Charles University

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Thermodynamics and mechanics of solids

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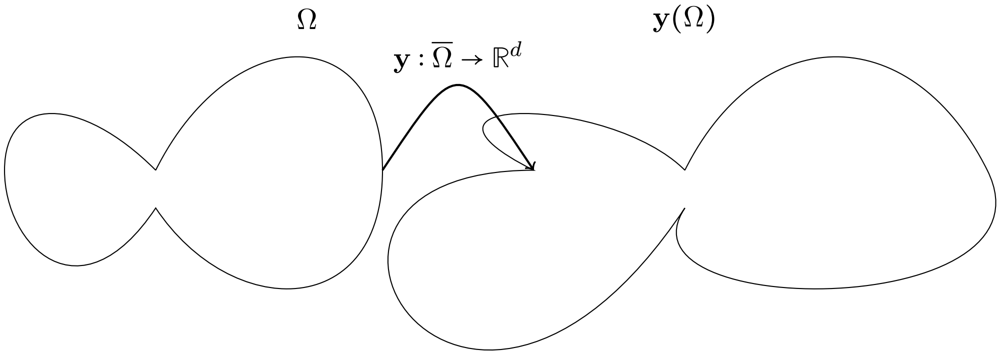
1 TODO

- include missing lecture about potential forces
- include missing lecture about rank one convexity

2 Geometry

2.1 Deformation

Suppose we are given an abstract body $\Omega \subset \mathbb{R}^d, d = 2, 3$. Choosing a particular state, we denote it the **reference configuration**. After the acting of forces, the body deforms into the **current, deformed configuration**.



The mapping that produces the deformation is called the **deformation**, denoted $\mathbf{y}$, i.e.

$$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^d.$$

Of large interest will be the **deformation gradient**

$$\mathbb{F}(\mathbf{x}) = \nabla y(\mathbf{x}), (\nabla y)_{ij} = \frac{\partial y^i}{\partial x^j},$$

on which we put some physically sound restrictions, such as

$$\det \mathbb{F} > 0.$$

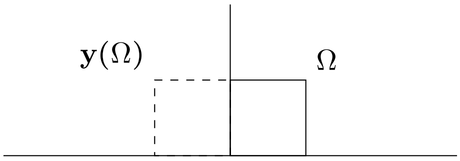
This means in particular that the determinant is nonzero, but also that the deformation preserves the orientation of bases:

$$(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0 \Rightarrow (\mathbb{F}\mathbf{e}_1 \times \mathbb{F}\mathbf{e}_2) \cdot \mathbb{F}\mathbf{e}_3 > 0.$$

Example. Suppose the deformation is given as

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x},$$

i.e., $\mathbb{F} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det \mathbb{F} = -1$. This is an example of a deformation that is *forbidden*.



Imagine it is a sheet of paper in a plane - you cannot reflect it without lifting it from the plane.

2.2 Displacement

Another useful way of describing the deformation is by using the **displacement vector u**:

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x},$$

so taking the gradient gives

$$\nabla \mathbf{u}(\mathbf{x}) = \mathbb{F}(\mathbf{x}) - \mathbb{I},$$

or in other words

$$\mathbb{F}(\mathbf{x}) = \mathbb{I} + \nabla \mathbf{u}(\mathbf{x}).$$

Remark. It is interesting that we do not place any restrictions on the determinant of the displacement gradient.

2.3 Changes of measures

We need to examine how the volume, area and lengths change under deformation. In what follows, for a set $\omega \subset \mathbb{R}^d$ in the reference configuration we denote $\omega^y \subset \mathbb{R}^d$ to be the deformed body in the current configuration, i.e.

$$\omega^y = \mathbf{y}(\omega).$$

2.3.1 Change of volume

Using the change of variable theorem we obtain (realize $\det \mathbb{F} > 0$)

$$\lambda(\omega^y) = \int_{\omega^y} 1 \, \mathrm{d}\mathbf{x}^y = \int_{\omega} \det \mathbb{F}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,,$$

so we write $\mathrm{d}\mathbf{x}^y = \det \mathbb{F} \, \mathrm{d}\mathbf{x}$. This motivates "our" definition of the determinant of the deformation gradient:[1]

$$\det \mathbb{F}(\mathbf{x}) = \lim_{r \to 0^+} \frac{\lambda(\mathbf{y}(B(\mathbf{x}, r))}{\lambda(B(\mathbf{x}, r))}, \tag{1}$$

where $B(\mathbf{x}, r)$ is a (closed) ball centered at $\mathbf{x}$ of radius r.

[1] This is in fact just the Lebesgue differentiation theorem.

2.3.2 Change of lengths

Suppose the line segment $\mathbf{x} + \Delta\mathbf{x}$ undergoes deformation. How does its length change? Taylor expansion yields:

$$\mathbf{y}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \mathbb{F}(\mathbf{x})\Delta\mathbf{x} + \text{h.o.t.},$$

where h.o.t. stands for higher order terms. Using the expansion, we can write

$$|\mathbf{y}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{y}(\mathbf{x})|^2 = (\Delta\mathbf{x})^\top \mathbb{F}^\top \mathbb{F} \Delta\mathbf{x} = (\Delta\mathbf{x})^\top \mathbb{C}(\mathbf{x})\Delta\mathbf{x},$$

where we have denoted

$$\mathbb{C}(\mathbf{x}) = \mathbb{F}^\top(\mathbf{x})\mathbb{F}(\mathbf{x}),$$

as the **Right Cauchy Green tensor**. Realize that in fact

$$\mathbb{C} : \overline{\Omega} \to \overline{\Omega}, \mathbb{C} : \mathbf{x} \mapsto \mathbb{F}^\top(\mathbb{F}\mathbf{x}),$$

and recall that $\mathbb{C}$ is in fact the metric tensor on $\mathbf{y}(\omega)$ (for admissable $\mathbf{y}$.)

Example. Let the deformation $\mathbf{y}$ be given as $\mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v}, \mathbf{v} \in \mathbb{R}^d, \mathbb{R} \in \mathrm{SO(d)}$, where [2]

$$\mathrm{SO(d)} = \left\{\mathbb{A} \in \mathbb{R}^{d\times d}, \mathbb{A}^\top\mathbb{A} = \mathbb{A}\mathbb{A}^\top = \mathbb{I}, \det\mathbb{A} = 1 \det\mathbb{A} > 0\right\}.$$

Then $\mathbb{F} = \mathbb{R}$, $\mathbb{C} = \mathbb{I}$.

2.3.3 Change of surfaces

For $\mathbb{A} \in \mathbb{R}^{d\times d}$ regular we define its **cofactor matrix** $\operatorname{cof}\mathbb{A}$ as

$$\operatorname{cof}\mathbb{A} = (\det\mathbb{A})\mathbb{A}^{-\top},$$

which is an interesting quantity whatsoever; we will be quite often using the following theorem

Theorem 1 (Piola's identity)**.** *Let* $\mathbf{y} \in C^2(\Omega; \mathbb{R}^d)$*, then* $\forall \mathbf{x} \in \Omega$:

$$\nabla \cdot (\operatorname{cof}\nabla\mathbf{y}(\mathbf{x})) = \mathbf{0}.$$

For a regular matrix $\mathbb{A}$, we also have the identity

$$\mathbb{A}^{-1} = \frac{1}{\det\mathbb{A}}(\operatorname{cof}\mathbb{A})^\top, \tag{2}$$

What about the determinant of the cofactor? Clearly

$$\det\operatorname{cof}\mathbb{A} = (\det\mathbb{A})^d \det\mathbb{A}^{-\top} = (\det\mathbb{A})^{d-1},$$

so we can also express equation 2 in a different way

$$\mathbb{A}^{-1} = \frac{(\operatorname{cof}\mathbb{A})^\top}{(\det\operatorname{cof}\mathbb{A})^{\frac{1}{d-1}}}. \tag{3}$$

From geometry, recall the change of variables for surface integration:

$$\int_{\partial\omega^y} \mathbf{n}^y(\mathbf{x}^y)\,\mathrm{d}S^y = \int_{\partial\omega} \operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})\,\mathrm{d}S\,,$$

[2]From the fact $\mathbb{A}$ is orthogonal automatically follows $\det\mathbb{A} = \pm 1$.

where $\mathbf{n}^y(\mathbf{x}^y)$ is the outward unit normal to the deformed boundary ω^y at the point $\mathbf{x}^y \in \omega^y$. Informally, we write $\mathbf{n}^y \, \mathrm{d}S^y = \operatorname{cof} \mathbb{F}\mathbf{n} \, \mathrm{d}S$. We can also explicitely express the normal to the deformed boundary as

$$\mathbf{n}^y(\mathbf{x}^y) = \frac{\operatorname{cof} \mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})}{|\operatorname{cof} \mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})|}, \mathbf{x} \in \partial\omega, \mathbf{y}(\mathbf{x}) \in \partial\omega^y. \tag{4}$$

Notice we are silently assuming

$$\mathbf{y}(\partial\omega) = \partial\mathbf{y}(\omega) = \partial\omega^y.$$

2.4 Affine transformations

An example of a deformation is the so called **affine transformation**.

Example. Suppose the folowing deformation:

$$\mathbf{y}(\mathbf{x}) = \mathbb{A}\mathbf{x} + \mathbf{v}, \mathbb{A} \in \mathbb{R}^{d \times d}, \mathbf{v} \in \mathbb{R}^d, \det \mathbb{A} > 0.$$

Clearly then $\mathbb{F}(\mathbf{x}) = \mathbb{A}$.

It is crucial to realize how $\mathbb{F}, \mathbb{F}^\top, \mathbb{F}^{-1}\mathbb{F}^{-\top}$ work.

- $\mathbb{F}$ takes a vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration* and maps it to the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ in the *current configuration*
- $\mathbb{F}^{-1}$ takes the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ from the *current configuration* and maps it to the vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration*
- $\mathbb{F}^\top$ is defined through: $\mathbb{F}\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbb{F}^\top \mathbf{w}$, and since $\mathbb{F}$ is defined on the reference configuration, $\mathbb{F}^\top$ must take something from the *current configuration* and return something from the *reference configuration.*
- $\mathbb{F}^{-\top}$ consequently takes something from the *reference configuration* and maps it to something from the *current configuration.*

What when $\mathbb{C} = \mathbb{I}$? Can we say something about $\mathbb{F}$? Write $\mathbb{C} = \mathbb{F}^\top\mathbb{F} = \mathbb{I}$, so $\mathbb{F}^\top = \mathbb{F}^{-1}, \det \mathbb{F} > 0$, meaning $\mathbb{F} \in \mathrm{SO(d)}$,*i.e.*, $\mathbb{F}(\mathbf{x}) = \mathbb{R}(\mathbf{x}), \mathbf{x} \in \Omega$, where $\mathbb{R}$ is a rotation tensor. Investigate the cofactor of the deformation gradient:

$$\operatorname{cof} \mathbb{F} = \det \mathbb{F}\mathbb{F}^{-\top} = \operatorname{cof} \mathbb{R} = 1\mathbb{R}(\mathbf{x}) = \mathbb{F}(\mathbf{x}),$$

and so $\operatorname{cof} \mathbb{F} = \mathbb{F}$. Recall Piola's identity:

$$\mathbf{0} = \nabla \cdot \operatorname{cof} \mathbb{F} = \nabla \cdot \mathbb{F}(\mathbf{x}) = \nabla \cdot \nabla\mathbf{y}(\mathbf{x}) = \triangle \, \mathbf{y}(\mathbf{x}).$$

We have the identity:

$$\frac{1}{2} \triangle |\nabla\mathbf{y}|^2 = |\nabla\nabla\mathbf{y}|^2 + \nabla\mathbf{y} : \nabla \triangle \mathbf{y},$$

but since $\triangle \mathbf{y} = 0$ from the above and $\triangle |\nabla\mathbf{y}|^2 = \triangle \operatorname{tr}(\mathbb{F}^\top\mathbb{F}) = \triangle \operatorname{tr}(\mathbb{I}) = 0$, we also have $|\nabla\nabla\mathbf{y}|^2 = 0$, meaning the deformation must have have the form

$$\mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v} \, \text{locally} \, ,$$

meaning that $\mathbf{y}$ is only piecewise affine. We will show that it must however be globally affine. Let $\mathbf{y}$ be piecewise affine. Since $\mathbf{y}$ is continous on the whole $\overline{\Omega}$, it must be continuous across the faces of the partition and in particular

$$\mathbb{R}_1\mathbf{x} + \mathbf{v}_1 = \mathbb{R}_2\mathbf{x} + \mathbf{v}_2,$$

with $\mathbb{R}_1, \mathbb{R}_2 \in \mathrm{SO(d)}$ being rotations, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$ some constant vectors and $\mathbf{x} \in \{\mathbf{x}\cdot\mathbf{n} = c\}$ is a vector from the interface. Denoting $\mathbf{n}, \mathbf{t}$ to be the normal and tangential vector to the interface, one has (realize $\mathbf{x} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + (\mathbf{x}\cdot\mathbf{t})\mathbf{t}$)

$$(\mathbb{R}_1 - \mathbb{R}_2)\mathbf{n}\mathbf{x}\cdot\mathbf{n} + (\mathbb{R}_1 - \mathbb{R}_2)\mathbf{t}\mathbf{x}\cdot\mathbf{t} = \mathbf{v}_2 - \mathbf{v}_1, \forall \mathbf{x} \in \{\mathbf{x}\cdot\mathbf{n} = c\},$$
$$c(\mathbb{R}_1 - \mathbb{R}_2)\mathbf{n} + (\mathbb{R}_1 - \mathbb{R}_2)\mathbf{t} = \mathbf{v}_2 - \mathbf{v}_1.$$

Looking on the LHS, we see that the first term in constant on the hyperplane, but the second one is not - the tangential vector depends on the position. However, the RHS is constant, which must mean

$$(\mathbb{R}_1 - \mathbb{R}_2)\mathbf{t} = 0,$$

or

$$\mathrm{Ker}\,(\mathbb{R}_1 - \mathbb{R}_2) = \mathrm{span}\,\{\mathbf{t}\} = \mathbf{n}^\perp.$$

But if that is true, all nontrivial bussiness is happening only for vectors that are perpendicular to $\mathbf{t}$,*i.e.*, that are in the direction of $\mathbf{n}$, so it must hold

$$\mathbb{R}_1 - \mathbb{R}_2 = (\mathbb{R}_1 - \mathbb{R}_2)(\mathbf{n}\otimes\mathbf{n}).$$

But that must mean $\mathbb{R}_1 = \mathbb{R}_2$: we see the rank of the RHS is 1, but the rank of the LHS is at least 2: one has

$$\dim\,\mathrm{Ker}\,(\mathbb{R}_1 - \mathbb{R}_2) + \mathrm{rank}\,(\mathbb{R}_1 - \mathbb{R}_2) = d,$$

and the dimension of the kernel is 1, as derived above. So, $\mathrm{rank}\,(\mathbb{R}_1 - \mathbb{R}_2) = d - 1 \geq 2$, for[3] $d \geq 3$. Finally, using one of the original equations

$$c(\mathbb{R}_1 - \mathbb{R}_2)\mathbf{n} + (\mathbb{R}_1 - \mathbb{R}_2)\mathbf{t} = \mathbf{v}_2 - \mathbf{v}_1,$$

and our fresh information $(\mathbb{R}_1 - \mathbb{R}_2)\mathbf{t} = \mathbf{0}, \mathbb{R}_1 = \mathbb{R}_2$, we conclude that also

$$\mathbf{v}_2 = \mathbf{v}_1.$$

In total, we have obtained $\mathbb{R}_1 = \mathbb{R}_2 \equiv \mathbb{R}, \mathbf{v}_1 = \mathbf{v}_2 \equiv \mathbf{v}$ and the transformation is affine.

