Partial differential equations II

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1 Winter semester addendum

1.0.1 Weak* convergence

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is not reflexive, we cannot extracting a convergent subsequence; however, we know the predual of $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is reflexive, i.e.

$$\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \approx (\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big))^*,$$

which means that balls in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ are weakly* compact. Moreover, $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is *separable*, from which it follows $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ with the weak* topology is metrizable and thus there exists s weakly * converging subsequence (from the balls).

Theorem 1. *Let the assumptions of the previous theorem hold and $\Omega \in C^{1,1}, \delta \in (0,1)$. Then $u \in L_2\Big((\delta,T);\, W^{2,2}(\Omega)\Big)$.*

Proof. Take the weak formulation in $t \in (\delta,T)$. WLOG further assume $d = 0$. Then

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi - bu\varphi - \mathbf{c}\cdot\nabla u\varphi - \int_\Omega \partial_t u\varphi = \int_\Omega (f - bu - \mathbf{c}\cdot\nabla u - \partial_t u)\varphi,$$

and the integrand of the last integral is in $\mathrm{L}_2(\Omega)$ for a.e. $t \in (\delta,T)$. We can thus use the elliptic regularity results and write:

$$\|u\|^2_{\mathrm{W}^{2,2}(\Omega)} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\partial_t u\|^2_{\mathrm{L}_2(\Omega)}),$$

integrating both sides $\int_\delta^T \mathrm{d}t$ yields

$$\|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{L}_2\left((0,T);\mathrm{W}^{1,2}(\Omega)\right)} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))}\Big)$$

□

Theorem 2. *If data are smooth and satisfy the compatibility condiitons, then the weak solutions to the parabolic equation are smooth.*

Proof. no. □

Remark (Compatibility condition). : Take the heat equation : $\partial_t u - \triangle u = f$ at time zero: $\triangle u(0) + f(0) = \partial_t u(0) \in \mathrm{W}_0^{1,2}(\Omega)$, so we need that $f(0) + \triangle u(0)$ has zero trace $\Rightarrow$ compatibility conditions.

Theorem 3 (Uniqueness of the solution to a hyperbolic equation)**.** *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that* $\mathbf{c} \in W^{1,\infty}(\Omega)$*. Then the weak solution to the hyperbolic equation is unique.*

Proof. It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the weak equation, multiply it by $\varphi \in V$ fixed and integrate in time and space:

$$< \partial_t u(t), \varphi > + \int_\Omega \int_0^t \mathbb{A}(s)\nabla u(s)\nabla\varphi \,\mathrm{d}s + \int_\Omega \int_0^t (bu(s) + \mathbf{c}\cdot\nabla u(s))\varphi - \int_\Omega \int_0^t u(s)\mathbf{d}(s)\cdot\nabla\varphi = 0,$$

next take $\varphi = u(t)$ as a test function and integrate $\int_0^\tau \mathrm{d}t\,, \tau \in (0,T)$. The duality term becomes

$$\int_0^\tau \frac{1}{2}\partial_t \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\,,$$

the remaining terms are (we are using Fubini theorem)

$$\int_0^\tau \int_0^t \int_\Omega \mathbb{A}\nabla u \cdot \nabla u(t)\,\mathrm{d}s\,\mathrm{d}t = \int_\Omega \int_0^\tau \int_s^\tau \nabla u(t)\,\mathrm{d}t\,\mathbb{A}(s)\nabla u(s)\,\mathrm{d}s\,,$$

denote $\partial_s w(s) = -u(s)$, then

□

2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d$ *open*, $p \in [1, +\infty], k \in N$. We define

$$\mathrm{W}^{\mathrm{k,p}}(\Omega) = \left\{ f \in \mathrm{L}_\mathrm{p}(\Omega)\,; D^\alpha f \in \mathrm{L}_\mathrm{p}(\Omega)\,, \forall |\alpha| \le k \right\},$$

with the norm

$$\|f\|^p_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \|f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)} + \sum_{0<|\alpha|\le k} \|D^\alpha f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)}.$$

Recall that:

- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is Banach $\forall p$ and Hilbert for $p = 2$.
- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is separable if $p < \infty$ and reflexive if $p > 1, p < \infty$.

Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.

2.1 Tools from functional analysis

Definition 1 (Regularization kernel)**.** The function η is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\operatorname{supp} \eta \subset \mathrm{U}(0,1)$
- $\eta \geq 0$
- η is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$

Definition 2 (Regularization of a function)**.** Let η be a regularization kernel. Set $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(x/\varepsilon), \varepsilon > 0$. We define the smoothing of f by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

Remark (Propertios of regularization)*.* The regularization has the following properties:

- $f \in \mathrm{L_p}(\Omega) \Rightarrow f_\varepsilon \to f \,\mathrm{in}\, \mathrm{L_p}(\Omega)$ and also a.e
- $f \in \mathrm{L}_\infty(\Omega) \Rightarrow f_\varepsilon \to f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathrm{U}(x,\varepsilon)} f_y \eta_\varepsilon(x-y)\,\mathrm{d}y$
- $\operatorname{supp} f_\varepsilon \subset \overline{U(\Omega,\varepsilon)}, f = 0 \,\mathrm{on}\, \mathrm{U}(x,\varepsilon) \Rightarrow f_\varepsilon(x) = 0$

Definition 3 ($\Omega' \subset\subset \Omega$)**.** $O \subset\subset \Omega$ means $\overline{O}$ is compact and $\overline{O} \subset \Omega$.

