4.4) we conclude that

$$y_{k'} = a_{n^*_{i+1}+1} - a^*_{i+1}$$

holds and that the D-sequence for the finite Y-sequence $1 = y_1 < y_2 < \cdots y_{k'} \le L'$ is

$$\underbrace{a^*_1, a^*_1, \ldots, a^*_1}_{s^*_1 - 2},\ \underbrace{a^*_2, a^*_2, \ldots, a^*_2}_{s^*_2 - 2},\ \ldots,\ \underbrace{a^*_{i+1}, \ldots, a^*_{i+1}}_{s^*_{i+1} - 2},$$

thereby completing the proof. □

Lemma 4.7: Suppose that the A-sequence satisfies the conditions a) and b) of Theorem 4.1 and is given by the sequence of pairs

$$(t_1, s^*_1),\ (t_2, s^*_2),\ \ldots \text{ (where } t_i \ge 0,\ s^*_i \ge 3,\ i = 1, 2, \ldots).$$

Then the corresponding Y-sequence also satisfies the conditions a) and b) of Theorem 4.1:

a) If $t_1 = 0$, then the Y-sequence has the form

$$(s^*_1 - 2,\ t_2 + 3),\ (s^*_2 - 3,\ t_3 + 3),\ \ldots,\ (s^*_i - 3,\ t_{i+1} + 3),\ \ldots .$$

b) If $t_1 > 0$, then the Y-sequence has the form

$$(0,\ t_1 + 2),\ (s^*_1 - 3,\ t_2 + 3), \ldots, (s^*_{i-1} - 3,\ t_i + 3),\ \ldots .$$

Proof: This is an almost immediate consequence of Lemma 4.6. From the proof of Lemma 4.6 we know how the Y-sequence starts in the two different cases $t_1 = 0$ and $t_1 > 0$. Consider now the situation when in the D-sequence $d_1, d_2, \ldots$ of the Y-sequence we move from the element a^*_i to the element a^*_{i+1}.

Suppose that

$$y_k - y_{k-1} = a^*_i \text{ and } y_{k+1} - y_k = a^*_{i+1}. \tag{4.5}$$

We want to show that

$$y_{k+1} = (t_{i+1}+3)\, y_k - y_{k-1}. \tag{4.6}$$

Our definitions give

$$a^*_{i+1} = a_{n^*_i+1} + t_{i+1}(a_{n^*_i+1} - a^*_i). \tag{4.7}$$

From (4.3), (4.5), and (4.7) we obtain (4.6). □

Lemma 4.5 now follows immediately from Lemma 4.7. □

Consider now a finite A-sequence $\tilde{A}$: $1 = a_1 < a_2 < \cdots < a_m \leq L$ with respect to the bound L. If for every positive integer $n < m$ the condition a) of Theorem 4.1 holds, we say that *$\tilde{A}$ satisfies the condition a) of Theorem 4.1.* From our results we easily get the following

Corollary 4.8: Suppose that $\tilde{A}$: $1 = a_1 < a_2 < \cdots < a_m \leq L$ is a finite A-sequence with respect to the bound L and satisfies the condition a) of Theorem 4.1. Then also $\tilde{Y}$ satisfies the condition a) of Theorem 4.1 and $\tilde{A}_3 = \tilde{A}_1$, that is, the sequence $\tilde{A}_1, \tilde{A}_2, \ldots$ is purely periodic with period length 2.

Proof: We continue $\tilde{A}$ in such a way to an infinite A-sequence that $L < a_{m+1} < a_{m+2} < \cdots$ and that the conditions a) and b) of Theorem 4.1 hold. Then we use Lemmas 4.7, 4.5, and 4.2 to get the result. □

We can now prove the other half of Theorem 4.1.

Lemma 4.9: Suppose that we have an A-sequence A for which the sequence $A_1, A_2, \ldots$ is purely periodic. Then A satisfies the conditions a) and b) of Theorem 4.1.

Proof: Suppose that $A_1, A_2, \ldots$ is purely periodic but A does not satisfy a) and b) of Theorem 4.1. If only condition b) failed, the set $\{a_{n+1}-a_n \mid n \in N\}$ would be bounded, and there would be no sequence $Y = A_2$ corresponding to $A = A_1$ (by the remark after Theorem 1.1). It follows that for some n the condition a) of Theorem 4.1 does not hold. Let k be the minimal such n. We prove that the element a_{k+1} of the A-sequence A is not a member of any sequence A_i, $i > 1$. This clearly contradicts the pure periodicity of the sequence $A_1, A_2, \ldots$.

Consider the finite A-sequence $\tilde{A} = A \cap [1, L]$ with respect to the bound $L = a_{k+1} - 1$. From the definition of k it follows that $\tilde{A}$ satisfies the condition a) of Theorem 4.1 and therefore, by Corollary 4.8, starts a purely periodic sequence $\tilde{A}_1, \tilde{A}_2, \tilde{A}_1, \tilde{A}_2, \ldots$ where also $\tilde{A}_2$ satisfies the condition a) of Theorem 4.1.

Consider now also the finite A-sequence $\tilde{A}' = A \cap [1, L']$ with respect to the bound $L' = a_{k+1}$. Since $a_{k+1} \in \tilde{A}'$ and so $h(a_{k+1}) = 1$, it is clear that $a_{k+1} \notin \tilde{A}'_2$. Therefore the finite A-sequences $\tilde{A}_2$ and $\tilde{A}'_2$, *considered only as finite sequences,* are the same, and therefore $\tilde{A}'_2$, like $\tilde{A}_2$, satisfies the condition a) of Theorem 4.1.

It follows from Corollary 4.8 that starting form $\tilde{B}' = \tilde{A}'_2$ with respect to the bound $L' = a_{k+1}$, we get a purely periodic sequence $\tilde{B}'_1, \tilde{B}'_2, \tilde{B}'_1, \tilde{B}'_2, \ldots$ where also $\tilde{B}'_2$ satisfies the condition a) of Theorem 4.1.

We have already seen that $a_{k+1} \notin \tilde{B}'_1$. Therefore it is enough to show that $a_{k+1} \notin \tilde{B}_2$ to get the conclusion that $a_{k+1} \notin A_i \cap [1, L'] = \tilde{B}'_{i-1}$, $i > 1$ and so that $a_{k+1} \notin A_i$, $i \geq 1$, our contradiction. But $\tilde{B}'_2 \cap [1, L]$, with respect to the bound L, is obviously the same as $\tilde{A}_1$ (both are actually $= A_3 \cap [1, L]$) and therefore the same as $A_1 \cap [1, L]$. Therefore, if $a_{k+1} \in \tilde{B}'_2$, this would mean that $\tilde{B}'_2$ would be the same as $\tilde{A}'_1$ and therefore the condition a) of Theorem 4.1 would not hold for $\tilde{B}'_2$. However, we have already seen that $\tilde{B}'_2$ satisfies the condition a) of Theorem 4.1. This completes the proof. □

5. MINIMAL REPRESENTATIONS

Let B be any set of positive integers and $n \in \mathbf{N}$. We say that a representation

$$n = b_{(1)} + b_{(2)} + \dots + b_{(u)}, \tag{5.1}$$

with (not necessarily distinct) elements $b_{(i)} \in B$, $i = 1, 2, \dots, u$ is *minimal* if the number u is smallest possible. If n has some representation (5.1), then it trivially also has a minimal representation, but this is not necessarily unique.

With the elements taken from an A-sequence, it is natural to ask whether the representation (1.1) is minimal, or even uniquely minimal. I originally proved the latter case for the A-sequence of the second solution (3.4). However, Selmer has pointed out to me that this is only a special case of the following general result:

Theorem 5.1: For all sequences A satisfying the conditions of Theorem 4.1, the representation (1.1) of any positive integer is minimal. The representation is uniquely minimal if and only if the case (i) does not occur.

Proof: We collect equal terms in (1.1):

$$n = e_r a_r + e_{r-1} a_{r-1} + \dots + e_2 a_2 + e_1 a_1, \quad e_i \in \{0, 1, 2, \dots\}. \tag{5.2}$$

We consider (i) and (ii) of Theorem 4.1 as substitutions

$$\begin{aligned} &\text{(i)} \quad a_{t+1} = 2a_t - a_{t-1} \\ &\text{(ii)} \quad a_{t+1} = s_t a_t - a_{t-1},\ s_t \geq 3, \end{aligned}$$

and use them in the representation (5.2) to obtain alternative representations of n, with non-negative coefficients. A result of Hofmeister ([1, Satz 2], [12, Prop. 4.2]) implies that all such alternative representations can be obtained from (5.2) by suitable combinations of the substitutions (i) and (ii).

The substitution (ii) clearly increases the coefficient sum (the number of addends), whereas (i) leaves it unaltered. Theorem 5.1 is an immediate consequence when we have shown the "only if". For this purpose, choose t such that

$$a_{t+1} = 2a_t - a_{t-1},\ a_{t+2} \geq 3a_{t+1} - a_t > 2a_t$$

(case (ii) must occur). Let $n = a_{t+1} + a_{t-1} = 2a_t$. Then $n = a_{t+1} + a_{t-1}$ is of the form (5.2), but $n = 2a_t$ is not. □

6. A QUESTION OF KLÖVE

In 1984 Torleiv Klöve (Bergen, Norway) asked me the following question: *"Can you prove that for some A-sequence A the sequence $A_1, A_2, \dots$ is not periodic?"* At that time, having just found the first solution (2.3) to Tverberg's problem, my answer was "No". But now I have changed my mind.

In the proof of the following result, the last theorem in this paper, we have the pleasure to meet again our old friends the Fibonacci numbers.

Theorem 6.1: Consider the A-sequence formed by the powers of 2, that is, the sequence $a_n = 2^{n-1}$, n = 1, 2, ... The sequence A_1, A_2, ... is not periodic.

Proof: Using Theorem 1.1 and induction, the following results are easily obtained:

a) Suppose that for some A-sequence we have $a_1 = 1$, $a_2 = 2$ and

$$a_1+a_2+\ldots+a_{n-1} \leq a_{n+1}-a_n-1 < a_1+a_2+\ldots+a_{n-1}+a_n \tag{6.1}$$

for n = 2, 3, ... Then

$$y_1 = a_1 = 1 \; ; \; y_n = y_{n-1} + a_n \; , \; n = 2, 3, \ldots . \tag{6.2}$$

b) Suppose that for some A-sequence we have $a_1 = 1$, $a_2 = 3$, and

$$2a_1+a_2+\ldots+a_{n-1} \leq a_{n+1}-a_n-1 < 2a_1+a_2+\ldots+a_{n-1}+a_n \tag{6.3}$$

for n = 2, 3, ... Then

$$y_1 = a_1 = 1; \; y_n = y_{n-1} + a_{n-1}, \; n = 2, 3, \ldots . \tag{6.4}$$

c) The sequence (6.2) satisfies the condition (6.3).

d) The sequence (6.4) satisfies the condition (6.1) .

Lemma 6.2: Let A be the A-sequence formed by the powers of 2, that is, the sequence $a_n = 2^{n-1}$, n = 1, 2, ... The sequence A_1, A_2, ... satisfies the following conditions:

a) If n is odd, the A-sequence A_n starts as F_1, F_3, ..., F_{n+2}, $F_{n+4}-1$.

b) If n is even, the A-sequence A_n starts as F_2, F_4, ..., F_{n+2}, $F_{n+4}-1$.

Proof: This special sequence A clearly satisfies the condition (6.1). It follows that when n is odd, A_n satisfies the condition (6.1) and when n is even, A_n satisfies the condition (6.3). The rest follows easily by induction. □

Theorem 6.1 is now an immediate consequence of Lemma 6.2. □

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GENERALIZATIONS OF SEQUENCES OF LUCAS AND BELL

A.G. Shannon and A.F. Horadam

Lucas defined second order "primordial" and "fundamental" recurring sequences and Bell defined third order "basic" recurring sequences. This paper defines arbitrary order generalizations of these sequences and establishes some inter-relationships amongst them. Specific examples of second and third order cases are given to illustrate the principal results. The relationship of the generalization to an extension of Pascal's triangle is noted, and an application to the spread of infectious diseases is outlined.

1. INTRODUCTION

Lucas [16] studied the two sequences $\{U^{(2)}_{0,n}\}$ and $\{U^{(2)}_{2,n}\}$ (in our notation [22]). These sequences satisfy the second order homogenous linear recurrence relation

$$U^{(2)}_{s,n} = P_{21}U^{(2)}_{s,n-1} - P_{22}U^{(2)}_{s,n-2}, \qquad n > 2 \tag{1.1}$$

s = 0, 1, 2 where the P_{2i} are arbitrary integers, with initial conditions

$$\begin{aligned} U^{(2)}_{0,1} &= 2, & U^{(2)}_{0,2} &= \alpha_{21} + \alpha_{22} \\ U^{(2)}_{1,1} &= 1, & U^{(2)}_{1,2} &= 0 \\ U^{(2)}_{2,1} &= 0, & U^{(2)}_{2,2} &= 1 \end{aligned} \tag{1.2}$$

where α_{21}, α_{22} are roots of the auxiliary equation associated with the relation (1.1), namely

$$x^2 - P_{21}x + P_{22} = 0. \tag{1.3}$$

The first few terms of these sequences are given in Table 1 for the case $P_{21} = -P_{22} = 1$:

n	1	2	3	4	5	6	7	8
$U^{(2)}_{0n}$	2	1	3	4	7	11	18	29
$U^{(2)}_{1n}$	1	0	1	1	2	3	5	8
$U^{(2)}_{2n}$	0	1	1	2	3	5	8	13

Table 1

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 299–309.

Thus $\{U_{0n}^{(2)}\}$ and $\{U_{in}^{(2)}\}$, $i = 1, 2$, are the ordinary Lucas and Fibonacci sequences when $P_{21} = -P_{22} = 1$, though we acknowledge that usually $L_0 = 2$. More generally, in the notation of Horadam [13],

$$\{U_{0,n+1}^{(2)}\} \equiv \{v_n(P_{21}, P_{22})\} \text{ and } \{U_{2,n+2}^{(2)}\} \equiv \{u_n(P_{21}, P_{22})\}$$

are the "primordial" and "fundamental" sequences respectively of Lucas. We define r basic sequences of order r, $\{U_{s,n}^{(r)}\}$, $s = 1, 2, \ldots, r$, by the recurrence relation of order r

$$U_{s,n}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} U_{sn-j}^{(r)}, \qquad n > r, \tag{1.4}$$

and the initial terms

$$U_{s,n}^{(r)} = \delta_{sn}, \qquad n = 1, 2, \ldots, r, \tag{1.5}$$

where the P_{rj} are arbitrary integers and $\delta_{sn}$is the Kronecker delta. The adjective basic is used by analogy with the corresponding third order sequences of Bell [2]. To correspond to the second order primordial sequence of Lucas we define the primordial sequence of order r, $\{U_{0,n}^{(r)}\}$, as one which satisfies the recurrence relation (1.4) but has initial terms given by

$$U_{0,n}^{(r)} = \begin{cases} 0 & n < 1 \\ \sum_{j=1}^{r} \alpha_{rj}^{n-1} & 1 \le n \le r, \end{cases} \tag{1.6}$$

where the α_{rj} are the roots, assumed distinct, of the auxiliary equation associated with (1.4), namely,

$$f(x) \equiv x^r - \sum_{j=1}^{r} (-1)^{j+1} P_{rj} x^{r-j} = 0 \tag{1.7}$$

where

$$f(x) = \prod_{j=1}^{r} (x - \alpha_{rj}). \tag{1.8}$$

We shall restrict ourselves to this non-degenerate case (i.e., roots distinct), but the essential arguments survive when the zeros of f(x) are not distinct.

By analogy with Lucas' second order "fundamental" sequence, the fundamental sequence of order r is $\{U_{r,n+r}^{(r)}\}$. For notational convenience, we shall designate

$$\{u_n^{(r)}\} \equiv \{U_{r,n+r}^{(r)}\} \text{ and } \{v_n^{(r)}\} \equiv \{U_{0n+1}^{(r)}\}. \tag{1.9}$$

Generally in the literature only one basic second order sequence is mentioned, namely the fundamental one, but Gootherts [8] in his demonstration of the construction of a linear algebra with whole Fibonacci sequences as elements has shown the need for the two basic second order sequences as well as the fundamental one.

Our purpose here is to show how a general arbitrary order sequence $\{w_n^{(r)}\}$, defined in (2.1), can be expressed in terms of the corresponding r basic sequences. This is done in (2.5). We also show how the fundamental and primordial sequences are related in (3.4).