Definition 1 (Types of deformation)**.** The deformation $\mathbf{y} : \overline{\Omega} \to \mathbb{R}^d, \mathbb{F}(\mathbf{x}) = \nabla\mathbf{y}(\mathbf{x})$ is called

- homogenous, if $\mathbb{F}$ is constant in $\overline{\Omega}$,
- rigid, if it is homogenous and $\mathbb{F} = \mathbb{R} \in \mathrm{SO(d)}$,
- incompressible, if $\det\mathbb{F} = 1$.

3 Forces

Naturally, the deformation is caused by the presence of some forces. To capture the evolution of the shape of the body throughout the deformation even when the forces are known is kind of

[3] In the case $d = 2$ we have to argue differently, but the assertion is true.

complicated, as it is nonlinear from its very nature (we will see this in the upcoming chapter). Classical physics gives us some balance laws, that usually hold in inertial frames, *i.e.*, in time-constant frames. Hence, formulating fundamental balance laws will always be easier in the deformed, current configuration, rather then in the reference configuration. Let us so begin with the study of forces in the currrent configuration.

3.1 Forces in the deformed configuration

Recall that our assumption always is $\mathbf{y}(\overline{\Omega}) = \overline{\Omega^y} = \overline{\Omega}^y$, for $\Omega^y = \mathbf{y}(\Omega)$. We are thus able to define the **volume density of applied body forces**

$$\mathbf{f}^y : \overline{\Omega}^y \to \mathbb{R}^3$$

(in newtons per cubic meters, e.g. gravity). The same on the boundary

$$\mathbf{g}^y : \Gamma_N^y \to \mathbb{R}^3$$

(**surface density of applied contact forces** (in newtons per square meters = Pascals, e.g. hydrostatic pressure.)

3.1.1 Cauchy stress tensor

Lemma 1 (Stress principle of Euler and Cauchy)**.** *There exists a (Cauchy) stress vector function* $\mathbf{t}^y : \overline{\Omega}^y \times \mathcal{S}^{d-1} \to \mathbb{R}^d$ *with the following properties.*

1. *If* $\mathbf{x}^y \in \Gamma_N^y$, *then* $\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbf{g}^y(\mathbf{x}^y)$, *where* $\mathbf{n}^y$ *is the unit outer normal vector to* $\partial\Omega^y$ *at* $\mathbf{x}^y$.
2. $\forall \omega^y \subset \Omega^y$ *it holds that*
$$\int_{\omega^y} \mathbf{f}(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)\,\mathrm{d}S^y = \mathbf{0}.$$
(Balance of forces in static equilibrium.)
3. $\forall \omega^y \subset \Omega^y$ *it holds that*
$$\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)\,\mathrm{d}S^y = \mathbf{0}.$$
(Balance of momenta of forces in static equilibrium.)

Euler says (while thinking of the Newton's 3rd law) that the direct consequence of this is the existence of $\mathbb{T}^y(\mathbf{x}^y)$ *such that*

$$\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y, \tag{5}$$

where the tensorial quantity (field) $\mathbb{T}^y(\mathbf{x}^y)$ *is called the* ***Cauchy stress tensor****.*

3.1.2 Balance equations in the deformed configuration

Classical physics gives us 2 fundamental relations: Newtons second law for momenta and for angular momenta. We examine these in the continuum mechanics setting.

Using the second property together with 5 gives

$$\int_{\omega^y} \mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\,\mathrm{d}S^y = \int_{\omega^y} \mathbf{f}^y(\mathbf{x})^y\,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \nabla\cdot\mathbb{T}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y = 0, \tag{6}$$

so using the localization theorem we get

$$\mathbf{f}^y(\mathbf{x}^y) + \nabla\cdot\mathbb{T}^y(\mathbf{x}^y) = \mathbf{0}, \forall \mathbf{x}^y \in \Omega^y. \tag{7}$$

From the third property it follows

$$\begin{aligned}\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\,\mathrm{d}S^y &= \int_{\omega^y} \varepsilon_{ijk}x_j^y f_k^y \mathbf{e}_i\,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk}x_j^y(T_{km}^y n_m^y)\mathbf{e}_i\,\mathrm{d}S^y = \\ &= \int_{\omega^y} \varepsilon_{ijk}x_j^y f_k^y \mathbf{e}_i\,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk}\frac{\partial(x_j^y T_{km}^y)}{\partial x_m^y}\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y = \\ &= \int_{\omega^y} \varepsilon_{ijk}x_j^y f_k^y \mathbf{e}_i\,\mathrm{d}\mathbf{x}^y + \\ &+ \int_{\omega^y} \varepsilon_{ijk}x_j^y \frac{\partial T_{km}^y}{\partial x_m^y}\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk}\delta_{jm}T_{km}^y\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y = \mathbf{0},\end{aligned}$$

realize now

$$\int_{\omega^y} \varepsilon_{ijk}\left(x_j^y f_k^y + x_j^y \frac{\partial T_{km}^y}{\partial x_m^y}\right)\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y = \int_{\omega^y} \varepsilon_{ijk}x_j^y\left(f_k^y + \underbrace{\frac{\partial T_{km}^y}{\partial x_m^y}}_{=(\nabla\cdot\mathbb{T}^y)_k}\right)\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y = \mathbf{0},$$

because the balance of forces 7 holds. The balance of angular momenta thus reduces to

$$\int_{\omega^y} \varepsilon_{ijk}T_{kj}^y\mathbf{e}_i\,\mathrm{d}\mathbf{x}^y = \mathbf{0},$$

and using the localization theorem, we obtain

$$T_{kj}^y(\mathbf{x}^y) = T_{jk}^y(\mathbf{x}^y),\ i.e.\, \mathbb{T}^y(\mathbf{x}^y) = (\mathbb{T}^y(\mathbf{x}^y))^\top. \tag{8}$$

The **Cauchy stress tensor is symmetric.**

3.2 Forces in the undeformed configuration

We have obtained the equations in the deformed configuration, where they are easily formulated. On the other hand, that is a bit unconvenient - we solve the equations to find the deformed configuration, *i.e.*, the equations hold in the domain that is obtained as a solution to the equations themselves. This brings us to to the need to find a new way to write the down the balance laws - in the reference configuration. The construction is a bit synthetic, our intuition will be guided by the requirement to obtain similair equations as in the current configuration.

3.2.1 Piola-Kirchhoff stresses

Definition 2 (First Piola-Kirchhoff stress tensor)**.** Given the Cauchy stress tensor $\mathbb{T}^y(\mathbf{x}^\mathbf{y})$, we define the **First Piola Kirchhoff stress tensor**

$$\mathbb{T} : \overline{\Omega} \to \mathbb{R}^{3\times 3}, \mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{y}(\mathbf{x}))\,\mathrm{cof}\,\mathbb{F}(\mathbf{x}) = \det\mathbb{F}(\mathbf{x})\mathbb{T}^y(\mathbf{y}(\mathbf{x}))\mathbb{F}^{-\top}(\mathbf{x}).$$

Definition 3 (Second Piola-Kirchhoff stress tensor)**.** The quantity

$$\mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1}(\mathbf{y}(\mathbf{x}))\mathbb{T}(\mathbf{x}),$$

is called the **second Piola-Kirchhoff stress tensor.**

Remark. The first Piola-Kirchhoff stress tensor $\mathbb{T}$ is *not symmetric in general*, but the second

$$\mathbb{S} = \det \mathbb{F}\mathbb{F}^{-1}\mathbb{T}^y\mathbb{F}^{-\top}$$

is symmetric. Also, we see that not every matrix can serve as $\mathbb{T}$; it must hold $\mathbb{T}(\operatorname{cof}\mathbb{F})^{-1}$ is symmetric.

Remark. We have the following identity (contrary to the appearance, this is not a trivial computation and one has to use the Piola's identity):

$$\nabla_{\mathbf{x}} \cdot \mathbb{T}(\mathbf{x}) = \det \mathbb{F}(\mathbf{x}) \left(\nabla_{\mathbf{x}^y} \cdot \mathbb{T}^y(\mathbf{x}^y)\right)\Big|_{\mathbf{x}^y=\mathbf{y}(\mathbf{x})}. \tag{9}$$

3.2.2 Balance equations in the deformed configuration

Recall the balance of momentum

$$-\nabla \cdot \mathbb{T}^y(\mathbf{x}^y) = \mathbf{f}^y(\mathbf{x}^y) \text{ in } \Omega^y,$$

multiplying by $\det \mathbb{F} > 0$ yields

$$\det \mathbb{F}(\mathbf{x})\nabla \cdot (\mathbb{T}^y(\mathbf{x}^y)) = \det \mathbb{F}(\mathbf{x})\mathbf{f}^y(\mathbf{x}^y).$$

Using the Piola's identity we have shown 9, so the left hand side actually is $-\nabla \cdot \mathbb{T}(\mathbf{x})$. Seeing the similarity with the balance of forces in the current configuration, we are tempted to denote

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}^y(\mathbf{y}(\mathbf{x})) \det \mathbb{F}(\mathbf{x}),$$

as the volume density of body forces in the *reference configuration*; after this definition, the balance of forces becomes

$$-\nabla \cdot \mathbb{T}(\mathbf{x}) = \mathbf{f}(\mathbf{x}),$$

and since this equation is formulated in the reference configuration, we call it the **balance of forces in the reference configuration.**[4]

Viewed from an "integral" perspective, the total applied body force on the body can be written as

$$\int_{\omega^y} \mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y = \int_{\omega} \mathbf{f}^y(\mathbf{y}(\mathbf{x})) \det \mathbb{F}(\mathbf{x})\,\mathrm{d}x = \int_{\omega} \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

We can do the same for the balance of surface forces; the total contact force acting on the body is

$$\begin{aligned}\int_{\partial\omega^y} \mathbf{g}^y(\mathbf{x}^y)\,\mathrm{d}S^y &= \int_{\partial\omega^y} \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)\,\mathrm{d}S^y = \int_{\partial\omega^y} \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y(\mathbf{x}^y)\,\mathrm{d}S^y = \\ &= \int_{\partial\omega} \mathbb{T}^y(\mathbf{y}(\mathbf{x})) \operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})\,\mathrm{d}S = \int_{\partial\omega} \mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x})\,\mathrm{d}S = \int_{\partial\omega} \mathbf{g}^y(\mathbf{y}(\mathbf{x}))|\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})|\,\mathrm{d}S\,,\end{aligned}$$

[4] This title is used even though it is slightly misleading - the forces are still acting in the current configuration, but are expressed through the reference configuration.

so if we define

$$\mathbf{g}(\mathbf{x}) = \mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{g}^y(\mathbf{y}(\mathbf{x}))|\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})|,$$

as the contact force in the **reference configuration**, we formally have a similiar expression.

Remark. Let us summarize our stress measures.

- the Cauchy stress tensor: $\mathbb{T}^y = \mathbb{T}^y(\mathbf{x}^y)$. The Cauchy stress tensor is connected with the surface density of the contact forces in the *current configuration* given a point in the *current configuration.* In concrete terms

$$\mathbf{g}^y(\mathbf{x}^y) = \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y(\mathbf{x}^y),$$

- the First Piola-Kirchhoff stress tensor: $\mathbb{T} = \mathbb{T}(\mathbf{x}), \mathbb{T}(\mathbf{x}) = \det\mathbb{F}(\mathbf{x})\mathbb{T}^y(\mathbf{y}(\mathbf{x}))\operatorname{cof}\mathbb{F}(\mathbf{x})$. The First Piola-Kirchhoff stress tensor is connected with the surface density of the contact forces in the *current configuration* given a ppint in the *reference configuration.* In concrete terms

$$\mathbf{g}(\mathbf{x}) = \mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x}),$$

- the Second Piola-Kirchhoff stress tensor $\mathbb{S} = \mathbb{S}(\mathbf{x}), \mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1}(\mathbf{y}(\mathbf{x}))\mathbb{T}(\mathbf{x})$ works similairly to the First Piola-Kirchhoff tensor, but is symmetric.

Note carefully that in both cases we are talking about forces in the *current configuration*, but these can be expressed either through the *reference configuration* (with the help of $\mathbb{T}$) or through the *current configuration* itself (by using $\mathbb{T}^y$).

4 Elasticity

So far, our excursion to the world of solid mechanics has been fairly general. We have formulated fundamental mechanical laws independetly of the material that undergoes the deformation. In this chapter, we will discuss a class of materials known as **elastic solids.**

Definition 4 (Elasticity)**.** We say that a material is **elastic (or Cauchy elastic)** if the Cauchy stress tensor is determined only by the current configuration. In morece concrete terms, the material is elastic provided there is a response function $\tilde{\mathbb{T}}^D : \Omega \times \mathbb{R}^{3\times3}_{+} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ such that (Kružík and Roubíček, 2019)

$$\mathbb{T}^y(\mathbf{y}(\mathbf{x})) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}).$$

The response function is also called the **constitutive law**. Realize that this consideration is quite restrictive

- the response does not depend on any thermodynamical quantities (explicitely): temperature, dissipation, etc.,
- the response depends only on the gradient of the deformation, not on the deformation itself,
- the response function *does depend* on the reference configuration, but the stress *does not.*

Remark. If we know the material is elastic, we also have the information about the First Piola-Kirchhoff stress, as $\mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{x}^y)\operatorname{cof}\mathbb{F}(\mathbf{x})$, so

$$\mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{y}(\mathbf{x}))\operatorname{cof}\mathbb{F}(\mathbf{x}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\operatorname{cof}\mathbb{F}(\mathbf{x}) := \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}). \tag{10}$$

We will be using both $\tilde{\mathbb{T}}^D, \tilde{\mathbb{T}}$ from now on.