Lemma 1 (Approximation of Sobolev functions using regularization)**.** *Assume* $p \in [1,\infty), \Omega \subset \mathbb{R}^d$ *open*$, k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega' \subset\subset \Omega$*. Then it holds*

1. $\operatorname{dist}(\overline{\Omega}', \partial\Omega) = D > 0$
2. $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$ *in* $\Omega', \forall \varepsilon \in (0,D), \forall |\alpha| \leq k$
3. $f_\varepsilon \to f$ *in* $W^{k,p}(\Omega), \varepsilon \to 0^+$

Proof. 1. disjoint compact and closed set

2. WLOG $\frac{\partial f_\varepsilon}{\partial x^k} = \frac{\partial \int_{\mathbb{R}^d} f_y \eta_\varepsilon(x-y)\mathrm{d}y}{\partial x^k} = \int_\Omega f_y \frac{\partial \eta_\varepsilon}{\partial x^k}\,\mathrm{d}y = -\int_\Omega f(y)\frac{\partial \eta_\varepsilon}{\partial y^k}\,\mathrm{d}y = -\int_\Omega \frac{\partial f}{\partial y^k}\eta_\varepsilon(x-y)\,\mathrm{d}y = (D^\alpha f)_\varepsilon(x)$.
3. follows from 2) and the remark above applied to $f, D^\alpha f, |\alpha| \leq k$.

□

Lemma 2 (Partition of unity)**.** *Let* $E \subset \mathbb{R}^d, \mathcal{G}$ *opencovering. Then there exists a countable system* $\mathcal{F}$ *of nonnegative functions* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *such that* $0 \leq \varphi \leq 1$ *and*

1. $\mathcal{F}$ *is subordinate to* $\mathcal{G} : \forall \varphi \exists U \in \mathcal{Q} : \operatorname{supp} \varphi \subset U$
2. $\mathcal{F}$ *is locally finite:* $\forall K \subset E$ *compact*$, \operatorname{supp} \varphi \cap K \neq \emptyset$ *for at most finitely many* $\varphi \in \mathcal{F}$*.*

3. $\sum_{\varphi\in\mathcal{F}}\varphi(x) = 1, \forall x \in E$.

Proof. (Sketch) *Step 1 (E is compact)*:

E compact $\Rightarrow \exists N \in \mathbb{N} : U_j \in \mathcal{Q} \quad s.t. E \subset \bigcup_{j=1}^{m} U_j$. Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \cup_{j=1}^{m} K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set: $K_m = \left\{x \in U, \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m\right\}$. Then clearly $K_1 \subset K_2 \dots$, and they "converge monotonously to U. Next, find $\phi_j \in C_c(U_j), \phi_j > 0 \,\text{on}\, K_j$, e.g. $\phi_j = \theta(\operatorname{dist}(x, \partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally $\varphi_j = \frac{\psi_j}{\sum_j \psi_j}$.

Step 2 (E is open):

Use exhaustion argument, then finite $\to$ countable. ☐

2.2 Density of smooth functions

Theorem 4 (Density of smooth functions I)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}, p \in [1, \infty)$. Then $\left\{f \in C^\infty(\Omega), \operatorname{supp} f \quad bounded\right\} \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W^{k,p}}(\Omega), \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega) \cap \mathrm{W^{k,p}}(\Omega) \quad s.t \|u - v\|_{\mathrm{W^{k,p}}(\Omega)} < \varepsilon$. Using the exhaustion argument, define

$$\Omega_j = \left\{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\right\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1}, \cup_{j=1}^{\infty}\Omega_j = \Omega$. Next, set $U_j = \Omega_{j+1} \; \overline{\Omega_{j-1}}, j = 1, 2, \dots,$ where $\Omega_0 = \Omega_{-1} = \varnothing$. Using the partition of unity lemma, $\exists\{\varphi_j\}$ partition of unity subordinate to $\{U_j\}$. We can write $u = \sum_j u\varphi_j$, where $u\varphi_j \in \mathrm{W^{k,p}}(\Omega), \operatorname{supp} u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$ sufficiently small: set $v_j = (u\varphi_j)_{\varepsilon_j}$. By the properties of regularization, we now

$$\|u - u\varphi_j\|_{\mathrm{W^{k,p}}(\Omega)} < \frac{\varepsilon}{2^j},$$

by taking ε_j small enough. Set $v = \sum_j v_j$ and use the following trick:

Fix $N \in \mathbb{N}$ and estimate $\|v - u\|_{\mathrm{W^{k,p}}(\Omega)}$. Observe $u - v = \sum_{j=1}^{\infty}(u\varphi_j - v_j)$, so taking $x \in \Omega_N$ i have $(u - v)(x) = \sum_{j=1}^{N+1}(u\varphi_j - v_j)$. The norm of this is

$$\|u - v\|_{\mathrm{W^{k,p}}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{\mathrm{W^{k,p}}(\Omega)} < \varepsilon.$$

It only remains to let $N \to \infty$ and realize $\|u - v\|_{\mathrm{W^{k,p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W^{k,p}}(\Omega)}$ by Lévi's theorem: $\int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x \to \int_\Omega |D^\alpha| \,\mathrm{d}x$. ☐

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Recall $\Omega \in C^0$ means $\exists U_j, j = 1, \dots, m \quad open, \exists \alpha, \beta > 0, a_j : \overline{U(0,\alpha)} \to \mathbb{R}, \mathbb{A}_j : \mathbb{R}^d \to \mathbb{R}^d \quad aff.orthogonal, \quad suchthat \partial\Omega$ $\cup_{j=1}^{m} U_j, \partial\Omega \cap U_j = \left\{(x', a(x'), x' \in U(0, \alpha)\right\}$. Setting $G_j(x', b) = \mathbb{A}_j(x', a(x') + b)$ we moreover require $G_j(U(0, \alpha) \times (0, \beta)) \subset \Omega, G_j(U(0, \alpha) \times (-\beta, 0)) \subset \overline{\mathbb{R}^d/\Omega}$.

