תרגיל בית 2:

<u>מגישים:</u>

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שאלה 1:

סעיף 1.

$$MSE\left(\hat{\theta}\right)=E\left[\left(\hat{\theta}-\theta\right)^2\right]=E\left[\left(\hat{\theta}-E\left(\hat{\theta}\right)+E\left(\hat{\theta}\right)-\theta\right)^2\right]=$$

$$=E\left[\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)^2+\left(E\left(\hat{\theta}\right)-\theta\right)^2+2\left(E\left(\hat{\theta}\right)-\theta\right)\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)\right]=$$

$$=E\left[\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)^2\right]+E\left[\left(E\left(\hat{\theta}\right)-\theta\right)^2\right]+2E\left[\left(E\left(\hat{\theta}\right)-\theta\right)\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)\right]=$$

$$=Var\left(\hat{\theta}\right)+Bias\left(\hat{\theta},\theta\right)+2E\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)E\left[\left(E\left(\hat{\theta}\right)-\theta\right)\right]=$$

$$=Var\left(\hat{\theta}\right)+Bias\left(\hat{\theta}\right)$$

סעיף 2.

$$plug\ in\ estimator\ \ for\ \ p:\ \ \hat{p}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\sigma=\sqrt{Var\left(\hat{p}\right)}=\sqrt{Var\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)}=\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}Var\left(x_i\right)}=\sqrt{\frac{1}{n^2}n\cdot p\left(1-p\right)}=\sqrt{\frac{p\left(1-p\right)}{n}}$$

נגדיר $C_n=\left(\hat{p}_n-\varepsilon_n,\hat{p}_n+\varepsilon_n\right)$,

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$$P\left(\left|p-\hat{p}_n\right|\geq\varepsilon_n\right)\geq 2\exp\left(-\frac{2n\varepsilon_n{}^2}{d_n}\right) \quad , \quad d_n=(1-p)-p=1$$

$$\Leftrightarrow$$

$$1-P\left(\left|p-\hat{p}_n\right|\geq\varepsilon_n\right)\leq 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{1}\right)\Leftrightarrow$$

$$P\left(\left|p-\hat{p}_n\right|\leq\varepsilon_n\right)\leq 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{1}\right)\Leftrightarrow$$

$$P\left(p\in C_n\right)\leq 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{1}\right)=1-\alpha\Leftrightarrow$$

$$\alpha=2\exp\left(-\frac{2n\varepsilon_n{}^2}{d_n}\right)\Leftrightarrow\ \ln\left(\frac{\alpha}{2}\right)=-2n\varepsilon_n{}^2\Leftrightarrow$$

$$\varepsilon_n{}^2=\frac{\ln\left(\frac{2}{\alpha}\right)}{2n}\Rightarrow\varepsilon_n=\sqrt{\frac{\ln\left(\frac{2}{\alpha}\right)}{2n}}\Rightarrow$$

$$\textit{confidence of } 90\%\Leftarrow\varepsilon_n=\sqrt{\frac{\ln\left(\frac{2}{0.1}\right)}{2n}}=\frac{1.224}{\sqrt{n}}$$

$$\textit{confidence of } 95\%\Leftarrow\varepsilon_n=\sqrt{\frac{\ln\left(\frac{2}{0.05}\right)}{2n}}=\frac{1.358}{\sqrt{n}}$$

$$\textit{confidence of } 99\%\Leftarrow\varepsilon_n=\sqrt{\frac{\ln\left(\frac{2}{0.01}\right)}{2n}}=\frac{1.627}{\sqrt{n}}$$

$$\textit{plug in estimator for } p:\ \hat{p}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\sigma=\sqrt{Var\left(\hat{p}\right)}=\sqrt{Var\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)}=\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}Var\left(x_i\right)}=\sqrt{\frac{1}{n^2}n\cdot p\left(1-p\right)}=\sqrt{\frac{p\left(1-p\right)}{n}}$$

$$\textit{plug in estimator for } \left(p-q\right):\ \hat{r}=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-y_i\right)$$

$$\sigma=\sqrt{Var\left(\hat{r}\right)}=\sqrt{Var\left(\frac{1}{n}\sum_{i=1}^{n}\left(x_i-y_i\right)\right)}=\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}Var\left(x_i-y_i\right)}=$$

$$=\sqrt{\frac{1}{n^2}n\cdot\left\{p\left(1-p\right)-q\left(1-q\right)\right\}}=\sqrt{\frac{p\left(1-p\right)-q\left(1-q\right)}{n}}$$

נגדיר $\tilde{C}_n = \left(\hat{r}_n - \varepsilon_n, \hat{r}_n + \varepsilon_n\right)$, $r = p - q$,