4.1 Frame invariance principle

The principle of (material) frame invariance, or (material) frame indifference is closely connected with the notion of **objectivity**. Those terms concern the change of observer: a transformation $\mathbf{x} \mapsto \mathbb{R}\mathbf{x}$, for some $\mathbb{R} \in \mathrm{SO}(3)$. A certain invariance is fundamental to (classical) physics - recall the Galileo's principle of relativity. At the moment, we will deal ourselves with "objective vectors and tensors." That is, suppose we are given the stress vector $\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)$ and a different deformation

$$\mathbf{z}(\mathbf{x}) = \mathbb{R}\mathbf{y}(\mathbf{x}), \mathbb{R} \in \mathrm{SO}(3)\,,$$

(this deformation is only a rotation, so it can in fact be seen as a change of the frame in the current configuration.) Denote as expected $\mathbf{x}^z = \mathbf{z}(\mathbf{x})$; the priniciple of frame invariance then states

$$\mathbf{t}^z(\mathbf{x}^z, \mathbf{n}^z) = \mathbf{t}^z(\mathbb{R}\mathbf{x}^y, \mathbb{R}\mathbf{n}^y) \equiv \mathbb{R}\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y), \forall \mathbb{R} \in \mathrm{SO}(3)\,,$$

i.e., if we just rotate the current configuration, the traction vector also only rotates. Realize that this in fact means

$$\mathbf{t}^z(\mathbf{x}^z, \mathbf{n}^z) = \mathbf{t}^z(\mathbf{x}^z, \mathbb{R}\mathbf{n}^y) = \mathbb{T}^z(\mathbf{x}^z)\mathbb{R}\mathbf{n}^y = \mathbb{R}\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbb{R}\mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y,$$

where we have denoted $\mathbb{T}^z$ to be the Cauchy stress tensor in the configuration after the deformation $\mathbf{z}$. This however means

$$\mathbb{T}^z(\mathbf{x}^z) = \mathbb{T}^z(\mathbb{R}\mathbf{x}^y) = \mathbb{R}\mathbb{T}^y(\mathbf{x}^y)\mathbb{R}^\top.$$

This is the transformation of the Cauchy stress tensor under the change of observer. It can be shown [5] that the deformation gradient transforms as $\mathbb{R}\mathbb{F}$. On the level of constitutive laws, the frame invariance principle states:

$$\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{R}\mathbb{F}) = \mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{R}^\top, \forall \mathbb{R} \in \mathrm{SO}(3)\,, \forall \mathbb{F} \in \mathbb{R}^{3\times 3}_+, \forall \mathbf{x} \in \overline{\Omega},$$

which is closely related to the transformation properties of the Cauchy stress tensor. See that we are not transorming $\mathbf{x}$ - that is a vector from the *reference configuration*, which remains unchanged in the change of observer transformation. For $\tilde{\mathbb{T}}$ (defined in 10) one has

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \det(\mathbb{R}\mathbb{F})\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{R}\mathbb{F})(\mathbb{R}\mathbb{F})^{-\top} = \det(\mathbb{R}\mathbb{F})\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{R}^\top\mathbb{R}\mathbb{F}^{-\top} = \det\mathbb{F}\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{F}^{-\top} = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}),$$

thus

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}), \forall \mathbb{R} \in \mathrm{SO}(3)\,, \forall \mathbb{F} \in \mathbb{R}^{3\times 3}_+, \forall x \in \overline{\Omega},$$

Notice that the primary formulation of the frame indifference principle is through the constitutive law for the Cauchy stress tensor, an objective tensor, from which the transformation of the First Piola-Kirchhoff tensor is derived.

4.2 Isotropic material

The principle of material frame indifference is a consequence of the fundamental invariance of laws of classical physics - it holds for any materials whatsoever. Still, some materials have further

[5] Let $\mathbf{x}^z = \mathbb{R}\mathbf{x}^y$, then

$$F^z_{ij} = \frac{\partial x^z_i}{\partial x_j} = \frac{\partial x^z_i}{\partial x^y_k}\frac{\partial x^y_k}{\partial x_j} = \frac{\partial}{\partial x^y_k}(R_{im}x^y_m)F^y_{kj} = R_{ik}F^y_{kj}.$$

important properties that should be captured in the constitutive law. We examine now the property of **material symmetry**.

Take $\mathbf{x}_0 \in \overline{\Omega}$ general but fixed, take

$$\mathbf{v}(\mathbf{z}) = \mathbf{x}_0 + \mathbb{R}^\top(\mathbf{z} - \mathbf{x}_0)$$

for some $\mathbb{R} \in \mathrm{SO}(3)$, so $(\tilde{\mathbf{x}} := \mathbf{v}(\mathbf{z}))$

$$\mathbf{v}^{-1}(\tilde{\mathbf{x}}) = \mathbf{x}_0 + \mathbb{R}(\tilde{\mathbf{x}} - \mathbf{x}_0),$$

and define a *new deformation* $\tilde{\mathbf{y}} = \mathbf{y} \circ \mathbf{v}^{-1} : \mathbf{v}(\overline{\Omega}) \to \mathbf{y}(\overline{\Omega})$, as

$$\tilde{\mathbf{y}}(\tilde{\mathbf{x}}) = \mathbf{y}(\mathbf{x}_0 + \mathbb{R}(\tilde{\mathbf{x}} - \mathbf{x}_0)).$$

Compute

$$\tilde{\mathbb{F}}(\tilde{\mathbf{x}}) = \nabla_{\tilde{\mathbf{x}}}\tilde{\mathbf{y}} = \mathbb{F}(\tilde{\mathbf{x}})\mathbb{R},$$

and notice that $\mathbf{x}_0^{\tilde{y}} = \tilde{\mathbf{y}}(\mathbf{x}_0) = \mathbf{y}(\mathbf{x}_0) = \mathbf{x}_0^{y}$, from which it follows

$$\mathbb{T}^y(\mathbf{x}_0^y) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \mathbb{F}) = \mathbb{T}^{\tilde{y}}(\mathbf{x}_0^{\tilde{y}}) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \tilde{\mathbb{F}}(\mathbf{x}_0)) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \mathbb{F}(\mathbf{x}_0)\mathbb{R}).$$

We see that at the point $\mathbf{x}_0$, the response function has the property

$$\tilde{\mathbb{T}}^D(\mathbf{x}_0, \tilde{\mathbb{F}}(\mathbf{x}_0)) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \tilde{\mathbb{F}}(\mathbf{x}_0)\mathbb{R}),$$

which motivates our definition

Definition 5 (Isotropic material)**.** We cal the material **isotropic** if it holds

$$\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}\mathbb{R}), \forall \mathbb{R} \in \mathrm{SO}(3), \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}, \forall \mathbf{x} \in \overline{\Omega}.$$

Remark. For the First Piola-Kirchhoff we obtain:

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}\mathbb{R}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}\mathbb{R}) \operatorname{cof}(\mathbb{F}\mathbb{R}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{F}^{-\top}\mathbb{R} \det \mathbb{F} = \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F})\mathbb{R},$$

In total, for a isotropic elastic material the constitutive law for the First Piola-Kirchhoff stress tensor has the property

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}_1\mathbb{F}\mathbb{R}_2) = \mathbb{R}_1\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F})\mathbb{R}_2, \forall \mathbb{R}_1, \mathbb{R}_2 \in \mathrm{SO}(3), \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}, \forall \mathbf{x} \in \overline{\Omega},$$

so in particular

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}\mathbb{R}^\top) = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F})\mathbb{R}^\top,$$

which will prove very useful later on.

Remark. We stress that instead of material indifference, that granted us $\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F})$, and which is valid for all materials, the second property $\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}\mathbb{R}) = \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F})\mathbb{R}$ holds *only for isotropic materials.*

4.3 Hyperelastic materials

Definition 6. We say that a material is hyperelastic (sometimes called Green elastic) if there is a function $W:\overline{\Omega}\times\mathbb{R}^{3\times3}_{+}\to\mathbb{R}$ such that (Kružík and Roubíček, 2019)

$$\mathbb{T}(\mathbf{x})=\tilde{\mathbb{T}}(\mathbf{x},\mathbb{F})=\frac{\partial W(\mathbf{x},\mathbb{F})}{\partial\mathbb{F}}.$$

The function $W,[W]=\frac{J}{m^3}=\frac{Nm}{m^3}=\mathrm{Pa}$, is called **stored energy density.**

Remark. See that this simple is the formula for the First Piola-Kirchhoff stress; for the Cauchy stress tensor, it gets more complicated.

4.4 Properties of W

Physics puts some assumption on W :

1. $W\geq 0$,
2. $W(\mathbf{x},\mathbb{R}\mathbb{F})=W(\mathbf{x},\mathbb{F}),\forall\mathbb{R}\in\mathrm{SO}(3),\forall\mathbf{x}\in\overline{\Omega},\forall\mathbb{F}\in\mathbb{R}^{3\times3}_{+}$,
3. $W(\mathbf{x},\underbrace{\mathbb{R}\mathbb{U}}_{=\mathbb{F}})=W(\mathbf{x},\mathbb{U}),\mathbb{U}=\sqrt{\mathbb{C}}$, (matrices are from the polar decomposition)
4. $W(\mathbf{x},\mathbb{F})\to\infty$ if $\det\mathbb{F}\to 0_+$,
5. $\exists\alpha>0,\exists p,q,r\geq 1\ s.t.\ W(\mathbf{x},\mathbb{F})\geq\alpha(\|\mathbb{F}\|^p+\|\mathrm{cof}\,\mathbb{F}\|^q+(\det\mathbb{F})^r)-d,\forall\mathbf{x}\in\overline{\Omega},\forall\mathbb{F}\in\mathbb{R}^{3\times3}_{+}$.

Let us comment on these briefly.

1. means really that energy is nonnegative,
2. states that energy does not change under the change of observer; see below for details[6]
3. energy changes only when the the geometry of the domain changes,
4. it takes infinite energy to deform the body to a point,
5. mostly a mathematical assumption - this yields *coercivity*

The principle of frame indifference told us

$$\tilde{\mathbb{T}}(\mathbf{x},\mathbb{R}\mathbb{F})=\mathbb{R}\tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}),$$

so

$$\mathbb{R}^\top\frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}}=\frac{\partial W(\mathbf{x},\mathbb{F})}{\partial\mathbb{F}},$$

we *suppose we can Taylor expand*:

$$\begin{aligned}W\big(\mathbf{x},\mathbb{R}(\mathbb{F}+\tilde{\mathbb{F}})\big)=W\big(\mathbf{x},\mathbb{R}\mathbb{F}+\mathbb{R}\tilde{\mathbb{F}}\big)&=W(\mathbf{x},\mathbb{R}\mathbb{F})+\frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}}:\big(\mathbb{R}\tilde{\mathbb{F}}\big)+\text{h.o.t.}\\&=W(\mathbf{x},\mathbb{R}\mathbb{F})+\mathbb{R}^\top\frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}}:\tilde{\mathbb{F}}+\text{h.o.t.},\end{aligned}$$

[6] We could also show that if the hyperelastic solid is also isotropic, we also have $W(\mathbf{x},\mathbb{F}\mathbb{R})=W(\mathbf{x},\mathbb{F})$, and so $W\big(\mathbf{x},\mathbb{R}\mathbb{F}\mathbb{R}^\top\big)=W(\mathbf{x},\mathbb{F}),\forall\mathbb{R}\in\mathrm{SO(d)}$, so W is an isotropic scalar function, and from the fact $W(\mathbf{x},\mathbb{F})=\hat{W}(\mathbf{x},\mathbb{C})$, in fact follows $W(\mathbf{x},\mathbb{F})=\hat{W}(\mathbf{x},\mathbb{C})=\hat{W}\big(\mathbf{x},\mathbb{R}\mathbb{C}\mathbb{R}^\top\big)$, and so $\hat{W}$ is a scalar isotropic function of a symmetric positive definite tensorial argument. This allows one to represent W using only the invariants of $\mathbb{C}$.

where we have used

$$\mathbb{A} : (\mathbb{B}\mathbb{C}) = \sqrt{\operatorname{tr}(\mathbb{A}^\top \mathbb{B}\mathbb{C})} = \sqrt{\operatorname{tr}(\mathbb{C}^\top \mathbb{B}^\top \mathbb{A})} = \sqrt{\operatorname{tr}(\mathbb{B}^\top \mathbb{A}\mathbb{C}^\top)} = \left(\mathbb{B}^\top \mathbb{A}\right) : \mathbb{C}.$$

Using the principle of frame indifference, we in fact have shown

$$W\left(\mathbf{x}, \mathbb{R}(\mathbb{F} + \tilde{\mathbb{F}})\right) = W(\mathbf{x}, \mathbb{R}\mathbb{F}) + \frac{\partial W(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}} : \tilde{\mathbb{F}}.$$

If we expand now just $W\left(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\right)$, we obtain

$$W\left(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\right) = W(\mathbf{x}, \mathbb{F}) + \frac{\partial W(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}} : \tilde{\mathbb{F}},$$

so upon subtraction we obtain

$$W\left(\mathbf{x}, \mathbb{R}\left(\mathbb{F} + \tilde{\mathbb{F}}\right)\right) - W\left(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\right) = W(\mathbf{x}, \mathbb{R}\mathbb{F}) - W(\mathbf{x}, \mathbb{F}),$$

and above we have shown this quantity has a zero first order expansion,*i.e.*

$$\frac{\partial}{\partial \mathbb{F}} \left(W(\mathbf{x}, \mathbb{R}\mathbb{F}) - W(\mathbf{x}, \mathbb{F})\right) = \mathbb{0}.$$

from which it follows [7]

$$W(\mathbf{x}, \mathbb{R}\mathbb{F}) = W(\mathbf{x}, \mathbb{F}) + k(\mathbb{R}).$$

If we first take $\mathbb{F} = \mathbb{I}$, then

$$W(\mathbf{x}, \mathbb{R}) = W(\mathbf{x}, \mathbb{I}) + k(\mathbb{R}),$$

and then $\mathbb{F} = \mathbb{R}$,

$$W\left(\mathbf{x}, \mathbb{R}^2\right) = W(\mathbf{x}, \mathbb{R}\mathbb{R}) = W(\mathbf{x}, \mathbb{R}) + k(\mathbb{R}) = W(\mathbf{x}, \mathbb{I}) + 2k(\mathbb{R}),$$

so

$$W(\mathbf{x}, \mathbb{R}^n) = W(\mathbf{x}, \mathbb{I}) + nk(\mathbb{R}), \forall \mathbb{R} \in \mathrm{SO(d)}$$

Since the set of rotations is closed and bounded, it is compact, so there exists a convergent subsequence of $\{\mathbb{R}^n\}$. Moreover, we assume W to be continuous (we took the derivative...), so $\lim_{n\to\infty} W(\mathbf{x}, \mathbb{R}^n)$ exists and from the properties of W we get it is finite. But then $k(\mathbb{R}) = 0$, as otherwise $nk(\mathbb{R}) \to \infty$. All in all, we have shown

$$W(\mathbf{x}, \mathbb{R}\mathbb{F}) = W(\mathbf{x}, \mathbb{F}), \forall \mathbb{R} \in \mathrm{SO(d)}\,, \forall \mathbb{F} \in \mathbb{R}_+^{d\times d}, \forall x \in \overline{\Omega}. \tag{11}$$

This also implies the property

$$W(\mathbf{x}, \mathbb{F}) = W(\mathbf{x}, \mathbb{R}\mathbb{U}) = W(\mathbf{x}, \mathbb{U}),$$

as from the properties of the polar decompoisiton one has $\mathbb{R} \in \mathrm{SO(d)}\,, \mathbb{U} \in \mathbb{R}_+^{d\times d}$.

Remark. Evidently, W has a potential.

Definition 7 (Natural state of the body)**.** The natural state of the body is the state in which

$$W(\mathbf{x}, \mathbb{F}) = 0 \wedge \frac{\partial W}{\partial \mathbb{F}}(\mathbf{x}, \mathbb{F}) = \mathbb{0}. \tag{12}$$

[7] The set of matrices with positive determinant is connected.

Remark (Unnatural vegetable). Not all materials (bodies) have its natural states. Zum Beispiel, carrot does not have a natural state.

Definition 8 (Energy functional). Let us have $\partial\Omega = \Gamma_N \cup \Gamma_D, \Gamma_N \cap \Gamma_D = \varnothing$, where the parts of the boundary are those when Neumann/Dirichlet boudary conditions are prescribed. The energy functional of the material is the functional

$$I(\mathbf{y}) = \int_\Omega W(\mathbf{x}, \mathbb{F}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}S\,,$$

where the first part corresponds to the stored energy and the remaining terms are the work done by the external loads.

Remark. If $\mathbf{y}$ is the minimizer of I, then $I(t\boldsymbol{\varphi} + \mathbf{y}) \geq I(\mathbf{y}), \forall t, \boldsymbol{\varphi}$. If we denote

$$a(t) := I(t\boldsymbol{\varphi} + \mathbf{y}),$$

then it most hold

$$0 = a'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega W(\mathbb{F} + t\nabla\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}x\right)\bigg|_{t=0},$$

calculating the derivatives yields

$$\begin{aligned}
0 &= \int_\Omega \frac{\partial W(\mathbb{F})}{\partial \mathbb{F}} : \nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\varphi_i\right)\mathrm{d}\mathbf{x} - \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right)\varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_{\Gamma_N} \frac{\partial W(\mathbb{F})}{\partial F_{ij}}\varphi_i n_j\,\mathrm{d}S - \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right)\varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S\,,
\end{aligned}$$

so it must hold

$$-\frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right) = f_i \text{ in}\,\Omega, \frac{\partial W(\mathbb{F})}{\partial F_{ij}} n_j = g_i \,\text{on}\,\Gamma_N.$$

This is exactly

$$-\nabla \cdot \mathbb{T} = \mathbf{f}\,\text{in}\,\Omega, \mathbb{T}\mathbf{n} = \mathbf{g}, \text{ on}\,\Gamma_N.$$

This implies that $\mathbf{y}$ minimizes energy $\Leftrightarrow$ $\mathbf{y}$ is governed by the equations of classical mechanis.

Are there some other qualities of W? It is natural to assume

$$W(\mathbb{I}) = 0 \Rightarrow W(\mathbb{R}) = 0, \forall \mathbb{R} \in \mathrm{SO}(3)$$

and $W(\mathbb{F}) > 0$ whenever $\mathbb{F} \notin \mathrm{SO}(3)$ This however implies W is not convex! Assume

$$\mathbb{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbb{R}_2 = \mathbb{I},$$

then

$$W\left(\frac{1}{4}\mathbb{R}_1 + \frac{3}{4}\mathbb{R}_2\right) > \frac{1}{4}W(\mathbb{R}_1) + \frac{3}{4}W(\mathbb{R}_2) = 0.$$

Example (Minimizer does not exist). Assume $J(u) = \int_0^1 \left(1 - (u'(x))^2\right)^2 + u(x)^2\,\mathrm{d}x\,, u \in \mathrm{W}^{1,4}(0,1), u(0) = u(1) = 0$, and find the minimum of J. First of all, $J > 0$, so the minimum also. I can take u_k such that $u_k'(x) = 1$ on $(0,1/2)$ and $u_k'(x) = -1$ on $(1/2,1)$. Then $J(u_k) \to 0 \Rightarrow \inf J = 0$ but there is no minimizer.

Not everything is lost...

Definition 9 (Polyconvexity, 1977 J.M. Ball)**.** $W : \mathbb{R}^{3\times3} \to \mathbb{R} \cup \{\infty\}$ is polyconvex provided there exists convex and lower-semicontinuous function $h : \mathbb{R}^{19} \to \mathbb{R} \cup \{\infty\}$:

$$W(\mathbb{A}) = h(\mathbb{A}, \operatorname{cof} \mathbb{A}, \det \mathbb{A}).$$

Example.
- If W is convex and lower-semicontinuous then W is polyconvex.
- $W(\mathbb{A}) = \det \mathbb{A}$ is polyconvex but not convex.

Remark (Weak convergence in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$). Let $1 < p < \infty$ and $\{\mathbf{u_k}\} \subset \mathrm{L_p}(\Omega;\mathbb{R}^3)$. We say $\{\mathbf{u_k}\}$ converges weakly to $\mathbf{u}$ in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$ provided

$$\int_\Omega \mathbf{u_k} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}, \forall \boldsymbol{\varphi} \in \mathrm{L}_{p'}(\Omega;\mathbb{R}^3).$$

Theorem 2 (Magic)**.** *Assume that* $\mathbf{y}^k$ *converges weakly to* $\mathbf{y}$ *in* $W^{1,p}(\Omega;\mathbb{R}^3), \Omega \subset \mathbb{R}^3 \in C^{0,1}, p > 3$. *Then* $\det \nabla \mathbf{y}^k$ *converges weakly to* $\det \nabla \mathbf{y}$ *in* $L_{\frac{p}{3}}(\Omega)$. *Moreover* $\operatorname{cof} \nabla \mathbf{y}^l$ *converges weakly to* $\operatorname{cof} \nabla \mathbf{y}$ *in* $L_{\frac{p}{2}}(\Omega;\mathbb{R}^{3\times3})$.

Proof. Only in 2 dimensions. The determinant can be written as:

$$\det \nabla \mathbf{y} = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(y_1 \frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1 \frac{\partial y_2}{\partial x_1}\right) = \nabla \cdot \left(y_1, -\frac{\partial y_2}{\partial x_1}\right),$$

so then

$$\int_\Omega \det \nabla \mathbf{y}^k \varphi \, \mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1^k \frac{\partial y_2^k}{\partial x_2}\right)\varphi \, \mathrm{d}x - \int_\Omega \frac{\partial}{\partial x_2}\left(y_1^k \frac{\partial y_2^k}{\partial x_1}\right)\varphi \, \mathrm{d}x = -\int_\Omega y_1^k \frac{\partial y_2^k}{\partial x_2}\frac{\partial \varphi}{\partial x_1} \, \mathrm{d}x + \int_\Omega \frac{\partial \varphi}{\partial x_2} y_1^k \frac{\partial y_2^k}{\partial x_1} \, \mathrm{d}x,$$

and the result follows from the embedding theorems and strong convergence (strong times weak gives weak convergence). □

4.5 Rank-one convexity

Assume the following domain: $\Omega = (1,2) \times (0,4\pi) \times (1,2)$ and the deformation $\mathbf{y} : \overline{\Omega} \to \mathbb{R}^3, \mathbf{y}(x_1,x_2,x_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3), \mathbb{F} = \begin{bmatrix} \cos x_2 & -x_1 \sin x_2 & 0 \\ \sin x_2 & x_1 \cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can calculate $\det \mathbb{F} = x_1 \cos^2 x_2 + x_1 \sin^2 x_2 = x_1$. But even though the deformation has positive determinant, we still face self-penetration issues, i.e., $\mathbf{y}$ *is not injective.*

Theorem 3 (Ciarlet-Nečas condition)**.** *Let* $p > 3$ *and let* $\det \mathbb{F} > 0$ *a.e. in* $\Omega \subset \mathbb{R}^3, \mathbf{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$. *If*

$$\int_\Omega \det \mathbb{F}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \le \lambda(\mathbf{y}(\Omega))$$

then $\mathbf{y}$ *is injective almost everywhere in* Ω, *i.e.,* $\exists \omega \subset \Omega : \lambda(\omega) = 0, \mathbf{y}|_{\Omega/\omega}$ *is injective.*

Is the determinant condition of any use? Let us compute, assuming $\mathbf{y} = \mathbf{0}$ on $\partial\Omega$.

$$\int_\Omega \det \mathbb{F}\,\mathrm{d}\mathbf{x} = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1\frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1\frac{\partial y_2}{\partial x_1}\right)\mathrm{d}\mathbf{x} = \int_{\partial\Omega} y_1\frac{\partial y_2}{\partial x_2}n_1 - y_1\frac{\partial y_2}{\partial x_1}n_2\,\mathrm{d}S \underbrace{\Rightarrow}_{y=0 \text{ on } \partial\Omega} \int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

This is powerful! Assume that $\mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}$ on $\partial\Omega$, then

$$\int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \lambda(\Omega)\det\mathbb{F}.$$

Now let

$$I(\mathbf{y}) = \int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,,\mathbf{y} \in \mathrm{W}^{1,p}\left(\Omega;\mathbb{R}^3\right),\mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}.$$

Then I is constant[8] and it holds

$$I(\mathbf{y}) = \lambda(\Omega)\det\mathbb{F}.$$

5 Linearized elasticity

Recall the Right Cauchy-Green tensor: $\mathbb{C} = \mathbb{F}^\top\mathbb{F}$. Using it, we can define

Theorem 4 (Green-Lagrange strain tensor/Green-St.-Venaint strain tensor)**.** *Let $\mathbb{C}$ be the Right Cauchy-Green tensor. We define the Green-Lagrange strain tensor/Green-St.-Venaint strain tensor as*

$$\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I}).$$

Remark. The Green-St.-Venaint strain tensor can be rewritten as:

$$\mathbb{E} = \frac{1}{2}\left((\mathbb{I} + \nabla\mathbf{u})^\top(\mathbb{I} + \nabla\mathbf{u}) - \mathbb{I}\right) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) + \frac{1}{2}(\nabla\mathbf{u})^\top\nabla\mathbf{u} = \mathbb{e}(\mathbf{u}) + \frac{1}{2}\mathbb{C}(\nabla\mathbf{u}).$$

For the stored energy density, we can write

$$W(\mathbb{F}) = W(\mathbb{R}\mathbb{F}) = \overline{W}(\mathbb{C}(\mathbb{F})) = \hat{W}(\mathbb{E}(\mathbb{F})).$$

and also

$$W(\mathbb{F}) = \hat{W}(\mathbb{e}(\mathbf{u}) + \mathbb{C}(\nabla\mathbf{u})).$$

It is our assumption that

$$\hat{W}(\mathbb{0}) = 0, \hat{W}(\mathbb{E}) > 0 \,\mathrm{if}\, \mathbb{E} \neq \mathbb{0},$$

and also that

$$\mathbb{C}(\nabla\mathbf{u}) = \mathbf{0}.$$

Using Taylor expansion, we can write

$$\hat{W}(\mathbb{e}(\mathbf{u})) = \hat{W}(\mathbb{0}) + \frac{\partial\hat{W}}{\partial\mathbb{e}}(\mathbb{0})\mathbb{e}(\mathbf{u}) + \frac{1}{2}\frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{0})\mathbb{e}(\mathbf{u})\mathbf{e}(u) + \text{h.o.t.}.$$

Since $\hat{W}(\mathbb{0}) = \frac{\partial\hat{W}}{\partial\mathbb{0}}(\mathbb{0}) = 0$ the above (formal) manipulation leads us to the definition

Definition 10 (Tensor of elastic constants)**.**

$$\mathcal{C} = \frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{0}), C_{ijkl} = \frac{\partial^2\hat{W}}{\partial e_{ij}\partial e_{kl}}.$$

[8]All constant functionals are convex.

Remark. Since we assume $\hat{W}$ is smooth, we have some symmetries, and from the general 81 components of C_{ijkl} only 21 are unique.

With the notion of a tensor of elastic constants, we can then write the stored energy density as

$$w(\mathbb{e}) = \frac{1}{2}(\mathcal{C}\mathbb{e}) : \mathbb{e}.$$

Following our definition $\mathbb{T} = \frac{\partial \hat{W}}{\partial \mathbb{F}}$ we see

$$\sigma = \frac{\partial w(\mathbb{e})}{\partial \mathbb{e}} = \mathbb{C}\mathbb{e}, \sigma_{ij} = C_{ijkl}e_{kl}.$$

Is a useful notion of stress. It is denoted as the *Cauchy stress.* or in components

$$\sigma_{ij} = C_{ijkl}e_{kl}.$$

5.1 Equations

Rewritting the equations in the linearized elasticity setting we obtain the system

$$\begin{aligned} -\nabla \cdot \sigma = -\nabla \cdot (\mathcal{C}\mathbb{e}) &= \mathbf{f} \,\mathrm{in}\, \Omega \\ \sigma \mathbf{n} &= \mathbf{g} \,\mathrm{on}\, \Gamma_N, \\ \mathbf{u} &= \mathbf{0} \,\mathrm{on}\, \Gamma_D. \end{aligned}$$

The weak formulation can be obtained as

$$\int_\Omega \frac{\partial}{\partial x_j}\left(C_{ijkl}e_{kl}\right) v_i \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}\,, \forall \mathbf{v} \in \mathrm{W}^{1,2}\left(\Omega; \mathbb{R}^3\right), u = 0 \,\mathrm{on}\, \Gamma_D,$$

so

$$\int_\Omega C_{ijkl}e_{kl} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} C_{ijkl}e_{kl}v_i n_j \,\mathrm{d}S = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}\,,$$

which can be rewritten as

$$\underbrace{\int_\Omega \mathbb{C}\mathbb{e}(\mathbf{u}) \cdot \mathbb{e}(\mathbf{v}) \,\mathrm{d}\mathbf{x}}_{:=B(u,v)} = \underbrace{\int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \,\mathrm{d}S,}_{:=L(v)}$$

where we have denoted

$$\mathbb{e}(\mathbf{v}) = \mathrm{sym}\,(\nabla \mathbf{v}).$$

We are looking for

$$u \in V = \left\{u \in \mathrm{W}^{1,2}\left(\Omega; \mathbb{R}^3\right), \mathrm{tr}\, u = 0 \,\mathrm{on}\, \Gamma_D\right\} : B(u,v) = L(v) \forall v \in V,$$

and to prove the existence, we will use the Lax-Milgram lemma. Show that

- $L \in V^*$
- $B : V \times V \to \mathbb{R}$ is V-bounded and V-coercive

Realize that in order to show the properties, we would have to be able to control $\nabla \mathbf{u}$ by $\mathrm{sym}\,(\nabla \mathbf{u})$. Is that even possible?

Example. Let $u = 0$ on $\partial\Omega$. In particular, let us take $\mathbf{u} \in \mathcal{D}(\Omega;\mathbb{R}^n)$. Then

$$\exists C > 0 : \int_\Omega |\mathbb{e}(\mathbf{u})|^2 \,\mathrm{d}\mathbf{x} \geq c \int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,.$$

Can this hold? Make a quick test: Take $\mathbf{u}$ such that $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, so $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0$, so of course:

$$\nabla\mathbf{u} = -(\nabla\mathbf{u})^\top,$$

and $\nabla\mathbf{u}$ must have the form

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix},$$

where $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = 0$, but since $\mathbf{u} = \mathbf{0}$ at the boundary, it also holds that $\mathbf{u} = \mathbf{0}$ in Ω. Okay, so that not disprove the above inequality.

Let us try something else (although unsure what this means):

$$\begin{aligned}\int_\Omega |\mathbb{e}(\mathbf{u})|^2 \,\mathrm{d}x &= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \left(\frac{\partial u_j}{\partial x_i}\right)^2 \mathrm{d}x = \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\,\mathrm{d}x\,,\end{aligned}$$

where we used the symmetry property. Integrating by parts two times to obtain "$\partial_i u_i \partial_j u_j = (\partial_j u_j)^2$ [9]. All in all

$$\frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_i}{\partial x_i}\right)^2 \mathrm{d}x \geq 0.$$

Theorem 5 (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain ($\Omega \in C^{0,1}$). Then there exists $C > 0$ such that $\forall \mathbf{u} \in W^{1,2}((\Omega;\mathbb{R}^n))$ it holds*

$$\left(\|\mathbb{e}(\mathbf{u})\|^2_{L_2((\Omega;\mathbb{R}^{n\times n}))} + \|\mathbf{u}\|^2_{L_2((\Omega;\mathbb{R}^n))}\right) \geq c\|\mathbf{u}\|_{W^{1,2}((\Omega;\mathbb{R}^n))}.$$

Definition 11 (Axial vectors)**.** Let $\mathbb{A} = -\mathbb{A}^\top, \mathbb{A} \in \mathbb{R}^{n\times n}$. Then there is $b \in \mathbb{R}^n$ such that $\mathbb{A}\mathbf{v} = \mathbf{b}\times\mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{b}$ is called the axial vector of $\mathbb{A}$.

Remark ($\mathbb{R}^n$). This truly holds in $\mathbb{R}^n$, not only in $\mathbb{R}^3$. We only have to replace $\times$ by $\wedge$, the outter product.

Assume that $\mathbf{u} \in C^2(\Omega;\mathbb{R}^3)$. Then

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial e_{ik}}{\partial x_j}(\mathbf{u}) + \frac{\partial e_{ij}}{\partial x_k}(\mathbf{u}) - \frac{\partial e_{jk}}{\partial x_i}(\mathbf{u}).$$

If now $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, then $\mathbf{u}$ is an affine function, because $\frac{\partial^2 u_i}{\partial x_j \partial x_k}, \forall i,j,k \in \{1,2,3\}$. [10] It must thus hold

$$u_i(x) = a_i + b_{ij}x_j,$$

and $\frac{\partial u_i}{\partial x_j} = b_{ij} = -b_{ji}$, because $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, so it must be skew symmetric. The skew-symmetry also means it can be written

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d}\times\mathbf{x}.$$

[9] Sign does not change as we integrate 2 times. Also, we have homogenous Dirichlet

[10] Recall that Ω is simply connected.

If additionaly we assume that $\mathbf{u} = \mathbf{0}$ on some $\Gamma_D \subset \partial\Omega, \mathcal{H}(\Gamma_D) > 0$ and $\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d} \times \mathbf{x}$, then $\mathbf{u} = \mathbf{0}$ identically in Ω This moreover means that

$$\mathbf{u} \mapsto \|\mathbb{e}(\mathbf{u})\|_{\mathrm{L}_2((\Omega;\mathbb{R}^{n\times n}))}$$

is a norm on

$$V = \{\mathbf{w} \in \mathrm{W}^{1,2}\big((\Omega;\mathbb{R}^3)\big), \mathbf{w} = \mathbf{0} \,\mathrm{on}\, \Gamma_D\}$$

which is equivalent to the norm of $\mathrm{W}^{1,2}\big((\Omega;\mathbb{R}^3)\big)$.

Coming back to our equation $B(u,v) = L(v), \forall v \in V$, we have showed everything to use Lax-Milgram $\Rightarrow \exists! u \in V$. This also means the functional

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega (\mathbb{C}\mathbb{e}(\mathbf{v}) : \mathbb{e}(\mathbf{v}) - L(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,, \forall v \in V.$$

has an unique minimizer.

5.2 Convex analysis

We will deal with the analysis of the functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, f$ is convex.

Definition 12 (Epigraph of a set)**.** The epigraph of a function f is the set

$$\mathrm{epi}\, f = \{(x,y) : y \geq f(x)\}$$

Remark. With the notion of $\mathrm{epi}\, f$ we can work with sets instead of functions. Moreover, it holds

- $\mathrm{epi}\, f$ is closed $\Leftrightarrow$ f is lower-semicontinuous,
- f is convex $\Leftrightarrow$ $\mathrm{epi}\, f$ is convex

From one of the consequences of Hahn-Banach theorem (oddělovací věty), we obtain the existence of such $\xi \in \mathbb{R}^n$ (dependent of x) that for fixed x it yields

$$f(z) \geq f(x) + \xi \cdot (z - x), \forall z \in \mathbb{R}^n.$$

If f is differentiable at x, then

$$\xi = \nabla f(x).$$

But in general it does not have to be differentiable. This motivates the following definition

Definition 13 (Subgradient, subdifferential)**.** The function $\xi(x)$ such that

$$f(z) \geq f(x) + \xi(x) \cdot (z - x), \forall z \in \mathbb{R}^n,$$

is called the **subgradient** of f at x. The set of all subgradients of f at x is called the **subdifferential** of f at x and it is denoted $\partial f(x)$.

Let $f : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$, convex and lower semicontinuous [11], $f \neq \infty$. The function $\xi(\mathbb{X})$ such that

$$f(\mathbb{Y}) \geq f(\mathbb{X}) + \xi(\mathbb{X}) \cdot (\mathbb{Y} - \mathbb{X}), \forall \mathbb{Y} \in \mathbb{R}^{n\times m},$$

is called the subgradient of f at $\mathbb{X}$. The set of all subgradients of f at $\mathbb{X}$ is called the subdifferential and denoted $\partial f(\mathbb{X})$.

Remark.
- If $\partial f(\mathbb{X})$ is a singleton, then $\nabla f(\mathbb{X})$ exist.

[11] $f(x) \leq \liminf_{k\to\infty} f(x_k), x_k \to x$

- $\partial f(\mathbb{X})$ is convex
- $0 \in \partial f(x) \forall x \in \mathbb{R}^n$ is a condition for the minimizer.

Definition 14 (Indicator function)**.** Let $K \subset \mathbb{R}^{n\times m}$ be a closed convex nonempty set. The function $I_K(\mathbb{X})$ given as

$$I_K(\mathbb{X}) = \begin{cases} 0, & \text{if} \, \mathbb{X} \in K \\ +\infty, & \text{otherwise} \end{cases},$$

is called the indicator function of K

The indicator function is hepful for constraint minimization. If f is reasonably (at least finitely valued on K) then it holds:

$$\min_K f = \min_{\mathbb{R}^{n\times m}} (f + I_K).$$

Example (Unit interval)**.** Let $K = [0,1]$. What is $\partial I_K(x)$?

If $x \in (0,1)$, then $I_K(x) = 0$ so the only ξ such that $I_K(y) \geq 0 + \xi(y-x)$ holds is $\xi = 0$.

If $x = 0, x = 1$ then $\partial I_K(0) = (-\infty, 0], \partial I_K(1) = [0,\infty)$. This resembles a normal "vector", but in fact it is not a single vector and more a "cone" of vectors.

Definition 15 (Normal cone to a set)**.** Let K be closed convex nonempty set. The subdifferential of the indicator function I_K is called the normal cone to the set K and it is denoted by N_K.

Example. Minimize x^2 on $[1,2]$. We are looking for

$$\min_{[1,2]} x^2 = \min_{\mathbb{R}} \left(x^2 + I_{[1,2]}(x)\right).$$

It must hold at the minimum

$$0 \in \partial\left(x^2 + I_{[1,2]}(x)\right) \Leftrightarrow -\partial I_{[1,2]}(x) \subset \partial x^2 \Leftrightarrow \left(x^2\right)' \in -N_{[1,2]}(x)$$

Example. Take a square $K = [0,1] \times [0,1] \subset \mathbb{R}^2$. We know $K \in C^{0,1}$ so the outter normal exist at a.a. points on the boundary. The outter normal does not exist in the corners, but the normal cone does.

Definition 16 (Fenchel (convex) conjugate — Legendre transformation)**.** Let x^* be a slope i have chosen (it is given). I require

$$f(x) \geq x^* \cdot x - k, \forall x \in \mathbb{R}^{n\times m},$$

which means $k \geq x^* \cdot x - f(x), \forall x \mathbb{R}^{n\times m}$, and so we can define

$$f^*(x^*) := \sup_{x \in \mathbb{R}^{n\times m}} (x^* \cdot x - f(x)).$$

Remark. f^* is always convex even if f is not. But when f is convex and lower-semicontinuous, then

$$f^{**} = f, \text{ (biconjugate)}.$$

Theorem 6 (Fenchel identity)**.** *Let* $x^* \in \partial f(x)$*. Then*

$$x^* \cdot x = f(x) + f^*(x^*).$$

Proof. Let us assume that $x^* \in \partial f(x)$. Then it must hold

$$f(y) \geq f(x) + x^* \cdot (y - x), \forall y,$$

so

$$x^* \cdot x - f(x) \geq x^* \cdot y - f(y),$$

and taking the supremum over y yields[12]

$$x^* \cdot x - f(x) = \sup_y (x^* \cdot y - f(y)) = f^*(x^*).$$

We have thus obtained

$$x^* \cdot x = f(x) + f^*(x^*).$$

□

Remark (Minimization of $f \Leftrightarrow$ minimization of f^*). We see that it holds:

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*).$$

5.3 Problem of a man...

Assume a person is pulling a box of weight m of weight m of weight m of weight m by a spring. If he is pulling just a little, the box does not move, only the spring is deformed - but in a reversible, elastic way. To move the box, the man needs to pull at least with the force $\sigma_0 = mgc$, where c is some friction coefficient. When he us pulling with force greater than σ_0, the box is moving and does not require any extra force to be moved (the system to be deformed). The deformation can be decomposed as

$$\mathbb{e} = \tilde{\mathbb{e}} + \mathbb{p},$$

where $\mathbb{e}$ is the total strain, $\tilde{\mathbb{e}}$ is the elastic strain and $\mathbb{p}$ is the plastic strain.

5.4 von Mieses elatoplasticity

The elasticity part is described as

$$\begin{cases} -\nabla \cdot \sigma = \mathbf{f}, & \text{in the bulk} \\ \sigma \mathbf{n} = \mathbf{g}, & \text{on the boundary} \end{cases},$$

with some constituive relation $\sigma = \mathcal{C}\tilde{\mathbb{e}} = \mathcal{C}(\mathbb{e} - \mathbb{p})$. What about the plastic part?

$$\begin{cases} \dot{\mathbb{p}}(t) \in N_K(\sigma), \\ \mathbb{p}(0) = \mathbb{p}_0, \end{cases}$$

where K is a convex closed subset such that $0 \in K$. This means that the plastic deformation is zero inside K, i.e. for some stresses.

Remark. Very often, the deformation is considered ”incompressible”, i.e.,

$$\det \mathbb{F} = 1,$$

[12] The inequality becomes equality, as it can be reached by taking $y = x$.

which in linear case translates into

$$\operatorname{tr} \varepsilon = 0.$$

In most cases, the set K is given as

$$K = \{\sigma : \varphi(\sigma) \leq 0\},$$

where φ is the **yield function.** The set

$$\{\sigma | \varphi(\sigma) = 0\}$$

is called the **yield surface.** Very often we have

$$\varphi(\sigma) = |\sigma^D| - c_0,$$

where $|\cdot|$ denotes the Frobenius norm and

$$\sigma^D = \sigma - \frac{1}{3}(\operatorname{tr} \sigma)\mathbb{I},$$

is the *deviatoric part of the stress tensor.*

5.4.1 Plastic evolution

From the previous we have

$$\dot{\mathbb{p}} = \begin{cases} \mathbb{O}, & \text{if } \varphi(\sigma) < 0, \\ \frac{\lambda}{|\sigma^D|}\sigma^D, & \text{if } \varphi(\sigma) = 0, \lambda \geq 0 \end{cases}.$$

Also $\dot{\mathbb{p}} \in N_K(\sigma) = \partial I_K(\sigma)$ so

$$\sigma \in \partial I_K^*(\dot{\mathbb{p}}),$$

where

$$I_K^*(\dot{\mathbb{p}}) = \sup_{\mathbb{q} \in \mathbb{R}^{3\times 3}} (\dot{\mathbb{p}} : \mathbb{q} - I_K(\mathbb{q})) = \sup_{\mathbb{q} \in K} \dot{\mathbb{p}} : \mathbb{q},$$

is the Fenckel transformation of I_K, also called the **supporting function** of $\dot{\mathbb{p}}$. We are able to rewrite the supremum to take the form[13]

$$I_K^*(\dot{\mathbb{p}}) = \dot{\mathbb{p}} : \frac{c_0}{|\dot{\mathbb{p}}|}\dot{\mathbb{p}},$$

if however the second term lies in K. Realize now that if $\operatorname{tr} \dot{\mathbb{p}} = 0$ then

$$I_K^*(\dot{\mathbb{p}}) = c_0 |\dot{\mathbb{p}}|,$$

and if $\operatorname{tr} \dot{\mathbb{p}} \neq 0$, then $I_K^*(\dot{\mathbb{p}}) = +\infty$. If we now define the **dissipation potential** D as

$$D(\dot{\mathbb{p}}) = \begin{cases} c_0 |\dot{\mathbb{p}}|, & \text{if } \operatorname{tr} \dot{\mathbb{p}} = 0 \\ +\infty, & \text{otherwise} \end{cases},$$

we get the following condition

$$\sigma \in \partial D(\dot{\mathbb{p}}).$$

[13]To utilize Cauchy-Schwarz later.

Let us summarise a bit. For the stress tensor we have $\sigma = \mathcal{C}(\mathbb{e} - \mathbb{p}) \in D(\dot{\mathbb{p}})$. The general relation also yields $\sigma = \frac{\partial w(\tilde{\mathbb{e}})}{\partial \tilde{\mathbb{e}}} = \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \tilde{\mathbb{e}}}$, where $w(\tilde{\mathbb{e}}) = \frac{1}{2} C \tilde{\mathbb{e}} : \tilde{\mathbb{e}}$ is the free energy density. Using the chain rule we obtain the condition

$$\frac{\partial w(\mathbb{e} - \mathbb{p})}{\partial \mathbb{p}} \in \partial D(\dot{\mathbb{p}}).$$

In total, we are solving the following system

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D(\dot{\mathbb{p}}), & \text{in}\, \Omega (\textit{ flow rule}) \\ \mathbb{p}(0) = \mathbb{p}_0, & \text{in}\, \Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e} - \mathbb{p})) = \mathbf{f}, & \text{in}\, \Omega \\ \text{boundary condiitons}\,, & \text{on}\, \partial\Omega \end{cases} .$$

How to solve the system?

5.4.2 Discrete time setting

Let us take $t \in [0, T]$ and fix $\tau = \frac{T}{N}, N \in \mathbb{N}$ for some $N >> 1$. Assume that using some discrete scheme, we are able to calculate $\mathbb{p}$ at a certain time. Then we must solve

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D\left(\frac{\mathbb{p} - \dot{\mathbb{p}}_{\mathbb{k}-\mathbb{1}}}{\tau}\right), & \text{in}\, \Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k, & \text{in}\, \Omega \end{cases} .$$

Which are the E-L equations of the functional [14]

$$I(\mathbf{u}, \mathbb{p}) = \int_\Omega w(\mathbb{e}(\mathbf{u}) - \mathbb{p}) \,\mathrm{d}x + \tau \int_\Omega D\left(\frac{1}{\tau}(\mathbb{p} - \mathbb{p}_{k-1})\right) \mathrm{d}x - \int_\Omega \mathbf{f}_k \cdot \mathbf{u} \,\mathrm{d}x - \int_{\Gamma_N} \mathbf{g}_k \cdot u \,\mathrm{d}S\,.$$

Really, taking the variaton with respect to $\mathbf{u}$ gives us

$$-\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k,$$

and the variation with respect to $\mathbb{p}$ gives us

$$0 \in -\sigma + \partial D\left(\frac{1}{\tau}(\mathbb{p} - \mathbb{p}_{k-1})\right).$$

If we want to minimize this functional, *i.e.*, solve the equations, it must hold [15] $D(\mathbb{q}) \neq +\infty$ (for $\mathbb{q}$ being the argument). From our assumptions on the dissipation potential this however implies.

$$D(\mathbb{q}) = c_0 |\mathbb{q}|, \operatorname{tr} \mathbb{q} = 0,$$

and we say the evolution is **rate-independent**. We see that D is 1-homogenous:

$$D(\alpha \mathbb{q}) = \alpha D(\mathbb{q}).$$

Rewritting the functional now yields:

$$I(\mathbf{u}, \mathbb{p}) = \frac{1}{2} \int_\Omega \mathcal{C}(\mathbb{e}(\mathbf{u}) - \mathbb{p}) : (\mathbb{e}(\mathbf{u}) - \mathbb{p}) \,\mathrm{d}x + \int_\Omega c_0 |\mathbb{p} - \mathbb{p}_{k-1}| \,\mathrm{d}x - L_k(\mathbf{u}), \mathbb{p}(0) = \mathbb{p}_0,$$

[14] We have guessed it.
[15] If not, we have no chance of minimizing it.

where $L_k(\mathbf{u})$ is the loading (at the k-th time step.) The sought solution is the pair $(\mathbf{u}_k, \mathbb{p}_k)$ which satisfies

$$I(\mathbf{u}_k, \mathbb{p}_k) = \min_{\mathbf{u},\mathbb{p}} I(\mathbf{u}, \mathbb{p}).$$

5.5 Rheological models

5.5.1 Dashpots

Or *tlumič* in Czech. The stress is assumed to take the form

$$\sigma = \mathcal{D}\dot{\mathbb{e}}(\nabla \mathbf{u}), \sigma_{ij} = D_{ijkl}\dot{e}_{kl}(\nabla \mathbf{u}),$$

where $\mathcal{D}$ is the **tensor of viscosity constants.** [16]

5.5.2 Kelvin-Voigt mateial

The response of some materials can be modelled as a "parallel composition of a spring and a dashpot." Then, the total stress is

$$\sigma = \sigma_p + \sigma_e,$$

that is the sum of the plastic and the elastic stresses. The strain is of course the same:

$$\mathbb{e} = \mathbb{e}_p = \mathbb{e}_e.$$

The governing equations thus are

$$\begin{aligned} -\nabla \cdot (\mathcal{C}\mathbb{e}(\mathbf{u}) + \mathcal{D}\dot{\mathbb{e}}(\mathbf{u})) &= \mathbf{f}, \text{ in } \Omega \\ (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}})\mathbf{n} &= \mathbf{0}, \text{ on } \Gamma_N \\ \mathbf{u} &= \mathbf{0}, \text{ on } \Gamma_D \\ \mathbb{e}(t=0) &= \mathbb{e}_0, \text{ in } \Omega. \end{aligned}$$

Let us obtain the energy *formally* balance. As usual, multiply the first equation by $\dot{\mathbf{u}}$ and integrate $\int_\Omega \mathrm{d}\mathbf{x}$.

$$\int_\Omega -\nabla \cdot (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}}) \cdot \dot{\mathbf{u}} \,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \dot{\mathbf{u}} \,\mathrm{d}x\,,$$

using Gauss

$$\int_\Omega (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}}) : \nabla\dot{\mathbf{u}} \,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \dot{\mathbf{u}} \,\mathrm{d}x =^{17} \int_\Omega \mathcal{C}\mathbb{e} : \dot{\mathbb{e}} \,\mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}} : \dot{\mathbb{e}} \,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{u} \,\mathrm{d}x\,,$$

and now we rewrite

$$= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\right) \mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u}) \,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \dot{\mathbf{u}} \,\mathrm{d}x\,,$$

[16] People say viscosity stresses or viscous stress. This is used, but nonetheless it is wrong.
[17] It holds $\dot{\mathbb{e}}(\mathbf{u}) = \mathbb{e}(\dot{\mathbf{u}})$.

and integrate in time:

$$\int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

Remember that

$$w(\mathbb{e}(\mathbf{u})) = \frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u}),$$

so we have obtained

$$\int_\Omega w(\mathbb{e}(\mathbf{u}(T)))\,\mathrm{d}x - \int_\Omega w(\mathbb{e}(\mathbf{u}(0)))\,\mathrm{d}x + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

5.5.3 Maxwell material

This is the case when we ”put the spring and the dashpot in serial composition”. The total stress is

$$\sigma = \sigma_p = \sigma_e,$$

and the total strain is

$$\varepsilon = \mathbb{e}_p + \mathbb{e}_e.$$

5.6 Internal parameters

A lot of materials can be described using some internal parameters $\mathbb{z}$ (scalars, vectos, tensors; we take the tensor case for generality); for example, plastic strain, fatique, damage, length of a crack, delamination.

The model

$$\sigma = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{e}} w(\mathbb{e}, \mathbb{z}),$$

with the flow rule

$$0 \in \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}).$$

is called the **generalized Kelvin-Voigt** model/material. From now on, we will be using φ for the stored energy density. There is some analogy:

- φ is the stored energy density = potential of stress
- ζ is the (pseudo)potential of dissipative forces.

To do anything, we need to obtain some energy balance, so test by $\dot{\mathbf{u}}$. Investigate the terms:

$$\sigma : \dot{\mathbb{e}} = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{e}},$$

realize now that from the flow rule it follows

$$(\partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{z}) + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z})) : \dot{\mathbb{z}} = 0,$$

so i can add it to the previous term and obtain

$$\sigma : \dot{\mathbb{e}} = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{z}} + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{z}} = 0,$$

Realize now that we have obtained

$$\partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z})\dot{\mathbb{e}} + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e}, \mathbb{z}),$$

and denoting the quantity

$$\xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) := \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{z}},$$

as the *rate of the dissipation* we obtain

$$\sigma : \dot{\mathbb{e}} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e}, \mathbb{z}) + \xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}).$$

What are the properties of ξ? First of all, we require

$$\xi \geq 0.$$

Assume ζ is a covex function:

$$\zeta(0,0) \geq \zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : (-\dot{\mathbb{e}}) + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : (-\dot{\mathbb{z}}).$$

Moreover, assume now $\zeta(0,0) = 0$. We have

$$\xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{z}} \geq \zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) \geq 0.$$

Finally, the total power balance becomes

$$\int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\rho|\dot{\mathbf{u}}|^2 \,\mathrm{d}x + \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e}, z)\,\mathrm{d}x + \int_\Omega \xi(\dot{\mathbb{e}}, \dot{z})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

and the total energy balance becomes

$$\int_\Omega \frac{1}{2}\rho\left(|\dot{\mathbf{u}}(T)|^2 - |\dot{\mathbf{u}}(0)|^2\right)\mathrm{d}x + \int_\Omega (\varphi(\mathbb{e}(T), z(T)) - \varphi(\mathbb{e}(0), z(0)))\,\mathrm{d}x + \int_0^T \int_\Omega \xi(\dot{\mathbb{e}}, \dot{z})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

6 Thermodynamics in the framework of GSM (generalized standard materials)

Having obtained some knowledge of thermodynamical quantites, we are ready to generalize the theory. We will see that the evolution of a specimen can be acquired by the knowledge of the stored energy density ψ and the dissipation "potential" ζ

Denote

$$\psi = \psi(\mathbb{e}, \mathbb{z}, \theta), \zeta = \zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}})$$

to be *the stored energy* and the *dissipation potential.* Here $\theta > 0$ denotes the absolute thermodynamic temperature. Let us denote

$$\sigma_{el} = \frac{\partial\psi}{\partial\mathbb{e}}, \sigma_{in} = \frac{\partial\psi}{\partial\mathbb{z}}, s = -\frac{\partial\psi}{\partial\theta},$$

as the elastic and inelastic stress and the entropy density. Moreover, define

$$w(\mathbb{e}, \mathbb{z}, \theta, s) = \psi(\mathbb{e}, \mathbb{z}, \theta) + \theta s$$

as the **internal energy density**. If we calculate the time derivative of the internal energy density we obtain:

$$\dot{w} = \frac{\partial}{\partial t}(\psi(\mathbb{e}, z, \theta) + \theta s) = \frac{\partial \psi}{\partial \mathbb{e}} : \dot{\mathbb{e}} + \frac{\partial \psi}{\partial \mathbb{z}} : \dot{\mathbb{z}} + \underbrace{\frac{\partial \psi}{\partial \theta}\dot{\theta} + \dot{\theta} s}_{=-s\dot{\theta}+\dot{\theta}s=0} + \theta \dot{s}.$$

We *postulate*:

$$\dot{w} = \sigma_{el} : \dot{\mathbb{e}} + \sigma_{in} : \dot{\mathbb{z}} + \xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) - \nabla \cdot \mathbf{j},$$

where $\mathbf{j}$ is the heat flux. From this postulate, we obtain

$$\xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) = \theta \dot{s} + \nabla \cdot \mathbf{j}. \tag{13}$$

A common modelling choice is the dependency

$$\mathbf{j} = \mathbf{j}(\theta, \mathbb{e}, \mathbb{z}, \nabla\theta) = -\mathbb{K}(\mathbb{e}, \mathbb{z}, \theta)\nabla\theta,$$

known as the *Fourier law.* Here

$$\mathbb{K} \in \{\mathbb{A} \in \mathbb{R}^{3\times 3} | \mathbb{A} > 0\},$$

is the *matrix of heat flux coefficients.* This is a classical example of a constitutive law.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s(t,\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{1}{\theta}(\xi - \nabla\cdot\mathbf{j})\,\mathrm{d}\mathbf{x} = \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}\mathbf{x} + \int_\Omega \frac{\nabla\cdot(\mathbb{K}\nabla\theta)}{\theta}\,\mathrm{d}\mathbf{x} = \\
&= \int_{\partial\Omega} \frac{\mathbb{K}\nabla\theta}{\theta}\cdot\mathbf{n}\,\mathrm{d}S - \int_\Omega \mathbb{K}\nabla\theta\cdot\nabla\left(\frac{1}{\theta}\right)\mathrm{d}\mathbf{x} + \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}\mathbf{x} = \\
&= \int_\Omega \left(\frac{\xi}{\theta} + \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2}\right)\mathrm{d}\mathbf{x} - \int_{\partial\Omega} \frac{\mathbf{j}}{\theta}\cdot\mathbf{n}\,\mathrm{d}S\,.
\end{aligned}$$

This relation is known as the *Clausius-Duhem inequality.*[18]

From the definition of s

$$s = -\frac{\partial\psi}{\partial\theta}(\theta, \mathbb{e}, \mathbb{z}),$$

it follows

$$\dot{s} = -\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}},$$

and so

$$\theta\dot{s} = \underbrace{-\frac{\partial^2\psi}{\partial\theta^2}\theta}_{:=C_V}\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : (\dot{\mathbb{e}}\theta) - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : (\dot{\mathbb{z}}\theta) = C_V\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : (\dot{\mathbb{e}}\theta) - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : (\dot{\mathbb{z}}\theta),$$

where we have identified

$$C_V = -\theta\frac{\partial^2\psi}{\partial\theta^2},$$

as the *heat capacity at the constant volume.* Coming back to 13, we read

$$C_V\dot{\theta} + \nabla\cdot\mathbf{j} = \xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}}.$$

[18] Although inequalty, there appears only the equality sign "=". I do not actually know what that means.

This is our *heat equation,* the right hand side are the sources. We could identify the derivatives of the potential with lets say some derivative of σ_{el}, but let us keep the "thermodynamics and mechanics separated."; although it does not really make sense. In total

$$
\begin{aligned}
C_V\dot{\theta} - \nabla\cdot(\mathbb{K}\nabla\theta) &= \xi + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}}, \\
\rho\ddot{\mathbf{u}} - \nabla\cdot(\sigma_{el} + \sigma_{in}) &= \mathbf{f}, \\
0 &\in \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{z}}\psi(\mathbb{e}, \mathbb{z}, \theta),
\end{aligned}
$$

plus of course some initial and boundary conditions.

7 Summary

At the end, the lecture is summarized.

It began with deformation:

$$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^3, \nabla\mathbf{y} = \mathbb{F}, \mathbb{C} = \mathbb{F}^\top\mathbb{F}, \det\mathbb{F} > 0.$$

and some quantities associated with these. A little excursion allowed as to define

$$W = W(\nabla\mathbf{y}) = W(\mathbb{F}),$$

to be the stored energy density. Note that later on, we have called it ψ. Coming back to deformation, we have defined various stress measures:

$$\mathbb{T}^y, \mathbb{T} = \mathbb{T}^y \operatorname{cof}\mathbb{F}, \mathbb{S} = \mathbb{F}^{-1}\mathbb{T}.$$

Wanting to show existence of solutions, we needeed the convexity of some functionals. A problem with rotations however meant we needed to lower our expectations and we had to discover polyconvexity and rank-1 convexity. This included *e.g.* Legendre-Hadamard condition.

Realizing we are stuck in full theory, we began exploring linearized elasticity. To show existence, we refreshed the Korn's inequality. And because that all seemed easy, a question about time dependence has been asked: is everything truly stationary?

No, it is not; that lead us to von Mises elastoplasticity and to a class of materials, such as Kelvin-Voigt or Maxwell materials. Generalizing this framework and also including some internal variables, we have given the foundations of (the thermodynamics of)gave generalized standard materials: this was especially elegant, as from the Helmholtz free energy and the dissipation potential, we were able to derive evolution equations for the important thermodynamical quantites. This included some energy/power estimates, balances and the notion of entropy and its rate.

8 (Some) tutorials

8.1 Change of observer

The requirement of material frame indifference yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}), \forall\mathbb{Q} \in \operatorname{orth}.$$

8.2 Change of reference configuration

The requiremenent of material symmetry yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{P}), \forall \mathbb{P} \in \mathcal{G},$$

where $\mathcal{G}$ is the symmetry group of the material.

8.3 Consequences of isotropic hyperelastic solid

Remark (Groups unim, orth)*.* The "biggest sensible" symmetry group is the unimodular group:

$$\text{unim} = \{\mathbb{P}, \det \mathbb{P} = \pm 1\}.$$

There exists another common group:

$$\text{orth}\,\{\mathbb{Q}, \mathbb{Q}\mathbb{Q}^\top = \mathbb{Q}^\top\mathbb{Q} = \mathbb{I}\} \subset \text{unim}\,.$$

We thus have $W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}), \forall \mathbb{Q} \in \text{orth}\,, \forall \mathbb{F}$.

Use *polar decomposition*: $\mathbb{F} = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R}, \mathbb{R} \in \text{orth}\,, \mathbb{U}, \mathbb{V}\,\text{positively definite}\,, \mathbb{U} = \sqrt{\mathbb{C}}, \mathbb{V} = \sqrt{\mathbb{B}}$.

To make use of it, we first use m.f.i. and then isotropy and then first use isotropy and then m.f.i. So from material frame indifference

$$W = \hat{F}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}\left(\mathbb{R}^\top\mathbb{R}\mathbb{U}\right) = \hat{W}(\mathbb{U}) = \overline{W}(\mathbb{C}),$$

where we have taken $\mathbb{Q} = \mathbb{R}^\top$. Note that this works universaly (without the need of isotropy), as it comes from the objectivity consideration.

From isotropy

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \overline{W}(\mathbb{C}) = \overline{W}\left((\mathbb{F}\mathbb{Q})^\top(\mathbb{F}\mathbb{Q})\right) = \overline{W}\left(\mathbb{Q}^\top\mathbb{F}^\top\mathbb{F}\mathbb{Q}\right) = \overline{W}\left(\mathbb{Q}^\top\mathbb{C}\mathbb{Q}\right), \forall Q \in \text{orth}\,, \forall \mathbb{C}\,\text{admissable}\,.$$

Now backwards using the second polar decomposition:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \hat{W}(\mathbb{V}\mathbb{R}\mathbb{Q}) = \hat{W}\left(\mathbb{V}\mathbb{R}\mathbb{R}^\top\right) = \hat{W}\left(\sqrt{\mathbb{B}}\right) = \tilde{W}(\mathbb{B}),$$

$$W = \tilde{W}(\mathbb{B}) = \tilde{W}\left(\mathbb{Q}\mathbb{F}(\mathbb{Q}\mathbb{F})^\top\right) = \tilde{W}\left(\mathbb{Q}\mathbb{B}\mathbb{Q}^\top\right).$$

So far, we have shown

$$\begin{aligned} W(t, \mathbf{X}) &= \overline{W}(\mathbb{C}(t, \mathbf{X}), \mathbf{X}) = \overline{W}\left(\mathbb{Q}\mathbb{C}\mathbb{Q}^\top\right), \\ W(t, \mathbf{X}) &= \tilde{W}(\mathbb{B}(t, \mathbf{X}), \mathbf{X}) = \tilde{W}\left(\mathbb{Q}\mathbb{B}\mathbb{Q}^\top\right), \end{aligned}$$

In HW, we will know

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}}, T_{iJ} = 2\frac{\partial \hat{W}}{\partial F_{iJ}}$$

and we can show

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}} = 2\mathbb{F}\frac{\partial \tilde{W}(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}, \mathbb{T}^y = \ldots, \mathbb{S} = 2\frac{\partial \overline{W}(\mathbb{C})}{\partial \mathbb{C}}.$$

Definition 17 (Isotropic functions)**.** We say the functions $\hat{a}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \hat{\mathbf{a}}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \hat{\mathbb{A}}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \alpha =$

$1,\dots,N$ are isotropic functions (of their respective arguments) if it holds

$$\begin{aligned}
\hat{a}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha) &= \hat{a}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),\\
\mathbb{Q}\hat{\mathbf{a}}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha) &= \hat{\mathbf{a}}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),\\
\mathbb{Q}\hat{\mathbb{A}}\mathbb{Q}^\top &= \hat{\mathbb{A}}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),\\
&.
\end{aligned}$$

So we see that $\overline{W}(\mathbb{C}),\tilde{W}(\mathbb{B})$ are **scalar isotropic functions of 1 tensorial (symmetric) argument.**

Theorem 7 (Representation theorem for scalar isotropic functions)**.** *Let* $\psi = \hat{\psi}(\mathbb{A}) = \hat{\psi}(\mathbb{Q}\mathbb{A}\mathbb{Q}^\top)$ *be a scalar isotropic function of a single symmetric tensorial variable. Then it must hold*

$$\hat{\psi}(\mathbb{A}) \equiv \hat{\psi}(\mathrm{I}_1(\mathbb{A}),\mathrm{I}_2(\mathbb{A}),\mathrm{I}_3(\mathbb{A})),$$

where

$$\begin{aligned}
\mathrm{I}_1(\mathbb{A}) &= \operatorname{tr}\mathbb{A},\\
\mathrm{I}_2(\mathbb{A}) &= \frac{1}{2}\big((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}\mathbb{A}^2\big),\\
\mathrm{I}_3(\mathbb{A}) &= \det\mathbb{A},
\end{aligned}$$

are the invariants of $\mathbb{A}$*.*

Proof. $\det(\mathbb{A}-\lambda\mathbb{I}) = -\lambda^3 + \lambda^2\,\mathrm{I}_1 - \lambda\,\mathrm{I}_2 + \mathrm{I}_3 = p_\lambda(\mathbb{A})$ We will prove a different assertion:

$\mathbb{A},\mathbb{B}$ are symmetric with the same invariants $\Leftrightarrow \exists\mathbb{Q} : \mathbb{A} = \mathbb{Q}\mathbb{B}\mathbb{Q}^\top$ " $\Leftarrow$ " is trivial, as then the matrices are similliar, so they have the same char. polynomial, so they have the same invariants. $\Rightarrow$ have same eigenvalues, so if i write the spectral decomposiiton, i can write

$$\mathbb{A} = \mathbb{Q}\Lambda\mathbb{Q}^\top, \mathbb{B} = \mathbb{Q}\Lambda\mathbb{R}^\top = \mathbb{R}\mathbb{Q}^\top\mathbb{A}\mathbb{Q}\mathbb{R}^\top.$$

Now suppose that the function is not a function of the invariants: $\hat{\psi} \neq \tilde{\psi}(\mathrm{I}_1,\mathrm{I}_2,\mathrm{I}_3)$. That means $\exists\mathbb{A}_,,\mathbb{A}_2$ such that $\mathrm{I}_1(\mathbb{A}_1) = \mathrm{I}_1(\mathbb{A}_2)$ and the same for the remaining invariants. Using the previous assertion, we have

$$\exists\mathbb{Q} : \mathbb{A}_1 = \mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top \Rightarrow \hat{\psi}(\mathbb{A}_1) = \hat{\psi}\big(\mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top\big) = \hat{\psi}(\mathbb{A}_2), \text{ but } \hat{\psi}(\mathbb{A}_1) \neq \hat{\psi}(\mathbb{A}_2).$$

□

Since using polar decomposition it can be shown the invariants of $\mathbb{B},\mathbb{C}$ are the same we recieve

$$W = \tilde{W}(\mathrm{I}_1(\mathbb{B}),\mathrm{I}_2(\mathbb{B}),\mathrm{I}_3(\mathbb{B})) = \overline{W}(\mathrm{I}_1(\mathbb{C}),\mathrm{I}_2(\mathbb{C}),\mathrm{I}_3(\mathbb{C})).$$

8.4 Representation in terms of principial stresses

... in terms of the eigenvalues $\mathbb{U},\mathbb{V}$. The invariants can be expressed as

$$\begin{aligned}
\mathrm{I}_1 &= \lambda_1 + \lambda_2 + \lambda_3,\\
\mathrm{I}_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3,\\
\mathrm{I}_3 &= \lambda_1\lambda_2\lambda_3.
\end{aligned}$$

Often in materials science the quantites can be expressed in these variables:

Example (Ogden materials).

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{n} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\lambda_k} + \lambda_3^{\alpha_k} - 3\right)$$

How to calculate e.g. $\mathbb{T}$ in this representation?

$$\mathbb{T} = 2\frac{\partial W(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)}{\partial \mathbb{B}}\mathbb{F} = 2\frac{\partial \hat{W}}{\partial \mathbb{B}}(\lambda_1(\mathbb{B}), \lambda_2(\mathbb{B}), \lambda_3(\mathbb{B}))2\frac{\partial \hat{W}}{\partial \lambda_i}\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}.$$

What (in the hell) is $\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}$? [19]

$$\mathbb{B}(s) = \sum_{\alpha=1}^{3} \omega_\alpha(s)\mathbf{g}_\alpha(s) \otimes \mathbf{g}_\alpha(s), \forall s \in I$$

where I is some open interval and $\{\mathbf{g}_\alpha\}$ is an ON eigenbasis of $\mathbb{B}$. Next, realize

$$\omega_1(s) = \mathbf{g}_1(s) \cdot \mathbb{B}(s)\mathbf{g}_1(s),$$

and differentiate this:

$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{g}_1}{\mathrm{d}s} \cdot \mathbb{B}\mathbf{g}_1 + \mathbf{g}_1\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}s}\mathbf{g}_1 + \mathbf{g}_1 \cdot \mathbb{B}\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}s} = \frac{1}{2} + +0.$$

8.5 Hyperelasticity with constraints

Very often, some other considerations have taken to be in account when describing some materials. Examples include

- *incompressibility*: $\det \mathbb{F} = 1$,
- *inextensibility*:$\mathbf{l} \cdot \mathbb{C}\mathbf{l} = 1$, for some $\mathbf{l} \in \mathbb{R}^3$ (*i.e.*, T.A. materials)

8.6 Rational thermodynamics

In rational thermodynamics, we *postulate*:

$$\mathbf{J}_\eta = \frac{1}{\theta_R}\mathbf{Q}, R_\eta = \frac{R}{\theta}, \tag{14}$$

i.e., the flux/production of entropy is the flux/production of heat divided by temperature. This *makes sense to assume.*

8.6.1 Clausius-Duhem inequality

Recall *balance of mass*:

$$\rho(t, \mathbf{x}) \det \mathbb{F} = \rho_R(\mathbf{X}),$$

balance of momentum:

$$\rho_r(\mathbf{x})\frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} = \nabla \cdot \mathbb{T} + \rho_R\mathbf{B}(t, \mathbf{X}),$$

[19] Recall the Daleckii-Krein theorem:

balance of internal energy:

$$\rho_r \frac{\partial e_r}{\partial t}(t, \mathbf{X}) = \underbrace{-\nabla \cdot \mathbf{Q}}_{=-\nabla\cdot(\det \mathbb{F}\mathbb{F}^{-1}\mathbf{q}(t,\mathbf{x}))} + \dot{\mathbb{F}} : \mathbb{T} + \rho_R R(t, \mathbf{X}),$$

where also

$$\dot{\mathbb{F}} : \mathbb{T} = \frac{1}{2}\mathbb{S} : \dot{\mathbb{C}},$$

and the *balance of entropy*:

$$\rho_R \frac{\partial \eta_R}{\partial t} = -\nabla \cdot \mathbf{J}_\eta + \rho_R R_\eta + \xi_R.$$

The definition of the Helmholtz free energy is: $\psi = e - \theta\eta$, or in the reference configuration:

$$\psi_R = e_R - \theta_R \eta_R.$$

Take the time derivative and calculate

$$\begin{aligned}\rho_R \dot{\psi}_R = \rho_R\big(\dot{e}_R - \dot{\theta}\eta - \theta\dot{\eta}\big) = \nabla \cdot \mathbf{Q} + \dot{\mathbb{F}} : \mathbb{T} + \rho_R\big(R - \dot{\theta}\eta\big) - \theta(-\nabla \cdot \mathbf{J}_\eta + \rho_R R_\eta + \xi_R) = \\ = -\nabla \cdot \mathbf{Q} + \dot{\mathbb{F}} : \mathbb{T} - \rho_R \dot{\theta}_R \eta + \frac{\theta_R}{\theta_R}\nabla \cdot (\mathbf{Q}) - \theta_R \xi_R + \theta_R \nabla\left(\frac{1}{\theta_R}\right) \cdot \mathbf{Q},\end{aligned}$$

where we have used 14 to cancel some terms. In total we obtain

$$\boxed{\rho_R\big(\dot{\psi}_R + \eta_R \dot{\theta}_R\big) - \dot{\mathbb{F}} : \mathbb{T} - \theta_R \mathbf{Q} \cdot \nabla\left(\frac{1}{\theta_R}\right) = -\theta_R \xi_R \leq 0. \qquad (15)}$$

Rational thermodynamics also *postulates* Clausius-Duhem inequality holds for all thermodynamically admissable proccesses.

8.6.2 Isothermal setting

Let us consider a special case - an *isothermal setting*, meaning $0 = \dot{\theta} = \nabla\left(\frac{1}{\theta_R}\right) = 0$. The Clausius-Duhem inequality then becomes:

$$\rho_R \dot{\psi}_R - \dot{\mathbb{F}} : \mathbb{T} \leq 0,$$

for all admissable proccesses. Realize that this is the same as:

$$\frac{\partial(\rho_R \psi_R)}{\partial t} - \frac{1}{2}\mathbb{S} : \dot{\mathbb{C}},$$

and that $\rho_R \psi_R = W$. By hyperelasticity $W = W(\mathbb{F})$ and so from material frame indifference $W = \overline{W}(\mathbb{C})$. Taking the time derivative means:

$$\left(\frac{\partial W}{\partial \mathbb{C}} - \frac{1}{2}\dot{\mathbb{T}}\right) : \dot{\mathbb{C}} \leq 0, \forall \mathbb{C}, \dot{\mathbb{C}}$$

and that can only be met when

$$2\frac{\partial W}{\partial \mathbb{C}} = \mathbb{S},$$

since $\mathbb{C}, \dot{\mathbb{C}}$ are independent.