Definition 4 (Shift operator)**.** For $u \in \mathrm{L_p}(\Omega), k \in \{1, \ldots, d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

Lemma 3 (Approximation property of the shift operator)**.** *For $u \in L_p(\Omega)$, it holds $\tau_h u \to u$ in $L_p(\Omega), h \to 0^+$.*

Theorem 5 (Density of smooth functions II)**.** *Let $\Omega \in C^0$ bounded, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W^{k,p}}(\Omega), \varepsilon > 0 \quad given$, i am looking for $v \in C^\infty_c(\mathbb{R}^d) \quad suchthat \|u - v\|_{\mathrm{W^{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple: covering of $\overline{\Omega}$, partition of unity. Clearly, $\Omega \subset \cup_{j=0}^m U_j$, where $U_0 = \Omega, U_j$ are from the definition of C^0 boundary. Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to this cover. Observe that $u\varphi_j \in \mathrm{W^{k,p}}(\Omega), \operatorname{supp} u\varphi_j \subset U_j$. Find

$$v_j \in \mathcal{D}(\mathbb{R}^d) \quad s.t. \|v_j - u\varphi_j\|_{\mathrm{W^{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

If i am able to do this, i am finished: just take

$$v = \sum_{j=0}^m v_j$$

.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset \Omega$, take $v_0 = (u\varphi_0)_\varepsilon$, so if we take $\varepsilon > 0$ small enough, i can use the previous lemma.

Case $j \in \{1, \ldots, m\}$. Set $w_j = u\varphi_j, \tau_\delta w_j(x', x_d) = w(x', x_d + \delta)$ (ignore $\mathbb{A}_{\mathbb{j}}$), observe $t_\delta u_j \in \mathrm{W^{k,p}}\big(U_j^\delta\big), U_j \subset\subset U_j^\delta$. Finally, set $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$ small enough. From the properties of the shift $\tau_\delta w_j$ is close to w_j in $\mathrm{L_p}(U_j \cap \Omega)$ and $D^\alpha \tau_\delta w_j = \tau_\delta(D^\alpha w_j)$ close to $D^\alpha w_j$ in $\mathrm{L_p}(U_j \cap \Omega)$. Finally, set $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$ small enough $\Rightarrow v_j \in \mathcal{D}(\mathbb{R}^d), \operatorname{supp} v_j \subset U_j$ by the previous lemma $\|v_j - \tau_\delta w_j\|_{\mathrm{W^{k,p}}(\Omega)}$ small.

□

Remark. Recall $C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \left\{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\right\}$.

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W^{k,p}}(\Omega)$, does there exist $\overline{u} \in \mathrm{W^{k,p}}\big(\mathbb{R}^d\big), \quad s.t. \overline{u}|_\Omega = u, \|\overline{u}\|_{\mathrm{W^{k,p}}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{\mathrm{W^{k,p}}(\Omega)}$?

The answer is **yes**, if Ω is nice enough.

Lemma 4. *Let $\alpha, \beta > 0, K \subset U(0, \alpha) \times [\alpha, \beta]$ be compact. Then*

$$\exists C > 0, \exists E : C^1(\overline{U(0,\alpha)} \times [0, \beta]) \to C^1(\overline{U(0,\alpha)} \times [-\beta, \beta]), \exists \tilde{K} \subset U(0, \alpha) \times [-\beta, b) \; compact$$

such that:

1. $\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \leq \|u\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))}$
2. *if* $\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Eu \subset \tilde{K}$

Proof. Use the following trick:

$$\overline{u}(x) = \begin{cases} u(x), & x_d > 0 \\ -3u(x_1, \dots, x_{d-1}, -x_d) + 4u(x_1, \dots, x_{d-1}, -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Is this extension C^1? Take some $a = (x_1, \dots, x_{d-1}, 0)$. Then

$$u(x \to a) = \begin{cases} u(a), & x_d > 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so $\overline{u}$ is continuous. Its derivative
$\partial_k \overline{u}, k = 1, \dots, d-1$ is the same as for u, where as

$$\partial_d \overline{u} = \begin{cases} \partial_d u, & x_d > 0 \\ -3\partial_d u(x_1, \dots, x_{d-1}, -x_d)(-1) + 4\partial_d u(x_1, \dots, x_{d-1}, -\frac{x_d}{2})(\frac{-1}{2}) = 3\partial_d u - 2\partial_d u, & x_d < 0, \end{cases}$$

so the the derivative is also continuous. Thus, we have $Eu = \overline{u} \in C^1 \subset \mathrm{W}^{1,p}(U(0,\alpha) \times (-\beta, \beta))$ and estimate of the norm $\|Eu\|_{\mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta))}$ is clear, as the wanted term is just some linear combination.

Mr. Prazak is not sure how this should be correctly finished and i am not also. □

Lemma 5 (Change of variables under C^1 diffeomorphisms)**.** *Let $U, V \subset \mathbb{R}^d$ be open, $\phi : U \to V$ be C^1 diffeomorphism. Let $\tilde{U} \subset\subset U$. Then*

$$\phi(\tilde{U}) \subset\subset V, \textit{ and} \exists C > 0 : \forall u \in C^1(V) : \|u \circ \phi\|_{W^{1,p}(\tilde{U})} \le C\|u\|_{W^{1,p}(\phi(\tilde{U}))}$$

Proof. $\|u \circ \phi\|^p_{\mathrm{L_p}(\tilde{U})} = \int_{\tilde{U}} (u\circ\phi)^p |\det \nabla\phi| \,\mathrm{d}x \le C_0^{-1} \int_{\tilde{U}} |u\circ\phi|^p |\det \nabla\phi| \,\mathrm{d}x$, where $\det \nabla\phi > 0 \,\mathrm{in}\, U$, so $\det \nabla\phi \ge C_0 > 0 \,\mathrm{in}\, \tilde{U}$. Together $\|u \circ \phi\|^p_{\mathrm{L_p}(\tilde{U})} = C_0^{-1} \int_{\phi(\tilde{U})} |u|^p \,\mathrm{d}x = C_0^{-1} \|u\|_{\mathrm{L_p}(\phi(\tilde{U}))}$ □

Lemma 6. *Let $\alpha, \beta > 0, K \subset U(0,\alpha)\times[0,\beta), K$ compact. Then there is $C > 0, E : C^1\left(\overline{U(0,\alpha) \times [0,\beta)}\right) \to C^1\left(\overline{U(0,\alpha)} \times [-\beta,\beta]\right), \tilde{K} \subset U(0,\alpha) \times [-\beta,\beta)$ compact such that*

