(although this is a bit inaccurate). Realize that since $u = 0$ outside of Ω, also u_j is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega_j)$. Now pick $\delta \in \left(0, \frac{\beta}{2}\right)$, where β is from the definition of $\mathrm{C}^{0,0}$ and set

$$S_j^\delta = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$
$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

i.e.,

$$"\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta\right\}\right)."$$

The trick is to shift the (support of) function u_j "into" Ω_j^δ

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\operatorname{supp}(\tau_\delta u_j) = \operatorname{supp}(u_j) - \delta,$$

from which it follows $\tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)$; we have only shifted the function u_j, but since we have also shifted S_j, qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j, \Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$, and the fact τ_δ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ *s.t.*

$$\|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to u_j, we are done once we approximate $\tau_\delta u_j$ by a function from $\mathrm{C}^\infty\left(\overline{\Omega}\right)$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $\mathrm{C}^\infty\left(\Omega_j^\delta\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right)$.

Show $\Omega \subset\subset \Omega_j^\delta$*:* We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k(\{(x', x_d) | x_d = a_k(x'), x' \in \mathrm{U}(0,\alpha)\}),$$

and the set Ω_j^δ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right).$$

Realize it suffices to show $\partial\Omega \notin \overline{S_j}$, as then it wont be excluded from $\mathbb{R}^d$ and thus will end up in Ω_j^δ. *Thanks to continuity of* a_j, we may write

$$\overline{S_j} = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

i.e., the " $<$ " have changed to " $\le$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d) | x_d = a_j(x'), x' \in \mathrm{U}(0,\alpha)\}`) \notin \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$