

Partial differential equations II

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1 Winter semester addendum

1.1 Weak* convergence

Since $L_\infty((0, T); L_2(\Omega))$ is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of $L_\infty((0, T); L_2(\Omega))$ is reflexive, i.e.

$$L_\infty((0, T); L_2(\Omega)) \cong \left(L_1((0, T); L_2(\Omega)) \right)^*,$$

which means that balls in $L_\infty((0, T); L_2(\Omega))$ are weakly* compact. Moreover, $L_1((0, T); L_2(\Omega))$ is *separable*, from which it follows $L_\infty((0, T); L_2(\Omega))$ with the weak* topology is metrizable and thus there exists a weakly* converging subsequence (from the balls).

Example (For people without Functional Analysis I). Let X be a linear normed space, $\{x_n\} \subset X$ a sequence in X . We say x_n converges weakly to $x \in X$ whenever

$$f(x_n) \rightarrow f(x), \forall f \in X^*.$$

Let X^* be the topological dual to X , $\{x_n\} \subset X^*$ a sequence in X^* . We say f_n converges weakly* to $f \in X^*$ whenever

$$f_n(x) \rightarrow f(x), \forall x \in X^*, \text{ i.e. } x(f_n) \rightarrow x(f),$$

where by $x(y), x \in X, y \in X^*$ we understand

$$\varepsilon_x : X^* \rightarrow \mathbb{K}, y \mapsto y(x).$$

Since $L_\infty((0, T); L_2(\Omega)) \cong \left(L_1((0, T); L_2(\Omega)) \right)^*$, every point $x \in L_\infty((0, T); L_2(\Omega))$ can be interpreted as a linear functional on $L_1((0, T); L_2(\Omega))$, so given $\{x_n\} \subset L_\infty((0, T); L_2(\Omega))$, we can interpret it as a $\{x_n\} \subset \left(L_1((0, T); L_2(\Omega)) \right)^*$, meaning given a weakly converging sequence in $L_\infty((0, T); L_2(\Omega))$, it is actually a weakly* converging sequence in $L_1((0, T); L_2(\Omega))$.

1.2 Regularity of parabolic problems

Theorem 1. *Let the assumptions of the previous theorem hold and $\Omega \in C^{1,1}, \delta \in (0, 1)$. Then $u \in L_2((\delta, T); W^{2,2}(\Omega))$.*

Proof. Take the weak formulation in $t \in (\delta, T)$. WLOG further assume $d = 0$. Then

$$\int_\Omega \mathbb{A} \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi - bu \varphi - \mathbf{c} \cdot \nabla u \varphi - \int_\Omega \partial_t u \varphi = \int_\Omega (f - bu - \mathbf{c} \cdot \nabla u - \partial_t u) \varphi,$$

and the integrand of the last integral is in $L_2(\Omega)$ for a.e. $t \in (\delta, T)$. We can thus use the elliptic regularity results and write:

$$\|u\|_{W^{2,2}(\Omega)}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{W^{1,2}(\Omega)}^2 + \|\partial_t u\|_{L_2(\Omega)}^2),$$

integrating both sides $\int_{\delta}^T dt$ yields

$$\|u\|_{L_2((\delta,T);L_2(\Omega))}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2((0,T);W^{1,2}(\Omega))}^2 + \|u\|_{L_2((\delta,T);L_2(\Omega))}^2)$$

□

Theorem 2. *If data are smooth and satisfy the compatibility conditions, then the weak solutions to the parabolic equation are smooth.*

Proof. no. □

Remark (Compatibility condition). : Take the heat equation : $\partial_t u - \Delta u = f$ at time zero: $\Delta u(0) + f(0) = \partial_t u(0) \in W_0^{1,2}(\Omega)$, so we need that $f(0) + \Delta u(0)$ has zero trace \Rightarrow compatibility conditions.

1.3 Uniqueness of solutions to hyperbolic problems

Theorem 3 (Uniqueness of the solution to a hyperbolic equation). *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that $\mathbf{c} \in W^{1,\infty}(\Omega)$. Then the weak solution to the hyperbolic equation is unique.*

Proof. It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the equation, multiply it by $\varphi \in V$ fixed and integrate over Ω for $t \in (0, T)$ fixed:

$$\langle \partial_{tt} u(t), \varphi \rangle + \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \varphi \, dx + \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \varphi \, dx - \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \varphi \, dx = 0.$$

Now, take a special test function

$$\psi(t) = \left(\int_t^s u(\tau) \, d\tau \right) \chi_{(0,s)}(t),$$

for some $s \in (0, T)$. Then $\partial_t \psi(t) = -u(t)$ on $t \in (0, s)$. Next, integrate the equation in time over $(0, s)$.

$$\int_0^s \langle \partial_{tt} u(t), \psi \rangle \, dt + \int_0^s \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s \langle \partial_{tt} u(t), \varphi \rangle \, dt = \langle \partial_t u(s), \psi(s) \rangle - \langle \partial_t u(0), \psi(0) \rangle - \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt,$$

and realize $\psi(s) = 0, \partial_t u(0) = 0$, so

$$- \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt + \int_0^s \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

but since $\partial_t \psi(t) = -u(t)$, we can actually write (time dependencies are omitted for brevity)

$$\int_0^s \langle \partial_t u, u \rangle \, dt + \int_0^s \int_{\Omega} -\mathbb{A} \nabla \partial_t \psi \cdot \nabla \psi - b \psi \partial_t \psi - \psi \mathbf{c} \cdot \nabla \partial_t \psi + \partial_t \psi \mathbf{d} \cdot \nabla \psi \, dx \, dt = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^s \frac{d}{dt} \left(\|u\|_{L_2(\Omega)}^2 - \int_{\Omega} \mathbb{A} \nabla \psi \cdot \nabla \psi + b \psi^2 + \psi \mathbf{c} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla \psi \, dx \right) dt = \\ &= \int_0^s \int_{\Omega} (\partial_t \mathbb{A}) \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 + \psi \partial_t \mathbf{c} \cdot \nabla \psi + \underbrace{\partial_t \psi}_{=-u(t)} \mathbf{c} \cdot \nabla \psi - \psi \partial_t \mathbf{d} \cdot \nabla \psi - \psi \mathbf{d} \cdot \nabla \underbrace{\partial_t \psi}_{=-u(t)} \, dx \, dt, \end{aligned}$$

and upon integration (recall $\psi(s) = 0$, from the definition of ψ it follows $\nabla \psi(0) = 0$, and $u(0) = 0$),

$$\begin{aligned} & \frac{1}{2} \left(\|u(s)\|_{L_2(\Omega)}^2 + \int_{\Omega} \mathbb{A}(0) \nabla \psi(0) \cdot \nabla \psi(0) + b(0) \psi(0)^2 + \psi(0) \mathbf{c}(0) \cdot \nabla \psi(0) + \psi(0) \mathbf{d}(0) \cdot \nabla \psi(0) \, dx \right) = \\ &= \int_0^s \int_{\Omega} \partial_t \mathbb{A} \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 - u \partial_t \mathbf{c} \cdot \nabla \psi - \psi \partial_t \mathbf{d} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla u \, dx \, dt. \end{aligned}$$

From this we obtain the following estimate:

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\psi(0)\|_{W^{1,2}(\Omega)}^2 \leq C \left(\int_0^s \|\psi\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\psi(0)\|_{L_2(\Omega)}^2,$$

where $C = C(\|\mathbb{A}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbb{A}\|_{L_{\infty}(\Omega)}, \|b\|_{L_{\infty}(\Omega)}, \|\partial_t b\|_{L_{\infty}(\Omega)}, \|\mathbf{c}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{c}\|_{L_{\infty}(\Omega)}, \|\mathbf{d}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{d}\|_{L_{\infty}(\Omega)})$.

Define now the test function $\chi(t) = \int_0^t u(\tau) \, d\tau$, and realize that in fact $\psi(t) = \chi(s) - \chi(t)$, $\chi(0) = 0$. Plugging this in the above inequality yields

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left(\int_0^s \|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\chi(s)\|_{L_2(\Omega)}^2,$$

and using

$$\|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 = \|\chi(t) - \chi(s)\|_{W^{1,2}(\Omega)}^2 \leq 2 \left(\|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \right),$$

and the definition of $\chi(t)$, from which it follows

$$\|\chi(s)\|_{L_2(\Omega)}^2 \leq \int_0^s \|u\|_{L_2(\Omega)}^2 \, dt,$$

we are allowed to write

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left(\int_0^s 2 \|\chi(s)\|_{W^{1,2}(\Omega)}^2 + 2 \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + 2 \|u\|_{L_2(\Omega)}^2 \, dt \right),$$

and so

$$\|u(s)\|_{L_2(\Omega)}^2 + (1 - 2sC) \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_1 \left(\int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right).$$

If we now choose $T_1 \in (0, T]$ small enough s.t. $1 - 2sC > 0$ for $s \in (0, T_1]$, we finally obtain

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_2 \left(\int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right), \forall s \in (0, T_1],$$

which implies $u = 0$ on $(0, T_1]$ by the Gronwall lemma: we have

$$\xi(t) \leq \int_0^t \xi(s) \, ds, \text{ for } a.a. \, t \in (0, T) \Rightarrow \xi(t) = 0 \, a.e..$$

for $\xi \in L_1((0, T))$ nonnegative¹. If we now bootstrap on $[T_1, 2T_1], [2T_1, 3T_1]$ etc., we obtain $u = 0$ on $(0, T]$. □

2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1, +\infty]$, $k \in \mathbb{N}$. We define

$$W^{k,p}(\Omega) = \left\{ f \in L_p(\Omega) ; D^\alpha f \in L_p(\Omega), \forall |\alpha| \leq k \right\},$$

with the norm

$$\|f\|_{W^{k,p}(\Omega)}^p = \|f\|_{L_p(\Omega)}^p + \sum_{0 < |\alpha| \leq k} \|D^\alpha f\|_{L_p(\Omega)}^p.$$

Recall that:

- $W^{k,p}(\Omega)$ is Banach $\forall p$ and Hilbert for $p = 2$.
- $W^{k,p}(\Omega)$ is separable if $p < \infty$ and reflexive if $p > 1, p < \infty$.

Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.

2.1 Tools from functional analysis

Definition 1 (Regularization kernel). The function η is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\text{supp } \eta \subset U(0, 1)$
- $\eta \geq 0$
- η is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x) dx = 1$

Definition 2 (Regularization of a function). Let η be a regularization kernel. Set²

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of $f \in L_1(\Omega)_{\text{loc}}$ by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

Remark (Properties of regularization). The regularization has the following properties:

- $f \in L_p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$ in $L_p(\Omega)$ and also a.e

¹In our case $\xi = \|u\|_{L_2(\Omega)}^2 + \|\chi\|_{W^{1,2}(\Omega)}^2$.

²Another common choice is $\eta_k = k^d \eta(kx)$, $k \in \mathbb{N}$.

- $f \in L_\infty(\Omega) \Rightarrow f_\varepsilon \rightarrow f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) \eta_\varepsilon(x-y) dy = \int_{U(x,\varepsilon)} f(y) \eta_\varepsilon(x-y) dy$
- $\text{supp } f_\varepsilon \subset \overline{U(\Omega, \varepsilon)}, f = 0 \text{ on } U(x, \varepsilon) \Rightarrow f_\varepsilon(x) = 0$

Definition 3 ($\Omega' \subset\subset \Omega$). $O \subset\subset \Omega$ means \overline{O} is compact and $\overline{O} \subset \Omega$.

Definition 4 (Shift operator). For $u \in L_p(\Omega), k \in \{1, \dots, d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h \mathbf{e}_k)$$

Lemma 1 (Approximation property of the shift operator). For $u \in L_p(\Omega)$, it holds $\tau_h u \rightarrow u$ in $L_p(\Omega), h \rightarrow 0^+$.

Lemma 2 (Partition of unity). Let $E \subset \mathbb{R}^d, \mathcal{G}$ be an open covering of E (possibly uncountable.) Then there exists a countable system \mathcal{F} of nonnegative functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ and

1. \mathcal{F} is subordinate to $\mathcal{G} : \forall \varphi \in \mathcal{F} \exists U \in \mathcal{G} : \text{supp } \varphi \subset U$
2. \mathcal{F} is locally finite³: $\forall K \subset E$ compact, $\text{supp } \varphi \cap K \neq \emptyset$ for at most finitely many $\varphi \in \mathcal{F}$.
3. $\sum_{\varphi \in \mathcal{F}} \varphi(x) = 1, \forall x \in E$.

Proof. (Sketch) *Step 1 (If E is compact):*

E compact $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}$ s.t. $E \subset \bigcup_{j=1}^m U_j$. Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \bigcup_{j=1}^m K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set:

$$K_m = \left\{ x \in U \mid \text{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m \right\}.$$

Then clearly $K_1 \subset K_2 \dots$, and they "converge monotonously to U ". Next, find $\phi_j \in C_c(U_j), \phi_j > 0$ on K_j , e.g. $\phi_j = \theta(\text{dist}(x, \partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

Step 2 (If E is open):

Approximate E by $K \subset E$ compact by the exhaustion argument, then the covering will enlarge from finite \rightarrow countable (nontrivial reasoning). \square

2.2 Density of smooth functions

Lemma 3 (Local approximation by smooth functions (using regularization)). Assume $p \in [1, \infty), \Omega \subset \mathbb{R}^d$ open, $k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$. Then it holds

1. $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ a.e. in $\Omega_\varepsilon, \forall |\alpha| \leq k$
2. $u_\varepsilon \rightarrow u$ in $W^{k,p}(\Omega)_{loc}, \varepsilon \rightarrow 0^+$

³In other words, φ_K is nonzero for at most finitely many $\varphi \in \mathcal{F} \Leftrightarrow$ points in K can be represented by finitely many functions $\varphi \in \mathcal{F}$.

Proof. First of all:

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\left(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y) dy\right) = \int_{\mathbb{R}^d} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy,$$

the integrable majorants are *e.g.* $\|\eta_\varepsilon\|_\infty |u| \chi_{U(0,\varepsilon)}(x) \in L_1(\Omega)$. Now picking $x \in \Omega_\varepsilon$ we realize $\forall y \in \mathbb{R}^d/\Omega : x-y \geq \text{dist}(x, \partial\Omega) \geq \varepsilon$, and so $\eta_\varepsilon(x-y) = 0$. Exchanging derivatives and using the definition of the weak derivative

$$\int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = (-1)^{|\alpha|} \int_{\Omega} u(y)D_y^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = (D^\alpha u)_\varepsilon.$$

Take $V \subset\subset \Omega$ open, then

$$\|u - u_\varepsilon\|_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u - D^\alpha u_\varepsilon\|_{L_p(V)} \rightarrow 0,$$

because $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \rightarrow D^\alpha u$ in $L_p(V)$, from the properties of regularization. \square

Theorem 4 (Global approximation by smooth functions). *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C = \{f \in C^\infty(\Omega), \text{supp } f \text{ bounded}\} \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, i.e.*

$$\overline{C \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

If moreover Ω is bounded, it holds:

$$\overline{C^\infty \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

Proof. Let $u \in W^{k,p}(\Omega), \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ s.t. $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$. For every $j \in \mathbb{N}$ define an open set

$$\Omega_j = \left\{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{j}\right\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$. Next, set

$$U_j = \Omega_{j+1} / \overline{\Omega_{j-1}}, j = 1, 2, \dots,$$

where $\Omega_0 = \Omega_{-1} = \emptyset$. Since Ω_j are open, U_j are also open and $\Omega \subset \bigcup_{j \in \mathbb{N}} U_j \Rightarrow \exists \{\varphi_j\}_{j \in \mathbb{N}}$ partition of unity subordinate to $\{U_j\}_{j \in \mathbb{N}}$. We can write $u = \sum_{j \in \mathbb{N}} u\varphi_j$, where $u\varphi_j \in W^{k,p}(\Omega), \text{supp } u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$: set $v_j = (u\varphi_j)_{\varepsilon_j}$ and fix an arbitrary $\delta > 0$. By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{\delta}{2^{j-1}},$$

for $\varepsilon_j > 0$ sufficiently small, which we now fix so the above inequality holds. To have a nice inequality, we actually want:

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{2^N}{2^{N+1} - 1} \frac{\delta}{2^{j-1}},$$

meaning of $N \in \mathbb{N}$ will be evident later.

Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $W^{k,p}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms (\mathcal{F} is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u - v\|_{W^{k,p}(\Omega)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for $m > N + 1$, i.e., $m - 1 > N$ it holds $U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}$, $\Omega_N \subset \Omega_{m-1}$ meaning $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \emptyset \Rightarrow \text{supp } u\varphi_j \cap \Omega_N = \text{supp } v_j \cap \Omega_N = \emptyset$, since $\text{supp } u\varphi_j \subset U_j$, $\text{supp } v_j \subset \text{supp } u\varphi_j \subset U_j$, $\forall j \geq m$. The norm of sum is

$$\|u - v\|_{W^{k,p}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{W^{k,p}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let $N \rightarrow \infty$ and realize

$$\|u - v\|_{W^{k,p}(\Omega_N)} \rightarrow \|u - v\|_{W^{k,p}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| dx = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) dx = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} dx = \int_{\mathbb{R}^d} |D^\alpha f| \chi_\Omega(x) dx = \int_\Omega |D^\alpha f| dx,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Altogether,

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows $v \in W^{k,p}(\Omega)$ (this was not totally evident) and thus $v \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ so indeed we have showed the desired density. \square

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Remark ($C^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $C^{k,\lambda}$ if: Ω is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0, 1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : B(0, \alpha) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R} \text{ s.t. } a_j \in C^{k,\lambda}(B(0, \alpha)), \exists \mathbb{A}_j \mathbb{R}^d \rightarrow \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \leq m : \partial\Omega \cap U_j = \mathbb{A}_j(\{(x', a_j(x')) \in \mathbb{R}^d | x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}\})$,
3. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \Omega$,
4. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$.

If $\lambda = 0$ we sometimes drop it and write $\Omega \in C^{k,0} \Leftrightarrow \Omega \in C^k$, if $k = 0, \lambda = 1$ we call $\Omega \in C^{0,1}$ to be a Lipschitz domain. *Remember that $\lambda(\Omega) < \infty$ is a part of the definition.*

Theorem 5 (Global approximation by smooth functions up to the boundary). *Let $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C_{\bar{\Omega}}^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in W^{k,p}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in C^\infty(\bar{\Omega})$ s.t. $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\bar{\Omega}$,
2. partition of unity,
3. approximation of u on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^m$ be from the definition of $C^{0,0}$ boundary. Then⁴

$$\bar{\Omega} \subset \bigcup_{j=0}^m U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\bar{\Omega}$, subordinate to $\{U_j\}_{j=0}^m$. Since

$$u = \sum_{j=0}^m u\varphi_j, \text{ on } \Omega$$

observe that $u_j := u\varphi_j \in W^{k,p}(\Omega)$, $\text{supp } u_j \subset \text{supp } \varphi_j \subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d/\Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \dots, m\}$.

Case $j = 0$. We have $\text{supp } u\varphi_0 \subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of Ω , it holds $u\varphi_0 \in W^{k,p}(\mathbb{R}^d)$. Since $W^{k,p}(\mathbb{R}^d) = W_0^{k,p}(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{W^{k,p}(\mathbb{R}^d)}}$, we can find $v_0 \in \mathcal{D}(\mathbb{R}^d)$ s.t.

$$\|v_0 - u\varphi_0\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{m+1}.$$

Case $j \in \{1, \dots, m\}$. We have a problem now: $\{U_j\}_{j=1}^m$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of Ω , where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate u_j on a larger *open* domain containing $\bar{\Omega}$ and then show this is also a good approximation of u_j on $\Omega \subset \bar{\Omega}$.

Set $w_j = u\varphi_j$, and denote

$$S_j = \mathbb{A}_j \left(\left\{ (x', x_d) \mid a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j = \mathbb{R}^d / \overline{S_j},$$

i.e.,

$${}^{\text{''}}\Omega_j = \Omega \cup \mathbb{A}_j \left(\left\{ (x', x_d) \mid x_d \leq a_j(x') - \frac{\beta}{2} \right\} \right),{}^{\text{''}}$$

⁴Our choice $U_0 = \Omega$ is important, as without it the definition of $C^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^m U_j$.

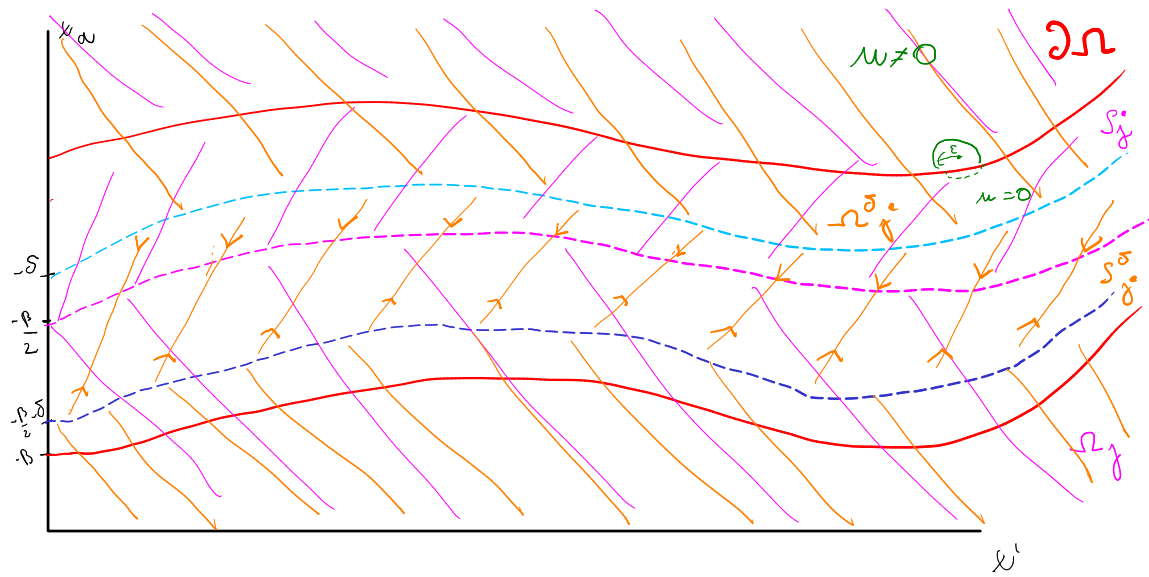


Figure 1: A cumbersome sketch of $\Omega_j, S_j, \Omega_j^\delta, S_j^\delta$

(although this is a bit inaccurate). Realize that since $u = 0$ outside of Ω , also u_j is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in W^{k,p}(\Omega_j)$. Now pick $\delta \in (0, \frac{\beta}{2})$, where β is from the definition of $C^{0,0}$ and set

$$S_j^\delta = \mathbb{A}_j \left(\left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

i.e.,

$${}^{\prime\prime}\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j \left(\left\{ (x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta \right\} \right).{}^{\prime\prime}$$

The trick is to shift the (support of) function u_j "into" Ω_j^δ

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\text{supp}(\tau_\delta u_j) = \text{supp}(u_j) - \delta,$$

from which it follows $\tau_\delta u_j \in W^{k,p}(\Omega_j^\delta)$; we have only shifted the function u_j , but since we have also shifted S_j , qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j$, $\Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$, and the fact τ_δ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in W^{k,p}(\Omega_j \cap \Omega_j^\delta)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ s.t.

$$\|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} \leq \|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega_j \cap \Omega_j^\delta)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to u_j , we are done once we approximate $\tau_\delta u_j$ by a function from $C^\infty(\overline{\Omega})$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $C^\infty(\overline{\Omega}) \subset C^\infty(\overline{\Omega})$.

Show $\Omega \subset \Omega_j^\delta$: We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k(\{(x', x_d) | x_d = a_k(x'), x' \in U(0, \alpha)\}),$$

and the set Ω_j^δ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j \left(\left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right).$$

Realize it suffices to show $\partial\Omega \not\subset \overline{S_j}$, as then it wont be excluded from \mathbb{R}^d and thus will end up in Ω_j^δ . Thanks to continuity of a_j , we may write

$$\overline{S_j} = \mathbb{A}_j \left(\left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

i.e., the " $<$ " have changed to " \leq ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d) | x_d = a_j(x'), x' \in U(0, \alpha)\}) \not\subset \mathbb{A}_j \left(\left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

which is equivalent to

$$\left((a_j \leq a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta) \right) \vee \left((a_j > a_j - \delta) \wedge (a_j \geq a_j - \frac{\beta}{2} - \delta) \right).$$

Our choice has been $\delta \in (0, \frac{\beta}{2})$, and $\beta > 0$ from the definition of $\Omega \in C^{0,0}$, so the second statement is clearly true $\forall j \in 1, \dots, m$. Consequently $\partial\Omega \notin \overline{S_j}$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

Approximation of $\tau_\delta u_j$. Since Ω_j^δ is open there $\exists w_j \in C^\infty(\Omega_j^\delta)$ such that

$$\|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega)} \leq \|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega_j^\delta)} < \frac{\varepsilon}{4(m+1)}.$$

What is more, since $\overline{\Omega} \subset \Omega_j^\delta$, we see $w_j \in C^\infty(\overline{\Omega})$ in fact. What is even more, we see $\text{supp } w_j \subset \text{supp } u_j - \delta$ so $\text{supp } \tau_{-\delta} w_j \subset \text{supp } u_j \subset \Omega$, and since w_j is defined on the whole \mathbb{R}^d , also $v_j := \tau_{-\delta} w_j$ is defined on the whole \mathbb{R}^d . Altogether, $v_j = \tau_{-\delta} w_j \in \mathcal{D}(\mathbb{R}^d)$, and from the properties of the shift operator, there $\exists \delta > 0$ s.t.

$$\|v_j - w_j\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{4(m+1)}.$$

Approximation of u .

Finally, let us set

$$v = \sum_{j=0}^m v_j.$$

Then $v \in \mathcal{D}(\mathbb{R}^d)$ and it holds

$$\begin{aligned} \|u - v\|_{W^{k,p}(\Omega)} &= \left\| \sum_{j=0}^m u_j - \sum_{j=0}^m v_j \right\|_{W^{k,p}(\Omega)} = \left\| \sum_{j=0}^m u_j - v_j \right\|_{W^{k,p}(\Omega)} \leq \sum_{j=0}^m \|u_j - v_j\|_{W^{k,p}(\Omega)} \leq \\ &\leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - u_j\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - w_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|w_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|\tau_\delta u_j - u_j\|_{W^{k,p}(\Omega)} \\ &< \frac{\varepsilon}{m+1} + 2 \sum_{j=1}^m \frac{\varepsilon}{4(m+1)} + \sum_{j=1}^m \frac{\varepsilon}{2(m+1)} = \varepsilon \end{aligned}$$

Finally,

$$\mathcal{D}(\mathbb{R}^d) \subset C_\Omega^\infty(\mathbb{R}^d) \subset C^\infty(\overline{\Omega}).$$

This proof may still have some flaws, but the author has decided to move on. □

Remark (What is $C_\Omega^\infty(\mathbb{R}^d)$). Recall

$$C_\Omega^\infty(\mathbb{R}^d) = \left\{ u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d) \right\}.$$

2.3 Extension of Sobolev functions

Problem of extension: For $u \in W^{k,p}(\Omega)$, does there exist

$$\bar{u} \in W^{k,p}(\mathbb{R}^d), \text{ s.t. } \bar{u}|_{\Omega} = u, \|\bar{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{W^{k,p}(\Omega)}$$

?

The answer is **yes**, if Ω is nice enough. Notice however this is not as simple as in the case of Lebesgue spaces, where we could just extend the function by zero trivially. We are dealing with derivatives, that are somehow regular, and if we extend a nonzero function by zero, it might mess up the regularity of the derivatives.

We will be using C^1 diffeomorphisms heavily, so we investigate some of their properties first.

Lemma 4 (Properties of C^1 diffeomorphisms). *Let $U, V \subset \mathbb{R}^d$ be open, $\Phi : U \rightarrow V$ be a C^1 diffeomorphism and let $\tilde{U} \subset \mathbb{R}^d$ s.t. $\tilde{U} \subset\subset U$. Then*

1. $\Phi(\tilde{U}) \subset\subset V$,
2. if moreover \tilde{U} is compact, then ⁵

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \leq C \|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

Proof. Ad 1.: No proof has been given.

Ad 2.: Just a change of variables formula:

$$\|u \circ \Phi\|_{L_p(\tilde{U})}^p = \int_{\tilde{U}} |u \circ \Phi|^p dx = \int_{\Phi(\tilde{U})} |u|^p |\det \nabla \Phi|^{-1} dx.$$

Since Φ is one-to-one, we know $|\det \nabla \Phi| > 0$ on U , and since $\Phi \in C^1(U)$ and $\tilde{U} \subset U \Rightarrow \Phi \in C^1(\tilde{U}) \Rightarrow \det \nabla \Phi \in C^0(\tilde{U})$, and since \tilde{U} is compact, $|\det \nabla \Phi| \geq C_1 > 0 \Leftrightarrow |\det \nabla \Phi|^{-1} \leq \frac{1}{C_1}$. In total

$$\|u \circ \Phi\|_{L_p(\tilde{U})}^p \leq \frac{1}{C_1} \int_{\Phi(\tilde{U})} |u|^p dx = C \|u\|_{L_p(\Phi(\tilde{U}))}^p.$$

As for the derivative, we have $\forall i \in \{1, \dots, d\}$:

$$\begin{aligned} \int_{\tilde{U}} |\partial_i(u \circ \Phi)|^p dx &\leq \int_{\tilde{U}} |\nabla(u \circ \Phi)|^p dx = \int_{\tilde{U}} |\nabla \Phi((\nabla u) \circ \Phi)|^p dx \leq \\ &\leq \|\nabla \Phi\| \int_{\tilde{U}} |(\nabla u) \circ \Phi|^p dx = \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p |\det \nabla \Phi|^{-1} dx \leq C \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p dx \leq \\ &\leq C \|\nabla u\|_{L_p(\Phi(\tilde{U}))}^p, \end{aligned}$$

where $\|\nabla \Phi\|$ is e.g. the operator norm of the matrix $\nabla \Phi$. □

Lemma 5 (Flat extension). *Let $\alpha, \beta > 0, K \subset\subset U(0, \alpha) \times [0, \beta]$ be compact. Then there $\exists C > 0$, a linear operator*

$$E : C^1((B(0, \alpha) \times [0, \beta])) \rightarrow C^1((B(0, \alpha) \times [-\beta, \beta])),$$

and the set $\tilde{K} \subset\subset U(0, \alpha) \times [-\beta, \beta]$ such that $\forall u \in C^1(B(0, \alpha) \times [0, \beta])$ it holds

⁵For \tilde{U} compact: $\tilde{U} \subset\subset V \Leftrightarrow \tilde{U} \subset V$.