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$$P\left(\left|r-\hat{r}_n\right| \ge \varepsilon_n\right) \ge 2\exp\left(-\frac{2n\varepsilon_n{}^2}{d_n}\right) \quad ,$$

$$\tilde{d}_n = \frac{1}{n}\sum_{i=1}^{n}\left[\left(1+1\right)-\left(0-1\right)\right]^2 = \frac{9}{n} \quad \left\{\text{since } \ 0-1 \le p-q \le 1+1\right\}$$

$$\Leftrightarrow$$

$$\Leftrightarrow$$

$$1-P\left(\left|r-\hat{r}_n\right| \ge \varepsilon_n\right) \le 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{\tilde{d}_n}\right) \Leftrightarrow$$

$$P\left(\left|r-\hat{r}_n\right| \le \varepsilon_n\right) \le 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{\tilde{d}_n}\right) \Leftrightarrow$$

$$P\left(p \in C_n\right) \le 1-2\exp\left(-\frac{2n\varepsilon_n{}^2}{\tilde{d}_n}\right) = 1-\alpha \Leftrightarrow$$

$$\alpha = 2\exp\left(-\frac{2n\varepsilon_n{}^2}{\tilde{d}_n}\right) \Leftrightarrow \ \tilde{d}_n \ln\left(\frac{\alpha}{2}\right) = -2n\varepsilon_n{}^2 \Leftrightarrow$$

$$\varepsilon_n{}^2 = \frac{\tilde{d}_n \ln\left(\frac{2}{\alpha}\right)}{2n} \Rightarrow \varepsilon_n = \sqrt{\frac{9\ln\left(\frac{2}{\alpha}\right)}{2n^2}}$$

$$\Rightarrow$$

$$\textit{confidence of } \ 90\% \Leftarrow \varepsilon_n = \sqrt{\frac{9\ln\left(\frac{2}{0.1}\right)}{2n^2}} = \frac{3.67}{n}$$

סעיף 3.

$$\hat{\theta}_{MLE} = \underset{\theta\in\Theta}{\arg\max}\left\{L(\theta)\right\}$$

$$L(\theta)=p(D|\theta)=p(X_1,X_2,...,X_N|\theta)\underset{iid}{=}\prod_{k=1}^{N}p(X_k|\theta)=\prod_{i=k}^{N}\left\{\binom{10}{X_k}\theta^{X_k}\cdot(1-\theta)^{(10-X_k)}\right\}\Rightarrow$$

$$l(\theta)=\log_e\left[L(\theta)\right]=\sum_{k=1}^{N}\log\left\{\binom{10}{X_k}\theta^{X_k}\cdot(1-\theta)^{(10-X_k)}\right\}=$$

$$=\sum_{k=1}^{N}\log\left\{\binom{10}{X_k}\right\}+\log\left(\theta^{X_k}\right)+\log\left((1-\theta)^{(10-X_k)}\right)=$$

$$=\sum_{k=1}^{N}\left\{\log\binom{10}{X_k}+X_k\log(\theta)+(10-X_k)\log(1-\theta)\right\}$$

$$\frac{\partial l(\theta)}{\partial\theta}=\sum_{k=1}^{N}\frac{\partial}{\partial\theta}\left\{\log\binom{10}{X_k}+X_k\log(\theta)+(10-X_k)\log(1-\theta)\right\}=$$

$$=\sum_{k=1}^{N}\left\{X_k\frac{1}{\theta}+\frac{X_k-10}{1-\theta}\right\}=0$$

$$\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\sum_{k=1}^{N}X_k=\frac{10N}{1-\theta}$$

$$\frac{1}{(1-\theta)\theta}\sum_{k=1}^{N}X_k=\frac{10N}{1-\theta}$$

$$\frac{1}{\theta}\sum_{k=1}^{N}X_k=10N\quad\Rightarrow\boxed{\hat{\theta}_{MLE}=\frac{10N}{\sum_{k=1}^{N}X_k}}$$

סעיף 4.

