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Direct transformation from geocentric coordinates to geodetic coordinates

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Abstract. The transformation from geocentric coordinates to geodetic coordinates is usually carried out by iteration. A closed-form algebraic method is proposed, valid at any point on the globe and in space, including the poles, regardless of the value of the ellipsoid's eccentricity.

Keywords: Coordinate transformation – Geocentric coordinates – Geodetic coordinates

1 Notation

a, b, e	=	semi-major axis, semi-minor axis, eccentricity of reference ellipsoid
X, Y, Z	=	Cartesian geocentric coordinates
λ, φ, h	=	geodetic longitude, geodetic latitude, geodetic height

2 Description of the method

The geocentric coordinates are related to the geodetic coordinates by the following formulae (see Fig. 1):

$$X = (h + n)\cos\varphi\cos\lambda \tag{1}$$

$$Y = (h + n)\cos\varphi\sin\lambda \tag{2}$$

$$Z = (h + n - e^2 n)\sin\varphi \tag{3}$$

where

$$n = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}} \tag{4}$$

2.1 Determination of h and φ

Let us introduce the strictly positive coefficient

$$k = \frac{QS}{PR} = \frac{h + n - e^2 n}{n}$$

which allows an *algebraic* computation, without ambiguity, of the values of h and φ. We show that k is a solution of an algebraic equation of degree 4. From the expression of k, we extract h

$$h = (k + e^2 - 1)n \tag{5}$$

and Eq. (3) gives

$$\sin\varphi = \frac{Z}{kn} \tag{6}$$

After substitution of $\sin\varphi$ from Eq. (6) into Eq. (4) squared, we extract n

$$n = \sqrt{a^2 + e^2\frac{Z^2}{k^2}} \tag{7}$$

Squaring Eqs. (1) and (2) and replacing h by its value given by Eq. (5), we obtain

$$X^2 + Y^2 = (h + n)^2\cos^2\varphi = (k + e^2)^2 n^2\cos^2\varphi$$

i.e.

$$\cos\varphi = \frac{\sqrt{X^2 + Y^2}}{(k + e^2)n} \tag{8}$$

$$X^2 + Y^2 = (k + e^2)^2 n^2(1 - \sin^2\varphi)$$

In this equation, by replacing $\sin\varphi$ by Eq. (6) and n by Eq. (7), we obtain

$$\frac{X^2 + Y^2}{(k + e^2)^2} + \frac{(1 - e^2)Z^2}{k^2} = a^2 \tag{9}$$

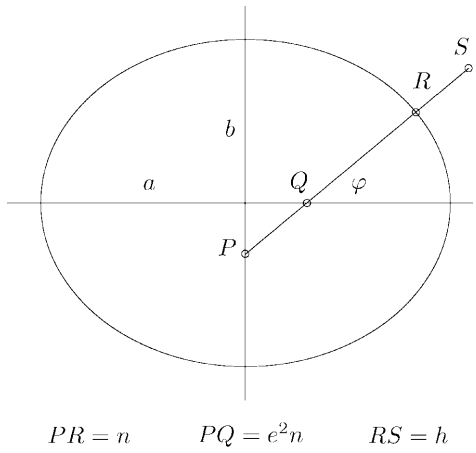

Fig. 1.

Our aim is to solve k from this expression. When we have solved k, h can be computed with Eqs. (5), (7) and (9) and φ with Eqs. (6) and (8). Now, if we write

$$p = \frac{X^2 + Y^2}{a^2}$$
$$q = \frac{1 - e^2}{a^2} Z^2$$

from Eq. (9) we deduce an algebraic equation of degree 4 in k

$$k^4 + 2e^2k^3 - (p + q - e^4)\, k^2 - 2e^2 q\, k - e^4 q = 0$$

Whatever the parameter u is, the above equation may be rewritten as

$$(k^2 + e^2k - u)^2 - [(p + q - 2u)\, k^2 + 2e^2(q - u)\, k + u^2 + e^4 q] = 0 \tag{10}$$

