Partial differential equations II

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Contents

1 Winter semester addendum

1.1 Weak* convergence

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is reflexive, i.e.

$$\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*,$$

which means that balls in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ are weakly* compact. Moreover, $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is *separable*, from which it follows $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ with the weak* topology is metrizable and thus there exists s weakly * converging subsequence (from the balls).

Example (For people without Functional Analysis I)**.** Let X be a linear normed space, $\{x_n\} \subset X$ a sequence in X. We say x_n converges weakly to $x \in X$ whenever

$$f(x_n) \to f(x), \forall f \in X^*.$$

Let $X*$ be the topological dual to X, $\{x_n\} \subset X^*$ a sequence in X. We say f_n converges weakly* to $f \in X^*$ whenever

$$f_n(x) \to f(x), \forall x \in X^*,\ i.e.\, x(f_n) \to x(f),$$

where by $x(y), x \in X, y \in X.*$ we understand

$$\varepsilon_x : X^* \to \mathbb{K}, y \mapsto y(x).$$

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, every point $x \in \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ can be interpreted as a linear functional on $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, so given $\{x_n\} \subset \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, we can interpret is as a $\{x_n\} \subset \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, meaning given a weakly converging sequence in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, it is actually a weakly* converging sequence in $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$.

1.2 Regularity of parabolic problems

Theorem 1. *Let the assumptions of the previous theorem hold and $\Omega \in C^{1,1}, \delta \in (0,1)$. Then $u \in L_2\Big((\delta,T);W^{2,2}(\Omega)\Big)$.*

Proof. Take the weak formulation in $t \in (\delta,T)$. WLOG further assume $d = 0$. Then

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi - bu\varphi - \mathbf{c}\cdot\nabla u\varphi - \int_\Omega \partial_t u\varphi = \int_\Omega (f - bu - \mathbf{c}\cdot\nabla u - \partial_t u)\varphi,$$

and the integrand of the last integral is in $\mathrm{L}_2(\Omega)$ for a.e. $t \in (\delta, T)$. We can thus use the elliptic regularity results and write:u

$$\|u\|^2_{\mathrm{W}^{2,2}(\Omega)} \leq C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\partial_t u\|^2_{\mathrm{L}_2(\Omega)}),$$

integrating both sides $\int_\delta^T \mathrm{d}t$ yields

$$\|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))} \leq C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{L}_2((0,T);\mathrm{W}^{1,2}(\Omega))} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))})$$

□

Theorem 2. *If data are smooth and satisfy the compatibility condiitons, then the weak solutions to the parabolic equation are smooth.*

Proof. no. □

Remark (Compatibility condition). : Take the heat equation : $\partial_t u - \triangle u = f$ at time zero: $\triangle u(0) + f(0) = \partial_t u(0) \in \mathrm{W}^{1,2}_0(\Omega)$, so we need that $f(0) + \triangle u(0)$ has zero trace $\Rightarrow$ compatibility conditions.

1.3 Uniqueness of solutions to hyperbolic problems

Theorem 3 (Uniqueness of the solution to a hyperbolic equation)**.** *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that* $\mathbf{c} \in W^{1,\infty}(\Omega)$*. Then the weak solution to the hyperbolic equation is unique.*

Proof. It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the equation, multiply it by $\varphi \in V$ fixed and integrate over Ω for $t \in (0,T)$ fixed:

$$< \partial_{tt} u(t), \varphi > + \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \varphi \,\mathrm{d}x - \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \varphi \,\mathrm{d}x = 0.$$

Now, take a special test function

$$\psi(t) = \left(\int_t^s u(\tau) \,\mathrm{d}\tau \right) \chi_{(0,s)}(t),$$

for some $s \in (0,T)$. Then $\partial_t \psi(t) = -u(t)$ on $t \in (0,s)$. Next, integrate the equation in time over $(0,s)$.

$$\int_0^s < \partial_{tt} u(t), \psi > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \,\mathrm{d}x \,\mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \,\mathrm{d}x \,\mathrm{d}t - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \,\mathrm{d}x \,\mathrm{d}t = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s < \partial_{tt} u(t), \varphi > \mathrm{d}t =< \partial_t u(s), \psi(s) > - < \partial_t u(0), \psi(0) > - \int_0^s < \partial_t u(t), \partial_t \psi(t) > \mathrm{d}t \,,$$

and realize $\psi(s) = 0, \partial_t u(0) = 0$, so

$$- \int_0^s < \partial_t u(t), \partial_t \psi(t) > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \,\mathrm{d}x \,\mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \,\mathrm{d}x \,\mathrm{d}t - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \,\mathrm{d}x \,\mathrm{d}t = 0,$$

but since $\partial_t\psi(t) = -u(t)$, we can actually write (time dependencies are omitted for brevity)

$$\int_0^s < \partial_t u, u > \mathrm{d}t + \int_0^s \int_\Omega -\mathbb{A}\nabla\partial_t\psi\cdot\nabla\psi - b\psi\partial_t\psi - \psi\mathbf{c}\cdot\nabla\partial_t\psi + \partial_t\psi\mathbf{d}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2_{\mathrm{L}_2(\Omega)} - \int_\Omega \mathbb{A}\nabla\psi\cdot\nabla\psi + b\psi^2 + \psi\mathbf{c}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla\psi\,\mathrm{d}x\Big)\mathrm{d}t =$$
$$= \int_0^s \int_\Omega (\partial_t\mathbb{A})\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 + \psi\partial_t\mathbf{c}\cdot\nabla\psi + \underbrace{\partial_t\psi}_{=-u(t)}\,\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi - \psi\mathbf{d}\cdot\nabla\underbrace{\partial_t\psi}_{=-u(t)})\,\mathrm{d}t\,\mathrm{d}x\,,$$

and upon integration (recall $\psi(s) = 0$, from the definition of ψ it follows $\nabla\psi(0) = 0$, and $u(0) = 0$,)

$$\frac{1}{2}\Big(\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathbb{A}(0)\nabla\psi(0)\cdot\nabla\psi(0) + b(0)\psi(0)^2 + \psi(0)\mathbf{c}(0)\cdot\nabla\psi(0) + \psi(0)\mathbf{d}(0)\nabla\psi(0)\,\mathrm{d}x\Big) =$$
$$= \int_0^s \int_\Omega \partial_t\mathbb{A}\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 - u\partial_t\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\,.$$

From this we obtain the following estimate:

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\psi(0)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C\Big(\int_0^s \|\psi\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big)\mathrm{d}t + \|\psi(0)\|^2_{\mathrm{L}_2(\Omega)},$$

where $C = C\Big(\|\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|b\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t b\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}\Big)$. Define now the test function $\chi(t) = \int_0^t u(\tau)\,\mathrm{d}\tau$, and realize that in fact $\psi(t) = \chi(s) - \chi(t), \chi(0) = 0$. Plugging this in the above inequalty yields

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s \|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big) + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)},$$

and using

$$\|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} = \|\chi(t) - \chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le 2\Big(\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)}\Big),$$

and the definition of $\chi(t)$, from which it follows

$$\|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le \int_0^s \|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\,,$$

we are allowed to write

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s 2\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big),$$

and so

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + (1 - 2sC)\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_1\Big(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big).$$

If we now choose $T_1 \in (0,T]$ small enough *s.t.* $1 - 2sC > 0$ for $s \in (0,T_1]$, we finally obtain

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_2\bigg(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\bigg), \forall s \in (0,T_1],$$

which implies $u = 0$ on $(0,T_1]$ by the Gronwall lemma: we have

$$\xi(t) \le \int_0^t \xi(s)\,\mathrm{d}s\,, \text{ for } a.a.\, t \in (0,T) \Rightarrow \xi(t) = 0\; a.e.\,.$$

for $\xi \in \mathrm{L}_1((0,T))$ nonnegative[1]. If we now boostrap on $[T_1, 2T_1], [2T_1, 3T_1]$ etc., we obtain $u = 0$ on $(0,T]$.

□

2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d$ open , $p \in [1,+\infty], k \in N$. We define

$$\mathrm{W}^{\mathrm{k,p}}(\Omega) = \Big\{f \in \mathrm{L}_\mathrm{p}(\Omega)\,;\, D^\alpha f \in \mathrm{L}_\mathrm{p}(\Omega)\,,\, \forall |\alpha| \le k\Big\},$$

with the norm

$$\|f\|^p_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \|f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)} + \sum_{0<|\alpha|\le k} \|D^\alpha f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)}.$$

Recall that:

- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is Banach $\forall p$ and Hilbert for $p = 2$.
- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is separable if $p < \infty$ and reflexive if $p > 1, p < \infty$.

Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.

2.1 Tools from functional analysis

Definition 1 (Regularization kernel)**.** The function η is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\operatorname{supp} \eta \subset \mathrm{U}(0,1)$
- $\eta \ge 0$
- η is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$

[1] In our case $\xi = \|u\|^2_{\mathrm{L}_2(\Omega)} + \|\chi\|^2_{\mathrm{W}^{1,2}(\Omega)}$.

Definition 2 (Regularization of a function)**.** Let η be a regularization kernel. Set[2]

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of $f \in \mathrm{L}_1(\Omega)_{\mathrm{loc}}$ by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

Remark (Propertios of regularization)*.* The regularization has the following properties:

- $f \in \mathrm{L_p}(\Omega) \Rightarrow f_\varepsilon \to f \,\mathrm{in}\, \mathrm{L_p}(\Omega)$ and also a.e
- $f \in \mathrm{L}_\infty(\Omega) \Rightarrow f_\varepsilon \to f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathrm{U}(x,\varepsilon)} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y$
- $\operatorname{supp} f_\varepsilon \subset \overline{U(\Omega,\varepsilon)}, f = 0 \,\mathrm{on}\, \mathrm{U}(x,\varepsilon) \Rightarrow f_\varepsilon(x) = 0$

Definition 3 ($\Omega' \subset\subset \Omega$)**.** $O \subset\subset \Omega$ means $\overline{O}$ is compact and $\overline{O} \subset \Omega$.

Definition 4 (Shift operator)**.** For $u \in \mathrm{L_p}(\Omega), k \in \{1,\dots,d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

Lemma 1 (Approximation property of the shift operator)**.** *For $u \in L_p(\Omega)$, it holds $\tau_h u \to u$ in $L_p(\Omega), h \to 0^+$.*

Lemma 2 (Partition of unity)**.** *Let $E \subset \mathbb{R}^d, \mathcal{G}$ be an open covering of E (possibly uncountable.) Then there exists a countable system $\mathcal{F}$ of nonnegative functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$ and*

1. *$\mathcal{F}$ is subordinate to $\mathcal{G}: \forall\varphi \in \mathcal{F}\exists U \in \mathcal{G}: \operatorname{supp}\varphi \subset U$*
2. *$\mathcal{F}$ is locally finite[3]: $\forall K \subset E$ compact$, \operatorname{supp}\varphi \cap K \neq \emptyset$ for at most finitely many $\varphi \in \mathcal{F}$.*
3. *$\sum_{\varphi\in\mathcal{F}} \varphi(x) = 1, \forall x \in E$.*

Proof. (Sketch) *Step 1 (If E is compact)*:

E compact $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}\, s.t.\, E \subset \bigcup_{j=1}^m U_j$. Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \cup_{j=1}^m K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set:

$$K_m = \left\{x \in U \middle|\, \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{m}, \|x\| \le m\right\}.$$

Then clearly $K_1 \subset K_2 \dots$, and they "converge monotonously to U. Next, find $\phi_j \in C_c(U_j), \phi_j > 0 \,\mathrm{on}\, K_j$, e.g. $\phi_j = \theta(\operatorname{dist}(x, \partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

Step 2 (If E is open):

Approximate E by $K \subset E$ compact by the exhaustion argument, then the covering will enlarge from finite $\to$ countable (nontrivial reasoning). □

[2]Another common choice is $\eta_k = k^d\eta(kx), k \in \mathbb{N}$.

[3]In other words, φ_K is nonzero for at most finitely many $\varphi \in \mathcal{F} \Leftrightarrow$ points in K can be represented by finitely many functions $\varphi \in \mathcal{F}$.

2.2 Density of smooth functions

Lemma 3 (Local approximation by smooth functions (using regularization)). *Assume* $p \in [1,\infty), \Omega \subset \mathbb{R}^d$ *open*, $k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega_\varepsilon = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. *Then it holds*

1. $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ *a.e. in* $\Omega_\varepsilon, \forall |\alpha| \le k$
2. $u_\varepsilon \to u$ *in* $W^{k,p}(\Omega)_{loc}, \varepsilon \to 0^+$

Proof. First of all:

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\left(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y\right) = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_\Omega u(y) D_x^\alpha \eta_\varepsilon(x-y)\,\mathrm{d}y\,,$$

the integrable majorants are *e.g.* $\|\eta_\varepsilon\|_\infty |u| \chi_{\mathrm{U}(0,\varepsilon)}(x) \in \mathrm{L}_1(\Omega)$. Now picking $x \in \Omega_\varepsilon$ we realize $\forall y \in \mathbb{R}^d/\Omega : x - y \ge \operatorname{dist}(x, \partial\Omega) \ge \varepsilon$, and so $\eta_\varepsilon(x-y) = 0$. Exchanging derivatives and using the definition of the weak derivative

$$\int_\Omega u(y) D_x^\alpha \eta_\varepsilon(x-y)\,\mathrm{d}y = (-1)^{|\alpha|}\int_\Omega u(y) D_y^\alpha \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_\Omega D_y^\alpha u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} D_y^\alpha u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = (D^\alpha u)_\varepsilon.$$

Take $V \subset\subset \Omega$ open, then

$$\|u - u_\varepsilon\|_{\mathrm{W}^{\mathrm{k,p}}(V)} = \sum_{|\alpha| \le k} \|D^\alpha u - D^\alpha u_\varepsilon\|_{\mathrm{L_p}(V)} \to 0,$$

because $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \to D^\alpha u \,\mathrm{in}\, \mathrm{L_p}(V)$, from the properties of regularization. ☐

Theorem 4 (Global approximation by smooth functions). *Let* $\Omega \subset \mathbb{R}^d$ *be open,* $k \in \mathbb{N}, p \in [1,\infty)$. *Then* $C = \left\{f \in C^\infty(\Omega), \operatorname{supp} f \text{ bounded}\right\} \bigcap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$, *i.e.*

$$\overline{C \bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

If moreover Ω *is bounded, it holds:*

$$\overline{C^\infty(\Omega) \bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

Proof. Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega) \bigcap \mathrm{W}^{\mathrm{k,p}}(\Omega)$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$. For every $j \in \mathbb{N}$ define an open set

$$\Omega_j = \left\{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\right\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$. Next, set

$$U_j = \Omega_{j+1}/\overline{\Omega_{j-1}}, j = 1, 2, \dots,$$

where $\Omega_0 = \Omega_{-1} = \varnothing$. Since Ω_j are open, U_j are also open and $\Omega \subset \bigcup_{j\in\mathbb{N}} U_j \Rightarrow \exists\{\varphi_j\}_{j\in\mathbb{N}}$ partition of unity subordinate to $\{U_j\}_{j\in\mathbb{N}}$. We can write $u = \sum_{j\in\mathbb{N}} u\varphi_j$, where $u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \operatorname{supp} u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$: set $v_j = (u\varphi_j)_{\varepsilon_j}$ and fix an arbitrary $\delta > 0$. By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(U)} < \frac{\delta}{2^{j-1}},$$

for any $U \subset\subset \Omega$, for $\varepsilon_j > 0$ sufficiently smalls. In fact, if needed, make the ε_j smaller so that

$$\operatorname{supp} v_j \subset \Omega_{j+2}/\overline{\Omega_{j-2}}.$$

That is completely possible: from the properties of regularization, we know

$$\operatorname{supp} v_j \subset \mathrm{B}(0,\varepsilon) + \overline{U_j},$$

and since

$$\overline{U_j} = \overline{\Omega_{j+1}}/\Omega_{j-1} \subset \overline{\Omega_{j+2}}/\Omega_{j-2} \subset \overline{\overline{\Omega_{j+2}}/\Omega_{j-2}} = \Omega_{j+2}/\overline{\Omega_{j-2}},$$

the compact set $\overline{U_j}$ is contained in the open set $\Omega_{j+2}/\overline{\Omega_{j-2}}$, meaning with ε_j small, the set $\overline{U_j} + \mathrm{B}(0,\varepsilon_j)$ will still be in $\Omega_{j+2}/\overline{\Omega_{j-2}}$. Also, let us take possibly ε_j even smaller to have a nice inequality: for fixed $N \in \mathbb{N}$:

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(U)} < \frac{1}{2-\left(\frac{1}{2}\right)^{N+1}} \frac{\delta}{2^{j-1}},$$

meaning of $N \in \mathbb{N}$ will be evident later.

Set

$$v = \sum_{j\in\mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms ($\mathcal{F}$ is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u-v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u-v)(x) = \sum_{j=1}^{N+2} (u\varphi_j - v_j),$$

because for $m > N+2$, *i.e.*, $m-2 > N$ it holds the functions u_m, v_m, have their supports in

- $\operatorname{supp} u_m \subset U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}$,
- $\sup v_m \subset \Omega_{m+2}/\overline{\Omega_{m-2}}$,

but $\Omega_{m-1} \subset \Omega_{m-2} \subset \Omega_N$, for $N < m-2$, meaning that the set Ω_N does not lie in the supports of those functions. The norm of sum is (recall $\Omega_N \subset\subset \Omega$, so the above estimate holds)

$$\|u-v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \le \sum_{j=1}^{N+2} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} < \delta \frac{1}{2-\left(\frac{1}{2}\right)^{N+1}} \sum_{j=1}^{N+2} \frac{1}{2^j} = \delta.$$

It only remains to let $N \to \infty$ and realize

$$\|u-v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \to \|u-v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N\in\mathbb{N}} \int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x = \sup_{N\in\mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \sup_{N\in\mathbb{N}} |D^\alpha f| \chi_{\Omega_N} \,\mathrm{d}x \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega}(x) \,\mathrm{d}x = \int_{\Omega} |D^\alpha f| \,\mathrm{d}x\,,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Alltogether,

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \delta, \forall \delta > 0$$

from which it follows $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$ (this was not totally evident) and thus $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega) \cap C^\infty(\Omega)$ so indeed we have showed the desired density. □

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Remark ($\mathrm{C}^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $\mathrm{C}^{k,\lambda}$ if: Ω is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0,1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : \mathrm{B}(0,\alpha) \subset \mathbb{R}^{d-1} :\to \mathbb{R}\, s.t.\, a_j \in \mathrm{C}^{k,\lambda}(\mathrm{B}(0,\alpha)), \exists \mathbb{A}_j \mathbb{R}^d \to \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \le m : \varnothing \ne \partial\Omega \cap U_j = \mathbb{A}_j\left(\left\{(x', a_j(x') \in \mathbb{R}^d | x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}\right\}\right)$,
3. $\forall j \le m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \Omega$,
4. $\forall j \le m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$.

If $\lambda = 0$ we sometimes drop it and write $\Omega \in \mathrm{C}^{k,0} \Leftrightarrow \Omega \in \mathrm{C}^k$, if $k = 0, \lambda = 1$ we call $\Omega \in \mathrm{C}^{0,1}$ to be a Lipschitz domain. *Remember that $\lambda(\Omega) < \infty$ is a part of the definition.*

Theorem 5 (Global approximation by smooth functions up to the boundary)**.** *Let $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1,\infty)$. Then $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in \mathrm{C}^\infty(\overline{\Omega})$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\overline{\Omega}$,
2. partition of unity,
3. approximation of u on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^m$ be from the definition of $\mathrm{C}^{0,0}$ boundary. Then[4]

$$\overline{\Omega} \subset \bigcup_{j=0}^m U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to $\{U_j\}_{j=0}^m$. Since

$$u = \sum_{j=0}^m u\varphi_j, \text{ on } \Omega$$

[4]Our choice $U_0 = \Omega$ is important, as without it the definition of $\mathrm{C}^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^m U_j$.

observe that $u_j := u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, $\operatorname{supp} u_j \subset \operatorname{supp} \varphi_j \subset\subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d / \Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \ldots, m\}$.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of Ω, it holds[5] $u\varphi_0 \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)$. Since $\mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right) = \mathrm{W}_0^{\mathrm{k,p}}\left(\mathbb{R}^d\right) = \overline{\mathcal{D}\left(\mathbb{R}^d\right)}^{\|\cdot\|_{\mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)}}$, we can find $v_0 \in \mathcal{D}\left(\mathbb{R}^d\right)$ *s.t.*

$$\|v_0 - u_0\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} = \|v_0 - u\varphi_0\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

Notice that naturally

$$\mathcal{D}\left(\mathbb{R}^d\right) \subset \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right),$$

and so $v_0 \in \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$.

Case $j \in \{1, \ldots, m\}$. We have a problem now: $\{U_j\}_{j=1}^m$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of Ω, where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate u_j on a larger *open* domain containing $\overline{\Omega}$ and then show this is also a good aproximation of u_j on $\Omega \subset \overline{\Omega}$.

Set $u_j = u\varphi_j$, and denote

$$\begin{aligned} S_j &= \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in \mathrm{U}(0, \alpha)\right\}\right), \\ \Omega_j &= \mathbb{R}^d / \overline{S_j}, \end{aligned}$$

i.e.,

$$\text{"}\Omega_j = \Omega \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d \le a_j(x') - \frac{\beta}{2}\right\}\right),\text{"}$$

(although this is a bit inaccurate). Realize that since $u = 0$ outside of Ω, also u_j is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega_j)$. Now pick $\delta \in \left(0, \frac{\beta}{2}\right)$, where β is from the definition of $\mathrm{C}^{0,0}$ and set

$$\begin{aligned} S_j^\delta &= \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0, \alpha)\right\}\right), \\ \Omega_j^\delta &= \mathbb{R}^d / \overline{S_j^\delta}, \end{aligned}$$

i.e.,

$$\text{"}\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta\right\}\right).\text{"}$$

The trick is to shift the (support of) function u_j "into" Ω_j^δ

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in \mathrm{U}(0, \alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\operatorname{supp}(\tau_\delta u_j) = \operatorname{supp}(u_j) - \mathrm{B}(0, \delta),$$

[5] This would not hold if the support were not compactly contained in Ω.

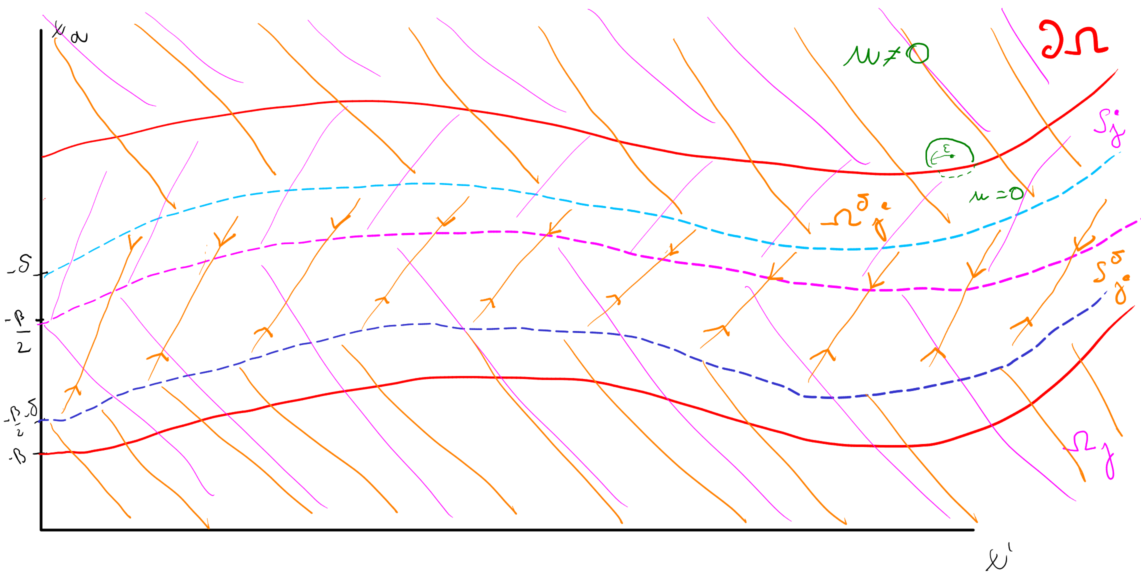

Figure 1: A cumbersome sketch of $\Omega_j, S_j, \Omega_j^\delta, S_j^\delta$

from which it follows $\tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)$; we have only shifted the function u_j, but since we have also shifted S_j, qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta$ and $\Omega \subset \Omega_j^\delta \cap \Omega_j$, and also $\Omega \subset \Omega_j$, $\Omega \subset \Omega_j^\delta \cap \Omega_j$, and the fact τ_δ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ *s.t.*

$$\|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \leq \|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to u_j, we are done once we approximate $\tau_\delta u_j$ by a function from $\mathrm{C}^\infty\left(\overline{\Omega}\right)$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $\mathrm{C}^\infty\left(\Omega_j^\delta\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right)$.

Show $\Omega \subset\subset \Omega_j^\delta$*:* We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^{m} \mathbb{A}_k(\{(x', x_d)| x_d = a_k(x'), x' \in \mathrm{U}(0,\alpha)\}),$$

and the set Ω_j^δ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right).$$

Realize it suffices to show $\partial\Omega \notin \overline{S_j}$, as then it wont be excluded from $\mathbb{R}^d$ and thus will end up in Ω_j^δ. *Thanks to continuity of* a_j, we may write

$$\overline{S_j} = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

i.e., the " $<$ " have changed to " $\leq$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d)| x_d = a_j(x'), x' \in \mathrm{U}(0,\alpha)\}`) \notin \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

which is equivalent to

$$\left((a_j \leq a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta)\right) \vee \left((a_j > a_j - \delta) \wedge (a_j \geq a_j - \frac{\beta}{2} - \delta)\right).$$

Our choice has been $\delta \in \left(0, \frac{\beta}{2}\right)$, and $\beta > 0$ from the definition of $\Omega \in \mathrm{C}^{0,0}$, so the second statement is clearly true $\forall j \in 1, \ldots, m$. Consequently $\partial\Omega \notin \overline{S}_j$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

Approximation of $\tau_\delta u_j$*.* Since Ω_j^δ is open, by the local approximation theorem there $\exists v_j \in \mathrm{C}^\infty\left(\mathbb{R}^d\right)$ such that for any $U \subset\subset \Omega_j^\delta$:

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{k,p}(U)} \frac{\varepsilon}{2(m+1)},$$

and so in particular, (as we have shown above $\Omega \subset \overline{\Omega} \subset \Omega_j^\delta$,)

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{2(m+1)}.$$

Realize that the function v_j is smooth on all $\mathbb{R}^d$, so in particular $v_j \in \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$.

Approximation of u.