$$\hat{F}_n=\frac{1}{n}\sum_{i=1}^{n}I\left\{X_i\le x\right\}\quad,\quad fixed\ F_X\left(x\right)$$

$$E\left(\hat{F}_n\right)=E\left(\frac{1}{n}\sum_{i=1}^{n}I\left\{X_i\le x\right\}\right)=\frac{1}{n}\sum_{i=1}^{n}E\left[I\left\{X_i\le x\right\}\right]=\frac{1}{n}\sum_{i=1}^{n}P\left(X_i\le x\right)=$$

$$=\frac{1}{n}\sum_{i=1}^{n}F_X\left(x\right)=\frac{1}{n}nF_X\left(x\right)=F_X\left(x\right)$$

$$Var\left(\hat{F}_n\right)=Var\left(\frac{1}{n}\sum_{i=1}^{n}I\left\{X_i\le x\right\}\right)=\frac{1}{n^2}\sum_{i=1}^{n}Var\left[I\left\{X_i\le x\right\}\right]=$$

$$=\frac{1}{n^2}\sum_{i=1}^{n}1^2\cdot P\left(X_i\le x\right)-1\cdot P^2\left(X_i\le x\right)=\frac{1}{n^2}\sum_{i=1}^{n}F_X\left(x\right)-F_X{}^2\left(x\right)=$$

$$=\frac{1}{n^2}n\left(F_X\left(x\right)-F_X{}^2\left(x\right)\right)=\frac{1}{n}\left(F_X\left(x\right)-F_X{}^2\left(x\right)\right)$$

$$\Rightarrow\ \text{using CLT}:$$

$$\frac{\hat{F}_n-E\left(\hat{F}_n\right)}{\sqrt{Var\left(\hat{F}_n\right)}}=\frac{\hat{F}_n-F_X\left(x\right)}{\sqrt{\frac{1}{n}\left(F_X\left(x\right)-F_X{}^2\left(x\right)\right)}}=\sqrt{n}\frac{\hat{F}_n-F_X\left(x\right)}{\sqrt{\left(F_X\left(x\right)-F_X{}^2\left(x\right)\right)}}\xrightarrow{D}N\left(0,1\right)$$

$$\Rightarrow$$

$$\sqrt{n}\left[\hat{F}_n-F_X\left(x\right)\right]\xrightarrow{D}N\left(0,\left(F_X\left(x\right)-F_X{}^2\left(x\right)\right)\right)$$

סעיף 5.

$$P_F\left(\sup_X\left|\hat{F}_n\left(x\right)-F\left(x\right)\right|>\varepsilon\right)\le C\left(k\right)\exp\left(-n\varepsilon^2\right)\ ,\ \forall\varepsilon>0$$

$$\alpha=C\left(k\right)\exp\left(-n\varepsilon^2\right)\ \Rightarrow\ln\left(\frac{\alpha}{C\left(k\right)}\right)=-n\varepsilon^2\quad\Rightarrow$$

$$\frac{1}{n}\ln\left(\frac{C\left(k\right)}{\alpha}\right)=\varepsilon^2\quad\Rightarrow\ \varepsilon=\sqrt{\frac{1}{n}\ln\left(\frac{C\left(k\right)}{\alpha}\right)}$$

סעיף 6.

$$\hat{F}_n = \frac{1}{n}\sum_{i=1}^{n} I\left\{X_i \le x\right\}$$

$$E\left(\hat{F}_n\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{X_i \le x\right\}\right) = \frac{1}{n}\sum_{i=1}^{n} E\left[I\left\{X_i \le x\right\}\right] = \frac{1}{n}\sum_{i=1}^{n} P\left(X_i \le x\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} F_X\left(x\right) = \frac{1}{n} nF_X\left(x\right) = F_X\left(x\right)$$

$$Var\left(\hat{F}_n\right) = Var\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{X_i \le x\right\}\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left[I\left\{X_i \le x\right\}\right] =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} 1^2 \cdot P\left(X_i \le x\right) - 1 \cdot P^2\left(X_i \le x\right) = \frac{1}{n^2}\sum_{i=1}^{n} F_X\left(x\right) - {F_X}^2\left(x\right) =$$

$$= \frac{1}{n^2} n\left(F_X\left(x\right) - {F_X}^2\left(x\right)\right) = \frac{1}{n}\left(F_X\left(x\right) - {F_X}^2\left(x\right)\right)$$

שאלה 2:

סעיף 1.

נמצא את המשערך עבור $E[K]$ -

$$E[K]=\int_{-\infty}^{\infty}KdF(K)=\int_{-\infty}^{\infty}KdF(K)=\frac{1}{N}\sum_{i=1}^{N}K_i$$

סעיף 2.

$$\hat{p}_3=P(K=3)=P(K\le 3)-P(K\le 2)=F(3)-F(2)$$

$$\hat{F}_N(k)=\frac{1}{N}\sum_{i=1}^{N}I_{\{K_i\le k\}}\quad\Rightarrow$$

$$\hat{p}_3=\frac{1}{N}\sum_{i=1}^{N}I_{\{K_i\le 3\}}-\frac{1}{N}\sum_{i=1}^{N}I_{\{K_i\le 2\}}=\frac{1}{N}\sum_{i=1}^{N}I_{\{K_i=3\}}$$

סעיף 3.

נגדיר $C_N(k)=\left(\hat{F}_N(k)-\varepsilon_n,\hat{F}_N(k)+\varepsilon_n\right),$,

$$P(F\in C_n)=P\left(\left|F-\hat{F}_N\right|\le\varepsilon_n\right)$$

$$P(F\notin C_n)\le P_F\left(\sup_k\left|\hat{F}_N(k)-F(k)\right|>\varepsilon\right)\underset{DKW}{\le}2\exp\left(-2n\varepsilon^2\right)$$

$$\Rightarrow$$

$$P(F\in C_n)\ge 1-2\exp\left(-2n\varepsilon^2\right)=1-\alpha$$

$$\varepsilon_n=\sqrt{\frac{\ln\left(\frac{2}{\alpha}\right)}{2N}}=\sqrt{\frac{\ln\left(\frac{2}{0.05}\right)}{2\cdot 100}}\cong 0.136$$

$$\Rightarrow\left|F(k)-\hat{F}_N(k)\right|\le\varepsilon_n$$

$$\Rightarrow-\varepsilon_n\le F(k)-\hat{F}_N(k)\le\varepsilon_n$$

$$\Rightarrow\hat{F}_N(k)-\varepsilon_n\le F(k)\le\varepsilon_n+\hat{F}_N(k)$$

$$\Rightarrow$$

$$\hat{F}_N(3)-\varepsilon_n-\left(\varepsilon_n+\hat{F}_N(2)\right)\le\hat{p}_3=F(3)-F(2)\le\varepsilon_n+\hat{F}_N(3)-\left(\hat{F}_N(2)-\varepsilon_n\right)$$

$$\hat{F}_N(3)-\varepsilon_n-\left(\varepsilon_n+\hat{F}_N(2)\right)\le\hat{p}_3\le\varepsilon_n+\hat{F}_N(3)-\left(\hat{F}_N(2)-\varepsilon_n\right)$$

$$\hat{F}_N(3)-\hat{F}_N(2)-2\varepsilon_n\le\hat{p}_3\le\hat{F}_N(3)-\hat{F}_N(2)+2\varepsilon_n$$

$$\frac{1}{100}\sum_{i=1}^{N=100}I_{\{K_i=3\}}-0.272\le\hat{p}_3\le\frac{1}{100}\sum_{i=1}^{N=100}I_{\{K_i=3\}}+0.272$$

סעיף 4.

$$K \sim Geom(p)$$

$$\hat{p}_{MLE} = \arg\max_{p \in P} L(p)$$

$$L(p) = P(K_1, K_2, ..., K_N \mid p) = \prod_{i=1}^{N} P(K_i \mid p) = \prod_{i=1}^{N} (1-p)^{K_i - 1} p$$

$$l(p) = \log(L(p)) = \sum_{i=1}^{N} (K_i - 1)\log(1-p) + \log(p)$$

$$\frac{\partial l(p)}{\partial p} = \sum_{i=1}^{N}\left[(K_i - 1)\frac{1}{p-1} + \frac{1}{p}\right] = 0 \Rightarrow$$

$$\frac{1}{p-1}\sum_{i=1}^{N}\left[(K_i - 1)\right] = -N\frac{1}{p}$$

$$p\left\{\sum_{i=1}^{N}[K_i] - N\right\} = -N(p-1) = N - Np$$

$$p\left(\sum_{i=1}^{N}[K_i]\right) = N$$

$$\Rightarrow \boxed{\hat{p}_{MLE} = \frac{N}{\left(\sum_{i=1}^{N}[K_i]\right)}}$$

$$\mu = \frac{1}{p} \Rightarrow \hat{\mu}_{MLE} = \frac{1}{\hat{p}_{MLE}} = \frac{\sum_{i=1}^{N}[K_i]}{N}$$

סעיף 5.

$$P(K \ \ is \ \ odd) = P\{K \mid k \bmod 2 = 1 \ , \ \mathrm{k} \in \mathrm{Natural}\} =$$

$$= \sum_{k}(1-p)^{k-1} \cdot p = \sum_{i=1}^{\infty}(1-p)^{2i-1-1} p = p\sum_{i=1}^{\infty}(1-p)^{2(i-1)} =$$

$$= \frac{p}{(1-p)^2} \cdot \frac{1}{1-(1-p)^2}$$

$$\Rightarrow \left[P(K \ \ is \ \ odd)\right]_{MLE} = \left[\frac{p}{(1-p)^2} \cdot \frac{1}{1-(1-p)^2}\right]_{p=\hat{p}_{MLE}}$$