The expression between square brackets will be a perfect square if its discriminant is null, i.e. if

$$e^4(q - u)^2 - (u^2 + e^4 q)\,(p + q - 2u) = 0 \tag{11}$$

or, after expansion

$$2u^3 - (p + q - e^4)\, u^2 - e^4 p\, q = 0$$

Let us introduce two new parameters r and s

$$r = \frac{p + q - e^4}{6}$$
$$s = e^4 \frac{p\, q}{4r^3}$$

then the previous equation in u is reduced, after division by $2r^3$, to

$$\frac{u^3}{r^3} - 3\frac{u^2}{r^2} - 2s = 0 \tag{12}$$

This is an equation of degree 3 in u/r which, since s is positive, has one unique real root which, in addition, is positive. Let us write this root in the form

$$\frac{u}{r} = 1 + m + t \tag{13}$$

Equation (12) becomes

$$t^3 + 3(mt - 1)(m + t) - 2(1 + s) + m^3 = 0$$

We may always impose the condition $mt = 1$, then after multiplying the previous equation by t^3, the equation is reduced to

$$t^6 - 2(1 + s)t^3 + 1 = 0$$

Both (positive) solutions of this equation of degree 2 in t^3 will give the same result for u, so we may choose either of them, for instance

$$t^3 = 1 + s + \sqrt{s(2 + s)}$$

whence

$$t = \sqrt[3]{1 + s + \sqrt{s(2 + s)}}$$

On account of Eq. (13) and since we have imposed $mt = 1$, then u results in

$$u = r\left(1 + t + \frac{1}{t}\right)$$

For this value of u, Eq. (11) is verified and if we let

$$v = \sqrt{u^2 + e^4 q}$$

and substitute the term $p + q - 2u$ of the expression between square brackets in Eq. (10) by its value extracted from Eq. (11), we obtain

$$(k^2 + e^2k - u)^2 - \left(e^2 \frac{q - u}{v} k + v\right)^2 = 0$$

i.e.

$$\left(k^2 + \frac{v - u + q}{v} e^2 k + v - u\right)\left(k^2 + \frac{u + v - q}{v} e^2 k - u - v\right) = 0$$

Since $v - u$, v and q are positive, with reference to the first expression between brackets in the above equation, the result cannot be null for a positive value of k. Let us write

$$w = e^2\, \frac{u + v - q}{2v}$$

Therefore the second expression between brackets will be null if

$$k^2 + 2wk - u - v = 0$$

Since $u + v$ is positive, the only positive solution of this equation of degree 2 in k is

$$k = \sqrt{u + v + w^2} - w$$

Now, the value of k being known, if we write

$$D = \frac{k\sqrt{X^2 + Y^2}}{k + e^2}$$

using Eq. (7) and replacing a^2 by its value given by Eq. (9) we obtain

$$n = \frac{\sqrt{D^2 + Z^2}}{k}$$

and then from Eq. (5)

$$h = \frac{k + e^2 - 1}{k}\sqrt{D^2 + Z^2}$$

In order to prevent computer problems near the poles, we do not compute φ from tangent with Eqs. (6) and (8) because with $\varphi = \pm 90$ deg, $\tan\varphi$ is not defined, neither from sinus with Eqs. (6) and (7) because $\varphi = \arcsin x$ is not accurate ($d\varphi/dx = \infty$) or real (if $|x| > 1$) with x near ± 1, but from tangent of the half value. Let us substitute $\sin\varphi$ and $\cos\varphi$ given by Eqs. (6) and (8) respectively, in the following identity

$$\tan\frac{\varphi}{2} = \frac{\sin\varphi}{1 + \cos\varphi} = \frac{\sin\varphi}{\cos\varphi + \sqrt{\sin^2\varphi + \cos^2\varphi}}$$

We find

$$\varphi = 2\arctan\frac{Z}{D + \sqrt{D^2 + Z^2}}$$

2.2 Determination of λ

Let us now give a formula for λ valid for $X = 0$. First , we substitute $\sin\lambda$ and $\cos\lambda$ extracted from Eqs. (1) and (2), respectively, in the following identity:

$$\tan\frac{\lambda}{2} = \frac{\sin\lambda}{1 + \cos\lambda} = \frac{\sin\lambda}{\cos\lambda + \sqrt{\sin^2\lambda + \cos^2\lambda}}$$

We find

$$\lambda = 2\arctan\frac{Y}{X + \sqrt{X^2 + Y^2}}$$

3 The algorithm

Starting from the Cartesian coordinates X, Y, Z, first we compute the value of k and D by the following sequence of formulae

$$p = \frac{X^2 + Y^2}{a^2}$$

$$q = \frac{1 - e^2}{a^2}Z^2$$

$$r = \frac{p + q - e^4}{6}$$

$$s = e^4\frac{p\,q}{4r^3}$$

$$t = \sqrt[3]{1 + s + \sqrt{s(2 + s)}}$$

$$u = r\left(1 + t + \frac{1}{t}\right)$$

$$v = \sqrt{u^2 + e^4 q}$$

$$w = e^2\,\frac{u + v - q}{2v}$$

$$k = \sqrt{u + v + w^2} - w$$

$$D = \frac{k\sqrt{X^2 + Y^2}}{k + e^2}$$

Table 1.