2. THE BASIC SEQUENCES

The fundamental nature of the sequence $\{u_n^{(r)}\}$ has been illustrated by d'Ocagne (Dickson [5]) who has shown in effect that any element $w_n^{(r)}$ of the set $\Omega = \Omega(P_{r1}, P_{r2}, \ldots, P_{rr})$ of all sequences of order r which satisfy the recurrence relation (1.4) can be expressed in terms of the fundamental sequence and the initial terms of $\{w_n^{(r)}\}$, namely,

$$w_n^{(r)} = \sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1} (-1)^{k-j} P_{rk-j} u_{n-k}^{(r)} \right) w_j^{(r)}, \quad n \geq 0 \tag{2.1}$$

in which $P_{r0} = 1$ for notational convenience. As an example, when r = 2,

$$\begin{aligned} w_n^{(2)} &= w_1^{(2)} u_{n-1}^{(2)} + w_0^{(2)}(u_n^{(2)} - P_{21}u_{n-1}^{(2)}) \\ &= w_1^{(2)} u_{n-1}^{(2)} - P_{22} w_0^{(2)} u_{n-2}^{(2)} \end{aligned}$$

which agrees with equation (3.14) of Horadam [12].

At this stage it is of interest to look at some elements of the four sequences when r = 3. This is done in Table 2 for $P_{31} = -P_{32} = P_{33} = 1$.

n	1	2	3	4	5	6	7	8
$U_{0,n}^{(3)}$	3	1	3	7	11	21	39	71
$U_{1,n}^{(3)}$	1	0	0	1	1	2	4	7
$U_{2,n}^{(3)}$	0	1	0	1	2	3	6	11
$U_{3,n}^{(3)}$	0	0	1	1	2	4	7	13

Table 2

Williams [29] extended Lucas' work to the three sequences $\{U_{s,n}^{(3)}\}$, s = 0, 1, 3, which satisfy the third order linear recurrence relation

$$U_{s,n}^{(3)} = P_{31}U_{s,n-1}^{(3)} - P_{32}U_{s,n-2}^{(3)} + P_{33}U_{s,n-3}^{(3)} \tag{2.2}$$

with appropriate initial conditions and where the P_{3i} are given complex numbers.

Williams [30] has also studied some congruence properties of generalized Lucas numbers defined in effect by

$$L_{s,n}^{(r)} = d^{-s} \sum_{j=1}^{r} \alpha_{rj}^n \zeta_r^{s(j-1)} \quad s = 0, 1, \ldots, r-1, \tag{2.3}$$

in which $\zeta_r = \exp(2\pi i/r)$, $i^2 = -1$,

is a modification of Carlitz [3], and d is some real number for r > 2. For r = 2, d is the difference between the roots of the auxiliary equation as usual. Thus, when r = 2 and s = 0,

$$L_{0,n}^{(2)} = \alpha_{21}^n + \alpha_{22}^n = U_{0,n+1}^{(2)} \, . \tag{2.4}$$

For r = 2, 3 Lucas and Williams restrict the coefficients in the recurrence relations to relatively prime integers. On the other hand, Horadam in his studies of generalized second order linear recursive sequences permits these coefficients to be arbitrary integers, and Williams [31] permits these coefficients to be arbitrary numbers.

Theorem 1:

$$w_{n+k}^{(r)} = \sum_{j=1}^{r} w_{n+j}^{(r)} U_{j,k}^{(r)} . \tag{2.5}$$

Proof: The result can be established by induction on k. For k = 1, 2, ..., r, the result is true since $U_{j,k}^{(r)} = \delta_{jk}$ then. Suppose the result is true for k = r + 1, r + 2, ... , s − 1. Then since $\{w_n^{(r)}\}$ satisfies (1.4)

$$\begin{aligned} w_{n+s}^{(r)} &= \sum_{j=1}^{r} (-1)^{j+1} P_{rj} w_{n+s-j}^{(r)} \\ &= \sum_{j=1}^{r} (-1)^{j+1} P_{rj} \sum_{i=1}^{r} w_{n+i-j}^{(r)} U_{i,s}^{(r)} \qquad \text{(inductive hypothesis)} \\ &= \sum_{i=1}^{r} \left(\sum_{j=1}^{r} (-1)^{j+1} P_{rj} w_{n+i-j}^{(r)} \right) U_{i,s}^{(r)} \\ &= \sum_{i=1}^{r} w_{n+i}^{(r)} U_{i,s}^{(r)} \end{aligned}$$

and so the result is true for all k, as required.

For example, when r = 2 and $P_{21} = -P_{22} = 1$, we get

$$w_{n+k}^{(2)} = w_{n+1}^{(2)} U_{1,k}^{(2)} + w_{n+2}^{(2)} U_{2,k}^{(2)}$$

which is equivalent to (8) of Horadam [11]. When r = 3 and $P_{31} = -P_{32} = P_{33} = 1$, we get the first few terms of $\{w_n^{(3)}\}$ as, for instance,

$$w_1, w_2, w_3, w_1 + w_2 + w_3,\ w_1 + 2w_2 + 2w_3,\ 2w_1 + 3w_2 + 4w_3, \ldots .$$

The last of these is

$$w_6^{(3)} = w_1^{(3)} U_{1,6}^{(3)} + w_2^{(3)} U_{2,6}^{(3)} + w_3^{(3)} U_{3,6}^{(3)}$$

which is a particular case of Theorem 1 when n = 0 and k = 6.

Theorem 1 shows the 'basic' nature of the sequences $\{U_{s,n}^{(r)}\}$ and is a generalization of the result that elements of second order sequences can be expressed in terms of Fibonacci sequences.

3. THE FUNDAMENTAL AND PRIMORDIAL SEQUENCES

The $\{u_n^{(r)}\}$ and $\{v_n^{(r)}\}$ can be related as follows. The ordinary generating function for $\{u_n^{(r)}\}$ is given (formally) by

$$\sum_{n=0}^{\infty} u_n^{(r)} x^n = (x^r f(1/x))^{-1}. \tag{3.1}$$

Proof: If

$$u(x) = \sum_{n=0}^{\infty} u_n^{(r)} x^n,$$

and if we let

$$u_n \equiv u_n^{(r)}, \quad \text{for notational convenience,}$$

then

$$u(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_r x^r + \cdots$$

$$-P_{r1} x u(x) = -P_{r1} u_0 x - P_{r1} u_1 x^2 - \ldots - P_{r1} u_{r-1} x^r - \ldots$$

$$\ldots$$

$$(-1)^r P_{rr} x^r u(x) = (-1)^r P_{rr} u_0 x^r + \ldots .$$

Thus

$$\left(1 - \sum_{j=1}^{r} (-1)^{j+1} P_{rj} x^j\right) u(x) = u_0,$$

or

$$x^r f\left(\tfrac{1}{x}\right) u(x) = 1$$

since

$$f(x) = x^r - \sum_{j=1}^{r} (-1)^{j+1} P_{rj} x^{r-j}$$

from equation (1.7). This gives the required results.

We also need the result from (1.6) that

$$v_n^{(r)} = \sum_{j=1}^{r} \alpha_{rj}^n \quad \text{for all } n \geq 0. \tag{3.2}$$

The result can be established inductively along the following lines.

Proof: From (1.4) and (1.9)

$$\begin{aligned} v_n^{(r)} &= \sum_{j=1}^{r} (-1)^{j+1} P_{rj} v_{n-j}^{(r)}, \quad n \geq r, \\ &= \sum_{j=1}^{r} (-1)^{j+1} P_{rj} \sum_{i=1}^{r} \alpha_{ri}^{n-j} \qquad \text{from (1.6)} \\ &= \sum_{i=1}^{r} \alpha_{ri}^{n-r} \sum_{j=1}^{r} (-1)^{j+1} P_{rj} \alpha_{ri}^{r-j} \\ &= \sum_{i=1}^{r} \alpha_{ri}^{n-r} \alpha_{ri}^{r} \qquad \text{from equation (1.7)} \\ &= \sum_{i=1}^{r} \alpha_{ri}^{n} \qquad \text{as required.} \end{aligned}$$

The ordinary generating function u(t) is related to the corresponding exponential generating function y(t),

$$y(t) = \sum_{n=0}^{\infty} u_n^{(r)} t^n / n!$$

by the known result:

$$u(t) = \int_0^\infty e^{-z}y(tz)dz \qquad \text{if } |\ \alpha_{rj}t\ | < 1.$$

The algebra associated with the ordinary generating function is known as the *Cauchy calculus*, whereas the algebra associated with the exponential generating function is known as the *Blissard* or *umbral* or *symbolic calculus* (Riordan [19]). We now show that the ordinary generating function u(x) for $u_n^{(r)}$ can be expressed as an exponential by

$$\sum_{n=0}^{\infty} u_n^{(r)}x^n = \exp\left(\sum_{m=1}^{\infty} v_m^{(r)}x^m/m\right). \tag{3.3}$$

<u>Proof:</u> From equation (3.1) and the expression (1.8)

$$u(x) = \prod_{j=1}^{r}(1 - \alpha_{rj}x)^{-1}.$$

Then

$$\begin{aligned}\log(u(x)) &= \log\prod_{j=1}^{r}(1 - \alpha_{rj}x)^{-1}, \qquad |\ \alpha_{rj}x\ | < 1,\\
&= -\log\prod_{j=1}^{r}(1 - \alpha_{rj}x)\\
&= -\sum_{j=1}^{r}\log(1 - \alpha_{rj}x)\\
&= \sum_{j=1}^{r}\sum_{m=1}^{\infty}\tfrac{1}{m}\alpha_{rj}x^m\\
&= \sum_{m=1}^{\infty}\tfrac{1}{m}\left(\sum_{j=1}^{r}\alpha_{rj}^m\right)x^m\\
&= \sum_{m=1}^{\infty}\tfrac{1}{m}v_m^{(r)}x^m \qquad \text{(by (3.2)).}\end{aligned}$$

Thus

$$u(x) = \exp\left(\sum_{m=1}^{\infty} v_m^{(r)}x^m/m\right),$$

which can be used to produce generalized Fibonacci polynomials [23].

<u>Theorem 2:</u>

$$u_n^{(r)} = \sum_{\Sigma j\lambda_j = n}\prod_j (v_j^{(r)}/j)^{\lambda_j}/\ \lambda_j!. \tag{3.4}$$

<u>Proof:</u>

$$\begin{aligned}\sum_{n=0}^{\infty}u_n^{(r)}x^n &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty}\tfrac{1}{m}v_m^{(r)}x^m\right)^n/n!\\
&= \sum_{n=0}^{\infty}\left(\sum_{\Sigma j\lambda_j=n}\frac{v_1^{(r)\lambda_1}\cdot v_2^{(r)\lambda_2}\cdots}{1^{\lambda_1}.2^{\lambda_2}\ldots\lambda_1!\lambda_2!\ldots}\right)x^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{\Sigma j\lambda_j=n}\prod_j(v_j^{(r)}/j)^{\lambda_j}/\lambda_j!\right)x^n.\end{aligned}$$

Comparison of coefficients of x^n gives the required result. For example, when r = 2, and n = 2, we have

$$\sum j\lambda_j = 2 \prod_j (v_j^{(2)}/j)^{\lambda_j}/\lambda_j! = \frac{(v_1^{(2)}/1)^2}{2!} + \frac{(v_2^{(2)}/2)^1}{1!}$$

$$= \tfrac{1}{2}P_{21}^2 + \tfrac{1}{2}(P_{21}^2 - 2P_{22})$$

$$= P_{21}^2 - P_{22}$$

$$= P_{21}u_1^{(2)} - P_{22}u_0^{(2)}$$

$$= u_2^{(2)}.$$

4. AN APPLICATION

Not surprisingly the elements of $\{U_{r,n}^{(r)}\}$ appear in Turner's generalization of Pascal triangles [25], and curiously Turner's own tables appear in a model of the spread of infectious diseases [24]. For example, we can postulate three stages of an infectious disease: (i) an initial stage of k periods during which those who are ill with the disease do not infect others; (ii) a mature stage of l periods when each person affects n(t) healthy people, where t represents time; and (iii) a recovery stage when each individual recovers r periods after initial infection. When k = 2, l = 4 = 15, the numbers infected in the first 14 periods will be the ordinary Fibonacci numbers, as in Table 3(b) when n(t) = 1.

A brief explanation is as follows. Suppose that n_j^i is the number of (i + 1)th generation infectives at the end of their (j + 1)th period. Consider as an example the common cold where k = 2, l = 3, r = 7 days as in Table 4. The cold takes about 2 days to develop, the symptoms persist for about 7 days (or 1 week with medicinal remedies), and the length of time for which a person remains infectious (not the same as the duration of the symptoms) is about 3 days.

The first seven elements of the sequence can be read from the table to yield

$$U_0 = n_0^0$$

$$U_1 = n_1^0$$

$$U_2 = n_2^0 + n_0^1$$

$$U_3 = n_3^0 + n_0^1 + n_1^1$$

$$U_4 = n_4^0 + n_0^1 + n_1^1 + n_2^1 + n_0^2$$

$$U_5 = n_5^0 + n_1^1 + n_2^1 + n_3^1 + 2n_0^2 + n_1^2$$

$$U_6 = n_6^0 + n_2^1 + n_3^1 + n_4^1 + 3n_0^2 + 2n_1^2 + n_2^2 + n_0^3$$

$$U_7 = n_3^1 + n_4^1 + n_5^1 + 2n_0^2 + 3n_1^2 + 2n_2^2 + n_3^2 + 3n_0^3 + n_1^3.$$

We could continue in this way, but it is notationally easier and hence more illuminating to take an average value for n(t). Then we get with $n^i = n^i_j$:

$$U_0 = 1 \qquad U_3 = 1 + 2n$$

$$U_1 = 1 \qquad U_4 = 1 + 3n + n^2$$

$$U_2 = 1 + n \qquad U_5 = 1 + 3n + 3n^2.$$

When n = 1, this is Table 3(a). The APL code for running this can be obtained from the authors.

14 8 DISEASE 2 3 7

1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	1	0	0	0	0	0	0	2
1	2	0	0	0	0	0	0	3
1	3	1	0	0	0	0	0	5
1	3	3	0	0	0	0	0	7
1	3	6	1	0	0	0	0	11
0	3	8	4	0	0	0	0	15
0	3	9	10	1	0	0	0	23
0	2	9	17	5	0	0	0	33
0	1	9	23	15	1	0	0	49
0	1	8	26	31	6	0	0	71
0	0	6	27	50	21	1	0	105
0	0	3	26	66	51	7	0	153

(a)

14 8 DISEASE 2 15 15

1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	1	0	0	0	0	0	0	2
1	2	0	0	0	0	0	0	3
1	3	1	0	0	0	0	0	5
1	4	3	0	0	0	0	0	8
1	5	6	1	0	0	0	0	13
1	6	10	4	0	0	0	0	21
1	7	15	10	1	0	0	0	34
1	8	21	20	5	0	0	0	55
1	9	28	35	15	1	0	0	89
1	10	36	56	35	6	0	0	144
1	11	45	84	70	21	1	0	233
1	12	55	120	126	56	7	0	377

(b)

Table 3

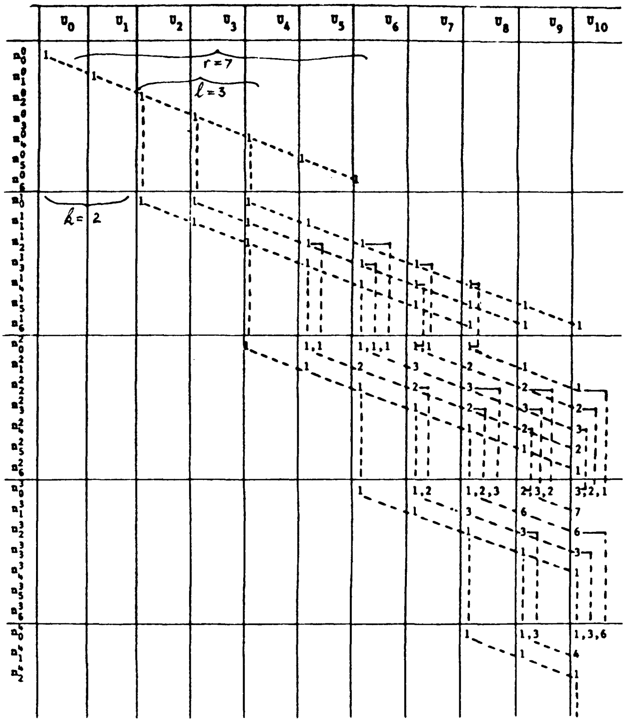

Table 4. Spread of Disease for k = 2, l = 3, r = 7

5. CONCLUDING COMMENTS

$u_n^{(r)}$ is (in the terminology of Macmahon [17]) the homogenous product sum of weight n of the quantities α_{rj}. It is the sum of a number of symmetric functions formed from a partition of the number n. The first three cases are (with $P'_{rj} = (-1)^{j+1}P_{rj}$ for convenience)

$$\begin{aligned} u_1^{(r)} &= P'_{r1} &&= \sum \alpha_{r1} \\ u_2^{(r)} &= P'^2_{r1} + P'_{r2} &&= \sum \alpha_{r1}^2 + \sum \alpha_{r1}\alpha_{r2} \\ u_3^{(r)} &= P'^3_{r1} + 2P'_{r1}P'_{r2} + P'_{r3} &&= \sum \alpha_{r1}^3 + \sum \alpha_{r1}^2\alpha_{r2} + \alpha_{r1}\alpha_{r2}\alpha_{r3}. \end{aligned}$$

