Slice closures of indexed languages and word equations with counting constraints

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Useful for decidability of any counting constraints?

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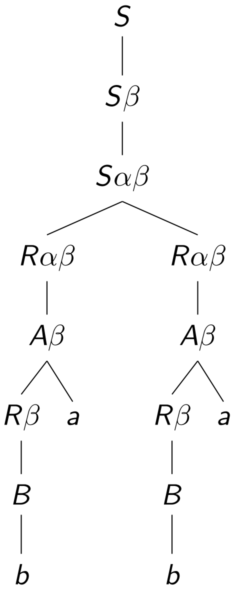

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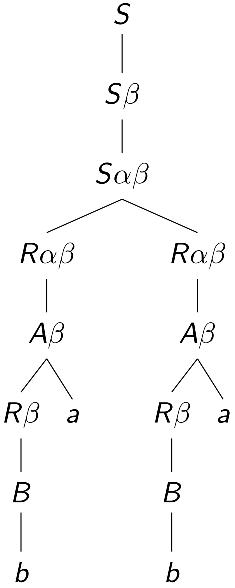

Context-free $X \to YZ$, $X \to x$ etc.

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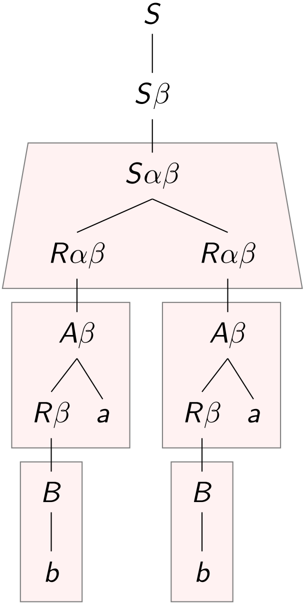

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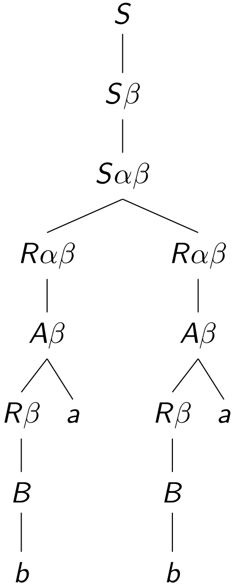

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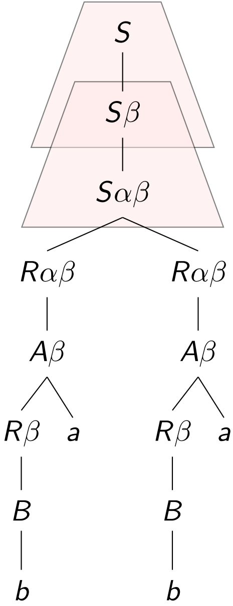

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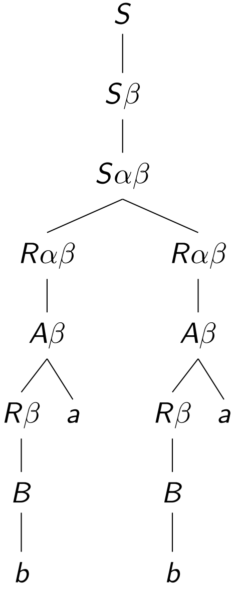

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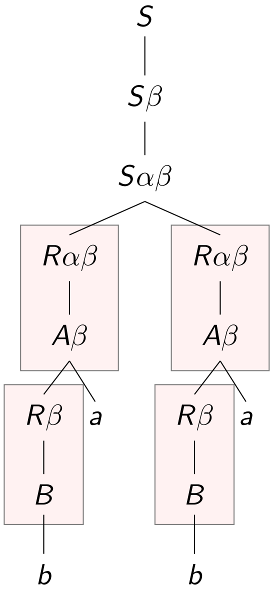

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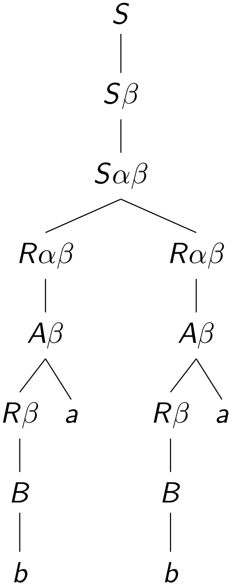