- $\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)), W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \le C$
- $u \in C^1\left(\overline{U(0,\alpha)} \times [0,\beta]\right), \operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Eu \subset \tilde{K}$

Proof. No proof. □

Lemma 7. *Let $U, V \subset \mathbb{R}^d$ open, $\Phi : U \to V, C^1$ diffeomoprhism, $\tilde{U} \subset\subset U$ compact. Then $\Phi(\tilde{U}) \subset\subset V$ and*

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \le C\|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

Proof. No proof. □

Theorem 6 (Extension of Sobolev functions)**.** *Let $\Omega \in C^{k-1,1}, k \in \mathbb{N}, p \in [1,\infty], V \subset \mathbb{R}^d$ open such that $\Omega \subset\subset V$. Then there is $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ bounded linear operator such that*

1. $\forall u \in W^{k,p}(\Omega) : Eu = u$ *a.e. in* Ω,

2. $\forall u \in W^{k,p}(\Omega) : \operatorname{supp} Eu \subset V$,

3. $\|E\| \le C, C = C(p, \Omega, V)$.

Proof. Only for $k = 1, \Omega \in C^1, p < \infty$. We know $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $\mathrm{W}^{1,p}(\Omega)$, we show existence of E for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ with properties 1),2),3) and then extend E to $\mathrm{W}^{1,p}(\Omega)$ by density.
Covering of Ω*:*

$$\overline{\Omega} \subset \Omega \cup \bigcup_{j=1}^{m} U_j$$

with $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ as in the definition of a C^1 domain. In particular, $a_j \in C^1(\mathrm{U}(0,\alpha))$.
Construction of E*:* We denote $\{\varphi_j\}_{j=0}^m$ partition of unity subordinate to $\{U_j\}_{j=1}^m$. For $j \in \{1, \dots, n\}$ we define $\phi_j : \mathrm{U}(0,\alpha) \times (-\beta, \beta) \to U_j$ by

$$\phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

Trivially ϕ_j is C^1 diffemorphism. Let us denote by $\tilde{E}$ the extension operator from the previous lemma. Then we have for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$. We define

$$Eu = \varphi_0 u + \sum_{j=1}^{m} \left(\eta \tilde{E}((\varphi_j u) \circ \phi_j)\right) \circ \phi_j^{-1},$$

where η is a cut-off function $\eta = 1 \,\mathrm{on}\, y_d \ge 0, \in (0,1) \,\mathrm{else}\,, = 0 \,\mathrm{on}\, y_d \le -h$, for some parameter $h > 0$ which will be defined later. We also take $\eta \in C^\infty$. Due to our construction,

$$\phi_j^{-1}(\mathrm{U}(0,\alpha) \times [-2h, \beta)) \subset \mathrm{U}(\Omega, \varepsilon) \subset \mathrm{U}(\Omega, 2\varepsilon) \subset V,$$

for some $\varepsilon > 0$.
Properties of E*:* It is clear that

- E is linear from its definition
- 1) holds, as ϕ_j and ϕ_j^{-1} cancel *somewhere*
- 2) holds for $h < \frac{\beta}{2}$
- 3) we use the previous lemma:

$$\begin{aligned}
\Bigg\| \underbrace{\left(\eta \tilde{E}(\varphi_j u \circ \phi_j)\right)}_{\operatorname{supp}() \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta)} \circ \phi_j^{-1} \Bigg\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &\le C \left\| \eta \tilde{E}(\varphi_j u \circ \phi_j) \right\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha) \times (-\beta,\beta))} \\
&\underbrace{\le}_{\text{previous lemma}} C \|\varphi_j u \circ \phi_j\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha) \times (0,\beta))} \\
&\underbrace{\le}_{\text{previous lemma}} C \|\varphi_j u\|_{\mathrm{W}^{1,p}(U_j \cap \Omega)} \le \|u\|_{\mathrm{W}^{1,p}(\Omega)} \Rightarrow \|E\| \le C.
\end{aligned}$$

So all the properties hold for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$. We need to show them also for $u \in \mathrm{W}^{1,p}(\Omega)$. Pick an arbitrary $u \in \mathrm{W}^{1,p}(\Omega)$, find $\{u_k\} \subset C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u_k \to u \,\text{in}\, \mathrm{W}^{1,p}(\Omega)$.
Ad 1): Since E is continuous, then $Eu_k \to Eu \,\text{in}\, \mathrm{W}^{1,p}(\mathbb{R}^d)$. Since $\Omega \subset \mathbb{R}^d \Rightarrow Eu = u \,\text{in}\, \mathrm{W}^{1,p}(\Omega)$.
Ad 2): $\operatorname{supp} Eu_k \subset \mathrm{U}(\Omega,\varepsilon) \Rightarrow \operatorname{supp} Eu \subset \overline{\mathrm{U}(\Omega,\varepsilon)} \subset V$.

□

Remark ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $C^{0,1}$ and $p \in (1,\infty), k > 1$.

2.4 Embedding theorems

Example. Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1,x_2) = \int_{-\infty}^{x_1} \partial_1 u(s,x_2)\,\mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1,s)\,\mathrm{d}s\,,$$

so

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u(x_1,x_2)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \le \int_{\mathbb{R}} |\partial_1 u(s,x_2)|\,\mathrm{d}s \int_{\mathbb{R}} |\partial_2 u(x_1,s)\,|\,\mathrm{d}s\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \Big(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}\lambda^2 \Big)^2,$$

so

$$\|u\|_{\mathrm{L}_2(\mathbb{R}^2)} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^2)}.$$

Lemma 8. *Let $d \ge 2$. Let $\hat{u_i} : \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative and measurable for $j \in \{1,\dots,d\}$. We define*

$$\hat{x_j} = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_d), \hat{\mathrm{dx_j}} = \mathrm{d}x_1 \dots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \dots \mathrm{d}x_d\,.$$

Consider the functions $u_j : \mathbb{R}^d \to \mathbb{R}, u_j(x) = \hat{u_j}(\hat{x_j})$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x)\,\mathrm{d}x \le \prod_{j=1}^d \Big(\int_{\mathbb{R}^{d-1}} (\hat{u_j}(\hat{x_j}))^{d-1} \hat{\mathrm{dx_j}} \Big)^{\frac{1}{d-1}}. \tag{1}$$

Proof. Induction by d.

