Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms ($\mathcal{F}$ is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for $m > N + 1$, *i.e.*, $m - 1 > N$ it holds $U_m = \Omega_{m+1} / \overline{\Omega_{m-1}}, \Omega_N \subset \Omega_{m-1}$ meaning $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \varnothing \Rightarrow \operatorname{supp} u\varphi_j \cap \Omega_N = \operatorname{supp} v_j \cap \Omega_N = \varnothing$, since $\operatorname{supp} u\varphi_j \subset U_j, \operatorname{supp} v_j \subset \operatorname{supp} u\varphi_j \subset U_j, \forall j \geq m$. The norm of sum is

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let $N \to \infty$ and realize

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} \,\mathrm{d}x \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega}(x) \,\mathrm{d}x = \int_{\Omega} |D^\alpha f| \,\mathrm{d}x\,,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Alltogether,

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$ (this was not totally evident) and thus $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega) \cap C^\infty(\Omega)$ so indeed we have showed the desired density. □

*Remark.* It is nice that we only require $\Omega$ to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

*Remark* ($\mathrm{C}^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $\mathrm{C}^{k,\lambda}$ if: $\Omega$ is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0, 1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : \mathrm{B}(0, \alpha) \subset \mathbb{R}^{d-1} :\to \mathbb{R}\; s.t.\; a_j \in \mathrm{C}^{k,\lambda}(\mathrm{B}(0, \alpha)), \exists \mathbb{A}_j \mathbb{R}^d \to \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \leq m : \varnothing \neq \partial\Omega \cap U_j = \mathbb{A}_j\left(\left\{(x', a_j(x') \in \mathbb{R}^d | x' \in \mathrm{U}(0, \alpha) \subset \mathbb{R}^{d-1}\right\}\right)$,
3. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in \mathrm{U}(0, \alpha), b \in (0, \beta)\}) \subset \Omega$,
4. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in \mathrm{U}(0, \alpha), b \in (0, \beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$.