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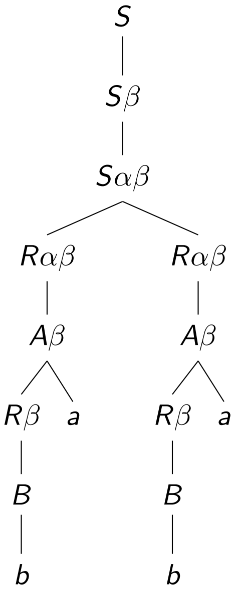

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- equivalent to Higher-Order Recursion Schemes (HORS) of order 2

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- $L \overset{?}{\subseteq}$ same number of a's and b's decidability open (Kobayashi 2019)

Semilinear sets

A set $S \subseteq \mathbb{N}^d$ is *linear* if it is of the form

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Presburger arithmetic

First-order logic over the structure $\langle \mathbb{N}; +, \leqslant, 0, 1 \rangle$. For example:

$$\exists y \colon x = y + y \quad \exists y \colon \exists z \colon \varphi(y) \wedge \psi(z) \wedge x \geqslant y + z$$

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E.g.: $\{a^{n^2} \mid n \in \mathbb{N}\}$ is an indexed language.

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$L \subseteq$ same number of a's and b's $\iff \Psi(L) \qquad\qquad \subseteq \{(n,n) \mid n \geqslant 0\}$

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For $M \subseteq \mathbb{N}^d$, denote by $\mathrm{Slice}(M)$ the *smallest* slice containing M.

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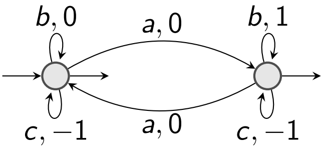
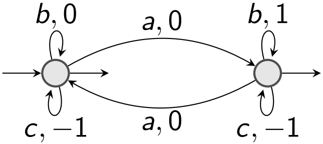
Slice closures encode more

Affine hull of $\Psi(L)$, upward closure $\Psi(u)\uparrow = \{u \in \mathbb{N}^d \mid \exists v \in \Psi(L) \colon u \geqslant v\}$.

Word equations and counting functions

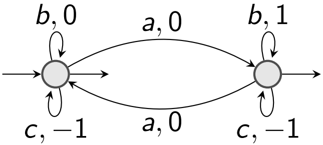


Word equations and counting functions



$f(w) =$ number of b's after a's, minus c's

Word equations and counting functions



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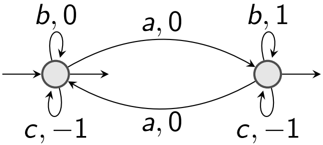

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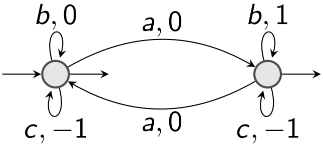

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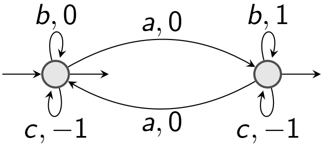

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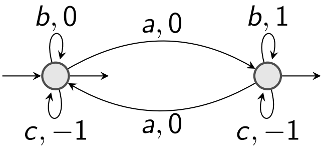

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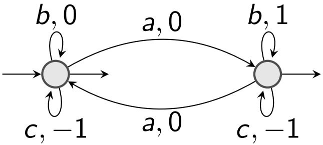

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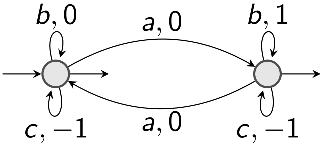

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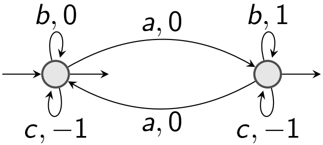

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- Linear combinations: $f(w) = \lambda_1 f_1(w) + \cdots + \lambda_n f_n(w)$

Corollary (to our main result)

Given a word equation, rational constraints, and counting function f, one can decide if there is a solution σ with $f(\mathsf{enc}(\sigma)) \neq 0$.