In general,

$$u_n^{(r)} = \sum_{\Sigma i\lambda = n} \alpha_{r1}^{\lambda 1}\alpha_{r2}^{\lambda 2} \cdots = \sum_{\Sigma \lambda_i = n} \prod_{i=1}^{r} \alpha_{ri}^{\lambda i} .$$

If we adapt Macmahon [17, pp. 2-4] we obtain directly the multinomial expression for $u_n^{(r)}$, namely

$$u_n^{(r)} = \sum_{\Sigma i\lambda_i = n} \frac{\left(\sum \lambda\right)!}{\lambda_1!\lambda_2!\ldots\lambda_n!} \prod_{i=1}^{r} P'^{\lambda i}_{ri} . \tag{5.1}$$

For example,

$$u_n^{(2)} = \sum_{\sum i\lambda_i = n} \frac{\left(\sum\lambda\right)!}{\lambda_1!\lambda_2!} P'^{\lambda_1}_{21} P'^{\lambda_2}_{22}$$

(where $\lambda_1 = s$, $\lambda_2 = m$),

$$= \sum_{s+2m=n} \binom{n-m}{m} P'^{n-2m}_{21} P'^{m}_{22}$$

which agrees with Barakat [1];

$$u_n^{(3)} = \sum_{\sum i\lambda_i = n} \frac{\left(\sum\lambda\right)!}{\lambda_1!\lambda_2!\lambda_3!} P'^{\lambda_1}_{31} P'^{\lambda_2}_{32} P'^{\lambda_3}_{33}$$

(where $\lambda_1 = s$, $\lambda_2 = m$, $\lambda_3 = t$)

$$= \sum_{s+2m+3t=n} \frac{(n-m-2t)!}{s!m!t!} P'^{s}_{31} P'^{m}_{32} P'^{t}_{33}$$

$$= \sum_{s+2m+3t=n} \binom{n-m-2t}{m+t} \binom{m+t}{t} P'^{n-2m-3t}_{31} P'^{m}_{32} P'^{t}_{33}$$

which agrees with Shannon [21]. Equation (4.1) reduces to Theorem 1 of Philippou [18] who also considered the arbitrary order case when the coefficients P'_{ri}, $i = 1, 2, \ldots, r$ are all unity. Results (2.15) and (2.16) of Horadam and Mahon [14] also follow as particular cases.

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DISTRIBUTION OF RESIDUES OF CERTAIN SECOND-ORDER LINEAR RECURRENCES MODULO P

Lawrence Somer

1. INTRODUCTION

Let (w) = w(a, b) be a second-order linear recurrence defined by the relation

$$w_{n+2} = aw_{n+1} + bw_n, \tag{1}$$

where the parameters a and b and the initial terms w_0, w_1 are all integers. Let $D = a^2 + 4b$ be the discriminant of w(a, b). Let

$$x^2 - ax - b$$

be the characteristic polynomial associated with w(a, b) and let r_1 and r_2 be its characteristic roots. Throughout this paper, p will denote an odd prime unless specified otherwise. Further, d will always denote a residue modulo p. We say that the recurrence (w) is *defective* modulo p if (w) has an incomplete system of residues modulo p. It was shown in [9] that w(a, 1) is defective modulo p for most primes p. In particular, the following theorem was proven:

Theorem 1: (i) If $p > 7$ and $p \not\equiv 1$ or 9 (mod 20), then all recurrences w(a, 1) for which $D = a^2 + 4 \not\equiv 0 \pmod p$ are defective modulo p.

(ii) If w(a, 1) is a recurrence such that $D \equiv 0 \pmod p$ and $w_0 \equiv 0$, $w_1 \equiv 1 \pmod p$, then $a \equiv \pm 2i \pmod p$, where $i^2 \equiv -1 \pmod p$, and w(a, 1) has a complete system of residues (mod p). In particular,

$$w_n \equiv ni^{n-1} \text{ or } w_n \equiv n(-i)^{n-1} \pmod p.$$

Let u(a, b), called the Lucas sequence of the first kind (LSFK), be the recurrence satisfying (1) with initial terms $u_0 = 0$ and $u_1 = 1$. In proving that all recurrences w(a, 1) were defective modulo p it was only necessary to show that u(a, 1) was defective modulo p. We will further refine the results given in Theorem 1 by finding upper and lower bounds for the number of distinct residues appearing in the recurrence u(a, 1) modulo p. We will also determine the possibilities for the number of occurrences of a residue d in a full period of the recurrence u(a, 1) modulo p. We will show that the residue d can appear at most four times in a full period of the LSFK u(a, 1) modulo p.

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2. PRELIMINARIES

The *preperiod* of u(a, b) modulo p is the number of non-repeating terms of (u) modulo p. The *period* of u(a, b) modulo p will be denoted by $\omega(a, b, p) = \omega(p)$. It is known that if $p \nmid b$, then u(a, b) is purely periodic modulo p. Unless stated otherwise, we will assume that $p \nmid b$ in the LSFK u(a, b). The *restricted period* of u(a, b) modulo p, denoted by $\alpha(a, b, p) = \alpha(p)$, is the least positive integer t such that $u_{n+t} \equiv su_n \pmod p$ for all non-negative integers n and some non-zero residue s. Then s is called the principal multiplier of (u) (mod p). Any non-zero residue s' such that $u_{n+t'} \equiv s'u_n \pmod p$ for some fixed positive integer t' and all non-negative integers n is called a general multiplier of (u) modulo p. It is known that $\beta(p) = \omega(p)/\alpha(p)$ is the exponent of the principal multiplier modulo p. It is further known that if s' is a general multiplier of (u) (mod p), then $s' \equiv s^i \pmod p$ for some i such that $0 \le i \le \beta(p) - 1$.

For the LSFK u(a, b), N(p) will denote the number of distinct residues which appear in (u) modulo p and A(d) will denote the number of times the residue d appears in a full period of (u) modulo p. If $k = \alpha(p)$, $A_i(d)$ will denote the number of times the residue d appears among the terms $u_{ki}, u_{ki+1}, \ldots, u_{ki+k-1}$ modulo p, where $0 \le i \le \beta(p) - 1$.

The Lucas sequence of the second kind (LSSK), denoted by v(a, b), is the recurrence satisfying (1) with initial terms $v_0 = 2$, $v_1 = a$. For the recurrences u(a, b) and v(a, b), it follows by the Binet formulas that

$$u_n = (r_1^n - r_2^n)/(r_1 - r_2); \; v_n = r_1^n + r_2^n. \tag{2}$$

Identities (3) and (4) below follow immediately from (2):

$$v_n^2 - Du_n^2 = 4(-b)^n. \tag{3}$$

$$u_{2n} = u_n v_n. \tag{4}$$

3. THE MAIN THEOREMS

Before stating our main theorems concerning the distribution of residues in the LSFK u(a, 1), we will give essentially the best-known result concerning the general recurrence w(a, b).

Theorem 2: Let w(a, b) be a recurrence and p be a prime which may divide b. Let h be the period of (w) modulo p. Let f and g be the period and preperiod respectively of u(a, b) modulo p. Let $R = f + g$. Then

$$| A(d) - h/p | \le (p - 1)(h/R)^{1/2}.$$

Theorem 2 is a special case of a theorem due to Niederreiter and proofs are given in [4, pages 449-450] and [5].

Our results for the specific recurrence u(a, 1) will give much sharper estimates for A(d) than that in Theorem 2 as well as bounds for N(p). For a given LSFK u(a, 1), our results will depend on knowledge of the values of $\alpha(p)$, $\beta(p)$, and (D/p), where (D/p) denotes the Legendre symbol. Theorems 3 and 4 will provide information on the values $\omega(p)$, $\alpha(p)$, and $\beta(p)$ can take for the LSFK u(a, 1) depending on whether $(D/p) = 0$, 1, or -1.

Theorem 3: Let u(a, b) be a LSFK. Then

$$\alpha(p) \mid p - (D/p). \tag{5}$$

Further, if $p \nmid D$, then

$$\alpha(p) \mid (p - (D/p))/2 \tag{6}$$

if and only if $(-b/p) = 1$. Moreover, if $(D/p) = 1$, then

$$\omega(p) \mid p - 1. \tag{7}$$

Proof: Proofs of (5) and (7) are given in [2, pages 44-45] and [1, pages 315-317]. Proofs of (6) are given in [3, page 441] and [1, pages 318-319].

Theorem 4: Consider the LSFK u(a, 1). If $(-1/p) = 1$, let $i \equiv \sqrt{-1}$ (mod p), where $0 \le i \le (p - 1)/2$.

(i) $\beta(a, 1, p) = 1, 2,$ or 4; $s(a, 1, p) \equiv 1, -1, i,$ or $-i$ (mod p).

(ii) $\beta(a, 1, p) = 1$ iff $\alpha(a, 1, p) \equiv 2$ (mod 4).

(iii) $\beta(a, 1, p) = 2$ iff $\alpha(a, 1, p) \equiv 0$ (mod 4).

(iv) $\beta(a, 1, p) = 4$ iff $\alpha(a, 1, p) \equiv 1$ (mod 2).

Proof: This is proved in [7, pages 325-326].

Let u(a, 1) be a LSFK. Our principal theorems, Theorems 5-10, will provide constraints for A(d) and various bounds for N(p) depending on the values of $\beta(p)$ and (D/p). Moreover, certain of the bounds for N(p) will be given in terms of $\alpha(p)$ and can be utilized when $\alpha(p)$ is known. These bounds will provide tighter constraints for N(p).

Theorem 5: Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 1$ and $p \nmid a$.

(i) $A(d) = 0, 1, 2,$ or 3.

(ii) $A(d) + A(-d) = 0, 2,$ or 4 if $d \not\equiv \pm 2/\sqrt{D}$ (mod p).

(iii) $A(d) + A(-d) = 1$ or 3 if $d \equiv \pm 2/\sqrt{D}$ (mod p).

(iv) $A(0) = 1$.

(v) If $a \equiv \pm 1$, then $A(1) = 3$ and $A(-1) = 1$.

(vi) If $A(d) + A(-d) = 4$, then $A(d) = 1$ or 3.

(vii) If $A(d) + A(-d) = 3$, then $A(d) = 1$ or 2.

Theorem 6: Let u(a, 1) be a LSFK. Suppose that $p \nmid a$ and $\beta(p) = 1$. Then $\alpha(p) \equiv 2$ (mod 4). Let $c_1 = 0$ if $a \equiv \pm 1$ (mod p) and $c_1 = 1$ if $a \not\equiv \pm 1$ (mod p).

(i) If $p \equiv 3$ (mod 4), then

$$N(p) \le (3p - 5)/4 + c_1. \tag{8}$$

(ii) If $p \equiv 1$ (mod 4), then

$$N(p) \le (3p - 7)/8 + c_1. \tag{9}$$

(iii) $\alpha(p)/2 + 1 \leq N(p) \leq (3\alpha(p) - 2)/4 + c_1.$ (10)

<u>Theorem 7:</u> Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 2$ and $p \nmid a$.

(i) $A(d) = A(-d)$.

(ii) $A(d) = 0, 1, 2, 3,$ or 4.

(iii) $A(d) = 1$ or 3 if and only if $d \equiv \pm 2/\sqrt{-D} \pmod{p}$.

(iv) $A_0(d) + A_0(-d) = A_1(d) + A_1(-d)$.

(v) $A(0) = 2$.

(vi) If $a \equiv \pm 1 \pmod{p}$, then $A(1) = A(-1) = 4$.

(vii) If $\alpha(p) = p + 1$ and $p \equiv 7 \pmod{8}$, then

$$A(2/\sqrt{-D}) = A(-2/\sqrt{-D}) = 1.$$

(viii) If $\alpha(p) = p + 1$ and $p \equiv 3 \pmod{4}$, then

$$A(2/\sqrt{-D}) = A(-2/\sqrt{-D}) = 3.$$

<u>Theorem 8:</u> Let u(a, 1) be a LSFK. Suppose that $p \nmid a$ and $\beta(p) = 2$. Then $\alpha(p) \equiv 0 \pmod{r}$. Let $c_2 = 3$ if $p \equiv 3 \pmod{8}$ and $c_2 = 7$ if $p \equiv 7 \pmod{8}$. Let $c_3 = -1$ if $a \equiv \pm 1 \pmod{p}$ and $c_3 = 1$ if $p \equiv 7 \pmod{8}$.

(i) $N(p) \equiv 1 \pmod{2}$.

(ii) If $(D/p) = -1$, then $p \equiv 3 \pmod{4}$ and

$$N(p) \leq (3p + c_2)/4. \tag{11}$$

(iii) If $(D/p) = 1$, then $p \equiv 1 \pmod{4}$ and

$$N(p) \leq (p - 1)/2 + c_3. \tag{12}$$

(iv) $\alpha(p)/2 + 1 \leq N(p) \leq \alpha(p) + c_3.$ (13)

(v) If $\alpha(p) = p + 1$, then $p \equiv 3 \pmod{4}$ and

$$N(p) = (3p + c_2)/4. \tag{14}$$

<u>Theorem 9:</u> Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 4$ and $p \nmid a$. Let $s \equiv \pm i \pmod{p}$ be the principal multiplier of (u) (mod p).

(i) $A(d) = A(s^j d)$ for $1 \leq j \leq 3$.

(ii) $A(d) = 0, 2,$ or 4.

(iii) $\sum_{j=0}^{3} A_m(s^j d) = \sum_{j=0}^{3} A_n(s^j d)$ for $0 \leq m < n \leq 3$.

(iv) $A(0) = 4$.

(v) If $a \equiv \pm 1 \pmod{p}$, then

$$A(1) = A(-1) = A(i) = A(-i) = 4.$$

Theorem 10: Let u(a, 1) be a LSFK. Suppose that $p \nmid aD$ and $\beta(p) = 4$. Then $p \equiv 1 \pmod 4$ and $\alpha(p) \equiv 1 \pmod 2$.

(i) $N(p) \equiv 1 \pmod 4$.

(ii) If $(D/p) = -1$, then

$$N(p) \leq p - 4c_4, \tag{15}$$

where $c_4 = 0$ if both $a \not\equiv \pm 1 \pmod p$ and $p \equiv 1$ or $9 \pmod{20}$, and $c_4 = 1$ otherwise.

(iii) If $(D/p) = 1$, then

$$N(p) \leq (p - 1)/2 - 4c_5 - 1, \tag{16}$$

where $c_5 = 0$ if $a \not\equiv \pm 1 \pmod p$ and $c_5 = 1$ if $a \equiv \pm 1 \pmod p$.

(iv) Let [x] denote the greatest integer less than or equal to x. Then

$$4[(\alpha(p) + 1)/4] + 1 < N(p) \leq 2\alpha(p) - c_6, \tag{17}$$

where $c_6 = 5$ if either it is both the case that $p \not\equiv 1$ or $9 \pmod{20}$ and $\alpha(p) = (p + 1)/2$, or it is the case that $a \equiv \pm 1 \pmod p$; and $c_6 = 1$ otherwise.

4. NECESSARY LEMMAS

The following lemmas will be needed for the proofs of Theorems 5-10.

Lemma 1: Let u(a, b) be a LSFK. Let s be the principal multiplier of (u) modulo p and let $k = \alpha(p)$. Then

$$u_{k-n} \equiv (-1)^{n+1} s u_n / b^n \pmod p \tag{18}$$

for $0 \leq n \leq k$.

Proof: This can be proved by an easy induction argument.