1. $d = 2 : \int_{\mathbb{R}^d} u_1(x_1,x_2)u_2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2)\hat{u}_2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \underbrace{=}_{\text{Fubini}} \int_{\mathbb{R}} \hat{u}_1(x_2)\,\mathrm{d}x_2 \int_{\mathbb{R}} \hat{u}_2\,\mathrm{d}x_1$.

2. $$\begin{aligned} d \to d+1 : \int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^{d} u_j(x)\,\mathrm{d}x_{d+1}\, u_{d+1}\,\mathrm{d}x\, \hat{\mathrm{dx}}_{\mathrm{d+1}} \\ &\underbrace{\le}_{\text{Holder}} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \int_{\mathbb{R}} (u_j(x))^d \,\mathrm{d}x_{d+1} \right)^{\frac{1}{d}} u_{d+1}(x) \hat{\mathrm{dx}}_{\mathrm{d+1}} \\ &\underbrace{\le}_{\text{Holder}} \left(\int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \int_{\mathbb{R}} u_j^d(x)\,\mathrm{d}x_{d-1} \right)^{\frac{1}{d-1}} \mathrm{d}x_{\hat{d}+1} \right)^{\frac{d-1}{d}} \left(\int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}x_{\hat{d}+1} \right)^{\frac{1}{d}} \\ &\underbrace{\le}_{\text{induction step}^1} \left(\int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \left(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d(x)\,\mathrm{d}x_{d+1}\,\mathrm{d}\hat{x}_j\,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{d-1}{d}\frac{1}{d-1}}. \end{aligned}$$

□

Theorem 7 (Gagliardo-Nirenberg)**.** *Let $p \in [1,d)$. Then $\forall u \in W^{1,p}(\mathbb{R}^d)$:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where $p^* = \frac{dp}{d-p}$.

Proof. Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$:

$$\forall j \in \{1,\dots,d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_d)\,\mathrm{d}s$$

independet of x_j, so

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots,s,\dots)\,\mathrm{d}s\,.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \leq \prod_{j=1}^{d} \underbrace{\left(\int_{\mathbb{R}} |\nabla u|(\dots,s,\dots)\,\mathrm{d}s\right)^{\frac{1}{d-1}}}_{u_j \text{ independent of } x_j},$$

so the integral

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}}\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \prod_{j=1}^{d} u)j\,\mathrm{d}x \underbrace{\leq}_{\text{previous lemma}} \left(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x)\,\mathrm{d}x_j\,\mathrm{d}\hat{x}_j\right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u|\,\mathrm{d}x\right)^{\frac{d}{d-1}}.$$

If $p \in (1,d)$, compute

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\mathbb{R}^d)} = \||u|^q\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{\mathrm{L}_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1}|\nabla u|\,\mathrm{d}x \underbrace{\leq}_{\text{Holder}} \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \|u\|^{q-1}_{\mathrm{L}_{(q-1)p'}(\mathbb{R}^d)}.$$

We want $\frac{(q-1)p'}{p-1} = \frac{qd}{d-1}$, so

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$q\frac{d}{d-1} = p^*.$$

$\Rightarrow$ statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\mathbb{R}^d)$. □

Remark.
- It is evident that nonzero constants are not in $\mathrm{W}^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.
- the set $\mathbb{R}^d$ is of course unbounded, so we have no ordering of $\mathrm{L_p}(\Omega)$ spaces.

- of course, we require no smoothness of the domain

Theorem 8. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1,d)$ the statement of the previous theorem holds.*

Proof. An immediate corollary of the previous theorem. □

Remark. In the proof of theorem we showed that $\forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\Omega)} \leq q\|\nabla u\|_{\mathrm{L_p}(\Omega)}\|u\|^{q-1}_{\mathrm{L}_{\frac{p(q-1)}{p-1}}(\Omega)},$$

for q such that $\frac{qd}{d-1} \leq p^*$.

Theorem 9 (Embedding theorem)**.** *Let $\Omega \subset C^{0,1}, p^* = \frac{dp}{1-p}$ If $p \in [1,d)$ then*

$$W^{1,p}(\Omega) \subset L_q(\Omega)\, \forall q \in [1,p^*].$$

Moreover, if $q < p^*$, then

$$W^{1,p}(\Omega) \subset\subset L_q(\Omega)\,.$$

If $p = d$, then

$$W^{1,p}(\Omega) \subset L_q(\Omega)\, \forall q < \infty,\, W^{1,p}(\Omega) \subset\subset L_q(\Omega)\, \forall 1 \leq q < \infty.$$

Proof. We would like to use the previous theorem + extension.

Ad continuity for $p < d : E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ the extension is continuous. We also know

- identity $I_1 : \mathrm{W}^{1,p}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\mathbb{R}^d)$ is continous,
- restriction $I_2 : \mathrm{L}_{p^*}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\Omega)$ is continuous,
- identity $I_3 : \mathrm{L}_{p^*}(\Omega) \to \mathrm{L}_q(\Omega)$ is continous.

Together, the mapping $id : \mathrm{W}^{1,p}(\Omega) : \mathrm{L}_q(\Omega), id = I_3 \circ I_2 \circ I_1 \circ E$ identity is continuous. If p=d, then $\mathrm{W}^{1,d}(\Omega) \subset \mathrm{W}^{1,r}(\Omega)\, \forall r \in [1,d)$, and $r^* \to \infty$ as $r \to d-$. For $q \in [1,\infty)$ find $r \in [1,d)$ s.t. $r^* > q$. Then

$$\mathrm{W}^{1,d}(\Omega) \subset \mathrm{W}^{1,r}(\Omega) \subset \mathrm{L}_{r^*}(\Omega) \subset \mathrm{L}_q(\Omega)\,,$$

using the previous results.