Finally, let us set

$$v = \sum_{j=0}^{m} v_j .$$

Then $v \in \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$ and it holds

$$\begin{aligned}\|u - v\|_{\mathrm{W}^{k,p}(\Omega)} &= \left\| \sum_{j=0}^{m} u_j - \sum_{j=0}^{m} v_j \right\|_{\mathrm{W}^{k,p}(\Omega)} = \left\| \sum_{j=0}^{m} u_j - v_j \right\|_{\mathrm{W}^{k,p}(\Omega)} \le \sum_{j=0}^{m} \|u_j - v_j\|_{\mathrm{W}^{k,p}(\Omega)} \le \\ &\le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - u_j\|_{\mathrm{W}^{k,p}(\Omega)} \le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - \tau_\delta u_j\|_{\mathrm{W}^{k,p}(\Omega)} + \sum_{j=1}^{m} \|\tau_\delta u_j - u_j\|_{\mathrm{W}^{k,p}(\Omega)} < \\ &< \frac{\varepsilon}{m+1} + 2 \sum_{j=1}^{m} \frac{\varepsilon}{2(m+1)} = \varepsilon\end{aligned}$$

This proof may still have some flaws, but the author has decided to move on. □

Remark (What is $\mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$). Recall

$$C^{\infty}_{\overline{\Omega}}(\mathbb{R}^d) = \left\{ u|_{\overline{\Omega}}, u \in C^{\infty}(\mathbb{R}^d) \right\}.$$

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W}^{k,p}(\Omega)$, does there exist

$$\overline{u} \in \mathrm{W}^{k,p}\left(\mathbb{R}^d\right),\ s.t.\ \overline{u}|_{\Omega} = u, \|\overline{u}\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \le C(\Omega)\|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

?

The answer is **yes**, if Ω is nice enough. Notice however this is not as simple as in the case of Lebesgue spaces, where we could just extend the function by zero trivailly. We are dealing with derivatives, that are somehow regular, and if we extend a nonzero function by zero, it might mess up the regularity of the derivatives.

We will be using C^1 diffeomorphisms heavily, so we investigate some of their properties first.

Lemma 4 (Properties of C^1 diffeomorphisms)**.** *Let $U, V \subset \mathbb{R}^d$ be open, $\Phi : U \to V$ be a C^1 diffeomorphism and let $\tilde{U} \subset \mathbb{R}^d$ s.t. $\tilde{U} \subset\subset U$. Then*

1. *$\Phi(\tilde{U}) \subset\subset V$,*
2. *if moreover $\tilde{U}$ is compact, then* [6]

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \le C\|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

[6] For $\tilde{U}$ compact: $\tilde{U} \subset\subset V \Leftrightarrow \tilde{U} \subset V$.

Proof. *Ad 1.*: No proof has been given.

Ad 2.: Just a change of variables formula:

$$\|u\circ\Phi\|^p_{\mathrm{L_p}(\tilde U)} = \int_{\tilde U} |u\circ\Phi|^p \,\mathrm{d}x = \int_{\Phi(\tilde U)} |u|^p |\det\nabla\Phi|^{-1}\,\mathrm{d}x\,.$$

Since Φ is one-to-one, we know $|\det\nabla\Phi|>0$ on U, and since $\Phi\in \mathrm{C}^1\left(U\right)$ and $\tilde U\subset U \Rightarrow \Phi\in\mathrm{C}^1\left(\tilde U\right)\Rightarrow$ $\det\nabla\Phi\in\mathrm{C}^0\left(\tilde U\right)$, and since $\tilde U$ is compact, $|\det\nabla\Phi|\geq C_1>0 \Leftrightarrow |\det\nabla\Phi|^{-1}\leq\frac{1}{C_1}$. In total

$$\|u\circ\Phi\|^p_{\mathrm{L_p}(\tilde U)} \leq \frac{1}{C_1}\int_{\Phi(\tilde U)}|u|^p\,\mathrm{d}x = C\|u\|^p_{\mathrm{L_p}(\Phi(\tilde U))}.$$

As for the derivative, we have $\forall i\in\{1,\dots,d\}$:

$$\begin{aligned}\int_{\tilde U}|\partial_i(u\circ\Phi)|^p\,\mathrm{d}x &\leq \int_{\tilde U}|\nabla(u\circ\Phi)|^p\,\mathrm{d}x = \int_{\tilde U}\left|\nabla\Phi\Big((\nabla u)\circ\Phi\Big)\right|^p\mathrm{d}x\leq\\ &\leq\|\nabla\Phi\|\int_{\tilde U}|(\nabla u)\circ\Phi|^p\,\mathrm{d}x = \|\nabla\Phi\|\int_{\Phi(\tilde U)}|\nabla u|^p|\det\nabla\Phi|^{-1}\,\mathrm{d}x\leq C\|\nabla\Phi\|\int_{\Phi(\tilde U)}|\nabla u|^p\,\mathrm{d}x\leq\\ &\leq C\|\nabla u\|^p_{\mathrm{L}_p(\Phi(\tilde U))},\end{aligned}$$

where $\|\nabla\Phi\|$ is *e.g.* the operator norm of the matrix $\nabla\Phi$. □

Lemma 5 (Flat extension)**.** *Let $\alpha,\beta>0, K\subset\subset U(0,\alpha)\times[0,\beta)$ be compact. Then there $\exists C>0$, a linear operator*

$$E: C^1\left((B(0,\alpha)\times[0,\beta])\right)\to C^1\left((B(0,\alpha)\times[-\beta,\beta])\right),$$

and the set $\tilde K\subset\subset U(0,\alpha)\times[-\beta,\beta)$ such that $\forall u\in C^1\left(B(0,\alpha)\times[0,\beta]\right)$ it holds

1. *$Eu=u$ on $B(0,\alpha)\times[0,\beta]$,*
2. *$\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))}\leq\|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}$. Actually,*

$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)),\,W^{1,p}(U(0,\alpha)\times(-\beta,\beta)))} = 2^{\frac{1}{p}}$$

3. *if* $\operatorname{supp} u\subset K$ *then* $\operatorname{supp} Eu\subset\tilde K$

Proof. (The set $\mathrm{U}(0,\alpha)\times[0,\beta)$ is a cylinder of radius α and height β)

The proof is constructive: for the assumed u we write ($x=(x',x_d)$, where $x'\in\mathrm{B}(0,\alpha)\subset\mathbb{R}^{d-1}, x_d\in[0,\beta]\subset\mathbb{R}$)

$$Eu(x',x_d)=\begin{cases}u(x',x_d), & x_d\geq0\\ -3u(x',-x_d)+4u(x',-\frac{x_d}{2}), & x_d<0.\end{cases}$$

Does Eu lie in $\mathrm{C}^1\left(\mathrm{B}(0,\alpha)\times[-\beta,\beta]\right)$? Since $u\in\mathrm{C}^1\left(\mathrm{B}(0,\alpha)\times[0,\beta]\right)$, it us continuous in the "lower cylinder", check only the transition through the origin plane: take some $a=(x',0)\in\mathrm{B}(0,\alpha)\times\{0\}$. Then

$$\lim_{x\to a}Eu(x)=\begin{cases}u(a), & x_d\geq0\\ -3u(a)+4u(a)=u(a), & x_d<0,\end{cases}$$

so Eu is continuous. The derivatives

$$\lim_{x\to a} \partial_d Eu(x',x_d) = \begin{cases} \partial_d u(x',0), & x_d \geq 0 \\ \left(-3\partial_d u(x',-x_d)(-1) + 4\partial_d u(x',-\frac{x_d}{2})(-\frac{1}{2})\right)\Big|_{x_d=0} = \partial_d u(x',0), & x_d < 0, \end{cases}$$

and also for any $i \in \{1,\dots,d-1\}$

$$\lim_{x\to a} \partial_i Eu(x_1,\dots,x_d) = \begin{cases} = \partial_i u(x_1,\dots,0), & x_d \geq 0 \\ = -3\partial_i u(x_1,\dots,0) + 4\partial_i u(x_1,\dots,0) = \partial_i u(x_1,\dots,0) \end{cases}$$

so the the derivative is also continuous. Thus, we have

$$Eu \in C^1 \subset \mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta))\,.$$

The first property is clear from the definition of Eu, the estimates of the norm: (all derivatives can in fact be assumed classical)

$$\begin{aligned} \|Eu\|^p_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} &= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = \\ &= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} \\ &+ + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} = \\ &= \|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|4u-3u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} + \\ &+ \sum_{|\alpha|=1} \|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1} \|D^\alpha (4u-3u)\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} \\ &= 2\|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + 2\sum_{|\alpha|=1} \|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} = 2\|u\|^p_{\mathrm{W^{k,p}}(\mathrm{U}(0,\alpha)\times(0,\beta))} \end{aligned}$$

and so

$$\|Eu\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = 2^{\frac{1}{p}}\, \|u\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))}.$$

Where we have used the obvious fact

$$\int_{\mathrm{U}(0,\alpha)\times(0,\beta)} |f|^p \,\mathrm{d}x = \int_{\mathrm{U}(0,\alpha)\times(-\beta,0)} |f|^p \,\mathrm{d}x\,,$$

We will assume (although this is *with* a loss of generality), that

$$Eu(x',x_d) = 0 \Leftrightarrow (u(x',x_d) = 0 \vee 3u(x',-x_d) = 0 \vee 4u(x',-\frac{x_d}{2}) = 0),$$

i.e., that support lies in

$$\begin{aligned} \operatorname{supp} Eu \subset \{(x',x_d) \in \mathrm{U}(0,\alpha)\times[0,\beta] | u(x',x_d) = 0\} \cup \{(x',x_d) \in \mathrm{U}(0,\alpha)\times[-\beta,0] | u(x',-x_d) \neq 0\} \cup \\ \cup \left\{(x',x_d) \in \mathrm{U}(0,\alpha)\times[-\beta,0] | u(x',-\frac{x_d}{2}) \neq 0\right\}. \end{aligned}$$

Denote $\Phi_1, \Phi_2 : \mathrm{U}(0,\alpha) \times (0,\beta) \to \mathrm{U}(0,\alpha) \times (-\beta,\beta)$ to be the mappings

$$\Phi_1(x', x_d) = (x', -x_d),$$
$$\Phi_2(x', x_d) = (x', -\frac{x_d}{2}),$$

then clearly Φ_1, Φ_2 are C^1 diffeomorphisms and

$$u(x', -x_d) = (u \circ \Phi_1)(x', x_d), u\left(x', -\frac{x_d}{2}\right) = (u \circ \Phi_2)(x', x_d),$$

i.e.,

$$u(x', x_d) = \left(u \circ \Phi_1^{-1}\right)(x', -x_d) = \left(u \circ \Phi_2^{-1}\right)\left(x', -\frac{x_d}{2}\right),$$

and so

$$\operatorname{supp} Eu \subset \operatorname{supp} u \cup \Phi_1^{-1}(\operatorname{supp} u) \cup \Phi_2^{-1}(\operatorname{supp} u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as $\operatorname{supp} u \subset\subset K$. Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\operatorname{supp} Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have $K \subset\subset \mathrm{U}(0,\alpha) \times [0,\beta) \Rightarrow K \subset\subset \mathrm{U}(0,\alpha) \times (-\beta,\beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset\subset \mathrm{U}(0,\alpha) \times (0,\beta)$, which really implies

$$\tilde{K} \subset\subset \mathrm{U}(0,\alpha) \times [-\beta,\beta).$$

$\square$

Let us prove the main result.

Theorem 6 (Extension of Sobolev functions)**.** *Let $\Omega \in C^{k-1,1}, k \in \mathbb{N}, p \in [1,\infty], V \subset \mathbb{R}^d$ open such that $\Omega \subset\subset V$. Then there is $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ bounded linear operator such that*

1. *$\forall u \in W^{k,p}(\Omega) : Eu = u$ a.e. in Ω,*
2. *$\forall u \in W^{k,p}(\Omega) : \operatorname{supp} Eu \subset V \supset\supset \Omega$,*
3. *$\|E\|_{\mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^d))} \le C, C = C(p, \Omega, V)$.*

Proof. **Will only be presented for $k = 1, \Omega \in C^1, p < \infty$.** The strategy is:

1. covering of $\overline{\Omega}$ & partition of unity
2. obtain a diffeomorphism from the fact $\Omega \in \mathrm{C}^{1,0}$,
3. suitable composition & cut off,
4. flat extension,
5. show existence of $E : \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d) \to \mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)$ with the desired properties,

6. extend via density

Covering of Ω*:* In the following, $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ are as in the definition of a $\mathrm{C}^{1,0}$ domain. Set $U_0 = \Omega$ and realize

$$\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j,$$

i.e., $\{U_j\}_{j=0}^m$ is an open covering of $\overline{\Omega}$. Denote $\{\varphi_j\}_{j=0}^m$ as the partition of unity subordinate to $\{U_j\}_{j=0}^m$.

Diffeomorphism & flat extension For $j \in \{1, \dots, m\}$ we define $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta, \beta) \to U_j$ by

$$\Phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

(A bit confusingly, we will however be "interpreting" Φ_j as $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta, \beta) \to \mathbb{R}^d$, with it being extended by zero on $\mathbb{R}^d / \overline{U_j}$, as we need Φ_j^{-1} to be defined on the whole $\mathbb{R}^d$.)

As $\Omega \in \mathrm{C}^{1,0}$, we know $a_j \in \mathrm{C}^1(\mathrm{B}(0,\alpha)) \subset \mathrm{C}^1(\mathrm{U}(0,\alpha))$, and so ϕ_j is a C^1 diffemorphism. Let us denote by $\tilde{E} : \mathrm{C}^1(\mathrm{B}(0,\alpha) \times [-\beta, \beta]) \to \mathrm{C}^1(\mathrm{B}(0,\alpha) \times [-2\beta, \beta])$ the extension operator from the Flat extension lemma. Then we for any $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$ define

$$Eu = \varphi_0 u + \sum_{j=1}^{m} \left(\eta \tilde{E}((\varphi_j u) \circ \Phi_j)\right) \circ \Phi_j^{-1},$$

where $\eta \in \mathrm{C}^\infty(\mathrm{U}(0,\alpha) \times \mathbb{R})$ is a cut-off function

$$\eta(y', y_d) \begin{cases} = 1 \,\text{on} & y_d \geq 0, \\ = 0 \,\text{on} & y_d \leq -h, \\ \in (0,1) & \text{else}. \end{cases},$$

for some parameter $h > 0$ which will be defined later. With this definition, $\forall x \in \Omega$ it holds

$$(Eu)(x) = (\varphi_0 u)(x) + \sum_{j=1}^{m} \left(\eta \tilde{E}(u\varphi_j \circ \Phi_j)\right) \underbrace{\left(\Phi_j^{-1}(x)\right)}_{\in \mathrm{U}(0,\alpha) \times (-\beta,\beta)} =$$

$$= (\varphi_0 u)(x) + \sum_{j=1}^{m} (u\varphi_j \circ \Phi_j)\left(\Phi_j^{-1}(x)\right) = (\varphi_0 u)(x) + \sum_{j=1}^{m} (u\varphi_j)(x) = \sum_{j=0}^{m} (u\varphi_j)(x) = u(x) \sum_{j=0}^{m} \varphi_j(x) = u(x),$$

since $\eta(y) = 1, \tilde{E}(\varphi_j u \circ \Phi_j)(y) = (\varphi_j u \circ \Phi_j)(y)$ for $y \in \mathrm{U}(0,\alpha) \times (-\beta, \beta)$, according to our definition of η and the properties of the extension $\tilde{E}$.

The motivation behind the cutoff is the following: we know $\operatorname{supp} \varphi_j \subset U_j$, so since $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta, \beta) \to U_j$, we have $\operatorname{supp} u\varphi_j \circ \Phi_j \subset \mathrm{U}(0,\alpha) \times (-\beta, \beta)$ and from the properties of the flat extension operator we also have $\operatorname{supp} \tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-2\beta, \beta)$. Since moreover $\operatorname{supp} \eta = \mathrm{U}(0,\alpha) \times (-2h, \infty)$, in total

$$\operatorname{supp} \eta \tilde{E}(u\varphi_j \circ \Phi_j) = \operatorname{supp} \eta \cap \operatorname{supp} \tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h, \infty) \cap (-\beta, \beta),$$

and also [7]

$$\operatorname{supp} Eu \subset \Phi_j\left(\operatorname{supp} \eta \tilde{E}(u\varphi_j \circ \Phi_j)\right).$$

[7] Generally, $\operatorname{supp} v \circ \Psi^{-1} = \overline{\{x \in \mathbb{R}^d | u(\Psi^{-1}(x)) \neq 0\}} = \overline{\{x \in \mathbb{R}^d | \Psi^{-1}(x) \in \operatorname{supp} u\}} = \overline{\{x \in \mathbb{R}^d | x \in \Psi(\operatorname{supp} u)\}} = \Psi(\operatorname{supp} u)$.

We need to prove $\operatorname{supp} Eu \subset V$, where V is some neighbourhood of Ω. If it holds

$$\operatorname{supp} \eta \tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta),$$

the desired property holds, as then

$$\operatorname{supp} Eu \subset \Phi_j\big(\operatorname{supp} \eta \tilde{E}(u\varphi_j \circ \Phi_j)\big) \subset \Phi_j(\mathrm{U}(0,\alpha) \times (-\beta,\beta)) \subset U_j \subset \overline{\Omega} \subset V.$$

In total,

$$\operatorname{supp} \eta \tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h,\infty) \cap (-\beta,\beta) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta).$$

So if h is such that $-2h > -\beta \Leftrightarrow h < \frac{\beta}{2}$, we can guarantee $\operatorname{supp} Eu \subset V$ and that is what we want.

Finally, E is clearly linear, its norm: (we are using the lemma about flat extension and the properties of C^1 diffeomorphisms together with the facts $\eta \le$ on $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, $\Phi_j(\mathrm{U}(0,\alpha) \times (0,\beta)) \subset U_j, \Phi_j^{-1}(\mathbb{R}^d) \subset$ [8], $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, [9])

$$\begin{aligned}
\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &= \big\|\big(\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big) \circ \Phi_j^{-1}\big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\big\|\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = \\
&= C\big\|\tilde{E}(u\varphi_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} \le C\|u\varphi_j \circ \Phi_j\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))} \le \\
&\le C\|u\varphi_j\|_{\mathrm{W}^{1,p}(U_j \cap \Omega)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},
\end{aligned}$$

from which it clearly follows $\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))} \le C$.

So all the properties hold for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$, let us show them also for $u \in \mathrm{W}^{1,p}(\Omega)$. Pick an arbitrary $u \in \mathrm{W}^{1,p}(\Omega)$, find $\{u_k\} \subset C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u_k \to u \text{ in } \mathrm{W}^{1,p}(\Omega)$. Since E is continous, we know $\lim_{k\to\infty} Eu_k$ exists. Let us set

$$Eu := \lim_{k\to\infty} Eu_k,$$

where Eu_k is defined above for $u_k \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$.

Ad 3): Clearly $E(\alpha u) = \lim_{k\to\infty} E(\alpha u_k) = \alpha Eu, E(u+v) = \lim_{k\to\infty} E(u_k+v_k) = \lim_{k\to\infty} Eu_k + Ev_k = Eu + Ev$, since E on the continous functions is linear. Also

$$\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} = \Big\|\lim_{k\to\infty} Eu_k\Big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} = \lim_{k\to\infty} \|Eu_k\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le \|E\| \lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} = \|E\|\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)},$$

(we are using $\{Eu_k\}$ has a limit); we see our above definition truly yields a continuous linear operator.

Ad 1): $\forall\, a.a.\, x \in \Omega : Eu(x) = \lim_{k\to\infty} Eu_k(x) = \lim_{k\to\infty} u_k(x) = u(x)$,

Ad 2): $\operatorname{supp} Eu_k \subset \mathrm{U}(\Omega,\varepsilon) \Rightarrow \operatorname{supp} Eu \subset \overline{\mathrm{U}(\Omega,\varepsilon)} \subset V$.

$$\operatorname{supp} Eu = \Big\{x \in \mathbb{R}^d \,|\, \lim_{k\to\infty} Eu_k \ne 0\Big\} \subset \bigcap_{k=1}^{\infty} \underbrace{\operatorname{supp} Eu_k}_{\subset V} \subset V.$$

We are done.

□

Remark ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $\Omega \in \mathrm{C}^{0,1}$ and $p \in (1,\infty), k \in \mathbb{N}$, but the construction of the extension must be different. "It seems the result is not known for $\Omega \in \mathrm{C}^{0,1}$ and $p = 1$, or $p = \infty$."

[8] We can somehow extend Φ_j^{-1} by zero from $U_j \subset \Omega \subset \mathbb{R}^d$ to be defined on the whole $\mathbb{R}^d$ (i guess)

[9] ...even though the assumptions to use those are not totally valid... but doc. Kaplicky is okay with that

2.4 Embedding theorems

From the definition of $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ it immediately follows $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L_p}(\Omega)$. Can we obtain $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L}_q(\Omega)$ for some $q > p$? The answer **yes**, if Ω is again nice enough (and there will also be some dependence on the dimension of $\mathbb{R}^d$.)

2.4.1 Theorems for $p \le d$

Example. Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2)\,\mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1, s)\,\mathrm{d}s\,,$$

so

$$\begin{aligned}\|u\|^2_{\mathrm{L}_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \\ &\le \left(\int_{\mathbb{R}}\int_{-\infty}^{x_1} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{-\infty}^{x_2} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\ &\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\ &\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) = \\ &= \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right) = \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)^2 = \\ &= \|\nabla u\|^2_{\mathrm{L}_1(\mathbb{R}^2)}\,,\end{aligned}$$

so

$$\|u\|_{\mathrm{L}_2(\mathbb{R}^2)} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^2)}.$$

This can be generalized in two ways:

- $d > 2$,
- less smoothness.

Lemma 6 (Gagliardo)**.** *Let $d \ge 2$. Let $\hat{u}_i : \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative and measurable for $j \in \{1, \ldots, d\}$. We define*

$$\hat{x}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d), \hat{\mathrm{dx}}_j = \mathrm{d}x_1 \ldots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \ldots \mathrm{d}x_d\,.$$

Consider the functions $u_j : \mathbb{R}^d \to \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x)\,\mathrm{d}x \le \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} \hat{\mathrm{dx}}_j\right)^{\frac{1}{d-1}}. \tag{1}$$

(Both integrals can be infinity.)

Proof. Induction by dimension: *The case* $d = 2$.:

$$\int_{\mathbb{R}^2} u_1(x_1,x_2)u_2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2)\hat{u}_2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \Big(\int_{\mathbb{R}} \hat{u}_1(x_2)\,\mathrm{d}x_2\Big)\Big(\int_{\mathbb{R}} \hat{u}_2(x_1)\,\mathrm{d}x_1\Big) =$$
$$= \Big(\int_{\mathbb{R}} \hat{u}_1(\hat{x}_1)\mathrm{d}\hat{\mathrm{x}}_1\Big)\Big(\int_{\mathbb{R}} \hat{u}_2(\hat{x}_2)\mathrm{d}\hat{\mathrm{x}}_2\Big).$$

(an equality in fact.) We have used Fubini once, which is permitted, as we have measurability + nonnegativity.

The case $d \to d+1$ Before we procceed, recall the "generalized Holder", all functions are nonnegative

$$\int_\Omega \prod_{j=1}^d f_j\,\mathrm{d}x \le \prod_{j=1}^d \Big(\int_\Omega f_j^{p_j}\,\mathrm{d}x\Big)^{\frac{1}{p_j}},$$

where $\sum_{j=1}^d \frac{1}{p_j} = 1$. See that if we take $p_j = d$, then $\sum_{j=1}^d \frac{1}{d} = \frac{1}{d}\sum_{j=1}^d 1 = 1$, so

$$\int_\Omega \prod_{j=1}^d f_j\,\mathrm{d}x \le \prod_{j=1}^d \Big(\int_\Omega f_j^{d}\,\mathrm{d}x\Big)^{\frac{1}{d}}.$$

Let us compute:

$$\int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \prod_{j=1}^d u_j \overbrace{u_{d+1}}^{=\hat{u}(\hat{x}_{d+1})} \overbrace{\mathrm{d}x_1\ldots\mathrm{d}x_d}^{=\mathrm{d}\hat{\mathrm{x}}_{d+1}}\,\mathrm{d}x_{d+1} = \int_{\mathbb{R}^d} \underbrace{\Big(\int_{\mathbb{R}} \prod_{j=1}^d u_j\,\mathrm{d}x_{d+1}\Big)}_{\text{Holder}} u_{d+1}\mathrm{d}\hat{\mathrm{x}}_{d+1} \le$$
$$\le \int_{\mathbb{R}^d} \underbrace{\Big(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d\,\mathrm{d}x_{d+1}\Big)^{\frac{1}{d}} u_{d+1}}_{\text{Holder}}\,\mathrm{d}\hat{\mathrm{x}}_{d+1} \le \Big(\int_{\mathbb{R}^d} u_{d+1}^d\,\mathrm{d}\hat{x}_{d+1}\Big)^{\frac{1}{d}} \Bigg(\int_{\mathbb{R}^d} \Big(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x)\,\mathrm{d}x_{d+1}\Big)^{\frac{1}{d}\frac{d}{d-1}}\,\mathrm{d}\hat{x}_{d+1}\Bigg)^{\frac{d-1}{d}} \le$$
$$\le \Big(\int_{\mathbb{R}^d} u_{d+1}^d\,\mathrm{d}\hat{x}_{d+1}\Big)^{\frac{1}{d}} \Bigg(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \Big(\int_{\mathbb{R}} u_j^d\,\mathrm{d}x_{d+1}\Big)^{\frac{1}{d-1}\frac{d-1}{1}}\,\mathrm{d}\hat{x}_j\,\mathrm{d}\hat{x}_{d+1}\Bigg)^{\frac{d-1}{d}\frac{1}{d-1}} =$$
$$= \Big(\int_{\mathbb{R}^d} u_{d+1}^d\,\mathrm{d}\hat{x}_{d+1}\Big)^{\frac{1}{d}} \prod_{j=1}^d \Big(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} u_j^d\,\mathrm{d}x_{d+1}\,\mathrm{d}\hat{x}_j\,\mathrm{d}\hat{x}_{d+1}\Big)^{\frac{1}{d}} = \Big(\int_{\mathbb{R}^d} u_{d+1}^d\,\mathrm{d}\hat{x}_{d+1}\Big)^{\frac{1}{d}} \prod_{j=1}^d \Big(\int_{\mathbb{R}^d} u_j^d\,\mathrm{d}\hat{x}_j\Big)^{\frac{1}{d}} =$$
$$= \prod_{j=1}^{d+1} \Big(\int_{\mathbb{R}^d} u_j^d\,\mathrm{d}\hat{x}_j\Big)^{\frac{1}{d}},$$

where the induction step is taken for the function

$$v_j = \Big(\int_{\mathbb{R}} u_j^d\,\mathrm{d}x_{d+1}\Big)^{\frac{1}{d-1}},$$

that is clearly nonnegative and measurable. ∎

Remark. Sometimes, the lemma is stated as: $\hat{u}_i \in \mathrm{L}_\infty\left(\mathbb{R}^{d-1}\right), \operatorname{supp} \hat{u}_i$ is compact $\forall i \in \{1, \ldots, d\}$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d |u_j(x)| \, \mathrm{d}x \le \prod_{j=1}^{} \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_j(\hat{x}_j)|^{d-1} \mathrm{d}\hat{\mathrm{x}}_\mathrm{i} \right)^{\frac{1}{d-1}} = \prod_{j=1}^d \|\hat{u}_j\|_{\mathrm{L}_{d-1}(\mathbb{R}^{d-1})}.$$

The difference is that in our version, we have nonnegativity in the assumptions and do not requiry compact supports and essential boundedness, as we work with integrals that are possibly infinite.

Theorem 7 (Gagliardo-Nirenberg)**.** *Let* $p \in [1, d)$. *Then* $\forall u \in W^{1,p}(\mathbb{R}^d)$*:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where $p^* = \frac{dp}{d-p}$.

Proof. Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$, then use density, as $\mathrm{W}^{1,p}(\mathbb{R}^d) = \mathrm{W}_0^{1,p}(\mathbb{R}^d)$.

$$\forall j \in \{1, \ldots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_d) \, \mathrm{d}s \,.$$

This estimate is independent of $j \in \{1, \ldots, d\}$, so it holds

$$|u(x)| \le \int_{\mathbb{R}} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s \,.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \le \left(\int_{\mathbb{R}^d} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s \right)^{\frac{d}{d-1}} = \prod_{j=1}^d \left(\int_{\mathbb{R}} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s \right)^{\frac{1}{d-1}}.$$

Denote

$$u_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) = \left(\int_{\mathbb{R}^d} |\nabla u| \overbrace{(\ldots, x_j, \ldots)}^{=x \text{ in fact}} \mathrm{d}x_j \right)^{\frac{1}{d-1}},$$

which is a function independent of $x_j, u_j \equiv u_j(\hat{x}_j)$. So the integral (the $\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)$ norm)

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} \, \mathrm{d}x \le \int_{\mathbb{R}^d} \prod_{j=1}^d u_j \, \mathrm{d}x \underbrace{\le}_{\text{Gagliardo lemma}} \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} u_j^{d-1}(\hat{x}_j) \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d-1}} =$$

$$= \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) \, \mathrm{d}x_j \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u| \, \mathrm{d}x \right)^{\frac{d}{d-1}},$$

and so

$$\|u\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)}^{\frac{d}{d-1}} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^d)}^{\frac{d}{d-1}},$$

meaning ($1^* = \frac{d}{d-1}$)

$$\|u\|_{\mathrm{L}_{1^*}(\mathbb{R}^d)} \le 1 \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^d)}.$$

If now $p \in (1,d)$, we investigate for what q can we obtain estimate of $\| |u|^q \|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)}$:

$$\|u\|^q_{L_{\frac{qd}{d-1}}(\mathbb{R}^d)} = \||u|^q\|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \le \|\nabla(|u|^q)\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1}|\nabla u|\,\mathrm{d}x \underbrace{\le}_{\text{Holder}} q\|\nabla u\|_{L_p(\mathbb{R}^d)} \left\||u|^{q-1}\right\|_{L_{p'}(\mathbb{R}^d)} =$$

$$= q\|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|^{q-1}_{L_{(q-1)p'}(\mathbb{R}^d)}.$$

We want $(q-1)p' = \frac{qd}{d-1}$, so we can divide both sides:

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$\frac{q}{d-1} = \frac{p}{d-p} \Leftrightarrow \frac{qd}{d-1} = \frac{dp}{d-p} = p^*, q = \frac{p(d-1)}{d-p}$$

and thus

$$\|u\|^q_{L_{p^*}(\mathbb{R}^d)} \le q\|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|^{q-1}_{L_{p^*}(\mathbb{R}^d)} \Leftrightarrow \|u\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

$\Rightarrow$ statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$: let $u \in W^{1,p}(\mathbb{R}^d)$, be arbitrary. Then $\exists \{u_k\} \subset \mathcal{D}(\mathbb{R}^d) : u_k \to u$ in $W^{1,p}(\mathbb{R}^d)$. Moreover, we have showed that $\forall k \in \mathbb{N}$:

$$\|u_k\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)},$$

so passing to the (strong) limit and using the continuity of the norm indeed yields

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

We are done.

□

Remark.
- It is evident that nonzero constants are not in $W^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.
- the set $\mathbb{R}^d$ is of course unbounded, so we have no ordering of $L_p(\Omega)$ spaces.
- of course, we require no smoothness of the domain

Theorem 8. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W^{1,p}_0(\Omega), \forall p \in [1,d)$ the statement of the previous theorem holds.*

Proof. An immediate corollary of the previous theorem: we have showed the inequality for $u \in \mathcal{D}(\mathbb{R}^d)$, but WLOG it holds also for $u \in \mathcal{D}(\Omega)$ (i can keep the integrals over $\mathbb{R}^d$, but in the end only values from Ω count) and since $W^{1,p}_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$, i can again extend it on the whole $W^{1,p}_0(\Omega)$. □

Remark. In the proof of theorem we showed that $\forall u \in \mathrm{W}^{1,p}\big(\mathbb{R}^d\big)$ it holds

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\Omega)} \leq q\|\nabla u\|_{\mathrm{L}_{\mathrm{p}}(\Omega)}\|u\|^{q-1}_{\mathrm{L}_{\frac{p(q-1)}{p-1}}(\Omega)},$$

for q such that $\frac{qd}{d-1} \leq p^*$.

Definition 5 (Continuous & compact embeddings)**.** Let X, Y be linear normed spaces. We say

- X is continuously embedded into Y, $X \hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and

$$\forall x \in X : \|x\|_Y \leq C\|x\|_X.$$

- X is compactly embedded into Y, $X \hookrightarrow\hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and

$$\forall A \subset X \text{ bounded} : \overline{A}^Y \text{ is compact in } Y.$$

This is the same as saying[10] $X \subset Y$ (is a subspace) and the identity $\mathrm{id} : X \to Y$ is

- a bounded linear operator, $\mathrm{id} \in \mathcal{L}(X, Y)$
- is a compact linear operator, $\mathrm{id} \in \mathcal{K}(X, Y)$

Theorem 9 (Embedding theorem for $p \leq d$)**.** *Let $\Omega \in C^{0,1}, p^* = \frac{dp}{d-p}$. Then*

- *if $p \in [1, d)$, then*
$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)\; \forall q \in [1, p^*],$$
- *if $p \in [1, d)$, then*
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)\,, \forall q \in [1, p^*),$$
- *if $p = d$, then*
$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega)\; \forall q \in [1, \infty),$$
- *if $p = d$, then*
$$W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)\; \forall q \in [1, \infty),$$

  (the same as above, i.e., every continuous embedding is also compact.)

Proof. We would like to use the previous lemmas + extension.
Ad continuity for $p < d$:
Recall that the composition of continuous operators yields a continuous operator. In our case:

- the extension operator $E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}\big(\mathbb{R}^d\big)$ is continuous
- identity $I_1 : \mathrm{W}^{1,p}\big(\mathbb{R}^d\big) \to \mathrm{L}_{p^*}\big(\mathbb{R}^d\big)$ is continous (Gagliardo-Nirenberg: $\|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$.)

[10] Really, we have $\|x\|_Y = \|\mathrm{id}\, x\|_Y \leq \|\mathrm{id}\|_{\mathcal{L}(X,Y)}\|x\|_X$, and if $A \subset X$ is bounded, than from the definition of $\mathrm{id} \in \mathcal{K}(X, Y) : \mathrm{id}(A) = A \subset Y$ is relatively compact in Y.

- restriction $I_2 : \mathrm{L}_{p^*}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\Omega)$ is continuous (monotonicity of the L. integral: $\Omega \subset \mathbb{R}^d \Rightarrow \|u\|_{\mathrm{L}_{p^*}(\Omega)} \leq \|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)}$.)
- identity $I_3 : \mathrm{L}_{p^*}(\Omega) \to \mathrm{L}_q(\Omega)$ is continous (embedding of Lebesgue spaces: Ω is bounded $\Rightarrow \mathrm{L}_{p^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\, \forall q \in [1, p^*]$)

Together, the mapping

$$\mathrm{id} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\Omega)\,, \mathrm{id} = I_3 \circ I_2 \circ I_1 \circ E$$

is continuous, and so $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1, p^*]$.

Ad continuity for $p = d$:

If $p = d$, we have (this holds generally) $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega)\, \forall r \in [1, d)$, (embedding of Lebesgue spaces, $\mathrm{L}_d(\Omega) \hookrightarrow \mathrm{L}_r(\Omega)\,, \forall r \in [1, d)$). Notice $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d-$, which means we can for all $q \in [1, \infty)$ find $r \in [1, d)\,\mathrm{s.t.}\, r^* > q$. Consequently,

$$\forall q \in [1, \infty) \exists r \in [1, d)\; s.t.\, \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,.$$

Notice also that $\forall r \in [1, d)$ we always have

$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega)$$

(that's just renaming p with r in embedding for $p < d$). Then it holds

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,,$$

and so $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1, \infty)$.

Ad compactness for $p < d$:

It suffices to show $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0, 1)$ is relatively compact in $\mathrm{L}_q(\Omega)$, which is, since $\mathrm{L}_q(\Omega)$ is complete, equivalent to U being totally bounded in $\mathrm{L}_q(\Omega)$.[11] Extend the functions to $\mathrm{W}^{1,p}(\mathbb{R}^d)$ using the extension operator E, so $EU \subset \mathrm{W}^{1,p}(\mathbb{R}^d)$. Take some yet undetermined $\delta > 0$ and denote by $(EU)_\delta$ the set of regularized functions from EU with some kernel η. Our next strategy is the following:

1. show $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0, R))$,
2. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_1(\Omega)$,
3. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)$.

Since the supports of the functions from EU are uniformly bounded[12], we know [13] $\exists R > 0$ *s.t.*

$$\forall v \in (EU)_\delta : \mathrm{supp}\, v \subsetneq \mathrm{B}(0, R) \subset \mathbb{R}^d.$$

[11] A metric space P is totally bounded if there exists a finite $\varepsilon-$net: a finite open covering of P by balls centered in P of radii smaller than ε.

[12] From the properties of extension, we know $\forall u \in \mathrm{W}^{1,p}(\Omega) : \mathrm{supp}\, Eu \subset V$ with V open *s.t.* $\Omega \subset\subset V$.

[13] Properties of mollification include $\mathrm{supp}(Eu)_\delta \subset \mathrm{B}(0, \delta) + \mathrm{supp}\, Eu$.

(Also remember that $\Omega \subset \mathrm{B}(0,R)$.) Moreover, from the properties of mollification[14] it follows:

$$(EU)_\delta \subset \mathrm{C}^1\left((\mathrm{B}(0,R))\right).$$

Next up, calculate for $v \in (EU)_\delta$ the norm $\|v\|_{\mathrm{C}^1(\mathrm{B}(0,1))}$. Realize that in fact $v = (Eu)_\delta$ for some $u \in U$, and so

$$|\int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,$$

as $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$. Also

$$\begin{aligned}|\nabla \int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y| = |\int_{\mathbb{R}^d} Eu(y)\nabla_x\eta_\delta(x-y)\,\mathrm{d}y| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\nabla\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le \\ \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,\end{aligned}$$

using the same arguments. In total

$$\forall v \in (EU)_\delta : \|v\|_{\mathrm{C}^1(\mathrm{B}(0,R))} \le C,$$

or in other words, all functions from $(EU)_\delta$ are uniformly bounded in $\mathrm{C}^1(\mathrm{B}(0,R)) \Rightarrow$ they are uniformly bounded and uniformly equicontinuous (that is implied by uniform boundedness of the derivatives). Thus we can use Arzela-Ascoli theorem and state

$$(EU)_\delta \subset\subset \mathrm{C}^0(\mathrm{B}(0,R)).$$

Since $\mathrm{C}^0(\mathrm{B}(0,R))$ is complete, this also means $(EU)_\delta$ is totally bounded in $\mathrm{C}^0(\mathrm{B}(0,R))$. Using the fact ($\mathrm{B}(0,R)$ is compact) [15]

$$\mathrm{C}^0(\mathrm{B}(0,R)) \hookrightarrow\hookrightarrow \mathrm{L}_1(\mathrm{U}(0,R)),$$

we also see that $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0,R))$.

Next, take an arbitrary $u \in U$ and compute (we are using the fact $Eu = u$ *a.e.* in $\Omega, \Omega \subset \mathrm{U}(0,R)$.)

$$\begin{aligned}\|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} &\le \Big\|\overbrace{Eu}^{:=v}-(Eu)_\delta\Big\|_{\mathrm{L}_1(\mathrm{U}(0,R))} = \int_{\mathrm{U}(0,R)}|v-v_\delta|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}|v(x)-\int_{\mathbb{R}^d}v(y)\eta_\delta(x-y)\,\mathrm{d}y|\,\mathrm{d}x = \\ &= \int_{\mathrm{U}(0,R)}|v(x)-\int_{\mathbb{R}^d}v(x-y)\eta_\delta(y)\,\mathrm{d}y|\,\mathrm{d}x = \left[x\mapsto x+y\right] = \int_{\mathrm{U}(0,R)}|\int_{\mathbb{R}^d}v(x+y)-v(x)\underbrace{\eta_\delta(y)}_{\le 1}\,\mathrm{d}y|\,\mathrm{d}x \le \\ &\le \int_{\mathrm{U}(0,R)}|\int_{\mathbb{R}^d}(v(x+y)-v(y))\eta_\delta(y)\,\mathrm{d}y|\,\mathrm{d}x \le \int_{\mathrm{U}(0,R)}\int_{\mathbb{R}^d}\frac{|v(x+y)-v(x)|}{|y|}|\eta_\delta(y)||y|\,\mathrm{d}y\,\mathrm{d}x \le \\ &\underbrace{\le}_{\text{Fubini}} \int_{\mathbb{R}^d}\int_{\mathrm{U}(0,R)}\frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y\,.\end{aligned}$$

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[14]We have $(EU)_\delta \subset \mathrm{C}^\infty(\mathbb{R}^d)$.

[15]Take some sequence $\{f_k\} \subset \mathrm{C}^0(\mathrm{B}(0,R))$ that is bounded, then

$$\lim_{k\to\infty}\int_{\mathrm{U}(0,R)}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}\lim_{k\to\infty}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}|f|\,\mathrm{d}x = \int_{\mathrm{B}(0,R)}|f|\,\mathrm{d}x \le \infty,$$

the majorant being *e.g.* $\max_{k\in\mathbb{N}}\|f_k\|_\infty$. Hence every bounded sequence in $\mathrm{C}^0(\mathrm{B}(0,R))$ has a converging (sub)sequence in $\mathrm{L}_1(\mathrm{U}(0,R))$.

Estimate the inner integral: assume v is smooth, $v \in \mathcal{D}(\mathrm{U}(0,R))$ and write

$$\int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x = \int_{\mathrm{U}(0,R)} \frac{1}{|y|}\Big|\int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(v(x+sy))}_{\nabla v(x+sy)\cdot y}\,\mathrm{d}s\Big|\,\mathrm{d}x \underbrace{\leq}_{\text{C-S}} \int_{\mathrm{U}(0,R)} \frac{1}{|y|}\int_0^1 |\nabla v(x+sy)||y|\,\mathrm{d}s\,\mathrm{d}x =$$

$$= \int_{\mathrm{U}(0,R)} \int_0^1 |\nabla v(x+sy)|\,\mathrm{d}s\,\mathrm{d}x = \int_0^1 \int_{\mathrm{U}(0,R)} |v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s\,.$$

The last integral can be further manipulated by using the change of variables $z := x+sy \in \{x+sy | x \in \mathrm{U}(0,R)\} = \mathrm{U}(0,R)+sy = \mathrm{U}(sy,R)$. Since $\nabla v \in \mathcal{D}(\mathrm{U}(0,R))$, the integral is nonzero only for $z \in \mathrm{U}(sy,R) \cap \mathrm{U}(0,R) \subset \mathrm{U}(0,R)$ so we can write

$$\int_0^1 \int_{\mathrm{U}(0,R)} |\nabla v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s = \int_0^1 \int_{\mathrm{U}(sy,R)\cap \mathrm{U}(0,R)} |\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \leq \int_0^1 \int_{\mathrm{U}(0,R)} |\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \leq$$

$$\leq \int_0^1 \|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))} (\lambda(\mathrm{U}(0,R)))^{\frac{1}{p'}}\,\mathrm{d}s \leq C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))},$$

and so we have shown

$$\forall v \in \mathcal{D}(\mathrm{U}(0,1)) : \int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x \leq C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

Now, take[16] $v \in \mathrm{W}_0^{1,p}(\mathrm{U}(0,R))$, then $\exists \{v_k\} \subset \mathcal{D}(\mathrm{U}(0,R)) : v_k \to v \,\mathrm{in}\, \mathrm{W}_0^{1,p}(\mathrm{U}(0,R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y)-v_k(x)|}{|y|}\,\mathrm{d}x \leq C(R)\|\nabla v_k\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \to C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

Putting it all together:

$$\|u-(Eu)_\delta\|_{\mathrm{L_1}(\Omega)} \leq \int_{\mathbb{R}^d}\int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y \leq C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \int_{\mathbb{R}^d} \underbrace{|y|}_{\leq\delta}\eta_\delta(y)\,\mathrm{d}y \leq$$

$$\leq C(R)\delta\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}\int_{\mathbb{R}^d}\eta_\delta(y)\,\mathrm{d}y = C(R)\delta\|v\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} = C(R)\delta\|Eu\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} \leq C_1\delta\|u\|_{\mathrm{W}^{1,p}(\Omega)} \leq$$

$$\leq C_1\delta.$$

where we have used the properties of the reg. kernel η_δ, the extension operator E and the fact $u \in U$.

Now fix $\varepsilon > 0$, find $\{(Eu_k)_\delta\}_{k=1}^m$ a finite $\frac{\varepsilon}{2}$-net in $(EB)_\delta$ in $\mathrm{L_1}(\mathrm{U}(0,R))$ (which is possible, since we have total boundedness in $\mathrm{L_1}(\mathrm{U}(0,R))$.) We will show $\{u_k\}_{k=1}^m$ is a (finite) ε–net in $\mathrm{L_1}(\Omega)$.

Up to now, $\delta > 0$ has been undetermined; now comes the time - set

$$\delta > 0 \,\mathrm{s.t.}\, C_1\delta < \frac{\varepsilon}{4}.$$

Fix an arbitrary $u \in U$, and find $j \in \{1,\dots,m\}$ *s.t.* $\|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L_1}(\mathrm{U}(0,R))} < \frac{\varepsilon}{2}$. Compute

$$\|u-u_j\|_{\mathrm{L_1}(\Omega)} \leq \|u-(Eu)_\delta\|_{\mathrm{L_1}(\Omega)} + \|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L_1}(\Omega)} + \|(Eu_j)_\delta - u_j\|_{\mathrm{L_1}(\Omega)} \leq C_1\delta + \frac{\varepsilon}{2}C_1\delta < \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2} < \varepsilon,$$

[16] Recall we have $v \in (EU)_\delta$ and so $\operatorname{supp} v \subsetneq \mathrm{B}(0,R)$, meaning it is "zero on $\mathrm{S}(0,R)$" - in the sense of traces.

where we have used the above estimate and the fact $\Omega \subset \mathrm{B}(0,R)$. Thus, we have shown U is totally bounded in $\mathrm{L}_1(\Omega)$ and so

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_1(\Omega)\,.$$

It remains to show the validity for a general $q \in [1,p^*)$. Using the interpolation theorem on Lebesgue spaces [17] we obtain

$$\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|^{\theta}_{\mathrm{L}_1(\Omega)} \|u\|^{1-\theta}_{\mathrm{L}_{p^*}(\Omega)},$$

where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Let us now show U is totally bounded in $\mathrm{L}_q(\Omega)$, *i.e.*, $\forall \epsilon > 0$ there exists a finite $\epsilon-$ net in U in $\mathrm{L}_q(\Omega)$. Pick $\{u_j\}_{j=1}^m \subset U$ that is an $\beta > 0$ net in $\mathrm{L}_1(\Omega)$, where β will be determined later. Then it holds

$$\|u-u_j\|_{\mathrm{L}_q(\Omega)} \le \|u-u_j\|^{\theta}_{\mathrm{L}_1(\Omega)} \|u-u_j\|^{1-\theta}_{\mathrm{L}_{p^*}(\Omega)} \le \beta^{\theta} \|u-u_j\|^{1-\theta}_{\mathrm{L}_{p^*}(\Omega)}.$$

Since we have already shown $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_{p^*}(\Omega)$, we know (again u, u_j are in U)

$$\|u-u_j\|^{1-\theta}_{\mathrm{L}_{p^*}(\Omega)} \le C_2 \|u-u_j\|^{1-\theta}_{\mathrm{W}^{1,p}(\Omega)} \le C_2 2^{1-\theta},$$

and so

$$\|u-u_j\|_{\mathrm{L}_q(\Omega)} \le \beta^{\theta} C_2 2^{1-\theta}.$$

We see that if we choose β *s.t.*

$$\beta < \left(\frac{\epsilon}{C_2 2^{1-\theta}}\right)^{\frac{1}{\theta}},$$

we obtain

$$\|u-u_j\|_{\mathrm{L}_q(\Omega)} \le \epsilon,$$

i.e., $\{u_j\}_{j=1}^m$ is a ϵ–set in $\mathrm{L}_q(\Omega)$. Thus $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1,p^*)$.

Ad compactness for $p = d$

Finally, let us show the last result. It holds

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega)\,, \forall r \in [1,d).$$

Moreover, we have just shown

$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_s(\Omega)\,, \forall s \in [1,r^*).$$

Notice that $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d^-$, so $\forall q \in [1,\infty)$ fixed $\exists r \in [1,d) : r^* > q$, *i.e.*, $q \in [1,r^*)$. But then

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1,\infty).$$

And realize that implies $\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\, \forall q \in [1,\infty)$: if $\{u_n\}$ is bounded in $\mathrm{W}^{1,d}(\Omega)$, then it is bounded in $\mathrm{W}^{1,r}(\Omega)$, as the identity between those spaces is continuous, and the above compact embedding tells us $\{u_{n_k}\}$ is convergent in $\mathrm{L}_q(\Omega)$ for some $\{n_k\}$. In total, $\{u_n\} \subset \mathrm{W}^{1,d}(\Omega)$ has a subsequence $\{u_{n_k}\}$ convergent in $\mathrm{L}_q(\Omega)$. We are done. □

[17] In the case $q \in [r,s)$ it holds $\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|^{\theta}_{\mathrm{L}_r(\Omega)} \|u\|^{1-\theta}_{\mathrm{L}_s(\Omega)}$, $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$.

2.4.2 Theorems for $p > d$

We know we will encounter Holder spaces. Let us recall some of their properties in a remark.

Remark (Properties of Holder spaces). Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $k \in \mathbb{N}_0, \lambda \in [0,1]$. The norm on the space $\mathrm{C}^{0,1}\left(\overline{\Omega}\right)$ is defined as

$$\|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} = \|u\|_{\mathrm{C}^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x \neq y, x, y, \in \overline{\Omega}} \frac{|D^\alpha u(x) - D^\alpha(y)|}{|x-y|^\lambda},$$

and the space

$$\mathrm{C}^{k,\lambda}\left(\overline{\Omega}\right) := \left\{u \in \mathrm{C}^k\left(\overline{\Omega}\right) | \|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} \leq \infty\right\},$$

where we identify

$$\mathrm{C}^{k,0}\left(\overline{\Omega}\right) = \mathrm{C}^k\left(\overline{\Omega}\right).$$

Moreover, we have the following embeddings: $\forall \alpha \in [0,1]$ it holds

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\beta}(\Omega), \forall \beta \in [1,\alpha],$$

and

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [1,\alpha).$$

A fresh start of a new chapter calls for a fresh new lemma.

Lemma 7 (Morrey). *Let $u \in \mathcal{D}\left(\mathbb{R}^d\right)$. Then $\forall x_1, x_2 \in \mathbb{R}^d, \forall \mu \in (0,1]$ it holds*

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)},$$

with

$$[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \sup_{\rho > 0} \int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \,\mathrm{d}y\,.$$

Proof. Pick arbitrary but fixed $x_1, x_2 \in \mathbb{R}^d$. Denote by C_ρ the closed cube with a side of length ρ *s.t.* x_1 and x_2 lie on opposite faces. Then $\rho \leq |x_1 - x_2| \leq \rho\sqrt{d}$ (it is not closer then the height and not further then the diagonal.)

Let us begin by first computing the deviation of $u(x_i)$ from the mean value of u on C_ρ:

$$\left|\frac{1}{\lambda(C_\rho)} \int_{C_\rho} u(x) \,\mathrm{d}x - u(x_i)\right| = \left|\int_{C_\rho} \frac{u(x) - u(x_i)}{\rho^d} \,\mathrm{d}x\right| \leq \int_{C_\rho} \frac{|u(x) - u(x_i)|}{\rho^d} \,\mathrm{d}x\,.$$

What other can we use when estimating differences than Newton - Leibniz, right? Since $u \in \mathcal{D}\left(\mathbb{R}^d\right) \subset \mathrm{C}^1\left(\mathbb{R}^d\right)$ it holds for $i \in \{1,2\}$ and $\forall x \in C_\rho$:

$$\begin{aligned}|u(x) - u(x_i)| &\leq \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} u(x_i + s(x - x_i)) \,\mathrm{d}s\right| = \left|\int_0^1 \nabla u(x_i + s(x - x_i)) \cdot (x - x_i) \,\mathrm{d}s\right| \leq \\ &\leq \int_0^1 |\nabla u(x_i + s(x - x_i))| |x - x_i| \,\mathrm{d}s \leq \rho\sqrt{d} \int_0^1 |\nabla u(x_i + s(x - x_i))| \,\mathrm{d}s\,.\end{aligned}$$

Notice that it is important $x \in C_\rho$ and x_1, x_2 are on the opposite sides. With this estimate for $|u(x) - u(x_i)|$ and Fubini we can write for the deviattion

$$|\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)| \le \sqrt{d}\int_{C_\rho}\frac{1}{\rho^{d-1}}\int_0^1 |\nabla u(x_i + s(x - x_i)|\,\mathrm{d}s\,\mathrm{d}x = \sqrt{d}\int_0^1\int_{C_\rho}\frac{|\nabla u(x_i + s(x - x_i))|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s\,.$$

This calls for a sensible change of variables. Denote $z = x_i + s(x - x_i)$, then under this transformation the cube C_ρ becomes

$$z \in x_i + s(C_\rho - x_i) = x_i(1-s) + C_{s\rho} = x_i(1-s) + [0, s\rho]^d := C^i_{s\rho}$$

which, since x_i is taken from the faces and $s \le 1$, is a cube with its "origin" somewhere in C_ρ and a side of length $s\rho \le \rho$, which implies $C^i_{s\rho} \subset [0,R]^d$ for some R. The integral then becomes (clearly $|\det \nabla_x z| = s^d$),

$$\sqrt{d}\int_0^1\int_{C_\rho}\frac{|\nabla u(x_i + s(x - x_i)|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s = \sqrt{d}\int_0^1\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{s^d\rho^{d-1}}\,\mathrm{d}z\,\mathrm{d}s = \sqrt{d}\rho^\mu\int_0^1 s^{\mu-1}\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z\,\mathrm{d}s\,,$$

where we are being a bit suggestively imaginative. The "deviation estimate" then becomes

$$|\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)| \le \sqrt{d}\rho^\mu\int_0^1 s^{\mu-1}\underbrace{\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z}_{\le[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}}\,\mathrm{d}s \le \sqrt{d}\rho^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}\int_0^1 s^{\mu-1}\,\mathrm{d}s = \frac{\sqrt{d}}{\mu}\rho^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)},$$

where we used that $C^i_{s\rho} \subset [0,R]^d$ and that $0 \in \mathbb{R}^d$, and so we could estimate the integral over $C^i_{s\rho}$ by $[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}$. Triangle inequality and the fact $\rho \le |x_1 - x_2|$ concludes our proof:

$$|u(x_1) - u(x_2)| \le |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_1)| + |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_2)| \le \frac{2\sqrt{d}}{\mu}\rho^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \le \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}.$$

$\square$

Remark. It is sufficient when $u \in \mathrm{C}^1_0(\mathbb{R}^d)$.

Gagliardo had in fact two lemmas, so let us even the game for Morrey.

Lemma 8. *Let $p \in (d, \infty)$, and let $\mu = 1 - \frac{d}{p}$. Then $\forall u \in \mathcal{D}(\mathbb{R}^d)$ it holds*

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} \le \left(1 + \frac{4\sqrt{d}}{\mu}\right)\|u\|_{W^{1,p}(\mathbb{R}^d)},$$

where

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}|u(x)| + \sup_{x,y\in\mathbb{R}^d, x\ne y}\frac{|u(x) - u(y)|}{|x-y|^\mu}.$$

Proof. We prove the assertion by estimating both terms in the above norm. To obtain those specific constants, we will pay some more attention to our procceeding. Let us also state the trivial: $\mu = 1 - \frac{d}{p} \in (0,1)$ for $p \in (d, \infty)$.

Begin with the differences: choose an arbitrary $\rho > 0$ and compute

$$\int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \,\mathrm{d}y \le \left(\int_{[0,\rho]^d} \left(\frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\right)^p \mathrm{d}y\right)^{\frac{1}{p}} \big(\lambda\big([0,\rho]^d\big)\big)^{\frac{p-1}{p}} = \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \frac{\rho^{\frac{d(p-1)}{p}}}{\rho^{d-1+\mu}} = \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

because $\frac{d(p-1)}{p} - d + 1 - \mu = \frac{dp-d}{p} - d + 1 - 1 + \frac{d}{p} = \frac{dp}{p} - d = 0$. Taking the suprema yields

$$[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \le \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

Going with this into the first Morrey lemma, we see

$$|u(x_1) - u(x_2)| \le \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

meaning

$$\sup_{x_1, x_2 \in \mathbb{R}^d, x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\mu} \le \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

To estimate the infinity norm, we can actually exploit the above result aswell: pick $x \neq y \in \mathbb{R}^d$ and write

$$|u(x)| - |u(y)| \le |u(x) - u(y)| \le \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

and so

$$|u(x)| \le |u(y)| + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

Now fix $\rho \ge 2|x - y| > 0$, integrate both sides w.r.t y over $[0,\rho]^d$ and obtain

$$|u(x)|\rho^d \le \int_{[0,\rho]^d} |u(y)| \,\mathrm{d}y + \frac{2\sqrt{d}}{\mu} \rho^d \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

which upon using Holder in the integral becomes

$$|u(x)|\rho^d \le \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} \rho^{\frac{d(p-1)}{p}} + \frac{2\sqrt{d}}{\mu} \rho^d \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

Since we have lost y, we can in fact choose $\rho = 1$, and upon taking the supremum write

$$\sup_{x \in \mathbb{R}^d} |u(x)| \le \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

and so in total

$$\|u\|_{\mathrm{C}^{0,\mu}(\mathbb{R}^d)} \le \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{4\sqrt{d}}{\mu} \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)} = \left(1 + \frac{4\sqrt{d}}{\mu}\right) \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

□

Remark. Since $\mathcal{D}\big(\mathbb{R}^d\big)$ is dense in $\mathrm{W}^{1,p}\big(\mathbb{R}^d\big)$, the above lemma holds also for $u \in \mathrm{W}^{1,p}\big(\mathbb{R}^d\big)$. We have to be careful to pick a good representant though.

Theorem 10 (Embedding theorems for $p > d$)**.** *Let $\Omega \in C^{0,1}, d \in \mathbb{N}, p > d$, i.e., $p \in (d, \infty]$. Denote $\mu^* = 1 - \frac{d}{p} \in (0,1)$. Then*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$

and

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*).$$

Proof. Ad continuous:

Since $\Omega \in \mathrm{C}^{0,1}$, we are able to use the extension theorem; recall $\forall u \in \mathrm{W}^{1,p}(\Omega) : \operatorname{supp} Eu \subset V$, where $\overline{\Omega} \subset V$. Let us deal with the case $p \in (d, \infty)$ first. We have shown

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} \leq \|u\|_{\mathrm{C}^{0,\mu^*}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}, \forall u \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right).$$

Realize that in fact

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})},$$

as $\partial\Omega$ is a set of zero Lebesgue measure and so in fact $Eu = u \text{ on } \overline{\Omega}$, which together with the obvious fact $Eu \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$ gives

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\Omega)}.$$

This exactly means

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right).$$

Realize also that

$$\mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$

and so

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*].$$

If now $p = \infty$, realize that (by embedding of Lebesgue spaces)

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{W}^{1,q}(\Omega), \forall q \in [1, \infty).$$

From the previous result it follows

$$\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in \left[0, 1 - \frac{d}{q}\right],$$

and notice that $1 - \frac{d}{q} \to 1$ as $q \to \infty$. This means that $\forall q \in [1, \infty)$ it holds

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, 1 - \frac{d}{q}].$$

Since $q \in [1, \infty)$ was arbitrary, we conclude it must be

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, 1].$$

[18]

Ad compactness This will be a bit of a cheating: we know

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [0, \mu^*],$$

and for Holder spaces it also holds

$$\mathrm{C}^{0,\beta}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \beta),$$

which means if we choose $\beta = \mu^*$, we in fact obtain

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*).$$

Using the same arguments as in the case of the compact embedding for $p < d$, we can conclude

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*).$$

We are done. □

Remark. Note that in the case of $p > d$, from

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$

and[19]

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{L}_\infty(\Omega), \forall \alpha \in [0, 1]$$

it follows[20]

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega), \forall p > d.$$

But that of course means (Ω is bounded) that

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty],$$

whenever $p > d$.

---

[18] The norm

$$\|u\|_{\mathrm{C}^{0,1-\frac{d}{q}}\left(\overline{\Omega}\right)} = \sup_{x \in \overline{\Omega}} |u(x)| + \sup_{x_1 \neq x_2 \in \overline{\Omega}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{1-\frac{d}{q}}},$$

actually allows one to pass to the limit $q \to \infty$, since the supremum is independent of the exponent in the denonimanotor, the function on the RHS is continuous. So if we know $\|u\|_{\mathrm{C}^{0,1-\frac{d}{q}}\left(\overline{\Omega}\right)} \leq C\|u\|_{\mathrm{W}^{1,\infty}(\Omega)}, \forall q \in [1, \infty)$, we can pass to the limit on the LHS and obtain $\|u\|_{\mathrm{C}^{0,1}\left(\overline{\Omega}\right)} \leq C\|u\|_{\mathrm{W}^{1,\infty}(\Omega)}$, which is the only missing possibility.

[19] Clearly,

$$\|u\|_{\mathrm{L}_\infty(\Omega)} \leq \|u\|_{\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right)} = \|u\|_{\mathrm{L}_\infty(\Omega)} + \sup_{x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{\alpha}}.$$

[20] Again, composition of a compact and continuous (linear) operators yields a compact operator independently of the order.

Remark (Summary). Let us summarize the obtained results. If $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, then upon denoting

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p},$$

it holds

- if $p < d$, then
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*],$$
  and
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*),$$
- if $p = d$, then
$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty)$$
  and
$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty),$$
- if $p > d$, then
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$
  and
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$
  which also imply
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty(\Omega),$$
  and
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega).$$

Remark (Summary - embedding into Lebesgue spaces). It can be guiding to look only at the embeddings into some Lebesgue spaces. Let $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, denote

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p}.$$

Then it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega),$$

where q is given:

- if $p < d$, then $q \in [1, p^*]$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

Also, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega),$$

where q is given as

- if $p < d$, then $q \in [1, p^*)$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

2.5 Trace theorems

There are many many troubles with the boundary. Another one we have yet not encountered arises with *e.g.* homogenous Dirichlet boundary coniditons: it should hold

$$u = 0 \,\mathrm{on}\, \partial\Omega,$$

but typically, u is an element of some Sobolev space on Ω, and $\lambda_d(\partial\Omega) = 0$. We cannot sensibly talk about pointwise values on a set of measure zero, they can be arbitrary. This can be dealt with provided - as expected - when Ω, *i.e.*, $\partial\Omega$ is benevolent enough.

Realize also that for Ω at least $\mathrm{C}^{0,0}$ we also have the density of $\mathrm{C}^{\infty}_{\overline{\Omega}}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\Omega)$, meaning the values on $\partial\Omega$ are well defined at least for a dense subset of $\mathrm{W}^{1,p}(\Omega)$. It should not be too difficult to extend to the whole $\mathrm{W}^{1,p}(\Omega)$, right...?

Moreover, in the case $p > d, \Omega \in \mathrm{C}^{0,1}$ we already know $\forall u \in \mathrm{W}^{1,p}(\Omega)$ there exists $u^* \in \mathrm{C}^0\left(\overline{\Omega}\right)$, such that $u = u^*$ *a.e.* in Ω, and so in these cases, the values $u^* \restriction_{\partial\Omega}$ are well defined. What if $p \leq d$?

Remark (The space $\mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega)$). In the winter semester, we have defined

$$\mathrm{L_p}(\partial\Omega) \equiv \mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega),$$

i.e., the lebesgue spaces are taken w.r.t the $d-1$ dimensional (normalized complete) Hausdorff measure $\mathcal{H}_{d-1}$.

Theorem 11 (Continuous trace theorem)**.** *Let $\Omega \in C^{0,1}, p \in [1,\infty]$, denote $p^{\#} = \frac{dp-p}{d-p}$. Then there is a continuous linear operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L_q(\partial\Omega)$, *with q being*

- *if $p < d$, then $q \in [1, p^{\#}]$,*
- *if $p = d$, then $q \in [1, \infty)$,*
- *if $p > d$, then $q \in [1, \infty]$.*

Moreover, $\forall u \in C^{\infty}\left(\overline{\Omega}\right)$ it holds

$$\mathrm{tr}\, u = u \restriction_{\partial\Omega},$$

meaning $\mathrm{tr} : W^{1,p}(\Omega) \to L_p(\partial\Omega)$ *is an extension of* $\tilde{\mathrm{tr}} : C^{\infty}\left(\overline{\Omega}\right) \to C^0(\partial\Omega)$.

Proof. The strategy is the following

1. define $\tilde{\mathrm{tr}}$ for smooth functions,
2. obtain estimates for $\tilde{\mathrm{tr}}$ using embedding theorems,
3. extend $\tilde{\mathrm{tr}}$ to the whole space, which defines tr.

Case $p < d$:

As we have mentioned, the case for functions smooth up to the boundary is evident. Let us so define $\tilde{\mathrm{tr}} : \mathrm{C}^{\infty}_{\overline{\Omega}}(\mathbb{R}^d) \to \mathrm{C}^0(\partial\Omega)$, by

$$\tilde{\mathrm{tr}}\, u = u \restriction_{\partial\Omega}.$$

Then clearly $\tilde{\mathrm{tr}}$ is a well defined linear continuous [21] operator.

(We are using the notation from the definition of a $\mathrm{C}^{0,1}$ domain). Let us for clarity define (and also recall)

$$\begin{aligned}
G_j &= \mathbb{A}_j(\{(x', a_j(x')|x' \in \mathrm{U}(0,\alpha)\}),\\
G_j^+ &= \mathbb{A}_j(\{x', a_j(x') + b|x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}),\\
G_j^- &= \mathbb{A}_j(\{x', a_j(x') - b|x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}).
\end{aligned}$$

Within this notation, $G_j \subset \partial\Omega, G_j^+ \subset \Omega, G_j^- \subset \mathbb{R}^d/\overline{\Omega}$ and $U_j = G_j \cup G_j^+ \cup G_j^-$. Moreover, $\{U_j\}_{j=1}^m$ are open sets *s.t.* $\partial\Omega \subset \bigcup_{j=1}^m U_j$. Denote $\{\varphi_j\}_{j=1}^m \subset \mathcal{D}\left(\mathbb{R}^d\right)$ to be the partition of unity subordinate to this (open) covering. Realize moreover that since $\mathcal{H}_{d-1}(\partial\Omega) < \infty$, it holds

$$\mathrm{L}_{p^\#}(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, p^\#].$$

So if we are able to show

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

we have the rest of the estimates for free.

Take $u \in \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right)$, and denote $u_j = u\varphi_j, j \in \{1,\dots,m\}$. Let for the moment $p > 1$, so $p^\# = \frac{dp-p}{d-p} > \frac{d-1}{d-1} = 1$. As $u_j \in \mathrm{C}^\infty\left(\mathbb{R}^d\right)$, it holds $u_j \in \mathrm{C}^1\left(\overline{G_j^+}\right)$. Moreover, since $\operatorname{supp} u_j \subset U_j$ and as $U_j \cap \mathbb{A}_j(\{(x', a_j(x') + b|x' \in \mathrm{U}(0,\alpha), b \in [0,\infty)\}) = \varnothing$, it holds

$$u_j(\mathbb{A}_j(x', a_j(x') + \beta)) = 0.$$

With those qualities, if we denote (for an arbitrary $j \in \{1,\dots,m\}$; this will become $\|u_j\|_{\mathrm{L}_{p^\#}(\partial\Omega)}$)

$$v(x') = |u_j(\mathbb{A}_j(x', a_j(x')))|^{p^\#} = |u_j(\mathbb{A}_j(x', a_j(x')))|^{\frac{dp-p}{d-p}},$$

we can write (recall $\mathbb{A}_j$ is orthogonal and $x' \in \mathrm{U}(0,\alpha)$)

$$\begin{aligned}
|v(x')| &= \left||u_j(\mathbb{A}_j(x', a_j(x')))|^{\frac{dp-p}{d-p}} - |u_j(\mathbb{A}_j(x', a_j(x') + \beta))|^{\frac{dp-p}{d-p}}\right| \le \left|\int_{a_j(x')+\beta}^{a_j(x')} \frac{\partial}{\partial s}|u_j(\mathbb{A}_j(x', s))|\,\mathrm{d}s\right| =\\
&= \left|\int_{a_j(x')+\beta}^{a_j(x')} p^\#|u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}}\operatorname{sign}(u_j(\mathbb{A}_j(x', s)))\nabla u_j(\mathbb{A}_j(x', s)) \cdot \mathbb{A}_j(x', 1)\,\mathrm{d}s\right| \le\\
&\le p^\# \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}}|\nabla u_j(\mathbb{A}_j(x', s))|\underbrace{|\mathbb{A}_j(x', 1)|}_{=|(x',1)|}\mathrm{d}s \le\\
&\le p^\#\sqrt{1+\alpha^2}\int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}}|\nabla u_j(x', s)|\,\mathrm{d}s\,.
\end{aligned}$$

[21] Even though $\mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) = \left\{u\restriction_{\overline{\Omega}} | u \in \mathrm{C}^\infty\left(\mathbb{R}^d\right)\right\}$ is not a normed space, it is a Frechet space, and so the notion of continuity is that of a topological vector space.

Integrate this inequality over $\mathrm{U}(0,\alpha)$ and write (recall the definition of G_j^+, we are using Fubini)

$$\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' \le p^{\#}\sqrt{1+\alpha^2}\int_{\mathrm{U}(0,\alpha)}\int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x',s))|^{\frac{dp-d}{d-p}}|\nabla u_j(\mathbb{A}_j(x',s))|\,\mathrm{d}s\,\mathrm{d}x' \le$$
$$\le p^{\#}\sqrt{1+\alpha^2}\int_{\mathbb{A}_j(\{(x',a_j(x')+b)|x'\in\mathrm{U}(0,\alpha),b\in[0,\beta]\})} |u_j|^{\frac{dp-d}{d-p}}|\nabla u_j(x)|\,\mathrm{d}x =$$
$$= p^{\#}\sqrt{1+\alpha^2}\int_{G_j^+} |u_j(x)|^{\frac{dp-d}{d-p}}|\nabla u(x)|\,\mathrm{d}x \le p^{\#}\sqrt{1+\alpha^2}\|\nabla u\|_{\mathrm{L}_p\left(G_j^+\right)}\left\||u_j|^{\frac{dp-d}{d-p}}\right\|_{\mathrm{L}_{p'}\left(G_j^+\right)},$$

and since $\frac{dp-d}{d-p}p' = \frac{dp-d}{d-p}\frac{p}{p-1} = \frac{dp}{d-p} = p^*$, we have

$$\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' = p^{\#}\sqrt{1+\alpha^2}\|\nabla u\|_{\mathrm{L}_p\left(G_j^+\right)}\|u_j\|_{\mathrm{L}_{p^*}\left(G_j^+\right)}^{\frac{dp-d}{d-p}}.$$

Since $G_j^+\in\mathrm{C}^{0,1}$, the last term can be estimated using the continuous embedding theorems:

$$\|u_j\|_{\mathrm{L}_{p^*}\left(G_j^+\right)}^{\frac{dp-d}{d-p}} \le C\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d}{d-p}},$$

whereas the integral on the LHS actually is

$$\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' = \int_{\mathrm{U}(0,\alpha)} |u_j(\mathbb{A}_j(x',a_j(x')))|^{p^{\#}}\,\mathrm{d}x = \int_{\mathbb{A}_j(\{(x',a_j(x'))|x'\in\mathrm{U}(0,\alpha)\})} |u_j(x)|^{p^{\#}}\,\mathrm{d}x = \|u_j\|_{\mathrm{L}_{p^{\#}}(G_j)}^{p^{\#}},$$

and so we write

$$\|u_j\|_{\mathrm{L}_{p^{\#}}(G_j)}^{p^{\#}} \le C\|\nabla u_j\|_{\mathrm{L}_p\left(G_j^+\right)}\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d}{d-p}} \ge C\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d+d-p}{d-p}} = C\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{p^{\#}},$$

and so

$$\|u_j\|_{\mathrm{L}_{p^{\#}}(G_j)} \le C\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}.$$

This has been done for $p>1$, but in fact taking the limit $p\to 1^+$ is allowed here (without a proof). Hence the above estimate holds $\forall p\in[1,d)$.

The estimates have so far been local - let us glue them together. Recall $\partial\Omega = \bigcup_{j=1}^m G_j, G_j^+\subset\Omega$, and so

$$\|u\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} = \left\|\sum_{j=1}^m u_j\right\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(\operatorname{supp} u_j\cap\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(G_j)} \le$$
$$\le C\sum_{j=1}^m \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where we have used the fact $0\le\varphi_j\le 1$ in the last inequality. And so we have shown

$$\|u\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \le \|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall u\in\mathrm{C}_{\overline{\Omega}}^{\infty}\left(\mathbb{R}^d\right).$$

Now let $u\in\mathrm{W}^{1,p}(\Omega)$ be arbitrary. Since $\Omega\in\mathrm{C}^{0,1}$, there $\exists\{u_k\}\subset\mathrm{C}_{\overline{\Omega}}^{\infty}\left(\mathbb{R}^d\right)\, s.t.\, u_k\to u \,\mathrm{in}\, \mathrm{W}^{1,p}(\Omega)$. Set

$$\operatorname{tr} u = \lim_{k\to\infty}\tilde{\operatorname{tr}}\, u_k.$$

The definition is sensible, as $\tilde{\mathrm{tr}}$ is continuous and $\{u_k\}$ converges. Also, from the arithmetic of the limits and linearity of $\tilde{\mathrm{tr}}$ we see tr is linear. Next, check

$$\|\mathrm{tr}\, u\|_{\mathrm{L}_q(\partial\Omega)} = \left\|\lim_{k\to\infty} \tilde{\mathrm{tr}} u_k\right\|_{\mathrm{L}_q(\partial\Omega)} = \lim_{k\to\infty} \|u_k \restriction_{\partial\Omega}\|_{\mathrm{L}_q(\partial\Omega)} \le C \lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\Omega)} = C\|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall q \in [1, p^{\#}].$$

and so inded $\mathrm{tr} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, p^{\#}]$ and it surely is bounded.

Case $p = d$

In this case, we have $\mathrm{id} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega)\,, \mathrm{W}^{1,r}(\Omega)), \forall r \in [1, d)$ and the previous result tells us $\mathrm{tr} \in \mathcal{L}\big(\mathrm{W}^{1,r}(\Omega)\,, \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1, r^{\#}]$. Observe that $r^{\#} = \frac{dr-r}{d-r} \to \infty$ as $r \to d^-$, meaning $\forall q \in [1, \infty)$ there exists $r \in [1, d)\; s.t.\; r^{\#} > q$, $i.e.\; q \in [1, r^{\#})$, which in fact means

$$\mathrm{tr} \in \mathcal{L}\big(\mathrm{W}^{1,r}(\Omega)\,, \mathrm{L}_q(\partial\Omega)\big), \forall r \in [1, d), \forall q \in [1, \infty).$$

But then $\mathrm{tr} \circ \mathrm{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is a continous linear operator $\forall q \in [1, \infty)$, as it as a composition of continuous linear operators.

Case $p > d$

This is the easiest case: we know that $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*]$, so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right) \subset \mathrm{C}^0\left(\partial\Omega\right) \subset \mathrm{L}_\infty\left(\partial\Omega\right),$$

and so in total

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right),$$

which together with the fact $(\mathcal{H}_{d-1}(\partial\Omega) < \infty)$

$$\mathrm{L}_\infty\left(\partial\Omega\right) \hookrightarrow \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, \infty),$$

concludes the proof. □

Remark. These results are very similiar to the results obtained in the case of embedding theorems - and it is not surprising, since it might seem we have in fact shown the embeddings of some Sobolev spaces into some Lebesgue spaces. One needs to be careful however, as it *is not an embedding* - we are taking Ω open, so $\partial\Omega \notin \Omega$. It only makes sense in the case of $p > d$.

The chapter will be concluded by stating the compact analogue to the embedding theorems.

Theorem 12 (Compact trace theorem). *Let* $d \in \mathbb{N}, \Omega \in C^{0,1}$, *denote* $p^{\#} = \frac{dp-p}{d-p}$, *and let* tr *be the trace operator from the previous theorem. Then*

$$\mathrm{tr} \in \mathcal{K}\big(W^{1,p}(\Omega)\,, L_q(\partial\Omega)\big),$$

where q *is*

- *if* $p < d$, *then* $q \in [1, p^{\#})$,
- *if* $p = d$, *then* $q \in [1, \infty)$,
- *if* $p > d$, *then* $q \in [1, \infty]$,

Proof. *Case $p < d$:* Let us adopt the custom that when talking about the properties of $u \in \mathrm{W}^{1,p}(\Omega)$, in the space $\mathrm{L}_q(\partial\Omega)$, we are always talking about the properties of $\operatorname{tr} u$ in $\mathrm{L}_q(\partial\Omega)$...

It will be pretty similiar to the continuous case, so let us skip only to the key estimate. We are not apriori sure in which space we will be able to comfortably show the compactness, so let first $q \in [1,\infty)$ and compute. Then we might use some interpolation estimates...

$$\int_{\mathrm{U}(0,\alpha)} |u_j(\mathbb{A}_j(x', a_j(x')))|^q \,\mathrm{d}x' \le \left| \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \frac{\partial}{\partial s} |u_j(\mathbb{A}_j(x', s))|^q \,\mathrm{d}s \,\mathrm{d}x' \right| \le$$
$$\le q\sqrt{1+\alpha^2} \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))| |\nabla u_j(\mathbb{A}_j(x', s))| \,\mathrm{d}s \,\mathrm{d}x' \le$$
$$\le q\sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{q-1} |\nabla u_j(x)| \,\mathrm{d}x \le q\sqrt{1+\alpha^2} \|\nabla u_j\|_{\mathrm{L}_q\left(G_j^+\right)} \|u_j\|_{\mathrm{L}_{q'(q-1)}\left(G_j^+\right)}^{q-1},$$

where $q'(q-1) = \frac{q}{q-1}(q-1) = q$, so all in all

$$\|u_j\|_{\mathrm{L}_q(G_j)} \le C(q,\Omega) \|\nabla u_j\|_{\mathrm{L}_q\left(G_j^+\right)}^{\frac{1}{q}} \|u_j\|_{\mathrm{L}_q\left(G_j^+\right)}^{1-\frac{1}{q}} \le C(q,\Omega) \|u_j\|_{\mathrm{W}^{1,q}\left(G_j^+\right)}^{\frac{1}{q}} \|u_j\|_{\mathrm{L}_q\left(G_j^+\right)}^{1-\frac{1}{q}}$$

which leads to

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} = \left\| \sum_{j=1}^m u_j \right\|_{\mathrm{L}_q(\partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(\operatorname{supp} u_j \cap \partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(G_j)} \le$$
$$\le C \sum_{j=1} \|u_j\|_{\mathrm{W}^{1,q}\left(G_j^+\right)}^{\frac{1}{q}} \|u_j\|_{\mathrm{L}_q\left(G_j^+\right)}^{1-\frac{1}{q}} \le C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)}^{\frac{1}{q}} \|u\|_{\mathrm{L}_q(\Omega)}^{1-\frac{1}{q}}.$$

Denote $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$; we will show $\operatorname{tr} U$ is totally bounded in $\mathrm{L}_q(\partial\Omega)$ for *some* $q \in [1,\infty)$. Let $\varepsilon > 0$ be given. Realize that $\forall p \in [1,d)$ it is $p < p^*$, and so it always holds $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\mathrm{p}(\Omega)$. For the moment, we pick $q = p$. Denote $\{u_k\}_{k=1}^m$ to be the δ−net from U in $\mathrm{L}_\mathrm{p}(\Omega)$, where δ will be chosen suitably later. Let now $u \in U$ be arbitrary and find $u_i \in U$ *s.t.* $\|u - u_i\|_{\mathrm{L}_\mathrm{p}(\Omega)} \le \delta$. Using the estimate from above we have

$$\|u - u_i\|_{\mathrm{L}_p(\partial\Omega)} \le C_1 \|u - u_i\|_{\mathrm{W}^{1,p}(\Omega)}^{\frac{1}{q}} \|u - u_i\|_{\mathrm{L}_p(\Omega)}^{1-\frac{1}{q}} \le C_1 2^{\frac{1}{q}} \delta^{1-\frac{1}{q}},$$

where we have used the fact $u, u_i \in U$. We see that upon choosing

$$\delta < \left(\frac{\varepsilon}{C_1 2^{\frac{1}{q}}} \right)^{\frac{1}{1-q}},$$

we in fact have

$$\|u - u_i\|_{\mathrm{L}_\mathrm{p}(\partial\Omega)} < \varepsilon,$$

and so $\{\operatorname{tr} u_i\}_{i=1}^m \subset U$ is a ε−net in $\mathrm{L}_\mathrm{p}(\partial\Omega)$, meaning

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_\mathrm{p}(\partial\Omega)\big).$$

Since now $\mathrm{L}_p(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega)$, $\forall q \in [1,p]$, we have also shown

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1,p].$$

The remaining case is when $q \in (p, p^{\#})$. As in the case of compact embedding of Sobolev spaces, a suitable interpolation theorem will do the job for us. It holds

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \leq \|u\|^{\theta}_{\mathrm{L}_p(\partial\Omega)} \|u\|^{1-\theta}_{\mathrm{L}_{p^{\#}}(\partial\Omega)},$$

where $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^{\#}}$. Let now $\{u_i\}_{i=1}^m \subset U$ be such that $\{\mathrm{tr}\, u_i\}_{i=1}^m$ is the β−net in $\mathrm{L_p}(\partial\Omega)$ whose existence we have just proven $\forall \beta > 0$. Recall also that since U is bounded and $\mathrm{tr} \in \mathcal{L}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big) \forall q \in [1, p^{\#}]$, there exists $0 < C_2 < \infty$ *s.t.*

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \leq C_2, \forall u \in U.$$

and so $\|u\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \leq C_2, \forall u \in U$ in particular.

Finally, $\forall \in U$ it holds

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \leq \|u - u_i\|^{\theta}_{\mathrm{L_p}(\partial\Omega)} \|u - u_i\|^{1-\theta}_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \leq \beta^{\theta} (2C_2)^{1-\theta},$$

so if we choose

$$\beta < \left(\frac{\varepsilon}{(2C_2)^{1-\theta}} \right)^{\frac{1}{\theta}},$$

then

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \leq \varepsilon, \forall u \in U.$$

which concludes the proof $\{\mathrm{tr}\, u_i\}_{i=1}^m$ is an ε−net in U in $\mathrm{L}_q(\partial\Omega)$ also for $q \in (p, p^{\#})$. In total, we have showed the compactness of the trace operator for all $q \in [1, p^{\#})$.

Case $p = d$

In this case $\mathrm{tr} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega), \mathrm{L}_q(\partial\Omega)), \forall q \in [1, \infty)$. We also know $\mathrm{id} \in \mathcal{L}\big(\mathrm{W}^{1,d}(\Omega), \mathrm{W}^{1,r}(\Omega)\big), \forall r \in [1, d)$ and that

$$\mathrm{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1, r^{\#}).$$

Repating the same arguments, we see $r^{\#} \to \infty$ as $r \to d^-$, meaning $\forall q \in [1, \infty)$ there exists $r \in [1, d)$ *s.t.* $r^{\#} > q$, and consequently this implies

$$\forall q \in [1, \infty) \exists r \in [1, d) \; s.t. \; \mathrm{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big).$$

But then the operator $\mathrm{tr} \circ \mathrm{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is compact $\forall q \in [1, \infty)$, as it is a composition of a (linear) continuous and compact operator.

Case $p > d$

This case is again trivial: from the embedding theorems, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu),$$

so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right) \subset \mathrm{C}^0\left(\partial\Omega\right) \subset \mathrm{L}_\infty\left(\partial\Omega\right),$$

meaning

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right),$$

which together with the embedding

$$\mathrm{L}_q(\partial\Omega) \hookrightarrow \mathrm{L}_\infty(\partial\Omega), \forall q \in [1, \infty).$$

completes the proof for any $q \in [1, \infty]$. We are done once again.

□

2.6 Fine properties of Sobolev spaces

In this last chapter, we present some other useful properties of Sobolev spaces.

2.6.1 Composition of Sobolev functions

In the chapter about extension of functions from Sobolev spaces we have seen: $U, V \subset \mathbb{R}^d$ open, $u \in \mathrm{W}^{1,p}(U), \Phi : U \to V$ a C^1-diffemorphism $\Rightarrow u \circ \Phi \in \mathrm{W}^{1,p}(V)$. Is there any other class of mappings that guarantee the composition remains in some Sobolev spaces?

Theorem 13 (Derivative of superposition)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1, \infty]$, $u : \Omega \to \mathbb{R}^d$ be in $W^{1,p}(\Omega)$. Denote for an arbitrary $a \in \mathbb{R}$ the set*

$$\Omega_a = \{x \in \Omega | u(x) = a\}.$$

Then

1. $\nabla u = 0$ *on* Ω_a,
2. *if* $f \in C^{0,1}\left(\mathbb{R}^d\right)$ *s.t.* $\|f\|_{L_\infty(\Omega)} < \infty$ *(it is a Lipschitz continuous function), then* $f \circ u - f(0) \in W^{1,p}(\Omega)$ *and it holds*

$$\nabla(f \circ u) = \begin{cases} (f' \circ u)\nabla u, & a.e.\ in \{u \notin S\}, \\ 0, & a.e.\ in \{u \in S\} \end{cases},$$

   where

$$S = \{s \in \mathbb{R} | f'(s) \text{ does not exist}\}$$

   (in the strong sense).

Proof. The proof has been presented for the case $f \in \mathrm{C}^1(\mathbb{R}), \|f\|_{\mathrm{L}_\infty(\mathbb{R})} < \infty$. In the general setting, one can play with *a.e.* convergence and Rademacher theorem.

Denote $L = \|f\|_{\mathrm{L}_\infty(\mathbb{R})}$. Since f is continuous, $f \circ u$ is measurable. Moreover, from the Lipschitzity on the whole $\mathbb{R}$ it follows

$$|(f \circ u)(x) - f(0)| \le L|u(x) - 0| = L|u(x)|,$$

and since $u \in \mathrm{L}_\mathrm{p}(\Omega)$, also $f \circ u - f(0) \in \mathrm{L}_\mathrm{p}(\Omega)$. To obtain the wanted result, we need to show $\nabla(f \circ u)$ exists, so let us move on to the second claim. The strategy is to work with some regularization and then pass to the limit with the support size. In concrete terms, pick $\varphi \in \mathcal{D}(\Omega)$, and regularization u_ε *s.t.* $\mathrm{dist}(\mathrm{supp}\,\varphi, \partial\Omega) > 2\varepsilon$. Then we can write

$$-\int_\Omega (f \circ u_\varepsilon)(x)\partial_i\varphi(x)\,\mathrm{d}x = \int_\Omega (f' \circ u_\varepsilon)(x)\partial_i u_\varepsilon(x)\varphi\,\mathrm{d}x,$$

using per partes and chain rules (those are strong derivatives). Now we would like to pass to the limit $\varepsilon \to 0^+$. Take the LHS:

$$\lim_{\varepsilon\to 0^+} \left| \int_\Omega (f\circ u_\varepsilon - f\circ u)\partial_i\varphi \,\mathrm{d}x \right| \le \lim_{\varepsilon\to 0^+} \int_\Omega |f\circ u_\varepsilon - f\circ u||\partial_i\varphi| \,\mathrm{d}x \le L\|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap \operatorname{supp}\varphi} |u_\varepsilon - u| \,\mathrm{d}x = 0,$$

as $u_\varepsilon \to u$ in $\mathrm{L}_1(\Omega\cap\operatorname{supp}\varphi)$ (notice from the theorem on mollification we actually obtain this convergence in $\mathrm{L}_1(\Omega_\varepsilon)$, but we have made a good choice of φ) The RHS can be manipulated

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \left| \int_\Omega (f'\circ u_\varepsilon)\partial_i u_\varepsilon \varphi - (f'\circ u)\partial_i u\varphi \,\mathrm{d}x \right| \le \lim_{\varepsilon\to 0^+} \int_\Omega |(f'\circ u_\varepsilon - f'\circ u)||\partial_i u||\varphi| \,\mathrm{d}x + \lim_{\varepsilon\to 0^+} \int_\Omega |f'\circ u_\varepsilon||\partial_i u_\varepsilon - \partial_i u||\varphi| \,\mathrm{d}x \le \\
&\le \|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\operatorname{supp}\varphi} |f'\circ u_\varepsilon - f'\circ u||\partial_i u| \,\mathrm{d}x + L\|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\operatorname{supp}\varphi} |\partial_i u_\varepsilon - \partial_i u| \,\mathrm{d}x = \\
&= \|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_\Omega |f'\circ u_\varepsilon - f'\circ u||\partial_i u| \,\mathrm{d}x\,,
\end{aligned}$$

where we have used $\nabla u_\varepsilon \to \nabla u$ in $\mathrm{L}_1(\Omega\cap\operatorname{supp}\varphi)$. As for the second integral, recall that $u_\varepsilon \to u$ *a.e.* in $\Omega\cap\operatorname{supp}\varphi$ and that f' is globally continous. Thus with the majorant

$$|f'\circ u_\varepsilon - f'\circ u||\partial_i u| \le L\|u\|_{\mathrm{W}^{1,p}(\Omega)}|u_\varepsilon - u|,$$

that is integrable on the bounded set $\operatorname{supp}\varphi\cap\Omega$ we see the second integral is zero aswell. All in all, we have shown

$$-\int_\Omega (f\circ u)\partial_i\varphi \,\mathrm{d}x = \int_\Omega (f'\circ u)\partial_i u\varphi \,\mathrm{d}x\,, \forall\varphi\in\mathcal{D}(\Omega)\,,$$

which is exactly what we need to state that the ∂_i-weak derivative of $f\circ u$ exists and is equal to $(f'\circ u)\partial_i u$. Finally, the following estimate holds:

$$\Bigg\| \underbrace{\nabla(f\circ u)}_{=\nabla(f\circ u - f(0))} \Bigg\|_{\mathrm{L_p}(\Omega)} = \|(f'\circ u)\nabla u\|_{\mathrm{L_p}(\Omega)} \le L\|\nabla u\|_{\mathrm{L_p}(\Omega)} \le L\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

and so

$$f\circ u - f(0) \in \mathrm{W}^{1,p}(\Omega)\,.$$

Let us now deal with the first assertion. We will first show $\nabla u = 0$ *a.e.* on Ω_0. For that, choose a special function

$$f(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and so $f\circ u = u^+$. If we now set

$$f_\varepsilon(x) = \begin{cases} \left(x^2 + \varepsilon^2\right)^{1/2} - \varepsilon, & x > 0, \\ 0, & x \le 0, \end{cases}$$

with the derivative being

$$f'_\varepsilon(x) = \begin{cases} \frac{x}{(x^2+\varepsilon^2)^{1/2}}, & x > 0, \\ 0, x \le 0 \end{cases}$$

and so $\lim_{\varepsilon\to 0^+} f_\varepsilon(x) = f(x)$, and $\lim_{\varepsilon\to 0^+} f'_\varepsilon(x) = \chi_{\mathbb{R}^+}(x)$ *a.e.* in $\mathbb{R}$, meaning

$$\lim_{\varepsilon\to 0^+} f_\varepsilon \circ u = u^+, \lim_{\varepsilon\to 0^+} f'_\varepsilon \circ u = \chi_{x\in\Omega|u(x)>0},$$

a.e. in Ω. Using the preivously obtained general expression and passing to the limit yields

$$-\int_\Omega (f\circ u)\nabla\varphi \,\mathrm{d}x = \int_\Omega \nabla u\varphi\chi_{\{x\in\Omega|u(x)>0\}}\,\mathrm{d}x\,,$$

and realizing[22] $u^- = (-u)^+$, we also have

$$-\int_\Omega (f\circ u)\nabla\varphi \,\mathrm{d}x = -\int_\Omega \nabla u\varphi\chi_{\{x\in\Omega|u(x)<0\}}\,\mathrm{d}x\,,$$

meaning (in the weak sense)

$$\nabla u^+ = \nabla u\chi_{\{x\in\Omega|u(x)>0\}}, \nabla u^- = -\nabla u\chi_{\{x\in\Omega|u(x)<0\}}.$$

But since $u = u^+ - u^-$, it holds $\nabla u = \nabla u^+ - \nabla u^-$ *a.e.*. We see $\nabla u^- = \nabla u^+ = 0$, on Ω_0, and so $\nabla u = 0$ on *a.e.* on Ω_0. Taking $\tilde{u} = u + c$ for $c \in \mathbb{R}$ arbitrary, we see $\nabla\tilde{u} = \nabla u = 0$ *a.e.* on Ω_0, meaning $\nabla\tilde{u} = 0$ *a.e.* on $\{x\in\Omega|\tilde{u} - c = 0\} = \Omega_c$. Since $\tilde{u}, c$ were arbitrary, we are done with this proof.

If $u \notin S$, the above presentation works just fine. If $u(x) \in S$, we know $\nabla u = 0$ *a.e.* on Ω_u, and since $\lambda(S) = 0$ by Rademacher theorem, we have also the above conclusion. This argumentation is sloppy, but has not been presented in full detail.

□

2.6.2 Difference quotients

There is some interplay between the weak derivative and difference quotients, which can be useful in some proofs, *e.g.*, regularity results for elliptic problems. We have seen that if a function posseses classical derivatives, it also has weak derivatives and they conincide *a.e.*. Is something there something to be said about the opossite implication?

Definition 6 (Difference quotient)**.** Let $u : \mathbb{R}^d \to \mathbb{R}, h \in \mathbb{R}$. For each $i \in \{1,\dots,d\}$ we define the difference quotient as

$$\Delta_h^i u(x) = \frac{u(x+he_i) - u(x)}{h},$$

where e_i is the canoncal base vector of $\mathbb{R}^d$.

Remark. If $\partial_i u(x)$ exists, then clearly

$$\partial_i u(x) = \lim_{h\to 0} \Delta_h^i u(x).$$

Let us jump straight to the main result.

[22] Recall $u^+ = \max(0,u), u^- = -\min(0,u)$ are both nonnegative functions, $u = u^+ - u^-, |u| = u^+ + u^-$. It holds trivially $u^- = -\min(0,u) = \max(0,-u) = (-u)^+$.

Theorem 14. *Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1, \infty]$ and $u \in L_p(\Omega)$. Denote $\forall \delta > 0$ the set*

$$\Omega_\delta = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Then it holds

1. *If moreover $u \in W^{1,p}(\Omega)$, then $\forall i \in \{1, \dots, d\}, \delta \in (0,1), h \in \left(0, \frac{\delta}{2}\right)$*

$$\left\|\Delta_h^i u\right\|_{L_p(\Omega_\delta)} \le \|\partial_i u\|_{L_p(\Omega)}.$$

2. *Let $p \in (1, \infty]$ and let there exist $\{C_i\}_{i=1}^d$ constants s.t. $\forall i \in \{1, \dots, d\}, \delta \in (0,1), h \in \left(0, \frac{\delta}{2}\right)$ it holds*

$$\left\|\Delta_h^i u\right\|_{L_p(\Omega_\delta)} \le C_i.$$

Then $u \in W^{1,p}(\Omega)$ and moreover $\forall i \in \{1, \dots, d\}$ we have

$$\|\partial_i u\|_{L_p(\Omega)} \le C_i.$$

Proof. Ad 1. Let first $p \in [1, \infty)$. *What else do we use to estimate differences then..., right?* If needed, extend u by zero outside of Ω and set $u_\varepsilon = u \star \omega_\varepsilon$. Recall that $u_\varepsilon \to u$ in $\mathrm{W}^{1,p}\left(\Omega_{\frac{\delta}{2}}\right)$. Realize that since $u_\varepsilon \in \mathrm{C}^1\left(\mathbb{R}^d\right)$, we have

$$\Delta_h^i u_\varepsilon(x) = \frac{u_\varepsilon(x + he_i) - u_\varepsilon(x)}{h} = \frac{1}{h} \int_0^h \partial_t u_\varepsilon(x + te_i) \,\mathrm{d}t\,.$$

Let us compute $\left\|\Delta_h^i\right\|_{\mathrm{L}_p(\Omega_\delta)}$. We have

$$\left|\Delta_h^i u_\varepsilon\right|^p = \left|\frac{1}{h} \int_0^h \partial_t u_\varepsilon(x + te_i) \,\mathrm{d}t\right|^p \le \frac{1}{h^p}\left(\int_0^h |\partial_t u_\varepsilon(x + te_i)| \,\mathrm{d}t\right)^p \le \frac{1}{h^p}\left(\left(\int_0^h |\partial_t u_\varepsilon(x + te_i)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^h 1 \,\mathrm{d}t\right)^{\frac{p-1}{p}}\right)^p =$$
$$= \frac{1}{h^p} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p \,\mathrm{d}t \, h^{p-1} = \frac{1}{h} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p \,\mathrm{d}t\,,$$

and so integrating over Ω_δ yields

$$\left\|\Delta_h^i u_\varepsilon\right\|_{\mathrm{L_p}(\Omega_\delta)}^p \le \frac{1}{h} \int_{\Omega_\delta} \int_0^h |\partial_t u_\varepsilon(x + te_i)|^p \,\mathrm{d}t \,\mathrm{d}x = \frac{1}{h} \int_0^h \int_{\Omega_\delta} |\partial_t u_\varepsilon(x + te_i)|^p \,\mathrm{d}x \,\mathrm{d}t\,,$$

denote $z = x + te_i$, then $z \in \Omega_\delta + te_i$, meaning the points have shifted towards Ω in one direction by at most $t \le h < \frac{\delta}{2}$. If we integrate over $\Omega_{\frac{\delta}{2}}$ instead, we make the domain larger and so

$$\left\|\Delta_h^i u_\varepsilon\right\|_{\mathrm{L_p}(\Omega_\delta)}^p \le \frac{1}{h} \int_0^h \int_{\Omega_{\frac{\delta}{2}}} |\partial_t u_\varepsilon(z)|^p \,\mathrm{d}z \,\mathrm{d}t \le \frac{1}{h} h \|\partial_t u_\varepsilon\|_{\mathrm{L_p}\left(\Omega_{\frac{\delta}{2}}\right)}^p,$$

so if we pass to the limit $\varepsilon \to 0^+$ (we are in Ω_δ), we see

$$\left\|\Delta_h^i u\right\|_{\mathrm{L_p}(\Omega_\delta)} \le \|\partial_t u\|_{\mathrm{L_p}\left(\Omega_{\frac{\delta}{2}}\right)} \le \|\partial_t u\|_{\mathrm{L_p}(\Omega)}.$$

If now $p = \infty$, the things are not that simple - Ω is not bounded apriori, so we have no ordering of Lebesgue spaces. Let us fix that: take some $R > 0$ and denote $\Omega^R = \Omega \cap \mathrm{U}(0,R)$. If $u \in \mathrm{W}^{1,\infty}(\Omega)$, it must be $u \in \mathrm{W}^{1,p}\left(\Omega^R\right)$ for all $p \in [1,\infty)$. On Ω^R_δ we have from the above

$$\left\|\Delta^i_h u\right\|_{\mathrm{L_p}\left(\Omega^R_\delta\right)} \le \left\|\partial_t u\right\|_{\mathrm{L_p}(\Omega^R)},$$

and since Ω^R_δ is bounded, we can pass to the limit $p \to \infty$ and write

$$\left\|\Delta^i_h u\right\|_{\mathrm{L_\infty}\left(\Omega^R_\delta\right)} \le \left\|\partial_t u\right\|_{\mathrm{L_\infty}(\Omega^R)},$$

and passing to the limit $R \to \infty$ we obtain

$$\left\|\Delta^i_h u\right\|_{\mathrm{L_\infty}(\Omega_\delta)} \le \left\|\partial_t u\right\|_{\mathrm{L_\infty}(\Omega)}.$$

Ad 2. Once again extend u by zero outside of Ω and let for the moment $p \in (1,\infty)$. Fix $\{h_n\} \subset \left(0, \frac{\delta}{2}\right)$ $s.t.$ $h_n \to 0$ and consider the functions

$$v^i_n = \Delta^i_{h_n} u \chi_{\Omega_{2h_n}}.$$

Clearly

$$\left\|v_n\right\|^i_{\mathrm{L_p}(\Omega)} = \left\|\Delta^i_{h_n} u\right\|_{\mathrm{L_p}(\Omega_{2h_n})} \le \left\|\Delta^i_{h_n} u\right\|_{\mathrm{L_p}(\Omega_\delta)} \le C_i,$$

and from the reflexivity of $\mathrm{L_p}(\Omega)$, $p \in (1,\infty)$, we know $\exists v_i \in \mathrm{L_p}(\Omega)$ such that

$$v^i_n \rightharpoonup v^i,$$

as $n \to \infty$. Since the norm is weak lower semicontinuous, we also have the estimate

$$\left\|v_i\right\|_{\mathrm{L_p}(\Omega)} \le C_i.$$

It remains to show now that v_i are weak derivatives of u. Before we procced, let us mention the following identity: let $\varphi \in \mathcal{D}(\Omega)$, then

$$\int_{\mathbb{R}^d} \Delta^i_h u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} u\Delta^i_{-h}\varphi \,\mathrm{d}x.$$

Really, it holds **THIS NEEDS TO BE FINISHED**

$$\int_{\mathbb{R}^d} \Delta^i_h u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{u(x+he_i) - u(x)}{h}\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{\varphi(x)u(x+he_i) - \varphi(x)u(x)}{h} \,\mathrm{d}x,$$

change the variables $y = x + he_i$, then the above integral is equal to

$$\int_{\mathbb{R}^d} \frac{\varphi(y-he_i)u(y) - \varphi(y-he_i)u(y-he_i)}{h} \,\mathrm{d}y =$$

Using this, we can obtain

$$\begin{aligned}\int_\Omega v_i\varphi \,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega v^i_n\varphi \,\mathrm{d}x = \lim_{n\to\infty}\int_{\Omega_{2h_n}} \Delta^i_{h_n} u\varphi \,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^d} \Delta^i_{h_n} u\varphi \,\mathrm{d}x = \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} u\Delta^i_{-h_n}\varphi \,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^d} u\frac{\varphi(x-h_ne_i) - \varphi(x)}{h_n}u(x) \,\mathrm{d}x = -\int_{\mathbb{R}^d} u\partial_i\varphi \,\mathrm{d}x = \\ &= -\int_\Omega u\partial_i\varphi \,\mathrm{d}x\end{aligned}$$

which really means $v_i = \partial_i u$ in the weak sense. The fact we can integrate over $\mathbb{R}^d$ instead of Ω_{2h_n} comes from the fact $\operatorname{supp}\varphi \subset \Omega_{2h_n}$ for some n large enough, and since the whole integrand is zero on any larger set then Ω_{2h_n}, meaning we without a doubt expand the integration domain to the whole space. All in all, we have shown $v_i = \partial_i u$ weakly and $\|v_i\|_{\mathrm{L_p}(\Omega)} \le C_i$, so we are done with the case $p \in (1,\infty)$.

If $p = \infty$, we can repeat the same arguments as in the above case: create a bounded domain $\Omega_R = \Omega \cap \mathrm{U}(0,R)$, realize that

$$\|v_i\|_{\mathrm{L_p}(\Omega^R)} \le C_i, \forall p \in (1,\infty),$$

and then pass to the limits $R \to \infty, p \to \infty$.

$\square$

Remark. We have thus shown that if $p \in (1,\infty)$, then

$$\Delta_h^i u \rightharpoonup \partial_i u, h \to 0^+.$$

2.6.3 Representation of duals

3 Nonlinear elliptic equations - compactness methods

So far we have dealt with only linear equations, *i.e.*, linear operators. When dealing with nonlinear operators, things get much more involved. This chapter will be dedicated to the study of nonlinear problems using compactness methods, which will especially allow us to solve problems with nonlinearities in the non-leading terms. **Be especially careful about the assumptions on the operators - they do not need to be linear.**

Let us start with some technical, mostly known lemmas.

3.1 Nemytskii operators

Lemma 9 (Fatou)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable, let $\{f_n\}$ be a sequence of measurable nonnegative functions. Then it holds*

$$\int_\Omega \liminf_{n\to\infty} f_n \,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x\,.$$

(Both integrals can be infinity.)

Theorem 15 (Iegorov)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable and of finite Lebesgue measure, let $\{f_n\}$ and f be measurable and finite a.e. in Ω. Then the following statements are equivalent*

1. *$f_n(x) \to f(x), \forall$ a.a. $x \in \Omega$,*
2. *$\forall \varepsilon > 0 \exists G \subset \Omega$ open s.t. $\lambda(G) < \varepsilon$ and $f_n \rightrightarrows f$ on Ω/G.*

Theorem 16 (Vitali)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable and of finite Lebesgue measure, let $\{f_n\}$ and f be measurable s.t. $f_n(x) \to f(x)$for a.a. $x \in \Omega$. Let moreover be true*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall n \in \mathbb{N}, \forall H \subset \Omega\, s.t.\, \lambda(H) < \delta \Rightarrow \int_H |f_n| \,\mathrm{d}x \le \varepsilon.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x\,.$$

Remark (What is $f(\cdot, x)$?). In the following definitions and throughout the chapter in general, we use the "$\cdot$" to denote variables in the following sense. Just to make it clear, this means the following. If U, V, W are some sets and $f : U \times V \to W$, we write

- $f(u, \cdot) : V \to W$, as the mapping $V \ni v \mapsto f(u, v) \in W$ for some $u \in U$ fixed,
- $f(\cdot, v) : U \to W$, as the mapping $U \ni u \mapsto f(u, v) \in W$ for some $v \in V$ fixed.

Definition 7 (Caratheodory function)**.** Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}$. We say the function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is Caratheodory if

1. $f(x, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is continuous for almost all $x \in \Omega$,
2. $f(\cdot, y) : \Omega \to \mathbb{R}$ is measurable for all $y \in \mathbb{R}^N$.

Definition 8 (Nemytskii operator)**.** Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Caratheodory. For $\mathbf{u} : \Omega \to \mathbb{R}^N, x \in \Omega$ we define the Nemytskii operator N_f: as

$$N_f : \mathbf{u}(x) \mapsto f(x, \mathbf{u}(x)),$$

meaning N_f is an operator between function spaces, whose image is the function $N_f \mathbf{u} : \Omega \to \mathbb{R}$ *s.t.*

$$N_f \mathbf{u}(x) = f(x, \mathbf{u}(x)), x \in \Omega.$$

Remark. We write $\mathbf{u}$ in bold to stress $u : \Omega \to \mathbb{R}^N$ is a vector valued mapping, even though we might drop the notation later on.

Remark. This definition is similiar to the one used in the Caratheodory theory of ordinary differential equations, but not the same.

Theorem 17 (On Nemytskii operators)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}, f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Caratheodory. Then*

1. *if $\mathbf{u} : \Omega \to \mathbb{R}^N$ is measurable, then $N_f \mathbf{u} : \Omega \to \mathbb{R}$ is also measurable,*
2. *if there are numbers $p_i \in [1, \infty), i \in \{1, \dots, N\}$, an exponent $p \in [1, \infty)$, a function g s.t. $g \in L_p(\Omega)$, and a constant $C \geq 0$ such that for almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$ it holds*

$$|f(x, y)| \leq g(x) + C \sum_{i=1}^{N} |y_i|^{\frac{p_i}{p}},$$

then the Nemytskii operator $N_f : \mathbf{u} \mapsto N_f \mathbf{u}$ is continuous from $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \dots L_{p_N}(\Omega)$ to $L_p(\Omega)$ and moreover it maps sets bounded in $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \dots L_{p_N}(\Omega)$ to sets bounded in $L_p(\Omega)$. We write

$$N_f \in \mathcal{C}(L_{p_1}(\Omega), \dots, L_{p_N}(\Omega); L_p(\Omega)).$$

Proof. No proof ☐

Remark. The information about boundedness of images of bounded sets is not trivial - *the Nemytskii operator is not linear in general.*

3.2 Fixed point theorems

The majority of the problems we will be dealing with will be solved by using some fixed points theorem. Here we state two of them (without proofs).

Definition 9 (Compact operator)**.** Let X, Y be normed linear spaces, $M \subset X$. The mapping $F : M \to Y$ is called a compact operator on M into Y provided

1. F is continuous,
2. $F(M \cap K) \subset Y$ is relatively compact in Y for any bounded $K \subset X$.

Remark.

- We have no linearity of F! So continuity cannot follow from compactness and we have to assume it explicitely. In general, compactness $\Rightarrow$ boundedness $\neq$ continuity for nonlinear operators.
- one might wonder why just not state $F(K) \subset Y$ is relatively compact in Y for $K \subset M$ bounded. First of all, this is pretty much the same, but in the applications, this formulation will be more suitable for us. Secondly, in general M need not be a subspace of X, let alone a normed linear space;in such cases the notion of boundedness of $K \subset M$ is not defined at all. On the other hand, boundedness of $K \subset X$ can be measured in the metric of X.
- The definition is from *Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2*

Theorem 18 (Brouwer fixed point theorem)**.** *Let $N \in \mathbb{N}$ and $K \subset \mathbb{R}^N$, be a nonempty convex closed bounded subset of $\mathbb{R}^N$. Assume further that $F : K \to K$ is continuous. Then F has a fixed point in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. No proof. □

Theorem 19 (Schauder fixed point theorem)**.** *Let X be a linear normed space and $K \subset X$ be a nonempty convex closed bonded subset of X. Assume further that $F : K \to K$ is compact on K into K and $F(K) \subset K$. Then there is fixed point of F in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. No proof. □

Remark.

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bouded, it is automatically compact, and since $F : K \to K$ is continuous, F is compact[23]. For Schauder, we have to assume this extra.
- realize again that in Schauder, the set K does not have to be a subspace of X.
- proof of Brouwer with N=1 is easy, based on Darboux property.

[23] Image of a compact set under continuous mapping is a compact set

3.3 Problem protypes

In this chapter some nonlinear elliptic equations are discussed. The strategy of solving them (*i.e.*, proving a solution exists) will be always the same - define a suitable operator and show it has a fixed point.

Example. Suppose the following problem:

$$\begin{cases} -\triangle u + g(u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

where $f \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$, $g : \mathbb{R} \to \mathbb{R}$ is continuous and has a controlled grow *s.t.* $\exists C > 0, \exists \alpha \in (0,1]$:

$$\forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

Even though g might be nonlinear, we assume only (sub)linear growth of it - so this is kind of cheating.

Theorem 20 (Existence)**.** *Let $\Omega \in C^{1,1}$, $f \in \left(W_0^{1,2}(\Omega)\right)^*$, g is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi + g(u)\varphi \,\mathrm{d}x = < f, \varphi >_{(W_0^{1,2}(\Omega))^*} .$$

If $f \in L_2(\Omega)$, then the solution $u \in W^{2,2}(\Omega)$.

Proof. We define $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ such that[24]

$$Sw = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = < f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x .$$

Clearly, when $Sw = w$, *i.e.*, S has a fixed point and then

$$Sw = w \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla w \cdot \nabla \varphi \,\mathrm{d}x = < f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x ,$$

and so the solution w to our problem exists. S is defined in an implicit way, but it might be guiding to think of it, written in strong formulation, as

$$Sw = -(\triangle)^{-1}(f - g(w),$$

meaning Sw is the solution u to (modulo boundary conditions)

$$-\triangle u = f - g(w),$$

also meaning Sw is the solution u to the Poisson equation with the RHS "build from w."

First, let us show S is well defined:

$$\left| \int_\Omega g(w)\varphi \,\mathrm{d}x + < f, \varphi > \right| \leq \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} \|\varphi\|_{\mathrm{W}^{1,2}(\Omega)} + \|\varphi\|_{\mathrm{L}_2(\Omega)} \|g(w)\|_{\mathrm{L}_2(\Omega)},$$

[24]Meaning the image Sw of w is the function u such that the integral equality holds. We could also assume $S : \mathrm{L}_2(\Omega) \to \mathrm{W}^{1,2}(\Omega)$, but for compactness we need the origin and target spaces to be the same.

and

$$\int_\Omega |g(w)|^2 \,\mathrm{d}x \le \int_\Omega C^2(1+|w|^\alpha)^2 \,\mathrm{d}x \le \int_\Omega 2C^2\big(1+|w|^{2\alpha}\big) \,\mathrm{d}x \le \int_\Omega 2C^2\big(1+|w|^2\big) \,\mathrm{d}x < \infty,$$

where we used the Young inequality[25], $\alpha \le 1$ and the fact S is defined on $\mathrm{L}_2(\Omega)$, meaning $w \in \mathrm{L}_2(\Omega)$. With that information, we see the last integral is finite and so truly $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ is well defined.

Also, the estimate $|g(w)| \le C(1+|w|^\alpha), \alpha \in (0,1]$ together with the fact $C(1+w^\alpha) \in \mathrm{L}_2(\Omega)$ for $w \in \mathrm{L}_2(\Omega)$ means the Nemytskii operator

$$w \mapsto g(w) \in \mathcal{C}(\mathrm{L}_2(\Omega)\,;\mathrm{L}_2(\Omega)),$$

is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$.

Let us now show that S is continuous as a whole. Define $I_g \in \big(\mathrm{W}_0^{1,2}(\Omega)\big)^*$ as

$$I_g(\varphi) = \int_\Omega g(w)\varphi \,\mathrm{d}x - < f, \varphi > .$$

We are interested in the continuity of the (nonlinear!) mapping $g \mapsto I_g$ as a mapping from $\mathrm{L}_2(\Omega)$ to $\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*$. Let $\{g_k\} \subset \mathrm{L}_2(\Omega)$ be a sequence *s.t.* $g_k \to g$ in $\mathrm{L}_2(\Omega)$. Then

$$\|I_{g_k} - I_g\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} = \sup_{\varphi \in \mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} |(I_{g_k} - I_g)(\varphi)| = \sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \left|\left(\int_\Omega g_k\varphi \,\mathrm{d}x - < f,\varphi > - \int_\Omega g\varphi \,\mathrm{d}x + < f, \varphi >\right)\right| \le$$

$$\sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \int_\Omega |g - g_k||\varphi| \,\mathrm{d}x \le \|g - g_k\|_{\mathrm{L}_2(\Omega)} \to 0,$$

so we have shown $g \to g_k$ in $\mathrm{L}_2(\Omega) \Rightarrow I_g \to I_{g_k}$ in $\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*$, and so $(g \mapsto I_g) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*\Big)$. Finally, define the operator $L : \big(\mathrm{W}_0^{1,2}(\Omega)\big)^* \to \mathrm{W}_0^{1,2}(\Omega)$, as

$$Lh = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{d}x =< h, \varphi >,$$

or in other words, L maps $h \mapsto u$, where u solves

$$\begin{cases} -\triangle u = h, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega \end{cases}.$$

or in other other, informal, words

$$Lh = -(\triangle)^{-1} h.$$

Realize that this approach been studiud in the chapter about Fredholm theory extensively. In this simple case however, recall the apriori estimates that the solution satisfies (that is guaranteed from the properties of the Poisson equation)

$$\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} \le C\|h\|_{(\mathrm{W}_0^{1,2}(\Omega))^*},$$

which together with the linearity of L checks $h \mapsto u$ is continuous. Putting it all together:

[25] In the form $(a+b)^2 \le 2\big(a^2+b^2\big)$.

- $(w \mapsto g(w)) \in \mathcal{C}(\mathrm{L}_2(\Omega)\,;\mathrm{L}_2(\Omega))$,
- $(g(w) \mapsto f - g(w)) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*\Big)$, [26]
- $(f - g(w) \mapsto u) \in \mathcal{C}\Big(\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*; \mathrm{W}_0^{1,2}(\Omega)\Big)$,

In total, the composition is continuous and yields S.

Next, we would like to show S is a compact (nonlinear) operator. We start with showing S maps bounded sets in $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_2(\Omega)$; for that we need apriori estimates: test the weak formulation with u: (we are doing this a bit more careful that might seem needed, but come on, it is the first example...)

$$
\begin{aligned}
\underbrace{\|\nabla u\|^2_{\mathrm{L}_2(\Omega)}}_{C_p^2\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)}\leq} &\leq \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \|g(w)\|_{\mathrm{L}_2(\Omega)}\|u\|_{\mathrm{L}_2(\Omega)} \leq \\
&\leq \frac{\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}}{\sqrt{2\varepsilon_1}}\sqrt{2\varepsilon_1}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{\|g(w)\|_{\mathrm{L}_2(\Omega)}}{\sqrt{2\varepsilon_2}}\sqrt{2\varepsilon_2}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} \leq \\
&\leq (\varepsilon_1+\varepsilon_2)\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{1}{4\varepsilon_1}\|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \frac{1}{4\varepsilon_2}\|g(w)\|^2_{\mathrm{L}_2(\Omega)} \leq \\
&\leq (\varepsilon_1+\varepsilon_2)\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{1}{4\min(\varepsilon_1,\varepsilon_2)}\Big(\|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|g(w)\|^2_{\mathrm{L}_2(\Omega)}\Big),
\end{aligned}
$$

and since $\|g(w)\|^2_{\mathrm{L}_2(\Omega)} \leq \big\|C\big(1+|w|^\alpha\big)\big\|^2_{\mathrm{L}_2(\Omega)} \leq 4C^2\Big((\lambda(\Omega))^2 + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big)$ and Poincare inequality, we can write

$$
\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq \frac{1}{4\big(C_p^2 - (\varepsilon_1+\varepsilon_2)\big)\min(\varepsilon_1,\varepsilon_2)}\Big(\|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + 4\Big((\lambda(\Omega))^2 + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big)\Big),
$$

which, for $\varepsilon_1, \varepsilon_2$ sufficiently small, just an estimate of the type

$$
\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq C\Big(1 + \|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big).
$$

As we can see, if $w \in \mathrm{L}_2(\Omega)$ is bounded, also $u \in \mathrm{W}_0^{1,2}(\Omega)$ is bounded, so S maps bounded sets from $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega)$. Since moreover $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_2(\Omega)$, this means S is compact from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$.

To conclude the proof, we need that there exists $K \subset \mathrm{L}_2(\Omega)$ closed convex bounded nonempty *s.t.* $S(K) \subset K$, so we can use Schauder. That clearly will be the case when we show $S\big(\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)\big) \subset \mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)$ for some $R > 0$. But this is simple - the estimate

$$
\|u\|^2_{\mathrm{L}_2(\Omega)} \leq \|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq C\left(\underbrace{1 + \|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*}}_{:=C_1} + \|w\|^2_{\mathrm{L}_2(\Omega)}\right),
$$

[26] In the sense we interpret the RHS as a point in the dual space. Precisely, we should write something like $\Big(g(w) \mapsto \Big(\varphi \in \mathrm{W}_0^{1,2}(\Omega) \mapsto < f, \varphi > - \int_\Omega g(w)\varphi\,\mathrm{d}x\Big)\Big) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*\Big)$.

tells us if the R were to exist, it must hold

$$R^2 \le CC_1 + CR^2 \Leftrightarrow R^2(1-C) \le CC_1.$$

If $1-C>0$, then just take $R \le \sqrt{\frac{CC_1}{1-C}}$, if $1-C<0$, any $R>0$ will do.

And so we see such an R exists in all cases $\Rightarrow$ the image of a ball is in a ball for some $R \Rightarrow S$ is compact on $\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)$ into $\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R) \Rightarrow$ it has a fixed point by Schauder $\Leftrightarrow$ the solution exists.

For the regularity part of the assertion, realize that u_0 solves

$$\begin{cases} -\triangle u_0 = f - g(u_0), & \text{in}\,\Omega \\ u_0 = 0, & \text{on}\,\partial\Omega. \end{cases},$$

and if $f \in \mathrm{L}_2(\Omega)$, then $f - g(u_0) \in \mathrm{L}_2(\Omega)$ and so from the regularity theory for elliptic equations we get

$$u \in \mathrm{W}^{2,2}(\Omega).$$

□

Theorem 21 (Uniqueness)**.** *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be weak solutions to the above problem. Let $f \in \left(W_0^{1,2}(\Omega)\right)^*$ and let g be continuous. Let either extra*

1. *g is nondecreasing, or*
2. *$g \in C^1(\mathbb{R}), \|g'\|_{L_\infty(\mathbb{R})}$ small.*

Then $u_1 = u_2$.

Proof. In the linear case, $u_1 - u_2$ would be a solution to the problem with zero data. In the nonlinear case, we have now certainity $u_1 - u_2$ solves anything, so we will have to be more careful.

Subtract the equations for u_1, u_2 and test the weak formulation with $u_1 - u_2$.:

$$\int_\Omega |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2)\,\mathrm{d}x = 0.$$

In the first case, the second term is nonnegative, so for the whole thing to be zero it has to hold

$$0 = \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \ge C_p \|u_1 - u_2\|^2_{\mathrm{W}^{1,2}(\Omega)} \Rightarrow u_1 - u_2 = 0 \in \mathrm{W}_0^{1,2}(\Omega).$$

In the second case, the second does not have a sign, but we can estimate it

$$\left|\int_\Omega (g(u_1) - g(u_2)(u_1 - u_2))\,\mathrm{d}x\right| \le \int_\Omega \|g'\|_{\mathrm{L}_\infty(\mathbb{R})} \big|u_1 - u_2|^2\big|\,\mathrm{d}x \le \|g'\|_{\mathrm{L}_\infty(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)},$$

meaning

$$\|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \pm \|g\|_{\mathrm{L}_\infty(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} = (1 \pm \|g\|_{\mathrm{L}_\infty(\mathbb{R})} C_p)\|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \le 0.$$

If the term is positive we again obtain $u_1 = u_2$ in $\mathrm{W}_0^{1,2}(\Omega)$ from the Poincare inequality, if it is negative, we require

$$\|g'\|_{\mathrm{L}_\infty(\mathbb{R})} C_p < 1,$$

and then we have the same result. □

Example. Suppose the following problem

$$\begin{cases} -\triangle u + b(\nabla u) = f, & \text{in}\,\Omega \\ u = 0, & \text{on}\,\partial\Omega. \end{cases}$$

where $f \in (\mathrm{W}_0^{1,2}(\Omega))^*$, and the function $b : \mathbb{R}^d \to \mathbb{R}$ is continuous and (essentialy) bounded. The weak formulation is:

$$\text{find}\, u \in \mathrm{W}_0^{1,2}(\Omega)\ s.t.: \forall\varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla\varphi + b(\nabla u)\varphi\,\mathrm{d}x = < f, \varphi >_{(\mathrm{W}_0^{1,2}(\Omega))^*},$$

Theorem 22. *Let $f \in \left(W_0^{1,2}(\Omega)\right)^*, \Omega \in C^{0,1}, b : \mathbb{R}^d \to \mathbb{R}$ continuous and essentialy bounded. Then there is a weak solution to the above problem.*

Proof. As in the previous: define the operator $S : \mathrm{W}_0^{1,2}(\Omega) \to \mathrm{W}_0^{1,2}(\Omega)$, such as

$$Sw = u \Leftrightarrow \forall\varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla\varphi\,\mathrm{d}x = < f, \varphi > - \int_\Omega b(\nabla w)\varphi\,\mathrm{d}x\,,$$

or equivalently

$$Sw = u \Leftrightarrow u\,\text{is the solution to}\ \begin{cases} -\triangle u = f - b(\nabla w) & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases},$$

or informally

$$Sw = -(\triangle)^{-1}(f - b(\nabla w)),$$

modulo boundary conditions. Once again, if we are able to show S has a fixed point, this information is equivalent to the considered problem having a solution.

Let us first show S is well defined. Holder yields

$$\|\nabla u\|_{\mathrm{L}_2(\Omega)}\|\nabla\varphi\|_{\mathrm{L}_2(\Omega)} \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|\varphi\|_{\mathrm{W}_0^{1,2}(\Omega)} + \|b\|_{\mathrm{L}_\infty(\mathbb{R}^d)}\|\varphi\|_{\mathrm{L}_1(\Omega)},$$

use Poincare inequality and $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow \mathrm{L}_1(\Omega)$, to obtain

$$\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} \le \underbrace{C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|b\|_{\mathrm{L}_\infty(\mathbb{R}^d)}\Big)}_{:=R},$$

and so we see that the target space really is $\mathrm{W}_0^{1,2}(\Omega)$. Also, the above estimate is extra nice, as it does not depend on w whatsoever. If we denote the RHS by $R > 0$, we immediately see $\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} = \|Sw\|_{\mathrm{W}_0^{1,2}(\Omega)} \le R$, meaning that for sure $S\Big(\mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)\Big) \subset \mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)$. Let us show other properties needed for compactness.

First, $\nabla : \mathrm{W}_0^{1,2}(\Omega) \to \mathrm{L}_2(\Omega;\mathbb{R}^d)\,, w \mapsto \nabla w$ is linear and bounded,

$$\|\nabla w\|_{\mathrm{L}_2(\Omega)} \le \|w\|_{\mathrm{W}_0^{1,2}(\Omega)},$$

and so continuous, the map $\mathbf{y} \mapsto b(\mathbf{y})$ satisfies the growth

$$|b(\mathbf{y})| \le \underbrace{\|b\|_{\mathrm{L}_\infty(\mathbb{R}^d)}}_{\text{"}=g(x)\text{"}} + 0 \cdot \sum_{i=1}^d |y_i|^{2/2},$$

and since $\lambda(\Omega) < \infty$, we have (in particular) $\|b\|_{L_\infty(\mathbb{R}^d)} \in L_2(\Omega)$. Putting this together means $N_b : \nabla w \mapsto b(\nabla w)$ is continuous from[27] $L_2(\Omega;\mathbb{R}^d)$ to $L_2(\Omega;\mathbb{R})$,

$$(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}\big(L_2(\Omega;\mathbb{R}^d);L_2(\Omega;\mathbb{R})\big)$$

by Nemytskii. Next, the mapping

$$b(\nabla w) \mapsto b(\nabla w) + f$$

from $L_2(\Omega)$ to $\big(W_0^{1,2}(\Omega)\big)^*$ in the sense

$$b(\nabla w) \mapsto \Big(W_0^{1,2}(\Omega) \ni \varphi \mapsto < f,\varphi >_{(W_0^{1,2}(\Omega))^*} + \int_\Omega b(\nabla w)\varphi\,\mathrm{d}x\Big),$$

is also continous: let $\{b_k\} \subset L_2(\Omega)$ $s.t.$ $b_k \to b$, in $L_2(\Omega)$, then

$$\sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \left|< f,\varphi > - \int_\Omega b_k(\nabla)\varphi\,\mathrm{d}x - < f,\varphi > + \int_\Omega b(\nabla w)\varphi\,\mathrm{d}x\right| \le \sup_{U_{W_0^{1,2}(\Omega)}(0,1)} \int_\Omega |b(\nabla w) - b_k(\nabla w)||\varphi|\,\mathrm{d}x \le$$
$$\le \|b_k - b\|_{L_2(\Omega)} \to 0,$$

and so the mapping $b(\nabla w) \mapsto b(\nabla w) + f$ is continuous from $L_2(\Omega)$ to $L_2(\Omega)$. Finally, the mapping $L : \big(W_0^{1,2}(\Omega)\big)^* \to W_0^{1,2}(\Omega)$,

$$Lh = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla\varphi\,\mathrm{d}x =< h,\varphi >,$$

i.e.,

$$Lh = u \Leftrightarrow u\,\text{solves} \begin{cases} -\triangle u = h, & \text{in}\,\Omega,\\ u = 0, & \text{on}\,\partial\Omega,\end{cases}$$

is also continuous between the spaces - it is just again the solution operator to the Poisson equation with a RHS $h \in \big(W_0^{1,2}(\Omega)\big)^*$. Altogether

- $(w \mapsto \nabla w) \in \mathcal{C}\big(W_0^{1,2}(\Omega);L_2(\Omega;\mathbb{R}^d)\big)$,
- $(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}\big(L_2(\Omega;\mathbb{R}^d);L_2(\Omega;\mathbb{R})\big)$,
- $(b(\nabla w) \mapsto f - b(\nabla w)) \in \mathcal{C}\Big(L_2(\Omega);\big(W_0^{1,2}(\Omega)\big)^*\Big)$,
- $(f - b(\nabla w) \mapsto u) \in \mathcal{C}\Big(\big(W_0^{1,2}(\Omega)\big)^*;W_0^{1,2}(\Omega)\Big)$,

and so S as a composition of those mappings is continuous from $W_0^{1,2}(\Omega)$ to $W_0^{1,2}(\Omega)$. It remains to show S is compact: we already have continuity, consider now $\{w_k\}_{k\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ bounded. Then, by the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L_1(\Omega)$, there $\exists\{u_k\} \subset \{w_k\} \subset W_0^{1,2}(\Omega)$ (also bounded) $s.t.$

[27] We should be more precise and write it is continuous from $L_2(\Omega;\mathbb{R}^d)$ to $L_2(\Omega;\mathbb{R})$, as $b : \mathbb{R}^d \to \mathbb{R}$.

$u_k \to u$ in $\mathrm{L}_1(\Omega)$. To conclude, use the following trick (as when showing uniqueness to the previous problem): subtract equation for u_k from equation for u_l and test with $u_l - u_k$

$$\int_\Omega \nabla u_l \cdot \nabla(u_l - u_k)\,\mathrm{d}x + \int_\Omega b(\nabla u_l)(u_l - u_k)\,\mathrm{d}x - < f, u_l - u_k >$$
$$- \left(\int_\Omega \nabla u_k \cdot \nabla(u_l - u_k)\,\mathrm{d}x + \int_\Omega b(\nabla u_k)(u_l - u_k)\,\mathrm{d}x - < f, u_l - u_k \right),$$

meaning

$$C\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \le \|\nabla(u_l - u_k)\|^2_{\mathrm{L}_2(\Omega)} \le \int_\Omega |b(\nabla u_l) - b(\nabla u_k)\,||u_l - u_k|\,\mathrm{d}x \le 2\|b\|_{\mathrm{L}_\infty(\Omega)}\|u_l - u_k\|_{\mathrm{L}_1(\Omega)},$$

so

$$\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \le C\|u_l - u_k\|_{\mathrm{L}_q(\Omega)}.$$

And we are finished - the sequence $\{u_n\}$ converges in $\mathrm{L}_1(\Omega)$, so is Cauchy in $\mathrm{L}_1(\Omega)$, and the above inequality proves that also $\{u_n\}$ is Cauchy in a complete space $\mathrm{W}_0^{1,2}(\Omega)$, meaning $\{u_n\}$ converges in $\mathrm{W}_0^{1,2}(\Omega)$. All in all, $\{u_n\} \subset \mathrm{W}_0^{1,2}(\Omega)$ is a convergent subsequence of a bounded sequence $\{w_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$, meaning S is compact on $\mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)$ into $\mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)$ and so must possess a fixed point by the Schauder fixed point theorem. □

4 Nonlinear elliptic equations - monotone operator theory

The method of compact perturbations allowed us to solve some nonlinear problems. This relied on the fact that nonlinearities were present mainly in the nonleading terms and so we were able to use our theory we developed for linear equations (continuity of the solution operators, regularity results, etc.) In this chapter we examine a different approach that will allow us to solve a particular class of problems even with nonlinearities in the leading term.

Consider the following problem:

$$-\sum_{i=1}^{d} \partial_i(a_i(x, u(x), \nabla u(x))) + a_0(x, u(x), \nabla u(x)) = f(x), \text{ in } \Omega$$
$$u = u_0, \text{ on } \partial\Omega,$$

or written more concisely

$$-\nabla \cdot (\mathbf{a}(x, u(x), \nabla u(x))) - a_0(x, u(x), \nabla u(x)) = f(x), \text{ in } \Omega,$$
$$u = u_0, \text{ on } \Omega,$$

where $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$,

$$\mathbf{a}(x, u(x), \nabla u(x)) = (a_1(x, u(x), \nabla u(x)), \dots, a_d(x, u(x), \nabla u(x)))),$$

is some *nonlinear* vector function. The data of the problem are

- $\Omega \in C^{0,1}$,

- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, i \in \{0, \dots, d\}$ are Caratheodory in $x \in \Omega$ and $(z, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^d$,
- *growth condition*: $\exists C > 0, r \in (1, \infty), h \in \mathrm{L}_{r'}(\Omega) : \forall i \in \{0, \dots, d\}, \forall\ a.a.\, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d :$

$$|a_i(x, z, \mathbf{p})| \le C\Big(|z|^{r-1} + |\mathbf{p}|^{r-1}\Big) + h(x),$$

- $f \in \left(\mathrm{W}_0^{1,r}(\Omega)\right)^*$,

and the unknown is $u : \Omega \to \mathbb{R}$.

Remark. The Nemytskii operator[28] $(u, \mathbf{p}) \mapsto a_i(\cdot, u, \mathbf{p})$ is continuous from $(\mathrm{L}_r(\Omega))^{d+1}$ to $\mathrm{L}_{r'}(\Omega)$. by Nemystkii theorem. Truly, generally, the estimate

$$|f(x, \mathbf{y})| \le g(x) + C\sum_{j=1}^{d} |y_i|^{\frac{p_i}{p}}, g \in \mathrm{L}_\mathrm{p}(\Omega)\,,$$

guarantess continuity from $\mathrm{L}_{p_1}(\Omega) \times \dots \times \mathrm{L}_{p_d}(\Omega)$ to $\mathrm{L}_\mathrm{p}(\Omega)$. So in our case, the target space is $\mathrm{L}_{r'}(\Omega)$, since $h \in \mathrm{L}_{r'}(\Omega)$, and the exponent from the origin space can be found by

$$\text{"}p_i = \frac{p_i}{p}p,\text{"}$$

i.e., taking the exponents we in fact have the estimates for and multiplying them by the target space exponent. In our case $p_i = (r-1)r' = r - 1\frac{r}{r-1} = r$.

Definition 10 (Coercivity)**.** We say that $\{a_i\}_{i=0}^d = \mathbf{a}$ is coercive if $\exists C_1 > 0, \exists C_2(x) \in \mathrm{L}_1(\Omega)$: $\forall\ a.e.\, x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1}$:

$$\sum_{i=1}^{d} a_i(x, z, \mathbf{p})p_i + a_0(x, z, \mathbf{p}) \ge C_1|\mathbf{p}|^r - C_2(x),$$

i.e.,

$$\mathbf{a}(x, z, \mathbf{p}) \cdot \mathbf{p} + a_0(x, z, \mathbf{p}) \ge C_1|\mathbf{p}|^r - C_2(x)$$

Definition 11 (Monotonicity)**.** We say that $\{a_i\}_{i=0}^d = \mathbf{a}$ is monotone if $\forall\ a.a.\, x \in \Omega, \forall (z_1, \mathbf{p}_1), (z_2, \mathbf{p}_2) \in \mathbb{R}^{d+1}$:

$$(\mathbf{a}(x, z_1, \mathbf{p}_1) - \mathbf{a}(x, z_2, \mathbf{p}_2)) \cdot (\mathbf{p}_1 - \mathbf{p}_2) + (a_0(x, z_1, \mathbf{p}_1) - a_0(x, z_2, \mathbf{p}_2))(z_1 - z_2) \ge 0.$$

Very similiarly we define strict monotonicity.

Definition 12 (Weak solution)**.** We say that $u \in \mathrm{W}^{1,r}(\Omega)$ is a weak solution to the above problem provided

$$\operatorname{tr} u = \operatorname{tr} u_0 \text{ on } \partial\Omega$$

and

$$\int_\Omega \mathbf{a}(\cdot, u, \nabla u) \cdot \nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \,\mathrm{d}x = \langle f, \varphi \rangle, \forall \varphi \in \mathrm{W}_0^{1,r}(\Omega)\,.$$

Now comes the proof of existence and uniqueness – let us prepare for that with a useful lemma.

[28] Again meaning $N_{a_i}(u, \mathbf{p})(x) = a_i(x, u(x), \mathbf{p}(x))$.

Lemma 10. *Let $g : B(0,R) \subset \mathbb{R}^n \to \mathbb{R}^N$ be continuous, $N \in \mathbb{N}, R > 0$, and $\forall c \in S(0,R) : g(c) \cdot c \geq 0$. Then, there is $c_0 \in B(0,R) : g(c_0) = 0$.*

Proof. By contradiction. Let $g \neq 0$ in $\mathrm{U}(0,R)$. Let us define

$$h(x) = -R\frac{g(x)}{|g(x)|}.$$

Then $h \in C(\mathrm{B}(0,R)), h(\mathrm{B}(0,R)) \subset \mathrm{S}(0,R)$, so by Brouwer there $\exists x_0 \in \mathrm{B}(0,R) : h(x_0 = x_0 \Rightarrow -R\frac{g(x_0)}{|g(x_0)|} = x_0$. Take the dot product with x_0 and write

$$\underbrace{-R\frac{g(x_0)\cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in \mathrm{S}(0,R),$$

so that is a contradiction. □

Theorem 23 (Existence and uniqueness)**.** *Let $\Omega \in C^{0,1}, u_0 \in W^{1,r}(\Omega), r \in (1,\infty), \{a_i\}_{i=1}^d$ be Caratheodory, coercive and m and let them also satisfy the growth conditions. Finally, let $f \in \left(W^{1,r}(\Omega)\right)^*$. Then, there is a weak solution to the problem. If, moreover, $\{a_i\}_{i=1}^d$ is strictly monotone, then the weak solution is unique.*

Proof. The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

Galerkin: Since $\mathrm{W}_0^{1,r}(\Omega)$ is separable $\Rightarrow \exists\{w_i\}_{i=1}^\infty$ that is a dense[29] linearly independent subset of $\mathrm{W}_0^{1,r}(\Omega)$. We search for $n \in \mathbb{N}$ such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where $\alpha_j \in \mathbb{R}$ and u^n satisfy

$$\forall j \in \{1,\dots,n\} : \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla w_j + a_0(\cdot,u^n,\nabla u^n) w_j \,\mathrm{d}x =< f, w_j > .$$

We claim such $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$ exist $\forall n \in \mathbb{N}$ by the previous lemma. We define a vector function

$$F(\alpha^n) := \{\int_\Omega a\cdot\nabla w_j + a_0 w_j \,\mathrm{d}x - < f, w_j >\}_{j=1}^n,$$

[29] It can be chosen such that it is itself dense, not only its span

from Nemystkii $F:\mathbb{R}^n \to \mathbb{R}^n, F$ is continuous on $\mathbb{R}^n$. Moreover, it holds

$$
\begin{aligned}
F(\alpha^n)\cdot\alpha^n &\geq \int_\Omega a(\cdot,u^n,\nabla u^n)\nabla(u^n-u_0) + a_0(u^n-u_0)\,\mathrm{d}x - < f, u^n - u_0 > \\
&\underbrace{\geq}_{\text{coercivity}} \int_\Omega C_1|\nabla u^n|^r - (C_2(\cdot) + |a||\nabla u_0| + |a_0||u_0|)\,\mathrm{d}x - \|u^n\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*)} - \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*)},
\end{aligned}
$$

together with the fact

$$
\|\nabla u^n\|^r_{\mathrm{L}_r(\Omega)} \geq \Big(|\|\nabla(u-u_0)\|_{\mathrm{L}_r(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big)^r \geq \|\nabla(u^n-u_0)\|^r_{\mathrm{L}_q(\Omega)} - \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} \geq C\|u^n-u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)},
$$

Next, realize that $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}$ is a norm equivalent to $|\alpha^n|$ (Euclidian norm). So that means $\exists K_1(n) > 0 : \forall\alpha\in\mathbb{R}^n : K_1(n)|\alpha^n| \leq \|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)}$. For $|\alpha^n| = R, R > 0$ determined later estimate $F(\alpha^n)\cdot\alpha^n \geq c\|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)} - \tilde{c}\Big(\|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + 1 + \|u_0\|^r_{\mathrm{L}_r(\Omega)} + \|f\|^{r'}_{(\mathrm{W}_0^{1,r}(\Omega))^*}\Big)$ (which is not a trivial computation). And so $\exists R > 0, \forall\alpha^n \in \mathrm{S}(0,R) \subset \mathbb{R}^n : F(\alpha^n)\cdot\alpha^n > 0$, so from the previous lemma $\exists\alpha^n \in \mathrm{S}(0,R) : F(\alpha^n) = 0$, and we fix these α^n. **Uniform estimates** They follow from the previous manipulation:

$$
\|u^n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),
$$

and

$$
\|u^n\|_{\mathrm{W}^{1,r}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),
$$

$$
\forall j \in \{0,\dots,d\} : \|a_j(\cdot,u^n,\nabla u^n)\|^{r'}_{\mathrm{L}_{r'}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),
$$

Limit passage From the separability of the spaces, we can extract sequences (not renamed):

$$
u^n \rightharpoonup u \,\mathrm{in}\, \mathrm{W}^{1,r}(\Omega)\,, a_j \rightharpoonup \alpha_j \,\mathrm{in}\, \mathrm{L}_{r'}(\Omega)\,.
$$

We pass to the limit in the estimates and are able to write:

$$
\forall j \in \mathbb{N} : \int_\Omega \alpha\cdot\nabla w_j + \alpha_0 w_j\,\mathrm{d}x =< f, w_j >,
$$

and from density of $\{w_j\}_{j\in\mathbb{N}}$ in $\mathrm{W}^{1,r}(\Omega)$ we have

$$
\forall\varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega \alpha\cdot\nabla\varphi + \alpha_0\varphi\,\mathrm{d}x =< f.\varphi > .
$$

Identification of α's We want to show $\alpha_j = a_j(\cdot,u,\nabla u), j \in \{0,\dots,d\}$. For that, we use the *Minty trick*:

$$
\begin{aligned}
0 &\leq \int_\Omega (a(\cdot,u^n,\nabla u^n) - a(\cdot,v,V))\cdot(\nabla u^n - V) + (a_0(\cdot,u^n,\nabla u^n) - a_0(\cdot,v,V))\cdot(u^n - v) \\
&\leq \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla u^n + a_0(\cdot,u^n,\nabla u^n)\cdot u^n\,\mathrm{d}x + \\
&- \int_\Omega (a(\cdot,u^n,\nabla u^n)V + a_0(\cdot,u^n,\nabla u^n)v - a(\cdot,v,V) + a_0(\cdot,v,V)\cdot(u^n - v))\,\mathrm{d}x\,.
\end{aligned}
$$

Denote

$$I^n = \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla(u^n-u_0)+a_0(\cdot,u^n,\nabla u^n)\cdot(u^n-u_0))\,\mathrm{d}x+\int_\Omega a(\cdot,u^n,\nabla u^n)\cdot u_0+a_0(\cdot,u^n,\nabla u^n)u_0\,\mathrm{d}x\,,$$

by using the equation we obtain

$$I^n = < f,u^n-u_0 > + \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot u_0+a_0(\cdot,u^n,\nabla u^n)u_0\,\mathrm{d}x \to < f,u-u_0 > + \int_\Omega \alpha\nabla u_0+\alpha_0 u_0\,\mathrm{d}x = \int_\Omega \alpha\nabla u+\alpha_0 u\,\mathrm{d}x\,,$$

as the rest has subtracted. In total, we have

$$0 \le \int_\Omega (\alpha - a(\cdot,v,V))\cdot(\nabla u - V) + (\alpha_0 - a_0(\cdot,v,V))(u-v)\,\mathrm{d}x\,.$$

So far, v, V have been arbitrary. If we take

$$V = \nabla u - \lambda\psi, \psi \in \mathrm{L}_r(\Omega)\,, v = u,$$

then $0 \le \int_\Omega (\alpha - a(\cdot,\nabla u+\lambda\psi))\lambda\psi\,\mathrm{d}x\,,$ so if we take $\lambda > 0$ and pass to the limit $\lambda \to 0_+$ (using Nemytskii theorem) we can write

$$0 \le \int_\Omega (\alpha - a(\cdot,u,\nabla u))\psi\,\mathrm{d}x\,.$$

Since ψ was arbitrary, we could have taken $\psi \to -\psi$, which in total means

$$0 = \int_\Omega (\alpha - a(\cdot,u,\nabla u))\psi\,\mathrm{d}x$$

Finally, from the previous results, we obtain

$$\forall\varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega a(\cdot,u,\nabla u)\nabla\varphi + a_0(\cdot,u,\nabla u)\varphi\,\mathrm{d}x = < f,\varphi >,$$

and we are almost done. It only remains to show the traces are correct, bt since $u^n \rightharpoonup u$ in $\mathrm{W}^{1,r}(\Omega)$ and from the continuity of the traces, we obtain

$$\operatorname{tr} u = \operatorname{tr} u_0.$$

Uniqueness: Let u_1, u_2 be two solutions. Use strict monotonicity, subtract the weak formulation and test with $u_2 - u_1$:

$$\int_\Omega \underbrace{(a(\cdot,u_2,\nabla u_2) - a(\cdot,u_1,\nabla u_1))\cdot\nabla(u_2-u_1) + (a_0(\cdot,u_2,\nabla u_2) - a_0(\cdot,u_1,\nabla u_1))(u_2-u_1)}_{:=T}\,\mathrm{d}x = 0,$$

where $T \ge 0$, so from strict monotonicity we obtain $T = 0$ a.e. in Ω but that means $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$, a.e. in $\Omega \Rightarrow u_1 = u_2$ in $\mathrm{W}^{1,r}(\Omega)$. □

Example (Nonlinearities vs weak convergence)**.** Let $g_n(x) = \sin(nx)$, then $g \rightharpoonup 0$ in $\mathrm{L}_2((0,4))$ (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx)\varphi\,\mathrm{d}x \ge \int_2^4 \sin^2(nx)\,\mathrm{d}x \to \frac{1}{2} \ne 0, \forall\varphi \in \mathrm{L}_2((0,4))\,,$$

so $\{u_n^2\} = \{\sin^2(nx)\}$ **does not converge weakly to** $0 = 0^2$.

Remark. The method of the presented proof is ***very important.***

Theorem 24. *Let* $\Omega \in C^{0,1}$. *Let* $X = W_0^{1,r}(\Omega), r \in (1,\infty)$ *with equivalent norm* $|||u||| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$. *Then*

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \ s.t. : \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_\Omega \mathbf{F} \cdot \nabla\varphi \,\mathrm{d}x \,, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

Proof. We solve the problem

$$\begin{cases} -\nabla \cdot \left(|\nabla u|^{r-2}\nabla u\right) = \Phi, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega \end{cases}. \tag{2}$$

Such $u \in \mathrm{W}_0^{1,r}(\Omega)$ exists and is unique by the above theorem. In this case: $a(x,z,p) = |p|^{r-2}p, a_0() = 0$. Coercivity is clear, monotonicity will be shown in the tutorials[30]. Write the weak formulation of the above problem:

$$\forall\varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega |\nabla u|^{r-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \Phi(\varphi).$$

Set $\mathbf{F} = |\nabla u|^{r-2}\nabla u$, and test the weak formulation with u itself:

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^r = \Phi(u) \le \|\Phi\|_{X^*}\|\nabla u\|_{\mathrm{L}_r(\Omega)}.$$

If now $\|\nabla u\|_{\mathrm{L}_r(\Omega)} = 0$, then $\Phi = 0$ and we are finished, if it is nonzero, then

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} \le \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} = \left\||\nabla u|^{r-1}\right\|_{\mathrm{L}_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \le \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{\mathrm{B}_X(0,1)} |\Phi(\varphi)| = \sup_{\mathrm{B}_X(0,1)} |\int_\Omega \mathbf{F} \cdot \nabla\varphi| \,\mathrm{d}x \le \sup_{\mathrm{B}_X(0,1)} \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}\|\nabla\varphi\|_{\mathrm{L}_r(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)},$$

so $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}$. □

5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset \mathrm{W}^{1,r}(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega F(\cdot, u, \nabla u) \,\mathrm{d}x \,,$$

with the basic assumptions $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + \mathrm{W}_0^{1,r}(\Omega), u_0 \in \mathrm{W}^{1,r}(\Omega), F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in \mathrm{L}_1(\Omega), \text{a.e.}\, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x,z,p) \ge C_1|p|^r - c_2(x).$$

[30] This was a lie

Remark. From the assumptions it follows $\int_\Omega F(\cdot,u,\nabla u)\,\mathrm{d}x$ is defined $\forall u \in \mathrm{W}^{1,r}(\Omega)$.

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

Lemma 11. *Let $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + W_0^{1,r}(\Omega), u_0 \in W^{1,r}(\Omega), F$ Caratheodory. Moreover, let the following condition hold*

$$\exists C>0, h \in L_1(\Omega) : \forall\ a.a\,x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x,z,p)|+|\partial_z F(x,z,p)| \le C(|z|^r + |p|^r)+|h(x)|, F(x,\cdot,\cdot) \in C^1\big(\mathbb{R}^{d+1}\big).$$

Let now $u \in u_0 + W_0^{1,r}(\Omega)$ be a local minimizer of I over X, i.e.,

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_\Omega F(\cdot,u,\nabla u)\,\mathrm{d}x \le \int_\Omega F(\cdot,u+v,\nabla(u+v))\,\mathrm{d}x\,, F(\cdot,u,\nabla u) \in L_1(\Omega)\,.$$

Then u is the weak solution to the ***Euler-Lagrange equations****:*

$$\begin{cases} -\nabla\cdot(\nabla_p F(\cdot,u,\nabla u) + \partial_z F(\cdot,u,\nabla u)) = 0, & in\,\Omega \\ u = u_0, & on\,\partial\Omega \end{cases},$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \nabla_p F(\cdot,u,\nabla u)\cdot\nabla\varphi + \partial_z F(\cdot,u,\nabla u)\varphi\,\mathrm{d}x = 0, \operatorname{tr} u = \operatorname{tr} u_0\ on\,\partial\Omega.$$

Proof. First $\operatorname{tr} u = \operatorname{tr} u_0$ holds, so we are fine. Now fix $\varphi \in \mathcal{D}(\Omega)$ and define

$$\iota : \mathbb{R} \to \mathbb{R}^*, \iota(\tau) = \int_\Omega \underbrace{F(\cdot,u+\tau\varphi,\nabla(u+\tau\varphi))}_{:=l(\tau,\cdot)}\,\mathrm{d}x\,.$$

Now ι has a local minimum in 0. We show that $\iota'(0)$ exists and is equal to the of Euler-Lagrange equations.

- $l(\tau,\cdot)$ is measurable for τ from some neighbourhood of 0.
- $l(\tau,\cdot)$ is differentiable

Moreover

$$\begin{aligned}\partial_\tau l(\tau,\cdot) &= \partial_z F(\cdot,u+\tau\varphi,\nabla(u+\tau\varphi))\varphi + \nabla_p F(\cdot,u+\tau\varphi,\nabla(u+\tau\varphi))\cdot\nabla\varphi = \\ &= \partial_z F(\cdot,u+\tau\varphi,\nabla(u+\tau\varphi))\varphi + \nabla_p F(\cdot,u+\tau\varphi,\nabla(u+\tau\varphi))\cdot\nabla\varphi.\end{aligned}$$

Also

$$i(0) = \int_\Omega F(\cdot,u,\nabla u)\,\mathrm{d}x < \infty$$

and

$$|\partial_\tau l(\tau,\cdot)| \le (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla\varphi|^r) + |h(x)|)(|\varphi| + |\nabla\varphi|) \in \mathrm{L}_1(\Omega)\,.$$

Altogether, $\iota(\tau)$ is finite on $(-1,1), \iota'(\tau)$ exists and

$$\iota'(0) = \int_\Omega \partial_z F(\cdot,u,\nabla u)\varphi + \nabla_p F(\cdot,u,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,.$$

$\square$

Example. Let

$$F(x,z,p) = \frac{1}{r}(1) + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x,z,p) = \left(\frac{r}{2}\frac{1}{r}2\left(1+|p|^2\right)^{\frac{r-2}{2}}\right)p - G = \left(1+|p|^2\right)^{\frac{r-2}{2}}p - G, \partial_z F(x,z,p) = -g.$$

We have

$$|\left(1+|p|^{2\frac{r-2}{2}}\right)p| \le \left(1+|p|^2\right)^{\frac{r-2}{2}}\left(1+|p|^2\right)^{\frac{1}{2}} = \left(1+|p|^2\right)^{\frac{r-1}{2}} \le C(1+|p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla\cdot\left(\left(1+|\nabla u|^2\right)^{\frac{r-2}{2}}\nabla u\right) = -\nabla\cdot G + g, & \text{in}\,\Omega \\ u = u_0, & \text{on}\,\partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall\varphi\in\mathcal{D}(\Omega): \int_\Omega \left(1+|\nabla u|^2\right)^{\frac{r-2}{2}}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega (G\cdot\nabla\varphi + g\varphi)\,\mathrm{d}x\,.$$

Remark. We have $\{u_n\}\subset X$ s.t. $\lim_{n\to\infty} I(u_n) = \inf_X I$. Then use

- compacntess: $u_n \to u$ in some sonse (weak convergence)
- weak lower semicontinuity $I(u) \le \liminf_{n\to\infty} I(u_n)$

Lemma 12. *Let $F:\mathbb{R}^N\to\mathbb{R}, F\in C^1\left(\mathbb{R}^N\right), N\in\mathbb{N}$. Then*

1. *F is (strictly) convex $\Leftrightarrow$ ∇F is (strictly) monotone*
2. *If F is (strictly) convex, then*

$$\forall\xi_1,\xi_2\in\mathbb{R}^N, \xi_1\ne\xi_2: F(\xi_1) - F(\xi_2) \ge \nabla F(\xi_2)\cdot(\xi_1-\xi_2).$$

Proof. Fix $\xi_1,\xi_2,\xi_1\ne\xi_2$, define $\varphi(t) = F(\xi_2 + t(\xi_1-\xi_2))$. Then $\varphi\in C^1(\mathbb{R})$ and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1-\xi_2))\cdot(\xi_1-\xi_2).$$

So

$$" \Rightarrow ": (\nabla F(\xi_1) - \nabla F(\xi_2))\cdot(\xi_1-\xi_2) = \varphi'(1) - \varphi'(0) \underbrace{\ge}_{\varphi\text{convex or strictly convex}} 0.$$

And $" \Leftarrow "$: Fix $t_1 > t_2$ and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1-\xi_2)) - \nabla F(\xi_2 + t_2(\xi_1-\xi_2)))\cdot(\xi_1-\xi_2)(t_1-t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1-\xi_2)(t_1-t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2))\cdot(\eta_1-\eta_2)$$

and we are in the same situation as before. For 2) we already know F (strictly) convex $\Rightarrow \varphi$ (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \to \varphi'(0),$$

as $t \to 0_+$. And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

Theorem 25. *Let $M, N \in \mathbb{N}, \Omega$ open, $F : \Omega \times \mathbb{R}^{N+M} \to \mathbb{R}$ Caratheodory, F convex in $p \in \mathbb{R}^n$, i.e. $\forall$ a.e. $x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot)$ is convex and $\exists c_2 \in L_1(\Omega), \forall$ a.e. $x \in \Omega, \forall z \in \mathbb{R}^M, \forall p \in \mathbb{R}^N : F(x, z, p) \geq c_2(x)$. Let $u_n \to u$ in $L_1(\Omega), U_n \rightharpoonup U$ in $L_1(\Omega)$. Then*

$$\int_\Omega F(\cdot, u, U) \, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega F(\cdot, u_n, U_n) \, \mathrm{d}x \, .$$

Proof. The proof will be given only if moreover $\forall$ *a.e.* $x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot) \in C^1(\mathbb{R}^N)$. Idea: by the previous lemma:

$$\int_\Omega F(\cdot, u_n, U_n) \, \mathrm{d}x \geq \int_\Omega (F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U) \cdot (U_n - U)) \, \mathrm{d}x \, ,$$

and we have uniform convergence in the first term and second term and weak convergence in $\mathrm{L}_1(\Omega)$ in the last term. If Ω is bounded, we can find $K_k \subset K_{k+1} \subset \Omega \, s.t. \, \lambda(\Omega \setminus \bigcup_{k\in\mathbb{N}} K_k) = 0$, and moreover $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$ are compact, $u_n \rightrightarrows u$ on K_k, $\|u\|_{\mathrm{L}_\infty(K_k)} + \|U\|_{\mathrm{L}_\infty(K_k)} \leq k$ up to a subsequence. We can now extract a subsequence $u_n \to u$ *a.e.* and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E}_k \, s.t. \, u_n \rightrightarrows u \, \text{on} \, \tilde{E}_k \wedge \lambda\big(\Omega \setminus \tilde{E}_k\big) < \frac{1}{k}.$$

Now define

$$\hat{E}_k = \bigcup_{j=1}^k \tilde{E}_j, E_k = \hat{E}_k \cap \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}\},$$

and E_k satisfy [31]

$$\lambda\left(\Omega \setminus \bigcup_k E_k\right) = 0.$$

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \leq k \wedge |U(x)| \leq k\}$$

and we also have $\lambda(\Omega \setminus \bigcup_k F_k) = 0$. FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\left(\Omega \setminus \bigcup_k K_k\right) = 0.$$

□

Remark.

- if $U_n \to U$ strongly $\Rightarrow u_n \to u, U_n \to U$ *a.e.* (up to a subsequence) and the claim follows from the Fatou lemma. [32]

[31] "This is homework", says doc. Kaplicky

[32] For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions $F - c_2 \geq 0, F - c_2 \in \mathrm{L}_1(\Omega)$

- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \, \text{in} \, \mathrm{L_p}(\Omega) \Rightarrow \int_\Omega |\nabla u|^p \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x\,.$$

6 Evolutionary equations

6.1 Embedding theorems for Sobolev-Bochner spaces

Lemma 13 (Arzela-Ascoli)**.** *Let X, Y be Banach spaces, $X \subset\subset Y$. Then*

$$C^1([0,T];X) \subset\subset C([0,T];Y).$$

Lemma 14 (Ehrling)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3$. Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + C\|u\|_{V_3}.$$

Proof. By contradicition, assume

$$\exists \varepsilon > 0 \, s.t. \, \forall n \in N \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n\|u_n\|_{V_3}.$$

WLOG we can assume $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$: truly, the inequaility is 1-homogenous and holds if $u_n = 0$. In particular we see $\|u_n\|_{V_3} < \frac{1}{n}$, so $u_n \to 0$ in V_3. Moreover, $\{u_n\}$ is bounded in V_1 and since $V_1 \subset\subset V_2$ there exists $\{u_{n_k}\} \subset \{u_n\}$ *s.t.*: $u_{n_k} \to u$ in V_2 strongly. Since $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$, also $\|u\|_{V_2} = 1$. Finally, taking the limit passage yields $0 \geq \|u\|_{V_3}$ and so $u = 0$ in V_3 and also in V_2. But that is a contradiction with the fact $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$. □

Theorem 26 (Aubin-Lions)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3, p \in [1,\infty)$. Then the space*

$$\mathcal{U} = \{u \in L_p\Big((0,T);V_1\Big), \partial_t u \in L_1\Big((0,T);V_3\Big)\},$$

with the norm

$$|\|u\|| = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

satisfies

$$\mathcal{U} \subset\subset L_p\Big((0,T);V_2\Big).$$

Proof. Strategy: I want to fix $M \subset \mathcal{U}$ bounded and show that it is precompact in $\mathrm{L_p}\Big((0,T);V_2\Big)$. That will be done in the following way:

1. Mollify M by convolution
2. use Arzela-Ascoli
3. show compactness in $\mathrm{L_p}\Big((0,T);V_3\Big)$
4. apply Ehrling lemma and show compactness in $\mathrm{L_p}\Big((0,T);V_2\Big)$.

Fix $M \subset \mathcal{U}$ bounded. Then $\exists C^* > 0 : \forall u \in M : |\|u\|| \geq C^*$.

Next, take

$$\varphi : \mathbb{R} \to [0, \infty), \varphi \in C^\infty(\mathbb{R}), \operatorname{supp} \varphi \subset (-1, 0), \int_{\mathbb{R}} \varphi \, \mathrm{d}x = 1,$$

a regularization kernel, then $\forall \delta > 0$ define $\varphi_\delta(t) :== \frac{1}{\delta} \varphi\big(\frac{t}{\delta}\big)$.

Now, extend functions from M to $(0, 2T)$ in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0, T) \\ u(2T - t), & t \in (T, 2T) \end{cases}.$$

Now mollify: for $\delta > 0, \delta < T$ fixed define

$$M_\delta = \{ (\tilde{u} \star \varphi_\delta)\Big|_{(0,T)} |u \in M\}.$$

From the properties of regularization it follows $M_\delta \subset C^1([0, T]; V_1) \underbrace{\subset\subset}_{\text{A.A.}} C([0, T]; V_2) \subset \mathrm{L}_\mathrm{p}\Big((0, T); V_2\Big)$.

Now estimate the distance of M and M_δ in $\mathrm{L}_\mathrm{p}\Big((0, T); V_3\Big)$: for

$$u \in M, t \in (0, T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^{0} \tilde{u}(t - s)\varphi_\delta)(s) \, \mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t - s))\varphi_\delta(s) \, \mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t - s)) \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t - s)) \int_{-\delta}^{s} \varphi_\delta(\sigma) \, \mathrm{d}\sigma \Big|_{s=-\delta}^{0} - \int_{-\delta}^{0} \frac{\mathrm{d}}{\mathrm{d}s} (\tilde{u}(t) - \tilde{u}(t - s)) \int_{-\delta}^{s} \varphi_\delta(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}s \,,$$

since the first bracket is 0 and by denoting the first term in the second integrand by $\tilde{u'}(t - s)$ this becomes (using Fubini)

$$= - \int_{-\delta}^{0} \int_{\sigma}^{0} \tilde{u'}(t - s) \, \mathrm{d}s \, \varphi_\sigma(\sigma) \, \mathrm{d}\sigma \,,$$

and we see

$$\|\tilde{u}(t) - \tilde{u_\delta}(t)\|_{V_3} \leq \int_{-\delta}^{0} \int_{\sigma}^{0} \left\|\tilde{u'}(t - s)\right\|_{V_3} \mathrm{d}s \, \varphi_\sigma(\sigma) \, \mathrm{d}\sigma \,.$$

$\mathrm{L}_1\Big((0, T); V_3\Big)$ estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3} \, \mathrm{d}t \leq \int_0^T \int_{-\delta}^{0} \int_{\sigma}^{0} \left\|u'(\tilde{t - s})\right\|_{V_3} \mathrm{d}s \, \varphi_\delta(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}t \leq 2\delta \|u'\|_{\mathrm{L}_1((0,T);V_3)} \leq 2\delta C^*$$

$\mathrm{L}_\infty\Big((0, T); V_3\Big)$ estimate:

$$\|u - u_\delta\|_{\mathrm{L}_\infty((O,T);V_3)} \leq 2\|u'\|_{\mathrm{L}_1((0,T);V_3)} \leq 2C^*$$

It remains to show $M_\delta \subset\subset \mathrm{L}_\mathrm{p}\Big((0, T); V_2\Big)$:

$$\|u - u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_3)} \leq \|u - u_\delta\|_{\mathrm{L}_1((0,T);V_3)}^{1/p} \|u - u_\delta\|_{\mathrm{L}_\infty((O,T);V_3)}^{1-1/p} \leq 2C^* \delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le \mu \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_1)} + C_\mu \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_3)}.$$

This means

$$\forall u \in M : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le C^* + C\mu 2C^* \delta^{1/p}.$$

Now fix $\varepsilon > 0$ and find

$$\mu > 0 : \mu C^* < \frac{\varepsilon}{2}, \delta > 0, C_\mu 2C^* \delta^{1/p} < \frac{\varepsilon}{2} \Rightarrow \forall u \in M : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} < \varepsilon.$$

This means $\exists \{w_k\}_{k=1}^N \subset M : \{(w_k)_\delta\}_{k=1}^n$ is ε–net in M in $\mathrm{L_p}\Big((0,T);V_2\Big)$. If we now fix $u \in M$, then

$$\exists K \in \{1, \dots, N\} : \left\| u_{\delta - (w_K)_\delta} \right\|_{\mathrm{L_p}((0,T);V_2)} < \varepsilon.$$

□

Remark. The pair $(\mathcal{U}, |||\cdot|||)$ is a Banach space.

6.2 Nonlinear parabolic equations

We will be dealing with the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot a(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = f & \text{in } (0,T) \times \Omega, \\ u = u_0, & \text{on } \{0\} \times \Omega, \\ u = 0, & \text{on } (0,T) \times \partial\Omega \end{cases} .$$

The unknown is the function $u : (0,T) \times \Omega \to \mathbb{R}$, and we are given $\Omega \in C^{0,1}, T > 0, Q_T = (0,T) \times \Omega, f : Q \to \mathbb{R}$ or $f : (0,T) \to X$ a Banach space, $u_0 : \Omega \to \mathbb{R}, a : \Omega \times \mathbb{R}\mathbb{R}^d \to \mathbb{R}^d, a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions: $\exists r > 1, \exists C > 0 :$ *a.e.* $x \in \Omega, \forall (z,p) \in \mathbb{R}^{d+1} : |a_0(x,z,p)| + |a(x,z,p)| \le C(1 + |z|^{r-1} + |p|^{r-1})$ and $\exists C_1, C_2 > 0, q \in (1, \max(2,r))$ *a.e.* $x \in \Omega, \forall (z,p) \in \mathbb{R}^{d+1} : a(x,z,p)p + a_0(\dots)z \ge C_1|p|^r - C_2(1 + |z|q)$.

Theorem 27. *Let $\Omega \in C^{0,1}, a, a_0$ satisfy growth conditions and coercivity, let $\{a_i\}_{i=0}^d$ be monotone. Denote $V = W_0^{1,r}(\Omega) \cap L_2(\Omega)$. Then $\forall f \in L_{r'}\Big((0,T);V^*\Big), u_0 \in L_2(\Omega) \, \exists u \in L_r\Big((0,T);V\Big)$ s.t. $\partial_t u \in L_{r'}\Big((0,T);V^*\Big), u \in C([0,T];L_2(\Omega)), u(0) = u_0$ and moreover*

$$a.e.\, t \in (0,T), \forall \varphi \in V :< \partial_t u, \varphi > + \int_\Omega a(\cdot, u, \nabla u)\nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \, \mathrm{d}x =< f, \varphi > .$$

Finally, the solution is unique.

Proof. The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences

4. limit passage
5. identification of the limits

Rothe method: Fix $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$ and approximate the derivative with

$$\partial_t u(t,x) \approx \frac{1}{h}(u(t,x) - u(t-h,x)).$$

Define $u_0 = u_0, u_{k+1} \in V$ as a solution of

$$\frac{1}{h}(u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1,}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \,\mathrm{in}\, \Omega, u_{k+1} = 0 \,\mathrm{on}\, \partial\Omega.$$

Define

$$f_{k+1} := \int_{kh}^{(k+1)h} f \,\mathrm{d}t\,,$$

then the weak formulation becomes

$$\int_\Omega \frac{u_{k+1} - u_k}{h}\varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla\varphi a_0(\cdot, u_{k+1}, \nabla u_{k+1})\varphi \,\mathrm{d}x =< f_{k+1}, \varphi > .$$

We *claim without a proof* that the solutions $\{u_k\}_{k=0}^n \subset V$ exist.

To obtain a-priori estimates, tes the equation with u_{k+1}. This yields:

$$\int_\Omega |u_{k+1}|^2 - u_k u_{k+1} \,\mathrm{d}x = \int_\Omega \frac{1}{2}|u_{k+1}|^2 + \frac{1}{2}(u_{k+1} - u_k) - \frac{|u_k|^2}{2} \,\mathrm{d}x \Rightarrow \sum_{k=0}^{j-1} \int_\Omega |u_{k+1}|^2 - u_{k+1}u_k \,\mathrm{d}x = \frac{1}{2}\|u_j\|^2_{\mathrm{L}_2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} (u_{k+}$$

so

$$\int_\Omega a(\dots)\nabla \cdot u_{k+1} + a_0(\dots)u_{k+1} \,\mathrm{d}x \ge C_1 \int_\Omega |\nabla u_{k+1}|^r \,\mathrm{d}x - C_2 \int_\Omega (1 + |u_{k+1}|^q) \,\mathrm{d}x\,,$$

$< f_{k+1}, u_{k+1} >\le \|f_{k+1}\|_{V^*}\Big(\|u_{k+1}\|_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|_{\mathrm{L}_2(\Omega)}\Big) \le \varepsilon\Big(\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)}\Big) + C\Big(\|f_{k+1}\|^{r'}_{V^*} + \|f_{k+1}\|^2_{V^*}\Big).$

So together $\|u_j\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1}\Big[(u_{k+1} - u_k)^2 + h\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)}\Big] \le C\Big(\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\Big(\|f\|^{r'}_{V^*}\|f\|^2_{V^*}\Big)\Big)$

□

Let us now define $u^n(t) = u_k$ for $t \in (h(k-1), hk)$, then

$$\|u^n\|^2_{\mathrm{L}_\infty((O,T);\mathrm{L}_2(\Omega))} + \|u^n\|^2_{\mathrm{L}_2\left((O,T);\mathrm{W}_0^{1,r}(Omega)\right)} < C(\,\mathrm{data}\,).$$

Now set $\tilde{u}^n(t) = u_{k-1} + \frac{t - t_{k-1}}{h}(u_k - u_{k-1})$ for $t \in (t_{k-1}, t_k)$ and

$$k \in \{1, \dots, n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the quation to the form

$$\int_\Omega \partial_t \tilde{u}^n \varphi + a(\cdot, u^n, \nabla u^n) \cdot \nabla\varphi + a_0(\cdot, u^n, \nabla u^n)\varphi \,\mathrm{d}x =< f^n, \varphi >,$$

where $f^n(t) := f_k$ in in
$$(t_{k-1}, t_k), k \in \{1, \dots, \}.$$
We are now ready to use growth and apriori estimates:
$$\|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} \le C(\,\mathrm{data}\,).$$
For the norm of the time derivative:
$$\sup_{\varphi \in \mathrm{S_V}(0,1)} < \partial_t \tilde{u}^n(t), \varphi >= \sup_{\varphi \in \mathrm{S_V}(0,1)} < f^n, \varphi > - \int_\Omega (a(\cdot, u^n, \nabla u^n) \cdot \nabla f + a_0(\cdot, u^n, \nabla u^n)\varphi)\,\mathrm{d}x\,,$$
at any $t \in (0,T)$. So using Holder:
$$\|\partial_t \tilde{u}^n(t)\|_{V^*} \le \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)},$$
and integrating
$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{r'}\,\mathrm{d}t \le C\Bigg(\int_0^T \|f^n\|_{V^*}^{r'} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}, \mathrm{d}t\Bigg) \le TC(\,\mathrm{data}\,)$$

7 Semigroup theory

We consider the equation

$$\begin{aligned} u' &= Au, A \,\text{is linear} \\ u(0) &= u_0, \end{aligned}$$

where $u : [0,\infty) \to \mathbb{R}$. We know that for example if $Au = au, a \in \mathbb{R}$ then
$$u(t) = u_0 e^{at}.$$
If $\mathbf{u} : [0,\infty) \to \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d\times d}$, then
$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{A}^k t^k.$$
This can be extended to $u : [0,\infty) \to X, X$ a banach space, $A \in \mathcal{L}(X)$, then
$$u(t) = \exp(tA)u_0,$$
where the operator exponential is the same. This works well for unbounded operators, but suppose now
$$X = \mathrm{L}_2(\Omega)\,, Au = \triangle\, u.$$
We *guess* the solution should be
$$u(t) = \exp(\triangle\, t)u_0,$$
but what is
$$\exp(\triangle\, t)?$$

Definition 13 (Linear operator and its domain)**.** Let X be a Banach space over $\mathbb{K}$. Linear operator on X is a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace of X and $A : \mathcal{D}(A) \to X$ is linear.

Definition 14. A family $\{S(T)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if

1. $S(0) = \mathrm{id}$
2. $\forall s, t \geq 0 : S(t)S(s) = S(t+s)$.

If moreover $\forall x \in X : S(t)x \to x$, as $t \to 0_+$, we call $\{S(t)\}$ a c_0- semigroup (strongly continuous).

Remark. $\{s(t)\}_{t\in\mathbb{R}}$ with the two conditions is an Abelian group $(\{S(t)\}_{t\in\mathbb{R}}, \circ)$ with

$$(S(t))^{-1} = S(-t). \tag{3}$$

Remark (X = Banach)*.* **In the following, X is always a Banach space.**

Lemma 15. *Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup in X. Then*

1. $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$,
2. $\forall x \in X, t \mapsto S(t)x \in C([0,\infty); X)$.

Proof. $1 \Rightarrow 2$. Fix $t > 0, x \in X$ compute

$$\lim_{h\to 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t)(S(h)x - x)\|_X \leq \lim_{h\to 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \to 0.$$

now compute $\lim_{h\to 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t-h)(x - S(h)x)\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \to 0$. □

Definition 15 (Infinitesimal generator)**.** A linear operator $(A, \mathcal{D}(A))$ is called a infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$, if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h\to 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X | \lim_{h\to 0_+} \frac{S(h)x - x}{x} \text{ exists in } X \right\},$$

Theorem 28. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ in X. Then*

1. $x \in \mathcal{D}(A) \Rightarrow \forall t \geq 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)x)$,
2. $x \in X \wedge t \geq 0 \Rightarrow x_t = \int_0^t S(s)x \,\mathrm{d}s \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x$.

Proof. Fix $x \in \mathcal{D}(A), t \geq 0$. Calculate

$$\lim_{h\to 0_+} \frac{S(h)S(t)x - S(t)x}{h} =^{33} \lim_{h\to 0_+} S(t)\frac{S(h)x - x}{h} = S(t)Ax,$$

[33] $S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$

(convergence is in the norm of the Banach space X). This means $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$, moreover, if $t > 0$:

$$\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right),$$

estimate,

$$\left\|\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right)\right\| = \lim_{h\to 0_+} \left\|\frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t$$

as $S(t)$ is continous and $S(0) = \mathrm{id}$. Clearly, $t \mapsto S(t)x$ is $C^1([0,\infty))$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h\to 0_+} \frac{1}{h}(S(h)x_t - x_t) = \lim_{h\to 0_+} \frac{1}{h}\left(\int_h^{t+h} S(s)x \,\mathrm{d}s - \int_0^t S(s)x \,\mathrm{d}s\right),$$

realize that

$$S(h)x_t = \int_0^t S(s+h)x \,\mathrm{d}s = \int_h^{t+h} S(s)x \,\mathrm{d}s\,,$$

so the previous computation continues as follows

$$= \lim_{h\to 0_+} \frac{1}{h}\left(\int_t^{t+h} S(s)x \,\mathrm{d}s - \int_0^h S(s)x \,\mathrm{d}s\right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

Definition 16 (Closed operator)**.** We say that a linear operator $(A, \mathcal{D}(A))$ is closed if $\forall \{u_n\} \subset \mathcal{D}(A) : u_n \to u \wedge Au_n \to v$, for some $u, v \in X$, then it most hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that $\{(x, Ax) | x \in \mathcal{D}(A)\} \subset X \times X$ is closed in $(X \times X, \|\cdot\|_1)$.

Example. Let $\Omega \in C^{1,1}, X = \mathrm{L}_2(\Omega), \mathcal{D}(A) = \mathrm{W}^{2,2}(\Omega) \cap \mathrm{W}_0^{1,2}(\Omega), Au = \triangle u$. Then $(A, \mathcal{D}(A)$ is closed. Really, take $\{u_n\} \subset \mathrm{L}_2(\Omega) : u_n \to u$ in $\mathrm{L}_2(\Omega)$ for some $u \in \mathrm{L}_2(\Omega)$. Suppose $Au_n \to v$ in $\mathrm{L}_2(\Omega), v \in \mathrm{L}_2(\Omega)$. Suppose the following equation: find

$$u_n \; s.t. - \triangle u_n = Au_n, u_n \,\mathrm{on}\, \partial\Omega.$$

From the regularity theory for elliptic problems, we know that $\|u_n\|_{\mathrm{W}^{2,2}(\Omega)} \le C\|Au_n\|_{\mathrm{L}_2(\Omega)} \le C$, so we can extract $u_{n_k} \rightharpoonup u$ in $\mathrm{W}^{2,2}(\Omega)$. Realize moreover

$$\int_\Omega \triangle u_n \varphi \,\mathrm{d}x = \int_\Omega u_n \triangle \varphi \,\mathrm{d}x\,, \forall \varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_\Omega v\varphi \,\mathrm{d}x = \int_\Omega u \triangle \varphi \,\mathrm{d}x = \int_\Omega \triangle u\varphi \,\mathrm{d}x\,,$$

which means $\triangle u = v \; a.e. \;\mathrm{in}\, \Omega$ and that $u \in \mathcal{D}(A), Au = v$.

Theorem 29. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0} \subset X$. Then*

1. *$\mathcal{D}(A)$ is dense in X,*
2. *$(A, \mathcal{D}(A)$ is closed.*

Proof. Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t}\int_0^t S(s)x \,\mathrm{d}s \underbrace{\in \mathcal{D}(A)}_{\text{prev. thm}}, \frac{x_t}{t} \to x \,\mathrm{in}\, X,$$

Ad 2.: Take $\{x_n\} \subset \mathcal{D}(A) : x_n \to x$ in $X, Ax \to v \,\mathrm{in}\, X$. Compute[34]

$$\frac{(S(h) - \mathrm{id})x_n}{h} = \frac{1}{h}\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}(S(s)x_n)\,\mathrm{d}s = \frac{1}{h}\int_0^h AS(s)x_n \,\mathrm{d}s = \frac{1}{h}\int_0^h S(s) \underbrace{Ax_n}_{\to v}, \text{ so taking the limit yields } \frac{(S(h) - \mathrm{id})x}{h} =$$

Altogether, $x \in \mathcal{D}(A), Ax = v$. □

Lemma 16. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of c_0-semigroups $\{S(t)\}_{t\geq 0}, \{\tilde{S}(t)\}_{t\geq 0}$. Then*

$$\{S(t)\}_{t\geq 0} = \{\tilde{S}(t)\}_{t\geq 0}.$$

Proof. We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix $x \in \mathcal{D}(A), t > 0$. Then $g(s) := S(s)\tilde{S}(t - x)x$ satisfies $g \in C^1([0,t]; X), g'(s) = S'(s)\tilde{S}(t - s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$, as A, S commute. This means $g(0) = g(1)$ and from this it follows $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X, S$ continous $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$, and since $t \geq 0$ was arbitrary, we are done. □

Definition 17 (Resolvent of a linear operator)**.** Let $(A, \mathcal{D}(A))$ be a linear (possibly unbounded) operator on X. We define

1. resolvent set

$$\rho(A) = \left\{\lambda \in \mathbb{K} | \lambda \,\mathrm{id} - A \,\text{is invertible and}\, (\lambda \mathrm{id} - A)^{-1} \in \mathcal{L}(X)\right\},$$

2. resolvent operator $R(\lambda, A) : X \to \mathcal{D}(A) : R(\lambda, A) = (\lambda \mathrm{id} - A)^{-1}$, for

$$\lambda \in \rho(A).$$

Remark. If $(A, \mathcal{D}(A)$ is a closed linear operator: $\lambda \in \rho(A) \Leftrightarrow \lambda\mathrm{id} - A$ is a bijection of $\mathcal{D}(A)$ onto X.

Lemma 17. *Let $(A, \mathcal{D}(A))$ be a linear operator on X. It holds*

1. *$\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x$,*
2. *$\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x$,*

[34]This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that X is a Banach space and work with some functionals and Bochner integrals or whatever

3. $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda) R(\lambda, A) R(\eta, A)$, *and* $R(\lambda, A) R(\eta, A) = R(\eta, A) R(\lambda, A)$,

4. *If moreover* $(A, \mathcal{D}(A)$ *is the infinitesimal generator of a* c_0*-semigroup* $\{S(t)\}_{t\geq 0}$ *s.t.* $\forall t \geq 0 :$ $\|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$, *then*

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t)\, \mathrm{d}t \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

Remark. The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

Proof. Ad 1.:

$$AR(\lambda, A)x = (A - \lambda \mathrm{id}) \underbrace{R(\lambda, A)}_{=(\lambda \mathrm{id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\mathrm{id} - (\lambda \mathrm{id} - A))R(\eta, A) = R(\lambda, A)(\eta \mathrm{id} - A - \lambda \mathrm{id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For $\lambda \neq \eta$ we also have

$$R(\lambda, A)R(\eta A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG asume $\omega = 0$, meaning $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$. Denote $\tilde{S}(t) = e^{-\omega t} S(t)$. Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x\, \mathrm{d}t\,.$$

First of all, this is well defined as

$$\left\|\tilde{R}x\right\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X\, \mathrm{d}T = \frac{M}{\lambda}\|x\|_X,$$

and so $\left\|\tilde{R}\right\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}, \tilde{R} \in \mathcal{L}(X)$. Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda \tilde{R}x - x \Leftrightarrow \mathrm{id} = (\lambda \mathrm{id} - A)\tilde{R}.$$

For $x \in X, h > 0$ fixed compute

$$\begin{aligned}
\frac{1}{h}\left(S(h)\tilde{R}x - \tilde{R}x\right) &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t} S(t+h)x - e^{-\lambda t} S(t)x\, \mathrm{d}t\right) = \\
&= \frac{1}{h}\left(\int_h^\infty e^{-\lambda(t-h)} S(t)x\, \mathrm{d}t - \int_0^\infty e^{-\lambda t} S(t)x\, \mathrm{d}t\right) = \\
&= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x\, \mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t} S(t)x\, \mathrm{d}t = \\
&= e^{-\lambda t}\frac{e^{\lambda h} - 1}{h} \to \lambda e^{-\lambda t}, \text{ as } h \to 0_+
\end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t)e^{-\lambda t}\frac{e^{h\lambda}-1}{h}S(t)x \to \lambda e^{-\lambda t}S(t)x \,\text{on}\, (0,\infty)\,\text{as}\, h \to 0_+.$$

The norm of this can be estimated as $\left\|\lambda e^{-\lambda t}S(t)x\right\| \le Ce^{-\lambda t}M\|x\|_X \in \mathrm{L}_1((0,\infty))$. Altogether, we obtain $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda\mathrm{id} - A)\tilde{R}x = x$.

To procced further, we need the following theorem:

$$x \in \mathcal{D}(A), A\,\text{closed} : A\tilde{R}x = A\left(\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \int_0^\infty e^{-\lambda t}\underbrace{AS(t)}_{=S(t)A}x\,\mathrm{d}t = \tilde{R}Ax,$$

which has been stated but not proved [35]. Finally, we can write: $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda\mathrm{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$. Moreover, we have also shown the mapping is a bijection. □

Definition 18 (Contraction semigroup)**.** We say that $\{S(t)\}_{t\ge 0}$ is a contraction semigroup if

$$\forall t \ge: \|S(t)\|_{\mathcal{L}(X)} \le 1.$$

Theorem 30 (Hille-Yosida)**.** *Let $M \ge 1, \omega \in \mathbb{R}$. A linear $(A, \mathcal{D}(A))$ on a Banach space X generates a c_0-semigroup (meaning it is its infinitesimal generator) satysfing $\forall t \ge 0 : \|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$* ***if and only if***

1. *$(A, \mathcal{D}(A))$ is closed,*
2. *$\mathcal{D}(A)$ is dense in X,*
3. *$\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{(\lambda-\omega)^n}$.*

Proof. If $M = 1, \omega = 0$, then $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda^n}$. " $\Rightarrow$ " has been proven, now show the other direction. The plan is to

1. approximate A by $\{A_n\} \subset \mathcal{L}(X)$,
2. construct S_n for A_n as previously,
3. estimate and limit passage.

Approximation: See the analogy: $a \in \mathbb{R} : \frac{n}{n-a} \to 1$, we would like $nR(n, A) \to \mathrm{id}$. Calculate the norm of $nAR(n, A) = n(nR(n, A) - \mathrm{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$, (This approx. is called the Yosida approximation.) For $x \in \mathcal{D}(A)$ fixed:

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \le \|R(n, A)\|_{\mathcal{L}(X)}\|Ax\|_X \le \frac{1}{n}\|Ax\|_X \to 0\,\text{as}\, n \to \infty.$$

If

$$y \in X : \|nR(n, A)y - y\|_X \le \|nR(n, A)(y - x)\|_X + \|nR(n, A)x - x\|_X + \|x - y\|_X \le 2\|y - x\| + \underbrace{\|nR(n, a)x - x\|_X}_{\to 0},$$

[35] It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

but $\|y-x\|_X$ can be made arbitrarily small from density of $\mathcal{D}(A)$ in X, so in fact

$$nR(n,A)y \to y \,\text{in}\, X, \forall y \in X.$$

And so $nR(n,A)$ really approximates id.

Using this gives us

$$\forall x \mathcal{D}(A) : A_n x = nAR(n,A)x = n\,\overbrace{R(n,A)A}^{=AR(n,A)}\,x \to Ax \,\text{in}\, X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \le \left\| \sum_{k=0}^{\infty} \frac{1}{k!}(tA_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!}\left(-nt\mathrm{id} + n^2 tR(n,A)\right)^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!}(-nt\mathrm{id})^k \sum_{k=0}^{\infty} \frac{\left(n^2 tR(n,A)\right)^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives $\le e^{-nt}\,\mathrm{id}\,\sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n,A)\|_X^k \le e^{-nt}e^n t = 1$, as $\|nR(n,A)\|^k \le 1$. This means $\{S_n(t)\}_{\mathcal{L}(X)} \le 1$.

Now show that this converges: fix $x \in \mathcal{D}(A)$, compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_n(s)S(m)(t-s)x)\,\mathrm{d}s \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x\,\mathrm{d}s \right\|_X \underbrace{\le}_{\|S_l\|_{\mathcal{L}(X)} \le 1} t\|(A_n - A_m)$$

and since X is Banach, it is convergent also. Finally, for $y \in X$, we have

$$\|S_n(t)y - S_m(t)y\|_X \le \|S_n(t)(y-x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x-y)\|_X \le 2\|x-y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that $\{S_n(t)y\}$ is Cauchy uniformly on $[0,T], T > 0 \Rightarrow \exists S(t) : S_n(t)y \to S(t)y \forall y \in X, t > 0$. And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain $\{S(t)\}_{t\ge0}$ is a c_0-semigroup.

It remains to answer this question. Is $(A, \mathcal{D}(A))$ the infinitesimal generator of $\{S(t)\}_{t\ge0}$? Let $(\tilde{A}, \mathcal{D}(\tilde{A})$ be the infinitesimal generator of $\{S(t)\}_{t\ge0}$. Compute

$$S_n(t)x - x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} S_n(s)x\,\mathrm{d}s = \int_0^t S_n(x)A_n x\,\mathrm{d}s\,,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \le \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \to 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \,\mathrm{d}s\,.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t\to 0_+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A}\,\mathrm{on}\,\mathcal{D}(A).$$

The opposite inclusion is simple: fix $\lambda > 0 : \lambda \in \rho(A) \cap \rho\big(\tilde{A}\big)$, and so $\lambda\mathrm{id} - A : \mathcal{D}(A) \to X$ is onto, but also $\lambda\mathrm{id} - A = \lambda\mathrm{id} - \tilde{A}$ on $\mathcal{D}(A)$, and so $\lambda\mathrm{id} - \tilde{A} : \mathcal{D}(A) \to X$ is onto. From the previous theorem, we know $\lambda\mathrm{id} - \tilde{A}$ is one-to-one, so $\mathcal{D}\big(\tilde{A}\big) = \mathcal{D}(A)$. Altogether, $A = \tilde{A}, \mathcal{D}(A) = \mathcal{D}(\tilde{A})$.

□

8 (Some) exercises

8.1 4.3.2025

Example (Coefficients for smooth extension)**.** Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\bigg(x', -\frac{x_d}{j}\bigg)c_j, x_d < 0.$$

for $u \in \mathcal{D}\big(\mathbb{R}^d\big)$. Find $\{c_j\}_{j=1}^{k=1}$ in such a way that $Eu \in C^k\big(\mathbb{R}^d\big)$. Moreover, take $d = 1$.

Proof. For $k = 0, j = 1$ we take $c_1 = 1, c_j = 0, j \neq 1$. For $k = 1$, compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k=1} (-1)^n \frac{\partial_{d^n} u\Big(x', \frac{x_d}{j}\Big)}{j^n} c_j, x_d < 0.$$

If we take $x_d = 0$ in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0)\bigg(-\frac{1}{j}\bigg)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \bigg(-\frac{1}{j}\bigg)^n, \forall n \in \{0, \ldots, k\}.$$

That is a linear system of $k+1$ equations. Is it solvable? □

8.2 8.4.2025

Example (Laplace)**.** Let $a_0 = 0, a(\cdot, z, p) = p$. Then $|a(\ldots)| \leq |p|$,growth can be accomplished for $r = 2$, $a(\ldots) \cdot p \geq |p|^2$. We can thus apply the theorem to our laplace equation

Example. Let $a_0 = 0, a(\cdot, z, p) = p\,\mathrm{atan}\left(1 + |p|^2\right)$. Then it is clearly Caratheodory, it is bounded $|a(\ldots)| \leq |p|\frac{\pi}{2}$, so the growth conditions yield, it is coercive as $\mathrm{atan}\left(1 + |p|^2\right) \geq \frac{\pi}{4}|p|^2$, and it is monotone

$$\left(\mathrm{atan}\left(1 + |p_1|^2\right)p_1 - \mathrm{atan}\left(1 + |p_2|^2\right)p_2\right)(p_1 - p_2) = \int_0^1 \sum_{j=1}^{d} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{atan}\Big(1 + |p_2 + s(p_1 - p_2)|^2\Big)(p_2 + s(p_1 - p_2))\,\mathrm{d}s\,(p_1 - p_2)_j$$