Lemma 2: Let u(a, b) be a LSFK. Let n and c be positive integers such that $n + c \leq \alpha(p) - 1$. Let $k = \alpha(p)$. Then

$$(u_{n+c}/u_n)\,(u_{k-n}/u_{k-n-c}) \equiv (-b)^c \pmod p. \tag{19}$$

Proof: This follows from congruence (18) in Lemma 1. Another proof is given in [9, page 123].

Lemma 3: Consider the LSFK u(a, b), Let c be a fixed integer such that $1 \leq c \leq \alpha(p) - 1$. Then the ratios u_{n+c}/u_n are all distinct modulo p for $1 \leq n \leq \alpha(p) - 1$.

Proof: This is proved in [9, pages 120-121].

Lemma 4: Let u(a, 1) be a LSFK and let $k = \alpha(p)$.

(i) Suppose $\beta(p) = 1$ or 2. Then $k \equiv 0 \pmod 2$ and

$$u_n \not\equiv \pm u_{n+2c} \pmod p$$

for any positive integers n and c such that either $n + 2c \leq k/2$ or it is the case that $n \geq k/2$ and $n + 2c \leq k - 1$.

(ii) Suppose $\beta(p) = 4$. Then $k \equiv 1 \pmod 2$ and

$$u_n \not\equiv \pm u_{n+2c} \pmod p$$

for any positive integers n and c such that $n + 2c \leq k - 1$.

Proof: (i) This is proved in Lemma 7(i) of [9] for the case in which $\beta(p) = 2$ and $p \equiv 3 \pmod 4$. The proofs of the other cases are completely similar.

(ii) This is proved in Lemma 7 (ii) of [9] for the case in which $\beta(p) = 4$ and $(D/p) = -1$. The proofs of the cases in which $(D/p) = 0$ or 1 are completely similar.

Lemma 5: Let u(a, b) be a LSFK. Let s be the principal multiplier of (u) modulo p. Let $t = \beta(p)$. Then

$$N(p) \equiv 1 \pmod t.$$

Proof: The general multipliers of (u) modulo p are s^i for $0 \leq i \leq t - 1$. It thus follows that if $d \not\equiv 0 \pmod p$, then the residues d, sd, ..., $s^{t-1}d$ are all distinct modulo p. Thus, the residue $d \not\equiv 0$ appears in (u) modulo p if and only if the residues $s^j d$ appear for $1 \leq j \leq t - 1$. Counting the residue 0, the result now follows.

Lemma 6: Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 2$. Let $k = \alpha(p)$. Let N_1 be the largest integer t such that there exist integers $n_1, n_2, \ldots, n_t$ for which $1 \leq n_i \leq [k/2]$ and $u_{n_i} \not\equiv \pm u_{n_j} \pmod p$ if $1 \leq i < j \leq [k/2]$. Then

$$N(p) = 2N_1 + 1. \tag{20}$$

Proof: Since -1 is the principal multiplier of (u) modulo p, the residue $-d$ appears in (u) modulo p if and only if d appears in (u) modulo p. Moreover, it follows from Lemma 1 and the fact that -1 is a principal multiplier of (u) modulo p that if $d \not\equiv 0 \pmod p$ and d appears in (u) (mod p), then $d \equiv \pm u_{n_i} \pmod p$ for some i such that $1 \leq i \leq N_1$. Including the residue 0, we see that (20) holds.

Lemma 7: Let u(a, 1) be a LSFK. Suppose $\beta(p) = 4$. Let s be the principal multiplier of (u) modulo p. Let $k = \alpha(p)$. Then $k \equiv 1 \pmod 2$. Let N_2 be the largest integer t such that there exist integers $n_1, n_2, \ldots, n_t$ for which $1 \leq n_i \leq (k-1)/2$ and $u_{n_i} \not\equiv s^m u_{n_j} \pmod p$ if $1 \leq i < j \leq (k-1)/2$ and $0 \leq m \leq 3$. Then

$$N(p) = 4N_2 + 1.$$

Proof: By Theorem 4, $k \equiv 1 \pmod 2$. The residue d appears in (u) modulo p if and only if the residue $s^m d$ appears for $1 \le m \le 3$. By Lemma 1, $u_{k-n} \equiv \pm s u_n \pmod p$ for $1 \le n \le (k-1)/2$. Since s is the principal multiplier of (u) (mod p), it follows that if $d \not\equiv 0 \pmod p$, then the residue d appears in (u) (mod p) if and only if at least one of the residues d, $-d$, sd, or $-sd$ appears among the terms $u_1, u_2, \ldots, u_{(k-1)/2} \pmod p$. Including the residue 0, the result now follows.

Lemma 8: Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 1$ or 2. Let $k = \alpha(p)$. Then $k \equiv 0 \pmod 2$. Let $A'(d)$ denote the number of times the residue d appears among the terms $u_1, u_2, \ldots, u_{[k/2]}$ modulo p. Let N_1 be defined as in Lemma 6.

(i) $A'(d) + A'(-d) \le 2$.

(ii) $[(k+2)/4] \le N_1 \le k/2$.

Proof: (i) This follows from Lemma 4 (i) upon using the pigeonhole principle.

(ii) This follows from (i) and the definition of N_1.

Lemma 9: Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 4$. Let s be the principal multiplier of (u) modulo p. Let $k = \alpha(p)$. Then $k \equiv 1 \pmod 2$. Let $A'(d)$ be defined as in Lemma 6. Let N_2 be defined as in Lemma 7.

(i) $\sum_{j=0}^{3} A'(s^j d) \le 2$.

(ii) $[(k+1)/4] \le N_2 \le (k-1)/2$.

Proof: The fact that $k \equiv 1 \pmod 2$ follows from Theorem 4.

(i) Suppose that there exist positive integers f, g, h, each less than or equal to $(k-1)/2$, such that

$$u_f \equiv s^{i_1} d,\ u_g \equiv s^{i_2} d, \text{ and } u_h \equiv s^{i_3} d \pmod p, \tag{21}$$

where $0 \le i_j \le 3$ for $j = 1, 2, 3$. If $u_g \equiv \pm u_f$ and $u_h \equiv \pm u_f \pmod p$, then by Lemma 4 (ii), $g - f$ is an odd integer and $h - g$ is an odd integer. Thus, $h - f$ is an even integer, which contradicts Lemma 4 (ii). Thus, we can assume without loss of generality that $u_g \equiv \pm u_f$ and $u_h \equiv \pm s u_f \pmod p$. By Lemma 4

(ii), $g - f$ is an odd integer. Moreover, by Lemma 1,

$$u_{k-h} \equiv \pm s^2 u_f \equiv \pm u_f \equiv \pm u_g \pmod p.$$

However, it then follows that $k - h - g$ is an even integer or that $k - h - f$ is an even integer, which contradicts Lemma 4 (ii). Thus, there cannot exist distinct integers satisfying (21), and assertion (i) now follows.

(ii) This follows from (i).

Lemma 10: Let u(a, b) be a LSFK. Suppose that $p \nmid b$. Let s be the principal multiplier of (u) modulo p and s^j be a general multiplier of (u) (mod p), where $1 \le j \le \beta(p) - 1$. Then

$$A(d) = A(s^j d).$$

Proof: Let $t = \beta(p)$ and $k = \alpha(p)$. Since s^j is a general multiplier of (u) modulo p, if $u_n \equiv d \pmod p$, then $u_{n+jk} \equiv s^j d \pmod p$. Thus, $A(d) \le A(s^j d)$. Now, if $u_n \equiv s^j \pmod p$, then

$$u_{n+(t-j)k} \equiv s^{t-j} s^j d \equiv s^t d \equiv d \pmod p$$

and $A(s^j d) \le A(d)$. Thus, $A(d) = A(s^j d)$.

Lemma 11: Let u(a, b) be a LSFK. Suppose that $p \nmid b$. Let s be the principal multiplier of (u) modulo p. Let $t = \beta(p)$. Let $0 \le i < j \le t - 1$. Then

$$\sum_{n=0}^{t-1} A_i(s^n d) = \sum_{n=0}^{t-1} A_j(s^n d) .$$

Proof: Let $k = \alpha(p)$. Note that

$$u_{n+kj} \equiv s^{j-i} u_{n+ki} \pmod p$$

for $0 \le n \le k - 1$ and

$$u_{m+ki} \equiv s^{t-j+i} u_{m+kj} \pmod p$$

for $0 \le m \le k - 1$. The result immediately follows.

Lemma 12: Let u(a, 1) be a LSFK with discriminant D. Let $k = \alpha(p)$.

(i) If $\beta(p) = 1$, then $k \equiv 2 \pmod 4$ and

$$u_{k/2} \equiv \pm 2/\sqrt{D} \pmod p.$$

(ii) If $\beta(p) = 2$, then $k \equiv 0 \pmod 4$ and

$$u_{k/2} \equiv \pm 2/\sqrt{-D} \pmod p.$$

Proof: (i) Suppose $\beta(p) = 1$. Then $\alpha(p) \equiv 2 \pmod 4$ by Theorem 4. By the identity (3),

$$v_{k/2}^2 - Du_{k/2}^2 = 4(-1)^{k/2} = -4. \tag{22}$$

Now, $u_{k/2} \not\equiv 0 \pmod p$. Since

$$0 \equiv u_k = u_{k/2} v_{k/2} \pmod p \tag{23}$$

by (4), $v_{k/2} \equiv 0 \pmod p$. Thus, by (23),

$$Du_{k/2}^2 \equiv 4 \pmod p. \tag{24}$$

If p | D, then $\alpha(p) = p$ by Theorem 3, which is impossible. Thus, $D \not\equiv 0 \pmod p$ and the assertion follows from (24).

(ii) Suppose $\beta(p) = 2$. Then $\alpha(p) \equiv 0 \pmod 4$ by Theorem 4. By (3),

$$v_{k/2}^2 - Du_{k/2}^2 = 4(-1)^{k/2} = 4. \tag{25}$$

Now, $u_{k/2} \not\equiv 0 \pmod p$. Thus, $v_{k/2} \equiv 0 \pmod p$ by (4). Hence by (25),

$$-Du_{k/2}^2 \equiv 4 \pmod p. \tag{26}$$

Since $\alpha(p) \neq p$, $p \nmid D$. The result now follows.

<u>Lemma 13:</u> Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 2$. Let $k = \alpha(p)$. Then $k \equiv 0$ (mod 4). Suppose that

$$u_{m+2c-1}/u_m \equiv \pm 1 \pmod p$$

for some positive integers m and c such that $m \leq k - 1$ and $2c - 1 \leq k - 1$. Then there exist positive integers n and g such that $n + 2g - 1 \leq k/2$, $u_n \equiv \pm u_m \pmod p$, and

$$u_{n+2g-1}/u_n \equiv \pm 1 \pmod p.$$

<u>Proof:</u> This follows from Lemma 1 and the fact that -1 is the principal multiplier of (u) modulo p.

<u>Lemma 14:</u> Let u(a, 1) be a LSFK. Suppose that $\beta(p) = 2$. Let $k = \alpha(p)$. Let $s \equiv -1 \pmod p$ be the principal multiplier of (u) modulo p.

(i) Suppose that

$$u_{n+2r-1} \equiv \pm u_n \pmod p,$$

where n and r are integers such that $1 \leq n < n + 2r - 1 < k/2$. Then the only values of $2c - 1$ and m such that $1 \leq 2c - 1 \leq k - 1$, $1 \leq m \leq k - 1$, $u_m \equiv \pm u_n \pmod p$, and $u_{m+2c-1}/u_m \equiv \pm 1 \pmod p$ are

$$2c - 1 = 2r - 1;\ m = n \text{ or } m = k - n - 2r + 1; \tag{27}$$
$$2c - 1 = k - 2r + 1;\ m = n + 2r - 1 \text{ or } m = k - n; \tag{28}$$
$$2c - 1 = k - n - 2r + 1;\ m = n \text{ or } m = n + 2r - 1; \tag{29}$$
$$2c - 1 = 2n + 2r - 1;\ m = k - n - 2r + 1 \text{ or } m = k - n. \tag{30}$$

(ii) Suppose that

$$u_{k/2} \equiv \pm u_n \pmod p,$$

where $1 \leq n < k/2$ and $k/2 = n + 2r - 1$ for some positive integer r. Then the only values of $2c - 1$ and m such that $1 \leq 2c - 1 < k - 1$, $1 \leq m \leq k - 1$, $u_m \equiv \pm u_n \pmod p$, and $u_{m+2c-1}/u_m \equiv \pm 1 \pmod p$ are

$$2c - 1 = 2r - 1; \; m = n \text{ or } m = k; \tag{31}$$
$$2c - 1 = k - 2r + 1; \; m = k/2 \text{ or } m = k/2 + 2r - 1. \tag{32}$$

Proof: (i) It follows from Lemma 4 that if $u_f \equiv \pm u_g \pmod p$ and $u_f \equiv \pm u_n \pmod p$, where $1 \le f < g < k/2$, then $f = n$ and $g = n + 2r - 1$. It now follows from the fact that $s \equiv -1 \pmod p$ and from Lemma 1 that the only values for $2c - 1$ and m are the ones listed in (27) - (30).

(ii) This follows by an argument similar to that used in the proof of (i).

Lemma 15: Let $u(a, 1)$ be a LSFK with discriminant D. Suppose that $\beta(p) = 2$ and $\alpha(p) = p + 1$. Then $p \equiv 3 \pmod 4$ and $(-D/p) = 1$.

(i) If $p \equiv 7 \pmod 8$, then

$$N(p) = (3p + 7)/4$$

and

$$A(2/\sqrt{-D}) = A(-2/\sqrt{-D}) = 1.$$

(ii) If $p \equiv 3 \pmod 8$, then

$$N(p) = (3p + 3)/4$$

and

$$A(2/\sqrt{-D}) = A(-2/\sqrt{-D}) = 3.$$

Proof: The assertions that $p \equiv 3 \pmod 4$ and $(-D/p) = 1$ are proved in [7, pages 325-326].

(i) Suppose $p \equiv 7 \pmod 8$. By Lemma 6, $N(p) = 2N_1 + 1$, where N_1 is as defined in Lemma 6. Since $\alpha(p) = p + 1$, it follows from Lemma 3 that if $1 \le 2r - 1 \le p$, then there exists an integer m such that $1 \le m \le p$ and

$$U_{m+2r-1}/u_m \equiv 1 \pmod p. \tag{33}$$

It now follows from Lemma 14 that if $u_m \not\equiv \pm u_{(p+1)/2} \pmod p$, then there exist exactly four distinct positive odd integers $d_i \le p$ such that one can find positive integers $m_i \le p$, $1 \le i \le 4$, for which $u_{m_i} \equiv \pm u_m \pmod p$ and $u_{m_i+d_i}/u_{m_i} \equiv \pm 1 \pmod p$. In particular, by Lemma 13, we can assume that

$$1 \le m_1 < m_1 + d_1 < (p + 1)/2.$$

Further, it follows from Lemma 14 and Lemma 3 that if $u_{g+d_i}/u_g \equiv \pm 1$, where $1 \le i \le 4$, then $u_g \equiv \pm u_m \pmod p$.

Now assume that in (33), $u_m \equiv \pm u_{(p+1)/2} \pmod p$. It then follows from Lemma 14 that there exist exactly two distinct positive odd integers $e_i \le p$ such that one can find distinct positive integers $k_i \le p$, $i = 1, 2$, for which $u_{k_i} \equiv \pm u_m \pmod p$ and $u_{k_i+d_i}/u_{k_i} \equiv \pm 1 \pmod p$. As above, one can assume by Lemma 13 that

$$1 \le k_1 < k_1 + e_1 = (p + 1)/2.$$

It again follows from Lemma 14 and Lemma 3 that if $u_{n+d_i}/u_n \equiv \pm 1 \pmod p$, where $i = 1, 2$, then $u_n \equiv \pm u_m \pmod p$.

