Project 1 - Flow in a driven cavity and non conforming mesh coupling

Numerics for Fluids, Structures and Electromagnetics

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Introduction

We consider the following Stokes problem on the domain $\Omega \subset \mathbf{R}^2$ for the velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure $p : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \text{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^2$ and $\mathbf{g} : \partial\Omega \to \mathbb{R}^2$ are two given function. In particular assume that $\mathbf{f} \in [H^{-1}(\Omega)]^2 = \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in [H^{1/2}(\partial\Omega)]^2 = \mathbf{H}^{1/2}(\partial\Omega)$.

This problem corresponds to solving a flow of a incompressible fluid where viscous forces dominate inertial forces (i.e. low Reynolds number). The equations are essentially the linearized Navier-Stokes equations neglecting inertial terms.

1 Question 1

Suppose, ab absurdo, that there exists a solution $\mathbf{u}$ to problem 1 but $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \neq 0$. It results that :

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} = \int_{\Omega} \text{div}\mathbf{u} = \int_{\Omega} 0 = 0$$

We used the fact that $\mathbf{u}$ is a solution to the equation. Furthermore the divergence theorem was applied. This results however in a contradiction to the assumption. Hence this proves that $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$ is a necessary condition for the existence of a solution.

From a physical point of view this correspond to the fact that there can be no net flux outwards from the domain Ω as the flow has no source of fluid inside the domain.

Question 2

Weak formulation

Problem 1 has to be expressed in a weak formulation. To do so we multiply the first equation by a test function $\mathbf{v}$ and integrate over the domain Ω. Hence we get :

$$
\begin{aligned}
&\int_\Omega -\Delta \mathbf{u}\,\mathbf{v} + \int_\Omega \nabla p\,\mathbf{v} = \int_\Omega \mathbf{f}\,\mathbf{v} \\
\Leftrightarrow &\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\partial\Omega} (\nabla \mathbf{u} \cdot n)\,\mathbf{v} - \int_\Omega p\,\mathrm{div}\mathbf{v} + \int_{\partial\Omega} p\,(\mathbf{v}\cdot\mathbf{n}) = \int_\Omega \mathbf{f}\,\mathbf{v} \\
\Leftrightarrow &\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p\,\mathrm{div}\mathbf{v} = \int_\Omega \mathbf{f}\,\mathbf{v} + \int_{\partial\Omega} (\nabla \mathbf{g}\cdot\mathbf{n})\,\mathbf{v} - \int_{\partial\Omega} p\,(\mathbf{v}\cdot\mathbf{n})
\end{aligned}
$$

In the development above we did an integration by parts and then used the fact that $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$.

Analogously we multiply the second equation by a another test function (q) and integrate over Ω , using the same properties the equation becomes :

$$
-\int_\Omega q\,\mathrm{div}\mathbf{u} + \int_{\partial\Omega} q\,(\mathbf{u}\cdot\mathbf{n}) = 0
$$

Functional spaces

It remains the question which are the functional spaces to be imposed for the solution for our problem. In the first equation the gradient of the testfunction $\mathbf{v}$ appears. In addition we wan't make the integral for the test function vanish on the boundary. A natural choice is : $[H^1_0(\Omega)]^2 = \mathbf{H}^1_0(\Omega)$ This space is formally defined by :

$$
\mathbf{H}^1_0(\Omega) = \left\{\mathbf{u}\ :\ \mathbf{u} \in L^2(\Omega), \nabla\mathbf{u} \in L^2(\Omega),\ \text{and } \mathbf{u}|_{\partial\Omega} = 0\right\}
$$

To be changed: **Thanks to this definition on the space the two border integrals will vanish. For the second test functions, nothing special appears, nothing more than integral over the domain, so a natural space is $L^2(\Omega)$. On the other hand, for the spaces of solutions, to apply the Lax-Milgram theorem we need the same space functions. However in this case this is not possible. First we have a boundary condition on the border for u. So we consider the affine space $[H^1_g(\Omega)]^2 = \mathbf{H}^1_g(\Omega)$ define as :**

$$
\mathbf{H}^1_g(\Omega) = \left\{\mathbf{u}\ :\ u \in L^2(\Omega), \nabla\mathbf{u} \in L^2(\Omega),\ \text{and } \mathbf{u}|_{\partial\Omega} = g\right\}
$$

For the pression, we notice that the solution is unique up to a constant term. To fix this we can impose that $\int_\Omega p = 0$. Appart from this special issue the pression space is free. We choose $L_0^2(\Omega)$ as the space for pression, and we impose it also for q.

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega) \; : \; \int_\Omega p = 0 \right\}$$

The problem reads : Find $(\mathbf{u}, p) \in \mathbf{H}_g^1(\Omega) \times L_0^2(\Omega)$ such that $\forall (\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ the following holds :

$$\begin{aligned} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \,\mathrm{div} \mathbf{v} &= \int_\Omega \mathbf{f}\, \mathbf{v} \\ - \int_\Omega q \,\mathrm{div} \mathbf{u} &= 0 \end{aligned} \tag{2}$$

Mixed finite element element discrete formulation

As we have seen in class problem 2 verifies the **inf-sup** condition and admits a unique solution in Ω. No we will write the inifite dimensional problem in a finite dimensional abstract form :

Find $(\mathbf{u}_h, p) \in \mathbf{V}_h(\Omega) \times Q_h(\Omega)$ such that $\forall (\mathbf{v}_h, q) \in \mathbf{V}_{h,0}(\Omega) \times Q_{h,0}(\Omega)$ the following holds :

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) &= F(\mathbf{v}) \\ b(\mathbf{u}_h, q_h) &= 0 \end{aligned} \tag{3}$$

The descrete case now has to verify the **descrete inf-sup** has to be verified. A common stable choice is the mini-element which consist of an enreached $\mathbb{P}_1$ finite element space with bubble function and $\mathbb{P}_1$ FEM space for the pressure.

A priori error estimate

Question 3

We implement the problem with the functional spaces given in the previous section. The complete code can be seen in annex 1. To fix the constant part of the pressure (we don't have any pressure boundary conditions) we solve a slightly different problem :

Find $(\mathbf{u}_h^\epsilon, p) \in \mathbf{V}_h(\Omega) \times \mathbf{Q}_h(\Omega)$ such that $\forall (\mathbf{v}_h, q) \in \mathbf{V}_{h,0}(\Omega) \times \tilde{Q}_h(\Omega)$ the following holds :

$$\begin{aligned} a(\mathbf{u}_h^\epsilon, \mathbf{v}_h^\epsilon) - b(p_h^\epsilon, \mathbf{v}_h^\epsilon) &= F(\mathbf{v}) \\ b(p, \mathbf{v}) - c(p_h^\epsilon, q_h^\epsilon) &= 0 \end{aligned} \tag{4}$$

This equations are called the perturbed Stokes equations. It is simple to verify that the zero mean pressure condition is verified. .... show that there exists a unique solution to this problem and that the error of the perturbed problem is not to far away from the the actual problem we wanted solve:

As a side remark: This strategy is called the penalty method and helps to....

As we seen that problems mostly occurs close to the boundary, we redefine the mesh to be finer close to the boundary. The space will be divided in two disjoint regions, specifically we will define an inner square of length 0.8. Inside this inner square the meshsize will be quite large, but outside this square the mesh size will be more fine grained. We can modify the mesh size by setting the number of points along the inner and outer boundary.

This definition of the mesh gives us :

With this mesh the solution is given by :

Question 4

Question 5

J'ai compris.

Annex