1. $Eu = u$ on $B(0, \alpha) \times [0, \beta]$,

2. $\|Eu\|_{W^{1,p}(U(0,\alpha) \times (-\beta, \beta))} \leq \|u\|_{W^{1,p}(U(0,\alpha) \times (0, \beta))}$. *Actually,*

$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha) \times (0, \beta)), W^{1,p}(U(0,\alpha) \times (-\beta, \beta)))} = 2^{\frac{1}{p}}$$

3. if $\text{supp } u \subset K$ then $\text{supp } Eu \subset \tilde{K}$

Proof. (The set $U(0, \alpha) \times [0, \beta]$ is a cylinder of radius α and height β)

The proof is constructive: for the assumed u we write $(x = (x', x_d))$, where $x' \in B(0, \alpha) \subset \mathbb{R}^{d-1}$, $x_d \in [0, \beta] \subset \mathbb{R}$)

$$Eu(x', x_d) = \begin{cases} u(x', x_d), & x_d \geq 0 \\ -3u(x', -x_d) + 4u(x', -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Does Eu lie in $C^1(B(0, \alpha) \times [-\beta, \beta])$? Since $u \in C^1(B(0, \alpha) \times [0, \beta])$, it is continuous in the "lower cylinder", check only the transition through the origin plane: take some $a = (x', 0) \in B(0, \alpha) \times \{0\}$. Then

$$\lim_{x \rightarrow a} Eu(x) = \begin{cases} u(a), & x_d \geq 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so Eu is continuous. The derivatives

$$\lim_{x \rightarrow a} \partial_d Eu(x', x_d) = \begin{cases} \partial_d u(x', 0), & x_d \geq 0 \\ (-3\partial_d u(x', -x_d)(-1) + 4\partial_d u(x', -\frac{x_d}{2})(-\frac{1}{2}))|_{x_d=0} = \partial_d u(x', 0), & x_d < 0, \end{cases}$$

and also for any $i \in \{1, \dots, d-1\}$

$$\lim_{x \rightarrow a} \partial_i Eu(x_1, \dots, x_d) = \begin{cases} = \partial_i u(x_1, \dots, 0), & x_d \geq 0 \\ = -3\partial_i u(x_1, \dots, 0) + 4\partial_i u(x_1, \dots, 0) = \partial_i u(x_1, \dots, 0) & \end{cases}$$

so the derivative is also continuous. Thus, we have

$$Eu \in C^1 \subset W^{1,p}(U(0, \alpha) \times (-\beta, \beta)).$$

The first property is clear from the definition of Eu , the estimates of the norm: (all derivatives can in fact be assumed classical)

$$\begin{aligned} \|Eu\|_{W^{1,p}(U(0,\alpha) \times (-\beta, \beta))}^p &= \|Eu\|_{L^p(U(0,\alpha) \times (-\beta, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (-\beta, \beta))}^p = \\ &= \|Eu\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \|Eu\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p \\ &+ + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p = \\ &= \|u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \|4u - 3u\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p + \\ &+ \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + \sum_{|\alpha|=1} \|D^\alpha (4u - 3u)\|_{L^p(U(0,\alpha) \times (-\beta, 0))}^p \\ &= 2\|u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p + 2 \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U(0,\alpha) \times (0, \beta))}^p = 2\|u\|_{W^{k,p}(U(0,\alpha) \times (0, \beta))}^p \end{aligned}$$

and so

$$\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} = 2^{\frac{1}{p}} \|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}.$$

Where we have used the obvious fact

$$\int_{U(0,\alpha)\times(0,\beta)} |f|^p dx = \int_{U(0,\alpha)\times(-\beta,0)} |f|^p dx,$$

We will assume (although this is *with* a loss of generality), that

$$Eu(x', x_d) = 0 \Leftrightarrow (u(x', x_d) = 0 \vee 3u(x', -x_d) = 0 \vee 4u(x', -\frac{x_d}{2}) = 0),$$

i.e., that support lies in

$$\begin{aligned} \text{supp } Eu \subset & \{(x', x_d) \in U(0, \alpha) \times [0, \beta] | u(x', x_d) = 0\} \cup \{(x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -x_d) \neq 0\} \cup \\ & \cup \left\{ (x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -\frac{x_d}{2}) \neq 0 \right\}. \end{aligned}$$

Denote $\Phi_1, \Phi_2 : U(0, \alpha) \times (0, \beta) \rightarrow U(0, \alpha) \times (-\beta, \beta)$ to be the mappings

$$\begin{aligned} \Phi_1(x', x_d) &= (x', -x_d), \\ \Phi_2(x', x_d) &= (x', -\frac{x_d}{2}), \end{aligned}$$

then clearly Φ_1, Φ_2 are C^1 diffeomorphisms and

$$u(x', -x_d) = (u \circ \Phi_1)(x', x_d), u\left(x', -\frac{x_d}{2}\right) = (u \circ \Phi_2)(x', x_d),$$

i.e.,

$$u(x', x_d) = (u \circ \Phi_1^{-1})(x', -x_d) = (u \circ \Phi_2^{-1})\left(x', -\frac{x_d}{2}\right),$$

and so

$$\text{supp } Eu \subset \text{supp } u \cup \Phi_1^{-1}(\text{supp } u) \cup \Phi_2^{-1}(\text{supp } u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as $\text{supp } u \subset K$. Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\text{supp } Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have $K \subset U(0, \alpha) \times [0, \beta] \Rightarrow K \subset U(0, \alpha) \times (-\beta, \beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset U(0, \alpha) \times (0, \beta)$, which really implies

$$\tilde{K} \subset U(0, \alpha) \times [-\beta, \beta].$$

□

Let us prove the main result.

Theorem 6 (Extension of Sobolev functions). *Let $\Omega \in C^{k-1,1}$, $k \in \mathbb{N}$, $p \in [1, \infty]$, $V \subset \mathbb{R}^d$ open such that $\Omega \subset V$. Then there is $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$ bounded linear operator such that*

1. $\forall u \in W^{k,p}(\Omega) : Eu = u \text{ a.e. in } \Omega,$
2. $\forall u \in W^{k,p}(\Omega) : \text{supp } Eu \subset V \supset \supset \Omega,$
3. $\|E\|_{\mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^d))} \leq C, C = C(p, \Omega, V).$

Proof. **Will only be presented for $k = 1, \Omega \in C^1, p < \infty$.** The strategy is:

1. covering of $\overline{\Omega}$ & partition of unity
2. obtain a diffeomorphism from the fact $\Omega \in C^{1,0}$,
3. suitable composition & cut off,
4. flat extension,
5. show existence of $E : C_{\Omega}^{\infty}(\mathbb{R}^d) \rightarrow W^{k,p}(\mathbb{R}^d)$ with the desired properties,
6. extend via density

Covering of Ω : In the following, $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ are as in the definition of a $C^{1,0}$ domain. Set $U_0 = \Omega$ and realize

$$\overline{\Omega} \subset \bigcup_{j=0}^m U_j,$$

i.e., $\{U_j\}_{j=0}^m$ is an open covering of $\overline{\Omega}$. Denote $\{\varphi_j\}_{j=0}^m$ as the partition of unity subordinate to $\{U_j\}_{j=0}^m$.

Diffeomorphism & flat extension For $j \in \{1, \dots, m\}$ we define $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$ by

$$\Phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

(A bit confusingly, we will however be "interpreting" Φ_j as $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow \mathbb{R}^d$, with it being extended by zero on $\mathbb{R}^d \setminus \overline{U_j}$, as we need Φ_j^{-1} to be defined on the whole \mathbb{R}^d .)

As $\Omega \in C^{1,0}$, we know $a_j \in C^1(B(0, \alpha)) \subset C^1(U(0, \alpha))$, and so ϕ_j is a C^1 diffeomorphism. Let us denote by $\tilde{E} : C^1(B(0, \alpha) \times [-\beta, \beta]) \rightarrow C^1(B(0, \alpha) \times [-2\beta, \beta])$ the extension operator from the Flat extension lemma. Then we for any $u \in C_{\Omega}^{\infty}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$ define

$$Eu = \varphi_0 u + \sum_{j=1}^m \left(\eta \tilde{E}((\varphi_j u) \circ \Phi_j) \right) \circ \Phi_j^{-1},$$

where $\eta \in C^{\infty}(U(0, \alpha) \times \mathbb{R})$ is a cut-off function

$$\eta(y', y_d) \begin{cases} = 1 & \text{on } y_d \geq 0, \\ = 0 & \text{on } y_d \leq -h, \\ \in (0, 1) & \text{else.} \end{cases}$$

for some parameter $h > 0$ which will be defined later. With this definition, $\forall x \in \Omega$ it holds

$$\begin{aligned} (Eu)(x) &= (\varphi_0 u)(x) + \sum_{j=1}^m \left(\eta \tilde{E}(u \varphi_j \circ \Phi_j) \right) \underbrace{(\Phi_j^{-1}(x))}_{\in U(0, \alpha) \times (-\beta, \beta)} = \\ &= (\varphi_0 u)(x) + \sum_{j=1}^m (u \varphi_j \circ \Phi_j)(\Phi_j^{-1}(x)) = (\varphi_0 u)(x) + \sum_{j=1}^m (u \varphi_j)(x) = \sum_{j=0}^m (u \varphi_j)(x) = u(x) \sum_{j=0}^m \varphi_j(x) = u(x), \end{aligned}$$

since $\eta(y) = 1$, $\tilde{E}(\varphi_j u \circ \Phi_j)(y) = (\varphi_j u \circ \Phi_j)(y)$ for $y \in U(0, \alpha) \times (-\beta, \beta)$, according to our definition of η and the properties of the extension \tilde{E} .

The motivation behind the cutoff is the following: we know $\text{supp } \varphi_j \subset U_j$, so since $\Phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$, we have $\text{supp } u\varphi_j \circ \Phi_j \subset U(0, \alpha) \times (-\beta, \beta)$ and from the properties of the flat extension operator we also have $\text{supp } \tilde{E}(u\varphi_j \circ \Phi_j) \subset U(0, \alpha) \times (-2\beta, \beta)$. Since moreover $\text{supp } \eta = U(0, \alpha) \times (-2h, \infty)$, in total

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) = \text{supp } \eta \cap \text{supp } \tilde{E}(u\varphi_j \circ \Phi_j) = U(0, \alpha) \times (-2h, \infty) \cap (-\beta, \beta),$$

and also ⁶

$$\text{supp } Eu \subset \Phi_j(\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j)).$$

We need to prove $\text{supp } Eu \subset V$, where V is some neighbourhood of Ω . If it holds

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) \subset U(0, \alpha) \times (-\beta, \beta),$$

the desired property holds, as then

$$\text{supp } Eu \subset \Phi_j(\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j)) \subset \Phi_j(U(0, \alpha) \times (-\beta, \beta)) \subset U_j \subset \bar{\Omega} \subset V.$$

In total,

$$\text{supp } \eta \tilde{E}(u\varphi_j \circ \Phi_j) = U(0, \alpha) \times (-2h, \infty) \cap (-\beta, \beta) \subset U(0, \alpha) \times (-\beta, \beta).$$

So if h is such that $-2h > -\beta \Leftrightarrow h < \frac{\beta}{2}$, we can guarantee $\text{supp } Eu \subset V$ and that is what we want.

Finally, E is clearly linear, its norm: (we are using the lemma about flat extension and the properties of C^1 diffeomorphisms together with the facts $\eta \leq 1$ on $U(0, \alpha) \times (-\beta, \beta)$, $\Phi_j(U(0, \alpha) \times (0, \beta)) \subset U_j, \Phi_j^{-1}(\mathbb{R}^d) \subset {}^7 U(0, \alpha) \times (-\beta, \beta)$, ⁸)

$$\begin{aligned} \|Eu\|_{W^{1,p}(\mathbb{R}^d)} &= \|(\eta \tilde{E}(\varphi u_j \circ \Phi_j)) \circ \Phi_j^{-1}\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|\eta \tilde{E}(\varphi u_j \circ \Phi_j)\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} = \\ &= C \|\tilde{E}(u\varphi_j \circ \Phi_j)\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} \leq C \|u\varphi_j \circ \Phi_j\|_{W^{1,p}(U(0, \alpha) \times (0, \beta))} \leq \\ &\leq C \|u\varphi_j\|_{W^{1,p}(U_j \cap \Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

from which it clearly follows $\|E\|_{\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^d))} \leq C$.

So all the properties hold for $u \in C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d)$, let us show them also for $u \in W^{1,p}(\Omega)$. Pick an arbitrary $u \in W^{1,p}(\Omega)$, find $\{u_k\} \subset C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d) : u_k \rightarrow u$ in $W^{1,p}(\Omega)$. Since E is continuous, we know $\lim_{k \rightarrow \infty} Eu_k$ exists. Let us set

$$Eu := \lim_{k \rightarrow \infty} Eu_k,$$

where Eu_k is defined above for $u_k \in C_{\bar{\Omega}}^{\infty}(\mathbb{R}^d)$.

Ad 3): Clearly $E(\alpha u) = \lim_{k \rightarrow \infty} E(\alpha u_k) = \alpha Eu$, $E(u + v) = \lim_{k \rightarrow \infty} E(u_k + v_k) = \lim_{k \rightarrow \infty} Eu_k + Ev_k = Eu + Ev$, since E on the continuous functions is linear. Also

⁶Generally, $\text{supp } v \circ \Psi^{-1} = \overline{\{x \in \mathbb{R}^d | u(\Psi^{-1}(x)) \neq 0\}} = \overline{\{x \in \mathbb{R}^d | \Psi^{-1}(x) \in \text{supp } u\}} = \overline{\{x \in \mathbb{R}^d | x \in \Psi(\text{supp } u)\}} = \Psi(\text{supp } u)$.

⁷We can somehow extend Φ_j^{-1} by zero from $U_j \subset \Omega \subset \mathbb{R}^d$ to be defined on the whole \mathbb{R}^d (i guess)

⁸...even though the assumptions to use those are not totally valid... but doc. Kaplicky is okay with that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} = \left\| \lim_{k \rightarrow \infty} Eu_k \right\|_{W^{1,p}(\mathbb{R}^d)} = \lim_{k \rightarrow \infty} \|Eu_k\|_{W^{1,p}(\mathbb{R}^d)} \leq \|E\| \lim_{k \rightarrow \infty} \|u_k\|_{W^{1,p}(\mathbb{R}^d)} = \|E\| \|u\|_{W^{1,p}(\mathbb{R}^d)},$$

(we are using $\{Eu_k\}$ has a limit); we see our above definition truly yields a continuous linear operator.

Ad 1): $\forall a.a. x \in \Omega : Eu(x) = \lim_{k \rightarrow \infty} Eu_k(x) = \lim_{k \rightarrow \infty} u_k(x) = u(x)$,

Ad 2): $\text{supp } Eu_k \subset U(\Omega, \varepsilon) \Rightarrow \text{supp } Eu \subset \overline{U}(\Omega, \varepsilon) \subset V$.

$$\text{supp } Eu = \left\{ x \in \mathbb{R}^d \mid \lim_{k \rightarrow \infty} Eu_k \neq 0 \right\} \subset \underbrace{\bigcap_{k=1}^{\infty} \text{supp } Eu_k}_{\subset V} \subset V.$$

We are done. □

Remark ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $\Omega \in C^{0,1}$ and $p \in (1, \infty)$, $k \in \mathbb{N}$, but the construction of the extension must be different. "It seems the result is not known for $\Omega \in C^{0,1}$ and $p = 1$, or $p = \infty$."

2.4 Embedding theorems

From the definition of $W^{k,p}(\Omega)$ it immediately follows $W^{k,p}(\Omega) \subset L_p(\Omega)$. Can we obtain $W^{k,p}(\Omega) \subset L_q(\Omega)$ for some $q > p$? The answer **yes**, if Ω is again nice enough (and there will also be some dependence on the dimension of \mathbb{R}^d .)

Example. Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2) ds = \int_{-\infty}^{x_2} \partial_2 u(x_1, s) ds,$$

so

$$\begin{aligned} \|u\|_{L_2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} |u(x_1, x_2)|^2 dx_1 dx_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 dx_1 dx_2 \leq \\ &\leq \left(\int_{\mathbb{R}} \int_{-\infty}^{x_1} |\partial_1 u(s, x_2)| ds dx_2 \right) \left(\int_{\mathbb{R}} \int_{-\infty}^{x_2} |\partial_2 u(x_1, s)| ds dx_2 \right) \leq \\ &\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_1 u(s, x_2)| ds dx_2 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_2 u(x_1, s)| ds dx_2 \right) \leq \\ &\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(s, x_2)| ds dx_2 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(x_1, s)| ds dx_2 \right) = \\ &= \left(\int_{\mathbb{R}^2} |\nabla u| dx \right) \left(\int_{\mathbb{R}^2} |\nabla u| dx \right) = \left(\int_{\mathbb{R}^2} |\nabla u| dx \right)^2 = \\ &= \|\nabla u\|_{L_1(\mathbb{R}^2)}^2, \end{aligned}$$

so

$$\|u\|_{L_2(\mathbb{R}^2)} \leq \|\nabla u\|_{L_1(\mathbb{R}^2)}.$$

This can be generalized in two ways:

- $d > 2$,
- less smoothness.

Lemma 6 (Gagliardo). *Let $d \geq 2$. Let $\hat{u}_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be nonnegative and measurable for $j \in \{1, \dots, d\}$. We define*

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), d\hat{x}_j = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_d.$$

Consider the functions $u_j : \mathbb{R}^d \rightarrow \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x) dx \leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} d\hat{x}_j \right)^{\frac{1}{d-1}}. \quad (1)$$

(Both integrals can be infinity.)

Proof. Induction by dimension: *The case $d = 2$:*

$$\begin{aligned} \int_{\mathbb{R}^2} u_1(x_1, x_2) u_2(x_1, x_2) dx_1 dx_2 &= \int_{\mathbb{R}^2} \hat{u}_1(x_2) \hat{u}_2(x_1) dx_1 dx_2 = \left(\int_{\mathbb{R}} \hat{u}_1(x_2) dx_2 \right) \left(\int_{\mathbb{R}} \hat{u}_2(x_1) dx_1 \right) = \\ &= \left(\int_{\mathbb{R}} \hat{u}_1(\hat{x}_1) d\hat{x}_1 \right) \left(\int_{\mathbb{R}} \hat{u}_2(\hat{x}_2) d\hat{x}_2 \right). \end{aligned}$$

(an equality in fact.) We have used Fubini once, which is permitted, as we have measurability + nonnegativity.

The case $d \rightarrow d+1$ Before we proceed, recall the "generalized Holder", all functions are non-negative

$$\int_{\Omega} \prod_{j=1}^d f_j dx \leq \prod_{j=1}^d \left(\int_{\Omega} f_j^{p_j} dx \right)^{\frac{1}{p_j}},$$

where $\sum_{j=1}^d \frac{1}{p_j} = 1$. See that if we take $p_j = d$, then $\sum_{j=1}^d \frac{1}{d} = \frac{1}{d} \sum_{j=1}^d 1 = 1$, so

$$\int_{\Omega} \prod_{j=1}^d f_j dx \leq \prod_{j=1}^d \left(\int_{\Omega} f_j^d dx \right)^{\frac{1}{d}}.$$

Let us compute:

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^d u_j \overbrace{u_{d+1}}^{=\hat{u}(\hat{x}_{d+1})} \overbrace{dx_1 \dots dx_d}^{=\hat{d}x_{d+1}} dx_{d+1} = \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}} \prod_{j=1}^d u_j dx_{d+1} \right)}_{\text{Holder}} u_{d+1} d\hat{x}_{d+1} \leq \\
&\leq \underbrace{\int_{\mathbb{R}^d} \left(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d}}}_{\text{Holder}} u_{d+1} d\hat{x}_{d+1} \leq \left(\int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \left(\int_{\mathbb{R}^d} \left(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x) dx_{d+1} \right)^{\frac{1}{d} \frac{d}{d-1}} d\hat{x}_{d+1} \right)^{\frac{d-1}{d}} \leq \\
&\leq \left(\int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d-1} \frac{d-1}{1}} d\hat{x}_j d\hat{x}_{d+1} \right)^{\frac{d-1}{d} \frac{1}{d-1}} = \\
&= \left(\int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d dx_{d+1} d\hat{x}_j d\hat{x}_{d+1} \right)^{\frac{1}{d}} = \left(\int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left(\int_{\mathbb{R}^d} u_j^d d\hat{x}_j \right)^{\frac{1}{d}} = \\
&= \prod_{j=1}^{d+1} \left(\int_{\mathbb{R}^d} u_j^d d\hat{x}_j \right)^{\frac{1}{d}},
\end{aligned}$$

where the induction step is taken for the function

$$v_j = \left(\int_{\mathbb{R}} u_j^d dx_{d+1} \right)^{\frac{1}{d-1}},$$

that is clearly nonnegative and measurable. Realize also $dx_{d+1} d\hat{x}_j d\hat{x}_{d+1} = d\hat{x}_d$, as the only "missing variable" is exactly x_d . \square

Remark. Sometimes, the lemma is stated as $\hat{u}_i \in L_\infty(\mathbb{R}^{d-1})$, $\text{supp } \hat{u}_i$ is compact $\forall i \in \{1, \dots, d\}$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d |u_j(x)| dx \leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_j(\hat{x}_j)|^{d-1} d\hat{x}_j \right)^{\frac{1}{d-1}} = \prod_{j=1}^d \|\hat{u}_j\|_{L_{d-1}(\mathbb{R}^{d-1})}.$$

The difference is that in our version, we have nonnegativity in the assumptions and do not require compact supports and essential boundedness, as we work with integrals that are possibly infinite.

Theorem 7 (Gagliardo-Nirenberg). *Let $p \in [1, d)$. Then $\forall u \in W^{1,p}(\mathbb{R}^d)$:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where $p^* = \frac{dp}{d-p}$.

Proof. Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$:

$$\forall j \in \{1, \dots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d) ds$$

independet of x_j , so

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \leq \prod_{j=1}^d \underbrace{\left(\int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds \right)^{\frac{1}{d-1}}}_{u_j \text{ independent of } x_j},$$

so the integral

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} dx \leq \int_{\mathbb{R}^d} \prod_{j=1}^d u_j dx \stackrel{\text{previous lemma}}{\leq} \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) dx_j d\hat{x}_j \right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u| dx \right)^{\frac{d}{d-1}}.$$

If $p \in (1, d)$, compute

$$\|u\|_{L^{\frac{qd}{d-1}}(\mathbb{R}^d)}^q = \| |u|^q \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1} |\nabla u| dx \stackrel{\text{Holder}}{\leq} \|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|_{L_{(q-1)p'}(\mathbb{R}^d)}^{q-1}.$$

We want $\frac{(q-1)p'}{p-1} = \frac{qd}{d-1}$, so

$$q \left(\frac{p}{p-1} - \frac{d}{d-1} \right) = \frac{p}{p-1}, \Leftrightarrow q \frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p} p.$$

Also

$$q \frac{d}{d-1} = p^*.$$

\Rightarrow statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$. \square

Remark. • It is evident that nonzero constants are not in $W^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.

- the set \mathbb{R}^d is of course unbounded, so we have no ordering of $L_p(\Omega)$ spaces.
- of course, we require no smoothness of the domain

Theorem 8. Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1, d)$ the statement of the previous theorem holds.

Proof. An immediate corollary of the previous theorem. \square

Remark. In the proof of theorem we showed that $\forall u \in W^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|_{L^{\frac{qd}{d-1}}(\Omega)}^q \leq q \|\nabla u\|_{L_p(\Omega)} \|u\|_{L^{\frac{p(q-1)}{p-1}}(\Omega)}^{q-1},$$

for q such that $\frac{qd}{d-1} \leq p^*$.

Definition 5 (Continuous & compact embeddings). Let X, Y be linear normed spaces. \hookrightarrow

Theorem 9 (Embedding theorem). *Let $\Omega \subset C^{0,1}$, $p^* = \frac{dp}{1-p}$. If $p \in [1, d)$ then*

$$W^{1,p}(\Omega) \subset L_q(\Omega) \quad \forall q \in [1, p^*].$$

Moreover, if $q < p^$, then*

$$W^{1,p}(\Omega) \subset\subset L_q(\Omega).$$

If $p = d$, then

$$W^{1,p}(\Omega) \subset L_q(\Omega) \quad \forall q < \infty, \quad W^{1,p}(\Omega) \subset\subset L_q(\Omega) \quad \forall 1 \leq q < \infty.$$

Proof. We would like to use the previous theorem + extension.

Ad continuity for $p < d$: $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ the extension is continuous. We also know

- identity $I_1 : W^{1,p}(\mathbb{R}^d) \rightarrow L_{p^*}(\mathbb{R}^d)$ is continuous,
- restriction $I_2 : L_{p^*}(\mathbb{R}^d) \rightarrow L_{p^*}(\Omega)$ is continuous,
- identity $I_3 : L_{p^*}(\Omega) \rightarrow L_q(\Omega)$ is continuous.

Together, the mapping $id : W^{1,p}(\Omega) \rightarrow L_q(\Omega)$, $id = I_3 \circ I_2 \circ I_1 \circ E$ identity is continuous. If $p=d$, then $W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \quad \forall r \in [1, d)$, and $r^* \rightarrow \infty$ as $r \rightarrow d^-$. For $q \in [1, \infty)$ find $r \in [1, d)$ s.t. $r^* > q$. Then

$$W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \subset L_{r^*}(\Omega) \subset L_q(\Omega),$$

using the previous results.

Ad compactness: We show $W^{1,p}(\Omega) \subset\subset L_q(\Omega)$ using Arzela-Ascoli and then it will get technical: show compactness in smooth functions, then show compactness in $L_1(\Omega)$, then approximate the norm of $L_q(\Omega)$ using the obtained quantities.

Consider $B = \overline{U_{W^{1,p}(\Omega)}(0,1)}$ and extend it to EB . Fix $\delta > 0$ and let η be a regularization kernel. Then $\exists R > 0 : \text{supp}(EB)_\delta \subset \overline{U(0,R)} \subset \mathbb{R}^d$ (i.e. all the functions from EB have the support contained in the ball). Moreover, $(EB)_\delta \subset C^1(\overline{U(0,R)})$. Actually, it is bounded in $C^1(\overline{U(0,R)})$. $\underbrace{\subset\subset}_{\text{Arzela-Ascoli}} C(\overline{U(0,R)})$ (uniform equicontinuity comes from uniform boundedness of the gradients, $\nabla(u * \eta_\delta) = u * \nabla\eta_\delta$.) Altogether $(EB)_\delta$ is relatively compact in

$$C(\overline{U(0,R)}) \xrightarrow[\text{the space } C(\overline{U(0,R)}) \text{ is complete}]{\Rightarrow} \text{bounded in } C(\overline{U(0,R)}) \xrightarrow[\text{bounded domain}]{\Rightarrow} \text{bounded in } L_1(U(0,R)).$$

Next, take

$$\begin{aligned} u \in B : \|u - (Eu)_\delta\|_{L_q(\Omega)} &\leq \|Eu - (Eu)_\delta\|_{L_q(U(0,R))} = \int_{U(0,R)} |v - v_\delta| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} v(x+y) - v(x) \eta_\delta(y) dy \right| dx \leq \\ &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} |\eta_\delta(y)| |y| dy \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy}_{\text{Fubini}}. \end{aligned}$$

Estimate the inner integral: assume v is smooth and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|} \left| \int_0^1 \underbrace{\frac{d}{ds}(v(x+sy))}_{\nabla v(x+sy) \cdot y} ds \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_0^1 |\nabla v|(x+sy) ds dx}_{\text{Cauchy Schwartz}} \leq \underbrace{C(R) \left(\int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}}_{\text{Holder}}.$$

Now, take $v \in W_0^{1,p}(U(0, R))$, then $\exists \{v_k\} \subset \mathcal{D}(U(0, R)) : v_k \rightarrow v$ in $W^{1,p}(U(0, R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} dx \leq C(R) \left(\int_{\mathbb{R}^d} |\nabla v_k|^p dx \right)^{\frac{1}{p}} \rightarrow C(R) \left(\int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}.$$

So finally

$$\|u - (Eu)_\delta\|_{L_q(\Omega)} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy \leq \underbrace{C(R) \delta}_{|y| \leq \delta} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}} dx \leq C_1 \delta.$$

Fix $\varepsilon > 0$, find finite $\frac{\varepsilon}{2}$ -net in $(EB)_\delta$ in $L_1(U(0, R))$ (that is possible since we have total boundedness in $L_1(U(0, R))$.) Set $\delta > 0$ s.t. $C_1 \delta \leq \frac{\varepsilon}{4}$.⁹ Denote the $\frac{\varepsilon}{2}$ -net as $\{Eu_k\}_{k=1}^m, m \in \mathbb{N}$. We show $\{u_k\}_{k=1}^m$ is a ε -net in B . Fix $u \in B$, find $j \in \{1, \dots, m\} : \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(U(0, R))} \leq \frac{\varepsilon}{2}$. Compute

$$\|u - u_j\|_{L_1(\Omega)} \leq \|u - (Eu)_\delta\|_{L_1(\Omega)} + \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(\Omega)} + \|(Eu_j)_\delta - u_j\|_{L_1(\Omega)} \leq 2C_1 \delta + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus, we have shown

$$W^{1,p}(\Omega) \subset\subset L_1(\Omega).$$

It remains to show the validity for a general q . Let $q \in [1, p^*) : \|v\|_{L_q(\Omega)} \leq \|v\|_{L_1(\Omega)}^\alpha \|v\|_{L_{p^*}(\Omega)}^{1-\alpha}$, for $\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}, \alpha \in (0, 1]$. Is B totally bounded in $L_q(\Omega)$? Let us compute

$$\|u - u_j\|_{L_q(\Omega)} \leq \|u - u_j\|_{L_1(\Omega)}^\alpha \underbrace{\|u - u_j\|_{L_{p^*}(\Omega)}^{1-\alpha}}_{\leq C, W^{1,p}(\Omega) \subset L_{p^*}(\Omega)} \leq C\varepsilon^\alpha.$$

□

2.5 Trace theorems

2.6 Composition of sobolev functions

2.7 Difference quotients

3 Nonlinear elliptic equations as compact perturbations

Theorem 10 (Nemytskii). *Let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}, N \in \mathbb{N}, \Omega \subset \mathbb{R}^d$ measurable, f Caratheodory. Then*

1. *if $u : \Omega \rightarrow \mathbb{R}^N$ is measurable then $f(\cdot, u)$ is also measurable*

⁹The order of the choices is not precise...

2. If there is $p_i \in [1, +\infty), i \in \{1, \dots, N\}, q \in [1, \infty), g \in L_q(\Omega), C > 0$ such that for almost all

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x, y)| \leq g(x) + c \sum_{i=1}^N |y_i|^{p_i/q}$$

, then $u \mapsto f(\cdot, u)$ is continuous from $L_{p_i}(\Omega) \times \dots \times L_{p_N}(\Omega)$ to $L_q(\Omega)$. Moreover, it maps bounded sets to bounded sets.

Proof. No proof □

Definition 6 (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2). Let X, Y be normed linear spaces, $M \subset X$. The mapping $F : M \rightarrow Y$ is called a compact operator on M into Y if F is continuous and $F(M \cap K)$ is relatively compact in Y for any bounded $K \subset X$.

Remark. We have no linearity of F ! So continuity cannot follow from compactness (we have compactness \Rightarrow boundedness \neq continuity for nonlinear operators)

Theorem 11 (Brouwer fixed point theorem). Let $K \subset \mathbb{R}^N, N \in \mathbb{N}$ be a nonempty convex closed bounded. Assume that $F : K \rightarrow K$ is continuous. Then F has a fixed point in K , i.e.,

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. No proof □

Theorem 12 (Schauder fixed point theorem). Let $K \subset X$ be a nonempty convex closed bounded subset of a linear normed space X . Assume that F is compact on K into K and $F(K) \subset K$. Then there is fixed point of F in K .

Proof. No proof □

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bounded, it is automatically compact, and since $F : K \rightarrow K$ is continuous, F is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with $N=1$ is easy, based on Darboux property.

3.0.1 Problem prototypes

In this chapter some nonlinear elliptic equations are discussed.

Example. Suppose the following problem:

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$g : \mathbb{R} \rightarrow \mathbb{R}, f \in (W_0^{1,2}(\Omega))^*, \text{ continuous, } \exists \alpha \in [0, 1) : \forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

Theorem 13 (Existence). Let $\Omega \in C^{1,1}, f \in (W_0^{1,2}(\Omega))^*, g$ is as above. Then there is a weak solution to the above problem, i.e., it holds:

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*}.$$

If $f \in L_2(\Omega)$, then the solution $u \in W^{2,2}(\Omega)$.

Proof. We define $S : L_2(\Omega) \rightarrow L_2(\Omega)$ such that

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx.$$

S is well defined:

$$|\text{RHS}| \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W^{1,2}(\Omega)} + \|\varphi\|_{L_2(\Omega)} \|g(w)\|_{L_2(\Omega)},$$

and

$$\int_{\Omega} |g(w)|^2 \, dx \leq \int_{\Omega} C(1 + |w|^\alpha)^2 \, dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) \, dx \leq \int_{\Omega} C(1 + |w|^2) \, dx \leq \infty,$$

where we used the Young inequality and $\alpha \leq 1$. We have thus shown the mapping $w \mapsto g(w)$ is continuous from $L_2(\Omega)$ to $L_2(\Omega)$ by Nemytskii. Next, S is continuous:

- $w \mapsto g(w)$ is continuous from $L_2(\Omega)$ to $L_2(\Omega)$
- $w \mapsto (\varphi W_0^{1,2}(\Omega) \rightarrow \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx)$ is continuous from $L_2(\Omega)$ to $(W_0^{1,2}(\Omega))^*$
- $F \rightarrow u$, where u is the weak solution of

$$\begin{cases} -\Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

, is linear and continuous from $(W_0^{1,2}(\Omega))^*$ to $W_0^{1,2}(\Omega)$.

In total, the composition is continuous and yields S . Next, we would like to show S is compact. We start with showing S maps bounded sets in $L_2(\Omega)$ to bounded sets in $W_0^{1,2}(\Omega)$; for that we need apriori estimates: test the weak formulation with u :

$$\|\nabla u\|_{L_2(\Omega)}^2 \leq \varepsilon \|u\|_{W^{1,2}(\Omega)}^2 + C \left(\|f\|_{(W^{1,2}(\Omega))^*}^2 + \|g(w)\|_{L_2(\Omega)}^2 \right) \underset{\text{Young}}{\leq} C \left(\|f\|_{(W_0^{1,2}(\Omega))^*} + 1 + \|w\|_{L_2(\Omega)}^2 \right),$$

from which follows S is compact from $L_2(\Omega)$ to $L_2(\Omega)$ by compact embedding. Now we need to show $S(U(0, R)) \subset U(0, R)$ for some $R > 0$. From the previous we know:

$$\frac{C}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \left(\|f\|_{(W_0^{1,2}(\Omega))^*} + \|g\|_{L_2(\Omega)}^2 \right),$$

so since

$$\tilde{C} \int_{\Omega} |g(w)|^2 \, dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) \, dx \underset{\text{Young}}{\leq} \int_{\Omega} \left(C + \frac{c}{4} |w|^2 \right) \, dx$$

we know

$$\frac{c}{2} \|u\|_{L_2(\Omega)}^2 \leq \frac{c}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + C\lambda(\Omega) + \frac{c}{4} \|w\|_{L_2(\Omega)}^2,$$

and thus

$$\|u\|_{L_2(\Omega)}^2 \leq \underbrace{\frac{2\tilde{C}}{c} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + \frac{2C}{c}}_{=\overline{C}} + \frac{1}{2} \|w\|_{L_2(\Omega)}^2.$$

so if $\overline{C} + \frac{1}{2}R^2 < R^2$, we are done ¹⁰. But such an R of course exists (says doc. Kaplicky) \Rightarrow the image of a ball is in a ball for some $R \Rightarrow S$ is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that u_0 solves
$$\begin{cases} -\Delta u_0 = f - g(u_0) \in L_2(\Omega) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

So from the regularity theory for elliptic equations we get

$$u \in W^{2,2}(\Omega).$$

□

Theorem 14 (Uniqueness). *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be weak solutions to the above problem. Let $f \in (W_0^{1,2}(\Omega))^*$, g be continuous. Let either*

1. *g is nondecreasing*
2. *$g \in C^1(\mathbb{R})$, $\|g'\|_\infty$ small.*

Then $u_1 = u_2$.

Proof. We subtract the equations for u_1, u_2 and test with $u_1 - u_2$:

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2) dx = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 \geq C \|u_1 - u_2\|_{W^{1,2}(\Omega)}^2 \Rightarrow u_1 - u_2 = 0.$$

$$|\int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2) dx| \leq \int_{\Omega} \|g'\|_\infty |u_1 - u_2|^2 dx \leq \|g'\|_\infty C_P \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 = 0 \Rightarrow u_1 = u_2,$$

whenever $C\|g'\|_\infty < 1$. □

Example. Suppose the following problem

$$\begin{cases} -\Delta u + b(\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f \in (W_0^{1,2}(\Omega))^*$, b is continuous and bounded. The weak formulation is

$$u \in W_0^{1,2}(\Omega) \wedge \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi + b(\nabla u) \varphi dx = \langle f, \varphi \rangle,$$

and the first apriori estimates (test with u)

$$\|\nabla u\|_{L_2(\Omega)} \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|u\|_{W_0^{1,2}(\Omega)} + \int_{\Omega} |u| dx \|b\|_{L_\infty(\Omega)}.$$

¹⁰The constants are most probably messed up.

Theorem 15. Let $f \in (W_0^{1,2}(\Omega))^*$, $\Omega \in C^{0,1}$, $b : \mathbb{R}^d \rightarrow \mathbb{R}$ continuous and bounded. Then there is a weak solution to the above problem.

Proof. $S : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$, $Sw = u$ iff u solves

$$\begin{cases} -\Delta u = f - b(\nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} b(\nabla w) \varphi \, dx.$$

Clearly, S is well defined and

$$\|Sw\|_{W_0^{1,2}(\Omega)} \leq C \underbrace{\left(\|f\|_{(W_0^{1,2}(\Omega))^*} + \|b\|_{L^\infty(\Omega)} \right)}_{:=R},$$

meaning $S(\overline{U(0,R)}) \subset \overline{U(0,R)}$. Moreover, S is continuous, as S is the composition of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show S is compact: we already have continuity, consider $\{w_k\}_{k \in \mathbb{N}} \subset W_0^{1,2}(\Omega)$ bounded. Then $\exists \{u_k\} \subset W_0^{1,2}(\Omega)$ bounded: $u_k \rightarrow u$ in $L_1(\Omega)$ by embedding up to a subsequence. Next, use the following trick: substitute equation for u_k from equation for u_l and test with $u_l - u_k$

$$C \|u_l - u_k\|_{W_0^{1,2}(\Omega)}^2 \leq \|\nabla(u_l - u_k)\|_{L_2(\Omega)}^2 \leq \int_{\Omega} |b(\nabla u_l) - b(\nabla u_k)| |u_l - u_k| \, dx \leq 2 \|b\|_{L^\infty(\Omega)} \|u_l - u_k\|_{L_1(\Omega)}.$$

All in all, S has a fixed point by Schauder, which is of course the weak solution. \square

But this says $\{u_k\}$ is Cauchy in $W_0^{1,2}(\Omega)$.

4 Nonlinear elliptic equations - monotone operator theory

Lemma 7. Let $g : B(0,R) \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$ be continuous, $N \in \mathbb{N}$, $R > 0$, and $\forall c \in S(0,R) : g(c) \cdot c \geq 0$. Then, there is $c_0 \in B(0,R) : g(c_0) = 0$.

Proof. By contradiction. Let $g \neq 0$ in $U(0,R)$. Let us define

$$h(x) = -R \frac{g(x)}{|g(x)|}.$$

Then $h \in C(\overline{B(0,R)})$, $h(B(0,R)) \subset S(0,R)$, so by Brouwer there $\exists x_0 \in B(0,R) : h(x_0) = x_0 \Rightarrow -R \frac{g(x_0)}{|g(x_0)|} = x_0$. Take the dot product with x_0 and write

$$\underbrace{-R \frac{g(x_0) \cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in S(0,R),$$

so that is a contradiction. \square

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i (a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x))) = f(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

The data are

- $\Omega \in C^{0,1}$,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, i \in \{1, \dots, d\}$ are Caratheodory in x and $(u, \nabla u)$.
- $f \in (W_0^{1,r}(\Omega))^*$,

and the unknown is $u : \Omega \rightarrow \mathbb{R}$.

Remark. The function $(u, p) \mapsto a_i(\cdot, u, p)$ is continuous from $(L_r(\Omega))^{d+1}$ to $L_{r'}(\Omega)$ by Nemystkii theorem.

Definition 7 (Coercivity). We say that $\{a_i\}_{i=0}^d$ are coercive if $\exists C_1 > 0, C_2 \in L_1(\Omega) : \text{a.e. } x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} :$

$$\sum_{i=1}^d a_i(x, z, p) p_i + a_0(x, z, p) \geq C_1 |p|^r - C_2(x), \text{ i.e. } a(x, z, p) \cdot p \geq C_1 |p|^r - C_2(x)$$

Definition 8 (Monotonicity). We say that $\{a_i\}_{i=0}^d = a$ is monotone if for almost all

$$x \in \Omega, \forall (z_1, p_1), (z_2, p_2) \in \mathbb{R}^{d+1} : (a(x, z_1, p_1) - a(x, z_2, p_2)) \cdot (p_1 - p_2) + (a_0(x, z_1, p_1) - a_0(x, z_2, p_2)) \cdot (z_1 - z_2) \geq 0.$$

Very similiarly we define strict monotonicity.

Definition 9 (Weak solution). We say that $u \in W^{1,r}(\Omega)$ is a weak solution to the above problem if

- $u = u_0$ in the sense of traces on $\partial\Omega$,

•

$$\int_{\Omega} a(\cdot, u, \nabla u) \cdot \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle, \forall \varphi \in W_0^{1,r}(\Omega).$$

Theorem 16 (Existence and uniqueness). *Let $\Omega \in C^{0,1}, u_0 \in W^{1,r}(\Omega), r \in (1, \infty), \{a_i\}_{i=1}^d$ be Caratheodory, coercive and monotone and let them also satisfy the growth conditions. Finally, let $f \in (W^{1,r}(\Omega))^*$. Then, there is a weak solution to the problem. If, moreover, $\{a_i\}_{i=1}^d$ is strictly monotone, then the weak solution is unique.*

Proof. The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

Galerkin: Since $W_0^{1,r}(\Omega)$ is separable $\Rightarrow \exists \{w_i\}_{i=1}^\infty$ that is a dense¹¹ linearly independent subset of $W_0^{1,r}(\Omega)$. We search for $n \in \mathbb{N}$ such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where $\alpha_j \in \mathbb{R}$ and u^n satisfy

$$\forall j \in \{1, \dots, n\} : \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla w_j + a_0(\cdot, u^n, \nabla u^n) w_j \, dx = \langle f, w_j \rangle.$$

We claim such $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$ exist $\forall n \in \mathbb{N}$ by the previous lemma. We define a vector function

$$F(\alpha^n) := \left\{ \int_{\Omega} a \cdot \nabla w_j + a_0 w_j \, dx - \langle f, w_j \rangle \right\}_{j=1}^n,$$

from Nemystkii $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F is continuous on \mathbb{R}^n . Moreover, it holds

$$\begin{aligned} F(\alpha^n) \cdot \alpha^n &\geq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \nabla(u^n - u_0) + a_0(u^n - u_0) \, dx - \langle f, u^n - u_0 \rangle \\ &\stackrel{\text{coercivity}}{\geq} \int_{\Omega} C_1 |\nabla u^n|^r - (C_2(\cdot) + |a| |\nabla u_0| + |a_0| |u_0|) \, dx - \|u^n\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*} - \|u_0\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*}, \end{aligned}$$

together with the fact

$$\|\nabla u^n\|_{L_r(\Omega)}^r \geq \left(\|\nabla(u - u_0)\|_{L_r(\Omega)} - \|\nabla u_0\|_{L_r(\Omega)} \right)^r \geq \|\nabla(u^n - u_0)\|_{L_q(\Omega)}^r - \|\nabla u_0\|_{L_r(\Omega)}^r \geq C \|u^n - u_0\|_{W^{1,r}(\Omega)}^r - \|\nabla u_0\|_{L_r(\Omega)}^r,$$

Next, realize that $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{W^{1,r}(\Omega)}$ is a norm equivalent to $|\alpha^n|$ (Euclidian norm). So that means $\exists K_1(n) > 0 : \forall \alpha \in \mathbb{R}^n : K_1(n) |\alpha^n| \leq \|u^n - u_0\|_{W^{1,r}(\Omega)}$. For $|\alpha^n| = R, R > 0$ determined later estimate $F(\alpha^n) \cdot \alpha^n \geq c \|u^n - u_0\|_{W^{1,r}(\Omega)} - \tilde{c} \left(\|\nabla u_0\|_{L_r(\Omega)}^r + 1 + \|u_0\|_{L_r(\Omega)}^r + \|f\|_{(W_0^{1,r}(\Omega))^*}^{r'} \right)$ (which is not a trivial computation). And so $\exists R > 0, \forall \alpha^n \in S(0, R) \subset \mathbb{R}^n : F(\alpha^n) \cdot \alpha^n > 0$, so from the previous lemma $\exists \alpha^n \in S(0, R) : F(\alpha^n) = 0$, and we fix these α^n . **Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|_{W^{1,r}(\Omega)}^r \leq C \left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

and

$$\|u^n\|_{W^{1,r}(\Omega)} \leq C \left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

$$\forall j \in \{0, \dots, d\} : \|a_j(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}^{r'} \leq C \left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*} \right),$$

Limit passage From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \text{ in } W^{1,r}(\Omega), a_j \rightharpoonup \alpha_j \text{ in } L_{r'}(\Omega).$$

¹¹It can be chosen such that it is itself dense, not only its span

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_{\Omega} \alpha \cdot \nabla w_j + \alpha_0 w_j \, dx = \langle f, w_j \rangle,$$

and from density of $\{w_j\}_{j \in \mathbb{N}}$ in $W^{1,r}(\Omega)$ we have

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} \alpha \cdot \nabla \varphi + \alpha_0 \varphi \, dx = \langle f, \varphi \rangle.$$

Identification of α 's We want to show $\alpha_j = a_j(\cdot, u, \nabla u), j \in \{0, \dots, d\}$. For that, we use the *Minty trick*:

$$\begin{aligned} 0 &\leq \int_{\Omega} (a(\cdot, u^n, \nabla u^n) - a(\cdot, v, V)) \cdot (\nabla u^n - V) + (a_0(\cdot, u^n, \nabla u^n) - a_0(\cdot, v, V)) \cdot (u^n - v) \\ &\leq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla u^n + a_0(\cdot, u^n, \nabla u^n) \cdot u^n \, dx + \\ &\quad - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) V + a_0(\cdot, u^n, \nabla u^n) v - a(\cdot, v, V) + a_0(\cdot, v, V)) \cdot (u^n - v) \, dx. \end{aligned}$$

Denote

$$I^n = \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla (u^n - u_0) + a_0(\cdot, u^n, \nabla u^n) \cdot (u^n - u_0) \, dx + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx,$$

by using the equation we obtain

$$I^n = \langle f, u^n - u_0 \rangle + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx = \langle f, u - u_0 \rangle + \int_{\Omega} \alpha \nabla u_0 + \alpha_0 u_0 \, dx = \int_{\Omega} \alpha \nabla u + \alpha_0 u \, dx,$$

as the rest has subtracted. In total, we have

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, v, V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot, v, V)) (u - v) \, dx.$$

So far, v, V have been arbitrary. If we take

$$V = \nabla u - \lambda \psi, \psi \in L_r(\Omega), v = u,$$

then $0 \leq \int_{\Omega} (\alpha - a(\cdot, \nabla u + \lambda \psi)) \lambda \psi \, dx$, so if we take $\lambda > 0$ and pass to the limit $\lambda \rightarrow 0_+$ (using Nemytskii theorem) we can write

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx.$$

Since ψ was arbitrary, we could have taken $\psi \rightarrow -\psi$, which in total means

$$0 = \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx$$

Finally, from the previous results, we obtain

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle,$$

and we are almost done. It only remains to show the traces are correct, bt since $u^n \rightharpoonup u$ in $W^{1,r}(\Omega)$ and from the continuity of the traces, we obtain

$$\text{tr } u = \text{tr } u_0.$$

Uniqueness: Let u_1, u_2 be two solutions. Use strict monotonicity, subtract the weak formulation and test with $u_2 - u_1$:

$$\int_{\Omega} \underbrace{(a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla (u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)}_{:=T} dx = 0,$$

where $T \geq 0$, so from strict monotonicity we obtain $T = 0$ a.e. in Ω but that means $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$, a.e. in $\Omega \Rightarrow u_1 = u_2$ in $W^{1,r}(\Omega)$. \square

Example (Nonlinearities vs weak convergence). Let $g_n(x) = \sin(nx)$, then $g \rightharpoonup 0$ in $L_2((0,4))$ (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx) \varphi dx \geq \int_2^4 \sin^2(nx) dx \rightarrow \frac{1}{2} \neq 0, \forall \varphi \in L_2((0,4)),$$

so $\{u_n^2\} = \{\sin^2(nx)\}$ **does not converge weakly to** $0 = 0^2$.

Remark. The method of the presented proof is **very important**.

Theorem 17. Let $\Omega \in C^{0,1}$. Let $X = W_0^{1,r}(\Omega)$, $r \in (1, \infty)$ with equivalent norm $\|u\| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$. Then

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t. : } \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_{\Omega} \mathbf{F} \cdot \nabla \varphi dx, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

Proof. We solve the problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{r-2} \nabla u) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}. \quad (2)$$

Such $u \in W_0^{1,r}(\Omega)$ exists and is unique by the above theorem. In this case: $a(x, z, p) = |p|^{r-2}p$, $a_0() = 0$. Coercivity is clear, monotonicity will be shown in the tutorials¹². Write the weak formulation of the above problem:

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi dx = \Phi(\varphi).$$

Set $\mathbf{F} = |\nabla u|^{r-2} \nabla u$, and test the weak formulation with u itself:

$$\|\nabla u\|_{L_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{L_r(\Omega)}.$$

If now $\|\nabla u\|_{L_r(\Omega)} = 0$, then $\Phi = 0$ and we are finished, if it is nonzero, then

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

¹²This was a lie

Realize now

$$\|\nabla u\|_{L^r(\Omega)}^{r-1} = \||\nabla u|^{r-1}\|_{L^{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{L^{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{L^{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{B_X(0,1)} |\Phi(\varphi)| = \sup_{B_X(0,1)} \left| \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx \right| \leq \sup_{B_X(0,1)} \|\mathbf{F}\|_{L^{r'}(\Omega)} \|\nabla \varphi\|_{L^r(\Omega)} = \|\mathbf{F}\|_{L^{r'}(\Omega)},$$

so $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{L^{r'}(\Omega)}$. □

5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset W^{1,r}(\Omega) \rightarrow \mathbb{R}, u \mapsto \int_{\Omega} F(\cdot, u, \nabla u) \, dx,$$

with the basic assumptions $\Omega \in C^{0,1}$, $r \in (1, \infty)$, $X = u_0 + W_0^{1,r}(\Omega)$, $u_0 \in W^{1,r}(\Omega)$, $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in L_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x, z, p) \geq C_1 |p|^r - c_2(x).$$

Remark. From the assumptions it follows $\int_{\Omega} F(\cdot, u, \nabla u) \, dx$ is defined $\forall u \in W^{1,r}(\Omega)$.

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

Lemma 8. *Let $\Omega \in C^{0,1}$, $r \in (1, \infty)$, $X = u_0 + W_0^{1,r}(\Omega)$, $u_0 \in W^{1,r}(\Omega)$, F Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall a, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x, z, p)| + |\partial_z F(x, z, p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x, \cdot, \cdot) \in C^1(\mathbb{R}^{d+1}).$$

Let now $u \in u_0 + W_0^{1,r}(\Omega)$ be a local minimizer of I over X , i.e.,

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_{\Omega} F(\cdot, u, \nabla u) \, dx \leq \int_{\Omega} F(\cdot, u+v, \nabla(u+v)) \, dx, F(\cdot, u, \nabla u) \in L_1(\Omega).$$

Then u is the weak solution to the **Euler-Lagrange equations**:

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u) + \partial_z F(\cdot, u, \nabla u)) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega \end{cases},$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi + \partial_z F(\cdot, u, \nabla u) \varphi \, dx = 0, \text{tr } u = \text{tr } u_0 \text{ on } \partial\Omega.$$

Proof. First $\text{tr } u = \text{tr } u_0$ holds, so we are fine. Now fix $\varphi \in \mathcal{D}(\Omega)$ and define

$$\iota : \mathbb{R} \rightarrow \mathbb{R}^*, \iota(\tau) = \int_{\Omega} \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:= \iota(\tau, \cdot)} \, dx.$$

Now ι has a local minimum in 0. We show that $\iota'(0)$ exists and is equal to the of Euler-Lagrange equations.

- $l(\tau, \cdot)$ is measurable for τ from some neighbourhood of 0.
- $l(\tau, \cdot)$ is differentiable

Moreover

$$\begin{aligned}\partial_\tau l(\tau, \cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi = \\ &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi.\end{aligned}$$

Also

$$i(0) = \int_{\Omega} F(\cdot, u, \nabla u) dx < \infty$$

and

$$|\partial_\tau l(\tau, \cdot)| \leq (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla \varphi|^r) + |h(x)|)(|\varphi| + |\nabla \varphi|) \in L_1(\Omega).$$

Altogether, $\iota(\tau)$ is finite on $(-1, 1)$, $\iota'(\tau)$ exists and

$$\iota'(0) = \int_{\Omega} \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi dx.$$

□

Example. Let

$$F(x, z, p) = \frac{1}{r}(1 + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x, z, p) = \left(\frac{r}{2} \frac{1}{r} 2(1 + |p|^2)^{\frac{r-2}{2}}\right)p - G = (1 + |p|^2)^{\frac{r-2}{2}}p - G, \quad \partial_z F(x, z, p) = -g.$$

We have

$$|(1 + |p|^2)^{\frac{r-2}{2}}p| \leq (1 + |p|^2)^{\frac{r-2}{2}}(1 + |p|^2)^{\frac{1}{2}} = (1 + |p|^2)^{\frac{r-1}{2}} \leq C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left((1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \right) = -\nabla \cdot G + g, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} (G \cdot \nabla \varphi + g\varphi) dx.$$

Remark. We have $\{u_n\} \subset X$ s.t. $\lim_{n \rightarrow \infty} I(u_n) = \inf_X I$. Then use

- compactness: $u_n \rightarrow u$ in some sense (weak convergence)
- weak lower semicontinuity $I(u) \leq \liminf_{n \rightarrow \infty} I(u_n)$

Lemma 9. Let $F : \mathbb{R}^N \rightarrow \mathbb{R}, F \in C^1(\mathbb{R}^N), N \in \mathbb{N}$. Then

1. F is (strictly) convex $\Leftrightarrow \nabla F$ is (strictly) monotone

2. If F is (strictly) convex, then

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2 : F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

Proof. Fix $\xi_1, \xi_2, \xi_1 \neq \xi_2$, define $\varphi(t) = F(\xi_2 + t(\xi_1 - \xi_2))$. Then $\varphi \in C^1(\mathbb{R})$ and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1 - \xi_2)) \cdot (\xi_1 - \xi_2).$$

So

$$" \Rightarrow " : (\nabla F(\xi_1) - \nabla F(\xi_2)) \cdot (\xi_1 - \xi_2) = \varphi'(1) - \varphi'(0) \underset{\varphi \text{ convex or strictly convex}}{\geq} 0.$$

And " \Leftarrow ": Fix $t_1 > t_2$ and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1 - \xi_2)) - \nabla F(\xi_2 + t_2(\xi_1 - \xi_2))) \cdot (\xi_1 - \xi_2)(t_1 - t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1 - \xi_2)(t_1 - t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2)) \cdot (\eta_1 - \eta_2)$$

and we are in the same situation as before. For 2) we already know F (strictly) convex $\Rightarrow \varphi$ (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \rightarrow \varphi'(0),$$

as $t \rightarrow 0_+$. And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

Theorem 18. Let $M, N \in \mathbb{N}, \Omega$ open, $F : \Omega \times \mathbb{R}^{N+M} \rightarrow \mathbb{R}$ Caratheodory, F convex in $p \in \mathbb{R}^n$, i.e. $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot)$ is convex and $\exists c_2 \in L_1(\Omega), \forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M, \forall p \in \mathbb{R}^N : F(x, z, p) \geq c_2(x)$. Let $u_n \rightarrow u$ in $L_1(\Omega), U_n \rightarrow U$ in $L_1(\Omega)$. Then

$$\int_{\Omega} F(\cdot, u, U) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} F(\cdot, u_n, U_n) dx.$$

Proof. The proof will be given only if moreover $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot) \in C^1(\mathbb{R}^N)$. Idea: by the previous lemma:

$$\int_{\Omega} F(\cdot, u_n, U_n) dx \geq \int_{\Omega} (F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U) \cdot (U_n - U)) dx,$$

and we have uniform convergence in the first term and second term and weak convergence in $L_1(\Omega)$ in the last term. If Ω is bounded, we can find $K_k \subset K_{k+1} \subset \Omega$ s.t. $\lambda(\Omega \setminus \bigcup_{k \in \mathbb{N}} K_k) = 0$, and moreover $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$ are compact, $u_n \rightarrow u$ on $K_k, \|u\|_{L_{\infty}(K_k)} + \|U\|_{L_{\infty}(K_k)} \leq k$ up to a subsequence. We can now extract a subsequence $u_n \rightarrow u$ a.e. and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E}_k \text{ s.t. } u_n \rightarrow u \text{ on } \tilde{E}_k \wedge \lambda(\Omega \setminus \tilde{E}_k) < \frac{1}{k}.$$

Now define

$$\hat{E}_k = \bigcup_{j=1}^k \tilde{E}_j, E_k = \hat{E}_k \cap \{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{k}\},$$

and E_k satisfy ¹³

$$\lambda\left(\Omega \bigcup_k E_k\right) = 0.$$

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \leq k \wedge |U(x)| \leq k\}$$

and we also have $\lambda(\Omega \bigcup_k F_k) = 0$. FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\left(\Omega \bigcup_k K_k\right) = 0.$$

□

Remark. • if $U_n \rightarrow U$ strongly $\Rightarrow u_n \rightarrow u, U_n \rightarrow U$ a.e. (up to a subsequence) and the claim follows from the Fatou lemma. ¹⁴

- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L_p(\Omega) \Rightarrow \int_{\Omega} |\nabla u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p dx.$$

Lemma 10 (Arzela-Ascoli). *Let X, Y be Banach spaces, $X \subset\subset Y$. Then*

$$C^1([0, T]; X) \subset\subset C([0, T]; Y).$$

Lemma 11 (Ehrling). *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3$. Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + C \|u\|_{V_3}.$$

Proof. By contradiction, assume

$$\exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{N} \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n \|u_n\|_{V_3}.$$

WLOG we can assume $\{u_n\} \subset S_{V_2}(0, 1)$: truly, the inequality is 1-homogenous and holds if $u_n = 0$. In particular we see $\|u_n\|_{V_3} < \frac{1}{n}$, so $u_n \rightarrow 0$ in V_3 . Moreover, $\{u_n\}$ is bounded in V_1 and since $V_1 \subset\subset V_2$ there exists $\{u_{n_k}\} \subset \{u_n\}$ s.t.: $u_{n_k} \rightarrow u$ in V_2 strongly. Since $\{u_n\} \subset S_{V_2}(0, 1)$, also $\|u\|_{V_2} = 1$. Finally, taking the limit passage yields $0 \geq \|u\|_{V_3}$ and so $u = 0$ in V_3 and also in V_2 . But that is a contradiction with the fact $\{u_n\} \subset S_{V_2}(0, 1)$. □

Theorem 19 (Aubin-Lions). *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3, p \in [1, \infty)$. Then the space*

$$\mathcal{U} = \{u \in L_p((0, T); V_1), \partial_t u \in L_1((0, T); V_3)\},$$

¹³"This is homework", says doc. Kaplicky

¹⁴For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions $F - c_2 \geq 0, F - c_2 \in L_1(\Omega)$

with the norm

$$\|u\| = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

satisfies

$$\mathcal{U} \subset\subset L_p((0,T);V_2).$$

Proof. Strategy: I want to fix $M \subset \mathcal{U}$ bounded and show that it is precompact in $L_p((0,T);V_2)$. That will be done in the following way:

1. Mollify M by convolution
2. use Arzela-Ascoli
3. show compactness in $L_p((0,T);V_3)$
4. apply Ehrling lemma and show compactness in $L_p((0,T);V_2)$.

Fix $M \subset \mathcal{U}$ bounded. Then $\exists C^* > 0 : \forall u \in M : \|u\| \geq C^*$.

Next, take

$$\varphi : \mathbb{R} \rightarrow [0, \infty), \varphi \in C^\infty(\mathbb{R}), \text{supp } \varphi \subset (-1, 0), \int_{\mathbb{R}} \varphi \, dx = 1,$$

a regularization kernel, then $\forall \delta > 0$ define $\varphi_\delta(t) := \frac{1}{\delta} \varphi(\frac{t}{\delta})$.

Now, extend functions from M to $(0, 2T)$ in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0, T) \\ u(2T - t), & t \in (T, 2T) \end{cases}.$$

Now mollify: for $\delta > 0, \delta < T$ fixed define

$$M_\delta = \{ (\tilde{u} \star \varphi_\delta) \Big|_{(0,T)} \mid u \in M \}.$$

From the properties of regularization it follows $M_\delta \subset C^1([0, T]; V_1) \underbrace{\subset\subset}_{\text{A.A.}} C([0, T]; V_2) \subset L_p((0, T); V_2)$.

Now estimate the distance of M and M_δ in $L_p((0, T); V_3)$: for

$$u \in M, t \in (0, T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^0 \tilde{u}(t-s) \varphi_\delta(s) \, ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \varphi_\delta(s) \, ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \frac{d}{ds} \int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) \, d\sigma \Big|_{s=-\delta}^0 - \int_{-\delta}^0 \frac{d}{ds} (\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) \, d\sigma \, ds,$$

since the first bracket is 0 and by denoting the first term in the second integrand by $\tilde{u}'(t-s)$ this becomes (using Fubini)

$$= - \int_{-\delta}^0 \int_{\sigma}^0 \tilde{u}'(t-s) \, ds \, \varphi_\delta(\sigma) \, d\sigma,$$

and we see

$$\|\tilde{u}(t) - \tilde{u}_\delta(t)\|_{V_3} \leq \int_{-\delta}^0 \int_{\sigma}^0 \|\tilde{u}'(t-s)\|_{V_3} ds \varphi_\sigma(\sigma) d\sigma.$$

$L_1((0, T); V_3)$ estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3} dt \leq \int_0^T \int_{-\delta}^0 \int_{\sigma}^0 \|u'(\tilde{t}-s)\|_{V_3} ds \varphi_\delta(\sigma) d\sigma dt \leq 2\delta \|u'\|_{L_1((0, T); V_3)} \leq 2\delta C^*$$

$L_\infty((0, T); V_3)$ estimate:

$$\|u - u_\delta\|_{L_\infty((0, T); V_3)} \leq 2\|u'\|_{L_1((0, T); V_3)} \leq 2C^*$$

It remains to show $M_\delta \subset L_p((0, T); V_2)$:

$$\|u - u_\delta\|_{L_p((0, T); V_3)} \leq \|u - u_\delta\|_{L_1((0, T); V_3)}^{1/p} \|u - u_\delta\|_{L_\infty((0, T); V_3)}^{1-1/p} \leq 2C^* \delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u - u_\delta\|_{L_p((0, T); V_2)} \leq \mu \|u - u_\delta\|_{L_p((0, T); V_1)} + C_\mu \|u - u_\delta\|_{L_p((0, T); V_3)}.$$

This means

$$\forall u \in M : \|u - u_\delta\|_{L_p((0, T); V_2)} \leq C^* + C_\mu 2C^* \delta^{1/p}.$$

Now fix $\varepsilon > 0$ and find

$$\mu > 0 : \mu C^* < \frac{\varepsilon}{2}, \delta > 0, C_\mu 2C^* \delta^{1/p} < \frac{\varepsilon}{2} \Rightarrow \forall u \in M : \|u - u_\delta\|_{L_p((0, T); V_2)} < \varepsilon.$$

This means $\exists \{w_k\}_{k=1}^N \subset M : \{(w_k)_\delta\}_{k=1}^N$ is ε -net in M in $L_p((0, T); V_2)$. If we now fix $u \in M$, then

$$\exists K \in \{1, \dots, N\} : \|u_\delta - (w_K)_\delta\|_{L_p((0, T); V_2)} < \varepsilon.$$

□

Remark. The pair $(\mathcal{U}, \|\cdot\|)$ is a Banach space.

We will be dealing with the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot a(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = f & \text{in } (0, T) \times \Omega, \\ u = u_0, & \text{on } \{0\} \times \Omega, \\ u = 0, & \text{on } (0, T) \times \partial\Omega \end{cases}.$$

The unknown is the function $u : (0, T) \times \Omega \rightarrow \mathbb{R}$, and we are given $\Omega \in C^{0,1}$, $T > 0$, $Q_T = (0, T) \times \Omega$, $f : Q \rightarrow \mathbb{R}$ or $f : (0, T) \rightarrow X$ a Banach space, $u_0 : \Omega \rightarrow \mathbb{R}$, $a : \Omega \times \mathbb{R} \mathbb{R}^d \rightarrow \mathbb{R}^d$, $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions: $\exists r > 1, \exists C > 0 : a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : |a_0(x, z, p)| + |a(x, z, p)| \leq C(1 + |z|^{r-1} + |p|^{r-1})$ and $\exists C_1, C_2 > 0, q \in (1, \max(2, r)) a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : a(x, z, p)p + a_0(\dots)z \geq C_1|p|^r - C_2(1 + |z|q)$.

Theorem 20. Let $\Omega \in C^{0,1}$, a, a_0 satisfy growth conditions and coercivity, let $\{a_i\}_{i=0}^d$ be monotone. Denote $V = W_0^{1,r}(\Omega) \cap L_2(\Omega)$. Then $\forall f \in L_{r^*}((0, T); V^*)$, $u_0 \in L_2(\Omega) \exists u \in L_r((0, T); V)$ s.t. $\partial_t u \in L_{r^*}((0, T); V^*)$, $u \in C([0, T]; L_2(\Omega))$, $u(0) = u_0$ and moreover

$$a.e. t \in (0, T), \forall \varphi \in V : \langle \partial_t u, \varphi \rangle + \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle.$$

Finally, the solution is unique.

Proof. The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

Rothe method: Fix $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$ and approximate the derivative with

$$\partial_t u(t, x) \approx \frac{1}{h} (u(t, x) - u(t - h, x)).$$

Define $u_0 = u_0, u_{k+1} \in V$ as a solution of

$$\frac{1}{h} (u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \text{ in } \Omega, u_{k+1} = 0 \text{ on } \partial\Omega.$$

Define

$$f_{k+1} := \int_{kh}^{(k+1)h} f \, dt,$$

then the weak formulation becomes

$$\int_{\Omega} \frac{u_{k+1} - u_k}{h} \varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla \varphi + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) \varphi \, dx = \langle f_{k+1}, \varphi \rangle.$$

We claim without a proof that the solutions $\{u_k\}_{k=0}^n \subset V$ exist.

To obtain a-priori estimates, test the equation with u_{k+1} . This yields:

$$\int_{\Omega} |u_{k+1}|^2 - u_k u_{k+1} \, dx = \int_{\Omega} \frac{1}{2} |u_{k+1}|^2 + \frac{1}{2} (u_{k+1} - u_k) - \frac{|u_k|^2}{2} \, dx \Rightarrow \sum_{k=0}^{j-1} \int_{\Omega} |u_{k+1}|^2 - u_{k+1} u_k \, dx = \frac{1}{2} \|u_j\|_{L_2(\Omega)}^2 - \frac{1}{2} \|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} (u_{k+1} - u_k) u_k$$

so

$$\int_{\Omega} a(\dots) \nabla \cdot u_{k+1} + a_0(\dots) u_{k+1} \, dx \geq C_1 \int_{\Omega} |\nabla u_{k+1}|^r \, dx - C_2 \int_{\Omega} (1 + |u_{k+1}|^q) \, dx,$$

$$\langle f_{k+1}, u_{k+1} \rangle \geq \|f_{k+1}\|_{V^*} \left(\|u_{k+1}\|_{W_0^{1,r}(\Omega)} + \|u_{k+1}\|_{L_2(\Omega)} \right) \leq \varepsilon \left(\|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r + \|u_{k+1}\|_{L_2(\Omega)}^2 \right) + C \left(\|f_{k+1}\|_{V^*}^{r'} + \|f_{k+1}\|_{V^*}^2 \right).$$

So together $\|u_j\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} \left[(u_{k+1} - u_k)^2 + h \|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r \right] \leq C \left(\|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \|u_{k+1}\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \left(\|f\|_{V^*}^{r'} \|f\|_{V^*}^2 \right) \right)$ □

Let us now define $u^n(t) = u_k$ for $t \in (h(k-1), hk)$, then

$$\|u^n\|_{L^\infty((0,T);L^2(\Omega))}^2 + \|u^n\|_{L^2((0,T);W_0^{1,r}(\Omega))}^2 < C(\text{data}).$$

Now set $\tilde{u}^n(t) = u_{k-1} + \frac{t-t_{k-1}}{h}(u_k - u_{k-1})$ for $t \in (t_{k-1}, t_k)$ and

$$k \in \{1, \dots, n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the equation to the form

$$\int_{\Omega} \partial_t \tilde{u}^n \varphi + a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi \, dx = \langle f^n, \varphi \rangle,$$

where $f^n(t) := f_k$ in in

$$(t_{k-1}, t_k), k \in \{1, \dots, n\}.$$

We are now ready to use growth and apriori estimates:

$$\|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} \leq C(\text{data}).$$

For the norm of the time derivative:

$$\sup_{\varphi \in S_V(0,1)} \langle \partial_t \tilde{u}^n(t), \varphi \rangle = \sup_{\varphi \in S_V(0,1)} \langle f^n, \varphi \rangle - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi) \, dx,$$

at any $t \in (0, T)$. So using Holder:

$$\|\partial_t \tilde{u}^n(t)\|_{V^*} \leq \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)},$$

and integrating

$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{r'} \, dt \leq C \left(\int_0^T \|f^n\|_{V^*}^{r'} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)} \, dt \right) \leq TC(\text{data})$$

6 Semigroup theory

We consider the equation

$$\begin{aligned} u' &= Au, A \text{ is linear} \\ u(0) &= u_0, \end{aligned}$$

where $u : [0, \infty) \rightarrow \mathbb{R}$. We know that for example if $Au = au, a \in \mathbb{R}$ then

$$u(t) = u_0 e^{at}.$$

If $\mathbf{u} : [0, \infty) \rightarrow \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d \times d}$, then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{A}^k t^k.$$

This can be extended to $u : [0, \infty) \rightarrow X$, X a Banach space, $A \in \mathcal{L}(X)$, then

$$u(t) = \exp(tA)u_0,$$

where the operator exponential is the same. This works well for unbounded operators, but suppose now

$$X = L_2(\Omega), Au = \Delta u.$$

We *guess* the solution should be

$$u(t) = \exp(\Delta t)u_0,$$

but what is

$$\exp(\Delta t)?$$

Definition 10 (Linear operator and its domain). Let X be a Banach space over \mathbb{K} . Linear operator on X is a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace of X and $A : \mathcal{D}(A) \rightarrow X$ is linear.

Definition 11. A family $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ is called a semigroup if

1. $S(0) = \text{id}$
2. $\forall s, t \geq 0 : S(t)S(s) = S(t+s).$

If moreover $\forall x \in X : S(t)x \rightarrow x$, as $t \rightarrow 0_+$, we call $\{S(t)\}$ a c_0 -semigroup (strongly continuous).

