Classical problems in continuum mechanics

Kamil Belan

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1 Curvilinear coordinates, tensor & vector calculus

How to write $\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u}$ etc. in polar, cylindrical and other coordinates? Notice that there are similiarities between change of coordinats $\mathbf{x} = \mathbf{x}(\boldsymbol{\gamma})$ and deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{x})$, the $\mathbb{B}$ tensor and the metric tensor $\mathbb{g}$.

1.1 Curvilinear coordinates

Let us have $x^1, x^2, \dots, x^n$ cartesian coordinates and a different set $\mathbf{x} = \mathbf{x}(\mathbf{y})$, for example $x = r\cos\varphi, y = r\sin\varphi, [x, y] = [x^1, x^2], [r, \varphi] = [y^1, y^2]$. That means every point on a plane can be described by using $[x^1, x^2]$ or $[r, \varphi]$. We are used to analysis in cartesian coordinates - how can i do it in a more general setting?

Remark. The name curvilinear coordinates come from the fact that the lines $y^k = \text{const}$ are not "straight lines"

Definition 1 (Coordinate lines)**.** Coordinate lines/curves are the curves

$$\boldsymbol{\gamma}^j(y^j) = \mathbf{x}(y^1, \dots, y^j, \dots, y^n).$$

1.1.1 Basis of a vector space

In cartesian coordinates: $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where the vectors are *tangent to the coordinate lines*, that is

$$\mathbf{e}_i = \frac{\partial \boldsymbol{\gamma}^i}{\partial x^i}. \tag{1}$$

In a curvilinear coordinate system, we can repeat the same construction. We can *define a vector tangent to the coordinate line*

$$\mathbf{g}_i(\mathbf{y}) = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}y^i}\left(y^i\right) = \frac{\partial \mathbf{x}}{\partial y^i}\left(y^1, \dots, y^i, \dots y^n\right) \tag{2}$$

The problem is that the vectors $\mathbf{g}_i$ are not constant in space! It is a vector field!.

1.1.2 Vector fields

A vector $\mathbf{v}$ is independent of a basis; i can write $\mathbf{v} = v^i \mathbf{e}_i = \nu^i \mathbf{g}_i$. (Note that in general $v^i \neq \nu^i$.) What about its derivatives?

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial\left(v^j \mathbf{e}_j\right)}{\partial x^i} = \frac{\partial v^j}{\partial x^i}\mathbf{e}_j,$$

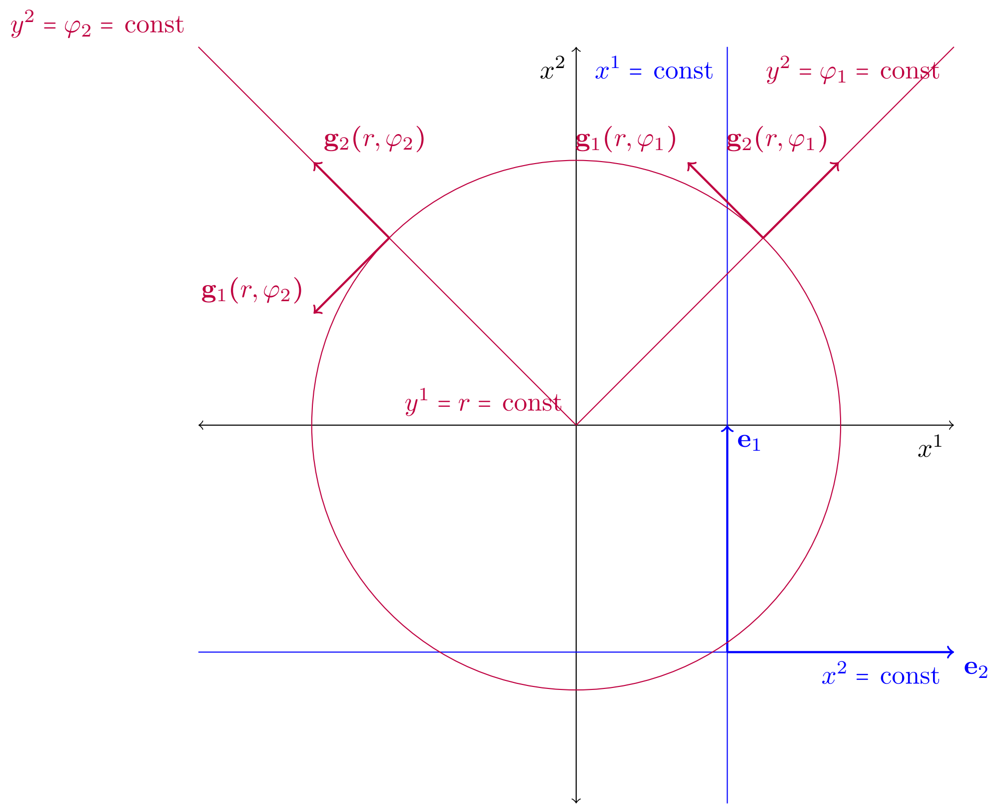

Figure 1: Coordinate lines and basis vectors in cartesian and polar coordinates
(*the length of the vectors is the same...*)

works perfectly fine in cartesian coordinates, as $\mathbf{e}_j = \text{const}$. In curvilinear setting

$$\frac{\partial \mathbf{v}}{\partial y^i} = \frac{\partial\left(v^j \mathbf{g}_j\right)}{\partial y^i} = \frac{\partial v^j}{\partial y^i}\mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial y^i}, \tag{3}$$

as generally $\frac{\partial \mathbf{g}_j}{\partial y^i} \neq \mathbf{0}$. We can identify the last term, as it must be a vector:

$$\frac{\partial \mathbf{g}_j}{\partial y^i} = \Gamma^k_{ji}\mathbf{g}_k,$$

where Γ^k_{ji} are the coefficients of the linear combinations. Thanks to the *commutation of the partial derivatives* [1], it holds

$$\Gamma^k_{ji} = \Gamma^k_{ij}, \tag{4}$$

i.e., Γ^k_{ij} is symmetric in ij. Well, that did not help *very* much, as we don't know Γ^k_{ji}, but at least we have the symmetry property. Going back to 3:

$$\frac{\partial \mathbf{v}}{\partial y^i} = \frac{\partial v^j \mathbf{g}_j}{\partial y^i} = \frac{\partial v^j}{\partial y^i}\mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial y^i} = \frac{\partial v^k}{\partial y^i}\mathbf{g}_k + v^j \Gamma^k_{ij}\mathbf{g}_k = \left(\frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j\right)\mathbf{g}_k. \tag{5}$$

In short $\frac{\partial \mathbf{v}}{\partial y^i} = \left(\frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j\right)\mathbf{g}_k$. Compare it to $\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial v^k}{\partial x^i}\mathbf{e}_k$. This leads us to the definition

Definition 2 (Covariant derivative of a vector field)**.** The quantity:

$$v^k|_i = \frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j, \tag{6}$$

is called the **covariant derivative of the vector field v**

1.1.3 Dot product

The number $\mathbf{v} \cdot \mathbf{u}$ is obtained in a special manner:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{e}_i \cdot u^j \mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{e}_j) v^i u^j = \delta_{ij} v^i u^j.$$

I can of course write the vectors in a different basis:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{g}_i \cdot u^j \mathbf{g}_j = (\mathbf{g}_i \cdot \mathbf{g}_j) v^i u^j = g_{ij} v^i u^j.$$

Definition 3 (Metric tensor)**.** The tensor $\mathbb{g}$ such that $\forall \mathbf{v} = v^i \mathbf{g}_i, \mathbf{u} = u^j \mathbf{g}_j$ it holds:

$$\mathbf{v} \cdot \mathbf{u} = g_{ij} v^i u^j, g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

is called the **metric tensor**.

[1] We are still in flat $\mathbb{R}^d$, i.e. euclidian space. No curvature, torsion, that would obstruct the comutation properties.

1.1.4 Dual space

The (vector) dual space is the space of all linear forms on the underlying vector space. In particular it is a vector basis itselfs, so $\forall \mathbf{l} \in V^* : \mathbf{l} = l_i \mathbf{e}^i$, where $\mathbf{e}^i$ is the i-th basis vector. The action of the forms can be described as

$$\mathbf{l}(\mathbf{v}) = l_i \mathbf{e}^i (v^j \mathbf{e}_j) = l_i v^j \mathbf{e}^i (\mathbf{e}_j), \forall \mathbf{v} \in V.$$

If it holds $\mathbf{e}^\mathbf{i}(\mathbf{e}_j) = \delta^i_j$, we call the basis $\mathbf{e}^i$ dual to $\mathbf{e}_j$. What about curvilinear setting? We can adopt the same definition

Definition 4. We call the basis $\mathbf{g}^j$ of V^* the dual basis to $\mathbf{g}_i$ iff

$$\mathbf{g}^j(\mathbf{g}_i) = \delta^j_i.$$

For the original basis we had $\mathbf{g}_i = \frac{\partial \gamma}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \mathbf{e}_j$, in the dual case (using the chain rule):

$$\delta^i_j = \frac{\partial y^i}{\partial y^j} = \frac{\partial y^i}{\partial x^k} \frac{\partial x^k}{\partial y^j},$$

so i can conclude

$$\mathbf{g}^j = \frac{\partial y^j}{\partial x^k} \mathbf{e}^k.$$

Recall that we have the *Riesz representation theorem*:

$$\forall \mathbf{l} \in V^*, \exists \quad unique \mathbf{u} \in V : \mathbf{l}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{u}, \forall \mathbf{v} \in V.$$

This implies

$$l_i \mathbf{g}^i (v^j \mathbf{g}_j) = u^m u^n g_{mn}, \quad i.e. l_i v^j \delta^i_j = u^m v^n g_{mn}, \quad i.e. l^i v_i = u^m v^i g_{mi}, \quad i.e. l_i = g_{im} u^m.$$

So $l_i = g_{im} u^m$, where $\mathbf{u}$ represents $\mathbf{l}$. It is common to write

$$l_i = g_{im} l^m.$$

1.1.5 Covector fields

How to compute $\frac{\partial \mathbf{l}}{\partial y^i}$? Just change the location of the index :)

$$\frac{\partial \mathbf{l}}{\partial y^i} = \frac{\partial (l_j \mathbf{g}^j)}{\partial y^i} = \frac{\partial l^j}{\partial y^i} \mathbf{g}^j + \frac{\partial \mathbf{g}^j}{\partial y^i} l_j.$$

Again, the last term must be expressable in the dual basis, so

$$\frac{\partial \mathbf{l}}{\partial y^i} = \frac{\partial l_j}{\partial y^i} \mathbf{g}^j + \tilde{\Gamma}^j_{im} l_j \mathbf{g}^m, \tag{7}$$

where again $\Gamma^{\tilde{j}}_{im} = \Gamma^{\tilde{j}}_{mi}$ are the coefficients of the linear combinations, "that are symmetric".

What is the relation between Γ^k_{im} and $\Gamma^{\tilde{k}}_{im}$? Recall $\delta^i_j = \mathbf{g}^i(\mathbf{g}_j)$, so differentiating can lead us to

$$\Gamma^j_{im} = -\Gamma^j_{im}. \tag{8}$$

Definition 5. Let $\mathbf{l}$ be a covector field. The quantity

$$\mathbf{l}_m|_j = \frac{\partial \mathbf{l}}{\partial y^j} - \Gamma^l{}_{jm} v_l, \tag{9}$$

is called **the covariant derivative of the covector field l**.

TODO $\mathbb{A} = A_{mn} \mathbf{g}^m \otimes \mathbf{g}^n$.

1.1.6 Direct expression of the Christoffel symbols

With the above relation, we can express $g_{mn}|_j$. Moreover, we can directly differentiate.

$$\frac{\partial g_{mn}}{\partial y^j} = \frac{\partial(\mathbf{g}_m \cdot \mathbf{g}_n)}{\partial y^j} = \frac{\partial \mathbf{g}_m}{\partial y^j} \cdot \mathbf{g}_n + \mathbf{g}_m \frac{\partial \mathbf{g}_n}{\partial y^j} = \Gamma^k{}_{mj} \mathbf{g}_k \cdot \mathbf{g}_n + \mathbf{g}_m \cdot \Gamma^k{}_{nj} \mathbf{g}_k = \Gamma^k{}_{mj} g_{kn} + \Gamma^k{}_{nj} g_{mk}$$

,

$$g_{mn}|_j = \frac{\partial g_{mn}}{\partial y^j} - g_{kn} \Gamma^k{}_{jm} - g_{mk} \Gamma^k{}_{jn}.$$