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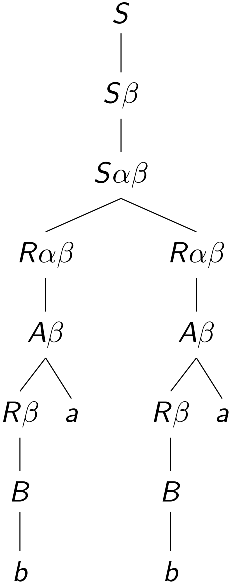
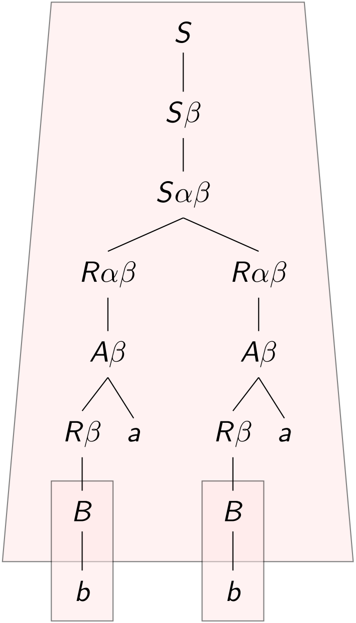
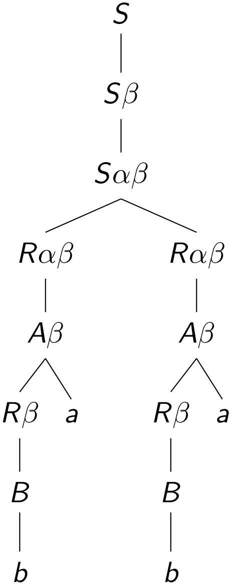


Building trees



Building trees

- Plugging



Building trees

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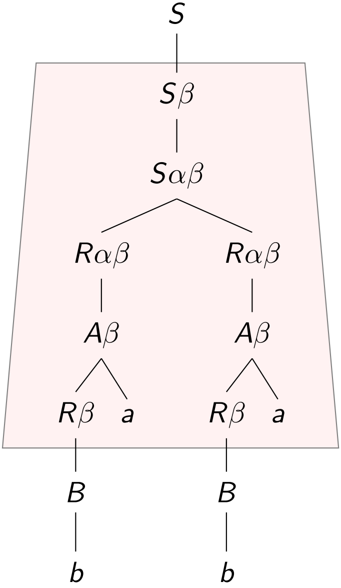

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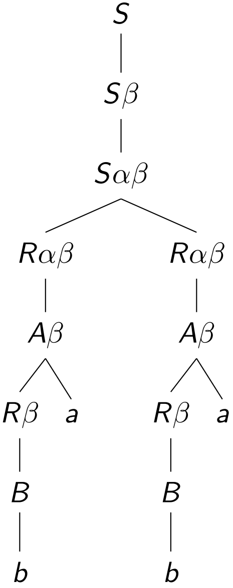

Building trees

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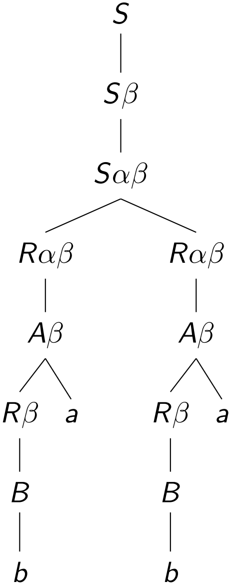
$\rightsquigarrow$ operator $\mathcal{D}$ on subsets of $\mathbb{N}^{N \cup T}$



Building trees

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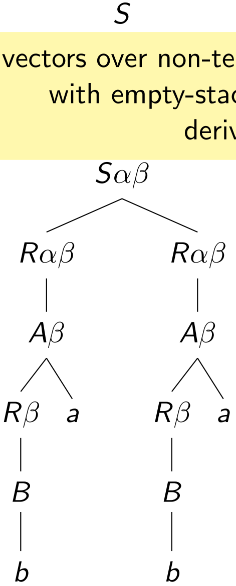
$$\text{derivable vectors} = \bigcup_{i \geqslant 0} \mathcal{D}^i(\varnothing)$$



vectors over non-terminals and terminals with empty-stack to empty-stack derivations

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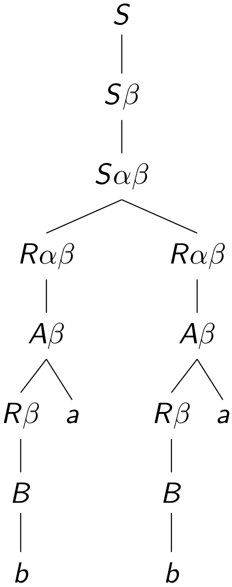

