

# Partial differential equations II

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## 1 Winter semester addendum

### 1.0.1 Weak\* convergence

Since  $L_\infty((0, T); L_2(\Omega))$  is not reflexive, we cannot extracting a convergent subsequence; however, we know the predual of  $L_\infty((0, T); L_2(\Omega))$  is reflexive, i.e.

$$L_\infty((0, T); L_2(\Omega)) \approx (L_1((0, T); L_2(\Omega)))^*,$$

which means that balls in  $L_\infty((0, T); L_2(\Omega))$  are weakly\* compact. Moreover,  $L_1((0, T); L_2(\Omega))$  is separable, from which it follows  $L_\infty((0, T); L_2(\Omega))$  with the weak\* topology is metrizable and thus there exists a weakly\* converging subsequence (from the balls).

**Theorem 1.** *Let the assumptions of the previous theorem hold and  $\Omega \in C^{1,1}$ ,  $\delta \in (0, 1)$ . Then  $u \in L_2((\delta, T); W^{2,2}(\Omega))$ .*

*Proof.* Take the weak formulation in  $t \in (\delta, T)$ . WLOG further assume  $d = 0$ . Then

$$\int_{\Omega} \mathbb{A} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi - bu \varphi - \mathbf{c} \cdot \nabla u \varphi - \int_{\Omega} \partial_t u \varphi = \int_{\Omega} (f - bu - \mathbf{c} \cdot \nabla u - \partial_t u) \varphi,$$

and the integrand of the last integral is in  $L_2(\Omega)$  for a.e.  $t \in (\delta, T)$ . We can thus use the elliptic regularity results and write:

$$\|u\|_{W^{2,2}(\Omega)}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{W^{1,2}(\Omega)}^2 + \|\partial_t u\|_{L_2(\Omega)}^2),$$

integrating both sides  $\int_{\delta}^T dt$  yields

$$\|u\|_{L_2((\delta, T); L_2(\Omega))}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2((0, T); W^{1,2}(\Omega))}^2 + \|u\|_{L_2((\delta, T); L_2(\Omega))}^2)$$

□

**Theorem 2.** *If data are smooth and satisfy the compatibility conditions, then the weak solutions to the parabolic equation are smooth.*

*Proof.* no.

□

*Remark* (Compatibility condition). : Take the heat equation :  $\partial_t u - \Delta u = f$  at time zero:  $\Delta u(0) + f(0) = \partial_t u(0) \in W_0^{1,2}(\Omega)$ , so we need that  $f(0) + \Delta u(0)$  has zero trace  $\Rightarrow$  compatibility conditions.

**Theorem 3** (Uniqueness of the solution to a hyperbolic equation). *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that  $\mathbf{c} \in W^{1,\infty}(\Omega)$ . Then the weak solution to the hyperbolic equation is unique.*

*Proof.* It is enough that if  $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$ . To do that, take the weak equation, multiply it by  $\varphi \in V$  fixed and integrate in time and space:

$$\langle \partial_t u(t), \varphi \rangle + \int_{\Omega} \int_0^t \mathbb{A}(s) \nabla u(s) \cdot \nabla \varphi \, ds + \int_{\Omega} \int_0^t (bu(s) + \mathbf{c} \cdot \nabla u(s)) \varphi - \int_{\Omega} \int_0^t u(s) \mathbf{d}(s) \cdot \nabla \varphi = 0,$$

next take  $\varphi = u(t)$  as a test function and integrate  $\int_0^\tau dt, \tau \in (0, T)$ . The duality term becomes

$$\int_0^\tau \frac{1}{2} \partial_t \|u(t)\|_{L_2(\Omega)}^2 \, dt,$$

the remaining terms are (we are using Fubini theorem)

$$\int_0^\tau \int_{\Omega} \int_0^t \mathbb{A} \nabla u \cdot \nabla u(t) \, ds \, dt = \int_{\Omega} \int_0^\tau \int_s^\tau \nabla u(t) \, dt \, \mathbb{A}(s) \nabla u(s) \, ds,$$

denote  $\partial_s w(s) = -u(s)$ , then

□

## 2 Sobolev spaces revisited

Let  $\Omega \subset \mathbb{R}^d$  open,  $p \in [1, +\infty]$ ,  $k \in \mathbb{N}$ . We define

$$W^{k,p}(\Omega) = \left\{ f \in L_p(\Omega); D^\alpha f \in L_p(\Omega), \forall |\alpha| \leq k \right\},$$

with the norm

$$\|f\|_{W^{k,p}(\Omega)}^p = \|f\|_{L_p(\Omega)}^p + \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L_p(\Omega)}^p.$$

Recall that:

- $W^{k,p}(\Omega)$  is Banach  $\forall p$  and Hilbert for  $p = 2$ .
- $W^{k,p}(\Omega)$  is separable if  $p < \infty$  and reflexive if  $p > 1, p < \infty$ .

*Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.*

### 2.1 Tools from functional analysis

**Definition 1** (Regularization kernel). The function  $\eta$  is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\text{supp } \eta \subset U(0, 1)$
- $\eta \geq 0$
- $\eta$  is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x) dx = 1$

**Definition 2** (Regularization of a function). Let  $\eta$  be a regularization kernel. Set  $\eta_\varepsilon(x) = \varepsilon^{-d} \eta(x/\varepsilon)$ ,  $\varepsilon > 0$ . We define the smoothing of  $f$  by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

*Remark* (Properties of regularization). The regularization has the following properties:

- $f \in L_p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  in  $L_p(\Omega)$  and also a.e
- $f \in L_\infty(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  a.e and \*-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) \eta_\varepsilon(x-y) dy = \int_{U(x, \varepsilon)} f(y) \eta_\varepsilon(x-y) dy$
- $\text{supp } f_\varepsilon \subset \overline{U(\Omega, \varepsilon)}$ ,  $f = 0$  on  $U(x, \varepsilon) \Rightarrow f_\varepsilon(x) = 0$

**Definition 3** ( $\Omega' \subset\subset \Omega$ ).  $\Omega' \subset\subset \Omega$  means  $\overline{\Omega'}$  is compact and  $\overline{\Omega'} \subset \Omega$ .

**Lemma 1** (Approximation of Sobolev functions using regularization). Assume  $p \in [1, \infty)$ ,  $\Omega \subset \mathbb{R}^d$  open,  $k \in \mathbb{N}$ ,  $u \in W^{k,p}(\Omega)$ ,  $\Omega' \subset\subset \Omega$ . Then it holds

1.  $\text{dist}(\overline{\Omega}', \partial\Omega) = D > 0$
2.  $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$  in  $\Omega'$ ,  $\forall \varepsilon \in (0, D)$ ,  $\forall |\alpha| \leq k$
3.  $f_\varepsilon \rightarrow f$  in  $W^{k,p}(\Omega)$ ,  $\varepsilon \rightarrow 0^+$

*Proof.* 1. disjoint compact and closed set

$$2. \text{ WLOG } \frac{\partial f_\varepsilon}{\partial x^k} = \frac{\partial \int_{\mathbb{R}^d} f_y \eta_\varepsilon(x-y) dy}{\partial x^k} = \int_{\Omega} f_y \frac{\partial \eta_\varepsilon}{\partial x^k} dy = - \int_{\Omega} f(y) \frac{\partial \eta_\varepsilon}{\partial y^k} dy = - \int_{\Omega} \frac{\partial f}{\partial y^k} \eta_\varepsilon(x-y) dy = (D^\alpha f)_\varepsilon(x).$$

3. follows from 2) and the remark above applied to  $f, D^\alpha f, |\alpha| \leq k$ .

□

**Lemma 2** (Partition of unity). *Let  $E \subset \mathbb{R}^d, \mathcal{G}$  opencovering. Then there exists a countable system  $\mathcal{F}$  of nonnegative functions  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $0 \leq \varphi \leq 1$  and*

1.  $\mathcal{F}$  is subordinate to  $\mathcal{G} : \forall \varphi \exists U \in \mathcal{Q} : \text{supp } \varphi \subset U$
2.  $\mathcal{F}$  is locally finite:  $\forall K \subset E$  compact,  $\text{supp } \varphi \cap K \neq \emptyset$  for at most finitely many  $\varphi \in \mathcal{F}$ .
3.  $\sum_{\varphi \in \mathcal{F}} \varphi(x) = 1, \forall x \in E$ .