Note that there are $(p + 1)/2$ odd integers less than or equal to p and that $(p + 1)/2 \equiv 0 \pmod 4$. It thus follows that if $1 \le 2r - 1 \le p$ and $u_{m+2r-1}/u_m \equiv \pm 1 \pmod p$, then $u_m \not\equiv \pm u_{(p+1)/2} \pmod p$. Since $u_{(p+1)/2} \equiv \pm 2/\sqrt{-D} \pmod p$ by Lemma 12 (ii), it follows from the fact that the principal multiplier of (u) modulo p is -1 and from Lemma 1 that

$$A(2/\sqrt{-D}) = A(-2/\sqrt{-D}) = 1.$$

It now follows from Lemma 14 that

$$N_1 = (p + 1)/2 - (1/4)(p + 1)/2 = (3p + 3)/8.$$

Thus,

$$N(p) = 2N_1 + 1 = 2((3p + 3)/8) + 1 = (3p + 7)/4.$$

(ii) Suppose $p \equiv 3 \pmod 8$. We note that there are $(p + 1)/2$ positive odd integers less than or equal to p and that $(p + 1)/2 \equiv 2 \pmod 4$. It thus follows from the arguments used in proving (i) that there exists a positive integer $m < (p + 1)/2$ such that $u_m \equiv \pm u_{(p+1)/2} \pmod p$. It then follows from the fact that the principal multiplier of (u) (mod p) is -1 and from Lemma 1 that

$$A(2/\sqrt{-D} = A(-2/\sqrt{-D}) = 3.$$

Now, $N(p) = 2N_1 + 1$. By the arguments used in the proof of (i),

$$N_1 = (p + 1)/2 - (1/2)(2) - (1/4)((p + 1)/2 - 2) = (3p - 1)/8.$$

Then

$$N(p) = 2N_1 + 1 = 2((3p - 1)/8) + 1 = (3p + 3)/8.$$

The proof is now complete.

5. PROOFS OF THE MAIN THEOREMS

We are finally ready for the proofs of Theorems 5-10.

Proof of Theorem 5: The fact that $(D/p) = 1$ is proved in [7, pages 325-326].

(i), (vi), and (vii) These follow from Lemma 8 (i) and Lemma 1.

(ii) and (iii) These follow from Lemmas 12 (i), 8 (i), and 1.

(iv) This follows by inspection.

(v) This follows from Lemmas 1 and 8 (i) upon noting that $u_1 \equiv \pm u_2 \equiv \pm 1 \pmod p$ if $a \equiv \pm 1 \pmod p$.

Proof of Theorem 6: The fact that $\alpha(p) \equiv 2 \pmod 4$ follows from Theorem 4. Note that $u_1 \equiv \pm u_2 \equiv \pm 1 \pmod p$ if $a \equiv \pm 1 \pmod p$.

(i) and (ii) The upper bounds follow by Theorem 3 and Lemmas 1 and 8 (ii).

(iii) The upper and lower bounds follow from Lemmas 8 (ii) and 1.

Remark: In Theorem 6, it can be shown that if $v_n(a, 1) = p$ for some odd prime n, then the upper bound in (10) is attained. For instance, this occurs in the Fibonacci and Lucas sequences when $p = 11$ or 29.

Proof of Theorem 7: Let s be the principal multiplier of u(a, 1) modulo p.

(i) This follows from Lemma 10.

(ii) This follows from Lemmas 8 (i) and 1 and the fact that $s \equiv -1 \pmod p$.

(iii) This follows from Lemmas 12 (ii), 8 (i), and 1 and the fact that $s \equiv -1 \pmod p$.

(iv) This follows from Lemma 11.

(v) This follows by inspection.

(vi) This follows from the fact that $s \equiv -1 \pmod p$ and from Lemmas 1 and 8 (i) upon noting that $u_1 \equiv \pm u_2 \equiv \pm 1 \pmod p$.

(vii) and (viii) These follow from Lemma 15.

Proof of Theorem 8: By Theorem 4, $\alpha(p) \equiv 0 \pmod 4$. Further, by Lemma 6, $N(p) = 2N_1 + 1$, where N_1 is as defined in Lemma 6. Note that $u_1 \equiv \pm u_2 \equiv \pm 1 \pmod p$ if $a \equiv \pm 1 \pmod p$.

(i) This follows from Lemma 5.

(ii) The fact that $p \equiv 3 \pmod 4$ if $(D/p) = -1$ follows from (3) and (4). The upper bound for N(p) follows from Theorem 3 and Lemma 15 upon noting that $\alpha(p) \leq (p + 1)/3$ if $\alpha(p) \neq p + 1$.

(iii) The fact that $p \equiv 1 \pmod 4$ if $(D/p) = 1$ follows from (3) and (4). The upper bound for N(p) follows from Theorem 3 and Lemma 8 (ii).

(iv) The upper and lower bounds follow from Lemma 8 (ii).

(v) This follows from Lemma 15 and Theorem 4.

Remark: In Theorem 8, it can be shown that if p is a Mersenne prime such that $(D/p) = -1$, then part (v) of this theorem holds and the upper bound in (11) is attained. For example, this occurs in the Fibonacci and Lucas sequences when $p = 3$, 7, or 127.

Proof of Theorem 9:

(i) This follows from Lemma 10.

(ii) This follows from (i), Lemmas 9 (i) and 1 and the fact that $s \equiv \pm i \pmod p$.

(iii) This follows from Lemma 11.

(iv) This follows by inspection.

(v) This follows from (i), Lemma 9 (i), and the fact that $s \equiv \pm i \pmod p$ upon noting that $u_1 \equiv \pm u_2 \equiv \pm 1 \pmod p$ if $a \equiv \pm 1 \pmod p$.

Proof of Theorem 10: The assertion that $p \equiv 1 \pmod 4$ follows from the facts that $\beta(p) = 4$ and the principal multiplier of (u) is congruent to $\pm i \pmod p$. The fact that $\alpha(p) \equiv 1 \pmod 2$ follows from Theorem 4.

(i) This follows by Lemma 5.

(ii) and (iii) These follow from Lemma 9 (ii) and Theorems 3 and 4.

(iv) This follows from Lemma 9 (ii).

Remark 1: In Theorem 10, it can be shown that if $u_n = p$ for some odd prime n, then the upper bound in (17) is attained. For instance, this occurs in the Fibonacci sequence when $p = 13$, 89, or 233.

Remark 2: Theorems 13-15 in [7] provide conditions for the hypotheses of Theorems 5-10 to be satisfied.

6. SPECIAL CASES

For completeness, we present Theorems 11 and 12 which detail the special cases we have not treated so far. For these Theorems p will designate a prime, not necessarily odd.

Theorem 11: Let u(a, 1) be a LSFK. Suppose $p \nmid D$.

(i) If $a \equiv 0 \pmod p$, then $\alpha(p) = 2$, $\beta(p) = 1$, $N(p) = 2$, $A(0) = A(1) = 1$, and $A(d) = 0$ if $d \not\equiv 0$ or $1 \pmod p$.

(ii) If $p = 2$ and $a \equiv 1 \pmod 2$, then $\alpha(p) = 3$, $\beta(p) = 1$, $N(p) = 2$, $A(0) = 1$, and $A(1) = 2$.

Proof: Parts (i) and (ii) follow by inspection.

Theorem 12: Let u(a, 1) be a LSFK. Suppose that $p \mid D$. Then $p = 2$ or $p \equiv 1 \pmod 4$. Further, $a \equiv \pm 2i \pmod p$, where $i^2 \equiv -1 \pmod p$. If $p = 2$, then $\alpha(p) = p$, $\beta(p) = 1$, $N(p) = p$, and $A(d) = 1$ for all residues d modulo p. If $p > 2$, then $\alpha(p) = p$, $\beta(p) = 4$, $N(p) = p$, and $A(d) = 4$ for all residues d modulo p.

Proof: This follows from part (ii) of Theorem 1.

Remark 1: If $D \equiv 0 \pmod p$, we see from Theorem 12 that the residues of u(a, 1) are equidistributed modulo p. See [4, page 463] for a comprehensive list of references on equidistributed linear recurrences.

Remark 2: A. Schinzel in his paper, "Special Lucas Sequences, Including the Fibonacci Sequence, Modulo a Prime", which is to appear in a volume honoring Paul Erdös has presented results similar to those appearing in this article. In particular, Schinzel's results lead to improvements of Theorem 1(i), Theorem 5(i), Theorem 9(ii), and Theorem 10(ii) and (iv).

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THE FIBONACCI TREE, HOFSTADTER AND THE GOLDEN STRING

Keith Tognetti, Graham Winley and Tony van Ravenstein

INTRODUCTION

"If a tree puts forth a new branch after one year, and always rests for a year, producing another new branch only in the year following, and if the same law applies to each branch, then, in the first year we should have only the principal shoot, in the second - two branches, in the third - three, then 5, 8, 13".

So observed the brilliant Polish mathematician, Hugo Steinhaus in that mathematical treasurehouse "Mathematical Snapshots", (see reference [7]) from which we have produced Figure 1.

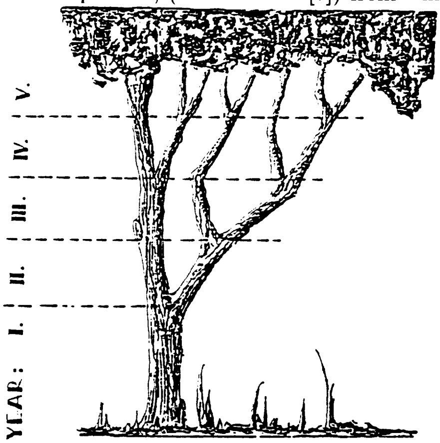

Figure 1. The Steinhaus Tree.

This appears to be the first recorded description of a tree with Fibonacci properties. Steinhaus observed that the numbers of branches as we go up the levels in the tree are the same as the populations in the successive generations of the Fibonacci rabbits.

Figure 2 is a reflection of the tree in Figure 1 showing vertex numbering as first proposed by Hofstadter [3, pp. 134-137]. We will refer to this as the **Fibonacci Tree.** In contrast what is referred to as a Fibonacci Tree by some computer scientists such as Horibe [4] is a different structure.

Additionally we have indicated the type of each vertex as "l" (for large) for those vertices of degree 3 and "s" (for small) for those vertices of degree 2.

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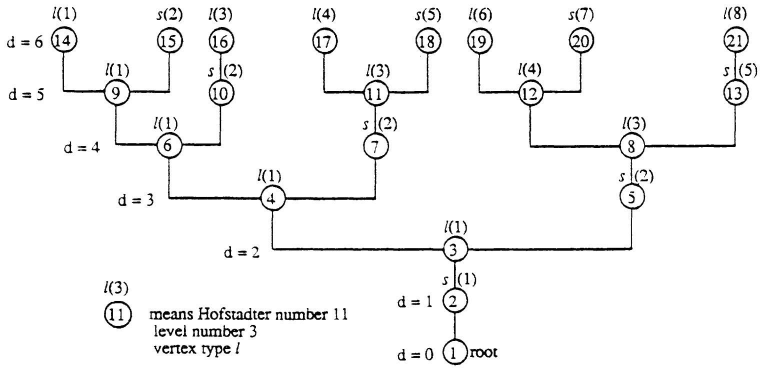

Figure 2. The Fibonacci Tree after Hofstadter.

There are some real world trees that are roughly approximated by the Fibonacci tree (see Stevens [8]) at some stages of their growth. However the main use of the Fibonacci tree is that it is sometimes a useful way of displaying the Fibonacci relationship when it occurs in other biological and computer structures.

Our purpose is to explore the properties of the Hofstadter numbering and then analyse the sequence of vertex types.

DEFINITIONS AND PROPERTIES

1. A tree has a unique chain between any pair of vertices hence we define the level d of a vertex as the number of edges in the path between that vertex and the origin vertex which we have numbered 1. Hence vertices 6, 7 and 8 are at the level 4 and vertex 1 which we have called our origin is at level 0.

Note - For $d > 0$ the number of vertices at the level d, is the Fibonacci number F_d. The total number of vertices in the tree below level d is F_{d+1}.

These results easily follow from the Fibonacci rabbit problem, when we relate type l vertices to large rabbits and type s vertices to small (immature) rabbits.

2. Level- Numbering

At each level we number the vertices in increasing order to the right starting with the number 1 for the leftmost vertex and finishing with F_d for the right most vertex. We have included the level number in brackets in Figure 2.

3. Hofstadter- Numbering

As with the level-numbering the label on each vertex is increased by unity as we go from left to right across a level. Additionally we increase the number by unity as we go up a level from the rightmost vertex on one level to the left most vertex on the next level. We start the numbering with the number one for the origin.

Thus we number the extreme left most vertex at level d with $F_{d+1} + 1$ and the right most vertex with F_{d+2}.

In this way for example we number the three vertices at level 4 with 6, 7, 8.

4. Property F- Vertex Generation Rules

A type s vertex at level d is always joined to a type l vertex at level d + 1.

A type l vertex at level d is always joined to a pair of vertices at level d + 1, in such a way that the left vertex in the pair is always of type l, and the right vertex in the pair is always of type s.

These are referred to as generation rules to emphasise that they tell us how vertices are related between levels.

We can now view each level as a generation and thus we can say that a type l vertex in generation d is the father of the pair of sons, ls, in generation d + 1. Similarly any type s vertex in generation d is the father of a type l vertex in generation d + 1.

It is seen that these rules correspond to those for the rabbit problem except that here the order is important.

It follows from Property F that, across a level, a type s vertex must occur singly and a type l vertex may occur singly or as a double but never in longer runs.

We can see that each vertex at a given level contributes exactly one type l vertex to the level above. That is there are as many type l vertices at level d as there are vertices of both types at level d − 1.

At level $d > 0$, of the total of F_d vertices, F_{d-1} are type l and F_{d-2} are type s, from which it is seen that the type l vertices are more numerous than type s. That is a fraction of F_{d-1}/F_d are type l vertices and F_{d-2}/F_d are type s vertices. As the level d increases these proportions approach τ and $1 - \tau = \tau^2$ respectively, where $\tau = (\sqrt{5} - 1)/2 = .618\ldots$. That is the ratio of type l to type s vertices at levels high up in the tree approaches $1+\tau$.

We must be careful not to confuse the successor of a vertex with its son. In contrast to the father son relation between levels, the successor relation applies to the sequence usually within a level except when the vertex is rightmost. Thus as we proceed across a level we see that the successor of a type s vertex is always a type l vertex. In contrast it is not clear what type follows a type l vertex. All we know is that a double occurrence of a type l is always followed by a type s.

5. If n is the Hofstadter number of a vertex at level d then we will use the symbol H(n) for the Hofstadter number of its father at level d − 1.

6. The Beta Sequence

For j = 1, 2, 3, ... we define

$$\beta_j = [(j + 1)\tau] - [j\tau],$$

where it is seen that the integer parts can differ by at most unity. Then the Beta Sequence is

$$\{\beta_j\} = 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, \ldots . \qquad (\beta)$$

7. Golden Sequences and Strings

Any list of symbols with a one to one correspondence with (β) will be referred to as a Golden Sequence. Thus with 1 replaced by "a" and 0 replaced by "b" in (β) we obtain the Golden sequence abaababaabaab... .

Let S(j) be the string made up of the first j symbols of a Golden sequence then we will refer to S(j) as a Golden string.

As usual we define XY as the concatenation of the two strings X and Y. Thus S(j)S(k) is the string of length j + k obtained by appending the Golden string S(k) to the Golden string S(j).

8. The type sequence $T_d(j)$ is simply the sequence of vertex types of length j, as we proceed across the level d from left to right. Thus for example

$$T_6(F_6) = \text{lsllslsl} .$$

It follows that $T_{d-1}(F_{d-1})T_{d-2}(F_{d-2})$ is the string of length F_d obtained by appending the entire type sequence at level d − 2 to that at level d − 1.

THE TYPE SEQUENCE

Lemma 1: $[(j + F_n)\tau] = F_{n-1} + [j\tau]$, for $0 < j < F_{n+1}$.

This is the particular case where $\alpha = \tau$, of the following result due to Fraenkel et al [2, pp. 443].

If p_{n-1}/q_{n-1} and p_n/q_n are convergents to α and $n > 0$ then for $0 < j < q_n$

$$[(j + q_{n-1})\alpha] = p_{n-1} + [j\alpha].$$

As special cases of Lemma 1 we have -

a) $[(F_n + 1)\tau] = F_{n-1}$, $n \geq 2$.

b) $F_{n-1} + 1 = [(F_n + 2)\tau] = [(F_n + 3)\tau]$.

G - The Hofstadter Recursive Tree

We now investigate the properties of the type sequence through the following recursive representation of a tree which Hofstadter defined in [3].

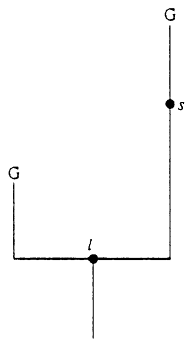

Figure 3. The Hofstadter Recursive Tree G.

Henceforth we will refer to this structure as G. It follows that the Fibonacci tree is simply, root -- s -- G.

The tree G is seen to be made up of a left tree which is simply a copy of itself shifted up one level and a right tree which is another copy shifted up two levels.

