Numerics for Fluids, Structures and Electromagnetics

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Project 1

Flow in a driven cavity and non conforming mesh coupling

The Stokes equations provide a mathematical model to describe the flow of a creeping fluid, namely a flow where advective forces are small compared to viscous forces. This project focuses on the two-dimensional stationary incompressible Stokes equations and the so called driven cavity (see Figure 1). This is a standard benchmark for fluid problems.

Let us thus consider the following Stokes problem on the domain $\Omega \subset \mathbb{R}^2$ for the velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure $p : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^2$ and $\mathbf{g} : \partial\Omega \to \mathbb{R}^2$ are two given functions. In particular, assume that $\mathbf{f} \in [H^{-1}(\Omega)]^2 = \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in [H^{1/2}(\partial\Omega)]^2 = \mathbf{H}^{1/2}(\partial\Omega)$.

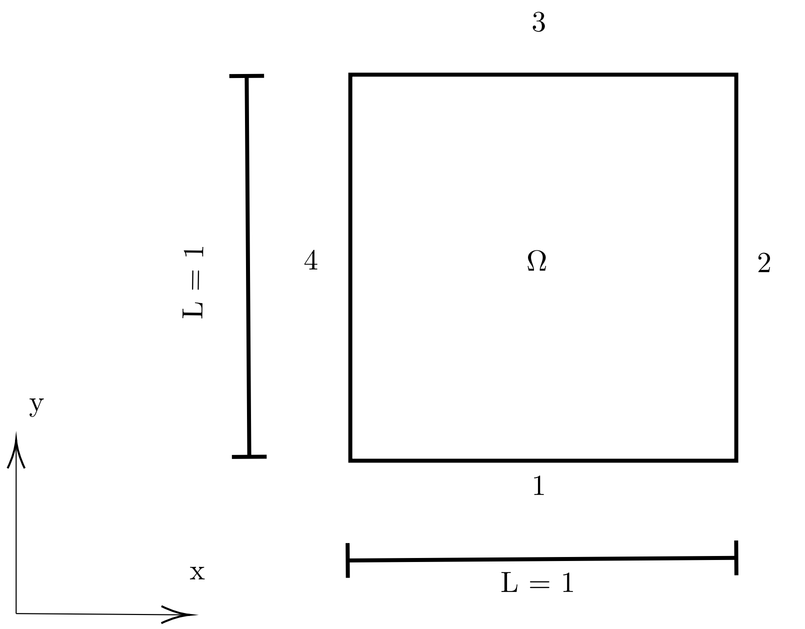

Figure 1: Geometry setup for the driven cavity problem.

1. Show that (1) admits a solution only under the following compatibility condition:

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0.$$

2. Write a weak formulation of (1) as well as a (mixed) finite element discrete formulation. In particular, discuss the choice of the functional spaces and of the FE spaces for the velocity $\mathbf{u}$ and the pressure p. Give an a priori estimate for the error on the velocity.

3. Implement and solve in `FreeFem++` the driven cavity problem. For the numerical experiments, set $\mathbf{f} = \mathbf{0}$ and

$$\mathbf{g} = \begin{cases} (1,0)^T & \text{on } \Gamma_3, \\ \mathbf{0} & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_4. \end{cases}$$

   Use a finer discretization close to the boundary, which is an area of interest for the problem at hand.

4. Another possible approach to reach higher accuracy is to use two different, non-conforming grids and to couple them through suitable Lagrange multipliers defined on their interfaces, as shown for example in Figure 2. Implement such a technique in `FreeFem++` and compare the results with the solution obtained with a fine confoming mesh, as you refine the grids.

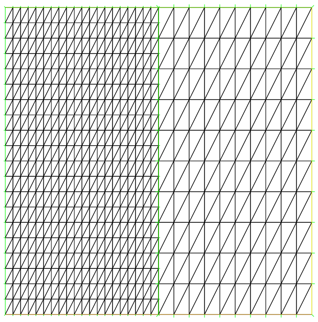

Figure 2: Example of non conforming grids

5. Motivate your choice of finite elements for the pressure. In particular, did you use a continous or a discontinuous field?

References

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[3] D. Boffi, F. Brezzi and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer.