

# Partial differential equations II

Kamil Belan

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# 1 Winter semester addendum

## 1.1 Weak\* convergence

Since  $L_\infty((0, T); L_2(\Omega))$  is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of  $L_\infty((0, T); L_2(\Omega))$  is reflexive, i.e.

$$L_\infty((0, T); L_2(\Omega)) \cong \left( L_1((0, T); L_2(\Omega)) \right)^*,$$

which means that balls in  $L_\infty((0, T); L_2(\Omega))$  are weakly\* compact. Moreover,  $L_1((0, T); L_2(\Omega))$  is *separable*, from which it follows  $L_\infty((0, T); L_2(\Omega))$  with the weak\* topology is metrizable and thus there exists a weakly\* converging subsequence (from the balls).

**Example** (For people without Functional Analysis I). Let  $X$  be a linear normed space,  $\{x_n\} \subset X$  a sequence in  $X$ . We say  $x_n$  converges weakly to  $x \in X$  whenever

$$f(x_n) \rightarrow f(x), \forall f \in X^*.$$

Let  $X^*$  be the topological dual to  $X$ ,  $\{x_n\} \subset X^*$  a sequence in  $X^*$ . We say  $f_n$  converges weakly\* to  $f \in X^*$  whenever

$$f_n(x) \rightarrow f(x), \forall x \in X, \text{ i.e. } x(f_n) \rightarrow x(f),$$

where by  $x(y), x \in X, y \in X^*$  we understand

$$\varepsilon_x : X^* \rightarrow \mathbb{K}, y \mapsto y(x).$$

Since  $L_\infty((0, T); L_2(\Omega)) \cong \left( L_1((0, T); L_2(\Omega)) \right)^*$ , every point  $x \in L_\infty((0, T); L_2(\Omega))$  can be interpreted as a linear functional on  $L_1((0, T); L_2(\Omega))$ , so given  $\{x_n\} \subset L_\infty((0, T); L_2(\Omega))$ , we can interpret it as a  $\{x_n\} \subset \left( L_1((0, T); L_2(\Omega)) \right)^*$ , meaning given a weakly converging sequence in  $L_\infty((0, T); L_2(\Omega))$ , it is actually a weakly\* converging sequence in  $L_1((0, T); L_2(\Omega))$ .

## 1.2 Regularity of parabolic problems

**Theorem 1.** *Let the assumptions of the previous theorem hold and  $\Omega \in C^{1,1}, \delta \in (0, 1)$ . Then  $u \in L_2((\delta, T); W^{2,2}(\Omega))$ .*

*Proof.* Take the weak formulation in  $t \in (\delta, T)$ . WLOG further assume  $d = 0$ . Then

$$\int_{\Omega} \mathbb{A} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi - b u \varphi - \mathbf{c} \cdot \nabla u \varphi - \int_{\Omega} \partial_t u \varphi = \int_{\Omega} (f - b u - \mathbf{c} \cdot \nabla u - \partial_t u) \varphi,$$

and the integrand of the last integral is in  $L_2(\Omega)$  for a.e.  $t \in (\delta, T)$ . We can thus use the elliptic regularity results and write:

$$\|u\|_{W^{2,2}(\Omega)}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{W^{1,2}(\Omega)}^2 + \|\partial_t u\|_{L_2(\Omega)}^2),$$

integrating both sides  $\int_\delta^T dt$  yields

$$\|u\|_{L_2((\delta,T);L_2(\Omega))}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2((0,T);W^{1,2}(\Omega))}^2 + \|u\|_{L_2((\delta,T);L_2(\Omega))}^2)$$

□

**Theorem 2.** *If data are smooth and satisfy the compatibility conditions, then the weak solutions to the parabolic equation are smooth.*

*Proof.* no. □

*Remark* (Compatibility condition). : Take the heat equation :  $\partial_t u - \Delta u = f$  at time zero:  $\Delta u(0) + f(0) = \partial_t u(0) \in W_0^{1,2}(\Omega)$ , so we need that  $f(0) + \Delta u(0)$  has zero trace  $\Rightarrow$  compatibility conditions.

### 1.3 Uniqueness of solutions to hyperbolic problems

**Theorem 3** (Uniqueness of the solution to a hyperbolic equation). *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that  $\mathbf{c} \in W^{1,\infty}(\Omega)$ . Then the weak solution to the hyperbolic equation is unique.*

*Proof.* It is enough that if  $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$ . To do that, take the equation, multiply it by  $\varphi \in V$  fixed and integrate over  $\Omega$  for  $t \in (0, T)$  fixed:

$$\langle \partial_{tt} u(t), \varphi \rangle + \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \varphi \, dx + \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \varphi \, dx - \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \varphi \, dx = 0.$$

Now, take a special test function

$$\psi(t) = \left( \int_t^s u(\tau) \, d\tau \right) \chi_{(0,s)}(t),$$

for some  $s \in (0, T)$ . Then  $\partial_t \psi(t) = -u(t)$  on  $t \in (0, s)$ . Next, integrate the equation in time over  $(0, s)$ .

$$\int_0^s \langle \partial_{tt} u(t), \psi \rangle \, dt + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s \langle \partial_{tt} u(t), \varphi \rangle \, dt = \langle \partial_t u(s), \psi(s) \rangle - \langle \partial_t u(0), \psi(0) \rangle - \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt,$$

and realize  $\psi(s) = 0, \partial_t u(0) = 0$ , so

$$- \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

but since  $\partial_t \psi(t) = -u(t)$ , we can actually write (time dependencies are omitted for brevity)

$$\int_0^s \langle \partial_t u, u \rangle \, dt + \int_0^s \int_\Omega -\mathbb{A} \nabla \partial_t \psi \cdot \nabla \psi - b \psi \partial_t \psi - \psi \mathbf{c} \cdot \nabla \partial_t \psi + \partial_t \psi \mathbf{d} \cdot \nabla \psi \, dx \, dt = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\begin{aligned}
& \frac{1}{2} \int_0^s \frac{d}{dt} \left( \|u\|_{L_2(\Omega)}^2 - \int_{\Omega} \mathbb{A} \nabla \psi \cdot \nabla \psi + b \psi^2 + \psi \mathbf{c} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla \psi \, dx \right) dt = \\
& = \int_0^s \int_{\Omega} (\partial_t \mathbb{A}) \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 + \psi \partial_t \mathbf{c} \cdot \nabla \psi + \underbrace{\partial_t \psi}_{=-u(t)} \mathbf{c} \cdot \nabla \psi - \psi \partial_t \mathbf{d} \cdot \nabla \psi - \psi \mathbf{d} \cdot \nabla \underbrace{\partial_t \psi}_{=-u(t)} \, dx \, dt,
\end{aligned}$$

and upon integration (recall  $\psi(s) = 0$ , from the definition of  $\psi$  it follows  $\nabla \psi(0) = 0$ , and  $u(0) = 0$ ),

$$\begin{aligned}
& \frac{1}{2} \left( \|u(s)\|_{L_2(\Omega)}^2 + \int_{\Omega} \mathbb{A}(0) \nabla \psi(0) \cdot \nabla \psi(0) + b(0) \psi(0)^2 + \psi(0) \mathbf{c}(0) \cdot \nabla \psi(0) + \psi(0) \mathbf{d}(0) \cdot \nabla \psi(0) \, dx \right) = \\
& = \int_0^s \int_{\Omega} \partial_t \mathbb{A} \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 - u \partial_t \mathbf{c} \cdot \nabla \psi - \psi \partial_t \mathbf{d} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla u \, dx \, dt.
\end{aligned}$$

From this we obtain the following estimate:

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\psi(0)\|_{W^{1,2}(\Omega)}^2 \leq C \left( \int_0^s \|\psi\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\psi(0)\|_{L_2(\Omega)}^2,$$

where  $C = C(\|\mathbb{A}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbb{A}\|_{L_{\infty}(\Omega)}, \|b\|_{L_{\infty}(\Omega)}, \|\partial_t b\|_{L_{\infty}(\Omega)}, \|\mathbf{c}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{c}\|_{L_{\infty}(\Omega)}, \|\mathbf{d}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{d}\|_{L_{\infty}(\Omega)})$ .

Define now the test function  $\chi(t) = \int_0^t u(\tau) \, d\tau$ , and realize that in fact  $\psi(t) = \chi(s) - \chi(t)$ ,  $\chi(0) = 0$ . Plugging this in the above inequality yields

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left( \int_0^s \|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\chi(s)\|_{L_2(\Omega)}^2,$$

and using

$$\|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 = \|\chi(t) - \chi(s)\|_{W^{1,2}(\Omega)}^2 \leq 2 \left( \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \right),$$

and the definition of  $\chi(t)$ , from which it follows

$$\|\chi(s)\|_{L_2(\Omega)}^2 \leq \int_0^s \|u\|_{L_2(\Omega)}^2 \, dt,$$

we are allowed to write

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left( \int_0^s 2 \|\chi(s)\|_{W^{1,2}(\Omega)}^2 + 2 \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + 2 \|u\|_{L_2(\Omega)}^2 \, dt \right),$$

and so

$$\|u(s)\|_{L_2(\Omega)}^2 + (1 - 2sC) \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_1 \left( \int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right).$$

If we now choose  $T_1 \in (0, T]$  small enough s.t.  $1 - 2sC > 0$  for  $s \in (0, T_1]$ , we finally obtain

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_2 \left( \int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right), \forall s \in (0, T_1],$$

which implies  $u = 0$  on  $(0, T_1]$  by the Gronwall lemma: we have

$$\xi(t) \leq \int_0^t \xi(s) \, ds, \text{ for } a.a. \, t \in (0, T) \Rightarrow \xi(t) = 0 \, a.e..$$

for  $\xi \in L_1((0, T))$  nonnegative<sup>1</sup>. If we now bootstrap on  $[T_1, 2T_1], [2T_1, 3T_1]$  etc., we obtain  $u = 0$  on  $(0, T]$ . □

## 2 Sobolev spaces revisited

Let  $\Omega \subset \mathbb{R}^d$  open,  $p \in [1, +\infty]$ ,  $k \in \mathbb{N}$ . We define

$$W^{k,p}(\Omega) = \left\{ f \in L_p(\Omega) ; D^\alpha f \in L_p(\Omega), \forall |\alpha| \leq k \right\},$$

with the norm

$$\|f\|_{W^{k,p}(\Omega)}^p = \|f\|_{L_p(\Omega)}^p + \sum_{0 < |\alpha| \leq k} \|D^\alpha f\|_{L_p(\Omega)}^p.$$

Recall that:

- $W^{k,p}(\Omega)$  is Banach  $\forall p$  and Hilbert for  $p = 2$ .
- $W^{k,p}(\Omega)$  is separable if  $p < \infty$  and reflexive if  $p > 1, p < \infty$ .

*Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.*

### 2.1 Tools from functional analysis

**Definition 1** (Regularization kernel). The function  $\eta$  is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\text{supp } \eta \subset U(0, 1)$
- $\eta \geq 0$
- $\eta$  is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x) dx = 1$

**Definition 2** (Regularization of a function). Let  $\eta$  be a regularization kernel. Set<sup>2</sup>

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of  $f \in L_1(\Omega)_{\text{loc}}$  by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

*Remark* (Properties of regularization). The regularization has the following properties:

- $f \in L_p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  in  $L_p(\Omega)$  and also a.e

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<sup>1</sup>In our case  $\xi = \|u\|_{L_2(\Omega)}^2 + \|\chi\|_{W^{1,2}(\Omega)}^2$ .

<sup>2</sup>Another common choice is  $\eta_k = k^d \eta(kx)$ ,  $k \in \mathbb{N}$ .

- $f \in L_\infty(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  a.e and \*-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) \eta_\varepsilon(x-y) dy = \int_{U(x,\varepsilon)} f(y) \eta_\varepsilon(x-y) dy$
- $\text{supp } f_\varepsilon \subset \overline{U(\Omega, \varepsilon)}, f = 0 \text{ on } U(x, \varepsilon) \Rightarrow f_\varepsilon(x) = 0$

**Definition 3** ( $\Omega' \subset\subset \Omega$ ).  $O \subset\subset \Omega$  means  $\overline{O}$  is compact and  $\overline{O} \subset \Omega$ .

**Definition 4** (Shift operator). For  $u \in L_p(\Omega), k \in \{1, \dots, d\}, h > 0$ , we introduce the shift operator

$$\tau_h u(x) = u(x + h \mathbf{e}_k)$$

**Lemma 1** (Approximation property of the shift operator). For  $u \in L_p(\Omega)$ , it holds  $\tau_h u \rightarrow u$  in  $L_p(\Omega), h \rightarrow 0^+$ .

**Lemma 2** (Partition of unity). Let  $E \subset \mathbb{R}^d, \mathcal{G}$  be an open covering of  $E$  (possibly uncountable.) Then there exists a countable system  $\mathcal{F}$  of nonnegative functions  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $0 \leq \varphi \leq 1$  and

1.  $\mathcal{F}$  is subordinate to  $\mathcal{G} : \forall \varphi \in \mathcal{F} \exists U \in \mathcal{G} : \text{supp } \varphi \subset U$
2.  $\mathcal{F}$  is locally finite<sup>3</sup>:  $\forall K \subset E$  compact,  $\text{supp } \varphi \cap K \neq \emptyset$  for at most finitely many  $\varphi \in \mathcal{F}$ .
3.  $\sum_{\varphi \in \mathcal{F}} \varphi(x) = 1, \forall x \in E$ .

*Proof.* (Sketch) *Step 1 (If  $E$  is compact):*

$E$  compact  $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}$  s.t.  $E \subset \bigcup_{j=1}^m U_j$ . Moreover,  $\exists K_j \subset U_j$  compact such that  $E \subset \bigcup_{j=1}^m K_j$ . That follows from the exhaustion argument: for  $U \subset \mathbb{R}^d$  open, you can approximate it by a compact set:

$$K_m = \left\{ x \in U \mid \text{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m \right\}.$$

Then clearly  $K_1 \subset K_2 \dots$ , and they "converge monotonously to  $U$ ". Next, find  $\phi_j \in C_c(U_j), \phi_j > 0$  on  $K_j$ , e.g.  $\phi_j = \theta(\text{dist}(x, \partial U_j))$ . Then use convolution:  $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$  small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

*Step 2 (If  $E$  is open):*

Approximate  $E$  by  $K \subset E$  compact by the exhaustion argument, then the covering will enlarge from finite  $\rightarrow$  countable (nontrivial reasoning).  $\square$

## 2.2 Density of smooth functions

**Lemma 3** (Local approximation by smooth functions (using regularization)). Assume  $p \in [1, \infty), \Omega \subset \mathbb{R}^d$  open,  $k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$ . Then it holds

1.  $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$  a.e. in  $\Omega_\varepsilon, \forall |\alpha| \leq k$
2.  $u_\varepsilon \rightarrow u$  in  $W^{k,p}(\Omega)_{loc}, \varepsilon \rightarrow 0^+$

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<sup>3</sup>In other words,  $\varphi_K$  is nonzero for at most finitely many  $\varphi \in \mathcal{F} \Leftrightarrow$  points in  $K$  can be represented by finitely many functions  $\varphi \in \mathcal{F}$ .

*Proof.* First of all:

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\left(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y) dy\right) = \int_{\mathbb{R}^d} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy,$$

the integrable majorants are *e.g.*  $\|\eta_\varepsilon\|_\infty |u| \chi_{U(0,\varepsilon)}(x) \in L_1(\Omega)$ . Now picking  $x \in \Omega_\varepsilon$  we realize  $\forall y \in \mathbb{R}^d/\Omega : x-y \geq \text{dist}(x, \partial\Omega) \geq \varepsilon$ , and so  $\eta_\varepsilon(x-y) = 0$ . Exchanging derivatives and using the definition of the weak derivative

$$\int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = (-1)^{|\alpha|} \int_{\Omega} u(y)D_y^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = (D^\alpha u)_\varepsilon.$$

Take  $V \subset\subset \Omega$  open, then

$$\|u - u_\varepsilon\|_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u - D^\alpha u_\varepsilon\|_{L_p(V)} \rightarrow 0,$$

because  $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \rightarrow D^\alpha u$  in  $L_p(V)$ , from the properties of regularization.  $\square$

**Theorem 4** (Global approximation by smooth functions). *Let  $\Omega \subset \mathbb{R}^d$  be open,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $C = \{f \in C^\infty(\Omega), \text{supp } f \text{ bounded}\} \cap W^{k,p}(\Omega)$  is dense in  $W^{k,p}(\Omega)$ , i.e.*

$$\overline{C \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

*If moreover  $\Omega$  is bounded, it holds:*

$$\overline{C^\infty \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

*Proof.* Let  $u \in W^{k,p}(\Omega), \varepsilon > 0$ . I want to show  $\exists v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$  s.t.  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ . For every  $j \in \mathbb{N}$  define an open set

$$\Omega_j = \left\{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{j}\right\}.$$

Clearly,  $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$ . Next, set

$$U_j = \Omega_{j+1} / \overline{\Omega_{j-1}}, j = 1, 2, \dots,$$

where  $\Omega_0 = \Omega_{-1} = \emptyset$ . Since  $\Omega_j$  are open,  $U_j$  are also open and  $\Omega \subset \bigcup_{j \in \mathbb{N}} U_j \Rightarrow \exists \{\varphi_j\}_{j \in \mathbb{N}}$  partition of unity subordinate to  $\{U_j\}_{j \in \mathbb{N}}$ . We can write  $u = \sum_{j \in \mathbb{N}} u\varphi_j$ , where  $u\varphi_j \in W^{k,p}(\Omega), \text{supp } u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$ . This is ready for convolution with  $\varepsilon_j > 0$ : set  $v_j = (u\varphi_j)_{\varepsilon_j}$  and fix an arbitrary  $\delta > 0$ . By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{\delta}{2^{j-1}},$$

for  $\varepsilon_j > 0$  sufficiently small, which we now fix so the above inequality holds. To have a nice inequality, we actually want:

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{2^N}{2^{N+1} - 1} \frac{\delta}{2^{j-1}},$$

meaning of  $N \in \mathbb{N}$  will be evident later.

Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then  $v \in C^\infty(\Omega)$ , (not clearly in  $W^{k,p}(\Omega)$  however) as  $\forall x \in \Omega$  the sum contains at most finitely many terms ( $\mathcal{F}$  is locally finite.)

Take the  $N \in \mathbb{N}$  and estimate the norm  $\|u - v\|_{W^{k,p}(\Omega)}$ . Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking  $x \in \Omega_N$  i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for  $m > N + 1$ , i.e.,  $m - 1 > N$  it holds  $U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}$ ,  $\Omega_N \subset \Omega_{m-1}$  meaning  $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \emptyset \Rightarrow \text{supp } u\varphi_j \cap \Omega_N = \text{supp } v_j \cap \Omega_N = \emptyset$ , since  $\text{supp } u\varphi_j \subset U_j$ ,  $\text{supp } v_j \subset \text{supp } u\varphi_j \subset U_j$ ,  $\forall j \geq m$ . The norm of sum is

$$\|u - v\|_{W^{k,p}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{W^{k,p}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let  $N \rightarrow \infty$  and realize

$$\|u - v\|_{W^{k,p}(\Omega_N)} \rightarrow \|u - v\|_{W^{k,p}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| dx = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) dx = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} dx = \int_{\mathbb{R}^d} |D^\alpha f| \chi_\Omega(x) dx = \int_\Omega |D^\alpha f| dx,$$

since  $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$ , and  $|D^\alpha f|$  is nonnegative, so the sequence under the integral is nondecreasing. Altogether,

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows  $v \in W^{k,p}(\Omega)$  (this was not totally evident) and thus  $v \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$  so indeed we have showed the desired density.  $\square$

*Remark.* It is nice that we only require  $\Omega$  to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

*Remark* ( $C^{k,\lambda}$  domain). Recall we call  $\Omega \subset \mathbb{R}^d$  to be of class  $C^{k,\lambda}$  if:  $\Omega$  is open and bounded,  $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0, 1], \alpha, \beta \in \mathbb{R}^+, \exists$  open sets  $U_j \subset \mathbb{R}^d, \exists a_j : B(0, \alpha) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R} \text{ s.t. } a_j \in C^{k,\lambda}(B(0, \alpha)), \exists \mathbb{A}_j \mathbb{R}^d \rightarrow \mathbb{R}^d$  affine orthogonal matrices such that

1.  $\partial\Omega \subset \bigcup_{j=1}^m U_j$ ,
2.  $\forall j \leq m : \partial\Omega \cap U_j = \mathbb{A}_j(\{(x', a_j(x')) \in \mathbb{R}^d | x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}\})$ ,
3.  $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \Omega$ ,
4.  $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$ .



If  $\lambda = 0$  we sometimes drop it and write  $\Omega \in C^{k,0} \Leftrightarrow \Omega \in C^k$ , if  $k = 0, \lambda = 1$  we call  $\Omega \in C^{0,1}$  to be a Lipschitz domain. *Remember that  $\lambda(\Omega) < \infty$  is a part of the definition.*

**Theorem 5** (Global approximation by smooth functions up to the boundary). *Let  $\Omega \in C^{0,0}$ ,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $C_{\bar{\Omega}}^\infty(\mathbb{R}^d)$  is dense in  $W^{k,p}(\Omega)$ .*

*Proof.* Let  $u \in W^{k,p}(\Omega)$ , and  $\varepsilon > 0$ , be given. We wish to find  $v \in C^\infty(\bar{\Omega})$  s.t.  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ .

The sketch is simple:

1. covering of  $\bar{\Omega}$ ,
2. partition of unity,
3. approximation of  $u$  on the covering sets,
4. glue it together.

Set  $U_0 = \Omega$ , and let  $\{U_j\}_{j=1}^m$  be from the definition of  $C^{0,0}$  boundary. Then<sup>4</sup>

$$\bar{\Omega} \subset \bigcup_{j=0}^m U_j,$$

Take  $\{\varphi_j\}$  to be the partition of unity on  $\bar{\Omega}$ , subordinate to  $\{U_j\}_{j=0}^m$ . Since

$$u = \sum_{j=0}^m u\varphi_j, \text{ on } \Omega$$

observe that  $u_j := u\varphi_j \in W^{k,p}(\Omega)$ ,  $\text{supp } u_j \subset \text{supp } \varphi_j \subset U_j$ . **Also, we define**  $u(x) = 0, \forall x \in \mathbb{R}^d/\Omega$ . The proofs differs in the cases  $j = 0$  and  $j \in \{1, \dots, m\}$ .

*Case  $j = 0$ .* We have  $\text{supp } u\varphi_0 \subset U_0 = \Omega$ . That means that after the extension of  $u\varphi_0$  by zero outside of  $\Omega$ , it holds  $u\varphi_0 \in W^{k,p}(\mathbb{R}^d)$ . Since  $W^{k,p}(\mathbb{R}^d) = W_0^{k,p}(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{W^{k,p}(\mathbb{R}^d)}}$ , we can find  $v_0 \in \mathcal{D}(\mathbb{R}^d)$  s.t.

$$\|v_0 - u\varphi_0\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{m+1}.$$

*Case  $j \in \{1, \dots, m\}$ .* We have a problem now:  $\{U_j\}_{j=1}^m$  covers  $\partial\Omega$ , which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of  $\partial\Omega$ , the kernel would pick up values from outside of  $\Omega$ , where  $u = 0$  and the mollification would not be a good approximation. Instead, we approximate  $u_j$  on a larger *open* domain containing  $\bar{\Omega}$  and then show this is also a good approximation of  $u_j$  on  $\Omega \subset \bar{\Omega}$ .

Set  $w_j = u\varphi_j$ , and denote

$$S_j = \mathbb{A}_j \left( \left\{ (x', x_d) \mid a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j = \mathbb{R}^d / \overline{S_j},$$

i.e.,

$${}^{\text{''}}\Omega_j = \Omega \cup \mathbb{A}_j \left( \left\{ (x', x_d) \mid x_d \leq a_j(x') - \frac{\beta}{2} \right\} \right),{}^{\text{''}}$$

---

<sup>4</sup>Our choice  $U_0 = \Omega$  is important, as without it the definition of  $C^{0,0}$  boundary only means  $\partial\Omega \subset \bigcup_{j=1}^m U_j$ .

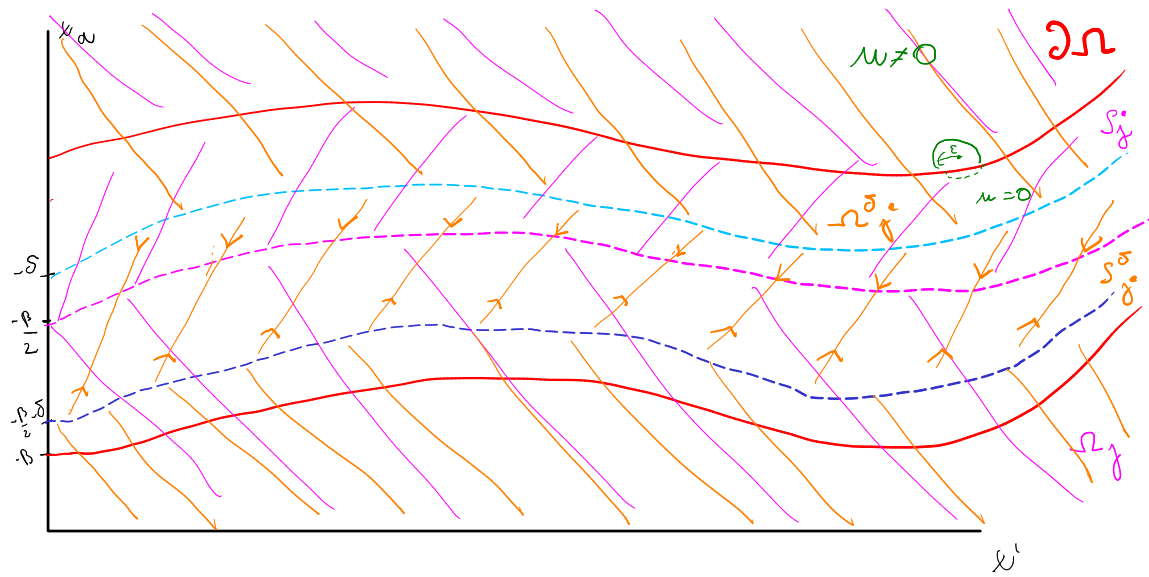


Figure 1: A cumbersome sketch of  $\Omega_j, S_j, \Omega_j^\delta, S_j^\delta$

(although this is a bit inaccurate). Realize that since  $u = 0$  outside of  $\Omega$ , also  $u_j$  is zero there and in particular it is zero on that "lower strip". Clearly then  $u_j \in W^{k,p}(\Omega_j)$ . Now pick  $\delta \in (0, \frac{\beta}{2})$ , where  $\beta$  is from the definition of  $C^{0,0}$  and set

$$S_j^\delta = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

i.e.,

$${}^{\prime\prime}\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j \left( \left\{ (x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta \right\} \right).{}^{\prime\prime}$$

The trick is to shift the (support of) function  $u_j$  "into"  $\Omega_j^\delta$

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\text{supp}(\tau_\delta u_j) = \text{supp}(u_j) - \delta,$$

from which it follows  $\tau_\delta u_j \in W^{k,p}(\Omega_j^\delta)$ ; we have only shifted the function  $u_j$ , but since we have also shifted  $S_j$ , qualitatively there is no difference. Since  $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j$ ,  $\Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$ , and the fact  $\tau_\delta$  is an isometry between Sobolev spaces, we also have  $u_j, \tau_\delta u_j \in W^{k,p}(\Omega_j \cap \Omega_j^\delta)$ . Moreover, from the properties of the shift operator it follows  $\exists \delta > 0$  s.t.

$$\|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} \leq \|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega_j \cap \Omega_j^\delta)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know  $\tau_\delta u_j$  is already close to  $u_j$ , we are done once we approximate  $\tau_\delta u_j$  by a function from  $C^\infty(\overline{\Omega})$ . Notice that if we show  $\overline{\Omega} \subset \Omega_j^\delta$ , then clearly  $C^\infty(\overline{\Omega}) \subset C^\infty(\overline{\Omega_j^\delta})$ .