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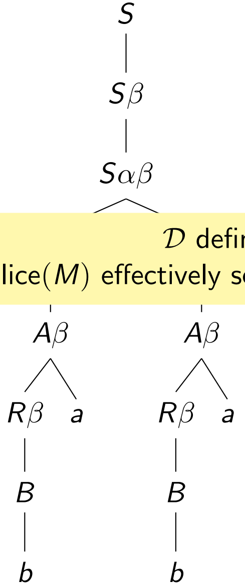

Building trees

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⇝ operator $\mathcal{D}$ on subsets of $\mathbb{N}^{N \cup T}$ with

$\mathcal{D}$ definable in Presburger arithmetic

Slice(M) effectively semilinear for semilinear M (Grabowski 1981)

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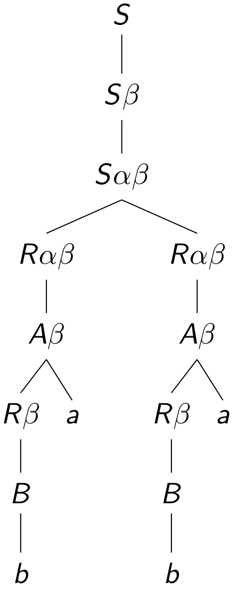

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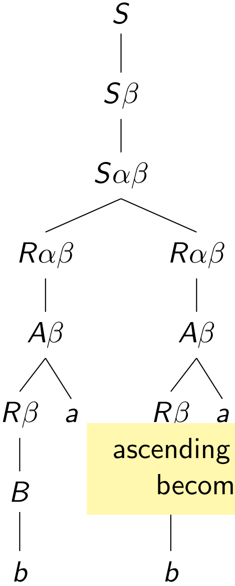

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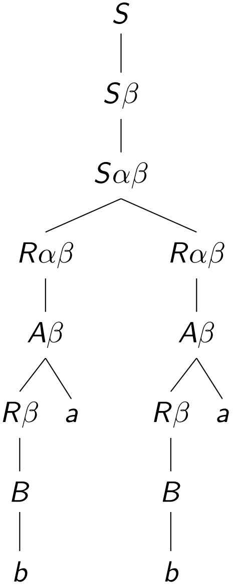
ascending chains of slices become stationary



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Key property of slices

Slice(derivable vectors $\cap\ \mathbb{N}^T$) = Slice(derivable vectors) $\cap\ \mathbb{N}^T$

Ingredient I: Ascending chains

Theorem (Eilenberg & Schützenberger 1969)

If $S_1 \subseteq S_2 \subseteq \cdots$ *are slices, then there is an* $i \geqslant 1$ *with* $S_i = S_{i+1} = \cdots$.

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For a slice $S \subseteq \mathbb{N}^d$, consider a linear set

$$L_i = \{u + \lambda_1 u_1 + \cdots + \lambda_n u_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$$

in S and consider the finite set $F_i = \{u, u + u_1, u + u_2, \ldots, u + u_n\}$. Then S is generated by $\bigcup_i F_i$. □

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If $S_1 \subseteq S_2 \subseteq \cdots$, then $\bigcup_j S_j$ is a slice, and thus finitely generated. These finitely many vectors must occur in some S_i, hence $S_i = S_{i+1} = \cdots$.

Ingredient II: Slice closures of semilinear sets

Theorem ($\approx$ Grabowski 1981)

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If both checks succeed, we know $R = \mathsf{Slice}(M)$.

Ingredient III: Slice closure of intersection

Key property of slices

For $M \subseteq \mathbb{N}^{N \cup T}$, we have $\mathsf{Slice}(M \cap \mathbb{N}^{T}) = \mathsf{Slice}(M) \cap \mathbb{N}^{T}$.

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$\supseteq$: $\mathbb{N}^T$ can only be produced from vectors in $\mathbb{N}^T$

Ingredient III: Slice closure of intersection

Key property of slices

For $M \subseteq \mathbb{N}^{N \cup T}$, we have $\mathsf{Slice}(M \cap \mathbb{N}^{T}) = \mathsf{Slice}(M) \cap \mathbb{N}^{T}$.

$\subseteq$: $\mathbb{N}^{T}$ is closed under the slice operation
$\supseteq$: $\mathbb{N}^{T}$ can only be produced from vectors in $\mathbb{N}^{T}$

Slices closure vs. other counting closures

Not true for existing closure operators (affine hull, Zariski closure)!

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Theorem (Ciobanu & Z., LICS 2024)

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Can Ciobanu, Diekert & Elder 2015 be strengthened, to avoid detour?