2nd Spanish Fluid Mechanics Conference
Barcelona (2 - 5 July 2023)

Regularized four-sided cavity flows: A spectral bifurcation benchmark implemented in Julia

Moritz Waldleben[1], Álvaro Meseguer[2], Oriol Batiste[2], and Arantxa Alonso[2]

moritz.waldleben@epfl.ch

[1]Section de Mathématiques, École Polytechnique Fédérale de Lausanne, EPFL (Switzerland)
[2]Departamento de Física, Universitat Politècnica de Catalunya, Barcelona, UPC (Spain)

This work presents a regularized version of the four-sided lid-driven cavity for incompressible fluids. The four-sided driven cavity is an extension of the simple lid-driven case where all lids move with the same velocity profile and parallel lids move in opposite directions. Lid-driven cavity flows are used to validate Navier-Stokes solvers. This study focuses on the regularized version to overcome corner singularities. The proposed method recovers near exponential convergence in spectral discretization schemes. The flow presents a variety of bifurcation scenarios and can make up for an amenable Navier-Stokes bifurcation benchmark. The algorithms are implemented in Julia, a high-performance language for scientific computing. A developed Julia module provides a reproducible example of spectral discretization for the proposed benchmark.

1 Introduction

Driven cavity flows are commonly used as benchmarks to validate Navier-Stokes solvers. These problems can test spatial discretization methodologies such as finite elements, finite differences, and spectral methods. They also assess a variety of boundary condition implementations and time-stepping schemes. The simple lid-driven cavity flow has received considerable attention. This flow is steady for high Reynolds numbers, with the first instability due to a supercritical Hopf bifurcation occurring within a large uncertainty interval (7500, 8100) [Kuhlmann (2018)].

More recent variants, such as the four-sided version of the cavity flow, have been proposed [Wahba (2009)]. Here the four lids are moving at the same speed (top-bottom and right-left lids moving rightwards-leftwards and upwards-downwards, respectively). Later works tested different numerical techniques and studied the bifurcations with linear stability analysis, arc-length continuation, and time-stepping [Perumal (2011), Cadou (2012), Chen (2013)]. This cavity has the computational advantage of exhibiting a variety of bifurcations at low or moderate Reynolds numbers.

Still, the problem suffers from corner singularities due to the discontinuous boundary conditions and affects the exponential convergence of spectral methods. This work presents a regularized version of the four-sided cavity flow to address the issue. A spectral Chebyshev discretization of the flow problem is implemented in Julia, a high-performance language for scientific computing. Julia is free, open-source, and provides good performance comparable to compiled C/Fortran codes, making it an attractive platform for scientific computing. A developed Julia module provides a reproducible example of the proposed cavity.

The regularized four-sided lid-driven cavity shows most of the primary bifurcation scenarios. The flow undergoes instabilities, such as pitchfork, saddle-node (fold), and Hopf. Predicting the precise location of the bifurcations could present an amenable benchmark when testing and comparing different schemes and implementations.

2 Mathematical formulation

We consider the two-dimensional and time-dependent Navier-Stokes equations in streamfunction formulation $\Psi(x,y,t)$ for incompressible fluids,

$$\partial_t \Delta\Psi = \frac{1}{\mathrm{Re}}\Delta^2\Psi + (\partial_x\Psi)\partial_y(\Delta\Psi) - (\partial_y\Psi)\partial_x(\Delta\Psi), \quad (1)$$

within the square domain $(x,y) \in [-1,1]\times[-1,1]$.

To discretize (1) using a spectral method, we define regularized boundary conditions in terms of exponential functions

$$u(\pm 1,y,t) = 0, \quad (2)$$
$$v(x,\pm 1,t) = 0, \quad (3)$$
$$u(x,\pm 1,t) = \pm(\mathrm{e}^{k_0(x-1)}-1)(\mathrm{e}^{-k_0(x+1)}-1)^2, \quad (4)$$
$$v(\pm 1,y,t) = \pm(\mathrm{e}^{k_0(y-1)}-1)(\mathrm{e}^{-k_0(y+1)}-1)^2. \quad (5)$$

The regularized boundary conditions impose a zero velocity at all four corners of the cavity. Figure 1 shows such a profile for the regularization parameter $k_0 = 10$.