Show  $\Omega \subset \Omega_j^\delta$ : We already know  $\Omega \subset \Omega_j^\delta$ , so it suffices to show  $\partial\Omega \subset \Omega_j^\delta$ . Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k(\{(x', x_d) | x_d = a_k(x'), x' \in U(0, \alpha)\}),$$

and the set  $\Omega_j^\delta$  is given as  $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta}$ , where

$$S_j = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right).$$

Realize it suffices to show  $\partial\Omega \not\subset \overline{S_j}$ , as then it wont be excluded from  $\mathbb{R}^d$  and thus will end up in  $\Omega_j^\delta$ . Thanks to continuity of  $a_j$ , we may write

$$\overline{S_j} = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

i.e., the " $<$ " have changed to " $\leq$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d) | x_d = a_j(x'), x' \in U(0, \alpha)\}) \not\subset \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

which is equivalent to

$$\left( (a_j \leq a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta) \right) \vee \left( (a_j > a_j - \delta) \wedge (a_j \geq a_j - \frac{\beta}{2} - \delta) \right).$$

Our choice has been  $\delta \in (0, \frac{\beta}{2})$ , and  $\beta > 0$  from the definition of  $\Omega \in C^{0,0}$ , so the second statement is clearly true  $\forall j \in 1, \dots, m$ . Consequently  $\partial\Omega \notin \overline{S_j}$  which leads to  $\partial\Omega \subset \Omega_j^\delta$ , and since also  $\Omega \subset \Omega_j^\delta$ , we have  $\overline{\Omega} \subset \Omega_j^\delta$ .

*Approximation of  $\tau_\delta u_j$ .* Since  $\Omega_j^\delta$  is open there  $\exists w_j \in C^\infty(\Omega_j^\delta)$  such that

$$\|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega)} \leq \|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega_j^\delta)} < \frac{\varepsilon}{4(m+1)}.$$

What is more, since  $\overline{\Omega} \subset \Omega_j^\delta$ , we see  $w_j \in C^\infty(\overline{\Omega})$  in fact. What is even more, we see  $\text{supp } w_j \subset \text{supp } u_j - \delta$  so  $\text{supp } \tau_{-\delta} w_j \subset \text{supp } u_j \subset \subset \Omega$ , and since  $w_j$  is defined on the whole  $\mathbb{R}^d$ , also  $v_j := \tau_{-\delta} w_j$  is defined on the whole  $\mathbb{R}^d$ . Altogether,  $v_j = \tau_{-\delta} w_j \in \mathcal{D}(\mathbb{R}^d)$ , and from the properties of the shift operator, there  $\exists \delta > 0$  s.t.

$$\|v_j - w_j\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{4(m+1)}.$$

*Approximation of  $u$ .*

Finally, let us set

$$v = \sum_{j=0}^m v_j.$$

Then  $v \in \mathcal{D}(\mathbb{R}^d)$  and it holds

$$\begin{aligned} \|u - v\|_{W^{k,p}(\Omega)} &= \left\| \sum_{j=0}^m u_j - \sum_{j=0}^m v_j \right\|_{W^{k,p}(\Omega)} = \left\| \sum_{j=0}^m u_j - v_j \right\|_{W^{k,p}(\Omega)} \leq \sum_{j=0}^m \|u_j - v_j\|_{W^{k,p}(\Omega)} \leq \\ &\leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - u_j\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - w_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|w_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|\tau_\delta u_j - u_j\|_{W^{k,p}(\Omega)} \\ &< \frac{\varepsilon}{m+1} + 2 \sum_{j=1}^m \frac{\varepsilon}{4(m+1)} + \sum_{j=1}^m \frac{\varepsilon}{2(m+1)} = \varepsilon \end{aligned}$$

Finally,

$$\mathcal{D}(\mathbb{R}^d) \subset C_\Omega^\infty(\mathbb{R}^d) \subset C^\infty(\overline{\Omega}).$$

*This proof may still have some flaws, but the author has decided to move on.* □

*Remark* (What is  $C_\Omega^\infty(\mathbb{R}^d)$ ). Recall

$$C_\Omega^\infty(\mathbb{R}^d) = \left\{ u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d) \right\}.$$

### 2.3 Extension of Sobolev functions

*Problem of extension:* For  $u \in W^{k,p}(\Omega)$ , does there exist

$$\bar{u} \in W^{k,p}(\mathbb{R}^d), \text{ s.t. } \bar{u}|_{\Omega} = u, \|\bar{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{W^{k,p}(\Omega)}$$

?

The answer is **yes**, if  $\Omega$  is nice enough. Notice however this is not as simple as in the case of Lebesgue spaces, where we could just extend the function by zero trivially. We are dealing with derivatives, that are somehow regular, and if we extend a nonzero function by zero, it might mess up the regularity of the derivatives.

We will be using  $C^1$  diffeomorphisms heavily, so we investigate some of their properties first.

**Lemma 4** (Properties of  $C^1$  diffeomorphisms). *Let  $U, V \subset \mathbb{R}^d$  be open,  $\Phi : U \rightarrow V$  be a  $C^1$  diffeomorphism and let  $\tilde{U} \subset \mathbb{R}^d$  s.t.  $\tilde{U} \subset \subset U$ . Then*

1.  $\Phi(\tilde{U}) \subset \subset V$ ,
2. if moreover  $\tilde{U}$  is compact, then <sup>5</sup>

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \leq C \|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

*Proof.* Ad 1.: No proof has been given.

Ad 2.: Just a change of variables formula:

$$\|u \circ \Phi\|_{L^p(\tilde{U})}^p = \int_{\tilde{U}} |u \circ \Phi|^p dx = \int_{\Phi(\tilde{U})} |u|^p |\det \nabla \Phi|^{-1} dx.$$

Since  $\Phi$  is one-to-one, we know  $|\det \nabla \Phi| > 0$  on  $U$ , and since  $\Phi \in C^1(U)$  and  $\tilde{U} \subset U \Rightarrow \Phi \in C^1(\tilde{U}) \Rightarrow \det \nabla \Phi \in C^0(\tilde{U})$ , and since  $\tilde{U}$  is compact,  $|\det \nabla \Phi| \geq C_1 > 0 \Leftrightarrow |\det \nabla \Phi|^{-1} \leq \frac{1}{C_1}$ . In total

$$\|u \circ \Phi\|_{L^p(\tilde{U})}^p \leq \frac{1}{C_1} \int_{\Phi(\tilde{U})} |u|^p dx = C \|u\|_{L^p(\Phi(\tilde{U}))}^p.$$

As for the derivative, we have  $\forall i \in \{1, \dots, d\}$ :

$$\begin{aligned} \int_{\tilde{U}} |\partial_i(u \circ \Phi)|^p dx &\leq \int_{\tilde{U}} |\nabla(u \circ \Phi)|^p dx = \int_{\tilde{U}} |\nabla \Phi((\nabla u) \circ \Phi)|^p dx \leq \\ &\leq \|\nabla \Phi\| \int_{\tilde{U}} |(\nabla u) \circ \Phi|^p dx = \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p |\det \nabla \Phi|^{-1} dx \leq C \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p dx \leq \\ &\leq C \|\nabla u\|_{L^p(\Phi(\tilde{U}))}^p, \end{aligned}$$

where  $\|\nabla \Phi\|$  is e.g. the operator norm of the matrix  $\nabla \Phi$ . □

**Lemma 5** (Flat extension). *Let  $\alpha, \beta > 0, K \subset \subset U(0, \alpha) \times [0, \beta]$  be compact. Then there  $\exists C > 0$ , a linear operator*

$$E : C^1((B(0, \alpha) \times [0, \beta])) \rightarrow C^1((B(0, \alpha) \times [-\beta, \beta])),$$

*and the set  $\tilde{K} \subset \subset U(0, \alpha) \times [-\beta, \beta]$  such that  $\forall u \in C^1(B(0, \alpha) \times [0, \beta])$  it holds*

---

<sup>5</sup>For  $\tilde{U}$  compact:  $\tilde{U} \subset \subset V \Leftrightarrow \tilde{U} \subset V$ .

1.  $Eu = u$  on  $B(0, \alpha) \times [0, \beta]$ ,

2.  $\|Eu\|_{W^{1,p}(U(0,\alpha) \times (-\beta, \beta))} \leq \|u\|_{W^{1,p}(U(0,\alpha) \times (0, \beta))}$ . *Actually,*

$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha) \times (0, \beta)), W^{1,p}(U(0,\alpha) \times (-\beta, \beta)))} = 2^{\frac{1}{p}}$$

3. if  $\text{supp } u \subset K$  then  $\text{supp } Eu \subset \tilde{K}$

*Proof.* (The set  $U(0, \alpha) \times [0, \beta]$  is a cylinder of radius  $\alpha$  and height  $\beta$ )

The proof is constructive: for the assumed  $u$  we write  $(x = (x', x_d)$ , where  $x' \in B(0, \alpha) \subset \mathbb{R}^{d-1}$ ,  $x_d \in [0, \beta] \subset \mathbb{R}$ )

$$Eu(x', x_d) = \begin{cases} u(x', x_d), & x_d \geq 0 \\ -3u(x', -x_d) + 4u(x', -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Does  $Eu$  lie in  $C^1(B(0, \alpha) \times [-\beta, \beta])$ ? Since  $u \in C^1(B(0, \alpha) \times [0, \beta])$ , it is continuous in the "lower cylinder", check only the transition through the origin plane: take some  $a = (x', 0) \in B(0, \alpha) \times \{0\}$ . Then

$$\lim_{x \rightarrow a} Eu(x) = \begin{cases} u(a), & x_d \geq 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so  $Eu$  is continuous. The derivatives

$$\lim_{x \rightarrow a} \partial_d Eu(x', x_d) = \begin{cases} \partial_d u(x', 0), & x_d \geq 0 \\ (-3\partial_d u(x', -x_d)(-1) + 4\partial_d u(x', -\frac{x_d}{2})(-\frac{1}{2}))|_{x_d=0} = \partial_d u(x', 0), & x_d < 0, \end{cases}$$

and also for any  $i \in \{1, \dots, d-1\}$

$$\lim_{x \rightarrow a} \partial_i Eu(x_1, \dots, x_d) = \begin{cases} = \partial_i u(x_1, \dots, 0), & x_d \geq 0 \\ = -3\partial_i u(x_1, \dots, 0) + 4\partial_i u(x_1, \dots, 0) = \partial_i u(x_1, \dots, 0) & \end{cases}$$

so the derivative is also continuous. Thus, we have

$$Eu \in C^1 \subset W^{1,p}(U(0, \alpha) \times (-\beta, \beta)).$$

The first property is clear from the definition of  $Eu$ , the estimates of the norm: (all derivatives can in fact be assumed classical)

$$\begin{aligned} \|Eu\|_{W^{1,p}(U(0,\alpha) \times (-\beta, \beta))}^p &= \|Eu\|_{L^p(U(0,\alpha) \times (-\beta, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (-\beta, \beta))}^p = \\ &= \|Eu\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \|Eu\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p \\ &+ + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p = \\ &= \|u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \|4u - 3u\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p + \\ &+ \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha (4u - 3u)\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p \\ &= 2\|u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + 2 \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p = 2\|u\|_{W^{k,p}(U(0,\alpha) \times (0, \beta))}^p \end{aligned}$$

and so

$$\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} = 2^{\frac{1}{p}} \|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}.$$

Where we have used the obvious fact

$$\int_{U(0,\alpha)\times(0,\beta)} |f|^p dx = \int_{U(0,\alpha)\times(-\beta,0)} |f|^p dx,$$

We will assume (although this is *with* a loss of generality), that

$$Eu(x', x_d) = 0 \Leftrightarrow (u(x', x_d) = 0 \vee 3u(x', -x_d) = 0 \vee 4u(x', -\frac{x_d}{2}) = 0),$$

i.e., that support lies in

$$\begin{aligned} \text{supp } Eu \subset & \{(x', x_d) \in U(0, \alpha) \times [0, \beta] | u(x', x_d) = 0\} \cup \{(x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -x_d) \neq 0\} \cup \\ & \cup \left\{ (x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -\frac{x_d}{2}) \neq 0 \right\}. \end{aligned}$$

Denote  $\Phi_1, \Phi_2 : U(0, \alpha) \times (0, \beta) \rightarrow U(0, \alpha) \times (-\beta, \beta)$  to be the mappings

$$\begin{aligned} \Phi_1(x', x_d) &= (x', -x_d), \\ \Phi_2(x', x_d) &= (x', -\frac{x_d}{2}), \end{aligned}$$

then clearly  $\Phi_1, \Phi_2$  are  $C^1$  diffeomorphisms and

$$u(x', -x_d) = (u \circ \Phi_1)(x', x_d), u\left(x', -\frac{x_d}{2}\right) = (u \circ \Phi_2)(x', x_d),$$

i.e.,

$$u(x', x_d) = (u \circ \Phi_1^{-1})(x', -x_d) = (u \circ \Phi_2^{-1})\left(x', -\frac{x_d}{2}\right),$$

and so

$$\text{supp } Eu \subset \text{supp } u \cup \Phi_1^{-1}(\text{supp } u) \cup \Phi_2^{-1}(\text{supp } u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as  $\text{supp } u \subset K$ . Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\text{supp } Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have  $K \subset U(0, \alpha) \times [0, \beta] \Rightarrow K \subset U(0, \alpha) \times (-\beta, \beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset U(0, \alpha) \times (0, \beta)$ , which really implies

$$\tilde{K} \subset U(0, \alpha) \times [-\beta, \beta].$$

□

Let us prove the main result.

**Theorem 6** (Extension of Sobolev functions). *Let  $\Omega \in C^{k-1,1}$ ,  $k \in \mathbb{N}$ ,  $p \in [1, \infty]$ ,  $V \subset \mathbb{R}^d$  open such that  $\Omega \subset V$ . Then there is  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$  bounded linear operator such that*

1.  $\forall u \in W^{k,p}(\Omega) : Eu = u \text{ a.e. in } \Omega,$
2.  $\forall u \in W^{k,p}(\Omega) : \text{supp } Eu \subset V \supset \supset \Omega,$
3.  $\|E\|_{\mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^d))} \leq C, C = C(p, \Omega, V).$

*Proof.* **Will only be presented for  $k = 1, \Omega \in C^1, p < \infty$ .** The strategy is:

1. covering of  $\overline{\Omega}$  & partition of unity
2. obtain a diffeomorphism from the fact  $\Omega \in C^{1,0}$ ,
3. suitable composition & cut off,
4. flat extension,
5. show existence of  $E : C_{\Omega}^{\infty}(\mathbb{R}^d) \rightarrow W^{k,p}(\mathbb{R}^d)$  with the desired properties,
6. extend via density

*Covering of  $\Omega$ :* In the following,  $U_j, a_j, \mathbb{A}_j, \alpha, \beta$  are as in the definition of a  $C^{1,0}$  domain. Set  $U_0 = \Omega$  and realize

$$\overline{\Omega} \subset \bigcup_{j=0}^m U_j,$$

i.e.,  $\{U_j\}_{j=0}^m$  is an open covering of  $\overline{\Omega}$ . Denote  $\{\varphi_j\}_{j=0}^m$  as the partition of unity subordinate to  $\{U_j\}_{j=0}^m$ .

*Diffeomorphism & flat extension* For  $j \in \{1, \dots, m\}$  we define  $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$  by

$$\Phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

(A bit confusingly, we will however be "interpreting"  $\Phi_j$  as  $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow \mathbb{R}^d$ , with it being extended by zero on  $\mathbb{R}^d \setminus \overline{U_j}$ , as we need  $\Phi_j^{-1}$  to be defined on the whole  $\mathbb{R}^d$ .)

As  $\Omega \in C^{1,0}$ , we know  $a_j \in C^1(B(0, \alpha)) \subset C^1(U(0, \alpha))$ , and so  $\phi_j$  is a  $C^1$  diffeomorphism. Let us denote by  $\tilde{E} : C^1(B(0, \alpha) \times [-\beta, \beta]) \rightarrow C^1(B(0, \alpha) \times [-2\beta, \beta])$  the extension operator from the Flat extension lemma. Then we for any  $u \in C_{\Omega}^{\infty}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$  define

$$Eu = \varphi_0 u + \sum_{j=1}^m \left( \eta \tilde{E}((\varphi_j u) \circ \Phi_j) \right) \circ \Phi_j^{-1},$$

where  $\eta \in C^{\infty}(U(0, \alpha) \times \mathbb{R})$  is a cut-off function

$$\eta(y', y_d) \begin{cases} = 1 & \text{on } y_d \geq 0, \\ = 0 & \text{on } y_d \leq -h, \\ \in (0, 1) & \text{else.} \end{cases}$$

for some parameter  $h > 0$  which will be defined later. With this definition,  $\forall x \in \Omega$  it holds

$$\begin{aligned} (Eu)(x) &= (\varphi_0 u)(x) + \sum_{j=1}^m \left( \eta \tilde{E}(u \varphi_j \circ \Phi_j) \right) \underbrace{(\Phi_j^{-1}(x))}_{\in U(0, \alpha) \times (-\beta, \beta)} = \\ &= (\varphi_0 u)(x) + \sum_{j=1}^m (u \varphi_j \circ \Phi_j)(\Phi_j^{-1}(x)) = (\varphi_0 u)(x) + \sum_{j=1}^m (u \varphi_j)(x) = \sum_{j=0}^m (u \varphi_j)(x) = u(x) \sum_{j=0}^m \varphi_j(x) = u(x), \end{aligned}$$



since  $\eta(y) = 1$ ,  $\tilde{E}(\varphi_j u \circ \Phi_j)(y) = (\varphi_j u \circ \Phi_j)(y)$  for  $y \in U(0, \alpha) \times (-\beta, \beta)$ , according to our definition of  $\eta$  and the properties of the extension  $\tilde{E}$ .

The motivation behind the cutoff is the following: we know  $\text{supp } \varphi_j \subset U_j$ , so since  $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$ , we have  $\text{supp } u\varphi_j \circ \Phi_j \subset U(0, \alpha) \times (-\beta, \beta)$  and from the properties of the flat extension operator we also have  $\text{supp } \tilde{E}(u\varphi_j \circ \Phi_j) \subset U(0, \alpha) \times (-2\beta, \beta)$ . Since moreover  $\text{supp } \eta = U(0, \alpha) \times (-2h, \infty)$ , in total

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) = \text{supp } \eta \cap \text{supp } \tilde{E}(u\varphi_j \circ \Phi_j) = U(0, \alpha) \times (-2h, \infty) \cap (-\beta, \beta),$$

and also <sup>6</sup>

$$\text{supp } Eu \subset \Phi_j(\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j)).$$

We need to prove  $\text{supp } Eu \subset V$ , where  $V$  is some neighbourhood of  $\Omega$ . If it holds

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) \subset U(0, \alpha) \times (-\beta, \beta),$$

the desired property holds, as then

$$\text{supp } Eu \subset \Phi_j(\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j)) \subset \Phi_j(U(0, \alpha) \times (-\beta, \beta)) \subset U_j \subset \bar{\Omega} \subset V.$$

In total,

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) = U(0, \alpha) \times (-2h, \infty) \cap (-\beta, \beta) \subset U(0, \alpha) \times (-\beta, \beta).$$

So if  $h$  is such that  $-2h > -\beta \Leftrightarrow h < \frac{\beta}{2}$ , we can guarantee  $\text{supp } Eu \subset V$  and that is what we want.

Finally,  $E$  is clearly linear, its norm: (we are using the lemma about flat extension and the properties of  $C^1$  diffeomorphisms together with the facts  $\eta \leq 1$  on  $U(0, \alpha) \times (-\beta, \beta)$ ,  $\Phi_j(U(0, \alpha) \times (0, \beta)) \subset U_j, \Phi_j^{-1}(\mathbb{R}^d) \subset {}^7 U(0, \alpha) \times (-\beta, \beta)$ , <sup>8</sup>)

$$\begin{aligned} \|Eu\|_{W^{1,p}(\mathbb{R}^d)} &= \|(\eta \tilde{E}(\varphi u_j \circ \Phi_j)) \circ \Phi_j^{-1}\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|\eta \tilde{E}(\varphi u_j \circ \Phi_j)\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} = \\ &= C \|\tilde{E}(u\varphi_j \circ \Phi_j)\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} \leq C \|u\varphi_j \circ \Phi_j\|_{W^{1,p}(U(0, \alpha) \times (0, \beta))} \leq \\ &\leq C \|u\varphi_j\|_{W^{1,p}(U_j \cap \Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

from which it clearly follows  $\|E\|_{\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^d))} \leq C$ .

So all the properties hold for  $u \in C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d)$ , let us show them also for  $u \in W^{1,p}(\Omega)$ . Pick an arbitrary  $u \in W^{1,p}(\Omega)$ , find  $\{u_k\} \subset C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d) : u_k \rightarrow u$  in  $W^{1,p}(\Omega)$ . Since  $E$  is continuous, we know  $\lim_{k \rightarrow \infty} Eu_k$  exists. Let us set

$$Eu := \lim_{k \rightarrow \infty} Eu_k,$$

where  $Eu_k$  is defined above for  $u_k \in C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d)$ .

Ad 3): Clearly  $E(\alpha u) = \lim_{k \rightarrow \infty} E(\alpha u_k) = \alpha Eu$ ,  $E(u + v) = \lim_{k \rightarrow \infty} E(u_k + v_k) = \lim_{k \rightarrow \infty} Eu_k + Ev_k = Eu + Ev$ , since  $E$  on the continuous functions is linear. Also

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<sup>6</sup>Generally,  $\text{supp } v \circ \Psi^{-1} = \overline{\{x \in \mathbb{R}^d | u(\Psi^{-1}(x)) \neq 0\}} = \overline{\{x \in \mathbb{R}^d | \Psi^{-1}(x) \in \text{supp } u\}} = \overline{\{x \in \mathbb{R}^d | x \in \Psi(\text{supp } u)\}} = \Psi(\text{supp } u)$ .

<sup>7</sup>We can somehow extend  $\Phi_j^{-1}$  by zero from  $U_j \subset \Omega \subset \mathbb{R}^d$  to be defined on the whole  $\mathbb{R}^d$  (i guess)

<sup>8</sup>...even though the assumptions to use those are not totally valid... but doc. Kaplicky is okay with that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} = \left\| \lim_{k \rightarrow \infty} Eu_k \right\|_{W^{1,p}(\mathbb{R}^d)} = \lim_{k \rightarrow \infty} \|Eu_k\|_{W^{1,p}(\mathbb{R}^d)} \leq \|E\| \lim_{k \rightarrow \infty} \|u_k\|_{W^{1,p}(\mathbb{R}^d)} = \|E\| \|u\|_{W^{1,p}(\mathbb{R}^d)},$$

(we are using  $\{Eu_k\}$  has a limit); we see our above definition truly yields a continuous linear operator.

Ad 1):  $\forall a.a. x \in \Omega : Eu(x) = \lim_{k \rightarrow \infty} Eu_k(x) = \lim_{k \rightarrow \infty} u_k(x) = u(x)$ ,

Ad 2):  $\text{supp } Eu_k \subset U(\Omega, \varepsilon) \Rightarrow \text{supp } Eu \subset \overline{U}(\Omega, \varepsilon) \subset V$ .

$$\text{supp } Eu = \left\{ x \in \mathbb{R}^d \mid \lim_{k \rightarrow \infty} Eu_k \neq 0 \right\} \subset \underbrace{\bigcap_{k=1}^{\infty} \text{supp } Eu_k}_{\subset V} \subset V.$$

We are done. □

*Remark* ( $\Omega \in C^{0,1}$  suffices). The theorem is still valid if we assume only  $\Omega \in C^{0,1}$  and  $p \in (1, \infty)$ ,  $k \in \mathbb{N}$ , but the construction of the extension must be different. "It seems the result is not known for  $\Omega \in C^{0,1}$  and  $p = 1$ , or  $p = \infty$ ."

## 2.4 Embedding theorems

From the definition of  $W^{k,p}(\Omega)$  it immediately follows  $W^{k,p}(\Omega) \subset L_p(\Omega)$ . Can we obtain  $W^{k,p}(\Omega) \subset L_q(\Omega)$  for some  $q > p$ ? The answer **yes**, if  $\Omega$  is again nice enough (and there will also be some dependence on the dimension of  $\mathbb{R}^d$ .)

### 2.4.1 Theorems for $p \leq d$

**Example.** Let  $u \in \mathcal{D}(\mathbb{R}^2)$ . Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2) ds = \int_{-\infty}^{x_2} \partial_2 u(x_1, s) ds,$$

so

$$\begin{aligned} \|u\|_{L_2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} |u(x_1, x_2)|^2 dx_1 dx_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 dx_1 dx_2 \leq \\ &\leq \left( \int_{\mathbb{R}} \int_{-\infty}^{x_1} |\partial_1 u(s, x_2)| ds dx_2 \right) \left( \int_{\mathbb{R}} \int_{-\infty}^{x_2} |\partial_2 u(x_1, s)| ds dx_2 \right) \leq \\ &\leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_1 u(s, x_2)| ds dx_2 \right) \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_2 u(x_1, s)| ds dx_2 \right) \leq \\ &\leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(s, x_2)| ds dx_2 \right) \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(x_1, s)| ds dx_2 \right) = \\ &= \left( \int_{\mathbb{R}^2} |\nabla u| dx \right) \left( \int_{\mathbb{R}^2} |\nabla u| dx \right) = \left( \int_{\mathbb{R}^2} |\nabla u| dx \right)^2 = \\ &= \|\nabla u\|_{L_1(\mathbb{R}^2)}^2, \end{aligned}$$

so

$$\|u\|_{L_2(\mathbb{R}^2)} \leq \|\nabla u\|_{L_1(\mathbb{R}^2)}.$$

This can be generalized in two ways:

- $d > 2$ ,
- less smoothness.

**Lemma 6** (Gagliardo). *Let  $d \geq 2$ . Let  $\hat{u}_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be nonnegative and measurable for  $j \in \{1, \dots, d\}$ . We define*

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), \mathrm{d}\hat{x}_j = \mathrm{d}x_1 \dots \mathrm{d}x_{j-1} \mathrm{d}x_{j+1} \dots \mathrm{d}x_d.$$

*Consider the functions  $u_j : \mathbb{R}^d \rightarrow \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$ . Then*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x) \mathrm{d}x \leq \prod_{j=1}^d \left( \int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} \mathrm{d}\hat{x}_j \right)^{\frac{1}{d-1}}. \quad (1)$$

*(Both integrals can be infinity.)*

*Proof.* Induction by dimension: *The case  $d = 2$ :*

$$\begin{aligned} \int_{\mathbb{R}^2} u_1(x_1, x_2) u_2(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 &= \int_{\mathbb{R}^2} \hat{u}_1(x_2) \hat{u}_2(x_1) \mathrm{d}x_1 \mathrm{d}x_2 = \left( \int_{\mathbb{R}} \hat{u}_1(x_2) \mathrm{d}x_2 \right) \left( \int_{\mathbb{R}} \hat{u}_2(x_1) \mathrm{d}x_1 \right) = \\ &= \left( \int_{\mathbb{R}} \hat{u}_1(\hat{x}_1) \mathrm{d}\hat{x}_1 \right) \left( \int_{\mathbb{R}} \hat{u}_2(\hat{x}_2) \mathrm{d}\hat{x}_2 \right). \end{aligned}$$

(an equality in fact.) We have used Fubini once, which is permitted, as we have measurability + nonnegativity.