*Proof.* (Sketch) *Step 1* ( $E$  is compact):

$E$  compact  $\Rightarrow \exists N \in \mathbb{N} : U_j \in \mathcal{Q}$  s.t.  $E \subset \bigcup_{j=1}^N U_j$ . Moreover,  $\exists K_j \subset U_j$  compact such that  $E \subset \bigcup_{j=1}^N K_j$ . That follows from the exhaustion argument: for  $U \subset \mathbb{R}^d$  open, you can approximate it by a compact set:  $K_m = \left\{ x \in U, \text{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m \right\}$ . Then clearly  $K_1 \subset K_2 \dots$ , and they "converge monotonously to  $U$ ". Next, find  $\phi_j \in C_c(U_j), \phi_j > 0$  on  $K_j$ , e.g.  $\phi_j = \theta(\text{dist}(x, \partial U_j))$ . Then use convolution:  $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$  small and take finally  $\varphi_j = \frac{\psi_j}{\sum_j \psi_j}$ .

*Step 2* ( $E$  is open):

Use exhaustion argument, then finite  $\rightarrow$  countable.

□

## 2.2 Density of smooth functions

**Theorem 4** (Density of smooth functions I). *Let  $\Omega \subset \mathbb{R}^d$  be open,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $\left\{ f \in C^\infty(\Omega), \text{supp } f \text{ bounded} \right\} \cap W^{k,p}(\Omega)$  is dense in  $W^{k,p}(\Omega)$ .*

*Proof.* Let  $u \in W^{k,p}(\Omega), \varepsilon > 0$ . I want to show  $\exists v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$  s.t.  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ . Using the exhaustion argument, define

$$\Omega_j = \left\{ x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{j} \right\}.$$

Clearly,  $\Omega_j \subset \Omega_{j+1}, \bigcup_{j=1}^\infty \Omega_j = \Omega$ . Next, set  $U_j = \Omega_{j+1} \cup \overline{\Omega_{j-1}}, j = 1, 2, \dots$ , where  $\Omega_0 = \Omega_{-1} = \emptyset$ . Using the partition of unity lemma,  $\exists \{\varphi_j\}$  partition of unity subordinate to  $\{U_j\}$ . We can write  $u = \sum_j u \varphi_j$ , where  $u \varphi_j \in W^{k,p}(\Omega), \text{supp } u \varphi_j \subset U_j \subset \Omega_{j+1} \subset \subset \Omega$ . This is ready for convolution with  $\varepsilon_j > 0$  sufficiently small: set  $v_j = (u \varphi_j)_{\varepsilon_j}$ . By the properties of regularization, we now

$$\|u - u \varphi_j\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2^j},$$

by taking  $\varepsilon_j$  small enough. Set  $v = \sum_j v_j$  and use the following trick:

Fix  $N \in \mathbb{N}$  and estimate  $\|v - u\|_{W^{k,p}(\Omega)}$ . Observe  $u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j)$ , so taking  $x \in \Omega_N$  i have  $(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j)$ . The norm of this is

$$\|u - v\|_{W^{k,p}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{W^{k,p}(\Omega)} < \varepsilon.$$

It only remains to let  $N \rightarrow \infty$  and realize  $\|u - v\|_{W^{k,p}(\Omega_N)} \rightarrow \|u - v\|_{W^{k,p}(\Omega)}$  by Lévi's theorem:  $\int_{\Omega_N} |D^\alpha f| dx \rightarrow \int_{\Omega} |D^\alpha f| dx$ .  $\square$

*Remark.* It is nice that we only require  $\Omega$  to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Recall  $\Omega \in C^0$  means  $\exists U_j, j = 1, \dots, m$  open,  $\exists \alpha, \beta > 0, a_j : \overline{U(0, \alpha)} \rightarrow \mathbb{R}, \mathbb{A}_j : \mathbb{R}^d \rightarrow \mathbb{R}^d$  aff.orthogonal, such that  $\partial\Omega \subset \cup_{j=1}^m U_j, \partial\Omega \cap U_j = \{(x', a(x')), x' \in U(0, \alpha)\}$ . Setting  $G_j(x', b) = \mathbb{A}_j(x', a(x') + b)$  we moreover require  $G_j(U(0, \alpha) \times (0, \beta)) \subset \Omega, G_j(U(0, \alpha) \times (-\beta, 0)) \subset \overline{\mathbb{R}^d / \Omega}$ .

**Definition 4** (Shift operator). For  $u \in L_p(\Omega), k \in \{1, \dots, d\}, h > 0$ , we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

**Lemma 3** (Approximation property of the shift operator). For  $u \in L_p(\Omega)$ , it holds  $\tau_h u \rightarrow u$  in  $L_p(\Omega), h \rightarrow 0^+$ .

**Theorem 5** (Density of smooth functions II). Let  $\Omega \in C^0$  bounded,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $C_\Omega^\infty(\mathbb{R}^d)$  is dense in  $W^{k,p}(\Omega)$ .

*Proof.* Let  $u \in W^{k,p}(\Omega), \varepsilon > 0$  given, i am looking for  $v \in C_c^\infty(\mathbb{R}^d)$  such that  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ .

The sketch is simple: covering of  $\overline{\Omega}$ , partition of unity. Clearly,  $\Omega \subset \cup_{j=0}^m U_j$ , where  $U_0 = \Omega, U_j$  are from the definition of  $C^0$  boundary. Take  $\{\varphi_j\}$  to be the partition of unity on  $\overline{\Omega}$ , subordinate to this cover. Observe that  $u\varphi_j \in W^{k,p}(\Omega), \text{supp } u\varphi_j \subset U_j$ . Find

$$v_j \in \mathcal{D}(\mathbb{R}^d) \quad \text{s.t.} \quad \|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{m+1}.$$

If i am able to do this, i am finished: just take

$$v = \sum_{j=0}^m v_j$$

*Case  $j = 0$ .* We have  $\text{supp } u\varphi_0 \subset \subset \Omega$ , take  $v_0 = (u\varphi_0)_\varepsilon$ , so if we take  $\varepsilon > 0$  small enough, i can use the previous lemma.

*Case  $j \in \{1, \dots, m\}$ .* Set  $w_j = u\varphi_j, \tau_\delta w_j(x', x_d) = w(x', x_d + \delta)$  (ignore  $\mathbb{A}_j$ ), observe  $t_\delta u_j \in W^{k,p}(U_j^\delta), U_j \subset \subset U_j^\delta$ . Finally, set  $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$  small enough. From the properties of the shift  $\tau_\delta w_j$  is close to  $w_j$  in  $L_p(U_j \cap \Omega)$  and  $D^\alpha \tau_\delta w_j = \tau_\delta(D^\alpha w_j)$  close to  $D^\alpha w_j$  in  $L_p(U_j \cap \Omega)$ . Finally, set  $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$  small enough  $\Rightarrow v_j \in \mathcal{D}(\mathbb{R}^d), \text{supp } v_j \subset U_j$  by the previous lemma  $\|v_j - \tau_\delta w_j\|_{W^{k,p}(\Omega)}$  small.  $\square$

*Remark.* Recall  $C_\Omega^\infty(\mathbb{R}^d) = \{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\}$ .

### 2.3 Extension of Sobolev functions

*Problem of extension:* For  $u \in W^{k,p}(\Omega)$ , does there exist  $\bar{u} \in W^{k,p}(\mathbb{R}^d)$ , s.t.  $\bar{u}|_\Omega = u$ ,  $\|\bar{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{W^{k,p}(\Omega)}$ ?

The answer is **yes**, if  $\Omega$  is nice enough.

**Lemma 4.** Let  $\alpha, \beta > 0, K \subset U(0, \alpha) \times [\alpha, \beta]$  be compact. Then

$$\exists C > 0, \exists E : C^1(\overline{U(0, \alpha)} \times [0, \beta]) \rightarrow C^1(\overline{U(0, \alpha)} \times [-\beta, \beta]), \exists \tilde{K} \subset U(0, \alpha) \times [-\beta, \beta] \text{ compact}$$

such that:

1.  $\|Eu\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} \leq \|u\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))}$
2. if  $\text{supp } u \subset K \Rightarrow \text{supp } Eu \subset \tilde{K}$

*Proof.* Use the following trick:

$$\bar{u}(x) = \begin{cases} u(x), & x_d > 0 \\ -3u(x_1, \dots, x_{d-1}, -x_d) + 4u(x_1, \dots, x_{d-1}, -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Is this extension  $C^1$ ? Take some  $a = (x_1, \dots, x_{d-1}, 0)$ . Then

$$u(x \rightarrow a) = \begin{cases} u(a), & x_d > 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so  $\bar{u}$  is continuous. Its derivative

$\partial_k \bar{u}, k = 1, \dots, d-1$  is the same as for  $u$ , where as

$$\partial_d \bar{u} = \begin{cases} \partial_d u, & x_d > 0 \\ -3\partial_d u(x_1, \dots, x_{d-1}, -x_d)(-1) + 4\partial_d u(x_1, \dots, x_{d-1}, -\frac{x_d}{2})(\frac{-1}{2}) = 3\partial_d u - 2\partial_d u, & x_d < 0, \end{cases}$$

so the the derivative is also continuous. Thus, we have  $Eu = \bar{u} \in C^1 \subset W^{1,p}(U(0, \alpha) \times (-\beta, \beta))$  and estimate of the norm  $\|Eu\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))}$  is clear, as the wanted term is just some linear combination.

*Mr. Przak is not sure how this should be correctly finished and i am not also.*  $\square$

**Lemma 5** (Change of variables under  $C^1$  diffeomorphisms). Let  $U, V \subset \mathbb{R}^d$  be open,  $\phi : U \rightarrow V$  be  $C^1$  diffeomorphism. Let  $\tilde{U} \subset U$ . Then

$$\phi(\tilde{U}) \subset V, \text{ and } \exists C > 0 : \forall u \in C^1(V) : \|u \circ \phi\|_{W^{1,p}(\tilde{U})} \leq C \|u\|_{W^{1,p}(\phi(\tilde{U}))}$$

*Proof.*  $\|u \circ \phi\|_{L_p(\tilde{U})}^p = \int_{\tilde{U}} (u \circ \phi)^p |\det \nabla \phi| dx \leq C_0^{-1} \int_{\tilde{U}} |u \circ \phi|^p |\det \nabla \phi| dx$ , where  $\det \nabla \phi > 0$  in  $U$ , so  $\det \nabla \phi \geq C_0 > 0$  in  $\tilde{U}$ . Together  $\|u \circ \phi\|_{L_p(\tilde{U})}^p = C_0^{-1} \int_{\phi(\tilde{U})} |u|^p dx = C_0^{-1} \|u\|_{L_p(\phi(\tilde{U}))}^p$   $\square$

**Lemma 6.** Let  $\alpha, \beta > 0, K \subset U(0, \alpha) \times [0, \beta], K$  compact. Then there is  $C > 0, E : C^1(\overline{U(0, \alpha)} \times [0, \beta]) \rightarrow C^1(\overline{U(0, \alpha)} \times [-\beta, \beta]), \tilde{K} \subset U(0, \alpha) \times [-\beta, \beta]$  compact such that

- $\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha) \times (0,\beta)), W^{1,p}(U(0,\alpha) \times (-\beta,\beta)))} \leq C$
- $u \in C^1(\overline{U(0,\alpha)} \times [0,\beta]), \text{supp } u \subset K \Rightarrow \text{supp } Eu \subset \tilde{K}$

*Proof.* No proof.  $\square$

**Lemma 7.** Let  $U, V \subset \mathbb{R}^d$  open,  $\Phi : U \rightarrow V, C^1$  diffeomorphism,  $\tilde{U} \subset\subset U$  compact. Then  $\Phi(\tilde{U}) \subset\subset V$  and

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \leq C \|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

*Proof.* No proof.  $\square$

**Theorem 6** (Extension of Sobolev functions). Let  $\Omega \in C^{k-1,1}, k \in \mathbb{N}, p \in [1, \infty], V \subset \mathbb{R}^d$  open such that  $\Omega \subset\subset V$ . Then there is  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$  bounded linear operator such that

1.  $\forall u \in W^{k,p}(\Omega) : Eu = u \text{ a.e. in } \Omega,$
2.  $\forall u \in W^{k,p}(\Omega) : \text{supp } Eu \subset V,$
3.  $\|E\| \leq C, C = C(p, \Omega, V).$

*Proof.* Only for  $k = 1, \Omega \in C^1, p < \infty$ . We know  $C^\infty_\Omega(\mathbb{R}^d)$  is dense in  $W^{1,p}(\Omega)$ , we show existence of  $E$  for  $u \in C^\infty_\Omega(\mathbb{R}^d)$  with properties 1), 2), 3) and then extend  $E$  to  $W^{1,p}(\Omega)$  by density.  
Covering of  $\Omega$ :

$$\overline{\Omega} \subset \Omega \cup \bigcup_{j=1}^m U_j$$

with  $U_j, a_j, \mathbb{A}_j, \alpha, \beta$  as in the definition of a  $C^1$  domain. In particular,  $a_j \in C^1(U(0, \alpha))$ .

*Construction of  $E$ :* We denote  $\{\varphi_j\}_{j=0}^m$  partition of unity subordinate to  $\{U_j\}_{j=1}^m$ . For  $j \in \{1, \dots, m\}$  we define  $\phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$  by

$$\phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

Trivially  $\phi_j$  is  $C^1$  diffeomorphism. Let us denote by  $\tilde{E}$  the extension operator from the previous lemma. Then we have for  $u \in C^\infty_\Omega(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$ . We define

$$Eu = \varphi_0 u + \sum_{j=1}^m (\eta \tilde{E}((\varphi_j u) \circ \phi_j)) \circ \phi_j^{-1},$$

where  $\eta$  is a cut-off function  $\eta = 1$  on  $y_d \geq 0, \in (0, 1)$  else,  $= 0$  on  $y_d \leq -h$ , for some parameter  $h > 0$  which will be defined later. We also take  $\eta \in C^\infty$ . Due to our construction,

$$\phi_j^{-1}(U(0, \alpha) \times [-2h, \beta)) \subset U(\Omega, \varepsilon) \subset U(\Omega, 2\varepsilon) \subset V,$$

for some  $\varepsilon > 0$ .

*Properties of  $E$ :* It is clear that

- $E$  is linear from its definition
- 1) holds, as  $\phi_j$  and  $\phi_j^{-1}$  cancel somewhere

- 2) holds for  $h < \frac{\beta}{2}$
- 3) we use the previous lemma:

$$\begin{aligned}
\left\| \underbrace{(\eta \tilde{E}(\varphi_j u \circ \phi_j))}_{\text{supp}(\cdot) \subset U(0, \alpha) \times (-\beta, \beta)} \circ \phi_j^{-1} \right\|_{W^{1,p}(\mathbb{R}^d)} &\leq C \|\eta \tilde{E}(\varphi_j u \circ \phi_j)\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} \\
&\stackrel{\text{previous lemma}}{\leq} C \|\varphi_j u \circ \phi_j\|_{W^{1,p}(U(0, \alpha) \times (0, \beta))} \\
&\stackrel{\text{previous lemma}}{\leq} C \|\varphi_j u\|_{W^{1,p}(U_j \cap \Omega)} \leq \|u\|_{W^{1,p}(\Omega)} \Rightarrow \|E\| \leq C.
\end{aligned}$$

So all the properties hold for  $u \in C_{\Omega}^{\infty}(\mathbb{R}^d)$ . We need to show them also for  $u \in W^{1,p}(\Omega)$ . Pick an arbitrary  $u \in W^{1,p}(\Omega)$ , find  $\{u_k\} \subset C_{\Omega}^{\infty}(\mathbb{R}^d) : u_k \rightarrow u$  in  $W^{1,p}(\Omega)$ .

Ad 1): Since  $E$  is continuous, then  $Eu_k \rightarrow Eu$  in  $W^{1,p}(\mathbb{R}^d)$ . Since  $\Omega \subset \mathbb{R}^d \Rightarrow Eu = u$  in  $W^{1,p}(\Omega)$ .

Ad 2):  $\text{supp } Eu_k \subset U(\Omega, \varepsilon) \Rightarrow \text{supp } Eu \subset \overline{U(\Omega, \varepsilon)} \subset V$ .

□

*Remark* ( $\Omega \in C^{0,1}$  suffices). The theorem is still valid if we assume only  $C^{0,1}$  and  $p \in (1, \infty), k > 1$ .

## 2.4 Embedding theorems

**Example.** Let  $u \in \mathcal{D}(\mathbb{R}^2)$ . Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2) ds = \int_{-\infty}^{x_2} \partial_2 u(x_1, s) ds,$$

so

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u(x_1, x_2)|^2 dx_1 dx_2 \leq \int_{\mathbb{R}} |\partial_1 u(s, x_2)| ds \int_{\mathbb{R}} |\partial_2 u(x_1, s)| ds dx_1 dx_2 \leq \left( \int_{\mathbb{R}^2} |\nabla u|^2 d\lambda^2 \right)^2,$$

so

$$\|u\|_{L_2(\mathbb{R}^2)} \leq \|\nabla u\|_{L_1(\mathbb{R}^2)}.$$

**Lemma 8.** Let  $d \geq 2$ . Let  $\hat{u}_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be nonnegative and measurable for  $j \in \{1, \dots, d\}$ . We define

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), d\hat{x}_j = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_d.$$

Consider the functions  $u_j : \mathbb{R}^d \rightarrow \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$ . Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x) dx \leq \prod_{j=1}^d \left( \int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} d\hat{x}_j \right)^{\frac{1}{d-1}}. \quad (1)$$

*Proof.* Induction by  $d$ .

$$1. \quad d = 2 : \int_{\mathbb{R}^d} u_1(x_1, x_2) u_2(x_1, x_2) dx_1 dx_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2) \hat{u}_2(x_1) dx_1 dx_2 \stackrel{\text{Fubini}}{=} \int_{\mathbb{R}} \hat{u}_1(x_2) dx_2 \int_{\mathbb{R}} \hat{u}_2 dx_1.$$



2.

$$\begin{aligned}
d \rightarrow d+1 : \int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^d u_j(x) dx_{d+1} u_{d+1} dx d\hat{x}_{d+1} \\
&\stackrel{\text{Holder}}{\leq} \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} (u_j(x))^d dx_{d+1} \right)^{\frac{1}{d}} u_{d+1}(x) d\hat{x}_{d+1} \\
&\stackrel{\text{Holder}}{\leq} \left( \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x) dx_{d+1} \right)^{\frac{1}{d-1}} d\hat{x}_{d+1} \right)^{\frac{d-1}{d}} \left( \int_{\mathbb{R}^d} u_{d+1}^d dx_{d+1} \right)^{\frac{1}{d}} \\
&\stackrel{\text{induction step}^1}{\leq} \left( \int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d(x) dx_{d+1} d\hat{x}_j d\hat{x}_{d+1} \right)^{\frac{d-1}{d} \frac{1}{d-1}}.
\end{aligned}$$

□

**Theorem 7** (Gagliardo-Nirenberg). *Let  $p \in [1, d)$ . Then  $\forall u \in W^{1,p}(\mathbb{R}^d)$ :*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where  $p^* = \frac{dp}{d-p}$ .

*Proof.* Estimate for  $u \in \mathcal{D}(\mathbb{R}^d)$ :

$$\forall j \in \{1, \dots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d) ds$$

independet of  $x_j$ , so

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds.$$

Next, consider  $p = 1, p^* = \frac{d}{d-1}$  and estimate:

$$|u|^{\frac{d}{d-1}} \leq \prod_{j=1}^d \underbrace{\left( \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds \right)^{\frac{1}{d-1}}}_{u_j \text{ independent of } x_j},$$

so the integral

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} dx \leq \int_{\mathbb{R}^d} \prod_{j=1}^d u_j dx \stackrel{\text{previous lemma}}{\leq} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) dx_j d\hat{x}_j \right)^{\frac{1}{d-1}} = \left( \int_{\mathbb{R}^d} |\nabla u| dx \right)^{\frac{d}{d-1}}.$$

If  $p \in (1, d)$ , compute

$$\|u\|_{L_{\frac{dp}{d-1}}^d(\mathbb{R}^d)}^q = \| |u|^q \|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1} |\nabla u| dx \stackrel{\text{Holder}}{\leq} \|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|_{L_{(q-1)p'}(\mathbb{R}^d)}^{q-1}.$$

We want  $\frac{(q-1)p'}{p-1} = \frac{qd}{d-1}$ , so

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$q\frac{d}{d-1} = p^*.$$

$\Rightarrow$  statement holds for  $u \in \mathcal{D}(\mathbb{R}^d)$ . To finish, use density of  $\mathcal{D}(\mathbb{R}^d)$  in  $W^{1,p}(\mathbb{R}^d)$ .  $\square$

*Remark.* • It is evident that nonzero constants are not in  $W^{1,p}(\mathbb{R}^d)$  and that also the inequality does not hold for them.

- the set  $\mathbb{R}^d$  is of course unbounded, so we have no ordering of  $L_p(\Omega)$  spaces.
- of course, we require no smoothness of the domain

**Theorem 8.** *Let  $\Omega \subset \mathbb{R}^d$  be open. Then  $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1, d)$  the statement of the previous theorem holds.*

*Proof.* An immediate corollary of the previous theorem.  $\square$

*Remark.* In the proof of theorem we showed that  $\forall u \in W^{1,p}(\mathbb{R}^d)$  it holds

$$\|u\|_{L_{\frac{qd}{d-1}}(\Omega)}^q \leq q \|\nabla u\|_{L_p(\Omega)} \|u\|_{L_{\frac{p(q-1)}{p-1}}(\Omega)}^{q-1},$$

for  $q$  such that  $\frac{qd}{d-1} \leq p^*$ .

**Theorem 9** (Embedding theorem). *Let  $\Omega \subset C^{0,1}, p^* = \frac{dp}{1-p}$ . If  $p \in [1, d)$  then*

$$W^{1,p}(\Omega) \subset L_q(\Omega) \quad \forall q \in [1, p^*].$$

Moreover, if  $q < p^*$ , then

$$W^{1,p}(\Omega) \subset\subset L_q(\Omega).$$

If  $p = d$ , then

$$W^{1,p}(\Omega) \subset L_q(\Omega) \quad \forall q < \infty, \quad W^{1,p}(\Omega) \subset\subset L_q(\Omega) \quad \forall 1 \leq q < \infty.$$

*Proof.* We would like to use the previous theorem + extension.

Ad continuity for  $p < d : E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$  the extension is continuous. We also know

- identity  $I_1 : W^{1,p}(\mathbb{R}^d) \rightarrow L_{p^*}(\mathbb{R}^d)$  is continous,
- restriction  $I_2 : L_{p^*}(\mathbb{R}^d) \rightarrow L_{p^*}(\Omega)$  is continuous,
- identity  $I_3 : L_{p^*}(\Omega) \rightarrow L_q(\Omega)$  is continous.

Together, the mapping  $id : W^{1,p}(\Omega) : L_q(\Omega)$ ,  $id = I_3 \circ I_2 \circ I_1 \circ E$  identity is continuous. If  $p=d$ , then  $W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \forall r \in [1, d)$ , and  $r^* \rightarrow \infty$  as  $r \rightarrow d^-$ . For  $q \in [1, \infty)$  find  $r \in [1, d)$  s.t.  $r^* > q$ . Then

$$W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \subset L_{r^*}(\Omega) \subset L_q(\Omega),$$

using the previous results.

Ad compactness: We show  $W^{1,p}(\Omega) \subset L_q(\Omega)$  using Arzela-Ascoli and then it will get technical: show compactness in smooth functions, then show compactness in  $L_1(\Omega)$ , then approximate the norm of  $L_q(\Omega)$  using the obtained quantities.

Consider  $B = U_{W^{1,p}(\Omega)}(0, 1)$  and extend it to  $EB$ . Fix  $\delta > 0$  and let  $\eta$  be a regularization kernel. Then  $\exists R > 0 : \text{supp}(EB)_\delta \subset \overline{U(0, R)} \subset \mathbb{R}^d$  (i.e. all the functions from  $EB$  have the support contained in the ball). Moreover,  $(EB)_\delta \subset C^1(\overline{U(0, R)})$ . Actually, it is bounded in  $C^1(\overline{U(0, R)})$ .  $\underbrace{\subset}_{\text{Arzela-Ascoli}} C(\overline{U(0, R)})$  (uniform equicontinuity comes from uniform boundedness of the gradients,  $\nabla(u * \eta_\delta) = u * \nabla \eta_\delta$ .) Altogether  $(EB)_\delta$  is relatively compact in

$$C(\overline{U(0, R)}) \xRightarrow[\text{the space } C(\overline{U(0, R)}) \text{ is complete}]{\text{Arzela-Ascoli}} \text{bounded in } C(\overline{U(0, R)}) \xRightarrow[\text{bounded domain}]{\text{}} \text{bounded in } L_1(U(0, R)).$$

Next, take

$$\begin{aligned} u \in B : \|u - (Eu)_\delta\|_{L_q(\Omega)} &\leq \|Eu - (Eu)_\delta\|_{L_q(U(0, R))} = \int_{U(0, R)} |v - v_\delta| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} v(x+y) - v(x) \eta_\delta(y) dy \right| dx \leq \\ &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} |\eta_\delta(y)| |y| dy \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy}_{\text{Fubini}}. \end{aligned}$$

Estimate the inner integral: assume  $v$  is smooth and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|} \left| \int_0^1 \frac{d}{ds} (v(x+sy)) ds \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_0^1 |\nabla v|(x+sy) ds dx}_{\text{Cauchy Schwartz}} \leq \underbrace{C(R) \left( \int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}}_{\text{Holder}}.$$

Now, take  $v \in W_0^{1,p}(U(0, R))$ , then  $\exists \{v_k\} \subset \mathcal{D}(U(0, R)) : v_k \rightarrow v$  in  $W^{1,p}(U(0, R))$ . So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} dx \leq C(R) \left( \int_{\mathbb{R}^d} |\nabla v_k|^p dx \right)^{\frac{1}{p}} \rightarrow C(R) \left( \int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}.$$

So finally

$$\|u - (Eu)_\delta\|_{L_q(\Omega)} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy \leq \underbrace{C(R) \delta \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla u|^p dx \right)^{\frac{1}{p}} dx}_{|y| \leq \delta} \leq C_1 \delta.$$

Fix  $\varepsilon > 0$ , find finite  $\frac{\varepsilon}{2}$ -net in  $(EB)_\delta$  in  $L_1(U(0, R))$  (that is possible since we have total boundedness in  $L_1(U(0, R))$ .) Set  $\delta > 0$  s.t.  $C_1 \delta \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}$ . Denote the  $\frac{\varepsilon}{2}$ -net as  $\{Eu_k\}_{k=1}^m$ ,  $m \in \mathbb{N}$ . We show  $\{u_k\}_{k=1}^m$  is

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<sup>2</sup>The order of the choices is not precise...

a  $\varepsilon$ -net in  $B$ . Fix  $u \in B$ , find  $j \in \{1, \dots, m\} : \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(U(0,R))}$ . Compute

$$\|u - u_j\|_{L_1(\Omega)} \leq \|u - (Eu)_\delta\|_{L_1(\Omega)} + \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(\Omega)} + \|(Eu_j)_\delta - u_j\|_{L_1(\Omega)} \leq 2C_1\delta + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus, we have shown

$$W^{1,p}(\Omega) \subset L_1(\Omega).$$

It remains to show the validity for a general  $q$ . Let  $q \in [1, p^*) : \|v\|_{L_q(\Omega)} \leq \|v\|_{L_1(\Omega)}^\alpha \|v\|_{L_{p^*}(\Omega)}^{1-\alpha}$ , for  $\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}$ ,  $\alpha \in (0, 1]$ . Is  $B$  totally bounded in  $L_q(\Omega)$ ? Let us compute

$$\|u - u_j\|_{L_q(\Omega)} \leq \|u - u_j\|_{L_1(\Omega)}^\alpha \underbrace{\|u - u_j\|_{L_{p^*}(\Omega)}^{1-\alpha}}_{\leq C, W^{1,p}(\Omega) \subset L_{p^*}(\Omega)} \leq C\varepsilon^\alpha.$$

□

## 2.5 Trace theorems

## 2.6 Composition of sobolev functions

## 2.7 Difference quotients

# 3 Nonlinear elliptic equations as compact perturbations

**Theorem 10** (Nemytskii). *Let  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $N \in \mathbb{N}$ ,  $\Omega \subset \mathbb{R}^d$  measurable,  $f$  Caratheodory. Then*

1. *if  $u : \Omega \rightarrow \mathbb{R}^N$  is measurable then  $f(\cdot, u)$  is also measurable*
2. *If there is  $p_i \in [1, +\infty)$ ,  $i \in \{1, \dots, N\}$ ,  $q \in [1, \infty)$ ,  $g \in L_q(\Omega)$ ,  $C > 0$  such that for almost all*

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x, y)| \leq g(x) + c \sum_{i=1}^N |y_i|^{p_i/q}$$

*, then  $u \mapsto f(\cdot, u)$  is continuous from  $L_{p_1}(\Omega) \times \dots \times L_{p_N}(\Omega)$  to  $L_q(\Omega)$ . Moreover, it maps bounded sets to bounded sets.*

*Proof.* No proof

□

**Definition 5** (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2). Let  $X, Y$  be normed linear spaces,  $M \subset X$ . The mapping  $F : M \rightarrow Y$  is called a compact operator on  $M$  into  $Y$  if  $F$  is continuous and  $F(M \cap K)$  is relatively compact in  $Y$  for any bounded  $K \subset X$ .

*Remark.* We have no linearity of  $F$ ! So continuity cannot follow from compactness (we have compactness  $\Rightarrow$  boundedness  $\neq$  continuity for nonlinear operators)

**Theorem 11** (Brouwer fixed point theorem). *Let  $K \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$  be a nonempty convex closed bounded. Assume that  $F : K \rightarrow K$  is continuous. Then  $F$  has a fixed point in  $K$ , i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

*Proof.* No proof □

**Theorem 12** (Schauder fixed point theorem). *Let  $K \subset X$  be a nonempty convex closed bounded subset of a linear normed space  $X$ . Assume that  $F$  is compact on  $K$  into  $K$  and  $F(K) \subset K$ . Then there is fixed point of  $F$  in  $K$ .*

*Proof.* No proof □

- for Brouwer,  $K \subset \mathbb{R}^N$  so since it is closed and bounded, it is automatically compact, and since  $F : K \rightarrow K$  is continuous,  $F$  is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with  $N=1$  is easy, based on Darboux property.

### 3.0.1 Problem prototypes

In this chapter some nonlinear elliptic equations are discussed.

**Example.** Suppose the following problem:

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$g : \mathbb{R} \rightarrow \mathbb{R}, f \in (W_0^{1,2}(\Omega))^*, \text{ continuous, } \exists \alpha \in [0, 1) : \forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

**Theorem 13** (Existence). *Let  $\Omega \in C^{1,1}$ ,  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g$  is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*}.$$

If  $f \in L_2(\Omega)$ , then the solution  $u \in W^{2,2}(\Omega)$ .

*Proof.* We define  $S : L_2(\Omega) \rightarrow L_2(\Omega)$  such that

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx.$$

$S$  is well defined:

$$|\text{RHS}| \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W^{1,2}(\Omega)} + \|\varphi\|_{L_2(\Omega)} \|g(w)\|_{L_2(\Omega)},$$

and

$$\int_{\Omega} |g(w)|^2 \, dx \leq \int_{\Omega} C(1 + |w|^\alpha)^2 \, dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) \, dx \leq \int_{\Omega} C(1 + |w|^2) \, dx \leq \infty,$$

where we used the Young inequality and  $\alpha \leq 1$ . We have thus shown the mapping  $w \mapsto g(w)$  is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$  by Nemytskii. Next,  $S$  is continuous:

- $w \mapsto g(w)$  is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$
- $w \mapsto (\varphi W_0^{1,2}(\Omega) \rightarrow \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx)$  is continuous from  $L_2(\Omega)$  to  $(W_0^{1,2}(\Omega))^*$

- $F \rightarrow u$ , where  $u$  is the weak solution of  $\begin{cases} -\Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$ , is linear and continuous from  $(W_0^{1,2}(\Omega))^*$  to  $W_0^{1,2}(\Omega)$ .

In total, the composition is continuous and yields  $S$ . Next, we would like to show  $S$  is compact. We start with showing  $S$  maps bounded sets in  $L_2(\Omega)$  to bounded sets in  $W_0^{1,2}(\Omega)$ ; for that we need apriori estimates: test the weak formulation with  $u$ :

$$\|\nabla u\|_{L_2(\Omega)}^2 \leq \varepsilon \|u\|_{W^{1,2}(\Omega)}^2 + C \left( \|f\|_{(W^{1,2}(\Omega))^*}^2 + \|g(w)\|_{L_2(\Omega)}^2 \right) \underset{\text{Young}}{\leq} C \left( \|f\|_{(W_0^{1,2}(\Omega))^*} + 1 + \|w\|_{L_2(\Omega)}^2 \right),$$

from which follows  $S$  is compact from  $L_2(\Omega)$  to  $L_2(\Omega)$  by compact embedding. Now we need to show  $S(U(0, R)) \subset U(0, R)$  for some  $R > 0$ . From the previous we know:

$$\frac{C}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \left( \|f\|_{(W_0^{1,2}(\Omega))^*} + \|g\|_{L_2(\Omega)}^2 \right),$$

so since

$$\tilde{C} \int_{\Omega} |g(w)|^2 dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) dx \underset{\text{Young}}{\leq} \int_{\Omega} \left( C + \frac{c}{4} |w|^2 \right) dx$$

we know

$$\frac{c}{2} \|u\|_{L_2(\Omega)}^2 \leq \frac{c}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + C\lambda(\Omega) + \frac{c}{4} \|w\|_{L_2(\Omega)}^2,$$

and thus

$$\|u\|_{L_2(\Omega)}^2 \leq \underbrace{\frac{2\tilde{C}}{c} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + 2\frac{C}{c}}_{=\bar{C}} + \frac{1}{2} \|w\|_{L_2(\Omega)}^2.$$

so if  $\bar{C} + \frac{1}{2}R^2 < R^2$ , we are done<sup>3</sup>. But such an  $R$  of course exists (says doc. Kaplicky)  $\Rightarrow$  the image of a ball is in a ball for some  $R \Rightarrow S$  is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that  $u_0$  solves  $\begin{cases} -\Delta u_0 = f - g(u_0) \in L_2(\Omega) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$

So from the regularity theory for elliptic equations we get

$$u \in W^{2,2}(\Omega).$$

□

**Theorem 14** (Uniqueness). *Let  $u_1, u_2 \in W_0^{1,2}(\Omega)$  be weak solutions to the above problem. Let  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g$  be continuous. Let either*

1.  $g$  is nondecreasing
2.  $g \in C^1(\mathbb{R})$ ,  $\|g'\|_{\infty}$  small.

*Then  $u_1 = u_2$ .*

---

<sup>3</sup>The constants are most probably messed up.

*Proof.* We subtract the equations for  $u_1, u_2$  and test with  $u_1 - u_2$ :

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2) dx = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1 - u_2)\|_{L_2(\Omega)} \geq C \|u_1 - u_2\|_{W^{1,2}(\Omega)}^2 \Rightarrow u_1 - u_2 = 0.$$

$$|\int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2) dx| \leq \int_{\Omega} \|g'\|_{\infty} |u_1 - u_2|^2 dx \leq \|g'\|_{\infty} C_P \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 = 0 \Rightarrow u_1 = u_2.$$

whenever  $C\|g'\|_{\infty} < 1$ .  $\square$

**Example.** Suppose the following problem

$$\begin{cases} -\Delta u + b(\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where  $f \in (W_0^{1,2}(\Omega))^*$ ,  $b$  is continuous and bounded. The weak formulation is

$$u \in W_0^{1,2}(\Omega) \wedge \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi + b(\nabla u) \varphi dx = \langle f, \varphi \rangle,$$

and the first apriori estimates (test with  $u$ )

$$\|\nabla u\|_{L_2(\Omega)} \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|u\|_{W_0^{1,2}(\Omega)} + \int_{\Omega} |u| dx \|b\|_{L_{\infty}(\Omega)}.$$

**Theorem 15.** Let  $f \in (W_0^{1,2}(\Omega))^*$ ,  $\Omega \in C^{0,1}$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}$  continuous and bounded. Then there is a weak solution to the above problem.

*Proof.*  $S : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$ ,  $Sw = u$  iff  $u$  solves

$$\begin{cases} -\Delta u = f - b(\nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi dx = \langle f, \varphi \rangle - \int_{\Omega} b(\nabla w) \varphi dx.$$

Clearly,  $S$  is well defined and

$$\|Sw\|_{W_0^{1,2}(\Omega)} \leq C \underbrace{\left( \|f\|_{(W_0^{1,2}(\Omega))^*} + \|b\|_{L_{\infty}(\Omega)} \right)}_{:=R},$$

meaning  $S(\overline{U(0, R)}) \subset \overline{U(0, R)}$ . Moreover,  $S$  is continuous, as  $S$  is the composition of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show  $S$  is compact: we already have continuity, consider  $\{w_k\}_{k \in \mathbb{N}} \subset W_0^{1,2}(\Omega)$  bounded. Then  $\exists \{u_k\} \subset W_0^{1,2}(\Omega)$  bounded:  $u_k \rightarrow u$  in  $L_1(\Omega)$  by embedding up to a subsequence. Next, use the following trick: substitute equation for  $u_k$  from equation for  $u_l$  and test with  $u_l - u_k$

$$C \|u_l - u_k\|_{W_0^{1,2}(\Omega)}^2 \leq \|\nabla(u_l - u_k)\|_{L_2(\Omega)}^2 \leq \int_{\Omega} |b(\nabla u_l) - b(\nabla u_k)| |u_l - u_k| dx \leq 2 \|b\|_{L_{\infty}(\Omega)} \|u_l - u_k\|_{L_1(\Omega)}.$$

All in all,  $S$  has a fixed point by Schauder, which is of course the weak solution.  $\square$

But this says  $\{u_k\}$  is Cauchy in  $W_0^{1,2}(\Omega)$ .

## 4 Nonlinear elliptic equations - monotone operator theory

**Lemma 9.** Let  $g : B(0, R) \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$  be continuous,  $N \in \mathbb{N}, R > 0$ , and  $\forall c \in S(0, R) : g(c) \cdot c \geq 0$ . Then, there is  $c_0 \in B(0, R) : g(c_0) = 0$ .

*Proof.* By contradiction. Let  $g \neq 0$  in  $U(0, R)$ . Let us define

$$h(x) = -R \frac{g(x)}{|g(x)|}.$$

Then  $h \in C(B(0, R))$ ,  $h(B(0, R)) \subset S(0, R)$ , so by Brouwer there  $\exists x_0 \in B(0, R) : h(x_0) = x_0 \Rightarrow -R \frac{g(x_0)}{|g(x_0)|} = x_0$ . Take the dot product with  $x_0$  and write

$$\underbrace{-R \frac{g(x_0) \cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in S(0, R),$$

so that is a contradiction. □

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i (a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x))) = f(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

The data are

- $\Omega \in C^{0,1}$ ,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, i \in \{1, \dots, d\}$  are Caratheodory in  $x$  and  $(u, \nabla u)$ .
- *growth condition:*  $\exists C > 0, r \in (1, \infty), h \in L_{r'}(\Omega) : \forall i \in \{0, \dots, d\}$ , for almost all  $x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |a_i(x, z, p)| \leq C(|z|^{r-1} + |p|^{r-1}) + h(x)$ ,
- $f \in (W_0^{1,r}(\Omega))^*$ ,

and the unknown is  $u : \Omega \rightarrow \mathbb{R}$ .

*Remark.* The function  $(u, p) \mapsto a_i(\cdot, u, p)$  is continuous from  $(L_r(\Omega))^{d+1}$  to  $L_{r'}(\Omega)$ . by Nemystkii theorem.

**Definition 6** (Coercivity). We say that  $\{a_i\}_{i=0}^d$  are coercive if  $\exists C_1 > 0, C_2 \in L_1(\Omega) : \text{a.e. } x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} :$

$$\sum_{i=1}^d a_i(x, z, p) p_i + a_0(x, z, p) \geq C_1 |p|^r - C_2(x), \text{ i.e. } a(x, z, p) \cdot p \geq C_1 |p|^r - C_2(x)$$

**Definition 7** (Monotonicity). We say that  $\{a_i\}_{i=0}^d = a$  is monotone if for almost all

$$x \in \Omega, \forall (z_1, p_1), (z_2, p_2) \in \mathbb{R}^{d+1} : (a(x, z_1, p_1) - a(x, z_2, p_2)) \cdot (p_1 - p_2) + (a_0(x, z_1, p_1) - a_0(x, z_2, p_2)) \cdot (z_1 - z_2) \geq 0.$$

Very similiarly we define strict monotonicity.



**Definition 8** (Weak solution). We say that  $u \in W^{1,r}(\Omega)$  is a weak solution to the above problem if

- $u = u_0$  in the sense of traces on  $\partial\Omega$ ,

- 

$$\int_{\Omega} a(\cdot, u, \nabla u) \cdot \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle, \forall \varphi \in W_0^{1,r}(\Omega).$$

**Theorem 16** (Existence and uniqueness). *Let  $\Omega \in C^{0,1}$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $r \in (1, \infty)$ ,  $\{a_i\}_{i=1}^d$  be Caratheodory, coercive and  $m$  and let them also satisfy the growth conditions. Finally, let  $f \in (W^{1,r}(\Omega))^*$ . Then, there is a weak solution to the problem. If, moreover,  $\{a_i\}_{i=1}^d$  is strictly monotone, then the weak solution is unique.*

*Proof.* The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

**Galerkin:** Since  $W_0^{1,r}(\Omega)$  is separable  $\Rightarrow \exists \{w_i\}_{i=1}^{\infty}$  that is a dense<sup>4</sup> linearly independent subset of  $W_0^{1,r}(\Omega)$ . We search for  $n \in \mathbb{N}$  such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where  $\alpha_j \in \mathbb{R}$  and  $u^n$  satisfy

$$\forall j \in \{1, \dots, n\} : \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla w_j + a_0(\cdot, u^n, \nabla u^n) w_j \, dx = \langle f, w_j \rangle.$$

We claim such  $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$  exist  $\forall n \in \mathbb{N}$  by the previous lemma. We define a vector function

$$F(\alpha^n) := \left\{ \int_{\Omega} a \cdot \nabla w_j + a_0 w_j \, dx - \langle f, w_j \rangle \right\}_{j=1}^n,$$

from Nemystkii  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F$  is continuous on  $\mathbb{R}^n$ . Moreover, it holds

$$F(\alpha^n) \cdot \alpha^n \geq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \nabla(u^n - u_0) + a_0(u^n - u_0) \, dx - \langle f, u^n - u_0 \rangle \underset{\text{coercivity}}{\geq} \int_{\Omega} C_1 |\nabla u^n|^r - (C_2(\cdot) + |a| |\nabla u_0| + |a_0| |u_0|) \, dx$$

together with the fact

$$\|\nabla u^n\|_{L^r(\Omega)}^r \geq \left( \|\nabla(u - u_0)\|_{L^r(\Omega)} - \|\nabla u_0\|_{L^r(\Omega)} \right)^r \geq \|\nabla(u^n - u_0)\|_{L^q(\Omega)}^r - \|\nabla u_0\|_{L^r(\Omega)}^r \geq C \|u^n - u_0\|_{W^{1,r}(\Omega)}^r - \|\nabla u_0\|_{L^r(\Omega)}^r,$$

---

<sup>4</sup>It can be chosen such that it is itself dense, not only its span

Next, realize that  $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{W^{1,r}(\Omega)}$  is a norm equivalent to  $|\alpha^n|$  (Euclidian norm). So that means  $\exists K_1(n) > 0 : \forall \alpha \in \mathbb{R}^n : K_1(n)|\alpha^n| \leq \|u^n - u_0\|_{W^{1,r}(\Omega)}$ . For  $|\alpha^n| = R, R > 0$  determined later estimate  $F(\alpha^n) \cdot \alpha^n \geq c\|u^n - u_0\|_{W^{1,r}(\Omega)} - \tilde{c}\left(\|\nabla u_0\|_{L^r(\Omega)}^r + 1 + \|u_0\|_{L^r(\Omega)}^r + \|f\|_{(W_0^{1,r}(\Omega))^*}^r\right)$  (which is not a trivial computation). And so  $\exists R > 0, \forall \alpha^n \in S(0, R) \subset \mathbb{R}^n : F(\alpha^n) \cdot \alpha^n > 0$ , so from the previous lemma  $\exists \alpha^n \in S(0, R) : F(\alpha^n) = 0$ , and we fix these  $\alpha^n$ .

**Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|_{W^{1,r}(\Omega)}^r \leq C\left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}\right),$$

and

$$\|u^n\|_{W^{1,r}(\Omega)} \leq C\left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}\right),$$

$$\forall j \in \{0, \dots, d\} : \|a_j(\cdot, u^n, \nabla u^n)\|_{L^{r'}(\Omega)}^r \leq C\left(1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}\right),$$

**Limit passage** From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \text{ in } W^{1,r}(\Omega), a_j \rightharpoonup \alpha_j \text{ in } L^{r'}(\Omega).$$

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_{\Omega} \alpha \cdot \nabla w_j + \alpha_0 w_j \, dx = \langle f, w_j \rangle,$$

and from density of  $\{w_j\}_{j \in \mathbb{N}}$  in  $W^{1,r}(\Omega)$  we have

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} \alpha \cdot \nabla \varphi + \alpha_0 \varphi \, dx = \langle f, \varphi \rangle.$$

**Identification of  $\alpha$ 's** We want to show  $\alpha_j = a_j(\cdot, u, \nabla u), j \in \{0, \dots, d\}$ . For that, we use the *Minty trick*:

$$\begin{aligned} 0 &\leq \int_{\Omega} (a(\cdot, u^n, \nabla u^n) - a(\cdot, v, V)) \cdot (\nabla u^n - V) + (a_0(\cdot, u^n, \nabla u^n) - a_0(\cdot, v, V)) \cdot (u^n - v) \\ &\leq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla u^n + a_0(\cdot, u^n, \nabla u^n) \cdot u^n \, dx + \\ &\quad - \int_{\Omega} (a(\cdot, u^n, \nabla u^n)V + a_0(\cdot, u^n, \nabla u^n)v - a(\cdot, v, V) + a_0(\cdot, v, V)) \cdot (u^n - v) \, dx. \end{aligned}$$

Denote

$$I^n = \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla (u^n - u_0) + a_0(\cdot, u^n, \nabla u^n) \cdot (u^n - u_0) \, dx + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx,$$

by using the equation we obtain

$$I^n = \langle f, u^n - u_0 \rangle + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx \rightarrow \langle f, u - u_0 \rangle + \int_{\Omega} \alpha \nabla u_0 + \alpha_0 u_0 \, dx = \int_{\Omega} \alpha \nabla u + \alpha_0 u \, dx,$$

as the rest has subtracted. In total, we have

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, v, V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot, v, V))(u - v) \, dx.$$

So far,  $v, V$  have been arbitrary. If we take

$$V = \nabla u - \lambda \psi, \psi \in L_r(\Omega), v = u,$$

then  $0 \leq \int_{\Omega} (\alpha - a(\cdot, \nabla u + \lambda \psi)) \lambda \psi \, dx$ , so if we take  $\lambda > 0$  and pass to the limit  $\lambda \rightarrow 0_+$  (using Nemytskii theorem) we can write

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx.$$

Since  $\psi$  was arbitrary, we could have taken  $\psi \rightarrow -\psi$ , which in total means

$$0 = \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx$$

Finally, from the previous results, we obtain

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle,$$

and we are almost done. It only remains to show the traces are correct, but since  $u^n \rightharpoonup u$  in  $W^{1,r}(\Omega)$  and from the continuity of the traces, we obtain

$$\text{tr } u = \text{tr } u_0.$$

**Uniqueness:** Let  $u_1, u_2$  be two solutions. Use strict monotonicity, subtract the weak formulation and test with  $u_2 - u_1$ :

$$\int_{\Omega} (a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla (u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)_{:=T} \, dx = 0,$$

where  $T \geq 0$ , so from strict monotonicity we obtain  $T = 0$  a.e. in  $\Omega$  but that means  $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$ , a.e. in  $\Omega \Rightarrow u_1 = u_2$  in  $W^{1,r}(\Omega)$ .  $\square$

**Example** (Nonlinearities vs weak convergence). Let  $g_n(x) = \sin(nx)$ , then  $g \rightarrow 0$  in  $L_2((0,4))$  (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx) \varphi \, dx \geq \int_2^4 \sin^2(nx) \, dx \rightarrow \frac{1}{2} \neq 0, \forall \varphi \in L_2((0,4)),$$

so  $\{u_n^2\} = \{\sin^2(nx)\}$  **does not converge weakly to**  $0 = 0^2$ .

*Remark.* The method of the presented proof is **very important**.

**Theorem 17.** Let  $\Omega \in C^{0,1}$ . Let  $X = W_0^{1,r}(\Omega)$ ,  $r \in (1, \infty)$  with equivalent norm  $\|u\| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$ . Then

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t. : } \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

*Proof.* We solve the problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{r-2} \nabla u) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Such  $u \in W_0^{1,r}(\Omega)$  exists and is unique by the above theorem. In this case:  $a(x, z, p) = |p|^{r-2}p, a_0() = 0$ . Coercivity is clear, monotonicity will be shown in the tutorials. Write the weak formulation of the above problem:

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \, dx = \Phi(\varphi).$$

Set

$$\mathbf{F} = |\nabla u|^{r-2} \nabla u,$$

and test the weak formulation with  $u$  itself:

$$\|\nabla u\|_{L_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{L_r(\Omega)}.$$

If now  $\|\nabla u\|_{L_r(\Omega)} = 0$ , then  $\Phi = 0$  and we are finished, if it is nonzero, then

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} = \| |\nabla u|^{r-1} \|_{L_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{L_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{L_{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{B_X(0,1)} |\Phi(\varphi)| = \sup_{B_X(0,1)} \left| \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx \right| \leq \sup_{B_X(0,1)} \|\mathbf{F}\|_{L_{r'}(\Omega)} \|\nabla \varphi\|_{L_r(\Omega)} = \|\mathbf{F}\|_{L_{r'}(\Omega)},$$

so

$$\|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

□

## 5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset W^{1,r}(\Omega) \rightarrow \mathbb{R}, u \mapsto \int_{\Omega} F(\cdot, u, \nabla u) \, dx,$$

with the basic assumptions  $\Omega \in C^{0,1}, r \in (1, \infty), X = u_0 + W_0^{1,r}(\Omega), u_0 \in W^{1,r}(\Omega), F : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in L_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x, z, p) \geq C_1 |p|^r - c_2(x).$$

*Remark.* From the assumptions it follows  $\int_{\Omega} F(\cdot, u, \nabla u) \, dx$  is defined  $\forall u \in W^{1,r}(\Omega)$ .

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

**Lemma 10.** *Let  $\Omega \in C^{0,1}, r \in (1, \infty), X = u_0 + W_0^{1,r}(\Omega), u_0 \in W^{1,r}(\Omega), F$  Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall a. a x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x, z, p)| + |\partial_z F(x, z, p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x, \cdot, \cdot) \in C^1(\mathbb{R}^{d+1}).$$

Let now  $u \in u_0 + W_0^{1,r}(\Omega)$  be a local minimizer of  $I$  over  $X$ , i.e.,

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_{\Omega} F(\cdot, u, \nabla u) dx \leq \int_{\Omega} F(\cdot, u+v, \nabla(u+v)) dx, F(\cdot, u, \nabla u) \in L_1(\Omega).$$

Then  $u$  is the weak solution to the **Euler-Lagrange equations**:

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u) + \partial_z F(\cdot, u, \nabla u)) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega \end{cases},$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi + \partial_z F(\cdot, u, \nabla u) \varphi dx = 0, \text{tr } u = \text{tr } u_0 \text{ on } \partial\Omega.$$

*Proof.* First  $\text{tr } u = \text{tr } u_0$  holds, so we are fine. Now fix  $\varphi \in \mathcal{D}(\Omega)$  and define

$$\iota : \mathbb{R} \rightarrow \mathbb{R}^*, \iota(\tau) = \int_{\Omega} \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:=l(\tau, \cdot)} dx.$$

Now  $\iota$  has a local minimum in 0. We show that  $\iota'(0)$  exists and is equal to the of Euler-Lagrange equations.

- $l(\tau, \cdot)$  is measurable for  $\tau$  from some neighbourhood of 0.
- $l(\tau, \cdot)$  is differentiable

Moreover

$$\begin{aligned} \partial_{\tau} l(\tau, \cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla \varphi = \\ &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla \varphi. \end{aligned}$$

Also

$$i(0) = \int_{\Omega} F(\cdot, u, \nabla u) dx < \infty$$

and

$$|\partial_{\tau} l(\tau, \cdot)| \leq (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla \varphi|^r) + |h(x)|)(|\varphi| + |\nabla \varphi|) \in L_1(\Omega).$$

Altogether,  $\iota(\tau)$  is finite on  $(-1, 1)$ ,  $\iota'(\tau)$  exists and

$$\iota'(0) = \int_{\Omega} \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi dx.$$

□

**Example.** Let

$$F(x, z, p) = \frac{1}{r}(1 + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x, z, p) = \left( \frac{r}{2} \frac{1}{r} 2(1 + |p|^2)^{\frac{r-2}{2}} \right) p - G = (1 + |p|^2)^{\frac{r-2}{2}} p - G, \partial_z F(x, z, p) = -g.$$

We have

$$|(1 + |p|^{\frac{r-2}{2}})p| \leq (1 + |p|^2)^{\frac{r-2}{2}} (1 + |p|^2)^{\frac{1}{2}} = (1 + |p|^2)^{\frac{r-1}{2}} \leq C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left( (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \right) = -\nabla \cdot G + g, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} (G \cdot \nabla \varphi + g\varphi) \, dx.$$

*Remark.* We have  $\{u_n\} \subset X$  s.t.  $\lim_{n \rightarrow \infty} I(u_n) = \inf_X I$ . Then use

- compactness:  $u_n \rightarrow u$  in some sense (weak convergence)
- weak lower semicontinuity  $I(u) \leq \liminf_{n \rightarrow \infty} I(u_n)$

**Lemma 11.** *Let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $F \in^1(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$ . Then*

1.  $F$  is (strictly) convex  $\Leftrightarrow \nabla F$  is (strictly) monotone
2. If  $F$  is (strictly) convex, then

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2 : F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

*Proof.* Fix  $\xi_1, \xi_2, \xi_1 \neq \xi_2$ , define  $\varphi(t) = F(\xi_2 + t(\xi_1 - \xi_2))$ . Then  $\varphi \in C^1(\mathbb{R})$  and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1 - \xi_2)) \cdot (\xi_1 - \xi_2).$$

So

$$" \Rightarrow " : (\nabla F(\xi_1) - \nabla F(\xi_2)) \cdot (\xi_1 - \xi_2) = \varphi'(1) - \varphi'(0) \underset{\varphi \text{ convex or strictly convex}}{\geq} 0.$$

And  $" \Leftarrow "$  : Fix  $t_1 > t_2$  and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1 - \xi_2)) - \nabla F(\xi_2 + t_2(\xi_1 - \xi_2))) \cdot (\xi_1 - \xi_2)(t_1 - t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1 - \xi_2)(t_1 - t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2)) \cdot (\eta_1 - \eta_2)$$

and we are in the same situation as before. For 2) we already know  $F$  (strictly) convex  $\Rightarrow \varphi$  (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \rightarrow \varphi'(0),$$

as  $t \rightarrow 0_+$ . And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

## 6 Exercises

### 6.1 4.3.2025

**Example** (Coefficients for smooth extension). Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for  $u \in \mathcal{D}(\mathbb{R}^d)$ . Find  $\{c_j\}_{j=1}^{k+1}$  in such a way that  $Eu \in C^k(\mathbb{R}^d)$ . Moreover, take  $d = 1$ .

*Proof.* For  $k = 0, j = 1$  we take  $c_1 = 1, c_j = 0, j \neq 1$ . For  $k = 1$ , compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k+1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take  $x_d = 0$  in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \dots, k\}.$$

That is a linear system of  $k + 1$  equations. Is it solvable? □

### 6.2 8.4.2025

**Example** (Laplace). Let  $a_0 = 0, a(\cdot, z, p) = p$ . Then  $|a(\dots)| \leq |p|$ , growth can be accomplished for  $r = 2, a(\dots) \cdot p \geq |p|^2$ . We can thus apply the theorem to our laplace equation

**Example.** Let  $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}(1 + |p|^2)$ . Then it is clearly Caratheodory, it is bounded  $|a(\dots)| \leq |p| \frac{\pi}{2}$ , so the growth conditions yield, it is coercive as  $\operatorname{atan}(1 + |p|^2) \geq \frac{\pi}{4} |p|^2$ , and it is monotone

$$(\operatorname{atan}(1 + |p_1|^2) p_1 - \operatorname{atan}(1 + |p_2|^2) p_2)(p_1 - p_2) = \int_0^1 \sum_{j=1}^d \frac{d}{ds} \operatorname{atan}(1 + |p_2 + s(p_1 - p_2)|^2) (p_2 + s(p_1 - p_2)) ds (p_1 - p_2)_j$$