Ad compactness: We show $\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_q(\Omega)$ using Arzela-Ascoli and then it will get technical: show compactness in smooth functions, then show compactness in $\mathrm{L}_1(\Omega)$, then approximate the norm of $\mathrm{L}_q(\Omega)$ using the obtained quantities.

Consider $B = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$ and extend it to EB. Fix $\delta > 0$ and let η be a regularization kernel. Then $\exists R > 0 : \operatorname{supp}(EB)_\delta \subset \overline{\mathrm{U}(0,R)} \subset \mathbb{R}^d$ (i.e. all the functions from EB have the support contained in the ball). Moreover, $(EB)_\delta \subset C^1\left(\overline{\mathrm{U}(0,R)}\right)$. Actually, it is bounded in $C^1\left(\overline{\mathrm{U}(0,R)}\right)$. $\underbrace{\subset\subset}_{\text{Arzela-Ascoli}}$ $C\left(\overline{\mathrm{U}(0,R)}\right)$ (uniform equicontinuity comes from uniform boundedness of the gradients, $\nabla(u * \eta_\delta) = u * \nabla\eta_\delta$.) Altogether $(EB)_\delta$ is relatively compact in

$$C\left(\overline{\mathrm{U}(0,R)}\right) \underbrace{\Rightarrow}_{\text{the space } C\left(\overline{\mathrm{U}(0,R)}\right) \text{ is complete}} \text{bounded in } C\left(\overline{\mathrm{U}(0,R)}\right) \underbrace{\Rightarrow}_{\text{bounded domain}} \text{bounded in } \mathrm{L}_1(\mathrm{U}(0,R))\,.$$

Next, take

$$u \in B : \|u - (Eu)_\delta\|_{\mathrm{L}_q(\Omega)} \le \|Eu - (Eu)_\delta\|_{\mathrm{L}_q(\mathrm{U}(0,R))} = \int_{\mathrm{U}(0,R)} |v - v_\delta| \,\mathrm{d}x = \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} v(x+y) - v(x) \eta_\delta(y) \,\mathrm{d}y \,| \,\mathrm{d}x \le$$
$$\le \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} |\eta_\delta(y)||y| \,\mathrm{d}y \,\mathrm{d}x \underbrace{\le}_{\text{Fubini}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} \,\mathrm{d}x \,|y| \eta_\delta(y) \,\mathrm{d}y \,.$$

Estimate the inner integral: assume v is smooth and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|} | \int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(v(x+sy))}_{\nabla v(x+sy)\cdot y} \,\mathrm{d}s \,| \,\mathrm{d}x \underbrace{\le}_{\text{Cauchy Schwartz}} \int_{\mathbb{R}^d} \int_0^1 |\nabla v|(x+sy) \,\mathrm{d}s \,\mathrm{d}x \underbrace{\le}_{\text{Holder}} C(R) \Big(\int_{\mathbb{R}^d} |\nabla v|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} .$$

Now, take $v \in \mathrm{W}_0^{1,p}(\mathrm{U}(0,R))$, then $\exists \{v_k\} \subset \mathcal{D}(\mathrm{U}(0,R)) : v_k \to v \,\mathrm{in}\, \mathrm{W}^{1,p}(\mathrm{U}(0,R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} \,\mathrm{d}x \le C(R) \Big(\int_{\mathbb{R}^d} |\nabla v_k|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \to C(R) \Big(\int_{\mathbb{R}^d} |\nabla v|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} .$$

So finally

$$\|u - (Eu)_\delta\|_{\mathrm{L}_q(\Omega)} \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} \,\mathrm{d}x \,|y| \eta_\delta(y) \,\mathrm{d}y \underbrace{\le}_{|y| \le \delta} C(R) \delta \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \mathrm{d}x \le C_1 \delta.$$

Fix $\varepsilon > 0$, find finite $\frac{\varepsilon}{2}$-net in $(EB)_\delta$ in $\mathrm{L}_1(\mathrm{U}(0,R))$ (that is possible since we have total boundedness in $\mathrm{L}_1(\mathrm{U}(0,R))$.) Set $\delta > 0 \,\mathrm{s.t.}\, C_1 \delta \frac{\varepsilon}{4}$. [2]. Denote the $\frac{\varepsilon}{2}$-net as $\{Eu_k\}_{k=1}^m, m \in \mathbb{N}$. We show $\{u_k\}_{k=1}^m$ is a ε-net in B. Fix $u \in B$, find $j \in \{1, \dots, m\} : \|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L}_1(\mathrm{U}(0,R))}$. Compute

$$\|u - u_j\|_{\mathrm{L}_1(\Omega)} \le \|u - (Eu)_\delta\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu)_\delta - (Eu_j)_\delta\right\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu_j)_\delta - u_j\right\|_{\mathrm{L}_1(\Omega)} \le 2C_1 \delta + \frac{\varepsilon}{2} \le \varepsilon.$$

Thus, we have shown

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_1(\Omega) \,.$$

It remains to show the validity for a general q. Let $q \in [1, p^*) : \|v\|_{\mathrm{L}_q(\Omega)} \le \|v\|_{\mathrm{L}_1(\Omega)}^\alpha \|v\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\alpha}$, for $\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}, \alpha \in (0,1]$. Is B totally bounded in $\mathrm{L}_q(\Omega)$? Let us compute

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \|u - u_j\|_{\mathrm{L}_1(\Omega)}^\alpha \underbrace{\|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\alpha}}_{\le C, \mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}_{p^*}(\Omega)} \le C \varepsilon^\alpha .$$

□

[2] The order of the choices is not precise...