*The case  $d \rightarrow d + 1$*  Before we proceed, recall the "generalized Holder", all functions are non-negative

$$\int_{\Omega} \prod_{j=1}^d f_j \mathrm{d}x \leq \prod_{j=1}^d \left( \int_{\Omega} f_j^{p_j} \mathrm{d}x \right)^{\frac{1}{p_j}},$$

where  $\sum_{j=1}^d \frac{1}{p_j} = 1$ . See that if we take  $p_j = d$ , then  $\sum_{j=1}^d \frac{1}{d} = \frac{1}{d} \sum_{j=1}^d 1 = 1$ , so

$$\int_{\Omega} \prod_{j=1}^d f_j \mathrm{d}x \leq \prod_{j=1}^d \left( \int_{\Omega} f_j^d \mathrm{d}x \right)^{\frac{1}{d}}.$$

Let us compute:

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^d u_j \overbrace{u_{d+1}}^{=\hat{u}(\hat{x}_{d+1})} \overbrace{dx_1 \dots dx_d dx_{d+1}}{=\hat{d}x_{d+1}} = \int_{\mathbb{R}^d} \underbrace{\left( \int_{\mathbb{R}} \prod_{j=1}^d u_j dx_{d+1} \right)}_{\text{Holder}} u_{d+1} \hat{d}x_{d+1} \leq \\
&\leq \underbrace{\int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d}} u_{d+1} \hat{d}x_{d+1}}_{\text{Holder}} \leq \left( \int_{\mathbb{R}^d} u_{d+1}^d \hat{d}x_{d+1} \right)^{\frac{1}{d}} \left( \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x) dx_{d+1} \right)^{\frac{1}{d} \frac{d}{d-1}} \hat{d}x_{d+1} \right)^{\frac{d-1}{d}} \leq \\
&\leq \left( \int_{\mathbb{R}^d} u_{d+1}^d \hat{d}x_{d+1} \right)^{\frac{1}{d}} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d-1} \frac{d-1}{1}} \hat{d}x_j \hat{d}x_{d+1} \right)^{\frac{d-1}{d} \frac{1}{d-1}} = \\
&= \left( \int_{\mathbb{R}^d} u_{d+1}^d \hat{d}x_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left( \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d dx_{d+1} \hat{d}x_j \hat{d}x_{d+1} \right)^{\frac{1}{d}} = \left( \int_{\mathbb{R}^d} u_{d+1}^d \hat{d}x_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left( \int_{\mathbb{R}^d} u_j^d \hat{d}x_j \right)^{\frac{1}{d}} = \\
&= \prod_{j=1}^{d+1} \left( \int_{\mathbb{R}^d} u_j^d \hat{d}x_j \right)^{\frac{1}{d}},
\end{aligned}$$

where the induction step is taken for the function

$$v_j = \left( \int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d-1}},$$

that is clearly nonnegative and measurable.  $\square$

*Remark.* Sometimes, the lemma is stated as:  $\hat{u}_i \in L_\infty(\mathbb{R}^{d-1})$ ,  $\text{supp } \hat{u}_i$  is compact  $\forall i \in \{1, \dots, d\}$ . Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d |u_j(x)| dx \leq \prod_{j=1}^d \left( \int_{\mathbb{R}^{d-1}} |\hat{u}_j(\hat{x}_j)|^{d-1} \hat{d}x_j \right)^{\frac{1}{d-1}} = \prod_{j=1}^d \|\hat{u}_j\|_{L_{d-1}(\mathbb{R}^{d-1})}.$$

The difference is that in our version, we have nonnegativity in the assumptions and do not require compact supports and essential boundedness, as we work with integrals that are possibly infinite.

**Theorem 7** (Gagliardo-Nirenberg). *Let  $p \in [1, d)$ . Then  $\forall u \in W^{1,p}(\mathbb{R}^d)$ :*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where  $p^* = \frac{dp}{d-p}$ .

*Proof.* Estimate for  $u \in \mathcal{D}(\mathbb{R}^d)$ , then use density, as  $W^{1,p}(\mathbb{R}^d) = W_0^{1,p}(\mathbb{R}^d)$ .

$$\forall j \in \{1, \dots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d) ds.$$

This estimate is independent of  $j \in \{1, \dots, d\}$ , so it holds

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds.$$

Next, consider  $p = 1, p^* = \frac{d}{d-1}$  and estimate:

$$|u|^{\frac{d}{d-1}} \leq \left( \int_{\mathbb{R}^d} |\nabla u|(\dots, s, \dots) ds \right)^{\frac{d}{d-1}} = \prod_{j=1}^d \left( \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds \right)^{\frac{1}{d-1}}.$$

Denote

$$u_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) = \left( \int_{\mathbb{R}^d} |\nabla u|(\overbrace{\dots, x_j, \dots}^{=x \text{ in fact}}) dx_j \right)^{\frac{1}{d-1}},$$

which is a function independent of  $x_j, u_j \equiv u_j(\hat{x}_j)$ . So the integral (the  $L_{\frac{d}{d-1}}(\mathbb{R}^d)$  norm)

$$\begin{aligned} \int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} dx &\leq \int_{\mathbb{R}^d} \prod_{j=1}^d u_j dx \quad \underbrace{\leq}_{\text{Gagliardo lemma}} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} u_j^{d-1}(\hat{x}_j) d\hat{x}_j \right)^{\frac{1}{d-1}} = \\ &= \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) dx_j d\hat{x}_j \right)^{\frac{1}{d-1}} = \left( \int_{\mathbb{R}^d} |\nabla u| dx \right)^{\frac{d}{d-1}}, \end{aligned}$$

and so

$$\|u\|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)}^{\frac{d}{d-1}} \leq \|\nabla u\|_{L_1(\mathbb{R}^d)}^{\frac{d}{d-1}},$$

meaning  $(1^* = \frac{d}{d-1})$

$$\|u\|_{L_{1^*}(\mathbb{R}^d)} \leq 1 \|\nabla u\|_{L_1(\mathbb{R}^d)}.$$

If now  $p \in (1, d)$ , we investigate for what  $q$  can we obtain estimate of  $\|u\|^q_{L_{\frac{d}{d-1}}(\mathbb{R}^d)}$ :

$$\begin{aligned} \|u\|_{L_{\frac{qd}{d-1}}(\mathbb{R}^d)}^q &= \| |u|^q \|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1} |\nabla u| dx \quad \underbrace{\leq}_{\text{Holder}} q \|\nabla u\|_{L_p(\mathbb{R}^d)} \| |u|^{q-1} \|_{L_{p'}(\mathbb{R}^d)} = \\ &= q \|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|_{L_{(q-1)p'}(\mathbb{R}^d)}^{q-1}. \end{aligned}$$

We want  $(q-1)p' = \frac{qd}{d-1}$ , so we can divide both sides:

$$q \left( \frac{p}{p-1} - \frac{d}{d-1} \right) = \frac{p}{p-1}, \Leftrightarrow q \frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p} p.$$

Also

$$\frac{q}{d-1} = \frac{p}{d-p} \Leftrightarrow \frac{qd}{d-1} = \frac{dp}{d-p} = p^*, q = \frac{p(d-1)}{d-p}$$

and thus

$$\|u\|_{L_{p^*}(\mathbb{R}^d)}^q \leq q \|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|_{L_{p^*}(\mathbb{R}^d)}^{q-1} \Leftrightarrow \|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

$\Rightarrow$  statement holds for  $u \in \mathcal{D}(\mathbb{R}^d)$ . To finish, use density of  $\mathcal{D}(\mathbb{R}^d)$  in  $W^{1,p}(\mathbb{R}^d)$ : let  $u \in W^{1,p}(\mathbb{R}^d)$ , be arbitrary. Then  $\exists \{u_k\} \subset \mathcal{D}(\mathbb{R}^d) : u_k \rightarrow u$  in  $W^{1,p}(\mathbb{R}^d)$ . Moreover, we have showed that  $\forall k \in \mathbb{N}$ :

$$\|u_k\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

so passing to the (strong) limit and using the continuity of the norm indeed yields

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

We are done. □

*Remark.* • It is evident that nonzero constants are not in  $W^{1,p}(\mathbb{R}^d)$  and that also the inequality does not hold for them.

- the set  $\mathbb{R}^d$  is of course unbounded, so we have no ordering of  $L_p(\Omega)$  spaces.
- of course, we require no smoothness of the domain

**Theorem 8.** *Let  $\Omega \subset \mathbb{R}^d$  be open. Then  $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1, d)$  the statement of the previous theorem holds.*

*Proof.* An immediate corollary of the previous theorem: we have showed the inequality for  $u \in \mathcal{D}(\mathbb{R}^d)$ , but WLOG it holds also for  $u \in \mathcal{D}(\Omega)$  (i can keep the integrals over  $\mathbb{R}^d$ , but in the end only values from  $\Omega$  count) and since  $W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$ , i can again extend it on the whole  $W_0^{1,p}(\Omega)$ . □

*Remark.* In the proof of theorem we showed that  $\forall u \in W^{1,p}(\mathbb{R}^d)$  it holds

$$\|u\|_{L^{\frac{qd}{d-1}}(\Omega)}^q \leq q \|\nabla u\|_{L_p(\Omega)} \|u\|_{L^{\frac{p(q-1)}{p-1}}(\Omega)}^{q-1},$$

for  $q$  such that  $\frac{qd}{d-1} \leq p^*$ .

**Definition 5** (Continuous & compact embeddings). Let  $X, Y$  be linear normed spaces. We say

- $X$  is continuously embedded into  $Y$ ,  $X \hookrightarrow Y$ , provided  $X \subset Y$  (is a subspace) and

$$\forall x \in X : \|x\|_Y \leq C \|x\|_X.$$

- $X$  is compactly embedded into  $Y$ ,  $X \hookrightarrow\hookrightarrow Y$ , provided  $X \subset Y$  (is a subspace) and

$$\forall A \subset X \text{ bounded} : \overline{A}^Y \text{ is compact in } Y.$$

This is the same as saying<sup>9</sup>  $X \subset Y$  (is a subspace) and the identity  $\text{id} : X \rightarrow Y$  is

- a bounded linear operator,  $\text{id} \in \mathcal{L}(X, Y)$
- is a compact linear operator,  $\text{id} \in \mathcal{K}(X, Y)$

**Theorem 9** (Embedding theorem for  $p \leq d$ ). *Let  $\Omega \in C^{0,1}, p^* = \frac{dp}{d-p}$ . Then*

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<sup>9</sup>Really, we have  $\|x\|_Y = \|\text{id} x\|_Y \leq \|\text{id}\|_{\mathcal{L}(X, Y)} \|x\|_X$ , and if  $A \subset X$  is bounded, than from the definition of  $\text{id} \in \mathcal{K}(X, Y) : \text{id}(A) = A \subset Y$  is relatively compact in  $Y$ .

- if  $p \in [1, d)$ , then

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega) \quad \forall q \in [1, p^*],$$

- if  $p \in [1, d)$ , then

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \quad \forall q \in [1, p^*),$$

- if  $p = d$ , then

$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega) \quad \forall q \in [1, \infty),$$

- if  $p = d$ , then

$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega) \quad \forall q \in [1, \infty),$$

(the same as above, i.e., every continuous embedding is also compact.)

*Proof.* We would like to use the previous lemmas + extension.

*Ad continuity for  $p < d$ :*

Recall that the composition of continuous operators yields a continuous operator. In our case:

- the extension operator  $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$  is continuous
- identity  $I_1 : W^{1,p}(\mathbb{R}^d) \rightarrow L_{p^*}(\mathbb{R}^d)$  is continous (Gagliardo-Nirenberg:  $\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^d)}$ .)
- restriction  $I_2 : L_{p^*}(\mathbb{R}^d) \rightarrow L_{p^*}(\Omega)$  is continuous (monotonicity of the L. integral:  $\Omega \subset \mathbb{R}^d \Rightarrow \|u\|_{L_{p^*}(\Omega)} \leq \|u\|_{L_{p^*}(\mathbb{R}^d)}$ .)
- identity  $I_3 : L_{p^*}(\Omega) \rightarrow L_q(\Omega)$  is continous (embedding of Lebesgue spaces:  $\Omega$  is bounded  $\Rightarrow L_{p^*}(\Omega) \hookrightarrow L_q(\Omega) \quad \forall q \in [1, p^*]$ )

Together, the mapping

$$\text{id} : W^{1,p}(\Omega) \rightarrow L_q(\Omega), \quad \text{id} = I_3 \circ I_2 \circ I_1 \circ E$$

is continuous, and so  $W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \quad \forall q \in [1, p^*]$ .

*Ad continuity for  $p = d$ :*

If  $p = d$ , we have (this holds generally)  $W^{1,d}(\Omega) \hookrightarrow W^{1,r}(\Omega) \quad \forall r \in [1, d)$ , (embedding of Lebesgue spaces,  $L_d(\Omega) \hookrightarrow L_r(\Omega), \quad \forall r \in [1, d)$ ). Notice  $r^* = \frac{rd}{r-d} \rightarrow \infty$  as  $r \rightarrow d^-$ , which means we can for all  $q \in [1, \infty)$  find  $r \in [1, d)$  s.t.  $r^* > q$ . Consequently,

$$\forall q \in [1, \infty) \exists r \in [1, d) \text{ s.t. } L_{r^*}(\Omega) \hookrightarrow L_q(\Omega).$$

Notice also that  $\forall r \in [1, d)$  we always have

$$W^{1,r}(\Omega) \hookrightarrow L_{r^*}(\Omega)$$

(that's just renaming  $p$  with  $r$  in embedding for  $p < d$ ). Then it holds

$$W^{1,d}(\Omega) \hookrightarrow W^{1,r}(\Omega) \hookrightarrow L_{r^*}(\Omega) \hookrightarrow L_q(\Omega),$$

and so  $W^{1,d}(\Omega) \hookrightarrow L_q(\Omega), \quad \forall q \in [1, \infty)$ .

*Ad compactness for  $p < d$ :*

It suffices to show  $U = U_{W^{1,p}(\Omega)}(0, 1)$  is relatively compact in  $L_q(\Omega)$ , which is, since  $L_q(\Omega)$  is complete, equivalent to  $U$  being totally bounded in  $L_q(\Omega)$ .<sup>10</sup> Extend the functions to  $W^{1,p}(\mathbb{R}^d)$  using the extension operator  $E$ , so  $EU \subset W^{1,p}(\mathbb{R}^d)$ . Take some yet undetermined  $\delta > 0$  and denote by  $(EU)_\delta$  the set of regularized functions from  $EU$  with some kernel  $\eta$ . Our next strategy is the following:

1. show  $(EU)_\delta$  is totally bounded in  $L_1(U(0, R))$ ,
2. show  $W^{1,p}(\Omega) \hookrightarrow L_1(\Omega)$ ,
3. show  $W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)$ .

Since the supports of the functions from  $EU$  are uniformly bounded<sup>11</sup>, we know<sup>12</sup>  $\exists R > 0$  s.t.

$$\forall v \in (EU)_\delta : \text{supp } v \subset B(0, R) \subset \mathbb{R}^d.$$

(Also remember that  $\Omega \subset B(0, R)$ .) Moreover, from the properties of mollification<sup>13</sup> it follows:

$$(EU)_\delta \subset C^1(B(0, R)).$$

Next up, calculate for  $v \in (EU)_\delta$  the norm  $\|v\|_{C^1(B(0,1))}$ . Realize that in fact  $v = (Eu)_\delta$  for some  $u \in U$ , and so

$$|\int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y) dy| \leq \|Eu\|_{L_p(\mathbb{R}^d)} \|\eta_\delta\|_{L_{p'}(\mathbb{R}^d)} \leq C \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|E\|_{\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^d))} \|u\|_{W^{1,p}(\Omega)} \leq C,$$

as  $U = U_{W^{1,p}(\Omega)}(0, 1)$ . Also

$$\begin{aligned} |\nabla \int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y) dy| &= |\int_{\mathbb{R}^d} Eu(y)\nabla_x \eta_\delta(x-y) dy| \leq \|Eu\|_{L_p(\mathbb{R}^d)} \|\nabla \eta_\delta\|_{L_{p'}(\mathbb{R}^d)} \leq \\ &\leq C \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|E\|_{\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^d))} \|u\|_{W^{1,p}(\Omega)} \leq C, \end{aligned}$$

using the same arguments. In total

$$\forall v \in (EU)_\delta : \|v\|_{C^1(B(0,R))} \leq C,$$

or in other words, all functions from  $(EU)_\delta$  are uniformly bounded in  $C^1(B(0, R)) \Rightarrow$  they are uniformly bounded and uniformly equicontinuous (that is implied by uniform boundedness of the derivatives). Thus we can use Arzela-Ascoli theorem and state

$$(EU)_\delta \subset\subset C^0(B(0, R)).$$

<sup>10</sup>A metric space  $P$  is totally bounded if there exists a finite  $\varepsilon$ -net: a finite open covering of  $P$  by balls centered in  $P$  of radii smaller than  $\varepsilon$ .

<sup>11</sup>From the properties of extension, we know  $\forall u \in W^{1,p}(\Omega) : \text{supp } Eu \subset V$  with  $V$  open s.t.  $\Omega \subset\subset V$ .

<sup>12</sup>Properties of mollification include  $\text{supp}(Eu)_\delta \subset \overline{\delta + \text{supp } Eu}$ .

<sup>13</sup>We have  $(EU)_\delta \subset C^\infty(\mathbb{R}^d)$ .



Since  $C^0(B(0, R))$  is complete, this also means  $(Eu)_\delta$  is totally bounded in  $C^0(B(0, R))$ . Using the fact  $(B(0, R))$  is compact<sup>14</sup>

$$C^0(B(0, R)) \hookrightarrow L_1(U(0, R)),$$

we also see that  $(Eu)_\delta$  is totally bounded in  $L_1(U(0, R))$ .

Next, take an arbitrary  $u \in U$  and compute (we are using the fact  $Eu = u$  a.e. in  $\Omega$ ,  $\Omega \subset U(0, R)$ .)

$$\begin{aligned} \|u - (Eu)_\delta\|_{L_1(\Omega)} &\leq \left\| \overbrace{Eu}^{:=v} - (Eu)_\delta \right\|_{L_1(U(0, R))} = \int_{U(0, R)} |v - v_\delta| dx = \int_{U(0, R)} |v(x) - \int_{\mathbb{R}^d} v(y) \eta_\delta(x - y) dy| dx = \\ &= \int_{U(0, R)} |v(x) - \int_{\mathbb{R}^d} v(x - y) \eta_\delta(y) dy| dx = [x \mapsto x + y] = \int_{U(0, R)} | \int_{\mathbb{R}^d} v(x + y) - v(x) \underbrace{\eta_\delta(y) dy}_{\leq 1} | dx \leq \\ &\leq \int_{U(0, R)} | \int_{\mathbb{R}^d} (v(x + y) - v(y)) \eta_\delta(y) dy | dx \leq \int_{U(0, R)} \int_{\mathbb{R}^d} \frac{|v(x + y) - v(x)|}{|y|} |\eta_\delta(y)| |y| dy dx \leq \\ &\stackrel{\text{Fubini}}{\leq} \int_{\mathbb{R}^d} \int_{U(0, R)} \frac{|v(x + y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy. \end{aligned}$$

Estimate the inner integral: assume  $v$  is smooth,  $v \in \mathcal{D}(U(0, R))$  and write

$$\begin{aligned} \int_{U(0, R)} \frac{|v(x + y) - v(x)|}{|y|} dx &= \int_{U(0, R)} \frac{1}{|y|} \left| \int_0^1 \underbrace{\frac{d}{ds}(v(x + sy))}_{\nabla v(x + sy) \cdot y} ds \right| dx \stackrel{\text{C-S}}{\leq} \int_{U(0, R)} \frac{1}{|y|} \int_0^1 |\nabla v(x + sy)| |y| ds dx = \\ &= \int_{U(0, R)} \int_0^1 |\nabla v(x + sy)| ds dx = \int_0^1 \int_{U(0, R)} |v(x + sy)| dx ds. \end{aligned}$$

The last integral can be further manipulated by using the change of variables  $z := x + sy \in \{x + sy | x \in U(0, R)\} = U(0, R) + sy = U(sy, R)$ . Since  $\nabla v \in \mathcal{D}(U(0, R))$ , the integral is nonzero only for  $z \in U(sy, R) \cap U(0, R) \subset U(0, R)$  so we can write

$$\begin{aligned} \int_0^1 \int_{U(0, R)} |\nabla v(x + sy)| dx ds &= \int_0^1 \int_{U(sy, R) \cap U(0, R)} |\nabla v(z)| dz ds \leq \int_0^1 \int_{U(0, R)} |\nabla v(z)| dz ds \leq \\ &\leq \int_0^1 \|\nabla v\|_{L_p(U(0, R))} (\lambda(U(0, R)))^{\frac{1}{p'}} ds \leq C(R) \|\nabla v\|_{L_p(U(0, R))}, \end{aligned}$$

and so we have shown

$$\forall v \in \mathcal{D}(U(0, 1)) : \int_{U(0, R)} \frac{|v(x + y) - v(x)|}{|y|} dx \leq C(R) \|\nabla v\|_{L_p(U(0, R))}.$$

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<sup>14</sup>Take some sequence  $\{f_k\} \subset C^0(B(0, R))$  that is bounded, then

$$\lim_{k \rightarrow \infty} \int_{U(0, R)} |f_k| dx = \int_{U(0, R)} \lim_{k \rightarrow \infty} |f_k| dx = \int_{U(0, R)} |f| dx = \int_{B(0, R)} |f| dx \leq \infty,$$

the majorant being e.g.  $\max_{k \in \mathbb{N}} \|f_k\|_\infty$ . Hence every bounded sequence in  $C^0(B(0, R))$  has a converging (sub)sequence in  $L_1(U(0, R))$ .

Now, take<sup>15</sup>  $v \in W_0^{1,p}(U(0, R))$ , then  $\exists \{v_k\} \subset \mathcal{D}(U(0, R)) : v_k \rightarrow v$  in  $W_0^{1,p}(U(0, R))$ . So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} dx \leq C(R) \|\nabla v_k\|_{L_p(U(0, R))} \rightarrow C(R) \|\nabla v\|_{L_p(U(0, R))}.$$

Putting it all together:

$$\begin{aligned} \|u - (Eu)_\delta\|_{L_1(\Omega)} &\leq \int_{\mathbb{R}^d} \int_{U(0, R)} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy \leq C(R) \|\nabla v\|_{L_p(U(0, R))} \int_{\mathbb{R}^d} \underbrace{|y|}_{\leq \delta} \eta_\delta(y) dy \leq \\ &\leq C(R) \delta \|\nabla v\|_{L_p(U(0, R))} \int_{\mathbb{R}^d} \eta_\delta(y) dy = C(R) \delta \|v\|_{W^{1,p}(U(0, R))} = C(R) \delta \|Eu\|_{W^{1,p}(U(0, R))} \leq C_1 \delta \|u\|_{W^{1,p}(\Omega)} \leq \\ &\leq C_1 \delta. \end{aligned}$$

where we have used the properties of the reg. kernel  $\eta_\delta$ , the extension operator  $E$  and the fact  $u \in U$ .

Now fix  $\varepsilon > 0$ , find  $\{(Eu_k)_\delta\}_{k=1}^m$  a finite  $\frac{\varepsilon}{2}$ -net in  $(EB)_\delta$  in  $L_1(U(0, R))$  (which is possible, since we have total boundedness in  $L_1(U(0, R))$ .) We will show  $\{u_k\}_{k=1}^m$  is a (finite)  $\varepsilon$ -net in  $L_1(\Omega)$ .

Up to now,  $\delta > 0$  has been undetermined; now comes the time - set

$$\delta > 0 \text{ s.t. } C_1 \delta < \frac{\varepsilon}{4}.$$

Fix an arbitrary  $u \in U$ , and find  $j \in \{1, \dots, m\}$  s.t.  $\|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(U(0, R))} < \frac{\varepsilon}{2}$ . Compute

$$\|u - u_j\|_{L_1(\Omega)} \leq \|u - (Eu)_\delta\|_{L_1(\Omega)} + \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(\Omega)} + \|(Eu_j)_\delta - u_j\|_{L_1(\Omega)} \leq C_1 \delta + \frac{\varepsilon}{2} C_1 \delta < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon,$$

where we have used the above estimate and the fact  $\Omega \subset B(0, R)$ . Thus, we have shown  $U$  is totally bounded in  $L_1(\Omega)$  and so

$$W^{1,p}(\Omega) \subset\subset L_1(\Omega).$$

It remains to show the validity for a general  $q \in [1, p^*)$ . Using the interpolation theorem on Lebesgue spaces<sup>16</sup> we obtain

$$\|u\|_{L_q(\Omega)} \leq \|u\|_{L_1(\Omega)}^\theta \|u\|_{L_{p^*}(\Omega)}^{1-\theta},$$

where  $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$ . Let us now show  $U$  is totally bounded in  $L_q(\Omega)$ , i.e.,  $\forall \epsilon > 0$  there exists a finite  $\epsilon$ -net in  $U$  in  $L_q(\Omega)$ . Pick  $\{u_j\}_{j=1}^m \subset U$  that is an  $\beta > 0$  net in  $L_1(\Omega)$ , where  $\beta$  will be determined later. Then it holds

$$\|u - u_j\|_{L_q(\Omega)} \leq \|u - u_j\|_{L_1(\Omega)}^\theta \|u - u_j\|_{L_{p^*}(\Omega)}^{1-\theta} \leq \beta^\theta \|u - u_j\|_{L_{p^*}(\Omega)}^{1-\theta}.$$

Since we have already shown  $W^{1,p}(\Omega) \hookrightarrow L_{p^*}(\Omega)$ , we know (again  $u, u_j$  are in  $U$ )

$$\|u - u_j\|_{L_{p^*}(\Omega)}^{1-\theta} \leq C_2 \|u - u_j\|_{W^{1,p}(\Omega)}^{1-\theta} \leq C_2 2^{1-\theta},$$

and so

$$\|u - u_j\|_{L_q(\Omega)} \leq \beta^\theta C_2 2^{1-\theta}.$$

<sup>15</sup>Recall we have  $v \in (EU)_\delta$  and so  $\text{supp } v \not\subset B(0, R)$ , meaning it is "zero on  $S(0, R)$ " - in the sense of traces.

<sup>16</sup>In the case  $q \in [r, s)$  it holds  $\|u\|_{L_q(\Omega)} \leq \|u\|_{L_r(\Omega)}^\theta \|u\|_{L_s(\Omega)}^{1-\theta}$ ,  $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$ .

We see that if we choose  $\beta$  s.t.

$$\beta < \left( \frac{\epsilon}{C_2 2^{1-\theta}} \right)^{\frac{1}{\theta}},$$

we obtain

$$\|u - u_j\|_{L_q(\Omega)} \leq \epsilon,$$

i.e.,  $\{u_j\}_{j=1}^m$  is a  $\epsilon$ -set in  $L_q(\Omega)$ . Thus  $W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, p^*)$ .

*Ad compactness for  $p = d$*

Finally, let us show the last result. It holds

$$W^{1,d}(\Omega) \hookrightarrow W^{1,r}(\Omega), \forall r \in [1, d).$$

Moreover, we have just shown

$$W^{1,r}(\Omega) \hookrightarrow L_s(\Omega), \forall s \in [1, r^*).$$

Notice that  $r^* = \frac{rd}{r-d} \rightarrow \infty$  as  $r \rightarrow d^-$ , so  $\forall q \in [1, \infty)$  fixed  $\exists r \in [1, d) : r^* > q$ , i.e.,  $q \in [1, r^*)$ . But then

$$W^{1,d}(\Omega) \hookrightarrow W^{1,r}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, \infty).$$

And realize that implies  $W^{1,d}(\Omega) \hookrightarrow L_q(\Omega) \forall q \in [1, \infty)$ : if  $\{u_n\}$  is bounded in  $W^{1,d}(\Omega)$ , then it is bounded in  $W^{1,r}(\Omega)$ , as the identity between those spaces is continuous, and the above compact embedding tells us  $\{u_{n_k}\}$  is convergent in  $L_q(\Omega)$  for some  $\{n_k\}$ . In total,  $\{u_n\} \subset W^{1,d}(\Omega)$  has a subsequence  $\{u_{n_k}\}$  convergent in  $L_q(\Omega)$ . We are done.  $\square$

#### 2.4.2 Theorems for $p > d$

We know we will encounter Holder spaces. Let us recall some of their properties in a remark.

*Remark* (Properties of Holder spaces). Let  $\Omega \subset \mathbb{R}^d$  be open and bounded,  $k \in \mathbb{N}_0, \lambda \in [0, 1]$ . The norm on the space  $C^{0,1}(\overline{\Omega})$  is defined as

$$\|u\|_{C^{k,\lambda}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x \neq y, x, y \in \overline{\Omega}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\lambda},$$

and the space

$$C^{k,\lambda}(\overline{\Omega}) := \left\{ u \in C^k(\overline{\Omega}) \mid \|u\|_{C^{k,\lambda}(\overline{\Omega})} \leq \infty \right\},$$

where we identify

$$C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega}).$$

Moreover, we have the following embeddings:  $\forall \alpha \in [0, 1]$  it holds

$$C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\beta}(\overline{\Omega}), \forall \beta \in [1, \alpha],$$

and

$$C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\beta}(\overline{\Omega}), \forall \beta \in [1, \alpha).$$

A fresh start of a new chapter calls for a fresh new lemma.

**Lemma 7** (Morrey). *Let  $u \in \mathcal{D}(\mathbb{R}^d)$ . Then  $\forall x_1, x_2 \in \mathbb{R}^d, \forall \mu \in (0, 1]$  it holds*

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)},$$

with

$$[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \sup_{\rho > 0} \int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} dy.$$

*Proof.* Pick arbitrary but fixed  $x_1, x_2 \in \mathbb{R}^d$ . Denote by  $C_\rho$  the closed cube with a side of length  $\rho$  s.t.  $x_1$  and  $x_2$  lie on opposite faces. Then  $\rho \leq |x_1 - x_2| \leq \rho\sqrt{d}$  (it is not closer then the height and not further then the diagonal.)

Let us begin by first computing the deviation of  $u(x_i)$  from the mean value of  $u$  on  $C_\rho$ :

$$\left| \frac{1}{\lambda(C_\rho)} \int_{C_\rho} u(x) dx - u(x_i) \right| = \left| \int_{C_\rho} \frac{u(x) - u(x_i)}{\rho^d} dx \right| \leq \int_{C_\rho} \frac{|u(x) - u(x_i)|}{\rho^d} dx.$$

What other can we use when estimating differences than Newton - Leibniz, right? Since  $u \in \mathcal{D}(\mathbb{R}^d) \subset C^1(\mathbb{R}^d)$  it holds for  $i \in \{1, 2\}$  and  $\forall x \in C_\rho$ :

$$\begin{aligned} |u(x) - u(x_i)| &\leq \left| \int_0^1 \frac{d}{ds} u(x_i + s(x - x_i)) ds \right| = \left| \int_0^1 \nabla u(x_i + s(x - x_i)) \cdot (x - x_i) ds \right| \leq \\ &\leq \int_0^1 |\nabla u(x_i + s(x - x_i))| |x - x_i| ds \leq \rho\sqrt{d} \int_0^1 |\nabla u(x_i + s(x - x_i))| ds. \end{aligned}$$

Notice that it is important  $x \in C_\rho$  and  $x_1, x_2$  are on the opposite sides. With this estimate for  $|u(x) - u(x_i)|$  and Fubini we can write for the deviation

$$\left| \frac{1}{\rho^d} \int_{C_\rho} u(x) dx - u(x_i) \right| \leq \sqrt{d} \int_{C_\rho} \frac{1}{\rho^{d-1}} \int_0^1 |\nabla u(x_i + s(x - x_i))| ds dx = \sqrt{d} \int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x - x_i))|}{\rho^{d-1}} dx ds.$$

This calls for a sensible change of variables. Denote  $z = x_i + s(x - x_i)$ , then under this transformation the cube  $C_\rho$  becomes

$$z \in x_i + s(C_\rho - x_i) = x_i(1 - s) + C_{s\rho} = x_i(1 - s) + [0, s\rho]^d := C_{s\rho}^i$$

which, since  $x_i$  is taken from the faces and  $s \leq 1$ , is a cube with its "origin" somewhere in  $C_\rho$  and a side of length  $s\rho \leq \rho$ , which implies  $C_{s\rho}^i \subset [0, R]^d$  for some  $R$ . The integral then becomes (clearly  $|\det \nabla_x z| = s^d$ ),

$$\sqrt{d} \int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x - x_i))|}{\rho^{d-1}} dx ds = \sqrt{d} \int_0^1 \int_{C_{s\rho}^i} \frac{|\nabla u(z)|}{s^d \rho^{d-1}} dz ds = \sqrt{d} \rho^\mu \int_0^1 s^{\mu-1} \int_{C_{s\rho}^i} \frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}} dz ds,$$

where we are being a bit suggestively imaginative. The "deviation estimate" then becomes

$$\left| \frac{1}{\rho^d} \int_{C_\rho} u(x) dx - u(x_i) \right| \leq \sqrt{d} \rho^\mu \int_0^1 s^{\mu-1} \underbrace{\int_{C_{s\rho}^i} \frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}} dz}_{\leq [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)}} ds \leq \sqrt{d} \rho^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} \int_0^1 s^{\mu-1} ds = \frac{\sqrt{d}}{\mu} \rho^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)},$$

where we used that  $C_{s\rho}^i \subset [0, R]^d$  and that  $0 \in \mathbb{R}^d$ , and so we could estimate the integral over  $C_{s\rho}^i$  by  $[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)}$ . Triangle inequality and the fact  $\rho \leq |x_1 - x_2|$  concludes our proof:

$$|u(x_1) - u(x_2)| \leq \left| \frac{1}{\rho^d} \int_{C_\rho} u(x) dx - u(x_1) \right| + \left| \frac{1}{\rho^d} \int_{C_\rho} u(x) dx - u(x_2) \right| \leq \frac{2\sqrt{d}}{\mu} \rho^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)}.$$

□

*Remark.* It is sufficient when  $u \in C_0^1(\mathbb{R}^d)$ .

Gagliardo had in fact two lemmas, so let us even the game for Morrey.

**Lemma 8.** *Let  $p \in (d, \infty)$ , and let  $\mu = 1 - \frac{d}{p}$ . Then  $\forall u \in \mathcal{D}(\mathbb{R}^d)$  it holds*

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu}\right) \|u\|_{W^{1,p}(\mathbb{R}^d)},$$

where

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |u(x)| + \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}.$$

*Proof.* We prove the assertion by estimating both terms in the above norm. To obtain those specific constants, we will pay some more attention to our proceeding. Let us also state the trivial:  $\mu = 1 - \frac{d}{p} \in (0, 1)$  for  $p \in (d, \infty)$ .

Begin with the differences: choose an arbitrary  $\rho > 0$  and compute

$$\int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} dy \leq \left( \int_{[0,\rho]^d} \left( \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \right)^p dy \right)^{\frac{1}{p}} (\lambda([0,\rho]^d))^{\frac{p-1}{p}} = \|\nabla u\|_{L_p(\mathbb{R}^d)} \frac{\rho^{\frac{d(p-1)}{p}}}{\rho^{d-1+\mu}} = \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

because  $\frac{d(p-1)}{p} - d + 1 - \mu = \frac{dp-d}{p} - d + 1 - 1 + \frac{d}{p} = \frac{dp}{p} - d = 0$ . Taking the suprema yields

$$[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} \leq \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

Going with this into the first Morrey lemma, we see

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

meaning

$$\sup_{x_1, x_2 \in \mathbb{R}^d, x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\mu} \leq \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

To estimate the infinity norm, we can actually exploit the above result aswell: pick  $x \neq y \in \mathbb{R}^d$  and write

$$|u(x)| - |u(y)| \leq |u(x) - u(y)| \leq \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

and so

$$|u(x)| \leq |u(y)| + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

Now fix  $\rho \geq 2|x - y| > 0$ , integrate both sides w.r.t  $y$  over  $[0, \rho]^d$  and obtain

$$|u(x)|\rho^d \leq \int_{[0, \rho]^d} |u(y)| dy + \frac{2\sqrt{d}}{\mu} \rho^d \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

which upon using Holder in the integral becomes

$$|u(x)|\rho^d \leq \|u\|_{L_p(\mathbb{R}^d)} \rho^{\frac{d(p-1)}{p}} + \frac{2\sqrt{d}}{\mu} \rho^d \|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

Since we have lost  $y$ , we can in fact choose  $\rho = 1$ , and upon taking the supremum write

$$\sup_{x \in \mathbb{R}^d} |u(x)| \leq \|u\|_{L_p(\mathbb{R}^d)} + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

and so in total

$$\|u\|_{C^{0, \mu}(\mathbb{R}^d)} \leq \|u\|_{L_p(\mathbb{R}^d)} + \frac{4\sqrt{d}}{\mu} \|\nabla u\|_{L_p(\mathbb{R}^d)} = \left(1 + \frac{4\sqrt{d}}{\mu}\right) \|u\|_{W^{1, p}(\mathbb{R}^d)}.$$

□

*Remark.* Since  $\mathcal{D}(\mathbb{R}^d)$  is dense in  $W^{1, p}(\mathbb{R}^d)$ , the above lemma holds also for  $u \in W^{1, p}(\mathbb{R}^d)$ . We have to be careful to pick a good representant though.

**Theorem 10** (Embedding theorems for  $p > d$ ). *Let  $\Omega \in C^{0, 1}$ ,  $d \in \mathbb{N}$ ,  $p > d$ , i.e.,  $p \in (d, \infty]$ . Denote  $\mu^* = 1 - \frac{d}{p} \in (0, 1)$ . Then*

$$W^{1, p}(\Omega) \hookrightarrow C^{0, \alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*],$$

and

$$W^{1, p}(\Omega) \hookrightarrow C^{0, \alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*].$$

*Proof.* *Ad continuous:*

Since  $\Omega \in C^{0, 1}$ , we are able to use the extension theorem; recall  $\forall u \in W^{1, p}(\Omega) : \text{supp } Eu \subset V$ , where  $\overline{\Omega} \subset V$ . Let us deal with the case  $p \in (d, \infty)$  first. We have shown

$$\|u\|_{C^{0, \mu^*}(\overline{\Omega})} \leq \|u\|_{C^{0, \mu^*}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right) \|u\|_{W^{1, p}(\mathbb{R}^d)}, \forall u \in W^{1, p}(\mathbb{R}^d).$$

Realize that in fact

$$\|u\|_{C^{0, \mu^*}(\overline{\Omega})} = \|Eu\|_{C^{0, \mu^*}(\overline{\Omega})},$$

as  $\partial\Omega$  is a set of zero Lebesgue measure and so in fact  $Eu = u$  on  $\overline{\Omega}$ , which together with the obvious fact  $Eu \in W^{1, p}(\mathbb{R}^d)$  gives

$$\|u\|_{C^{0, \mu^*}(\overline{\Omega})} = \|Eu\|_{C^{0, \mu^*}(\overline{\Omega})} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right) \|Eu\|_{W^{1, p}(\mathbb{R}^d)} \leq C \|u\|_{W^{1, p}(\Omega)}.$$

This exactly means

$$W^{1, p}(\Omega) \hookrightarrow C^{0, \mu^*}(\overline{\Omega}).$$

Realize also that

$$C^{0,\mu^*}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*],$$

and so

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\mu^*}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*].$$

If now  $p = \infty$ , realize that (by embedding of Lebesgue spaces)

$$W^{1,\infty}(\Omega) \hookrightarrow W^{1,q}(\Omega), \forall q \in [1, \infty).$$

From the previous result it follows

$$W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in \left[0, 1 - \frac{d}{q}\right],$$

and notice that  $1 - \frac{d}{q} \rightarrow 1$  as  $q \rightarrow \infty$ . This means that  $\forall q \in [1, \infty)$  it holds

$$W^{1,\infty}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in \left[0, 1 - \frac{d}{q}\right].$$

Since  $q \in [1, \infty)$  was arbitrary, we conclude it must be

$$W^{1,\infty}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, 1].$$

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*Ad compactness* This will be a bit of a cheating: we know

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega}), \forall \beta \in [0, \mu^*],$$

and for Holder spaces it also holds

$$C^{0,\beta}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \beta],$$

which means if we choose  $\beta = \mu^*$ , we in fact obtain

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\mu^*}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*].$$

Using the same arguments as in the case of the compact embedding for  $p < d$ , we can conclude

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*].$$

We are done. □

---

<sup>17</sup>The norm

$$\|u\|_{C^{0,1-\frac{d}{q}}(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |u(x)| + \sup_{x_1 \neq x_2 \in \overline{\Omega}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{1-\frac{d}{q}}},$$

actually allows one to pass to the limit  $q \rightarrow \infty$ , since the supremum is independent of the exponent in the denominator, the function on the RHS is continuous. So if we know  $\|u\|_{C^{0,1-\frac{d}{q}}(\overline{\Omega})} \leq C\|u\|_{W^{1,\infty}(\Omega)}, \forall q \in [1, \infty)$ , we can pass to the limit on the LHS and obtain  $\|u\|_{C^{0,1}(\overline{\Omega})} \leq C\|u\|_{W^{1,\infty}(\Omega)}$ , which is the only missing possibility.

*Remark.* Note that in the case of  $p > d$ , from

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*),$$

and<sup>18</sup>

$$C^{0,\alpha}(\overline{\Omega}) \hookrightarrow L_\infty(\Omega), \forall \alpha \in [0, 1]$$

it follows<sup>19</sup>

$$W^{1,p}(\Omega) \hookrightarrow L_\infty(\Omega), \forall p > d.$$

But that of course means ( $\Omega$  is bounded) that

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, \infty],$$

whenever  $p > d$ .

*Remark* (Summary). Let us summarize the obtained results. If  $d \in \mathbb{N}, \Omega \in C^{0,1}$ , then upon denoting

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p},$$

it holds

- if  $p < d$ , then

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, p^*],$$

and

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, p^*),$$

- if  $p = d$ , then

$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, \infty)$$

and

$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega), \forall q \in [1, \infty),$$

- if  $p > d$ , then

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*],$$

and

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu^*),$$

which also imply

$$W^{1,p}(\Omega) \hookrightarrow L_\infty(\Omega),$$

and

$$W^{1,p}(\Omega) \hookrightarrow L_\infty(\Omega).$$

---

<sup>18</sup>Clearly,

$$\|u\|_{L_\infty(\Omega)} \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L_\infty(\Omega)} + \sup_{x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}.$$

<sup>19</sup>Again, composition of a compact and continuous (linear) operators yields a compact operator independently of the order.



*Remark* (Summary - embedding into Lebesgue spaces). It can be guiding to look only at the embeddings into some Lebesgue spaces. Let  $d \in \mathbb{N}, \Omega \in C^{0,1}$ , denote

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p}.$$

Then it holds

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega),$$

where  $q$  is given:

- if  $p < d$ , then  $q \in [1, p^*]$ ,
- if  $p = d$ , then  $q \in [1, \infty)$ ,
- if  $p > d$ , then  $q \in [1, \infty]$ .

Also, it holds

$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega),$$

where  $q$  is given as

- if  $p < d$ , then  $q \in [1, p^*)$ ,
- if  $p = d$ , then  $q \in [1, \infty)$ ,
- if  $p > d$ , then  $q \in [1, \infty]$ .

## 2.5 Trace theorems

There are many many troubles with the boundary. Another one we have yet not encountered arises with *e.g.* homogenous Dirichlet boundary conditions: it should hold

$$u = 0 \text{ on } \partial\Omega,$$

but typically,  $u$  is an element of some Sobolev space on  $\Omega$ , and  $\lambda_d(\partial\Omega) = 0$ . We cannot sensibly talk about pointwise values on a set of measure zero, they can be arbitrary. This can be dealt with provided - as expected - when  $\Omega$ , *i.e.*,  $\partial\Omega$  is benevolent enough.

Realize also that for  $\Omega$  at least  $C^{0,0}$  we also have the density of  $C_{\overline{\Omega}}^\infty(\mathbb{R}^d)$  in  $W^{1,p}(\Omega)$ , meaning the values on  $\partial\Omega$  are well defined at least for a dense subset of  $W^{1,p}(\Omega)$ . It should not be too difficult to extend to the whole  $W^{1,p}(\Omega)$ , right...?

Moreover, in the case  $p > d, \Omega \in C^{0,1}$  we already know  $\forall u \in W^{1,p}(\Omega)$  there exists  $u^* \in C^0(\overline{\Omega})$ , such that  $u = u^*$  *a.e.* in  $\Omega$ , and so in these cases, the values  $u^*|_{\partial\Omega}$  are well defined. What if  $p \leq d$ ?

*Remark* (The space  $L_p(\mathcal{H}_{d-1}, \partial\Omega)$ ). In the winter semester, we have defined

$$L_p(\partial\Omega) \equiv L_p(\mathcal{H}_{d-1}, \partial\Omega),$$

*i.e.*, the lebesgue spaces are taken w.r.t the  $d-1$  dimensional (normalized complete) Hausdorff measure  $\mathcal{H}_{d-1}$ .

**Theorem 11** (Continuous trace theorem). *Let  $\Omega \in C^{0,1}, p \in [1, \infty]$ , denote  $p^\# = \frac{dp-p}{d-p}$ . Then there is a continuous linear operator  $\text{tr}: W^{1,p}(\Omega) \rightarrow L_q(\partial\Omega)$ , with  $q$  being*

- if  $p < d$ , then  $q \in [1, p^\#]$ ,
- if  $p = d$ , then  $q \in [1, \infty)$ ,
- if  $p > d$ , then  $q \in [1, \infty]$ .

Moreover,  $\forall u \in C^\infty(\overline{\Omega})$  it holds

$$\text{tr } u = u|_{\partial\Omega},$$

meaning  $\text{tr} : W^{1,p}(\Omega) \rightarrow L_p(\partial\Omega)$  is an extension of  $\tilde{\text{tr}} : C^\infty(\overline{\Omega}) \rightarrow C^0(\partial\Omega)$ .

*Proof.* The strategy is the following

1. define  $\tilde{\text{tr}}$  for smooth functions,
2. obtain estimates for  $\tilde{\text{tr}}$  using embedding theorems,
3. extend  $\tilde{\text{tr}}$  to the whole space, which defines  $\text{tr}$ .

*Case  $p < d$ :*

As we have mentioned, the case for functions smooth up to the boundary is evident. Let us so define  $\tilde{\text{tr}} : C^\infty_\Omega(\mathbb{R}^d) \rightarrow C^0(\partial\Omega)$ , by

$$\tilde{\text{tr}} u = u|_{\partial\Omega}.$$

Then clearly  $\tilde{\text{tr}}$  is a well defined linear continuous<sup>20</sup> operator.

(We are using the notation from the definition of a  $C^{0,1}$  domain). Let us for clarity define (and also recall)

$$\begin{aligned} G_j &= \mathbb{A}_j(\{(x', a_j(x')) | x' \in U(0, \alpha)\}), \\ G_j^+ &= \mathbb{A}_j(\{(x', a_j(x') + b | x' \in U(0, \alpha), b \in (0, \beta)\}), \\ G_j^- &= \mathbb{A}_j(\{(x', a_j(x') - b | x' \in U(0, \alpha), b \in (0, \beta)\}). \end{aligned}$$

Within this notation,  $G_j \subset \partial\Omega$ ,  $G_j^+ \subset \Omega$ ,  $G_j^- \subset \mathbb{R}^d/\overline{\Omega}$  and  $U_j = G_j \cup G_j^+ \cup G_j^-$ . Moreover,  $\{U_j\}_{j=1}^m$  are open sets *s.t.*  $\partial\Omega \subset \bigcup_{j=1}^m U_j$ . Denote  $\{\varphi_j\}_{j=1}^m \subset \mathcal{D}(\mathbb{R}^d)$  to be the partition of unity subordinate to this (open) covering. Realize moreover that since  $\mathcal{H}_{d-1}(\partial\Omega) < \infty$ , it holds

$$L_{p^\#}(\partial\Omega) \hookrightarrow L_q(\partial\Omega), \forall q \in [1, p^\#].$$

So if we are able to show

$$\|u\|_{L_{p^\#}(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)},$$

we have the rest of the estimates for free.

Take  $u \in C^\infty_\Omega(\mathbb{R}^d)$ , and denote  $u_j = u\varphi_j$ ,  $j \in \{1, \dots, m\}$ . Let for the moment  $p > 1$ , so  $p^\# = \frac{dp-p}{d-p} > \frac{d-1}{d-1} = 1$ . As  $u_j \in C^\infty(\mathbb{R}^d)$ , it holds  $u_j \in C^1(\overline{G_j^+})$ . Moreover, since  $\text{supp } u_j \subset U_j$  and as  $U_j \cap \mathbb{A}_j(\{(x', a_j(x') + b | x' \in U(0, \alpha), b \in [0, \infty)\}) = \emptyset$ , it holds

$$u_j(\mathbb{A}_j(x', a_j(x') + \beta)) = 0.$$

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<sup>20</sup>Even though  $C^\infty_\Omega(\mathbb{R}^d) = \{u|_{\overline{\Omega}} | u \in C^\infty(\mathbb{R}^d)\}$  is not a normed space, it is a Frechet space, and so the notion of continuity is that of a topological vector space.

With those qualities, if we denote (for an arbitrary  $j \in \{1, \dots, m\}$ ; this will become  $\|u_j\|_{L_{p^\#}(\partial\Omega)}$ )

$$v(x') = |u_j(\mathbb{A}_j(x', a_j(x')))|^{p^\#} = |u_j(\mathbb{A}_j(x', a_j(x') + \beta))|^{\frac{dp-p}{d-p}},$$

we can write (recall  $\mathbb{A}_j$  is orthogonal and  $x' \in U(0, \alpha)$ )

$$\begin{aligned} |v(x')| &= \left| |u_j(\mathbb{A}_j(x', a_j(x')))|^{\frac{dp-p}{d-p}} - |u_j(\mathbb{A}_j(x', a_j(x') + \beta))|^{\frac{dp-p}{d-p}} \right| \leq \left| \int_{a_j(x')+\beta}^{a_j(x')} \frac{\partial}{\partial s} |u_j(\mathbb{A}_j(x', s))| ds \right| = \\ &= \left| \int_{a_j(x')+\beta}^{a_j(x')} p^\# |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} \text{sign}(u_j(\mathbb{A}_j(x', s))) \nabla u_j(\mathbb{A}_j(x', s)) \cdot \mathbb{A}_j(x', 1) ds \right| \leq \\ &\leq p^\# \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(\mathbb{A}_j(x', s))| \underbrace{|\mathbb{A}_j(x', 1)|}_{=|(x', 1)|} ds \leq \\ &\leq p^\# \sqrt{1 + \alpha^2} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(x', s)| ds. \end{aligned}$$

Integrate this inequality over  $U(0, \alpha)$  and write (recall the definition of  $G_j^+$ , we are using Fubini)

$$\begin{aligned} \int_{U(0, \alpha)} |v(x')| dx' &\leq p^\# \sqrt{1 + \alpha^2} \int_{U(0, \alpha)} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(\mathbb{A}_j(x', s))| ds dx' \leq \\ &\leq p^\# \sqrt{1 + \alpha^2} \int_{\mathbb{A}_j(\{(x', a_j(x')+\beta)| x' \in U(0, \alpha), b \in [0, \beta]\})} |u_j|^{\frac{dp-d}{d-p}} |\nabla u_j(x)| dx = \\ &= p^\# \sqrt{1 + \alpha^2} \int_{G_j^+} |u_j(x)|^{\frac{dp-d}{d-p}} |\nabla u(x)| dx \leq p^\# \sqrt{1 + \alpha^2} \|\nabla u\|_{L_p(G_j^+)} \left\| |u_j|^{\frac{dp-d}{d-p}} \right\|_{L_{p^*}(G_j^+)}, \end{aligned}$$

and since  $\frac{dp-d}{d-p} p' = \frac{dp-d}{d-p} \frac{p}{p-1} = \frac{dp}{d-p} = p^*$ , we have

$$\int_{U(0, \alpha)} |v(x')| dx' = p^\# \sqrt{1 + \alpha^2} \|\nabla u\|_{L_p(G_j^+)} \|u_j\|_{L_{p^*}(G_j^+)}^{\frac{dp-d}{d-p}}.$$

Since  $G_j^+ \in C^{0,1}$ , the last term can be estimated using the continuous embedding theorems:

$$\|u_j\|_{L_{p^*}(G_j^+)}^{\frac{dp-d}{d-p}} \leq C \|u_j\|_{W^{1,p}(G_j^+)}^{\frac{dp-d}{d-p}},$$

whereas the integral on the LHS actually is

$$\int_{U(0, \alpha)} |v(x')| dx' = \int_{U(0, \alpha)} |u_j(\mathbb{A}_j(x', a_j(x')))|^{p^\#} dx = \int_{\mathbb{A}_j(\{(x', a_j(x'))| x' \in U(0, \alpha)\})} |u_j(x)|^{p^\#} dx = \|u_j\|_{L_{p^\#}(G_j)}^{p^\#},$$

and so we write

$$\|u_j\|_{L_{p^\#}(G_j)}^{p^\#} \leq C \|\nabla u_j\|_{L_p(G_j^+)} \|u_j\|_{W^{1,p}(G_j^+)}^{\frac{dp-d}{d-p}} \geq C \|u_j\|_{W^{1,p}(G_j^+)}^{\frac{dp-d+d-p}{d-p}} = C \|u_j\|_{W^{1,p}(G_j^+)}^{p^\#},$$

and so

$$\|u_j\|_{L_{p^\#}(G_j)} \leq C \|u_j\|_{W^{1,p}(G_j^+)}.$$

This has been done for  $p > 1$ , but in fact taking the limit  $p \rightarrow 1^+$  is allowed here (without a proof). Hence the above estimate holds  $\forall p \in [1, d)$ .

The estimates have so far been local - let us glue them together. Recall  $\partial\Omega = \bigcup_{j=1}^m G_j, G_j^+ \subset \Omega$ , and so

$$\begin{aligned} \|u\|_{L_{p^\#}(\partial\Omega)} &= \left\| \sum_{j=1}^m u_j \right\|_{L_{p^\#}(\partial\Omega)} \leq \sum_{j=1}^m \|u_j\|_{L_{p^\#}(\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{L_p(\text{supp } u_j \cap \partial\Omega)} = \sum_{j=1}^m \|u_j\|_{L_p(G_j)} \leq \\ &\leq C \sum_{j=1}^m \|u_j\|_{W^{1,p}(G_j^+)} \leq C \|u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where we have used the fact  $0 \leq \varphi_j \leq 1$  in the last inequality. And so we have shown

$$\|u\|_{L_{p^\#}(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}, \forall u \in C_{\bar{\Omega}}^\infty(\mathbb{R}^d).$$

Now let  $u \in W^{1,p}(\Omega)$  be arbitrary. Since  $\Omega \in C^{0,1}$ , there  $\exists \{u_k\} \subset C_{\bar{\Omega}}^\infty(\mathbb{R}^d)$  s.t.  $u_k \rightarrow u$  in  $W^{1,p}(\Omega)$ . Set

$$\text{tr } u = \lim_{k \rightarrow \infty} \tilde{\text{tr}} u_k.$$

The definition is sensible, as  $\tilde{\text{tr}}$  is continuous and  $\{u_k\}$  converges. Also, from the arithmetic of the limits and linearity of  $\tilde{\text{tr}}$  we see  $\text{tr}$  is linear. Next, check

$$\|\text{tr } u\|_{L_q(\partial\Omega)} = \left\| \lim_{k \rightarrow \infty} \tilde{\text{tr}} u_k \right\|_{L_q(\partial\Omega)} = \lim_{k \rightarrow \infty} \|u_k \upharpoonright \partial\Omega\|_{L_q(\partial\Omega)} \leq C \lim_{k \rightarrow \infty} \|u_k\|_{W^{1,p}(\Omega)} = C \|u\|_{W^{1,p}(\Omega)}, \forall q \in [1, p^\#].$$

and so indeed  $\text{tr} : W^{1,p}(\Omega) \rightarrow L_q(\partial\Omega)$ ,  $\forall q \in [1, p^\#]$  and it surely is bounded.

*Case  $p = d$*

In this case, we have  $\text{id} \in \mathcal{L}(W^{1,d}(\Omega), W^{1,r}(\Omega))$ ,  $\forall r \in [1, d)$  and the previous result tells us  $\text{tr} \in \mathcal{L}(W^{1,r}(\Omega), L_q(\partial\Omega))$ ,  $\forall q \in [1, r^\#]$ . Observe that  $r^\# = \frac{dr-r}{d-r} \rightarrow \infty$  as  $r \rightarrow d^-$ , meaning  $\forall q \in [1, \infty)$  there exists  $r \in [1, d)$  s.t.  $r^\# > q$ , i.e.  $q \in [1, r^\#]$ , which in fact means

$$\text{tr} \in \mathcal{L}(W^{1,r}(\Omega), L_q(\partial\Omega)), \forall r \in [1, d), \forall q \in [1, \infty).$$

But then  $\text{tr} \circ \text{id} : W^{1,d}(\Omega) \rightarrow L_q(\partial\Omega)$  is a continuous linear operator  $\forall q \in [1, \infty)$ , as it is a composition of continuous linear operators.

*Case  $p > d$*

This is the easiest case: we know that  $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ ,  $\forall \alpha \in [0, \mu^*]$ , so in particular

$$W^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \subset C^0(\partial\Omega) \subset L_\infty(\partial\Omega),$$

and so in total

$$W^{1,p}(\Omega) \hookrightarrow L_\infty(\partial\Omega),$$

which together with the fact  $(\mathcal{H}_{d-1}(\partial\Omega) < \infty)$

$$L_\infty(\partial\Omega) \hookrightarrow L_q(\partial\Omega), \forall q \in [1, \infty),$$

concludes the proof.  $\square$

*Remark.* These results are very similar to the results obtained in the case of embedding theorems - and it is not surprising, since it might seem we have in fact shown the embeddings of some Sobolev spaces into some Lebesgue spaces. One needs to be careful however, as it *is not an embedding* - we are taking  $\Omega$  open, so  $\partial\Omega \notin \Omega$ . It only makes sense in the case of  $p > d$ .

The chapter will be concluded by stating the compact analogue to the embedding theorems.

**Theorem 12** (Compact trace theorem). *Let  $d \in \mathbb{N}$ ,  $\Omega \in C^{0,1}$ , denote  $p^\# = \frac{dp-p}{d-p}$ , and let  $\text{tr}$  be the trace operator from the previous theorem. Then*

$$\text{tr} \in \mathcal{K}(W^{1,p}(\Omega), L_q(\partial\Omega)),$$

where  $q$  is

- if  $p < d$ , then  $q \in [1, p^\#)$ ,
- if  $p = d$ , then  $q \in [1, \infty)$ ,
- if  $p > d$ , then  $q \in [1, \infty]$ ,

*Proof.* *Case  $p < d$ :* Let us adopt the custom that when talking about the properties of  $u \in W^{1,p}(\Omega)$ , in the space  $L_q(\partial\Omega)$ , we are always talking about the properties of  $\text{tr } u$  in  $L_q(\partial\Omega)$ ...

It will be pretty similar to the continuous case, so let us skip only to the key estimate. We are not apriori sure in which space we will be able to comfortably show the compactness, so let first  $q \in [1, \infty)$  and compute. Then we might use some interpolation estimates...

$$\begin{aligned} \int_{U(0,\alpha)} |u_j(\mathbb{A}_j(x', a_j(x')))|^q dx' &\leq \left| \int_{U(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \frac{\partial}{\partial s} |u_j(\mathbb{A}_j(x', s))|^q ds dx' \right| \leq \\ &\leq q\sqrt{1+\alpha^2} \int_{U(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))| |\nabla u_j(\mathbb{A}_j(x', s))| ds dx' \leq \\ &\leq q\sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{q-1} |\nabla u_j(x)| dx \leq q\sqrt{1+\alpha^2} \|\nabla u_j\|_{L_q(G_j^+)} \|u_j\|_{L_{q'(q-1)}(G_j^+)}^{q-1}, \end{aligned}$$

where  $q'(q-1) = \frac{q}{q-1}(q-1) = q$ , so all in all

$$\|u_j\|_{L_q(G_j)} \leq C(q, \Omega) \|\nabla u_j\|_{L_q(G_j^+)}^{\frac{1}{q}} \|u_j\|_{L_q(G_j^+)}^{1-\frac{1}{q}} \leq C(q, \Omega) \|u_j\|_{W^{1,q}(G_j^+)}^{\frac{1}{q}} \|u_j\|_{L_q(G_j^+)}^{1-\frac{1}{q}}$$

which leads to

$$\begin{aligned} \|u\|_{L_q(\partial\Omega)} &= \left\| \sum_{j=1}^m u_j \right\|_{L_q(\partial\Omega)} \leq \sum_{j=1}^m \|u_j\|_{L_q(\text{supp } u_j \cap \partial\Omega)} \leq \sum_{j=1}^m \|u_j\|_{L_q(G_j)} \leq \\ &\leq C \sum_{j=1}^m \|u_j\|_{W^{1,q}(G_j^+)}^{\frac{1}{q}} \|u_j\|_{L_q(G_j^+)}^{1-\frac{1}{q}} \leq C_1 \|u\|_{W^{1,p}(\Omega)}^{\frac{1}{q}} \|u\|_{L_q(\Omega)}^{1-\frac{1}{q}}. \end{aligned}$$

Denote  $U = U_{W^{1,p}(\Omega)}(0,1)$ ; we will show  $\text{tr } U$  is totally bounded in  $L_q(\partial\Omega)$  for *some*  $q \in [1, \infty)$ . Let  $\varepsilon > 0$  be given. Realize that  $\forall p \in [1, d)$  it is  $p < p^*$ , and so it always holds  $W^{1,p}(\Omega) \hookrightarrow L_p(\Omega)$ . For the moment, we pick  $q = p$ . Denote  $\{u_k\}_{k=1}^m$  to be the  $\delta$ -net from  $U$  in  $L_p(\Omega)$ , where  $\delta$  will be

chosen suitably later. Let now  $u \in U$  be arbitrary and find  $u_i \in U$  s.t.  $\|u - u_i\|_{L_p(\Omega)} \leq \delta$ . Using the estimate from above we have

$$\|u - u_i\|_{L_p(\partial\Omega)} \leq C_1 \|u - u_i\|_{W^{1,p}(\Omega)}^{\frac{1}{q}} \|u - u_i\|_{L_p(\Omega)}^{1-\frac{1}{q}} \leq C_1 2^{\frac{1}{q}} \delta^{1-\frac{1}{q}},$$

where we have used the fact  $u, u_i \in U$ . We see that upon choosing

$$\delta < \left( \frac{\varepsilon}{C_1 2^{\frac{1}{q}}} \right)^{\frac{1}{1-\frac{1}{q}}},$$

we in fact have

$$\|u - u_i\|_{L_p(\partial\Omega)} < \varepsilon,$$

and so  $\{\text{tr } u_i\}_{i=1}^m \subset U$  is a  $\varepsilon$ -net in  $L_p(\partial\Omega)$ , meaning

$$\text{tr} \in \mathcal{K}(W^{1,p}(\Omega), L_p(\partial\Omega)).$$

Since now  $L_p(\partial\Omega) \hookrightarrow L_q(\partial\Omega)$ ,  $\forall q \in [1, p]$ , we have also shown

$$\text{tr} \in \mathcal{K}(W^{1,p}(\Omega), L_q(\partial\Omega)), \forall q \in [1, p].$$

The remaining case is when  $q \in (p, p^\#)$ . As in the case of compact embedding of Sobolev spaces, a suitable interpolation theorem will do the job for us. It holds

$$\|u\|_{L_q(\partial\Omega)} \leq \|u\|_{L_p(\partial\Omega)}^\theta \|u\|_{L_{p^\#}(\partial\Omega)}^{1-\theta},$$

where  $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^\#}$ . Let now  $\{u_i\}_{i=1}^m \subset U$  be such that  $\{\text{tr } u_i\}_{i=1}^m$  is the  $\beta$ -net in  $L_p(\partial\Omega)$  whose existence we have just proven  $\forall \beta > 0$ . Recall also that since  $U$  is bounded and  $\text{tr} \in \mathcal{L}(W^{1,p}(\Omega), L_q(\partial\Omega)) \forall q \in [1, p^\#]$ , there exists  $0 < C_2 < \infty$  s.t.

$$\|u\|_{L_q(\partial\Omega)} \leq C_2, \forall u \in U.$$

and so  $\|u\|_{L_{p^\#}(\partial\Omega)} \leq C_2, \forall u \in U$  in particular.

Finally,  $\forall \varepsilon \in U$  it holds

$$\|u - u_i\|_{L_q(\partial\Omega)} \leq \|u - u_i\|_{L_p(\partial\Omega)}^\theta \|u - u_i\|_{L_{p^\#}(\partial\Omega)}^{1-\theta} \leq \beta^\theta (2C_2)^{1-\theta},$$

so if we choose

$$\beta < \left( \frac{\varepsilon}{(2C_2)^{1-\theta}} \right)^{\frac{1}{\theta}},$$

then

$$\|u - u_i\|_{L_q(\partial\Omega)} \leq \varepsilon, \forall u \in U.$$

which concludes the proof  $\{\text{tr } u_i\}_{i=1}^m$  is an  $\varepsilon$ -net in  $U$  in  $L_q(\partial\Omega)$  also for  $q \in (p, p^\#)$ . In total, we have showed the compactness of the trace operator for all  $q \in [1, p^\#]$ .

*Case  $p = d$*

In this case  $\text{tr} \in \mathcal{L}(W^{1,d}(\Omega), L_q(\partial\Omega)), \forall q \in [1, \infty)$ . We also know  $\text{id} \in \mathcal{L}(W^{1,d}(\Omega), W^{1,r}(\Omega)), \forall r \in [1, d)$  and that

$$\text{tr} \in \mathcal{K}(W^{1,r}(\Omega), L_q(\partial\Omega)), \forall q \in [1, r^\#).$$

Repeating the same arguments, we see  $r^\# \rightarrow \infty$  as  $r \rightarrow d^-$ , meaning  $\forall q \in [1, \infty)$  there exists  $r \in [1, d)$  s.t.  $r^\# > q$ , and consequently this implies

$$\forall q \in [1, \infty) \exists r \in [1, d) \text{ s.t. } \text{tr} \in \mathcal{K}(W^{1,r}(\Omega), L_q(\partial\Omega)).$$

But then the operator  $\text{tr} \circ \text{id} : W^{1,d}(\Omega) \rightarrow L_q(\partial\Omega)$  is compact  $\forall q \in [1, \infty)$ , as it is a composition of a (linear) continuous and compact operator.

*Case  $p > d$*

This case is again trivial: from the embedding theorems, it holds

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \forall \alpha \in [0, \mu),$$

so in particular

$$W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega}) \subset C^0(\partial\Omega) \subset L_\infty(\partial\Omega),$$

meaning

$$W^{1,p}(\Omega) \hookrightarrow L_\infty(\partial\Omega),$$

which together with the embedding

$$L_q(\partial\Omega) \hookrightarrow L_\infty(\partial\Omega), \forall q \in [1, \infty).$$

completes the proof for any  $q \in [1, \infty]$ . We are done once again. □

## 2.6 Fine properties of Sobolev spaces

In this last chapter, we present some other useful properties of Sobolev spaces.

### 2.6.1 Composition of Sobolev functions

In the chapter about extension of functions from Sobolev spaces we have seen:  $U, V \subset \mathbb{R}^d$  open,  $u \in W^{1,p}(U)$ ,  $\Phi : U \rightarrow V$  a  $C^1$ -diffeomorphism  $\Rightarrow u \circ \Phi \in W^{1,p}(V)$ . Is there any other class of mappings that guarantee the composition remains in some Sobolev spaces?

**Theorem 13** (Derivative of superposition). *Let  $\Omega \subset \mathbb{R}^d$  open,  $p \in [1, \infty]$ ,  $u : \Omega \rightarrow \mathbb{R}^d$  be in  $W^{1,p}(\Omega)$ . Denote for an arbitrary  $a \in \mathbb{R}$  the set*

$$\Omega_a = \{x \in \Omega \mid u(x) = a\}.$$

*Then*

1.  $\nabla u = 0$  on  $\Omega_a$ ,

2. if  $f \in C^{0,1}(\mathbb{R}^d)$  s.t.  $\|f\|_{L^\infty(\Omega)} < \infty$  (it is a Lipschitz continuous function), then  $f \circ u - f(0) \in W^{1,p}(\Omega)$  and it holds

$$\nabla(f \circ u) = \begin{cases} (f' \circ u) \nabla u, & \text{a.e. in } \{u \notin S\}, \\ 0, & \text{a.e. in } \{u \in S\} \end{cases},$$

where

$$S = \{s \in \mathbb{R} \mid f'(s) \text{ does not exist}\}$$

(in the strong sense).

*Proof.* The proof has been presented for the case  $f \in C^1(\mathbb{R})$ ,  $\|f\|_{L^\infty(\mathbb{R})} < \infty$ . In the general setting, one can play with *a.e.* convergence and Rademacher theorem.

Denote  $L = \|f\|_{L^\infty(\mathbb{R})}$ . Since  $f$  is continuous,  $f \circ u$  is measurable. Moreover, from the Lipschitzity on the whole  $\mathbb{R}$  it follows

$$|(f \circ u)(x) - f(0)| \leq L|u(x) - 0| = L|u(x)|,$$

and since  $u \in L_p(\Omega)$ , also  $f \circ u - f(0) \in L_p(\Omega)$ . To obtain the wanted result, we need to show  $\nabla(f \circ u)$  exists, so let us move on to the second claim. The strategy is to work with some regularization and then pass to the limit with the support size. In concrete terms, pick  $\varphi \in \mathcal{D}(\Omega)$ , and regularization  $u_\varepsilon$  s.t.  $\text{dist}(\text{supp } \varphi, \partial\Omega) > 2\varepsilon$ . Then we can write

$$-\int_{\Omega} (f \circ u_\varepsilon)(x) \partial_i \varphi(x) \, dx = \int_{\Omega} (f' \circ u_\varepsilon)(x) \partial_i u_\varepsilon(x) \varphi \, dx,$$

using per partes and chain rules (those are strong derivatives). Now we would like to pass to the limit  $\varepsilon \rightarrow 0^+$ . Take the LHS:

$$\lim_{\varepsilon \rightarrow 0^+} \left| \int_{\Omega} (f \circ u_\varepsilon - f \circ u) \partial_i \varphi \, dx \right| \leq \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} |f \circ u_\varepsilon - f \circ u| |\partial_i \varphi| \, dx \leq L \|\varphi\|_{W^{1,\infty}(\Omega)} \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \cap \text{supp } \varphi} |u_\varepsilon - u| \, dx = 0,$$

as  $u_\varepsilon \rightarrow u$  in  $L_1(\Omega \cap \text{supp } \varphi)$  (notice from the theorem on mollification we actually obtain this convergence in  $L_1(\Omega_\varepsilon)$ , but we have made a good choice of  $\varphi$ ) The RHS can be manipulated

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\Omega} (f' \circ u_\varepsilon) \partial_i u_\varepsilon \varphi - (f' \circ u) \partial_i u \varphi \, dx \right| &\leq \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} |(f' \circ u_\varepsilon - f' \circ u)| |\partial_i u| |\varphi| \, dx + \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} |f' \circ u_\varepsilon| |\partial_i u_\varepsilon - \partial_i u| |\varphi| \, dx \leq \\ &\leq \|\varphi\|_{L^\infty(\Omega)} \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \cap \text{supp } \varphi} |f' \circ u_\varepsilon - f' \circ u| |\partial_i u| \, dx + L \|\varphi\|_{L^\infty(\Omega)} \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \cap \text{supp } \varphi} |\partial_i u_\varepsilon - \partial_i u| \, dx = \\ &= \|\varphi\|_{L^\infty(\Omega)} \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} |f' \circ u_\varepsilon - f' \circ u| |\partial_i u| \, dx, \end{aligned}$$

where we have used  $\nabla u_\varepsilon \rightarrow \nabla u$  in  $L_1(\Omega \cap \text{supp } \varphi)$ . As for the second integral, recall that  $u_\varepsilon \rightarrow u$  *a.e.* in  $\Omega \cap \text{supp } \varphi$  and that  $f'$  is globally continous. Thus with the majorant

$$|f' \circ u_\varepsilon - f' \circ u| |\partial_i u| \leq L \|u\|_{W^{1,p}(\Omega)} |u_\varepsilon - u|,$$

that is integrable on the bounded set  $\text{supp } \varphi \cap \Omega$  we see the second integral is zero aswell. All in all, we have shown

$$-\int_{\Omega} (f \circ u) \partial_i \varphi \, dx = \int_{\Omega} (f' \circ u) \partial_i u \varphi \, dx, \forall \varphi \in \mathcal{D}(\Omega),$$



which is exactly what we need to state that the  $\partial_i$ -weak derivative of  $f \circ u$  exists and is equal to  $(f' \circ u)\partial_i u$ . Finally, the following estimate holds:

$$\left\| \underbrace{\nabla(f \circ u)}_{=\nabla(f \circ u - f(0))} \right\|_{L^p(\Omega)} = \|(f' \circ u)\nabla u\|_{L^p(\Omega)} \leq L\|\nabla u\|_{L^p(\Omega)} \leq L\|u\|_{W^{1,p}(\Omega)},$$

and so

$$f \circ u - f(0) \in W^{1,p}(\Omega).$$

Let us now deal with the first assertion. We will first show  $\nabla u = 0$  *a.e.* on  $\Omega_0$ . For that, choose a special function

$$f(x) = \begin{cases} x, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

and so  $f \circ u = u^+$ . If we now set

$$f_\varepsilon(x) = \begin{cases} (x^2 + \varepsilon^2)^{1/2} - \varepsilon, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

with the derivative being

$$f'_\varepsilon(x) = \begin{cases} \frac{x}{(x^2 + \varepsilon^2)^{1/2}}, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

and so  $\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = f(x)$ , and  $\lim_{\varepsilon \rightarrow 0^+} f'_\varepsilon(x) = \chi_{\mathbb{R}^+}(x)$  *a.e.* in  $\mathbb{R}$ , meaning

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon \circ u = u^+, \quad \lim_{\varepsilon \rightarrow 0^+} f'_\varepsilon \circ u = \chi_{x \in \Omega | u(x) > 0},$$

*a.e.* in  $\Omega$ . Using the previously obtained general expression and passing to the limit yields

$$-\int_{\Omega} (f \circ u) \nabla \varphi \, dx = \int_{\Omega} \nabla u \varphi \chi_{\{x \in \Omega | u(x) > 0\}} \, dx,$$

and realizing<sup>21</sup>  $u^- = (-u)^+$ , we also have

$$-\int_{\Omega} (f \circ u) \nabla \varphi \, dx = -\int_{\Omega} \nabla u \varphi \chi_{\{x \in \Omega | u(x) < 0\}} \, dx,$$

meaning (in the weak sense)

$$\nabla u^+ = \nabla u \chi_{\{x \in \Omega | u(x) > 0\}}, \quad \nabla u^- = -\nabla u \chi_{\{x \in \Omega | u(x) < 0\}}.$$

But since  $u = u^+ - u^-$ , it holds  $\nabla u = \nabla u^+ - \nabla u^-$  *a.e.*. We see  $\nabla u^- = \nabla u^+ = 0$ , on  $\Omega_0$ , and so  $\nabla u = 0$  *a.e.* on  $\Omega_0$ . Taking  $\tilde{u} = u + c$  for  $c \in \mathbb{R}$  arbitrary, we see  $\nabla \tilde{u} = \nabla u = 0$  *a.e.* on  $\Omega_0$ , meaning  $\nabla \tilde{u} = 0$  *a.e.* on  $\{x \in \Omega | \tilde{u} - c = 0\} = \Omega_c$ . Since  $\tilde{u}, c$  were arbitrary, we are done with this proof.

If  $u \notin S$ , the above presentation works just fine. If  $u(x) \in S$ , we know  $\nabla u = 0$  *a.e.* on  $\Omega_u$ , and since  $\lambda(S) = 0$  by Rademacher theorem, we have also the above conclusion. This argumentation is sloppy, but has not been presented in full detail. □

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<sup>21</sup>Recall  $u^+ = \max(0, u)$ ,  $u^- = -\min(0, u)$  are both nonnegative functions,  $u = u^+ - u^-$ ,  $|u| = u^+ + u^-$ . It holds trivially  $u^- = -\min(0, u) = \max(0, -u) = (-u)^+$ .

### 2.6.2 Difference quotients

There is some interplay between the weak derivative and difference quotients, which can be useful in some proofs, *e.g.*, regularity results for elliptic problems. We have seen that if a function possesses classical derivatives, it also has weak derivatives and they coincide *a.e.*. Is something there something to be said about the opposite implication?

**Definition 6** (Difference quotient). Let  $u : \mathbb{R}^d \rightarrow \mathbb{R}, h \in \mathbb{R}$ . For each  $i \in \{1, \dots, d\}$  we define the difference quotient as

$$\Delta_h^i u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where  $e_i$  is the canonical base vector of  $\mathbb{R}^d$ .

*Remark.* If  $\partial_i u(x)$  exists, then clearly

$$\partial_i u(x) = \lim_{h \rightarrow 0} \Delta_h^i u(x).$$

Let us jump straight to the main result.

**Theorem 14.** Let  $\Omega \subset \mathbb{R}^d$  open,  $p \in [1, \infty]$  and  $u \in L_p(\Omega)$ . Denote  $\forall \delta > 0$  the set

$$\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}.$$

Then it holds

1. If moreover  $u \in W^{1,p}(\Omega)$ , then  $\forall i \in \{1, \dots, d\}, \delta \in (0, 1), h \in (0, \frac{\delta}{2})$

$$\|\Delta_h^i u\|_{L_p(\Omega_\delta)} \leq \|\partial_i u\|_{L_p(\Omega)}.$$

2. Let  $p \in (1, \infty]$  and let there exist  $\{C_i\}_{i=1}^d$  constants s.t.  $\forall i \in \{1, \dots, d\}, \delta \in (0, 1), h \in (0, \frac{\delta}{2})$  it holds

$$\|\Delta_h^i u\|_{L_p(\Omega_\delta)} \leq C_i.$$

Then  $u \in W^{1,p}(\Omega)$  and moreover  $\forall i \in \{1, \dots, d\}$  we have

$$\|\partial_i u\|_{L_p(\Omega)} \leq C_i.$$

*Proof.* Ad 1. Let first  $p \in [1, \infty)$ . What else do we use to estimate differences then..., right? If needed, extend  $u$  by zero outside of  $\Omega$  and set  $u_\varepsilon = u * \omega_\varepsilon$ . Recall that  $u_\varepsilon \rightarrow u$  in  $W^{1,p}(\Omega_{\frac{\varepsilon}{2}})$ . Realize that since  $u_\varepsilon \in C^1(\mathbb{R}^d)$ , we have

$$\Delta_h^i u_\varepsilon(x) = \frac{u_\varepsilon(x + he_i) - u_\varepsilon(x)}{h} = \frac{1}{h} \int_0^h \partial_t u_\varepsilon(x + te_i) dt.$$

Let us compute  $\|\Delta_h^i u_\varepsilon\|_{L_p(\Omega_\delta)}$ . We have

$$\begin{aligned} |\Delta_h^i u_\varepsilon|^p &= \left| \frac{1}{h} \int_0^h \partial_t u_\varepsilon(x + te_i) dt \right|^p \leq \frac{1}{h^p} \left( \int_0^h |\partial_t u_\varepsilon(x + te_i)| dt \right)^p \leq \frac{1}{h^p} \left( \left( \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p dt \right)^{\frac{1}{p}} \left( \int_0^h 1 dt \right)^{\frac{p-1}{p}} \right)^p \\ &= \frac{1}{h^p} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p dt h^{p-1} = \frac{1}{h} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p dt, \end{aligned}$$

and so integrating over  $\Omega_\delta$  yields

$$\|\Delta_h^i u_\varepsilon\|_{L_p(\Omega_\delta)}^p \leq \frac{1}{h} \int_{\Omega_\delta} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p dt dx = \frac{1}{h} \int_0^h \int_{\Omega_\delta} |\partial_t u_\varepsilon(x + te_i)|^p dx dt,$$

denote  $z = x + te_i$ , then  $z \in \Omega_\delta + te_i$ , meaning the points have shifted towards  $\Omega$  in one direction by at most  $t \leq h < \frac{\delta}{2}$ . If we integrate over  $\Omega_{\frac{\delta}{2}}$  instead, we make the domain larger and so

$$\|\Delta_h^i u_\varepsilon\|_{L_p(\Omega_\delta)}^p \leq \frac{1}{h} \int_0^h \int_{\Omega_{\frac{\delta}{2}}} |\partial_t u_\varepsilon(z)|^p dz dt \leq \frac{1}{h} h \|\partial_t u_\varepsilon\|_{L_p(\Omega_{\frac{\delta}{2}})}^p,$$

so if we pass to the limit  $\varepsilon \rightarrow 0^+$  (we are in  $\Omega_\delta$ ), we see

$$\|\Delta_h^i u\|_{L_p(\Omega_\delta)} \leq \|\partial_t u\|_{L_p(\Omega_{\frac{\delta}{2}})} \leq \|\partial_t u\|_{L_p(\Omega)}.$$

If now  $p = \infty$ , the things are not that simple -  $\Omega$  is not bounded apriori, so we have no ordering of Lebesgue spaces. Let us fix that: take some  $R > 0$  and denote  $\Omega_\delta^R = \Omega \cap \mathcal{U}(0, R)$ . If  $u \in W^{1,\infty}(\Omega)$ , it must be  $u \in W^{1,p}(\Omega_\delta^R)$  for all  $p \in [1, \infty)$ . On  $\Omega_\delta^R$  we have from the above

$$\|\Delta_h^i u\|_{L_p(\Omega_\delta^R)} \leq \|\partial_t u\|_{L_p(\Omega_\delta^R)},$$

and since  $\Omega_\delta^R$  is bounded, we can pass to the limit  $p \rightarrow \infty$  and write

$$\|\Delta_h^i u\|_{L_\infty(\Omega_\delta^R)} \leq \|\partial_t u\|_{L_\infty(\Omega_\delta^R)},$$

and passing to the limit  $R \rightarrow \infty$  we obtain

$$\|\Delta_h^i u\|_{L_\infty(\Omega_\delta)} \leq \|\partial_t u\|_{L_\infty(\Omega)}.$$

*Ad 2.* Once again extend  $u$  by zero outside of  $\Omega$  and let for the moment  $p \in (1, \infty)$ . Fix  $\{h_n\} \subset (0, \frac{\delta}{2})$  s.t.  $h_n \rightarrow 0$  and consider the functions

$$v_n^i = \Delta_{h_n}^i u \chi_{\Omega_{2h_n}}.$$

Clearly

$$\|v_n^i\|_{L_p(\Omega)} = \|\Delta_{h_n}^i u\|_{L_p(\Omega_{2h_n})} \leq \|\Delta_{h_n}^i u\|_{L_p(\Omega_\delta)} \leq C_i,$$

and from the reflexivity of  $L_p(\Omega)$ ,  $p \in (1, \infty)$ , we know  $\exists v_i \in L_p(\Omega)$  such that

$$v_n^i \rightharpoonup v^i,$$

as  $n \rightarrow \infty$ . Since the norm is weak lower semicontinuous, we also have the estimate

$$\|v_i\|_{L_p(\Omega)} \leq C_i.$$

It remains to show now that  $v_i$  are weak derivatives of  $u$ . Before we proceed, let us mention the following identity: let  $\varphi \in \mathcal{D}(\Omega)$ , then

$$\int_{\mathbb{R}^d} \Delta_h^i u \varphi dx = \int_{\mathbb{R}^d} u \Delta_{-h}^i \varphi dx.$$

Really, it holds **THIS NEEDS TO BE FINISHED**

$$\int_{\mathbb{R}^d} \Delta_h^i u \varphi \, dx = \int_{\mathbb{R}^d} \frac{u(x + he_i) - u(x)}{h} \varphi \, dx = \int_{\mathbb{R}^d} \frac{\varphi(x)u(x + he_i) - \varphi(x)u(x)}{h} \, dx,$$

change the variables  $y = x + he_i$ , then the above integral is equal to

$$\int_{\mathbb{R}^d} \frac{\varphi(y - he_i)u(y) - \varphi(y - he_i)u(y - he_i)}{h} \, dy =$$

Using this, we can obtain

$$\begin{aligned} \int_{\Omega} v_i \varphi \, dx &= \lim_{n \rightarrow \infty} \int_{\Omega} v_n^i \varphi \, dx = \lim_{n \rightarrow \infty} \int_{\Omega_{2h_n}} \Delta_{h_n}^i u \varphi \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Delta_{h_n}^i u \varphi \, dx = \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} u \Delta_{-h_n}^i \varphi \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} u \frac{\varphi(x - h_n e_i) - \varphi(x)}{h_n} \varphi \, dx = - \int_{\mathbb{R}^d} u \partial_i \varphi \, dx = \\ &= - \int_{\Omega} u \partial_i \varphi \, dx \end{aligned}$$

which really means  $v_i = \partial_i u$  in the weak sense. The fact we can integrate over  $\mathbb{R}^d$  instead of  $\Omega_{2h_n}$  comes from the fact  $\text{supp } \varphi \subset \Omega_{2h_n}$  for some  $n$  large enough, and since the whole integrand is zero on any larger set then  $\Omega_{2h_n}$ , meaning we without a doubt expand the integration domain to the whole space. All in all, we have shown  $v_i = \partial_i u$  weakly and  $\|v_i\|_{L_p(\Omega)} \leq C_i$ , so we are done with the case  $p \in (1, \infty)$ .

If  $p = \infty$ , we can repeat the same arguments as in the above case: create a bounded domain  $\Omega_R = \Omega \cap U(0, R)$ , realize that

$$\|v_i\|_{L_p(\Omega_R)} \leq C_i, \forall p \in (1, \infty),$$

and then pass to the limits  $R \rightarrow \infty, p \rightarrow \infty$ .

□

*Remark.* We have thus shown that if  $p \in (1, \infty)$ , then

$$\Delta_h^i u \rightharpoonup \partial_i u, h \rightarrow 0^+.$$

### 2.6.3 Representation of duals

## 3 Nonlinear elliptic equations - compactness methods

So far we have dealt with only linear equations, *i.e.*, linear operators. When dealing with nonlinear operators, things get much more involved. This chapter will be dedicated to the study of nonlinear problems using compactness methods, which will especially allow us to solve problems with nonlinearities in the non-leading terms. **Be especially careful about the assumptions on the operators - they do not need to be linear.**

Let us start with some technical, mostly known lemmas.

### 3.1 Nemytskii operators

**Lemma 9** (Fatou). *Let  $\Omega \subset \mathbb{R}^d$  be measurable, let  $\{f_n\}$  be a sequence of measurable nonnegative functions. Then it holds*

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n \, dx.$$

(Both integrals can be infinity.)

**Theorem 15** (Igorov). *Let  $\Omega \subset \mathbb{R}^d$  be measurable and of finite Lebesgue measure, let  $\{f_n\}$  and  $f$  be measurable and finite a.e. in  $\Omega$ . Then the following statements are equivalent*

1.  $f_n(x) \rightarrow f(x), \forall \text{ a.a. } x \in \Omega,$
2.  $\forall \varepsilon > 0 \exists G \subset \Omega \text{ open s.t. } \lambda(G) < \varepsilon \text{ and } f_n \rightrightarrows f \text{ on } \Omega/G.$

**Theorem 16** (Vitali). *Let  $\Omega \subset \mathbb{R}^d$  be measurable and of finite Lebesgue measure, let  $\{f_n\}$  and  $f$  be measurable s.t.  $f_n(x) \rightarrow f(x)$  for a.a.  $x \in \Omega$ . Let moreover be true*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall n \in \mathbb{N}, \forall H \subset \Omega \text{ s.t. } \lambda(H) < \delta \Rightarrow \int_H |f_n| \, dx \leq \varepsilon.$$

Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n \, dx = \int_{\Omega} f \, dx.$$

*Remark* (What is  $f(\cdot, x)$ ?). In the following definitions and throughout the chapter in general, we use the “.” to denote variables in the following sense. Just to make it clear, this means the following. If  $U, V, W$  are some sets and  $f : U \times V \rightarrow W$ , we write

- $f(u, \cdot) : V \rightarrow W$ , as the mapping  $V \ni v \mapsto f(u, v) \in W$  for some  $u \in U$  fixed,
- $f(\cdot, v) : U \rightarrow W$ , as the mapping  $U \ni u \mapsto f(u, v) \in W$  for some  $v \in V$  fixed.

**Definition 7** (Caratheodory function). Let  $\Omega \subset \mathbb{R}^d$  be measurable,  $N \in \mathbb{N}$ . We say the function  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  is Caratheodory if

1.  $f(x, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous for almost all  $x \in \Omega$ ,
2.  $f(\cdot, y) : \Omega \rightarrow \mathbb{R}$  is measurable for all  $y \in \mathbb{R}^N$ .

**Definition 8** (Nemytskii operator). Let  $\Omega \subset \mathbb{R}^d$  be measurable,  $N \in \mathbb{N}$  and  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  be Caratheodory. For  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^N, x \in \Omega$  we define the Nemytskii operator  $N_f$ : as

$$N_f : \mathbf{u}(x) \mapsto f(x, \mathbf{u}(x)),$$

meaning  $N_f$  is an operator between function spaces, whose image is the function  $N_f \mathbf{u} : \Omega \rightarrow \mathbb{R}$  s.t.

$$N_f \mathbf{u}(x) = f(x, \mathbf{u}(x)), x \in \Omega.$$

*Remark.* We write  $\mathbf{u}$  in bold to stress  $u : \Omega \rightarrow \mathbb{R}^N$  is a vector valued mapping, even though we might drop the notation later on.

*Remark.* This definition is similiar to the one used in the Caratheodory theory of ordinary differential equations, but not the same.

**Theorem 17** (On Nemytskii operators). *Let  $\Omega \subset \mathbb{R}^d$  be measurable,  $N \in \mathbb{N}$ ,  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  be Caratheodory. Then*

1. *if  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^N$  is measurable, then  $N_f \mathbf{u} : \Omega \rightarrow \mathbb{R}$  is also measurable,*
2. *if there are numbers  $p_i \in [1, \infty)$ ,  $i \in \{1, \dots, N\}$ , an exponent  $p \in [1, \infty)$ , a function  $g$  s.t.  $g \in L_p(\Omega)$ , and a constant  $C \geq 0$  such that for almost all  $x \in \Omega$  and all  $y \in \mathbb{R}^N$  it holds*

$$|f(x, y)| \leq g(x) + C \sum_{i=1}^N |y_i|^{\frac{p_i}{p}},$$

*then the Nemytskii operator  $N_f : \mathbf{u} \mapsto N_f \mathbf{u}$  is continuous from  $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \dots \times L_{p_N}(\Omega)$  to  $L_p(\Omega)$  and moreover it maps sets bounded in  $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \dots \times L_{p_N}(\Omega)$  to sets bounded in  $L_p(\Omega)$ . We write*

$$N_f \in \mathcal{C}(L_{p_1}(\Omega), \dots, L_{p_N}(\Omega); L_p(\Omega)).$$

*Proof.* No proof □

*Remark.* The information about boundedness of images of bounded sets is not trivial - the Nemytskii operator is not linear in general.

### 3.2 Fixed point theorems

The majority of the problems we will be dealing with will be solved by using some fixed points theorem. Here we state two of them (without proofs).

**Definition 9** (Compact operator). Let  $X, Y$  be normed linear spaces,  $M \subset X$ . The mapping  $F : M \rightarrow Y$  is called a compact operator on  $M$  into  $Y$  provided

1.  $F$  is continuous,
2.  $F(M \cap K) \subset Y$  is relatively compact in  $Y$  for any bounded  $K \subset X$ .

*Remark.* • We have no linearity of  $F$ ! So continuity cannot follow from compactness and we have to assume it explicitly. In general, compactness  $\Rightarrow$  boundedness  $\neq$  continuity for nonlinear operators.

- one might wonder why just not state  $F(K) \subset Y$  is relatively compact in  $Y$  for  $K \subset M$  bounded. First of all, this is pretty much the same, but in the applications, this formulation will be more suitable for us. Secondly, in general  $M$  need not be a subspace of  $X$ , let alone a normed linear space; in such cases the notion of boundedness of  $K \subset M$  is not defined at all. On the other hand, boundedness of  $K \subset X$  can be measured in the metric of  $X$ .

- The definition is from *Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2*

**Theorem 18** (Brouwer fixed point theorem). *Let  $N \in \mathbb{N}$  and  $K \subset \mathbb{R}^N$ , be a nonempty convex closed bounded subset of  $\mathbb{R}^N$ . Assume further that  $F : K \rightarrow K$  is continuous. Then  $F$  has a fixed point in  $K$ , i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

*Proof.* No proof. □

**Theorem 19** (Schauder fixed point theorem). *Let  $X$  be a linear normed space and  $K \subset X$  be a nonempty convex closed bounded subset of  $X$ . Assume further that  $F : K \rightarrow K$  is compact on  $K$  into  $K$  and  $F(K) \subset K$ . Then there is fixed point of  $F$  in  $K$ , i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

*Proof.* No proof. □

*Remark.* • for Brouwer,  $K \subset \mathbb{R}^N$  so since it is closed and bounded, it is automatically compact, and since  $F : K \rightarrow K$  is continuous,  $F$  is compact<sup>22</sup>. For Schauder, we have to assume this extra.

- realize again that in Schauder, the set  $K$  does not have to be a subspace of  $X$ .
- proof of Brouwer with  $N=1$  is easy, based on Darboux property.

### 3.3 Problem prototypes

In this chapter some nonlinear elliptic equations are discussed. The strategy of solving them (i.e., proving a solution exists) will be always the same - define a suitable operator and show it has a fixed point.

**Example.** Suppose the following problem:

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and has a controlled growth s.t.  $\exists C > 0, \exists \alpha \in (0, 1]$ :

$$\forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

Even though  $g$  might be nonlinear, we assume only (sub)linear growth of it - so this is kind of cheating.

**Theorem 20** (Existence). *Let  $\Omega \in C^{1,1}$ ,  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g$  is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi + g(u)\varphi \, dx = \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*}.$$

If  $f \in L_2(\Omega)$ , then the solution  $u \in W^{2,2}(\Omega)$ .

*Proof.* We define  $S : L_2(\Omega) \rightarrow L_2(\Omega)$  such that<sup>23</sup>

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} g(w)\varphi \, dx.$$

<sup>22</sup>Image of a compact set under continuous mapping is a compact set

<sup>23</sup>Meaning the image  $Sw$  of  $w$  is the function  $u$  such that the integral equality holds. We could also assume  $S : L_2(\Omega) \rightarrow W^{1,2}(\Omega)$ , but for compactness we need the origin and target spaces to be the same.

Clearly, when  $Sw = w$ , *i.e.*,  $S$  has a fixed point and then

$$Sw = w \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla w \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx,$$

and so the solution  $w$  to our problem exists.  $S$  is defined in an implicit way, but it might be guiding to think of it, written in strong formulation, as

$$Sw = -(\Delta)^{-1}(f - g(w),$$

meaning  $Sw$  is the solution  $u$  to (modulo boundary conditions)

$$-\Delta u = f - g(w),$$

also meaning  $Sw$  is the solution  $u$  to the Poisson equation with the RHS "build from  $w$ ."

First, let us show  $S$  is well defined:

$$\left| \int_{\Omega} g(w) \varphi \, dx + \langle f, \varphi \rangle \right| \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W_0^{1,2}(\Omega)} + \|\varphi\|_{L_2(\Omega)} \|g(w)\|_{L_2(\Omega)},$$

and

$$\int_{\Omega} |g(w)|^2 \, dx \leq \int_{\Omega} C^2(1 + |w|^{\alpha})^2 \, dx \leq \int_{\Omega} 2C^2(1 + |w|^{2\alpha}) \, dx \leq \int_{\Omega} 2C^2(1 + |w|^2) \, dx < \infty,$$

where we used the Young inequality<sup>24</sup>,  $\alpha \leq 1$  and the fact  $S$  is defined on  $L_2(\Omega)$ , meaning  $w \in L_2(\Omega)$ . With that information, we see the last integral is finite and so truly  $S : L_2(\Omega) \rightarrow L_2(\Omega)$  is well defined.

Also, the estimate  $|g(w)| \leq C(1 + |w|^{\alpha})$ ,  $\alpha \in (0, 1]$  together with the fact  $C(1 + w^{\alpha}) \in L_2(\Omega)$  for  $w \in L_2(\Omega)$  means the Nemytskii operator

$$w \mapsto g(w) \in \mathcal{C}(L_2(\Omega); L_2(\Omega)),$$

is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$ .

Let us now show that  $S$  is continuous as a whole. Define  $I_g \in (W_0^{1,2}(\Omega))^*$  as

$$I_g(\varphi) = \int_{\Omega} g(w) \varphi \, dx - \langle f, \varphi \rangle.$$

We are interested in the continuity of the (nonlinear!) mapping  $g \mapsto I_g$  as a mapping from  $L_2(\Omega)$  to  $(W_0^{1,2}(\Omega))^*$ . Let  $\{g_k\} \subset L_2(\Omega)$  be a sequence *s.t.*  $g_k \rightarrow g$  in  $L_2(\Omega)$ . Then

$$\begin{aligned} \|I_{g_k} - I_g\|_{(W_0^{1,2}(\Omega))^*} &= \sup_{\varphi \in U_{W_0^{1,2}(\Omega)}(0,1)} |(I_{g_k} - I_g)(\varphi)| = \sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \left| \left( \int_{\Omega} g_k \varphi \, dx - \langle f, \varphi \rangle - \int_{\Omega} g \varphi \, dx + \langle f, \varphi \rangle \right) \right| \leq \\ &\sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \int_{\Omega} |g - g_k| |\varphi| \, dx \leq \|g - g_k\|_{L_2(\Omega)} \rightarrow 0, \end{aligned}$$

---

<sup>24</sup>In the form  $(a + b)^2 \leq 2(a^2 + b^2)$ .



so we have shown  $g \rightarrow g_k$  in  $L_2(\Omega) \Rightarrow I_g \rightarrow I_{g_k}$  in  $(W_0^{1,2}(\Omega))^*$ , and so  $(g \mapsto I_g) \in \mathcal{C}(L_2(\Omega); (W_0^{1,2}(\Omega))^*)$ . Finally, define the operator  $L : (W_0^{1,2}(\Omega))^* \rightarrow W_0^{1,2}(\Omega)$ , as

$$Lh = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle h, \varphi \rangle,$$

or in other words,  $L$  maps  $h \mapsto u$ , where  $u$  solves

$$\begin{cases} -\Delta u = h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}.$$

or in other other, informal, words

$$Lh = -(\Delta)^{-1}h.$$

Realize that this approach been studiud in the chapter about Fredholm theory extensively. In this simple case however, recall the apriori estimates that the solution satisfies (that is guaranteed from the properties of the Poisson equation)

$$\|u\|_{W_0^{1,2}(\Omega)} \leq C\|h\|_{(W_0^{1,2}(\Omega))^*},$$

which together with the linearity of  $L$  checks  $h \mapsto u$  is continuous. Putting it all together:

- $(w \mapsto g(w)) \in \mathcal{C}(L_2(\Omega); L_2(\Omega))$ ,
- $(g(w) \mapsto f - g(w)) \in \mathcal{C}(L_2(\Omega); (W_0^{1,2}(\Omega))^*)$ ,<sup>25</sup>
- $(f - g(w) \mapsto u) \in \mathcal{C}((W_0^{1,2}(\Omega))^*; W_0^{1,2}(\Omega))$ ,

In total, the composition is continuous and yields  $S$ .

Next, we would like to show  $S$  is a compact (nonlinear) operator. We start with showing  $S$  maps bounded sets in  $L_2(\Omega)$  to bounded sets in  $W_0^{1,2}(\Omega) \hookrightarrow L_2(\Omega)$ ; for that we need apriori estimates: test the weak formulation with  $u$ : (we are doing this a bit more careful that might seem needed, but come on, it is the first example...)

$$\begin{aligned} \underbrace{\|\nabla u\|_{L_2(\Omega)}^2}_{C_p^2 \|u\|_{W_0^{1,2}(\Omega)}^2} &\leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|u\|_{W_0^{1,2}(\Omega)} + \|g(w)\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)} \leq \\ &\leq \frac{\|f\|_{(W_0^{1,2}(\Omega))^*}}{\sqrt{2\varepsilon_1}} \sqrt{2\varepsilon_1} \|u\|_{W_0^{1,2}(\Omega)} + \frac{\|g(w)\|_{L_2(\Omega)}}{\sqrt{2\varepsilon_2}} \sqrt{2\varepsilon_2} \|u\|_{W_0^{1,2}(\Omega)} \leq \\ &\leq (\varepsilon_1 + \varepsilon_2) \|u\|_{W_0^{1,2}(\Omega)}^2 + \frac{1}{4\varepsilon_1} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + \frac{1}{4\varepsilon_2} \|g(w)\|_{L_2(\Omega)}^2 \leq \\ &\leq (\varepsilon_1 + \varepsilon_2) \|u\|_{W_0^{1,2}(\Omega)}^2 + \frac{1}{4\min(\varepsilon_1, \varepsilon_2)} \left( \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + \|g(w)\|_{L_2(\Omega)}^2 \right), \end{aligned}$$

---

<sup>25</sup>In the sense we interpret the RHS as a point in the dual space. Precisely, we should write something like  $(g(w) \mapsto (\varphi \in W_0^{1,2}(\Omega) \mapsto \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx)) \in \mathcal{C}(L_2(\Omega); (W_0^{1,2}(\Omega))^*)$ .

and since  $\|g(w)\|_{L_2(\Omega)}^2 \leq \|C(1+|w|^\alpha)\|_{L_2(\Omega)}^2 \leq 4C^2((\lambda(\Omega))^2 + \|w\|_{L_2(\Omega)}^2)$  and Poincare inequality, we can write

$$\|u\|_{W_0^{1,2}(\Omega)}^2 \leq \frac{1}{4(C_p^2 - (\varepsilon_1 + \varepsilon_2)) \min(\varepsilon_1, \varepsilon_2)} \left( \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + 4((\lambda(\Omega))^2 + \|w\|_{L_2(\Omega)}^2) \right),$$

which, for  $\varepsilon_1, \varepsilon_2$  sufficiently small, just an estimate of the type

$$\|u\|_{W_0^{1,2}(\Omega)}^2 \leq C \left( 1 + \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + \|w\|_{L_2(\Omega)}^2 \right).$$

As we can see, if  $w \in L_2(\Omega)$  is bounded, also  $u \in W_0^{1,2}(\Omega)$  is bounded, so  $S$  maps bounded sets from  $L_2(\Omega)$  to bounded sets in  $W_0^{1,2}(\Omega)$ . Since moreover  $W_0^{1,2}(\Omega) \hookrightarrow L_2(\Omega)$ , this means  $S$  is compact from  $L_2(\Omega)$  to  $L_2(\Omega)$ .

To conclude the proof, we need that there exists  $K \subset L_2(\Omega)$  closed convex bounded nonempty *s.t.*  $S(K) \subset K$ , so we can use Schauder. That clearly will be the case when we show  $S(B_{L_2(\Omega)}(0, R)) \subset B_{L_2(\Omega)}(0, R)$  for some  $R > 0$ . But this is simple - the estimate

$$\|u\|_{L_2(\Omega)}^2 \leq \|u\|_{W_0^{1,2}(\Omega)}^2 \leq C \left( \underbrace{1 + \|f\|_{(W_0^{1,2}(\Omega))^*}^2}_{:=C_1} + \|w\|_{L_2(\Omega)}^2 \right),$$

tells us if the  $R$  were to exist, it must hold

$$R^2 \leq CC_1 + CR^2 \Leftrightarrow R^2(1 - C) \leq CC_1.$$

If  $1 - C > 0$ , then just take  $R \leq \sqrt{\frac{CC_1}{1-C}}$ , if  $1 - C < 0$ , any  $R > 0$  will do.

And so we see such an  $R$  exists in all cases  $\Rightarrow$  the image of a ball is in a ball for some  $R \Rightarrow S$  is compact on  $B_{L_2(\Omega)}(0, R)$  into  $B_{L_2(\Omega)}(0, R) \Rightarrow$  it has a fixed point by Schauder  $\Leftrightarrow$  the solution exists.

For the regularity part of the assertion, realize that  $u_0$  solves

$$\begin{cases} -\Delta u_0 = f - g(u_0), & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

and if  $f \in L_2(\Omega)$ , then  $f - g(u_0) \in L_2(\Omega)$  and so from the regularity theory for elliptic equations we get

$$u \in W^{2,2}(\Omega).$$

□

**Theorem 21** (Uniqueness). *Let  $u_1, u_2 \in W_0^{1,2}(\Omega)$  be weak solutions to the above problem. Let  $f \in (W_0^{1,2}(\Omega))^*$  and let  $g$  be continuous. Let either extra*

1.  *$g$  is nondecreasing*
2.  *$g \in C^1(\mathbb{R}), \|g'\|_{L_\infty(\mathbb{R})}$  small.*

*Then  $u_1 = u_2$ .*

*Proof.* In the linear case,  $u_1 - u_2$  would be a solution to the problem with zero data. In the nonlinear case, we have now certainty  $u_1 - u_2$  solves anything, so we will have to be more careful.

Subtract the equations for  $u_1, u_2$  and test the weak formulation with  $u_1 - u_2$ :

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2) dx = 0.$$

In the first case, the second term is nonnegative, so for the whole thing to be zero it has to hold

$$0 = \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 \geq C_p \|u_1 - u_2\|_{W^{1,2}(\Omega)}^2 \Rightarrow u_1 - u_2 = 0 \in W_0^{1,2}(\Omega).$$

In the second case, the second does not have a sign, but we can estimate it

$$\left| \int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2) dx \right| \leq \int_{\Omega} \|g'\|_{L_{\infty}(\mathbb{R})} |u_1 - u_2|^2 dx \leq \|g'\|_{L_{\infty}(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2,$$

meaning

$$\|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 \pm \|g\|_{L_{\infty}(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 = (1 \pm \|g\|_{L_{\infty}(\mathbb{R})} C_p) \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 \leq 0.$$

If the term is positive we again obtain  $u_1 = u_2$  in  $W_0^{1,2}(\Omega)$  from the Poincare inequality, if it is negative, we require

$$\|g'\|_{L_{\infty}(\mathbb{R})} C_p < 1,$$

and then we have the same result. □

**Example.** Suppose the following problem

$$\begin{cases} -\Delta u + b(\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

where  $f \in (W_0^{1,2}(\Omega))^*$ , and the function  $b : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous and (essentially) bounded. The weak formulation is:

$$\text{find } u \in W_0^{1,2}(\Omega) \text{ s.t. : } \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi + b(\nabla u) \varphi dx = \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*},$$

**Theorem 22.** *Let  $f \in (W_0^{1,2}(\Omega))^*$ ,  $\Omega \in C^{0,1}$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}$  continuous and essentially bounded. Then there is a weak solution to the above problem.*

*Proof.* As in the previous: define the operator  $S : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$ , such as

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi dx = \langle f, \varphi \rangle - \int_{\Omega} b(\nabla w) \varphi dx,$$

or equivalently

$$Sw = u \Leftrightarrow u \text{ is the solution to } \begin{cases} -\Delta u = f - b(\nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases},$$

or informally

$$Sw = -(\Delta)^{-1}(f - b(\nabla w)),$$

modulo boundary conditions. Once again, if we are able to show  $S$  has a fixed point, this information is equivalent to the considered problem having a solution.

Let us first show  $S$  is well defined. Holder yields

$$\|\nabla u\|_{L_2(\Omega)} \|\nabla \varphi\|_{L_2(\Omega)} \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W_0^{1,2}(\Omega)} + \|b\|_{L_\infty(\mathbb{R}^d)} \|\varphi\|_{L_1(\Omega)},$$

use Poincare inequality and  $W_0^{1,2}(\Omega) \hookrightarrow L_1(\Omega)$ , to obtain

$$\|u\|_{W_0^{1,2}(\Omega)} \leq \underbrace{C \left( \|f\|_{(W_0^{1,2}(\Omega))^*} + \|b\|_{L_\infty(\mathbb{R}^d)} \right)}_{:=R},$$

and so we see that the target space really is  $W_0^{1,2}(\Omega)$ . Also, the above estimate is extra nice, as it does not depend on  $w$  whatsoever. If we denote the RHS by  $R > 0$ , we immediately see  $\|u\|_{W_0^{1,2}(\Omega)} = \|Sw\|_{W_0^{1,2}(\Omega)} \leq R$ , meaning that for sure  $S(B_{W_0^{1,2}(\Omega)}(0, R)) \subset B_{W_0^{1,2}(\Omega)}(0, R)$ . Let us show other properties needed for compactness.