From this, it follows

$$g_{mn}|_j = 0. \tag{10}$$

This property is particularly useful, as it allows us to express the Christoffel symbols. Using cyclic permuation, we can write

$$\begin{aligned} A &= \frac{\partial g_{mn}}{\partial y^j} = \Gamma^k{}_{mj} g_{kn} + \Gamma^k{}_{nj} g_{mk}, \\ B &= \frac{\partial g_{jm}}{\partial y^j} = \Gamma^k{}_{jn} g_{kn} + \Gamma^k{}_{mn} g_{jk}, \\ C &= \frac{\partial g_{nj}}{\partial y^n} = \Gamma^k{}_{nm} g_{kj} + \Gamma^k{}_{jm} g_{nk}. \end{aligned}$$

Taking $A - B - C$ yields

$$\frac{\partial g_{mn}}{\partial y^j} - \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} = -2\Gamma^k{}_{nm} g_{jk},$$

multiplying by g^{jl} gives

$$-2\Gamma^k{}_{nm} \delta^l_k = g^{jl} \left(\frac{\partial g_{mn}}{\partial y^j} - \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} \right),$$

from which it follows

$$\Gamma^l{}_{nj} = \frac{1}{2} g^{lm} \left(\frac{\partial g_{mn}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} \right). \tag{11}$$

1.1.7 Interchangability of the derivatives

In euclidian space:

$$\frac{\partial^2 \mathbf{v}}{\partial y^j \partial y^i} = \frac{\partial^2 \left(v^k \mathbf{e}_k \right)}{\partial y^j \partial y^i} = \left(\frac{\partial^2 v^k}{\partial x^j \partial x^i} \right) \mathbf{e}_k,$$

when $\mathbf{e}_k$ are basis vectors of *cartesian coordinate system.* Will it hold even in curvilinear coordinate systems?

$$
\begin{aligned}
0 &= \frac{\partial^2 \mathbf{v}}{\partial x^j \partial x^k} - \frac{\partial^2 \mathbf{v}}{\partial x^j \partial x^k} = \text{apply the covariant derivative two times} = \\
&= \left(v^k|_{ij} - v^k|_{ji}\right)\mathbf{g}_k = \left(\frac{\partial \Gamma^i{}_{jm}}{\partial y^k} - \frac{\partial \Gamma^i{}_{km}}{\partial y^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}\right) v^m \mathbf{g}_i.
\end{aligned}
$$

Definition 6 (Riemann curvature tensor)**.** The *tensor*

$$
R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial y^k} - \frac{\partial \Gamma^i{}_{km}}{\partial y^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}, \tag{12}
$$

is called the **Riemann curvature tensor**

We see that if the Riemann curvature tensor is zero, then effectively, we are in the case of a flat euclidian space, as the derivatives commute (?). In other words, in flat euclidian space, the Riemann curvature tensor is always zero. If we flip this, we see that if we have a space with zero Riemann curvature tensor, *we have a chance* that the derivatives commute, i.e. that the structure is euclidian.

Example (Interpretation in continuum mechanics)**.**

$$
\begin{aligned}
\mathbf{x} &= \mathbf{x}(\mathbf{y}) \text{ vs } \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \\
\mathbf{g}_i &= \frac{\partial \mathbf{x}}{\partial y^i} \text{ vs } \mathbf{g}_i = \frac{\partial \boldsymbol{\chi}}{\partial X^i}, \text{ i.e. } (\mathbf{g}_m)^i = F^i{}_m = \frac{\partial \chi^i}{\partial X^m}, \\
g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j = (\mathbb{F}^\top \mathbb{F})_{ij} = (\mathbb{C})_{ij},
\end{aligned}
$$

So

$$
\mathbb{g} = \mathbb{C}. \tag{13}
$$

We can calculate:

$$
\frac{\partial \mathbf{g}_m}{\partial X^j} = \Gamma^k{}_{mj}\mathbf{g}_k,
$$

which is a system of equations for the basis vectors. There are some solvability conditions

$$
\partial_{JI}\mathbf{g}_m = \partial_{IJ}\mathbf{g}_m = \frac{\partial\left(\Gamma^k{}_{im}\mathbf{g}_k\right)}{\partial X^j} = \frac{\partial\left(\Gamma^k{}_{mj}\mathbf{g}_k\right)}{\partial X^i},
$$

this is equivalent to

$$
\partial_{JI}\mathbf{g}_m - \partial_{IJ}\mathbf{g}_n = 0 \Leftrightarrow \ldots R^i{}_{jkm} = 0.
$$

So this implies that all physically admissable deformations produce a deformed space with zero Riemann curvature.