$\sqrt{X^2 + Y^2}$ m	Z m	φ_{article} deg	h_{article} m	φ_{Bowring} deg	h_{Bowring} m
00 000 000.000	−6 359 593.314	−90.000 000 000	00 002 841.000	(See note)	(See note)
05 442 896.133	03 313 081.153	31.500 000 000	−0 000 394.000	31.500 000 000	−0 000 394.000
26 578 137.000	00 000 000.000	00.000 000 000	20 200 000.000	00.000 000 000	20 200 000.000
26 477 160.722	02 312 729.964	05.000 000 000	20 200 000.000	05.000 000 002	20 200 000.000
26 174 989.441	04 607 941.737	10.000 000 000	20 200 000.000	10.000 000 018	20 200 000.002
25 673 890.779	06 868 244.851	15.000 000 000	20 200 000.000	15.000 000 057	20 200 000.007
24 977 627.324	09 076 503.683	20.000 000 000	20 200 000.000	20.000 000 120	20 200 000.020
24 091 431.413	11 215 963.350	25.000 000 000	20 200 000.000	25.000 000 203	20 200 000.044
23 021 969.796	13 270 373.735	30.000 000 000	20 200 000.000	30.000 000 293	20 200 000.078
21 777 298.135	15 224 110.924	35.000 000 000	20 200 000.000	35.000 000 375	20 200 000.122
20 366 805.351	17 062 295.288	40.000 000 000	20 200 000.000	40.000 000 431	20 200 000.168
18 801 147.859	18 770 905.389	45.000 000 000	20 200 000.000	45.000 000 451	20 200 000.209
17 092 173.807	20 336 886.789	50.000 000 000	20 200 000.000	50.000 000 431	20 200 000.238
15 252 837.537	21 748 254.818	55.000 000 000	20 200 000.000	55.000 000 374	20 200 000.248

Note – The calculator cannot evaluate $\arctan(-\infty) = -90$ deg and even with this value of φ, h is indeterminate. Bowring (1985) has given a better expression for h, correct for every φ, namely $h = R\cos\varphi + Z\sin\varphi - a\sqrt{1 - e^2\sin^2\varphi}$. For $\varphi = -90$ deg, this formula gives $h = 2841.000$ m.

Next, we compute the geodetic coordinates λ, φ and h by

$$\lambda = 2\arctan\frac{Y}{X+\sqrt{X^2+Y^2}}$$
$$\varphi = 2\arctan\frac{Z}{D+\sqrt{D^2+Z^2}}$$
$$h = \frac{k+e^2-1}{k}\sqrt{D^2+Z^2}$$

4 Conclusion

In Table 1 we give some results obtained with both the above method and the approximation formulae of Bowring (1976), namely

$$\tan\varphi = \frac{Z+be'^2\sin^3\theta}{R-ae^2\cos^3\theta}$$
$$h = \frac{R}{\cos\varphi} - \frac{a}{\sqrt{1-e^2\sin^2\varphi}}$$

where

$$e^2 = 1-\frac{b^2}{a^2}$$
$$e'^2 = \frac{e^2}{1-e^2}$$
$$R = \sqrt{X^2+Y^2}$$
$$\tan\theta = \frac{a}{b}\frac{Z}{R}$$

For various points in space, with known latitude φ and height h, we computed the values of $\sqrt{X^2+Y^2}$ and Z using Eqs. (1)–(4). With these values as initial data, we next applied the present algorithm and Bowring's formulae in order to recover the known values of φ and h. The computation was carried out with $a = 6\,378\,137$ m and $e = 0.081819191$. The first line of the table corresponds to the South Pole at the height of 2841 m and the second, to a point in the Dead Sea depression of height -394 m and latitude 31.5 deg North. The remaining lines correspond to points at the constant height of 20 200 km, with latitudes taken every 5 degrees from 0 to +55 deg.

All the computations have been performed with a HP 32 SII, 12-digit scientific calculator.

Borkowski (1987), (1989), Heikkinen (1982) and Lapaine (1991) have proposed their own solutions based also on an equation of degree 4. The resulting formulae are more complicated than the method described here.

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