First,  $\nabla : W_0^{1,2}(\Omega) \rightarrow L_2(\Omega; \mathbb{R}^d)$ ,  $w \mapsto \nabla w$  is linear and bounded,

$$\|\nabla w\|_{L_2(\Omega)} \leq \|w\|_{W_0^{1,2}(\Omega)},$$

and so continuous, the map  $\mathbf{y} \mapsto b(\mathbf{y})$  satisfies the growth

$$|b(\mathbf{y})| \leq \underbrace{\|b\|_{L_\infty(\mathbb{R}^d)}}_{=:g(x)} + 0 \cdot \sum_{i=1}^d |y_i|^{2/2},$$

and since  $\lambda(\Omega) < \infty$ , we have (in particular)  $\|b\|_{L_\infty(\mathbb{R}^d)} \in L_2(\Omega)$ . Putting this together means  $N_b : \nabla w \mapsto b(\nabla w)$  is continuous from<sup>26</sup>  $L_2(\Omega; \mathbb{R}^d)$  to  $L_2(\Omega; \mathbb{R})$ ,

$$(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}(L_2(\Omega; \mathbb{R}^d); L_2(\Omega; \mathbb{R}))$$

by Nemytskii. Next, the mapping

$$b(\nabla w) \mapsto b(\nabla w) + f$$

from  $L_2(\Omega)$  to  $(W_0^{1,2}(\Omega))^*$  in the sense

$$b(\nabla w) \mapsto \left( W_0^{1,2}(\Omega) \ni \varphi \mapsto \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*} + \int_{\Omega} b(\nabla w) \varphi \, dx \right),$$

is also continous: let  $\{b_k\} \subset L_2(\Omega)$  s.t.  $b_k \rightarrow b$ , in  $L_2(\Omega)$ , then

$$\begin{aligned} \sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \left| \langle f, \varphi \rangle - \int_{\Omega} b_k(\nabla) \varphi \, dx - \langle f, \varphi \rangle + \int_{\Omega} b(\nabla w) \varphi \, dx \right| &\leq \sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \int_{\Omega} |b(\nabla w) - b_k(\nabla w)| |\varphi| \, dx \leq \\ &\leq \|b_k - b\|_{L_2(\Omega)} \rightarrow 0, \end{aligned}$$

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<sup>26</sup>We should be more precise and write it is continuous from  $L_2(\Omega; \mathbb{R}^d)$  to  $L_2(\Omega; \mathbb{R})$ , as  $b : \mathbb{R}^d \rightarrow \mathbb{R}$ .

and so the mapping  $b(\nabla w) \mapsto b(\nabla w) + f$  is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$ . Finally, the mapping  $L : (W_0^{1,2}(\Omega))^* \rightarrow W_0^{1,2}(\Omega)$ ,

$$Lh = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle h, \varphi \rangle,$$

i.e.,

$$Lh = u \Leftrightarrow u \text{ solves } \begin{cases} -\Delta u = h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is also continuous between the spaces - it is just again the solution operator to the Poisson equation with a RHS  $h \in (W_0^{1,2}(\Omega))^*$ . Altogether

- $(w \mapsto \nabla w) \in \mathcal{C}(W_0^{1,2}(\Omega); L_2(\Omega; \mathbb{R}^d))$ ,
- $(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}(L_2(\Omega; \mathbb{R}^d); L_2(\Omega; \mathbb{R}))$ ,
- $(b(\nabla w) \mapsto f - b(\nabla w)) \in \mathcal{C}(L_2(\Omega); (W_0^{1,2}(\Omega))^*)$ ,
- $(f - b(\nabla w) \mapsto u) \in \mathcal{C}((W_0^{1,2}(\Omega))^*; W_0^{1,2}(\Omega))$ ,

and so  $S$  as a composition of those mappings is continuous from  $W_0^{1,2}(\Omega)$  to  $W_0^{1,2}(\Omega)$ . It remains to show  $S$  is compact: we already have continuity, consider  $\{w_k\}_{k \in \mathbb{N}} \subset W_0^{1,2}(\Omega)$  bounded. Then  $\exists \{u_k\} \subset W_0^{1,2}(\Omega)$  bounded:  $u_k \rightarrow u$  in  $L_1(\Omega)$  by embedding up to a subsequence. Next, use the following trick: substitute equation for  $u_k$  from equation for  $u_l$  and test with  $u_l - u_k$

$$C \|u_l - u_k\|_{W_0^{1,2}(\Omega)}^2 \leq \|\nabla(u_l - u_k)\|_{L_2(\Omega)}^2 \leq \int_{\Omega} |b(\nabla u_l) - b(\nabla u_k)| |u_l - u_k| \, dx \leq 2 \|b\|_{L_{\infty}(\Omega)} \|u_l - u_k\|_{L_1(\Omega)}.$$

All in all,  $S$  has a fixed point by Schauder, which is of course the weak solution.  $\square$

But this says  $\{u_k\}$  is Cauchy in  $W_0^{1,2}(\Omega)$ .

## 4 Nonlinear elliptic equations - monotone operator theory

**Lemma 10.** Let  $g : B(0, R) \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$  be continuous,  $N \in \mathbb{N}, R > 0$ , and  $\forall c \in S(0, R) : g(c) \cdot c \geq 0$ . Then, there is  $c_0 \in B(0, R) : g(c_0) = 0$ .

*Proof.* By contradiction. Let  $g \neq 0$  in  $U(0, R)$ . Let us define

$$h(x) = -R \frac{g(x)}{|g(x)|}.$$

Then  $h \in C(B(0, R))$ ,  $h(B(0, R)) \subset S(0, R)$ , so by Brouwer there  $\exists x_0 \in B(0, R) : h(x_0) = x_0 \Rightarrow -R \frac{g(x_0)}{|g(x_0)|} = x_0$ . Take the dot product with  $x_0$  and write

$$\underbrace{-R \frac{g(x_0) \cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in S(0, R),$$

so that is a contradiction.  $\square$

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i (a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x))) = f(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

The data are

- $\Omega \in C^{0,1}$ ,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, i \in \{1, \dots, d\}$  are Caratheodory in  $x$  and  $(u, \nabla u)$ .
- $f \in (W_0^{1,r}(\Omega))^*$ ,

and the unknown is  $u : \Omega \rightarrow \mathbb{R}$ .

*Remark.* The function  $(u, p) \mapsto a_i(\cdot, u, p)$  is continuous from  $(L_r(\Omega))^{d+1}$  to  $L_{r'}(\Omega)$  by Nemystkii theorem.

**Definition 10** (Coercivity). We say that  $\{a_i\}_{i=0}^d$  are coercive if  $\exists C_1 > 0, C_2 \in L_1(\Omega) : \text{a.e. } x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} :$

$$\sum_{i=1}^d a_i(x, z, p) p_i + a_0(x, z, p) \geq C_1 |p|^r - C_2(x), \text{ i.e. } a(x, z, p) \cdot p \geq C_1 |p|^r - C_2(x)$$

**Definition 11** (Monotonicity). We say that  $\{a_i\}_{i=0}^d = a$  is monotone if for almost all

$$x \in \Omega, \forall (z_1, p_1), (z_2, p_2) \in \mathbb{R}^{d+1} : (a(x, z_1, p_1) - a(x, z_2, p_2)) \cdot (p_1 - p_2) + (a_0(x, z_1, p_1) - a_0(x, z_2, p_2)) \cdot (z_1 - z_2) \geq 0.$$

Very similiarly we define strict monotonicity.

**Definition 12** (Weak solution). We say that  $u \in W^{1,r}(\Omega)$  is a weak solution to the above problem if

- $u = u_0$  in the sense of traces on  $\partial\Omega$ ,

•

$$\int_{\Omega} a(\cdot, u, \nabla u) \cdot \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle, \forall \varphi \in W_0^{1,r}(\Omega).$$

**Theorem 23** (Existence and uniqueness). *Let  $\Omega \in C^{0,1}, u_0 \in W^{1,r}(\Omega), r \in (1, \infty), \{a_i\}_{i=1}^d$  be Caratheodory, coercive and monotone and let them also satisfy the growth conditions. Finally, let  $f \in (W^{1,r}(\Omega))^*$ . Then, there is a weak solution to the problem. If, moreover,  $\{a_i\}_{i=1}^d$  is strictly monotone, then the weak solution is unique.*

*Proof.* The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

**Galerkin:** Since  $W_0^{1,r}(\Omega)$  is separable  $\Rightarrow \exists \{w_i\}_{i=1}^\infty$  that is a dense<sup>27</sup> linearly independent subset of  $W_0^{1,r}(\Omega)$ . We search for  $n \in \mathbb{N}$  such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where  $\alpha_j \in \mathbb{R}$  and  $u^n$  satisfy

$$\forall j \in \{1, \dots, n\} : \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla w_j + a_0(\cdot, u^n, \nabla u^n) w_j \, dx = \langle f, w_j \rangle.$$

We claim such  $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$  exist  $\forall n \in \mathbb{N}$  by the previous lemma. We define a vector function

$$F(\alpha^n) := \left\{ \int_{\Omega} a \cdot \nabla w_j + a_0 w_j \, dx - \langle f, w_j \rangle \right\}_{j=1}^n,$$

from Nemystkii  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F$  is continuous on  $\mathbb{R}^n$ . Moreover, it holds

$$\begin{aligned} F(\alpha^n) \cdot \alpha^n &\geq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \nabla(u^n - u_0) + a_0(u^n - u_0) \, dx - \langle f, u^n - u_0 \rangle \\ &\stackrel{\text{coercivity}}{\geq} \int_{\Omega} C_1 |\nabla u^n|^r - (C_2(\cdot) + |a| |\nabla u_0| + |a_0| |u_0|) \, dx - \|u^n\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*} - \|u_0\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*}, \end{aligned}$$

together with the fact

$$\|\nabla u^n\|_{L^r(\Omega)}^r \geq \left( \|\nabla(u - u_0)\|_{L^r(\Omega)} - \|\nabla u_0\|_{L^r(\Omega)} \right)^r \geq \|\nabla(u^n - u_0)\|_{L^r(\Omega)}^r - \|\nabla u_0\|_{L^r(\Omega)}^r \geq C \|u^n - u_0\|_{W^{1,r}(\Omega)}^r - \|\nabla u_0\|_{L^r(\Omega)}^r,$$

Next, realize that  $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{W^{1,r}(\Omega)}$  is a norm equivalent to  $|\alpha^n|$  (Euclidian norm). So that means  $\exists K_1(n) > 0 : \forall \alpha \in \mathbb{R}^n : K_1(n) |\alpha^n| \leq \|u^n - u_0\|_{W^{1,r}(\Omega)}$ . For  $|\alpha^n| = R, R > 0$  determined later estimate  $F(\alpha^n) \cdot \alpha^n \geq c \|u^n - u_0\|_{W^{1,r}(\Omega)} - \tilde{c} \left( \|\nabla u_0\|_{L^r(\Omega)}^r + 1 + \|u_0\|_{L^r(\Omega)}^r + \|f\|_{(W_0^{1,r}(\Omega))^*}^{r'} \right)$  (which is not a trivial computation). And so  $\exists R > 0, \forall \alpha^n \in S(0, R) \subset \mathbb{R}^n : F(\alpha^n) \cdot \alpha^n > 0$ , so from the previous lemma  $\exists \alpha^n \in S(0, R) : F(\alpha^n) = 0$ , and we fix these  $\alpha^n$ . **Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|_{W^{1,r}(\Omega)}^r \leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

and

$$\|u^n\|_{W^{1,r}(\Omega)} \leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

$$\forall j \in \{0, \dots, d\} : \|a_j(\cdot, u^n, \nabla u^n)\|_{L^{r'}(\Omega)}^{r'} \leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

**Limit passage** From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \text{ in } W^{1,r}(\Omega), a_j \rightharpoonup \alpha_j \text{ in } L^{r'}(\Omega).$$

---

<sup>27</sup>It can be chosen such that it is itself dense, not only its span

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_{\Omega} \alpha \cdot \nabla w_j + \alpha_0 w_j \, dx = \langle f, w_j \rangle,$$

and from density of  $\{w_j\}_{j \in \mathbb{N}}$  in  $W^{1,r}(\Omega)$  we have

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} \alpha \cdot \nabla \varphi + \alpha_0 \varphi \, dx = \langle f, \varphi \rangle.$$

**Identification of  $\alpha$ 's** We want to show  $\alpha_j = a_j(\cdot, u, \nabla u), j \in \{0, \dots, d\}$ . For that, we use the *Minty trick*:

$$\begin{aligned} 0 &\leq \int_{\Omega} (a(\cdot, u^n, \nabla u^n) - a(\cdot, v, V)) \cdot (\nabla u^n - V) + (a_0(\cdot, u^n, \nabla u^n) - a_0(\cdot, v, V)) \cdot (u^n - v) \\ &\leq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla u^n + a_0(\cdot, u^n, \nabla u^n) \cdot u^n \, dx + \\ &\quad - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) V + a_0(\cdot, u^n, \nabla u^n) v - a(\cdot, v, V) + a_0(\cdot, v, V)) \cdot (u^n - v) \, dx. \end{aligned}$$

Denote

$$I^n = \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla (u^n - u_0) + a_0(\cdot, u^n, \nabla u^n) \cdot (u^n - u_0) \, dx + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx,$$

by using the equation we obtain

$$I^n = \langle f, u^n - u_0 \rangle + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx = \langle f, u - u_0 \rangle + \int_{\Omega} \alpha \nabla u_0 + \alpha_0 u_0 \, dx = \int_{\Omega} \alpha \nabla u + \alpha_0 u \, dx,$$

as the rest has subtracted. In total, we have

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, v, V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot, v, V)) (u - v) \, dx.$$

So far,  $v, V$  have been arbitrary. If we take

$$V = \nabla u - \lambda \psi, \psi \in L_r(\Omega), v = u,$$

then  $0 \leq \int_{\Omega} (\alpha - a(\cdot, \nabla u + \lambda \psi)) \lambda \psi \, dx$ , so if we take  $\lambda > 0$  and pass to the limit  $\lambda \rightarrow 0_+$  (using Nemytskii theorem) we can write

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx.$$

Since  $\psi$  was arbitrary, we could have taken  $\psi \rightarrow -\psi$ , which in total means

$$0 = \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx$$

Finally, from the previous results, we obtain

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle,$$



and we are almost done. It only remains to show the traces are correct, bt since  $u^n \rightharpoonup u$  in  $W^{1,r}(\Omega)$  and from the continuity of the traces, we obtain

$$\text{tr } u = \text{tr } u_0.$$

**Uniqueness:** Let  $u_1, u_2$  be two solutions. Use strict monotonicity, subtract the weak formulation and test with  $u_2 - u_1$ :

$$\int_{\Omega} \underbrace{(a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla (u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)}_{:=T} dx = 0,$$

where  $T \geq 0$ , so from strict monotonicity we obtain  $T = 0$  a.e. in  $\Omega$  but that means  $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$ , a.e. in  $\Omega \Rightarrow u_1 = u_2$  in  $W^{1,r}(\Omega)$ .  $\square$

**Example** (Nonlinearities vs weak convergence). Let  $g_n(x) = \sin(nx)$ , then  $g \rightharpoonup 0$  in  $L_2((0,4))$  (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx) \varphi dx \geq \int_2^4 \sin^2(nx) dx \rightarrow \frac{1}{2} \neq 0, \forall \varphi \in L_2((0,4)),$$

so  $\{u_n^2\} = \{\sin^2(nx)\}$  **does not converge weakly to**  $0 = 0^2$ .

*Remark.* The method of the presented proof is **very important**.

**Theorem 24.** Let  $\Omega \in C^{0,1}$ . Let  $X = W_0^{1,r}(\Omega)$ ,  $r \in (1, \infty)$  with equivalent norm  $\|u\| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$ . Then

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t. : } \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_{\Omega} \mathbf{F} \cdot \nabla \varphi dx, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

*Proof.* We solve the problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{r-2} \nabla u) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}. \quad (2)$$

Such  $u \in W_0^{1,r}(\Omega)$  exists and is unique by the above theorem. In this case:  $a(x, z, p) = |p|^{r-2}p$ ,  $a_0() = 0$ . Coercivity is clear, monotonicity will be shown in the tutorials<sup>28</sup>. Write the weak formulation of the above problem:

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi dx = \Phi(\varphi).$$

Set  $\mathbf{F} = |\nabla u|^{r-2} \nabla u$ , and test the weak formulation with  $u$  itself:

$$\|\nabla u\|_{L_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{L_r(\Omega)}.$$

If now  $\|\nabla u\|_{L_r(\Omega)} = 0$ , then  $\Phi = 0$  and we are finished, if it is nonzero, then

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

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<sup>28</sup>This was a lie

Realize now

$$\|\nabla u\|_{L^r(\Omega)}^{r-1} = \|\nabla u\|_{L^{\frac{r}{r-1}}(\Omega)}^{r-1} = \|\mathbf{F}\|_{L^{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{L^{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{B_X(0,1)} |\Phi(\varphi)| = \sup_{B_X(0,1)} \left| \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx \right| \leq \sup_{B_X(0,1)} \|\mathbf{F}\|_{L^{r'}(\Omega)} \|\nabla \varphi\|_{L^r(\Omega)} = \|\mathbf{F}\|_{L^{r'}(\Omega)},$$

so  $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{L^{r'}(\Omega)}$ . □

## 5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset W^{1,r}(\Omega) \rightarrow \mathbb{R}, u \mapsto \int_{\Omega} F(\cdot, u, \nabla u) \, dx,$$

with the basic assumptions  $\Omega \in C^{0,1}$ ,  $r \in (1, \infty)$ ,  $X = u_0 + W_0^{1,r}(\Omega)$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in L_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x, z, p) \geq C_1 |p|^r - c_2(x).$$

*Remark.* From the assumptions it follows  $\int_{\Omega} F(\cdot, u, \nabla u) \, dx$  is defined  $\forall u \in W^{1,r}(\Omega)$ .

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

**Lemma 11.** *Let  $\Omega \in C^{0,1}$ ,  $r \in (1, \infty)$ ,  $X = u_0 + W_0^{1,r}(\Omega)$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $F$  Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall a, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x, z, p)| + |\partial_z F(x, z, p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x, \cdot, \cdot) \in C^1(\mathbb{R}^{d+1}).$$

Let now  $u \in u_0 + W_0^{1,r}(\Omega)$  be a local minimizer of  $I$  over  $X$ , i.e.,

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_{\Omega} F(\cdot, u, \nabla u) \, dx \leq \int_{\Omega} F(\cdot, u+v, \nabla(u+v)) \, dx, F(\cdot, u, \nabla u) \in L_1(\Omega).$$

Then  $u$  is the weak solution to the **Euler-Lagrange equations**:

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u)) + \partial_z F(\cdot, u, \nabla u) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega \end{cases},$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi + \partial_z F(\cdot, u, \nabla u) \varphi \, dx = 0, \text{tr } u = \text{tr } u_0 \text{ on } \partial\Omega.$$

*Proof.* First  $\text{tr } u = \text{tr } u_0$  holds, so we are fine. Now fix  $\varphi \in \mathcal{D}(\Omega)$  and define

$$\iota : \mathbb{R} \rightarrow \mathbb{R}^*, \iota(\tau) = \int_{\Omega} \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:= \iota(\tau, \cdot)} \, dx.$$

Now  $\iota$  has a local minimum in 0. We show that  $\iota'(0)$  exists and is equal to the of Euler-Lagrange equations.

- $l(\tau, \cdot)$  is measurable for  $\tau$  from some neighbourhood of 0.
- $l(\tau, \cdot)$  is differentiable

Moreover

$$\begin{aligned}\partial_\tau l(\tau, \cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi = \\ &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi.\end{aligned}$$

Also

$$i(0) = \int_{\Omega} F(\cdot, u, \nabla u) \, dx < \infty$$

and

$$|\partial_\tau l(\tau, \cdot)| \leq (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla \varphi|^r) + |h(x)|)(|\varphi| + |\nabla \varphi|) \in L_1(\Omega).$$

Altogether,  $\iota(\tau)$  is finite on  $(-1, 1)$ ,  $\iota'(\tau)$  exists and

$$\iota'(0) = \int_{\Omega} \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi \, dx.$$

□

**Example.** Let

$$F(x, z, p) = \frac{1}{r}(1 + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x, z, p) = \left(\frac{r}{2} \frac{1}{r} 2(1 + |p|^2)^{\frac{r-2}{2}}\right)p - G = (1 + |p|^2)^{\frac{r-2}{2}}p - G, \quad \partial_z F(x, z, p) = -g.$$

We have

$$|(1 + |p|^2)^{\frac{r-2}{2}}p| \leq (1 + |p|^2)^{\frac{r-2}{2}}(1 + |p|^2)^{\frac{1}{2}} = (1 + |p|^2)^{\frac{r-1}{2}} \leq C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left( (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \right) = -\nabla \cdot G + g, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} (G \cdot \nabla \varphi + g\varphi) \, dx.$$

*Remark.* We have  $\{u_n\} \subset X$  s.t.  $\lim_{n \rightarrow \infty} I(u_n) = \inf_X I$ . Then use

- compactness:  $u_n \rightarrow u$  in some sense (weak convergence)
- weak lower semicontinuity  $I(u) \leq \liminf_{n \rightarrow \infty} I(u_n)$

**Lemma 12.** Let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $F \in C^1(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$ . Then

1.  $F$  is (strictly) convex  $\Leftrightarrow \nabla F$  is (strictly) monotone

2. If  $F$  is (strictly) convex, then

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2 : F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

*Proof.* Fix  $\xi_1, \xi_2, \xi_1 \neq \xi_2$ , define  $\varphi(t) = F(\xi_2 + t(\xi_1 - \xi_2))$ . Then  $\varphi \in C^1(\mathbb{R})$  and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1 - \xi_2)) \cdot (\xi_1 - \xi_2).$$

So

$$" \Rightarrow " : (\nabla F(\xi_1) - \nabla F(\xi_2)) \cdot (\xi_1 - \xi_2) = \varphi'(1) - \varphi'(0) \underset{\varphi \text{ convex or strictly convex}}{\geq} 0.$$

And " $\Leftarrow$ ": Fix  $t_1 > t_2$  and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1 - \xi_2)) - \nabla F(\xi_2 + t_2(\xi_1 - \xi_2))) \cdot (\xi_1 - \xi_2)(t_1 - t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1 - \xi_2)(t_1 - t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2)) \cdot (\eta_1 - \eta_2)$$

and we are in the same situation as before. For 2) we already know  $F$  (strictly) convex  $\Rightarrow \varphi$  (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \rightarrow \varphi'(0),$$

as  $t \rightarrow 0_+$ . And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

**Theorem 25.** Let  $M, N \in \mathbb{N}, \Omega$  open,  $F : \Omega \times \mathbb{R}^{N+M} \rightarrow \mathbb{R}$  Caratheodory,  $F$  convex in  $p \in \mathbb{R}^N$ , i.e.  $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot)$  is convex and  $\exists c_2 \in L_1(\Omega), \forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M, \forall p \in \mathbb{R}^N : F(x, z, p) \geq c_2(x)$ . Let  $u_n \rightarrow u$  in  $L_1(\Omega), U_n \rightarrow U$  in  $L_1(\Omega)$ . Then

$$\int_{\Omega} F(\cdot, u, U) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} F(\cdot, u_n, U_n) dx.$$

*Proof.* The proof will be given only if moreover  $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot) \in C^1(\mathbb{R}^N)$ . Idea: by the previous lemma:

$$\int_{\Omega} F(\cdot, u_n, U_n) dx \geq \int_{\Omega} (F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U) \cdot (U_n - U)) dx,$$

and we have uniform convergence in the first term and second term and weak convergence in  $L_1(\Omega)$  in the last term. If  $\Omega$  is bounded, we can find  $K_k \subset K_{k+1} \subset \Omega$  s.t.  $\lambda(\Omega \setminus \bigcup_{k \in \mathbb{N}} K_k) = 0$ , and moreover  $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$  are compact,  $u_n \rightarrow u$  on  $K_k, \|u\|_{L_{\infty}(K_k)} + \|U\|_{L_{\infty}(K_k)} \leq k$  up to a subsequence. We can now extract a subsequence  $u_n \rightarrow u$  a.e. and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E}_k \text{ s.t. } u_n \rightarrow u \text{ on } \tilde{E}_k \wedge \lambda(\Omega \setminus \tilde{E}_k) < \frac{1}{k}.$$

Now define

$$\hat{E}_k = \bigcup_{j=1}^k \tilde{E}_j, E_k = \hat{E}_k \cap \{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{k}\},$$

and  $E_k$  satisfy <sup>29</sup>

$$\lambda\left(\Omega \bigcup_k E_k\right) = 0.$$

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \leq k \wedge |U(x)| \leq k\}$$

and we also have  $\lambda(\Omega \bigcup_k F_k) = 0$ . FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\left(\Omega \bigcup_k K_k\right) = 0.$$

□

*Remark.* • if  $U_n \rightarrow U$  strongly  $\Rightarrow u_n \rightarrow u, U_n \rightarrow U$  a.e. (up to a subsequence) and the claim follows from the Fatou lemma. <sup>30</sup>

- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L_p(\Omega) \Rightarrow \int_{\Omega} |\nabla u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p dx.$$

**Lemma 13** (Arzela-Ascoli). *Let  $X, Y$  be Banach spaces,  $X \subset\subset Y$ . Then*

$$C^1([0, T]; X) \subset\subset C([0, T]; Y).$$

**Lemma 14** (Ehrling). *Let  $V_1, V_2, V_3$  be Banach spaces s.t.  $V_1 \subset\subset V_2 \subset V_3$ . Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + C \|u\|_{V_3}.$$

*Proof.* By contradiction, assume

$$\exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{N} \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n \|u_n\|_{V_3}.$$

WLOG we can assume  $\{u_n\} \subset S_{V_2}(0, 1)$ : truly, the inequality is 1-homogenous and holds if  $u_n = 0$ . In particular we see  $\|u_n\|_{V_3} < \frac{1}{n}$ , so  $u_n \rightarrow 0$  in  $V_3$ . Moreover,  $\{u_n\}$  is bounded in  $V_1$  and since  $V_1 \subset\subset V_2$  there exists  $\{u_{n_k}\} \subset \{u_n\}$  s.t.:  $u_{n_k} \rightarrow u$  in  $V_2$  strongly. Since  $\{u_n\} \subset S_{V_2}(0, 1)$ , also  $\|u\|_{V_2} = 1$ . Finally, taking the limit passage yields  $0 \geq \|u\|_{V_3}$  and so  $u = 0$  in  $V_3$  and also in  $V_2$ . But that is a contradiction with the fact  $\{u_n\} \subset S_{V_2}(0, 1)$ . □

**Theorem 26** (Aubin-Lions). *Let  $V_1, V_2, V_3$  be Banach spaces s.t.  $V_1 \subset\subset V_2 \subset V_3, p \in [1, \infty)$ . Then the space*

$$\mathcal{U} = \{u \in L_p((0, T); V_1), \partial_t u \in L_1((0, T); V_3)\},$$

<sup>29</sup>"This is homework", says doc. Kaplicky

<sup>30</sup>For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions  $F - c_2 \geq 0, F - c_2 \in L_1(\Omega)$

with the norm

$$\|u\| = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

satisfies

$$\mathcal{U} \subset\subset L_p((0,T);V_2).$$

*Proof.* Strategy: I want to fix  $M \subset \mathcal{U}$  bounded and show that it is precompact in  $L_p((0,T);V_2)$ . That will be done in the following way:

1. Mollify  $M$  by convolution
2. use Arzela-Ascoli
3. show compactness in  $L_p((0,T);V_3)$
4. apply Ehrling lemma and show compactness in  $L_p((0,T);V_2)$ .

Fix  $M \subset \mathcal{U}$  bounded. Then  $\exists C^* > 0 : \forall u \in M : \|u\| \geq C^*$ .

Next, take

$$\varphi : \mathbb{R} \rightarrow [0, \infty), \varphi \in C^\infty(\mathbb{R}), \text{supp } \varphi \subset (-1, 0), \int_{\mathbb{R}} \varphi \, dx = 1,$$

a regularization kernel, then  $\forall \delta > 0$  define  $\varphi_\delta(t) := \frac{1}{\delta} \varphi(\frac{t}{\delta})$ .

Now, extend functions from  $M$  to  $(0, 2T)$  in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0, T) \\ u(2T - t), & t \in (T, 2T) \end{cases}.$$

Now mollify: for  $\delta > 0, \delta < T$  fixed define

$$M_\delta = \{ (\tilde{u} \star \varphi_\delta) \Big|_{(0,T)} \mid u \in M \}.$$

From the properties of regularization it follows  $M_\delta \subset C^1([0, T]; V_1) \underbrace{\subset\subset}_{\text{A.A.}} C([0, T]; V_2) \subset L_p((0, T); V_2)$ .