Remark. $\{s(t)\}_{t \in \mathbb{R}}$ with the two conditions is an Abelian group $(\{S(t)\}_{t \in \mathbb{R}}, \circ)$ with

$$(S(t))^{-1} = S(-t). \tag{3}$$

Remark ($X = \text{Banach}$). **In the following, X is always a Banach space.**

Lemma 12. Let $\{S(t)\}_{t \geq 0}$ be a c_0 -semigroup in X . Then

1. $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t},$
2. $\forall x \in X, t \mapsto S(t)x \in C([0, \infty); X).$

Proof. $1 \Rightarrow 2$. Fix $t > 0, x \in X$ compute

$$\lim_{h \rightarrow 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t)(S(h)x - x)\|_X \leq \lim_{h \rightarrow 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \rightarrow 0.$$

now compute $\lim_{h \rightarrow 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t-h)(x - S(h)x)\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \rightarrow 0.$ \square

Definition 12 (Infinitesimal generator). A linear operator $(A, \mathcal{D}(A))$ is called a infinitesimal generator of the semigroup $\{S(t)\}_{t \geq 0}$, if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X \mid \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\},$$

Theorem 21. Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0 -semigroup $\{S(t)\}_{t \geq 0}$ in X . Then

1. $x \in \mathcal{D}(A) \Rightarrow \forall t \geq 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x),$
2. $x \in X \wedge t \geq 0 \Rightarrow x_t = \int_0^t S(s)x \, ds \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x.$

Proof. Fix $x \in \mathcal{D}(A), t \geq 0$. Calculate

$$\lim_{h \rightarrow 0_+} \frac{S(h)S(t)x - S(t)x}{h} \stackrel{15}{=} \lim_{h \rightarrow 0_+} S(t) \frac{S(h)x - x}{h} = S(t)Ax,$$

(convergence is in the norm of the Banach space X). This means $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$, moreover, if $t > 0$:

$$\lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \rightarrow 0_+} S(t-h) \left(\frac{x - S(h)x}{-h} - S(h)Ax \right),$$

estimate,

$$\left\| \lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax \right\| = \lim_{h \rightarrow 0_+} \left\| S(t-h) \left(\frac{x - S(h)x}{-h} - S(h)Ax \right) \right\| = \lim_{h \rightarrow 0_+} \left\| \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax \right\| = \lim_{h \rightarrow 0_+} \|S(t-h) - S(t)\| \|x - S(h)x\| + \lim_{h \rightarrow 0_+} \|S(t-h) - S(t)\| \|S(h)Ax\|$$

as $S(t)$ is continuous and $S(0) = \text{id}$. Clearly, $t \mapsto S(t)x$ is $C^1([0, \infty))$ and

$$\frac{d}{dt}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h \rightarrow 0_+} \frac{1}{h} (S(h)x_t - x_t) = \lim_{h \rightarrow 0_+} \frac{1}{h} \left(\int_h^{t+h} S(s)x \, ds - \int_0^t S(s)x \, ds \right),$$

realize that

$$S(h)x_t = \int_0^t S(s+h)x \, ds = \int_h^{t+h} S(s)x \, ds,$$

so the previous computation continues as follows

$$= \lim_{h \rightarrow 0_+} \frac{1}{h} \left(\int_t^{t+h} S(s)x \, ds - \int_0^h S(s)x \, ds \right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

Definition 13 (Closed operator). We say that a linear operator $(A, \mathcal{D}(A))$ is closed if $\forall \{u_n\} \subset \mathcal{D}(A) : u_n \rightarrow u \wedge Au_n \rightarrow v$, for some $u, v \in X$, then it must hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that $\{(x, Ax) | x \in \mathcal{D}(A)\} \subset X \times X$ is closed in $(X \times X, \|\cdot\|_1)$.

$$\stackrel{15}{=} S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$$

Example. Let $\Omega \in C^{1,1}, X = L_2(\Omega), \mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), Au = \Delta u$. Then $(A, \mathcal{D}(A))$ is closed. Really, take $\{u_n\} \subset L_2(\Omega) : u_n \rightarrow u$ in $L_2(\Omega)$ for some $u \in L_2(\Omega)$. Suppose $Au_n \rightarrow v$ in $L_2(\Omega), v \in L_2(\Omega)$. Suppose the following equation: find

$$u_n \text{ s.t. } -\Delta u_n = Au_n, u_n \text{ on } \partial\Omega.$$

From the regularity theory for elliptic problems, we know that $\|u_n\|_{W^{2,2}(\Omega)} \leq C\|Au_n\|_{L_2(\Omega)} \leq C$, so we can extract $u_{n_k} \rightharpoonup u$ in $W^{2,2}(\Omega)$. Realize moreover

$$\int_{\Omega} \Delta u_n \varphi \, dx = \int_{\Omega} u_n \Delta \varphi \, dx, \forall \varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_{\Omega} v \varphi \, dx = \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} \Delta u \varphi \, dx,$$

which means $\Delta u = v$ a.e. in Ω and that $u \in \mathcal{D}(A), Au = v$.

Theorem 22. Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0 -semigroup $\{S(t)\}_{t \geq 0} \subset X$. Then

1. $\mathcal{D}(A)$ is dense in X ,
2. $(A, \mathcal{D}(A))$ is closed.

Proof. Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t} \int_0^t S(s)x \, ds \in \underbrace{\mathcal{D}(A)}_{\text{prev. thm}}, \frac{x_t}{t} \rightarrow x \text{ in } X,$$

Ad 2.: Take $\{x_n\} \subset \mathcal{D}(A) : x_n \rightarrow x$ in $X, Ax \rightarrow v$ in X . Compute¹⁶

$$\frac{(S(h) - \text{id})x_n}{h} = \frac{1}{h} \int_0^h \frac{d}{ds}(S(s)x_n) \, ds = \frac{1}{h} \int_0^h AS(s)x_n \, ds = \frac{1}{h} \int_0^h S(s) \underbrace{Ax_n}_{\rightarrow v}, \text{ so taking the limit yields } \frac{(S(h) - \text{id})x}{h} =$$

Altogether, $x \in \mathcal{D}(A), Ax = v$. □

Lemma 13. Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of c_0 -semigroups $\{S(t)\}_{t \geq 0}, \{\tilde{S}(t)\}_{t \geq 0}$. Then

$$\{S(t)\}_{t \geq 0} = \{\tilde{S}(t)\}_{t \geq 0}.$$

Proof. We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix $x \in \mathcal{D}(A), t > 0$. Then $g(s) := S(s)\tilde{S}(t-s)x$ satisfies $g \in C^1([0, t]; X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$, as A, S commute. This means $g(0) = g(t)$ and from this it follows $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X, S$ continuous $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$, and since $t \geq 0$ was arbitrary, we are done. □

Definition 14 (Resolvent of a linear operator). Let $(A, \mathcal{D}(A))$ be a linear (possibly unbounded) operator on X . We define

¹⁶This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that X is a Banach space and work with some functionals and Bochner integrals or whatever

1. resolvent set

$$\rho(A) = \left\{ \lambda \in \mathbb{K} \mid \lambda \text{id} - A \text{ is invertible and } (\lambda \text{id} - A)^{-1} \in \mathcal{L}(X) \right\},$$

2. resolvent operator $R(\lambda, A) : X \rightarrow \mathcal{D}(A) : R(\lambda, A) = (\lambda \text{id} - A)^{-1}$, for

$$\lambda \in \rho(A).$$

Remark. If $(A, \mathcal{D}(A))$ is a closed linear operator: $\lambda \in \rho(A) \Leftrightarrow \lambda \text{id} - A$ is a bijection of $\mathcal{D}(A)$ onto X .

Lemma 14. *Let $(A, \mathcal{D}(A))$ be a linear operator on X . It holds*

1. $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x$,
2. $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x$,
3. $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$, and $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A)$,
4. *If moreover $(A, \mathcal{D}(A))$ is the infinitesimal generator of a c_0 -semigroup $\{S(t)\}_{t \geq 0}$ s.t. $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, then*

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t) dt \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

Remark. The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

Proof. Ad 1.:

$$AR(\lambda, A)x = (A - \lambda \text{id}) \underbrace{R(\lambda, A)}_{=(\lambda \text{id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\text{id} - (\lambda \text{id} - A))R(\eta, A) = R(\lambda, A)(\eta \text{id} - A - \lambda \text{id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For $\lambda \neq \eta$ we also have

$$R(\lambda, A)R(\eta, A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG assume $\omega = 0$, meaning $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$. Denote $\tilde{S}(t) = e^{-\omega t} S(t)$.

Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} \tilde{S}(t)x dt.$$

First of all, this is well defined as

$$\|\tilde{R}x\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X dt = \frac{M}{\lambda} \|x\|_X,$$

and so $\|\tilde{R}\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}$, $\tilde{R} \in \mathcal{L}(X)$. Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Leftrightarrow \text{id} = (\lambda\text{id} - A)\tilde{R}.$$

For $x \in X$, $h > 0$ fixed compute

$$\begin{aligned} \frac{1}{h}(S(h)\tilde{R}x - \tilde{R}x) &= \frac{1}{h} \left(\int_0^\infty e^{-\lambda t} S(t+h)x - e^{-\lambda t} S(t)x dt \right) = \\ &= \frac{1}{h} \left(\int_h^\infty e^{-\lambda(t-h)} S(t)x dt - \int_0^\infty e^{-\lambda t} S(t)x dt \right) = \\ &= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x dt - \frac{1}{h} \int_0^h e^{-\lambda t} S(t)x dt = \\ &= e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} \rightarrow \lambda e^{-\lambda t}, \text{ as } h \rightarrow 0_+ \end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t) e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} S(t)x \rightarrow \lambda e^{-\lambda t} S(t)x \text{ on } (0, \infty) \text{ as } h \rightarrow 0_+.$$

The norm of this can be estimated as $\|\lambda e^{-\lambda t} S(t)x\| \leq C e^{-\lambda t} M \|x\|_X \in L_1((0, \infty))$. Altogether, we obtain $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda\text{id} - A)\tilde{R}x = x$.

To proceed further, we need the following theorem:

$$x \in \mathcal{D}(A), A \text{ closed} : A\tilde{R}x = A \left(\int_0^\infty e^{-\lambda t} S(t)x dt \right) = \int_0^\infty e^{-\lambda t} \underbrace{AS(t)}_{=S(t)A} x dt = \tilde{R}Ax,$$

which has been stated but not proved¹⁷. Finally, we can write: $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda\text{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$. Moreover, we have also shown the mapping is a bijection. \square

Definition 15 (Contraction semigroup). We say that $\{S(t)\}_{t \geq 0}$ is a contraction semigroup if

$$\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

Theorem 23 (Hille-Yosida). *Let $M \geq 1, \omega \in \mathbb{R}$. A linear $(A, \mathcal{D}(A))$ on a Banach space X generates a c_0 -semigroup (meaning it is its infinitesimal generator) satisfying $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$ **if and only if***

1. $(A, \mathcal{D}(A))$ is closed,
2. $\mathcal{D}(A)$ is dense in X ,
3. $\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}$.

Proof. If $M = 1, \omega = 0$, then $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}$. " \Rightarrow " has been proven, now show the other direction. The plan is to

1. approximate A by $\{A_n\} \subset \mathcal{L}(X)$,

¹⁷It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

2. construct S_n for A_n as previously,
3. estimate and limit passage.

Approximation: See the analogy: $a \in \mathbb{R} : \frac{n}{n-a} \rightarrow 1$, we would like $nR(n, A) \rightarrow \text{id}$. Calculate the norm of $nAR(n, A) = n(nR(n, A) - \text{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$, (This approx. is called the Yosida approximation.) For $x \in \mathcal{D}(A)$ fixed:

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leq \|R(n, A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \frac{1}{n} \|Ax\|_X \rightarrow 0 \text{ as } n \rightarrow \infty.$$

If

$$y \in X : \|nR(n, A)y - y\|_X \leq \|nR(n, A)(y - x)\|_X + \|nR(n, A)x - x\|_X + \|y - x\|_X \leq 2\|y - x\|_X + \underbrace{\|nR(n, A)x - x\|_X}_{\rightarrow 0},$$

but $\|y - x\|_X$ can be made arbitrarily small from density of $\mathcal{D}(A)$ in X , so in fact

$$nR(n, A)y \rightarrow y \text{ in } X, \forall y \in X.$$

And so $nR(n, A)$ really approximates id .

Using this gives us

$$\forall x \in \mathcal{D}(A) : A_n x = n \overbrace{R(n, A)}^{=AR(n, A)} x = n R(n, A) A x \rightarrow A x \text{ in } X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \leq \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (t A_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id} + n^2 t R(n, A))^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id})^k \sum_{k=0}^{\infty} \frac{(n^2 t R(n, A))^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives $\leq e^{-nt} \text{id} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n, A)\|_X^k \leq e^{-nt} e^{nt} = 1$, as $\|nR(n, A)\|^k \leq 1$. This means $\{S_n(t)\}_{\mathcal{L}(X)} \leq 1$.

Now show that this converges: fix $x \in \mathcal{D}(A)$, compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{d}{ds} (S_n(s)S_m(t-s)x) ds \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x ds \right\|_X \leq \underbrace{t}_{\|S_t\|_{\mathcal{L}(X)} \leq 1} \|A_n - A_m\|_X$$

and since X is Banach, it is convergent also. Finally, for $y \in X$, we have

$$\|S_n(t)y - S_m(t)y\|_X \leq \|S_n(t)(y - x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x - y)\|_X \leq 2\|x - y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that $\{S_n(t)y\}$ is Cauchy uniformly on $[0, T]$, $T > 0 \Rightarrow \exists S(t) : S_n(t)y \rightarrow S(t)y \forall y \in X, t > 0$. And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain $\{S(t)\}_{t \geq 0}$ is a c_0 -semigroup.

It remains to answer this question. Is $(A, \mathcal{D}(A))$ the infinitesimal generator of $\{S(t)\}_{t \geq 0}$? Let $(\tilde{A}, \mathcal{D}(\tilde{A}))$ be the infinitesimal generator of $\{S(t)\}_{t \geq 0}$. Compute

$$S_n(t)x - x = \int_0^t \frac{d}{ds} S_n(s)x \, ds = \int_0^t S_n(s)A_n x \, ds,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \leq \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \rightarrow 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \, ds.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t \rightarrow 0+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A} \text{ on } \mathcal{D}(A)).$$

The opposite inclusion is simple: fix $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$, and so $\lambda \text{id} - A : \mathcal{D}(A) \rightarrow X$ is onto, but also $\lambda \text{id} - A = \lambda \text{id} - \tilde{A}$ on $\mathcal{D}(A)$, and so $\lambda \text{id} - \tilde{A} : \mathcal{D}(A) \rightarrow X$ is onto. From the previous theorem, we know $\lambda \text{id} - \tilde{A}$ is one-to-one, so $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Altogether, $A = \tilde{A}$, $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$. □

7 (Some) exercises

7.1 4.3.2025

Example (Coefficients for smooth extension). Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for $u \in \mathcal{D}(\mathbb{R}^d)$. Find $\{c_j\}_{j=1}^{k+1}$ in such a way that $Eu \in C^k(\mathbb{R}^d)$. Moreover, take $d = 1$.

Proof. For $k = 0, j = 1$ we take $c_1 = 1, c_j = 0, j \neq 1$. For $k = 1$, compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k+1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take $x_d = 0$ in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \dots, k\}.$$

That is a linear system of $k + 1$ equations. Is it solvable? □

7.2 8.4.2025

Example (Laplace). Let $a_0 = 0, a(\cdot, z, p) = p$. Then $|a(\dots)| \leq |p|$, growth can be accomplished for $r = 2$, $a(\dots) \cdot p \geq |p|^2$. We can thus apply the theorem to our laplace equation

Example. Let $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}(1 + |p|^2)$. Then it is clearly Caratheodory, it is bounded $|a(\dots)| \leq |p| \frac{\pi}{2}$, so the growth conditions yield, it is coercive as $\operatorname{atan}(1 + |p|^2) \geq \frac{\pi}{4}$, and it is monotone

$$(\operatorname{atan}(1 + |p_1|^2)p_1 - \operatorname{atan}(1 + |p_2|^2)p_2)(p_1 - p_2) = \int_0^1 \sum_{j=1}^d \frac{d}{ds} \operatorname{atan}(1 + |p_2 + s(p_1 - p_2)|^2)(p_2 + s(p_1 - p_2)) ds (p_1 - p_2)_j$$