Really, define the motion

$$\boldsymbol{\chi}(t, \mathbf{X}) = \mathbf{X}_0 = \exp\left((t - t_0)\mathbb{D}\right)\mathbb{F}_0(\mathbf{X} - \mathbf{X}_0),$$

then

$$\mathbb{F}(t, \mathbf{X}) = \exp\left((t - t_0)\mathbb{D}\right)\mathbb{F}_0,$$

so $\mathbb{F}(t_0, \mathbf{X}_0) = \mathbb{F}_0$. Time derivative can be computed to be:

$$\dot{\mathbb{F}}(t, \mathbf{X}) = \mathbb{D}\exp\left((t - t_0)\right)\mathbb{F},$$

and $\dot{\mathbb{F}}(t_0, \mathbf{X}_0) = \mathbb{D}\mathbb{F}_0$. This means

$$\mathbb{C}(t_0, \mathbf{X}_0) = \mathbb{F}_0^\top \mathbb{F}_0,$$

and

$$\dot{\mathbb{C}}(t_0, \mathbf{X}_0) = 2\mathbb{F}_0^\top \mathbb{D}\mathbb{F}_0,$$

We see that we can choose $\mathbb{C}(t_0, \mathbf{X}_0)$ and $\dot{\mathbb{C}}(t_0, \mathbf{X}_0)$ independetly.

Suppose we are given the constraint

$$f(\mathbb{C}) = 0,$$

which fits the conditions

$$e.g. \, \mathbf{l} \cdot \mathbb{C}\mathbf{l} - 1 = 0, \det \mathbb{C} - 1 = 0.$$

Thus the Clausis-Duhem inequality with constraints reduces to:

$$\left(\frac{\partial W}{\partial \mathbb{C}} - \frac{1}{2}\dot{\mathbb{S}}\right) : \dot{\mathbb{C}} \le 0, \forall \mathbb{C}, \dot{\mathbb{C}} \; s.t. \; f(\mathbb{C}) = 0. \tag{16}$$

The condition is "almost equivalent" to

$$\frac{\partial f}{\partial \mathbb{C}} : \dot{\mathbb{C}} = 0,$$

which is convenient, as we have the following theorem.

Theorem 8. *Let $\mathbb{A} \in \mathbb{R}^{n \times m}, \mathbf{b} \in \mathbb{R}^{n \times 1}, \boldsymbol{\alpha} \in \mathbb{R}^{m \times 1}, \beta \in \mathbb{R}$ such that*

$$\mathbb{A}\mathbf{x} + \mathbf{b} = 0,$$
$$\boldsymbol{\alpha} \cdot \mathbf{x} + \beta \le 0,$$

for some $\mathbf{x} \in \mathbb{R}^m$. Let S be the set of solutions of the equation and assume it is nonempty. Then the following are equivalent:

- *$\forall \mathbf{x} \in S$ the equation holds*
- *$\exists \lambda \in \mathbb{R}^n \neq 0 \; s.t. \; \boldsymbol{\alpha}^\top - \lambda^\top \mathbb{A} = 0, \beta - \lambda \cdot \mathbf{b} \le 0.$*

Remark. In our case, we have

$$\mathbf{b} = 0, \mathbb{A} = \frac{\partial f}{\partial \mathbb{C}}, \mathbf{x} = \dot{\mathbb{C}}.$$

Using this it can be shown that under this constraint the Cauchy stress must take the form[20]

[20] use $\mathbb{T}^y = \det \mathbb{F}\mathbb{F}^{-1}\mathbb{T}^y\mathbb{F}^{-\top}$, differentiate and realize $\frac{\partial f}{\partial \mathbb{C}} = \det \mathbb{C} rcg^{-\top}, \det \mathbb{F} = 1 = \det \mathbb{C}$, and plug this in.

$$\mathbb{T}^y = \lambda\mathbb{I} + 2\mathbb{F}\frac{\partial W}{\partial \mathbb{C}}\mathbb{F}^\top.$$

We usually identify $\lambda = p_{\text{th}}$, with the thermodynamically determined stress.

Theorem 9. *The followig statements are equivalent:*

- $\forall \mathbf{x} \in S = \{\mathbf{x} : \mathbb{A}\mathbf{x} + \mathbf{b} = \mathbf{0}\} : \boldsymbol{\alpha} \cdot \mathbf{x} + \beta \geq 0,$
- $\exists \boldsymbol{\lambda} \neq \mathbf{0}\, s.t.\, \boldsymbol{\alpha}^\top - \boldsymbol{\lambda}^\top \mathbb{A} = \mathbf{0} \wedge b - \lambda \cdot \beta \geq 0$

Proof. first ii) $\Rightarrow$ i): multiply the first row by $\mathbf{x}$: $\boldsymbol{\alpha} \cdot \mathbf{v} - \boldsymbol{\lambda} \cdot \mathbb{A}\mathbf{x} = 0$, sum it up with the second inequality and obtain

$$\boldsymbol{\alpha} \cdot \mathbf{x} + \beta - \boldsymbol{\lambda} \cdot (\mathbb{A}\mathbf{x} + \mathbf{b}) \geq 0,$$

so when $\mathbf{x} \in S, \mathbb{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$, and we obtain

$$\boldsymbol{\alpha} \cdot \mathbf{x} + \beta \geq 0.$$

Now i) $\Rightarrow$ ii). It suffices to show i)

$$\Rightarrow \exists \boldsymbol{\lambda} \neq \mathbf{0}\, s.t.\, \boldsymbol{\alpha}^\top - \boldsymbol{\lambda}^\top \mathbb{A} = 0,$$

since if $\boldsymbol{\alpha} \cdot \mathbf{x} + \beta \geq 0 \forall \mathbf{x} \in S$, then $\boldsymbol{\alpha} \cdot \mathbf{x} - \boldsymbol{\lambda} \cdot \mathbb{A}\mathbf{x} = 0$, where $\mathbb{A}\mathbf{x} = -\mathbf{b} \forall \mathbf{x} \in S$. This immediately implies the sought result. This proof is by contradiction: suppose

$$(\mathbb{A}\mathbf{x} + \mathbf{b} = 0 \Rightarrow \boldsymbol{\alpha} \cdot \mathbf{x} + \beta \geq 0) \wedge \exists \mathbf{x}_0\, s.t.\, \mathbb{A}\mathbf{x}_0 = \mathbf{0} \wedge \boldsymbol{\alpha} \cdot \mathbf{x}_0 \neq 0.$$

We can now take

$$\mathbb{A}(\mathbf{x} + \delta \mathbf{x}_0) + \mathbf{b} = \mathbf{0} \Rightarrow \boldsymbol{\alpha} \cdot (\mathbf{x} + \delta \mathbf{x}_0) + \beta \geq 0,$$

for an arbitrary $\delta \in \mathbb{R}$. But this is clearly not possible, as we can take $\delta < 0, |\delta| >> 1$ and surely the second relation will not be met. □

8.7 Inflation of a hyperelastic balloon

To prepare ourselves, first we examine the *biaxial deformation of a incompressible hyperelastic sheet.*

8.7.1 Biaxial deformation of a incompressible hyperelastic sheet

The deformation gradient is

$$\mathbb{F} = \begin{bmatrix} \lambda_1 & 00 & \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \mathbb{B} = \mathbb{F}\mathbb{F}^\top = \begin{bmatrix} \lambda_1^2 & 00 & \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}.$$

Moreover, assume the material is the incompressible Ogden:

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{N} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3\right),$$
$$\lambda_1\lambda_2\lambda_3 = 1.$$

Cauchy stress then would be

$$\mathbb{T}^y = -p\mathbb{I} + \frac{2}{J}\frac{\partial W(\mathbb{B})}{\partial \mathbb{B}}\mathbb{B} = -p\mathbb{I} + 2\sum_{j=1}^{3}\frac{\partial W}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial \mathbb{B}}\mathbb{B},$$

recall that

$$J = 1, \frac{\partial \lambda_j}{\partial \mathbb{B}} = \frac{1}{2\lambda_j}\mathbf{g}_j \otimes \mathbf{g}_j, \mathbb{B} = \sum_{j=1}^{3}\lambda_j^2(\mathbf{g}_j \otimes \mathbf{g}_j.)$$

We must calculate

$$\frac{\partial W}{\partial \lambda_j} = \sum_{k=1}^{N}\frac{\mu_k}{\lambda_k}\alpha_k \lambda_j^{\alpha_k - 1},$$

and so

$$\mathbb{T}^y = -p\mathbb{I} + 2\sum_{j=1}^{3}\sum_{k=1}^{N}\mu_k \frac{1}{2}\lambda_j^{\alpha_k-2}(\mathbf{g}_j \otimes \mathbf{g}_j)\sum_{l=1}^{3}\lambda_l^2(\mathbf{g}_l \otimes \mathbf{g}_l) = -p\mathbb{I} + \sum_{j=1}^{3}\sum_{k=1}^{N}\mu_k\lambda_j^{\alpha_k}(\mathbf{g}_j \otimes \mathbf{g}_j).$$

We now assume

$$T_{33} = 0,$$

called the *thin sheet assumption*, *i.e.*, plane-stress problem. This means

$$0 = -p + \sum_{k=1}^{N}\mu_k\lambda_3^{\alpha_k},$$

since $\mathbf{g}_j = \mathbf{e}_j$. The pressure thus is

$$p = \sum_{k=1}^{N}\mu_k\lambda_3^{\alpha_k}.$$

The remaining stresses are

$$T_{11} = -\sum_{k=1}^{N}\mu_k\lambda_3^{\alpha_k} + \sum_{k=1}^{N}\mu_k\lambda_1^{\alpha_k},$$

where

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2},$$

so

$$T_{11} = \sum_{k=1}^{N}\mu_k\left(\lambda_1^{\alpha_k} - (\lambda_1\lambda_2)^{-\alpha_k}\right),$$

and similiarly

$$T_{22} = \sum_{k=1}^{N}\mu_k\left(\lambda_2^{\alpha_k} - (\lambda_1\lambda_2)^{-\alpha_k}\right).$$

Alltogether,

$$\mathbb{T}^y = T_{11}(\mathbf{e}_1 \otimes \mathbf{e}_1) + T_{22}(\mathbf{e}_2 \otimes \mathbf{e}_2) \equiv \sigma_1(\mathbf{e}_1 \otimes \mathbf{e}_2) + \sigma_2(\mathbf{e}_2 \otimes \mathbf{e}_2).$$

8.7.2 Simplified approach for a balloon

Assume now

$$\sigma_1 = \sigma_2 \equiv \sigma,$$

meaning the the balloon is being stretched the same way in both directions. This is equivalent to

$$\lambda_1 = \lambda_2 \equiv \lambda,$$

i.e.,

$$\sigma = \sum_{k=1}^{N} \mu_k \big(\lambda^{\alpha_k} - \lambda^{-2\alpha_k}\big).$$

Let the thickness of the baloon be h and assume $h << 1$. We also define

$$\sigma_T = \sigma h,$$

as in fact the *surface tension.* The virtual work principle states

$$p_0 \delta V = \sigma_T \delta S,$$

where p_0 is the overpressure. This can be manioulated into

$$p_0 \frac{4}{3}\pi 3\pi r^2 \delta r = \sigma_T 2\pi r \delta r$$
$$p_0 r^2 = 2r\sigma_T,$$

and so

$$p_0 = \frac{2\sigma_T}{r} = 2\sigma_T K,$$

which is the *Laplace-Young* condition. The pressure in the baloon is the greater the less the radius the balloon has, or the greater the curvature K gets.

Subsituting for σ_T yields

$$p_0 = \frac{2\sigma h}{r} = 2\sigma \underbrace{\left(\frac{h}{H}\right)}_{=\lambda_3 = \frac{1}{\lambda^2}} \frac{H}{R} \underbrace{\left(\frac{R}{r}\right)}_{=\frac{1}{\lambda} = \frac{2\sigma}{\lambda^3}} = \frac{2H}{R\lambda^3} \sum_{k=1}^{N} \mu_k \big(\lambda^{\alpha_k} - \lambda^{-2\lambda_k}\big),$$

where H, R are the reference thickness and radius and h, r are the thickness and radius in the deformed configuration. so finally

$$p_0 = \frac{2H}{R} \sum_{k=1} \mu_k \big(\lambda^{\alpha_k - 3} - \lambda^{-2\lambda_k - 3}\big).$$

Plotting this for a rubber-like material, the dependency $p_0(\lambda)$ shows that first, starting from 0, p_0 is very steep, but suddenly at a one time the material expands very rapidly.

8.7.3 Exact solution

Denote now A, B, H to be the inner radius, outer radius and the thickness, the same for a, b, h. It will be advantegous to use spherical coordinates:

$$R \in [A, B],$$
$$\Theta \in [0, \pi),$$
$$\Phi \in [0, 2\pi),$$

The deformation is

$$\begin{aligned} r &= f(R)R, \\ \theta &= \Theta, \\ \varphi &= \Phi \end{aligned}$$

This gets sophisticated now, as

$$\mathbb{F} = \frac{\partial \xi^i}{\partial X^J} \mathbf{g}_i \otimes \mathbf{G}^J, \mathbf{g}_i = \frac{\partial \mathbf{x}\left(\xi^1, \xi^2, \xi^3\right)}{\partial \xi^i},$$

in curvilinear coordinates. It can be obtained:

$$\mathbb{F} = \frac{\partial (f(R)R)}{\partial R}\left(\mathbf{g}_r \otimes \mathbf{G}^R\right) + \mathbf{g}_\theta \otimes \mathbf{G}^\Theta + \mathbf{g}_\varphi \otimes \mathbf{G}^\Phi,$$

but every decent person works in coordinate *s.t.* $\|\mathbf{g}_r\| = 1$ etc. Calculation gives

$$\|\mathbf{G}_R\| = 1, \|\mathbf{G}_\Theta\| = R, \|\mathbf{G}_\Phi\| = R \sin \Theta,$$

and the inverse for the forms. Now we write the deformation gradient in the ”normalized” coordinates, without writing things like $\mathbf{g}_{\hat{r}}$.

$$\mathbb{F} = R f'(R)\left(\mathbf{g}_r \otimes \mathbf{G}^R\right) + f(R)\mathbb{I}.$$

After long and complicated calculations, it can be shown

$$p_0 = \int_{\lambda_a}^{\lambda_b} \frac{\tilde{W}'(\lambda)}{\lambda^3 - 1} \, \mathrm{d}\lambda \,,$$

where $\tilde{W} = W\left(\frac{1}{\lambda^2}, \lambda, \lambda\right)$.

References

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