Now estimate the distance of  $M$  and  $M_\delta$  in  $L_p((0, T); V_3)$ : for

$$u \in M, t \in (0, T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^0 \tilde{u}(t-s) \varphi_\delta(s) \, ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \varphi_\delta(s) \, ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \frac{d}{ds} \int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) \, d\sigma \Big|_{s=-\delta}^0 - \int_{-\delta}^0 \frac{d}{ds} (\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) \, d\sigma \, ds,$$

since the first bracket is 0 and by denoting the first term in the second integrand by  $\tilde{u}'(t-s)$  this becomes (using Fubini)

$$= - \int_{-\delta}^0 \int_{\sigma}^0 \tilde{u}'(t-s) \, ds \, \varphi_\delta(\sigma) \, d\sigma,$$

and we see

$$\|\tilde{u}(t) - \tilde{u}_\delta(t)\|_{V_3} \leq \int_{-\delta}^0 \int_{\sigma}^0 \|\tilde{u}'(t-s)\|_{V_3} ds \varphi_\sigma(\sigma) d\sigma.$$

$L_1((0, T); V_3)$  estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3} dt \leq \int_0^T \int_{-\delta}^0 \int_{\sigma}^0 \|u'(\tilde{t}-s)\|_{V_3} ds \varphi_\delta(\sigma) d\sigma dt \leq 2\delta \|u'\|_{L_1((0, T); V_3)} \leq 2\delta C^*$$

$L_\infty((0, T); V_3)$  estimate:

$$\|u - u_\delta\|_{L_\infty((0, T); V_3)} \leq 2\|u'\|_{L_1((0, T); V_3)} \leq 2C^*$$

It remains to show  $M_\delta \subset L_p((0, T); V_2)$ :

$$\|u - u_\delta\|_{L_p((0, T); V_3)} \leq \|u - u_\delta\|_{L_1((0, T); V_3)}^{1/p} \|u - u_\delta\|_{L_\infty((0, T); V_3)}^{1-1/p} \leq 2C^* \delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u - u_\delta\|_{L_p((0, T); V_2)} \leq \mu \|u - u_\delta\|_{L_p((0, T); V_1)} + C_\mu \|u - u_\delta\|_{L_p((0, T); V_3)}.$$

This means

$$\forall u \in M : \|u - u_\delta\|_{L_p((0, T); V_2)} \leq C^* + C_\mu 2C^* \delta^{1/p}.$$

Now fix  $\varepsilon > 0$  and find

$$\mu > 0 : \mu C^* < \frac{\varepsilon}{2}, \delta > 0, C_\mu 2C^* \delta^{1/p} < \frac{\varepsilon}{2} \Rightarrow \forall u \in M : \|u - u_\delta\|_{L_p((0, T); V_2)} < \varepsilon.$$

This means  $\exists \{w_k\}_{k=1}^N \subset M : \{(w_k)_\delta\}_{k=1}^N$  is  $\varepsilon$ -net in  $M$  in  $L_p((0, T); V_2)$ . If we now fix  $u \in M$ , then

$$\exists K \in \{1, \dots, N\} : \|u_\delta - (w_K)_\delta\|_{L_p((0, T); V_2)} < \varepsilon.$$

□

*Remark.* The pair  $(\mathcal{U}, \|\cdot\|)$  is a Banach space.

We will be dealing with the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot a(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = f & \text{in } (0, T) \times \Omega, \\ u = u_0, & \text{on } \{0\} \times \Omega, \\ u = 0, & \text{on } (0, T) \times \partial\Omega \end{cases}.$$

The unknown is the function  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$ , and we are given  $\Omega \in C^{0,1}$ ,  $T > 0$ ,  $Q_T = (0, T) \times \Omega$ ,  $f : Q \rightarrow \mathbb{R}$  or  $f : (0, T) \rightarrow X$  a Banach space,  $u_0 : \Omega \rightarrow \mathbb{R}$ ,  $a : \Omega \times \mathbb{R} \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions:  $\exists r > 1, \exists C > 0 : a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : |a_0(x, z, p)| + |a(x, z, p)| \leq C(1 + |z|^{r-1} + |p|^{r-1})$  and  $\exists C_1, C_2 > 0, q \in (1, \max(2, r)) a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : a(x, z, p)p + a_0(\dots)z \geq C_1|p|^r - C_2(1 + |z|q)$ .

**Theorem 27.** Let  $\Omega \in C^{0,1}$ ,  $a, a_0$  satisfy growth conditions and coercivity, let  $\{a_i\}_{i=0}^d$  be monotone. Denote  $V = W_0^{1,r}(\Omega) \cap L_2(\Omega)$ . Then  $\forall f \in L_{r'}((0, T); V^*)$ ,  $u_0 \in L_2(\Omega) \exists u \in L_r((0, T); V)$  s.t.  $\partial_t u \in L_{r'}((0, T); V^*)$ ,  $u \in C([0, T]; L_2(\Omega))$ ,  $u(0) = u_0$  and moreover

$$a.e. t \in (0, T), \forall \varphi \in V : \langle \partial_t u, \varphi \rangle + \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle.$$

Finally, the solution is unique.

*Proof.* The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

*Rothe method:* Fix  $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$  and approximate the derivative with

$$\partial_t u(t, x) \approx \frac{1}{h} (u(t, x) - u(t - h, x)).$$

Define  $u_0 = u_0, u_{k+1} \in V$  as a solution of

$$\frac{1}{h} (u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \text{ in } \Omega, u_{k+1} = 0 \text{ on } \partial\Omega.$$

Define

$$f_{k+1} := \int_{kh}^{(k+1)h} f \, dt,$$

then the weak formulation becomes

$$\int_{\Omega} \frac{u_{k+1} - u_k}{h} \varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla \varphi + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) \varphi \, dx = \langle f_{k+1}, \varphi \rangle.$$

We claim without a proof that the solutions  $\{u_k\}_{k=0}^n \subset V$  exist.

To obtain a-priori estimates, tes the equation with  $u_{k+1}$ . This yields:

$$\int_{\Omega} |u_{k+1}|^2 - u_k u_{k+1} \, dx = \int_{\Omega} \frac{1}{2} |u_{k+1}|^2 + \frac{1}{2} (u_{k+1} - u_k) - \frac{|u_k|^2}{2} \, dx \Rightarrow \sum_{k=0}^{j-1} \int_{\Omega} |u_{k+1}|^2 - u_{k+1} u_k \, dx = \frac{1}{2} \|u_j\|_{L_2(\Omega)}^2 - \frac{1}{2} \|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} (u_k - u_{k+1})$$

so

$$\int_{\Omega} a(\dots) \nabla \cdot u_{k+1} + a_0(\dots) u_{k+1} \, dx \geq C_1 \int_{\Omega} |\nabla u_{k+1}|^r \, dx - C_2 \int_{\Omega} (1 + |u_{k+1}|^q) \, dx,$$

$$\langle f_{k+1}, u_{k+1} \rangle \geq \|f_{k+1}\|_{V^*} \left( \|u_{k+1}\|_{W_0^{1,r}(\Omega)} + \|u_{k+1}\|_{L_2(\Omega)} \right) \leq \varepsilon \left( \|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r + \|u_{k+1}\|_{L_2(\Omega)}^2 \right) + C \left( \|f_{k+1}\|_{V^*}^{r'} + \|f_{k+1}\|_{V^*}^2 \right).$$

So together  $\|u_j\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} \left[ (u_{k+1} - u_k)^2 + h \|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r \right] \leq C \left( \|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \|u_{k+1}\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \left( \|f\|_{V^*}^{r'} \|f\|_{V^*}^2 \right) \right)$  □



Let us now define  $u^n(t) = u_k$  for  $t \in (h(k-1), hk)$ , then

$$\|u^n\|_{L^\infty((0,T);L^2(\Omega))}^2 + \|u^n\|_{L^2((0,T);W_0^{1,r}(\Omega))}^2 < C(\text{data}).$$

Now set  $\tilde{u}^n(t) = u_{k-1} + \frac{t-t_{k-1}}{h}(u_k - u_{k-1})$  for  $t \in (t_{k-1}, t_k)$  and

$$k \in \{1, \dots, n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the equation to the form

$$\int_{\Omega} \partial_t \tilde{u}^n \varphi + a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi \, dx = \langle f^n, \varphi \rangle,$$

where  $f^n(t) := f_k$  in in

$$(t_{k-1}, t_k), k \in \{1, \dots, n\}.$$

We are now ready to use growth and apriori estimates:

$$\|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} \leq C(\text{data}).$$

For the norm of the time derivative:

$$\sup_{\varphi \in S_V(0,1)} \langle \partial_t \tilde{u}^n(t), \varphi \rangle = \sup_{\varphi \in S_V(0,1)} \langle f^n, \varphi \rangle - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi) \, dx,$$

at any  $t \in (0, T)$ . So using Holder:

$$\|\partial_t \tilde{u}^n(t)\|_{V^*} \leq \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)},$$

and integrating

$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{r'} \, dt \leq C \left( \int_0^T \|f^n\|_{V^*}^{r'} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)} \, dt \right) \leq TC(\text{data})$$

## 6 Semigroup theory

We consider the equation

$$\begin{aligned} u' &= Au, A \text{ is linear} \\ u(0) &= u_0, \end{aligned}$$

where  $u : [0, \infty) \rightarrow \mathbb{R}$ . We know that for example if  $Au = au, a \in \mathbb{R}$  then

$$u(t) = u_0 e^{at}.$$

If  $\mathbf{u} : [0, \infty) \rightarrow \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d \times d}$ , then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{A}^k t^k.$$

This can be extended to  $u : [0, \infty) \rightarrow X$ ,  $X$  a Banach space,  $A \in \mathcal{L}(X)$ , then

$$u(t) = \exp(tA)u_0,$$

where the operator exponential is the same. This works well for unbounded operators, but suppose now

$$X = L_2(\Omega), Au = \Delta u.$$

We *guess* the solution should be

$$u(t) = \exp(\Delta t)u_0,$$

but what is

$$\exp(\Delta t)?$$

**Definition 13** (Linear operator and its domain). Let  $X$  be a Banach space over  $\mathbb{K}$ . Linear operator on  $X$  is a couple  $(A, \mathcal{D}(A))$ , where  $\mathcal{D}(A)$  is a subspace of  $X$  and  $A : \mathcal{D}(A) \rightarrow X$  is linear.

**Definition 14.** A family  $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$  is called a semigroup if

1.  $S(0) = \text{id}$
2.  $\forall s, t \geq 0 : S(t)S(s) = S(t+s).$

If moreover  $\forall x \in X : S(t)x \rightarrow x$ , as  $t \rightarrow 0_+$ , we call  $\{S(t)\}$  a  $c_0$ -semigroup (strongly continuous).

*Remark.*  $\{s(t)\}_{t \in \mathbb{R}}$  with the two conditions is an Abelian group  $(\{S(t)\}_{t \in \mathbb{R}}, \circ)$  with

$$(S(t))^{-1} = S(-t). \tag{3}$$

*Remark* ( $X = \text{Banach}$ ). **In the following,  $X$  is always a Banach space.**

**Lemma 15.** Let  $\{S(t)\}_{t \geq 0}$  be a  $c_0$ -semigroup in  $X$ . Then

1.  $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t},$
2.  $\forall x \in X, t \mapsto S(t)x \in C([0, \infty); X).$

*Proof.*  $1 \Rightarrow 2$ . Fix  $t > 0, x \in X$  compute

$$\lim_{h \rightarrow 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t)(S(h)x - x)\|_X \leq \lim_{h \rightarrow 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \rightarrow 0.$$

now compute  $\lim_{h \rightarrow 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t-h)(x - S(h)x)\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \rightarrow 0.$   $\square$

**Definition 15** (Infinitesimal generator). A linear operator  $(A, \mathcal{D}(A))$  is called a infinitesimal generator of the semigroup  $\{S(t)\}_{t \geq 0}$ , if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X \mid \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\},$$

**Theorem 28.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  in  $X$ . Then

1.  $x \in \mathcal{D}(A) \Rightarrow \forall t \geq 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x),$
2.  $x \in X \wedge t \geq 0 \Rightarrow x_t = \int_0^t S(s)x \, ds \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x.$

*Proof.* Fix  $x \in \mathcal{D}(A), t \geq 0$ . Calculate

$$\lim_{h \rightarrow 0_+} \frac{S(h)S(t)x - S(t)x}{h} \stackrel{31}{=} \lim_{h \rightarrow 0_+} S(t) \frac{S(h)x - x}{h} = S(t)Ax,$$

(convergence is in the norm of the Banach space  $X$ ). This means  $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$ , moreover, if  $t > 0$ :

$$\lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \rightarrow 0_+} S(t-h) \left( \frac{x - S(h)x}{-h} - S(h)Ax \right),$$

estimate,

$$\left\| \lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax \right\| = \lim_{h \rightarrow 0_+} \left\| S(t-h) \left( \frac{x - S(h)x}{-h} - S(h)Ax \right) \right\| = \lim_{h \rightarrow 0_+} \left\| \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax \right\| = \lim_{h \rightarrow 0_+} \|S(t-h) - S(t)\| \|x - S(h)x\| + \lim_{h \rightarrow 0_+} \|S(t-h) - S(t)\| \|S(h)Ax\|$$

as  $S(t)$  is continuous and  $S(0) = \text{id}$ . Clearly,  $t \mapsto S(t)x$  is  $C^1([0, \infty))$  and

$$\frac{d}{dt}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h \rightarrow 0_+} \frac{1}{h} (S(h)x_t - x_t) = \lim_{h \rightarrow 0_+} \frac{1}{h} \left( \int_h^{t+h} S(s)x \, ds - \int_0^t S(s)x \, ds \right),$$

realize that

$$S(h)x_t = \int_0^t S(s+h)x \, ds = \int_h^{t+h} S(s)x \, ds,$$

so the previous computation continues as follows

$$= \lim_{h \rightarrow 0_+} \frac{1}{h} \left( \int_t^{t+h} S(s)x \, ds - \int_0^h S(s)x \, ds \right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

**Definition 16** (Closed operator). We say that a linear operator  $(A, \mathcal{D}(A))$  is closed if  $\forall \{u_n\} \subset \mathcal{D}(A) : u_n \rightarrow u \wedge Au_n \rightarrow v$ , for some  $u, v \in X$ , then it must hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that  $\{(x, Ax) | x \in \mathcal{D}(A)\} \subset X \times X$  is closed in  $(X \times X, \|\cdot\|_1)$ .

$$\stackrel{31}{=} S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$$

**Example.** Let  $\Omega \in C^{1,1}, X = L_2(\Omega), \mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), Au = \Delta u$ . Then  $(A, \mathcal{D}(A))$  is closed. Really, take  $\{u_n\} \subset L_2(\Omega) : u_n \rightarrow u$  in  $L_2(\Omega)$  for some  $u \in L_2(\Omega)$ . Suppose  $Au_n \rightarrow v$  in  $L_2(\Omega), v \in L_2(\Omega)$ . Suppose the following equation: find

$$u_n \text{ s.t. } -\Delta u_n = Au_n, u_n \text{ on } \partial\Omega.$$

From the regularity theory for elliptic problems, we know that  $\|u_n\|_{W^{2,2}(\Omega)} \leq C\|Au_n\|_{L_2(\Omega)} \leq C$ , so we can extract  $u_{n_k} \rightharpoonup u$  in  $W^{2,2}(\Omega)$ . Realize moreover

$$\int_{\Omega} \Delta u_n \varphi \, dx = \int_{\Omega} u_n \Delta \varphi \, dx, \forall \varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_{\Omega} v \varphi \, dx = \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} \Delta u \varphi \, dx,$$

which means  $\Delta u = v$  a.e. in  $\Omega$  and that  $u \in \mathcal{D}(A), Au = v$ .

**Theorem 29.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0} \subset X$ . Then

1.  $\mathcal{D}(A)$  is dense in  $X$ ,
2.  $(A, \mathcal{D}(A))$  is closed.

*Proof.* Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t} \int_0^t S(s)x \, ds \in \underbrace{\mathcal{D}(A)}_{\text{prev. thm}}, \frac{x_t}{t} \rightarrow x \text{ in } X,$$

Ad 2.: Take  $\{x_n\} \subset \mathcal{D}(A) : x_n \rightarrow x$  in  $X, Ax \rightarrow v$  in  $X$ . Compute<sup>32</sup>

$$\frac{(S(h) - \text{id})x_n}{h} = \frac{1}{h} \int_0^h \frac{d}{ds}(S(s)x_n) \, ds = \frac{1}{h} \int_0^h AS(s)x_n \, ds = \frac{1}{h} \int_0^h S(s) \underbrace{Ax_n}_{\rightarrow v}, \text{ so taking the limit yields } \frac{(S(h) - \text{id})x}{h} =$$

Altogether,  $x \in \mathcal{D}(A), Ax = v$ . □

**Lemma 16.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of  $c_0$ -semigroups  $\{S(t)\}_{t \geq 0}, \{\tilde{S}(t)\}_{t \geq 0}$ . Then

$$\{S(t)\}_{t \geq 0} = \{\tilde{S}(t)\}_{t \geq 0}.$$

*Proof.* We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix  $x \in \mathcal{D}(A), t > 0$ . Then  $g(s) := S(s)\tilde{S}(t-s)x$  satisfies  $g \in C^1([0, t]; X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$ , as  $A, S$  commute. This means  $g(0) = g(t)$  and from this it follows  $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$ . Since  $\overline{\mathcal{D}(A)} = X, S$  continuous  $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$ , and since  $t \geq 0$  was arbitrary, we are done. □

**Definition 17** (Resolvent of a linear operator). Let  $(A, \mathcal{D}(A))$  be a linear (possibly unbounded) operator on  $X$ . We define

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<sup>32</sup>This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that  $X$  is a Banach space and work with some functionals and Bochner integrals or whatever

1. resolvent set

$$\rho(A) = \left\{ \lambda \in \mathbb{K} \mid \lambda \text{ id} - A \text{ is invertible and } (\lambda \text{ id} - A)^{-1} \in \mathcal{L}(X) \right\},$$

2. resolvent operator  $R(\lambda, A) : X \rightarrow \mathcal{D}(A) : R(\lambda, A) = (\lambda \text{ id} - A)^{-1}$ , for

$$\lambda \in \rho(A).$$

*Remark.* If  $(A, \mathcal{D}(A))$  is a closed linear operator:  $\lambda \in \rho(A) \Leftrightarrow \lambda \text{ id} - A$  is a bijection of  $\mathcal{D}(A)$  onto  $X$ .

**Lemma 17.** *Let  $(A, \mathcal{D}(A))$  be a linear operator on  $X$ . It holds*

1.  $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x$ ,
2.  $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x$ ,
3.  $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$ , and  $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A)$ ,
4. *If moreover  $(A, \mathcal{D}(A))$  is the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  s.t.  $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ , then*

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t) dt \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

*Remark.* The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

*Proof.* Ad 1.:

$$AR(\lambda, A)x = (A - \lambda \text{ id}) \underbrace{R(\lambda, A)}_{=(\lambda \text{ id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\text{id} - (\lambda \text{ id} - A))R(\eta, A) = R(\lambda, A)(\eta \text{ id} - A - \lambda \text{ id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For  $\lambda \neq \eta$  we also have

$$R(\lambda, A)R(\eta, A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG assume  $\omega = 0$ , meaning  $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$ . Denote  $\tilde{S}(t) = e^{-\omega t} S(t)$ .

Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x dt.$$

First of all, this is well defined as

$$\|\tilde{R}x\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X dt = \frac{M}{\lambda} \|x\|_X,$$

and so  $\|\tilde{R}\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}$ ,  $\tilde{R} \in \mathcal{L}(X)$ . Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Leftrightarrow \text{id} = (\lambda\text{id} - A)\tilde{R}.$$

For  $x \in X$ ,  $h > 0$  fixed compute

$$\begin{aligned} \frac{1}{h}(S(h)\tilde{R}x - \tilde{R}x) &= \frac{1}{h} \left( \int_0^\infty e^{-\lambda t} S(t+h)x - e^{-\lambda t} S(t)x dt \right) = \\ &= \frac{1}{h} \left( \int_h^\infty e^{-\lambda(t-h)} S(t)x dt - \int_0^\infty e^{-\lambda t} S(t)x dt \right) = \\ &= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x dt - \frac{1}{h} \int_0^h e^{-\lambda t} S(t)x dt = \\ &= e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} \rightarrow \lambda e^{-\lambda t}, \text{ as } h \rightarrow 0_+ \end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t) e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} S(t)x \rightarrow \lambda e^{-\lambda t} S(t)x \text{ on } (0, \infty) \text{ as } h \rightarrow 0_+.$$

The norm of this can be estimated as  $\|\lambda e^{-\lambda t} S(t)x\| \leq C e^{-\lambda t} M \|x\|_X \in L_1((0, \infty))$ . Altogether, we obtain  $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda\text{id} - A)\tilde{R}x = x$ .

To proceed further, we need the following theorem:

$$x \in \mathcal{D}(A), A \text{ closed} : A\tilde{R}x = A \left( \int_0^\infty e^{-\lambda t} S(t)x dt \right) = \int_0^\infty e^{-\lambda t} \underbrace{AS(t)}_{=S(t)A} x dt = \tilde{R}Ax,$$

which has been stated but not proved<sup>33</sup>. Finally, we can write:  $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda\text{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$ . Moreover, we have also shown the mapping is a bijection.  $\square$

**Definition 18** (Contraction semigroup). We say that  $\{S(t)\}_{t \geq 0}$  is a contraction semigroup if

$$\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

**Theorem 30** (Hille-Yosida). Let  $M \geq 1, \omega \in \mathbb{R}$ . A linear  $(A, \mathcal{D}(A))$  on a Banach space  $X$  generates a  $c_0$ -semigroup (meaning it is its infinitesimal generator) satisfying  $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$  **if and only if**

1.  $(A, \mathcal{D}(A))$  is closed,
2.  $\mathcal{D}(A)$  is dense in  $X$ ,
3.  $\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}$ .

*Proof.* If  $M = 1, \omega = 0$ , then  $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}$ . "  $\Rightarrow$  " has been proven, now show the other direction. The plan is to

1. approximate  $A$  by  $\{A_n\} \subset \mathcal{L}(X)$ ,

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<sup>33</sup>It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

2. construct  $S_n$  for  $A_n$  as previously,
3. estimate and limit passage.

*Approximation:* See the analogy:  $a \in \mathbb{R} : \frac{n}{n-a} \rightarrow 1$ , we would like  $nR(n, A) \rightarrow \text{id}$ . Calculate the norm of  $nAR(n, A) = n(nR(n, A) - \text{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$ , (This approx. is called the Yosida approximation.) For  $x \in \mathcal{D}(A)$  fixed:

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leq \|R(n, A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \frac{1}{n} \|Ax\|_X \rightarrow 0 \text{ as } n \rightarrow \infty.$$

If

$$y \in X : \|nR(n, A)y - y\|_X \leq \|nR(n, A)(y - x)\|_X + \|nR(n, A)x - x\|_X + \|y - x\|_X \leq 2\|y - x\|_X + \underbrace{\|nR(n, A)x - x\|_X}_{\rightarrow 0},$$

but  $\|y - x\|_X$  can be made arbitrarily small from density of  $\mathcal{D}(A)$  in  $X$ , so in fact

$$nR(n, A)y \rightarrow y \text{ in } X, \forall y \in X.$$

And so  $nR(n, A)$  really approximates  $\text{id}$ .

Using this gives us

$$\forall x \in \mathcal{D}(A) : A_n x = \overbrace{nAR(n, A)}^{=AR(n, A)} x = nR(n, A)Ax \rightarrow Ax \text{ in } X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \leq \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (tA_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id} + n^2 t R(n, A))^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id})^k \sum_{k=0}^{\infty} \frac{(n^2 t R(n, A))^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives  $\leq e^{-nt} \text{id} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n, A)\|_X^k \leq e^{-nt} e^{nt} = 1$ , as  $\|nR(n, A)\|^k \leq 1$ . This means  $\{S_n(t)\}_{\mathcal{L}(X)} \leq 1$ .

Now show that this converges: fix  $x \in \mathcal{D}(A)$ , compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{d}{ds} (S_n(s)S_m(t-s)x) ds \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x ds \right\|_X \leq \underbrace{t}_{\|S_t\|_{\mathcal{L}(X)} \leq 1} \|A_n - A_m\|_X$$

and since  $X$  is Banach, it is convergent also. Finally, for  $y \in X$ , we have

$$\|S_n(t)y - S_m(t)y\|_X \leq \|S_n(t)(y - x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x - y)\|_X \leq 2\|x - y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that  $\{S_n(t)y\}$  is Cauchy uniformly on  $[0, T]$ ,  $T > 0 \Rightarrow \exists S(t) : S_n(t)y \rightarrow S(t)y \forall y \in X, t > 0$ . And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain  $\{S(t)\}_{t \geq 0}$  is a  $c_0$ -semigroup.

It remains to answer this question. Is  $(A, \mathcal{D}(A))$  the infinitesimal generator of  $\{S(t)\}_{t \geq 0}$ ? Let  $(\tilde{A}, \mathcal{D}(\tilde{A}))$  be the infinitesimal generator of  $\{S(t)\}_{t \geq 0}$ . Compute

$$S_n(t)x - x = \int_0^t \frac{d}{ds} S_n(s)x \, ds = \int_0^t S_n(s)A_n x \, ds,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \leq \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \rightarrow 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \, ds.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t \rightarrow 0+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A} \text{ on } \mathcal{D}(A)).$$

The opposite inclusion is simple: fix  $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$ , and so  $\lambda \text{id} - A : \mathcal{D}(A) \rightarrow X$  is onto, but also  $\lambda \text{id} - A = \lambda \text{id} - \tilde{A}$  on  $\mathcal{D}(A)$ , and so  $\lambda \text{id} - \tilde{A} : \mathcal{D}(A) \rightarrow X$  is onto. From the previous theorem, we know  $\lambda \text{id} - \tilde{A}$  is one-to-one, so  $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$ . Altogether,  $A = \tilde{A}$ ,  $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$ . □

## 7 (Some) exercises

### 7.1 4.3.2025

**Example** (Coefficients for smooth extension). Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for  $u \in \mathcal{D}(\mathbb{R}^d)$ . Find  $\{c_j\}_{j=1}^{k+1}$  in such a way that  $Eu \in C^k(\mathbb{R}^d)$ . Moreover, take  $d = 1$ .

*Proof.* For  $k = 0, j = 1$  we take  $c_1 = 1, c_j = 0, j \neq 1$ . For  $k = 1$ , compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k+1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take  $x_d = 0$  in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \dots, k\}.$$

That is a linear system of  $k + 1$  equations. Is it solvable? □



## 7.2 8.4.2025

**Example** (Laplace). Let  $a_0 = 0, a(\cdot, z, p) = p$ . Then  $|a(\dots)| \leq |p|$ , growth can be accomplished for  $r = 2$ ,  $a(\dots) \cdot p \geq |p|^2$ . We can thus apply the theorem to our laplace equation

**Example.** Let  $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}(1 + |p|^2)$ . Then it is clearly Caratheodory, it is bounded  $|a(\dots)| \leq |p| \frac{\pi}{2}$ , so the growth conditions yield, it is coercive as  $\operatorname{atan}(1 + |p|^2) \geq \frac{\pi}{4}$ , and it is monotone

$$(\operatorname{atan}(1 + |p_1|^2)p_1 - \operatorname{atan}(1 + |p_2|^2)p_2)(p_1 - p_2) = \int_0^1 \sum_{j=1}^d \frac{d}{ds} \operatorname{atan}(1 + |p_2 + s(p_1 - p_2)|^2)(p_2 + s(p_1 - p_2)) ds (p_1 - p_2)_j$$