2.5 Trace theorems

2.6 Composition of sobolev functions

2.7 Difference quotients

3 Nonlinear elliptic equations as compact perturbations

Theorem 10 (Nemytskii). *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}, N \in N, \Omega \subset \mathbb{R}^d$ measurable, f Caratheodory. Then*

1. *if $u : \Omega \to \mathbb{R}^N$ is measurable then $f(\cdot, u)$ is also measurable*
2. *If there is $p_i \in [1, +\infty), i \in \{1, \dots, N\}, q \in [1, \infty), g \in L_q(\Omega), C > 0$ such that for almost all*

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x, y)| \le g(x) + c \sum_{i=1}^{N} |y_i|^{p_i/q}$$

   , then $u \mapsto f(\cdot, u)$ is continuous from $L_{p_i}(\Omega) \times \cdots \times L_{p_N}(\Omega)$ to $L_q(\Omega)$. Moreover, it maps bounded sets to bounded sets.

Proof. No proof □

Definition 5 (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2)**.** Let X, Y be normed linear spaces, $M \subset X$. The mapping $F : M \to Y$ is called a compact operator on M into Y if F is continuous and $F(M \cap K)$ is relatively compact in Y for any bounded $K \subset X$.

Remark. We have no linearity of F! So continuity cannot follow from compactness (we have compactness $\Rightarrow$ boundedness $\neq$ continuity for nonlinear operators)

Theorem 11 (Brouwer fixed point theorem)**.** *Let $K \subset \mathbb{R}^N, N \in \mathbb{N}$ be a nonempty convex closed bounded. Assume that $F : K \to K$ is continuous. Then F has a fixed point in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. No proof □

Theorem 12 (Schauder fixed point theorem)**.** *Let $K \subset X$ be a nonempty convex closed bonded subset of a linear normed space X. Assume that F is compact on K into K and $F(K) \subset K$. Then there is fixed point of F in K.*

Proof. No proof □

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bouded, it is automatically compact, and since $F : K \to K$ is continuous, F is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with N=1 is easy, based on Darboux property.

3.0.1 Problem protypes

In this chapter some nonlinear elliptic equations are discussed.

Example. Suppose the following problem:

$$\begin{cases} -\triangle u + g(u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

where

$$g : \mathbb{R} \to \mathbb{R}, f \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*, \text{continuous}\,, \exists\alpha \in [0,1) : \forall s \in \mathbb{R} : |g(s)| \le C(1+|s|^\alpha).$$

Theorem 13 (Existence)**.** *Let $\Omega \in C^{1,1}$, $f \in \left(W_0^{1,2}(\Omega)\right)^*$, g is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall\varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{d}x =< f, \varphi >_{(W_0^{1,2}(\Omega))^*} .$$

If $f \in L_2(\Omega)$, then the solution $u \in W^{2,2}(\Omega)$.

Proof. We define $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ such that

$$Sw = u \Leftrightarrow \forall\varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{d}x =< f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x\,.$$

S is well defined:

$$|\,\mathrm{RHS}\,| \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} \|\varphi\|_{\mathrm{W}^{1,2}(\Omega)} + \|\varphi\|_{\mathrm{L}_2(\Omega)} \|g(w)\|_{\mathrm{L}_2(\Omega)},$$

and

$$\int_\Omega |g(w)|^2 \,\mathrm{d}x \le \int_\Omega C(1+|w|^\alpha)^2 \,\mathrm{d}x \le \int_\Omega C\left(1+|w|^{2\alpha}\right) \mathrm{d}x \le \int_\Omega C\left(1+|w|^2\right) \mathrm{d}x \le \infty,$$

where we used the Young inequality and $\alpha \le 1$. We have thus shown the mapping $w \mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by Nemytskii. Next, S is continuous:

- $w \mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$
- $w \mapsto \left(\varphi \mathrm{W}_0^{1,2}(\Omega) \to < f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x\right)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$
- $F \to u$, where u is the weak solution of $\begin{cases} -\triangle u = F & in\Omega \\ u = 0 & on\partial\Omega, \end{cases}$, is linear and continuous from $(\mathrm{W}_0^{1,2}(\Omega))^*$ to $\mathrm{W}_0^{1,2}(\Omega)$.

In total, the composition is continuous and yields S. Next, we would like to show S is compact. We start with showing S maps bounded sets in $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega)$; for that we need apriori estimates: test the weak formulation with u:

$$\|\nabla u\|^2_{\mathrm{L}_2(\Omega)} \le \varepsilon\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + C\left(\|f\|^2_{(\mathrm{W}^{1,2}(\Omega))^*} + \|g(w)\|^2_{\mathrm{L}_2(\Omega)}\right) \underbrace{\le}_{\text{Younge}} C\left(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*)} + 1 + \|w\|^2_{\mathrm{L}_2(\Omega)}\right),$$

from which follows S is compact from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by compact embedding. Now we need to show $S(\mathrm{U}(0,R)) \subset \mathrm{U}(0,R)$ for some $R > 0$. From the previous we know:

$$\frac{C}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\Big(\|f\|_{(\mathrm{W}^{1,2}_0(\Omega))^*} + \|g\|^2_{\mathrm{L}_2(\Omega)}\Big),$$

so since

$$\tilde{C}\int_\Omega |g(w)|^2\,\mathrm{d}x \le \int_\Omega C\big(1+|w|^{2\alpha}\big)\,\mathrm{d}x \underbrace{\le}_{\text{Younge}} \int_\Omega \Big(C + \frac{c}{4}|w|^2\Big)\,\mathrm{d}x$$

we know

$$\frac{c}{2}\|u\|^2_{\mathrm{L}_2(\Omega)} \le \frac{c}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\|f\|^2_{(\mathrm{W}^{1,2}_0(\Omega))^*} + C\lambda(\Omega) + \frac{c}{4}\|w\|^2_{\mathrm{L}_2(\Omega)},$$

and thus

$$\|u\|^2_{\mathrm{L}_2(\Omega)} \le \underbrace{\frac{2\tilde{C}}{c}\|f\|^2_{(\mathrm{W}^{1,2}_0(\Omega))^*} + 2\frac{C}{c}}_{=\overline{C}} + \frac{1}{2}\|w\|^2_{\mathrm{L}_2(\Omega)}.$$

so if $\overline{C} + \frac{1}{2}R^2 < R^2$, we are done [3]. But such an R of course exists (says doc. Kaplicky) $\Rightarrow$ the image of a ball is in a ball for some $R \Rightarrow S$ is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that u_0 solves $\begin{cases} -\triangle u_0 = f - g(u_0) \in \mathrm{L}_2(\Omega) & \text{in}\,\Omega \\ u_0 = 0 & \text{on}\,\partial\Omega. \end{cases}$. So from the regularity theory for elliptic equations we get

$$u \in \mathrm{W}^{2,2}(\Omega)\,.$$

□

Theorem 14 (Uniqueness)**.** *Let $u_1, u_2 \in W^{1,2}_0(\Omega)$ be weak solutions to the above problem. Let $f \in \big(W^{1,2}_0(\Omega)\big)^*$, g be continuous. Let either*

