Progress Report

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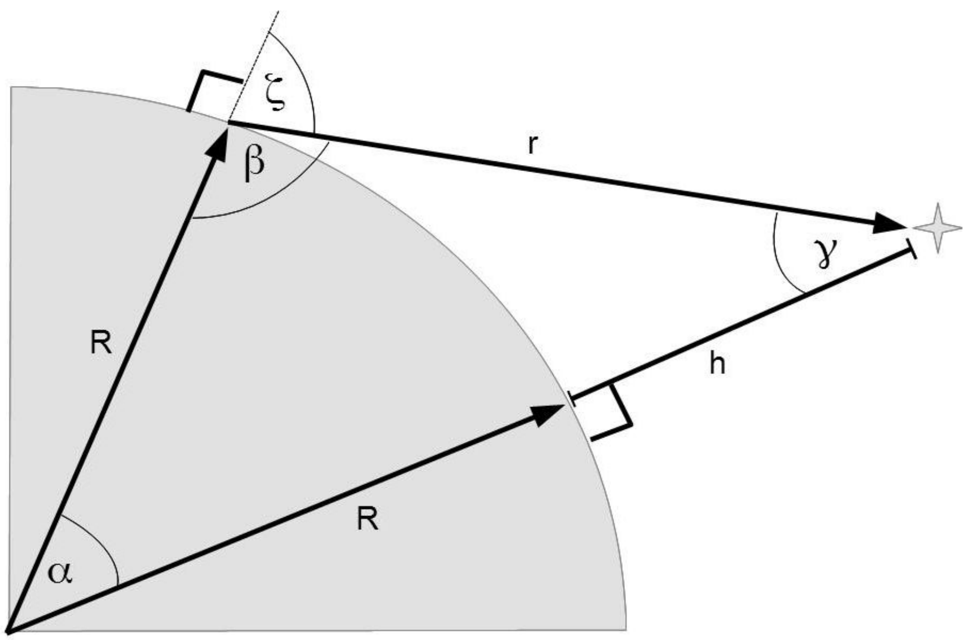
1 Introduction



Using the diagram above we were able to come up with exact equations for h,r, and ζ. Using the cosine law we have:

$$(R+h)^2 = r^2 + R^2 - 2rR\cos\beta \tag{1}$$

Taking the square root of both sid and rearranging we see:

$$h(r,\beta) = \sqrt{r^2 - 2rR\cos\beta + R^2} - R \tag{2}$$

And again from the diagram we see $\beta = \pi - \zeta$ and the final formula is given by:

$$h(r,\zeta) = \sqrt{r^2 + 2rR\cos\zeta + R^2} - R \tag{3}$$

Multiplying out the left hand side of equation 1 and rearranging we see:

$$r^2 - 2rR\cos\beta - h^2 - 2hR = r^2 + 2rR\cos\zeta - h^2 - 2hR = 0 \quad (4)$$

Applying the quadratic formula where $a = 1$, $b = 2R\cos\zeta$ and $c = -h^2 - 2hR$ we have the solutions:

$$r = \pm\sqrt{R^2\cos^2\zeta + h^2 + 2hR} - R\cos\zeta \quad (5)$$

Of which the only correct physical solution is given by:

$$r(h,\zeta) = \sqrt{R^2\cos^2\zeta + h^2 + 2hR} - R\cos\zeta \quad (6)$$

Again expanding the left hand side of equation 1 we see:

$$h^2 + 2hR + R^2 = r^2 + R^2 - 2rR\cos\beta = r^2 + R^2 + 2rR\cos\zeta \quad (7)$$

Rearranging the terms:

$$\cos\zeta = \frac{h^2 + 2hR - r^2}{2rR} \quad (8)$$

And finally:

$$\zeta(r,h) = \cos^{-1}\left\{\frac{h^2 + 2hR - r^2}{2rR}\right\} \quad (9)$$

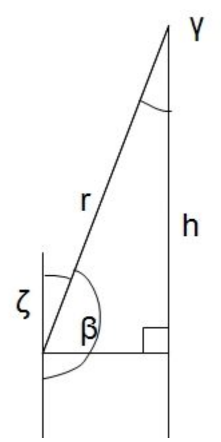
Equations (3), (6) and (9) are all exact solutions. We also used some approximations for simpler results. When $h << R$ and remembering $\beta = \pi - \zeta$ equation (1) reduces to:

$$R^2 = R^2 + r^2 + 2rR\cos\zeta \quad (10)$$

And it is easily seen that:

$$r = -2R\cos\zeta \quad (11)$$

Where ζ will always have an angle greater than 90° when the distance h is negligible.



When α is small, as shown in Figure 2 the portion of the circle's surface between the two points approximates a straight line, completing a triangle with r and h. For this triangle we can see that γ is equal to ζ, and h is given by

$$h(r, \zeta) = r \cos \zeta \tag{12}$$

When β is close to π, $\cos \beta = 1$ so the right-hand side of Equation 1 is a perfect square:

$$(R + h)^2 = (R + r)^2 \tag{13}$$

This implies that r and h must be equal.