This means that we can view the Fibonacci tree as follows. From level 3 on we can see a left tree which is simply a copy of the Fibonacci tree above level 1 shifted up one level and a right tree which is another copy of the Fibonacci tree above level 0 shifted up two levels.

It follows that when we observe the sequence of vertex types at any level after level 2 in the Fibonacci tree we see a left part which is identical to the complete sequence of vertex types one level below, together with a right part which is identical to the complete sequence of vertex types two levels below. Thus

<u>Lemma 2:</u> $T_d(F_d) = T_{d-1}(F_{d-1})\, T_{d-2}(F_{d-2})$, where $d \geq 3$ and $T_2(F_2) = l$ and $T_1(F_1) = s$.

It follows immediately from this that

<u>Lemma 3:</u> $T_d(F_d) = T_{d-2}(F_{d-2})\, T_{d-3}(F_{d-3})\, T_{d-2}(F_{d-2})$, where $d \geq 4$.

Lemma 2 shows us that at level d the first F_{d-1} terms are exactly the same as the first F_{d-1} terms on the previous level. Lemma 3 together with Lemma 2 shows us that at level d the last F_{d-2} terms are exactly the same as the first F_{d-2} terms at the beginning of this level.

Consequently, as we go up the tree we are simply uncovering more of the same sequence which we will show to be a Golden string.

Firstly we introduce the father function L(j). We define L(j) as the level number of the father of a vertex with level number j. As will be seen in Lemma 7, this is a well defined function independent of d.

<u>Lemma 4:</u> $H(F_d) = [(F_d + 1)\tau] = F_{d-1}$.

This follows from Property 3 and Lemma 1a.

<u>Lemma 5:</u> $L(F_d) = [(F_d + 1)\tau] = F_{d-1}$.

This follows from Property 2 and Lemma 1a.

<u>Lemma 6:</u> $L(F_{d-1} + k) = F_{d-2} + L(k)$, $1 \leq k \leq F_{d-2}$.

<u>Proof:</u> From Lemma 5 we see that the father of the vertex level-numbered F_{d-1} is numbered F_{d-2}. But from Lemma 3 the sequence of vertex types in the range $F_{d-1} + 1 \leq j \leq F_d$ is exactly the same as in the range $1 \leq k \leq F_{d-2}$ in which case the result follows.

<u>Lemma 7:</u> $L(j) = [(j + 1)\tau]$, $j \geq 1$.

Assume that the assertion is true for $1 \leq j \leq F_{d-1}$. Then from Lemma 6, over the range, $1 \leq k \leq F_{d-2}$ we have

$$L(F_{d-1} + k) = F_{d-2} + [(k + 1)\tau],$$

$$= [(F_{d-1} + k + 1)\tau], \text{ from Lemma 1.}$$

That is as the result is true for $1 < j < F_{d-1}$, it is true for $F_{d-1} < j < F_d$. In this way it is true for all values of j between consecutive Fibonacci numbers. This together with Lemma 6 proves our result.

Lemma 8: $H(j) = [(j + 1)\tau], j > 1$.

Proof: Suppose that we now renumber our vertices according to Hofstadter-numbering. This means that if we are at level d the vertex, previously level-numbered n, is now Hofstadter-numbered $j = F_{d+1} + n$.

Hence for $1 \leq n \leq F_d$

$H(F_{d+1} + n) = F_d + L(n)$

$= F_d + [(n + 1)\tau]$, from Lemma 7

$= [(F_{d+1} + n + 1)\tau]$, from Lemma 1.

GOLDEN STRING RELATIONSHIPS

Theorem 9: The type sequence at each level is a Golden string, that is $T_d(F_d) = S(F_d)$.

To prove this we show that β has a one to one correspondence with the type sequence-that is a zero corresponds to an "s" and a unit corresponds to an "l".

Through Lemma 7 and definition 6, it is seen that if $\beta_j = 0$ then vertex j and its predecessor have the same father, and if $\beta_j = 1$ then vertex j and its predecessor have different fathers.

We consider two cases:

Case a: The jth vertex is type s. Now from property F we know that such a type s vertex can be generated only from a type l vertex. We also know that the predecessor of vertex j is a type l and that both vertices have the same type l father. Consequently $\beta_j = 0$.

Case b: The jth vertex is type l. Now from property F we know that such a type l vertex can be generated in either of two ways.

Its father was type s and thus its predecessor, vertex $j - 1$, has a different father. Alternatively the father of vertex j was a type l vertex and again it follows that vertex $j - 1$ has a different father. Hence $\beta_j = 1$.

Thus the theorem follows.

Note

As the correspondence with the beta sequence is independent of the level it follows that the type sequence at each level simply uncovers more of the infinite Golden string

lsllslsllslls ... .

The results we have established for the type sequence now apply to the Golden string. In particular Lemma 2 becomes

Lemma 10: $S(F_j) = S(F_{j-1})\, S(F_{j-2})$, for $j \geq 3$, with $S(F_2) = l$ and $S(F_1) = s$. (S)

However, we should point out that using the definition of β_k we can readily obtain this result by expressing

$$S(F_j) = S(F_{j-1})\ X, \text{ where X is some string of length } F_{j-2}.$$

Generating the Golden String S(j)

Using (S) we can immediately generate Golden strings starting with the strings at level 3, S(2) = ls and level 2, S(1) = l.

Thus as we go from level d $-$ 1 to level d the string is increased in length by the fraction F_d/F_{d-1} which approaches $1 + \tau$ high up the tree.

It is also seen that if we are at level d for d $>$ 3, and we trace ancestors down the tree to level d = 3 which is simply the string "ls" then, all of the vertices in $S(F_{d-1})$ have the common ancestor "l" and all of the vertices in $S(F_{d-2})$ have the common ancestor "s".

This is the basis of our fast method for generating the Golden string - supposing we have generated the F_d vertices at level d, then we simply replace every type l vertex by $S(F_{d-1})$ and every type s vertex by $S(F_{d-2})$.

Knowing that $(F_{d-1})^2 + (F_{d-2})^2 = F_{2d-3}$, it is seen that the fractional increase in the length of our Golden string $S(F_d)$ with this fast technique is F_{2d-3}/F_d which at the top of the tree aproaches $(1 + \tau)^d/(1 + \tau)^3$.

Thus it is seen that any partition of a Golden string, into two Golden strings equal in length to successive Fibonacci numbers, generates another Golden string, if we replace the longer string by "l" and the shorter by "s".

THE HOFSTADTER RECURRENCE RELATIONSHIP

Lemma 11: In the Golden string S(j)

a) the number of type l vertices is $L(j) = [(j + 1)\tau]$.

b) the number of type s vertices is $[\{([j\tau]+1)\tau\}] = L(L(j-1))$.

Proof:

a) Consider $T_d(j)$ which we now know to be a Golden string of length j from Theorem 9. Now suppose

$$j = j_l + j_s$$

where j_l = number of type l vertices and j_s = number of type s vertices in this string.

Then the string of fathers of the vertices in the string $T_d(j)$ is from Lemma 7, $T_{d-1}([(j + 1)\tau])$. But from property F each of these is the father of a type l vertex in the next generation and thus $j_l = [(j + 1)\tau]$ as asserted.

b) Now each of the type l vertices in the string $T_{d-1}(j_l)$ generates a type s vertex in level d.

However, if in level d the (j + 1)th vertex is type s then the jth and (j + 1)th vertex form a pair of sons "ls" from a common father which is of course type l at the j_lth position at level d$-$1. Thus including this j_lth type l vertex generates one too many type s vertices within $T_d(j)$.

Now we know from case b) of Theorem 9 that

$$j_i - 1 = [(j + 1)\tau] - 1 = [j\tau]$$

and thus we can ignore this j_ith type l vertex in level d−1 by considering only fathers of vertices in the string $T_{d-1}(j_i - 1)$.

On the other hand if in level d the (j + 1)th vertex is type l then each type l vertex in $T_{d-1}(j_i - 1)$ does indeed generate a type s vertex within $T_d(j)$. But in this case from (a) of Theorem 9, $[(j + 1)\tau] = [j\tau]$. Thus in both cases

$$\begin{aligned} j_s &= [(j_i - 1)\tau] = L(L(j - 1)) \\ &= [\{([j\tau] + 1)\tau\}] = j - [(j + 1)\tau]. \end{aligned}$$

Part (a) of our lemma can be proved by noting that from definition 6

$$\sum_{i=1}^{j} \beta_i = [(j + 1)\tau] \,.$$

We also note that the ratio of type s to type l vertices in the Golden string S(j), is $j/[(j + 1)\tau] - 1$ and that this approaches τ as j increases.

Consider the recurrence relationship

$$U(n) = \begin{cases} 0, & n = 0, \\ n - U(U(n - 1)), & n \geq 1. \end{cases} \tag{R}$$

From Lemma 11, L(n) satisfies (R). Hence, from Lemmas 7 and 8, we can conclude the following.

<u>Theorem 12:</u> $U(n) = H(n) = L(n) = [(n + 1)\tau]$.

(R) was first described by Hofstadter [3] who claimed, but did not prove, that it was satisfied by H(n).

Meek, D. S. and van Rees G. H. J. [5] consider a generalisation of (R) namely

$$f_r(n) = n - f_r(f_r\ldots(f_r(n-1))\ldots), \quad n \geq 1,$$

where f_r is nested to r levels and thus $f_2(n)$ is our U(n). The solution in this case is given in terms of a Fibonacci base representation but was not simplified into a closed form solution.

Downey, P. J. and Griswold, R. E. [1] are concerned with a more general term

$$g_k(n) = n - g_k(g_k(n - k)), \quad n \geq 1,$$

where the nesting is the same as in (R) but the argument n − 1 has been extended to n − k. Thus $g_1(n)$ corresponds to our U(n) which is a particular case of [1, Lemma 2]. As a particular case of [1, Theorem 1] they proved that $U(n) = [(n + 1)\tau]$.

OTHER APPLICATIONS

On the Ancestory of a Male Bee

The Fibonacci tree is also relevant to the well known problem of tracing the antecedents of a honeybee. With "l" representing the female (queen) and "s" representing the male it is seen that if vertex generation rules (Property F) are reversed in direction then they reproduce the parent bee relationships.

Sequence of Gap Types

Suppose that n points are laid down in sequence around the circumference of a circle such that each point is at a fixed angle of τ revolutions from its predecessor. Now examine the sequence of points arranged in order of their clockwise arc length around the circumference from some origin and call a gap the arc length between any two successive points in this sequence. It is found that when n is a Fibonacci number the gaps come in only two sizes. This is a particular case of the Theorem first proved by Sós [6]. The sequence of gap types (with l for large and s for small) corresponds to a Golden string (see Tognetti et al [9], van Ravenstein et al [10], [11]).

ACKNOWLEDGEMENT

We sould like to express our gratitude to the referee for his patience, meticulous attention to detail and generosity.

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This delightful mathematical romp was written by a brilliant wide ranging Polish mathematican who amongst many other achievements first postulated what we now call the three gap theorem.

For an interesting account of his work see *Mathematical Monthly* June-July 1974, "Hugo Steinhaus - A Reminiscience and a Tribute" by Marc Kac.

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THREE NUMBER TREES - THEIR GROWTH RULES AND RELATED NUMBER PROPERTIES

J. C. Turner

1. INTRODUCTION

This paper reviews the discoveries we have made in recent years with regard to three kinds of tree graph, each kind being developed according to certain *growth* rules, and each having integers or rational numbers assigned to the tree nodes by means of *coloring* (or *weighting*) rules. We call such structures *number trees.*

The first kind [7], is a sequence of rooted trees which are generated by a Fibonacci-type recurrence construction method. As each tree is generated, its nodes are assigned the integers from a given coloring set. The resulting trees have many interesting structural properties; and certain sets derived from the node-weights on paths in the trees yield number-theoretic results (e.g. integer representation theorems). We have called these structures *convolution number trees.*

The second kind of tree [8] is grown and colored in an entirely different manner from the first. The colors are positive integers applied in monotone increasing sequence; and the tree growth rules are formulated in terms of the parity of the colors on the leaf nodes from which new growth arises. It is found that the frequencies of various types of integer (e.g. odd, even) which occur on the nodes of these trees at a given level from the tree root satisfy certain recurrence equations, which are derived, studied and generalised in the paper.

The third kind of tree [4] is a complete binary one, whose coloring rules can be defined in terms of continued fractions. In the resulting tree, each node bears a rational number in lowest fractional form. Many interesting relationships are given which are discovered by studying the rational numbers occurring on the nodes along paths traced from the root node upwards. Various Fibonacci identities are derived. A most remarkable property of this tree is that its branching rules model precisely the processes for tying of cylindrical braids; it forms the basis of an entirely new mathematical theory of knots.

Our general purpose in presenting results about three different sequences of number trees is to show how fruitful can be the process of linking together the concepts of 'combinatoric structure' (in this case sequentially defined trees) and 'number sequences' used to color the structures according to given rules. In [6] we give many more examples of this process, and coin the generic name *integer sequence geometry* for the methods and mathematics which arise from the process. Again, in [9] we have applied these same ideas to construct sequences of words, rather than labelled diagrams, and these too have led to many results of interest to those studying integer sequence problems. As we state in [6], the idea of studying numerically labelled diagrams is perhaps as old as recorded mathematical history. It is, for example, closely related to that involved in the study of figurate numbers, which practice dates back at least to the early Pythagoreans in the sixth century B.C. It is related, too, to the wide spectrum of work which involves numbered diagrams such as magic squares, Latin squares, Graeco-Latin squares, block

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 335–350.

designs, and so on. Our approach however, would seem to have a strong point of departure from other work in that we consciously seek methods of sequential construction and of labelling the resulting structures which will lead to interesting combinatoric properties and new results about integer sequences and integer or other number sets.

The next Section treats sequences of convolution number trees.

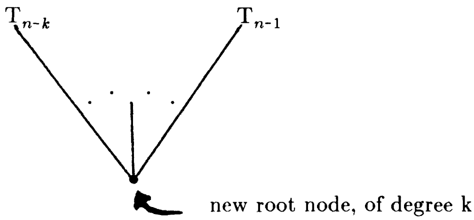
2. CONVOLUTION NUMBER TREES

Convolution trees are constructed sequentially, the nth tree in the sequence being formed by taking certain subsets of the previous $n-1$ trees and mounting them on a fork base. In [7] we introduced the following notation for kth order rooted trees:

$$T_n = \bigvee^{k} (T_{n-k}, \ldots, T_{n-1}).$$

This means that T_n is constructed from the k previously constructed trees by mounting them on a k-fork thus:

$T_n =$



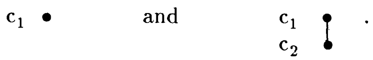
Given a sequence of colors $c = \{c_1, c_2, \ldots\}$, we can assign these sequentially to the nodes of the trees by the following *coloring rule* (drip-feed principle):

(i) assign color c_n to the root node of T_n;

(ii) use the previously colored trees unchanged when mounting them on the k-fork.

To complete the definition of a sequence of colored convolution trees, it is necessary to give the k initial trees of the sequence, with their colorings.

In [7] we studied the cases k = 2 (Fibonacci) and k = 3 (tribonacci) and then obtained some properties for a general sequence of the kth order convolution trees. To illustrate the types of results that can be obtained from number trees, we give the second-order case, using the initial trees:



Below we give the first four members of the resulting convolution tree sequence.

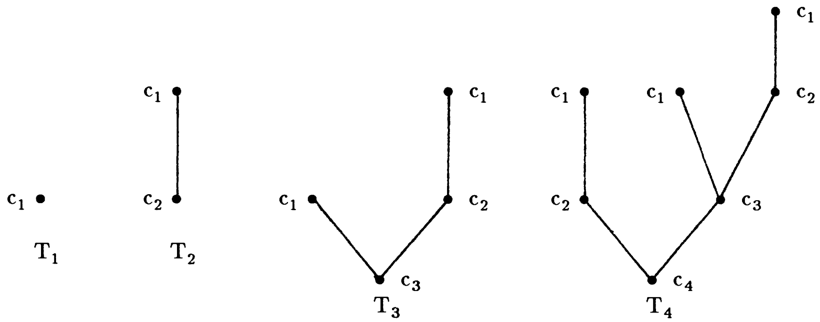


Figure 1. The First Four Fibonacci Convolution Trees.