1.2 Calculus

1.2.1 Gradient

Remember $(\nabla\varphi)_i = \frac{\partial\varphi}{\partial x^i}$, so that means

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{e}^i.$$

The gradient is a covector.

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{e}^i = \frac{\partial\varphi}{\partial\xi^j}\underbrace{\frac{\partial\xi^j}{\partial x^i}\mathbf{e}^i}_{=\mathbf{g}^j} = \frac{\partial\varphi}{\partial\xi^j}\mathbf{g}^j,$$

What about the gradient of a vector field? In cartesian coordinate system:

$$\nabla\mathbf{v} = \nabla\left(v^i\mathbf{e}_i\right) = \frac{\partial v^i}{\partial x^j}\mathbf{e}_i\otimes\mathbf{e}^j = \mathbf{v}\otimes\nabla.$$

In curvilinear coordinates:

$$\begin{aligned}
\nabla\left(v^i\mathbf{g}_i\right) &= \nabla\left(v^i\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right) = \frac{\partial}{\partial x^j}\left(v^i\frac{\partial x^m}{\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \left(\frac{\partial v^i}{\partial x^j}\frac{\partial x^m}{\partial\xi^i} + v^i\frac{\partial^2 x^m}{\partial x^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \left(\frac{\partial v^i}{\partial\xi^n}\frac{\partial\xi^n}{\partial x^j}\frac{\partial x^m}{\partial\xi^i} + v^i\frac{\partial^2 x^m}{\partial x^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial x^m}{\partial\xi^u}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial x^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial x^m}{\partial\xi^i}\right)\frac{\partial\xi^l}{\partial x^j}\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial x^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^l}{\partial x^j}\mathbf{e}^j\right) = \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\frac{\partial\mathbf{g}^i}{\partial\xi^l}\otimes\mathbf{g}^l = \\
&= \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\Gamma^s{}_{il}\mathbf{g}_s\otimes\mathbf{g}^l = \left(\frac{\partial v^s}{\partial\xi^l} + \Gamma^s{}_{il}v^i\right)\mathbf{g}_s\otimes\mathbf{g}^l = \\
&= v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l.
\end{aligned}$$

Until now, we have not discussed the fact $|\mathbf{g}_i| \neq 1$, which is a kind of a problem. Let us define

$$\mathbf{v} = v^i\mathbf{g}_i = v^i|\mathbf{g}_i|\frac{\mathbf{g}_i}{|\mathbf{g}_i|} = v^{\hat{i}}\mathbf{g}_{\hat{i}},$$

where we have defined

$$v^{\hat{i}} = |\mathbf{g}^i|v^i, \mathbf{g}_{\hat{i}} = \frac{\mathbf{g}_i}{|\mathbf{g}_i|}.$$

But! the differential formulas work for $v^i, \mathbf{g}_i$, **not for** $v^{\hat{i}}, \mathbf{g}_{\hat{i}}$**!**

$$\nabla\varphi = \frac{\partial\varphi}{\partial\xi^j}\mathbf{g}^j = |\mathbf{g}^i|\frac{\partial\varphi}{\partial\xi^i}\mathbf{g}^{\hat{i}}, \tag{14}$$

$$\nabla\mathbf{v} = v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l = |\mathbf{g}_s||\mathbf{g}^l|v^s|_l\mathbf{g}_{\hat{s}}\otimes\mathbf{g}^{\hat{l}}, \tag{15}$$

For the divergence of a vector field, we know: $\operatorname{tr}(\mathbf{u}\otimes\mathbf{v})$, so

$$\nabla\cdot\mathbf{v} = \operatorname{tr}(\nabla\mathbf{v}) = \operatorname{tr}\left(v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l\right) = v^s|_s.$$

The divergence of a tensor field a can be tricky, but be guided by the summation convention; for the tensor $\mathbb{A} = A^{is}\mathbf{g}_i \otimes \mathbf{g}_s$ we can define

$$\nabla \cdot \mathbb{A} = A^{is}|_s \mathbf{g}_i.$$

For the tensors of a different type, we need to change the position of the indices to obtain a bivector.