1. *g is nondecreasing*
2. *$g \in C^1(\mathbb{R})$, $\|g'\|_\infty$ small.*

Then $u_1 = u_2$.

Proof. We subtract the equations for u_1, u_2 and test with $u_1 - u_2$.:

$$\int_\Omega |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2)\,\mathrm{d}x = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1 - u_2)\|_{\mathrm{L}_2(\Omega)} \ge C\|u_1 - u_2\|^2_{\mathrm{W}^{1,2}(\Omega)} \Rightarrow u_1 - u_2 = 0.$$

$$|\int_\Omega (g(u_1) - g(u_2)(u_1 - u_2))\,\mathrm{d}x| \le \int_\Omega \|g'\|_\infty |u_1 - u_2|^2\,\mathrm{d}x \le \|g'\|_\infty C_P\|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} = 0 \Rightarrow u_1 = u_2.$$

whenever $C\|g'\|_\infty < 1$. □

[3] The constants are most probably messed up.

Example. Suppose the following problem

$$\begin{cases} -\triangle u + b(\nabla u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}$$

where $f \in (\mathrm{W}_0^{1,2}(\Omega))^*, b$ is continuous and bounded. The weak formulation is

$$u \in \mathrm{W}_0^{1,2}(\Omega) \wedge \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi + b(\nabla u)\varphi \,\mathrm{d}x =< f, \varphi >,$$

and the first apriori estimates (test with u)

$$\|\nabla u\|_{\mathrm{L}_2(\Omega)} \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \int_\Omega |u|\,\mathrm{d}x\,\|b\|_{\mathrm{L}_\infty(\Omega)}.$$

Theorem 15. *Let $f \in \left(W_0^{1,2}(\Omega)\right)^*, \Omega \in C^{0,1}, b : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded. Then there is a weak solution to the above problem.*

Proof. $S : \mathrm{W}_0^{1,2}(\Omega) \to \mathrm{W}_0^{1,2}(\Omega)\,, Sw = u$ iff u solves

$$\begin{cases} -\triangle u = f - b(\nabla w) & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x =< f, \varphi > - \int_\Omega b(\nabla w)\varphi\,\mathrm{d}x\,.$$

Clearly, S is well defined and

$$\|Sw\|_{\mathrm{W}_0^{1,2}(\Omega)} \le \underbrace{C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|b\|_{\mathrm{L}_\infty(\Omega)}\Big)}_{:=R},$$

meaning $S\Big(\overline{\mathrm{U}(0,R)}\Big) \subset \overline{\mathrm{U}(0,R)}$. Moreover, S]s continuous, as S is the composiiton of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show S is compact: we already have continuity, consider $\{w_k\}_{k\in\mathbb{N}} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded. Then $\exists\{u_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded: $u_k \to u$ in $\mathrm{L}_1(\Omega)$ by embedding up to a subsequence. Next, uss the following trick: substitue equation for u_k from equation for u_l and test with $u_l - u_k$

$$C\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \le \|\nabla(u_l - u_k)\|^2_{\mathrm{L}_2(\Omega)} \le \int_\Omega |b(\nabla u_l) - b(\nabla u_k)|\,|u_l - u_k|\,\mathrm{d}x \le 2\|b\|_{\mathrm{L}_\infty(\Omega)}\|u_l - u_k\|_{\mathrm{L}_1(\Omega)}.$$

All in all, S has a fixed point by Schauder, which is of course the weak solution. □

But this says $\{u_k\}$ is Cauchy in $\mathrm{W}_0^{1,2}(\Omega)$.

4 Nonlinear elliptic equations - monotone operator theory

Lemma 9. *Let $g : B(0,R) \subset \mathbb{R}^n \to \mathbb{R}^N$ be continuous, $N \in \mathbb{N}, R > 0$, and $\forall c \in S(0,R) : g(c) \cdot c \ge 0$. Then, there is $c_0 \in B(0,R) : g(c_0) = 0$.*

Proof. By contradiction. Let $g \neq 0$ in $\mathrm{U}(0, R)$. Let us define

$$h(x) = -R\frac{g(x)}{|g(x)|}.$$

Then $h \in C(\mathrm{B}(0, R)), h(\mathrm{B}(0, R)) \subset \mathrm{S}(0, R)$, so by Brouwer there $\exists x_0 \in \mathrm{B}(0, R) : h(x_{0=}x_0 \Rightarrow -R\frac{g(x_0)}{|g(x_0)|} = x_0$. Take the dot product with x_0 and write

$$\underbrace{-R\frac{g(x_0) \cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in \mathrm{S}(0, R),$$

so that is a contradiction. □

5 Exercises

5.1 4.3.2025

Example (Coefficients for smooth extension)**.** Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right)c_j, x_d < 0.$$

for $u \in \mathcal{D}\left(\mathbb{R}^d\right)$. Find $\{c_j\}_{j=1}^{k=1}$ in such a way that $Eu \in C^k\left(\mathbb{R}^d\right)$. Moreover, take $d = 1$.

Proof. For $k = 0, j = 1$ we take $c_1 = 1, c_j = 0, j \neq 1$. For $k = 1$, compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k=1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take $x_d = 0$ in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0)\left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \ldots, k\}.$$

That is a linear system of $k+1$ equations. Is it solvable? □