The convolution trees have many interesting graph properties, independent of their node colorings. For example, the *n*th tree in the sequence has the following properties (F_i is the *i*th Fibonacci integer):

Total number of nodes:	$\sum_{i=1}^{n} F_i = F_{n+2}-1$
Total number of edges:	$\sum_{i=1}^{n} F_i - 1$
Number of nodes of degree 1:	F_n
Number of nodes of degree 2:	$F_{n-1}+1$
Number of nodes of degree 3:	$F_n - 2$
Number of leaf nodes:	F_n
Height:	$n-1$.

$$\left.\begin{array}{l} F_n \\ F_{n-1}+1 \\ F_n-2 \end{array}\right\} \quad n > 2$$

If we now consider the distribution of the colors upon the nodes of the *n*th tree, we observe that the tree has:

F_n nodes with color c_1,
F_{n-1} nodes with color c_2,
... ...
F_1 nodes with color c_n.

Hence the ***weight*** of the tree (defined to be the sum of its colors) is equal to:

$$F_n c_1 + F_{n-1} c_2 + \ldots + F_1 c_n = (\mathbf{F} * \mathbf{c})_n,$$

the *n*th term in the convolution of the Fibonacci and color sequences. This result supplied the reason for us naming the graphs 'convolution number trees'; a general proof of it is easily found by the induction method.

From a consideration of the tree sequence it follows immediately that $(F * c)_n = (F * c)_{n-1} + (F * c)_{n-2} + c_n$. We call this the fundamental convolution property.

In [7], we studied the *level counting function*, defined as follows:

$$\binom{n}{m \mid i} \equiv \begin{array}{l}\text{the number of nodes in the } n\text{th tree} \\ \text{at level } m \text{ which have color } c_i.\end{array}$$

Connections between this number and the binomial coefficients are quickly found. Indeed, if a two-way table be drawn up for $\binom{n}{m \mid i}$, with m rows and n columns, Pascal's triangle appears, supported on the leading diagonal. The row sums are discovered to be powers of 2, and the column sums form the Fibonacci sequence. This table immediately gives the result (Lucas, 1876) that:

$$F_n = \sum_m \binom{m-1}{n-m}, \text{ with appropriate limits for m, n.}$$

And the table generalises (see [7]) in the *k*th order case, providing Pascal-T triangles with row sums being powers of *k* and column sums being terms of a *k*-nacci sequence.

A final demonstration of the use of convolution trees is to supply a graphic proof of the so-called Zeckendorf dual theorem on integer representations. In order to present this, we must first define the ***shade*** of a rooted tree.

Consider the set of root-to-leaf paths in a colored, rooted tree, taken in order from the leftmost one to the rightmost one. Define the weight w_i of the *i*th path to be the sum of the colors on its nodes. Then the ***shade*** of the tree is the set $\{w_1, w_2, \ldots\}$ of all path weights found in the tree. Note that this may be a multiset, with some of the weights being repeated.

Let us denote the shade of the *n*th Fibonacci convolution tree by Z_n. Inspection of Figure 1 shows that:

$$\begin{aligned} Z_1 &= \{c_1\}, \\ Z_2 &= \{c_1 + c_2\}, \\ Z_3 &= \{c_1 + c_3, c_1 + c_2 + c_3\}, \end{aligned}$$

and so on. Finally, let us say that the shade of the tree sequence is $\bigcup_n Z_n$.

Now it is a remarkable fact that if the Fibonacci sequence is used as c, for coloring the nodes, it is found that the shade of the resulting tree sequence is exactly the set of positive integers Z^+. Thus $Z_1 = \{1\}$, $Z_2 = \{2\}$, $Z_3 = \{3,4\}$, $Z_4 = \{5, 6, 7\}$, $Z_5 = \{8, 9, 10, 11, 12\}$ and so on, with $\bigcup_n Z_n = Z^+$. Thus each integer N occurs in the shade set uniquely as a sum of one or more distinct Fibonacci integers. Further, it is easy to see from the manner in which the convolution trees are constructed and colored that if F_i is one of the integers in the representation of N, and there is more than one integer in the representation, then one of F_{i-2}, F_{i-1}, F_{i+1}, F_{i+2} must also occur in the representation. In short, there is never a gap greater than one in the subscripts of those Fibonacci integers which occur in a representation.

Finally, we can remove $c_1 = F_1 = 1$ from each path weight, since it occurs on the leaf-node of every path: then the shade sets change to $Z'_1 = \{0\}$, $Z'_2 = \{1\}$, $Z'_3 = \{2,3\}$, and so on. And now the union $\bigcup_{n\geq 1} Z'_n$ covers Z^+ exactly, with each integer N being represented as a sum of members of the sequence $\{U_n\} = \{F_{n+1}\}$ with no gap greater than one occurring in the subscripts of the members in the sum.

That this can be done is precisely the well-known Zeckendorf dual theorem on integer representations in terms of members of $\{U_n\}$.

Thus a convolution tree sequence with Fibonacci colors on the nodes at once demonstrates the completeness with respect to the positive integers of the Fibonacci sequence of integers, and provides an algorithm for writing down a Fibonacci representation of any given integer N, the representation to have the properties required in the Zeckendorf dual theorem.

Many other properties of shades and integer representations are derived in [7], [6], and [10], with reference to convolution trees of various kinds.

In the next Section we define tree sequences which are produced using entirely different growth rules and coloring procedures, but which also have shade sets which partition Z^+.

3. NUMBER TREES WITH PARITY-DRIVEN GROWTH

In [8] we defined and studied a general family of tree sequences, with the tree growth rules being formulated in terms of the parity of the integer colors on the leaf nodes from which new growth arises. Each time a new layer of growth occurs (when the next tree in the sequence is formed) the new leaves are colored by an integer in such a way that the shade of the new tree is a subset of Z^+ in natural order. Thus the coloring rule actually *forces* the tree sequence to have shade equal to Z^+. With this constraint, a sequence of number trees is formed which has many interesting integer sequence properties. Moreover consideration of the root-to-leaf paths in the trees leads to integer representation theorems. The interplay between the notions of parity of node coloring, as used to determine new growth, and the resulting recurrences which are satisfied by the counts of various types of colored node which occur in the trees, is fascinating.

We illustrate these ideas by describing the simplest nontrivial member of the family, which turns out to have many Fibonacci properties.

The (2, 1)-trees (Fibonacci)

The pair of integers (2, 1) are parameter values which determine this member of the family of tree sequences. The values in the pair determine the *growth rules* thus: given the *n*th tree T_n, then to form the next tree T_{n+1} in the sequence, take T_n, as base and make:

(i) each leaf of T_n which bears an *odd* integer grow a 2-fork,
The 2 is the first parameter value in the pair.

(odd)

(ii) each leaf of T_n which bears an *even* integer grow a 1-fork,
The 1 is the second parameter value.

(even)

Thus T_{n+1} is simply T_n together with a new layer of growth, of 2-forks and 1-forks, growing from T_n's leaves. To complete T_{n+1}, the new leaves have to be colored (i.e. assigned an integer each).

The general *coloring rules* are given in [8], and they ensure that the shade of the tree sequence is Z^+. (We define the initial tree to be 1•). We give below diagrams of the first five trees for the (2, 1)-tree sequence, and then invite the reader to check that the shade sets are the integers in natural order. In note (ii) below we state the coloring rule that has been followed, and ask that this be checked against the diagrams.

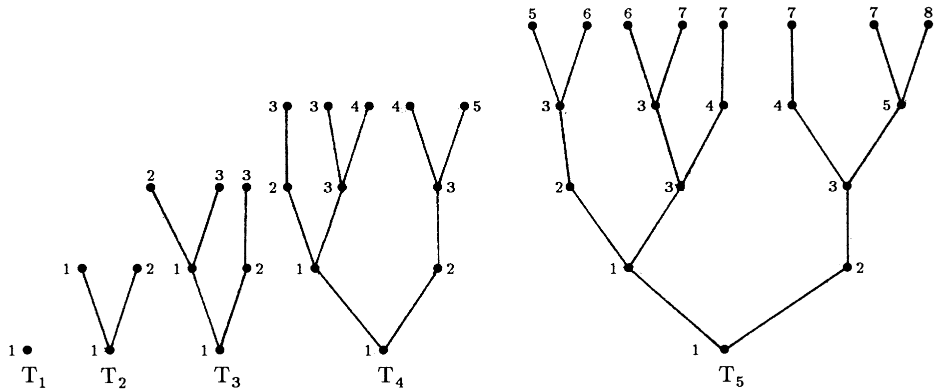


Figure 2. The First Five Fibonacci Parity-Driven Growth Trees

The reader should check from the diagrams that:

(i) the first five shade sets are $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10, 11\}$, $\{12, 13, 14, 15, 16, 17, 18, 19\}$;

(ii) the *coloring rule* for the new leaf nodes of T_{n+1} is the following, reading from left to right on the tree diagram:

a) the first leaf node in T_{n+1} takes the same integer as the last leaf node in T_n;

b) when starting to label leaf nodes of a new fork, use the last integer from the previous fork;

c) when labelling within a fork, assign integers in natural increasing order.

Some properties of the (2, 1)-trees

The following is a list of some of the properties of the Fibonacci (2, 1)-trees. These and other results about the trees are proved in [8].

(i) Number U_n of odd leaves in T_n : $U_n = F_n$.

(ii) Number V_n of even leaves in T_n : $V_n = F_{n-1}$.

(iii) Number W_n of leaves in T_n : $W_n = F_{n+1}$.

(iv) Total number of nodes (t_n) in T_n : $t_n = F_{n+3} - 2$.

(v) The leftmost edge of T_n bears the sequence $\{F_n\}$, from the root up to the leaf.

(iv) The rightmost edge of T_n bears the sequence $\{F_{n+1}\}$.

(vii) The nth shade set Z_n, of tree T_n, is $Z_n = \{F_{n+2}-1, \ldots, F_{n+3}-2\}$, with increments of 1, and the number of elements in Z_n is F_{n+1}.

(viii) The weight of the nth shade set is:

$$W(Z_n) = \tfrac{1}{2}F_{n+1}(F_{n+4} - 3).$$

(ix) The weight of the shade sequence, up to and including Z_n, is:

$$\sum_{i=1}^{n} W(Z_i) = \binom{F_{n+3}-1}{2}.$$

This is easily obtained using (vii). Then using (viii) and (ix) we quickly obtain the identity

$$\sum_{i=1}^{n} F_{i+1}F_{i+4} = F_{n+3}^2 - 4.$$

Other number trees treated in [8]

Now that an example has been given, an indication can be made of the generalisation to (p,q)-trees. A p-fork is a root with p branches from it; similarly a q-fork has q branches from a root node. The growth rules for producing (p,q)-trees are:

To form T_{n+1}, take T_n as base and then

(i) grow a p-fork from each odd integer leaf of T_n, and

(ii) grow a q-fork from each even integer leaf of T_n.

The new leaf-nodes are then to be colored by the same rule as was given above for the (2, 1)-trees.

After deriving properties of the special cases of tree sequences with parameters (1, 1), (2, 1) and (2, 3) respectively, in [8] we obtained results for two general cases, viz.

(i) (p,q)-trees with both p and q even integers;

(ii) (p,q)-trees with both p and q odd integers.

The total number, W_n, of leaves in T_n was shown to satisfy the following recurrences:

Case (i): $W_{n+1} = \tfrac{1}{2}(p+q)W_n$, $n > 1$, with $W_1 = 1$.

Case (ii): $W_{n+2} = \tfrac{1}{2}(p+q)W_{n+1} + \tfrac{1}{2}(p-q)W_n$, $n > 2$, with $W_1 = 1$, $W_2 = p$.

Thus for case (i), W_n is a G.P. with common ratio $r = \tfrac{1}{2}(p+q)$: and for case (ii) W_n is a G.P. with common ratio p if p=q, and satisfies a second-order linear recurrence otherwise. A simple example of (ii) is the (3, 1)-tree sequence, in which W_n follows the Pellian sequence {1, 3, 7, 17, 41, ...}.

Analysis of cases of (p,q)-trees where p and q differ in parity is more complex, and only the special cases of (2, 1)-trees (Fibonacci) and (2, 3)-trees (Pellian) were treated in [8].

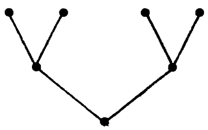
Integer representations

Because the shade of any (p,q)-tree sequence is exactly Z^+, it follows that for any chosen integer N, which occurs, say, in the shade Z_n of tree T_n, there is a unique root-to-leaf path in T_n whose node colors add up to N. Thus the node colors on the path constitute a unique representation of N, whose properties reflect those of the successive sets of colors on the leaf nodes of $T_1, T_2, T_3, \ldots, T_n$. In [8], to illustrate this we give an integer representation theorem which is derived from the right-half subtrees of the members of the (2, 1)-tree sequence.

In the next Section we describe a totally different kind of number tree sequence, [4]. This sequence was discovered when modelling the process of constructing a circular braid which has p parts and b bights. It will be seen later that to each such braid there corresponds a sequence of rational numbers p_1/b_1, p_2/b_2, ..., p/b, each reflecting a stage in the construction of the braid. Moreover, these rational numbers will be the colors assigned to nodes on a root-to-leaf path on a tree, of a tree-sequence which we are about to define. We shall call this a *rational number tree sequence.*

4. A SEQUENCE OF RATIONAL NUMBER TREES

The *growth rule* for our rational number trees (RNT's) is very simple. We obtain T_{n+1} by using T_n as base, then on each leaf-node of T_n we grow a 2-fork. Thus, beginning with T_1 as a single node, T_2 is a 2-fork, T_3 is of the form



and so on. It will be seen that the trees are complete binary in form.

We shall give the *coloring rule*, which assigns a rational number to each new leaf-node of T_{n+1}, directly in terms of continued fractions. This was not the approach used in [4], where the concepts of RNTs and braid-modelling were developed in some considerable detail; but it is convenient to use it here, to shorten the exposition.

Consider a leaf-node of T_{n+1}. Associated with it is a unique root-to-leaf path, which is a sequence of right-branches (Rs) and left-branches (Ls), starting from the root of the tree. Thus the path is uniquely coded by a word, or string, expressed in the symbols {R,L}. An example of a word representing a path to a leaf-node is: RR ... RLL ... LRR ... RLL ... L. Let us now compute the lengths of the subwords which are composed of just one symbol (R or L) in sequence. This gives us a sequence λ_1, λ_2, ..., λ_k, where

λ_1 is the number of Rs which begin the word; (N.B. $\lambda_1 = 0$ if the word begins with an L).

λ_2 is the number of Ls which directly follow the starting string of Rs;

and so on.

The *coloring rule* is as follows:

If λ_1, λ_2, ..., λ_k is the sequence of numbers (lengths of subwords of one symbol) just described, then assign to the leaf-node at the top of the path the rational number given by the continued fraction $[\lambda_1; \lambda_2, \ldots, \lambda_k+1]$.

With these rules, the following tree sequence (to n = 4) results:

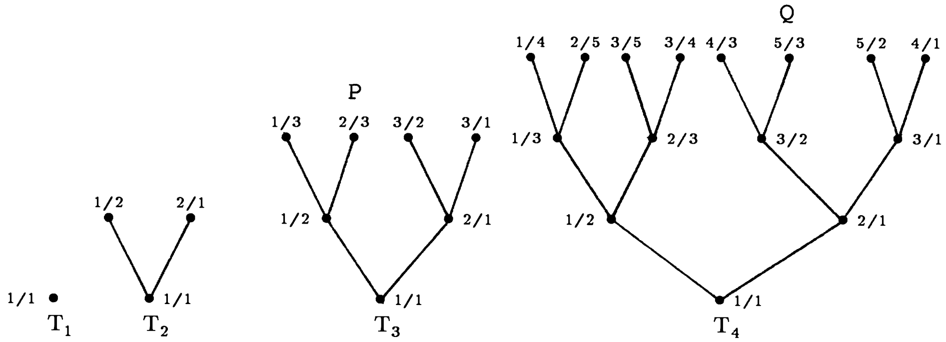


Figure 3. The First Four Rational Number Trees

To illustrate the use of the coloring rule, we shall compute the rational numbers for P (in T_3) and Q (in T_4), thus:

the path to P has code word LR, then $\lambda_1 = 0$, since the word begins with an L, $\lambda_2 = 1$, $\lambda_3 = 1$. Then the rational fraction to assign to P is

$$[0;\, 1,\, 2] = 0 + \frac{1}{1+\frac{1}{2}} = 2/3.$$

The path to Q has code work RLR, giving the λ-vector (1,1,1); hence the rational number to assign to Q is:

$$[1;\, 1,\, 2] = 0 + \frac{1}{1+\frac{1}{2}} = 5/3.$$

Properties of the RNTs

Many interesting results may be discovered about the occurrences of integer sequences, as numerators or denominators of the rational numbers on nodes of paths in the trees. We have derived many such results in [4]. A very simple example is to observe the Fibonacci sequences occurring on the following two paths:

	LRLR ...	$\Rightarrow$	1/1,	1/2,	2/3,	3/5,	...
and	RLRL ...	$\Rightarrow$	1/1,	2/1,	3/2,	5/3,	... .