1.2.2 Laplace-Beltrami operator

$$\triangle \varphi = \frac{1}{\sqrt{\det \mathbb{g}}} \frac{\partial}{\partial \xi^i} \left(\sqrt{\det \mathbb{g}} g^{ij} \frac{\partial \varphi}{\partial \xi^j} \right),$$

on one hand:

$$\triangle \varphi = \nabla \cdot \nabla \varphi = (\nabla \varphi)^i |_i = \left(g^{ij} \frac{\partial \varphi}{\partial \xi^j} \right) \Big|_i =$$

where we have rised the index $(\nabla\varphi)^i = g^{ij}(\nabla\varphi)_j = g^{ij}\frac{\partial \varphi}{\partial \xi^j}$, so using the covariant derivative definition

$$\nabla \cdot \nabla \varphi = \frac{\partial}{\partial \xi^i} \left(g^{ij} \frac{\partial \varphi}{\partial \xi^j} \right) + \Gamma^i{}_{il} g^{lj} \frac{\partial \varphi}{\partial \xi^j},$$

on the other

$$\begin{aligned}
\triangle \varphi &= \frac{1}{\sqrt{\det \mathbb{g}}} \left(\frac{\partial}{\partial \xi^i} \left(\sqrt{\det \mathbb{g}} \right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} \frac{\partial g^{ij}}{\partial \xi^i} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j} \right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}} \left(\frac{1}{2\sqrt{\det \mathbb{g}}} \frac{\partial}{\partial \xi^i} (\det \mathbb{g}) g^{ij} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} \frac{\partial g^{ij}}{\partial \xi^i} + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j} \right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}} \left(\frac{1}{2} \operatorname{tr} \left(\mathbb{g}^{-1} \frac{\partial \mathbb{g}}{\partial \xi^i} \right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} - \sqrt{\det \mathbb{g}} \left(\Gamma^j{}_{kn} g^{in} - \Gamma^i{}_{km} g^{mj} \right) + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j} \right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}} \left(\frac{1}{2} \left(g^{mn} \frac{\partial g_{mn}}{\partial \xi^i} \right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} - \right)
\end{aligned}$$

1.2.3 Bipolar coordinates

Define $\boldsymbol{\xi} = [\alpha, \beta]$, where

$$\alpha + i\beta = \log \frac{y + i(x+a)}{y + i(x-a)}.$$

This can be inversed and write

$$x = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta},$$
$$y = \frac{a \sin \beta}{\cosh \alpha - \cos \beta},$$

moreover,

$$(x - a\coth\alpha)^2 + y^2 = \frac{a^2}{\sinh^2\alpha},$$
$$x^2 + (y - a\cot\beta)^2 = \frac{a^2}{\sin^2\beta}.$$

Calculate *everything* for this coordinate system.

In general $\mathbf{g}_i = \frac{\partial x^j}{\partial \xi^i}\mathbf{e}_j$, so in our case

$$\begin{aligned}
\mathbf{g}_\alpha &= \frac{\partial x}{\partial \alpha}\mathbf{e}_x + \frac{\partial y}{\partial \alpha}\mathbf{e}_y \\
&= \left(\frac{(a\cosh\alpha)(\cosh\alpha - \cos\beta) - a\sinh\alpha\sinh\alpha}{(\cosh\alpha - \cos\beta)^2}\right)\mathbf{e}_x + \left(\frac{a\cos\beta(\cosh\alpha - \cos\beta) - a\sin\beta\sinh\alpha}{(\cosh\alpha - \cos\beta)^2}\right)\mathbf{e}_y = \\
&= \frac{a}{(\cosh\alpha - \cos\beta)^2}((1 - \cosh\alpha\cos\beta)\mathbf{e}_x - (\sin\beta\sinh\alpha)\mathbf{e}_y), \\
\mathbf{g}_\beta &= \frac{\partial x}{\partial \beta}\mathbf{e}_x + \frac{\partial y}{\partial \beta}\mathbf{e}_y \\
&= \cdots = \\
&= \frac{a}{(\cosh\alpha - \cos\beta)^2}(-(\sin\beta\sinh\alpha)\mathbf{e}_x + (-1 + \cosh\alpha\cos\beta)\mathbf{e}_y).
\end{aligned}$$