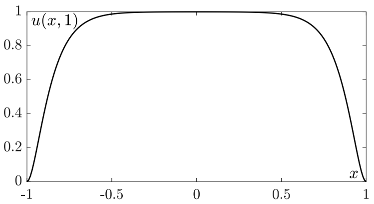

Figure 1: Regularized horizontal velocity profile (4) at the top wall $y = 1$ for $k_0 = 10$.

3 Computational methods and Results

The cavity is discretized using a two-dimensional Chebyshev grid $[\cos(\frac{i\pi}{m}), \cos(\frac{j\pi}{n})]$ where $i = 0, 1, ..., m$ and $j = 0, 1, ..., n$.

To incorporate the boundary condition, the streamfunction is set to be zero at the lids. The first inner grid points are defined through the derivative of the streamfunction (2-5).

For the steady-state flows, the zeros of equation (1) have to be computed. As the outer grid points are explicitly known through the boundary conditions, a reduced system of equations $F(\psi, \mathrm{Re}) = 0$ can be formulated. $\psi \in \mathbb{R}^{(m-1)\times(n-1)}$ corresponds now to the inner grid points.

We apply a Newton-Raphson algorithm to solve this system of non-linear equations, and discretize the associated Jacobian using finite differences. This Newton method can then be used in a pseudo-arclength continuation algorithm to track the solutions as a function of the Reynolds number.

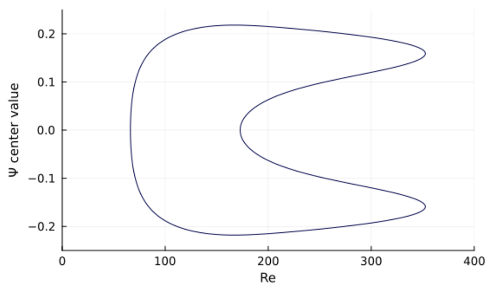

Figure 2: Branch of asymmetric solutions on a 32×32 grid obtained with pseudo-arclength continuation

Figure 2 shows the asymmetric branches obtained by the continuation algorithm and figure 3 illustrates one of these solutions. The pitchfork bifurcations seem to agree with the ones found in [Wahba (2009), Chen (2013)] whereas the locations of the saddle nodes are different as a result of the regularization process. Furthermore, it has been vindicated that there is a Hopf bifurcation occurring at a Reynolds number of around 350.

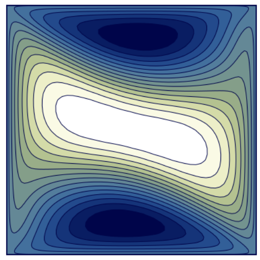

Figure 3: Asymmetric solution for 64×64 at Reynolds 100

4 Conclusions

We have shown some primary results of the regularized four-sided cavity flow. The problem has been discretized with a Chebyshev spectral method and provides a way to accurately determine a rich number of bifurcation scenarios. The computed pitchfork bifurcations seem to coincide with the un-regularized version. Additionally, it could be used as a benchmark to test other discretization schemes.

Acknowledgement

This research was supported by the Spanish Ministerio de Ciencia e Innovación, under grant No. PID2020-114043GB-I00.

References

Wahba E. M. (2009) “Multiplicity of states for two-sided and four-sided lid driven cavity flows“, *Comput. Fluids*, **38**, 247–253

Arumuga Perumal D., Dass A. K. (2011) “Multiplicity of steady solutions in two-dimensional lid-driven cavity flows by Lattice Boltzmann Method“, *Comput. Math. with Appl.*, **61**, 3711–3721

Cadou J. M., Guevel Y. and Girault G. (2012) “Numerical tools for the stability analysis of 2D flows: application to the two- and four-sided lid- driven cavity“, *Fluid Dyn. Res.*, **44** (3), 031403

Chen, K. T., Tsai C. C. and Luo W. J. (2013) “Multiplicity Flow Solutions in A Four-sided Lid-driven Cavity“, *Appl. Mech. Mater.*, **368-370**, 838–843

Kuhlmann H., Romanò F. (2018) “The Lid-Driven Cavity“, *Comput. Methods Appl. Sci.*