A generalisation of this arises from a study of what we call *a-step ladders* in $T_{L,n}$, the left-subtree stemming from the root of T_n. Notice that the rational numbers on the nodes of this subtree are all ≤ 1.

We define these ladders as follows:

Let P_i be a node of $T_{L,n}$, and q_i its associated rational number. Then the *a-step ladder* of length *am* from P_i is defined by:

$L(i,m,a) \equiv L^a\, R^a \ldots L^a$ (or R^a), which begins at P_i and has m multiple-steps, such as L^a and R^a, taken alternately.

<u>Proposition:</u> Let the rational numbers on the ladder $L(i, m, a)$, on the nodes which initiate each multiple-step, be the sequence:

$$q_i = p_i/b_i,\quad q_{i+1} = p_{i+1}/b_{i+1}, \ldots, q_{i+m} = p_{i+m}/b_{i+m}.$$

Then the following relations hold:

$$\left.\begin{aligned} ap_{i+m} &= v_{m+1}(p_{i+1} - p_i) + au_{m+1}p_i \\ ab_{i+m} &= v_{m+1}(b_{i+1} - b_i) + au_{m+1}b_i \end{aligned}\right\}$$

where $\{u_j\}$ and $\{v_j\}$ are two sequences satisfying the following conditions:

$$u_{j+2} = au_{j+1} + u_j, \quad \text{with } u_1 = u_2 = 1,$$

and $$v_j = u_{j+1} - u_j, \quad \text{for } j \geq 1.$$

The proof involves only elementary properties of Fibonacci sequences, but is somewhat lengthy and will be omitted. We make the following remarks about consequences of the proposition:

(i) If $a = 1$, and $q_i = \dfrac{F_i}{F_{i+1}}$, the relation given for p_{i+m} yields the well-known Fibonacci identity:

$$F_{i+m} = F_m F_{i-1} + F_{m+1} F_i\ .$$

(ii) Interesting forms of the above relations occur when:

(a) $p_{i+1} - p_i = ja$, with j any chosen integer,

and (b) m is a multiple of i.

(iii) We can show that the limit of p_{i+1}/p_i as $i \to \infty$ is equal to the largest root of the equation $x^2 - ax - 1 = 0$. Then, in terms of recurring continued fractions, we can express this as:

$$\begin{aligned} \lim_{i\to\infty}(p_{i+1}/p_i) &= \frac{a + \sqrt{a^2 + 4}}{2} \\ &= [a;a,a,a, \ldots] = [a;\bar{a}]. \end{aligned}$$

It follows that $\sqrt{a^2 + 4} = a + 2[0;\bar{a}]$ (positive root).

We shall mention only one further property of the rational number tree sequence, before going on to describe its relationship with the tying of braids. This we call the *fundamental property* of the RNT sequence.

Proposition: Let S_n be the set of rational numbers assigned to the nodes of T_n, and let n tend to infinity. Then $S_n \rightarrow Q^+_*$, the set of all positive rational numbers in their lowest form.

The proof will not be given in detail; but it should be evident that the words in L,R, as generated by the root-to-leaf paths in the complete binary trees of the sequence $\{T_n\}$, and the coloring rule given above, will generate all possible continued fractions of the form $[\lambda_1; \lambda_2, \ldots, \lambda_k+1]$, for $k = 1,2, \ldots$ And a well-known theorem states that any rational number can be expressed uniquely as a simple terminating continued fraction having 1 as the last quotient. Now $[\lambda_1; \lambda_2, \ldots, \lambda_k+1] = [\lambda_1; \lambda_2, \ldots, \lambda_k, 1]$. So it is clear that any element of S_n is in Q^+_*; and any element of Q^+_* will be in S_N for some N.

We see, then, that the RNT sequence generates all the rational numbers, and all are in their lowest forms.

We turn now to a description of the process of tying cylindrical braids and show how this is related to the RNT sequence.

5. CYLINDRICAL BRAIDS AND THE RATIONAL NUMBER TREES

The classical theory of braids was introduced by E. Artin [1] in 1925. He began with a general definition of an n-string braid, and defined a group operation for combining two braids. He was then able to develop a theory of braid groups, and to compute braid invariants from them. Classical knot and braid theory is very much concerned with classification problems, using topological invariants derived from groups or polynomials. The theory that we are about to describe (we shall merely introduce it, with an elementary example) has been discovered recently by A. G. Schaake and is described in [4]. It is concerned rather with modelling the actual process of tying braids, and it turns out that the mathematical theories required are those of recurrence equations, continued fractions and the solution of diophantine equations. We shall see how in the case of our example, forming a braid by tying it around a cylinder corresponds to taking an upward walk from the root of the rational number tree to one of the higher nodes.

The diagrams below show a 2 part-3 bight turk's head knot (commonly called a trefoil); on the left it is shown flat, whereas on the right it is shown tied around a cylinder.

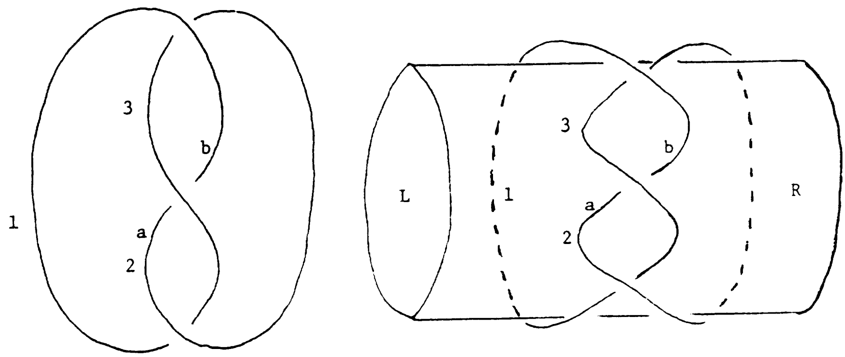


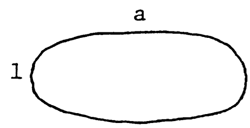
Figure 4. The 2/3 Turk's Head Knot.

A *bight* is a turn, or change in direction (L → R to R → L, or vice-versa) in the string of a knot. We have numbered the three left-hand bights; and these are the only ones counted when classifying a regular cylindrical knot (there is always an equal number of right-hand bights). The two parts on the middle, upward, L → R pass of the string are those labelled *a* and *b*. Notice that each pass, whether from L → R or from R → L has two parts. Hence the name *2/3 braid.*

The six diagrams on the next page show how the knot can be formed in six stages. Each stage requires either a L → R pass of the string, or else a R → L pass. On the first three passes, no string crossing has to be made; whereas on the final three passes there occur in succession an under-, an over- and an under-crossing. The letters S, W refer respectively to the *Standing* (or fixed) portion of the string and the *Working* (or moving) portion of the string.

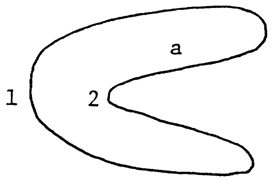
If at the end of the sixth stage the working end is glued to the standing end, the result will be the 2/3 turk's head braid.

If S and W were to be joined at the end of stage 2, the knot completed at that point would be as shown:



This is the 1/1 braid (topologically equivalent to the *unknot*).

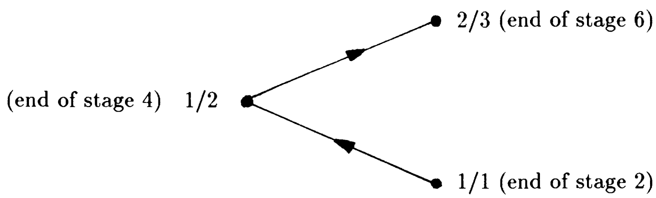
And if S and W were joined at the end of stage 4, the knot completed would be as shown next:



This is the 1/2 braid (also topologically equivalent to the unknot).

Thus on the way to producing the 2/3 braid we have passed through points at which first the 1/1 braid, and then the 1/2 braid, would have been formed.

Referring back to the diagram of the rational number tree (Figure 3), we find a path from the root thus:



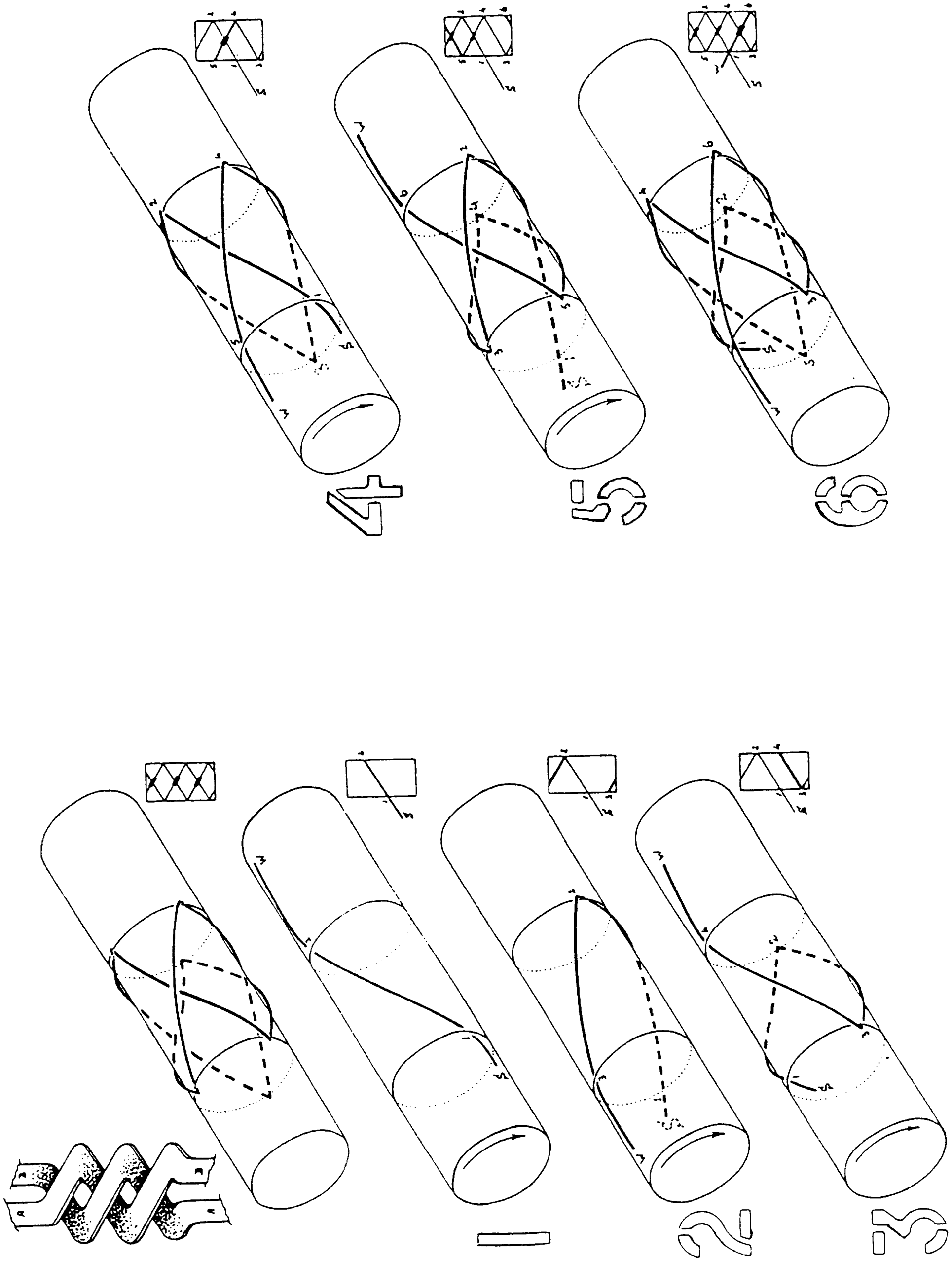


Figure 5. The Six Stages in the Formation of the 2/3 Braid

The correspondence of this path to the braid-tying process is now evident with regard to the equivalences of braids formed at even-numbered stages and the values of the rational numbers at the nodes. The correspondence goes much further than that, however. If we refer to the diagrams of Figure 5, we may note that before commencing stage 3 (and taking the second L → R step) we did so *without first taking the working part over the standing part.* Whereas, in stage 4, after almost completing a 1/2 braid, we *did take W across S* before commencing stage 5.

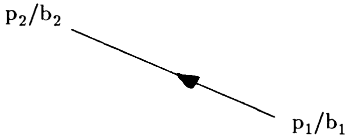
This gives a second kind of correspondence, namely:

(i) taking a left branch in the RNT corresponds to W *not crossing* S just before a new left-hand bight is formed; whereas

(ii) taking a right branch in the RNT corresponds to W *crossing* S just before a new left-hand bight is formed.

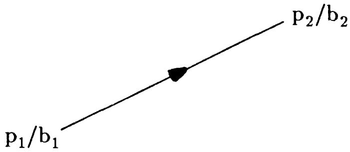
A further correspondence, which is demonstrated and proved in [4], leads us into simple diophantine theory. The following diagrams and equations show the relationships which must hold between the numbers of parts and bights of two consecutively produced braids. The two possible cases which correspond respectively to a left- and a right-branch in the rational number tree are:

Left branch *equation satisfied*



$\Delta p_2 b_1 - p_1 \Delta b_2 = -1$, where
Δ is the first difference operator

Right branch *equation satisfied*



$\Delta p_2 b_1 - p_1 \Delta b_2 = +1$.

Thus if the values for p_1/b_1 are given, the braid p_2/b_2 can only be formed by making a series of passes of string, L → R then R → L, and so on, around the cylinder, and continuing from the braid p_1/b_1 (actually making an enlargement from it) *if* the values of Δp_2 and Δb_2 jointly satisfy (and are the first solutions of) one or other of the above two equations.

It is seen that in order to produce a braid with a given number of parts (p) and of bights (b) we may find the node p/b in the rational number tree, trace the path from the root 1/1 to the node p/b, and then make the appropriate number of L → R and R → L passes of string around a cylinder, taking care to make W cross S (or not cross S) according to the rules given above. It must be noted, however, that this is not the whole story; more information is needed in order to know when under- and over-crossings are to be made as the braid tying proceeds. We call this further information, which is a certain sequence of unders and overs for each string pass, the *coding run* for the braid. In [5] we show how coding run algorithms can be worked out for any required braid corresponding to a node in RNT.

6. SUMMARY

In this paper we have shown how three different kinds of number tree sequence can be produced, by giving rules for growing the members of the sequences and for coloring the nodes of each tree as it is formed.

In the first two cases, the nodes were colored by positive integers; and in these cases the properties of the trees which were of interest were (i) numerical counts and relationships reflecting structural properties of the trees and (ii) properties of sets of integers (shade sets) arising as colors of nodes on root-to-leaf paths in the trees. From the shade sets it was possible to infer integer representation theorems relating to the integers in the coloring sequence and the structures of the shade sets.

In the third case, th: RNT, the nodes were colored by rational numbers, with the coloring rules being defined in terms of continued fractions. Some number properties relating to the structure of these trees were derived, and Fibonacci identities deduced. Finally, it was shown how the root-to-leaf paths in the RNT correspond in several vital aspects to the process of tying cylindrical braids. In short, we may claim that the RNT is a model of an infinite class of cylindrical braids (bearing in mind that it only supplies the information for the string runs around the cylinder; the coding runs have to be calculated separately). In [4] we have called this the *regular knot class*, and have used it as a basis for a classification of braids according to their tying rules. This work is truly applied integer sequence theory, the object set being the universe of knots. We have been unable to trace any but hints of earlier mathematical treatments of braid tying in the literature (for example in the encyclopaedic works [2], [3]), and these hints refer to elementary and apparently purely-empirically derived formulae. We find this remarkable, since knot and braid tying has been a widely-practised craft, and art, for millennia.

We hope that our general aim, of demonstrating the usefulness of examining combinatoric structures (trees in this paper) which have been produced and node-colored simultaneously by joint sets of rules, has been achieved.

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G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers, 351–357.

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Fibonacci Numbers and Their Applications

Proceedings of 'The First International Conference on Fibonacci Numbers and Their Applications'
University of Patras, Greece, August 1984

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