We can see that $\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = 0$ and so

$$\mathbb{g} = \left(\frac{a}{\cosh\alpha - \cos\beta}\right)^2 \mathbb{I}, \mathbb{g}^{-1} = \left(\frac{\cosh\alpha - \cos\beta}{a}\right)^2 \mathbb{I}.$$

Coming back to Laplace-Beltrami operator, we can calculate

$$\left(\sqrt{\det\mathbb{g}}\mathbb{g}^{-1}\right) = \left(\frac{a}{\cosh\alpha - \cos\beta}\right)^2 \left(\frac{\cosh\alpha - \cos\beta}{\alpha}\right)^2 \mathbb{I} = \cdots = \mathbb{I},$$

and calculating a bit more yields

$$\triangle\varphi \to \left(\frac{\cosh\alpha - \cos\beta}{a}\right)^2 \triangle_{\alpha\beta}\varphi.$$

Remark (Relation to complex analysis). This can be seen as a conformal transformation

$$\gamma = f(z),$$

where

$$\gamma = \alpha + i\beta,$$
$$z = x + iy,$$

Let us write

$$f(z) = f^x(x,y) + i f^y(x,y) \Leftrightarrow \mathbf{f}(\mathbf{x}) = [f^x(\mathbf{x}), f^y(\mathbf{x})], z = x + iy,$$

and compute

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ \frac{\partial f^y}{\partial x} & \frac{\partial f^y}{\partial y} \end{bmatrix}.$$

Recall Cauchy-Riemmann conditions:

$$\frac{\partial f^x}{\partial x} = \frac{\partial f^y}{\partial y},$$
$$\frac{\partial f^x}{\partial y} = -\frac{\partial f^y}{\partial x},$$

using which the gradient becomes:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ -\frac{\partial f^x}{\partial y} & \frac{\partial f^y}{\partial y} \end{bmatrix},$$

which is an **orthogonal matrix:**

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top = \left(\left(\frac{\partial f^x}{\partial x}\right)^2 + \left(\frac{\partial f^y}{\partial y}\right)^2\right)\mathbb{I}.$$

Realize that all this structure comes just from the fact that the transformation is given through a holomorfic function.

1.2.4 Compatibility conditions in linearised elasticity

$$R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial \xi^k} - \frac{\partial \Gamma^i{}_{km}}{\partial \xi^j} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km},$$

and we know

$$R^i{}_{jkm} = 0 \Leftrightarrow \mathbb{C} = \mathbb{F}^\top\mathbb{F}, \mathbb{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}.$$

All this works in fully *nonlinear setting!*. In the classical lecture, we have been able to obtain compatibility condition in *linearised elasticity*: $\nabla \times \varepsilon = \mathbb{0}, \mathbb{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$.

Consider the following setting:

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}), \\ \mathbf{u} &= \chi(\mathbf{X}) - \mathbf{X}, \\ \nabla\mathbf{u} &= \mathbb{F} - \mathbb{I}, \\ \mathbb{F} &= \mathbb{I} + \nabla\mathbf{u}, \end{aligned}$$

then

$$\mathbb{C} = \mathbb{F}^\top\mathbb{F} = \left(\mathbb{I} + (\nabla\mathbf{u})^\top\right)(\mathbb{I} + \nabla\mathbf{u}) = \mathbb{I} + 2\varepsilon + \text{h.o.t.},$$

and so

$$\mathbb{g}^{-1} = \mathbb{I} - 2\varepsilon.$$

The Christoffel symbols are

$$\begin{aligned} \Gamma^l{}_{nj} &= \frac{1}{2}g^{lm}\left(\frac{\partial g_{mn}}{\partial X^j} + \frac{\partial g_{jm}}{\partial X^n} - \frac{\partial g_{nj}}{\partial X^m}\right) \\ &\approx \frac{1}{2}(\mathbb{I} - 2\varepsilon)^{lm}\left(\frac{\partial}{\partial X^j}(\mathbb{I} + 2\varepsilon)_{mn} + \frac{\partial}{\partial X^n}(\mathbb{I} + 2\varepsilon)_{jm} - \frac{\partial}{\partial X^m}(\mathbb{I} + 2\varepsilon)_{nj}\right), \\ &\approx \delta^{lm}\left(\frac{\partial \varepsilon_{mn}}{\partial X^j} + \frac{\partial \varepsilon_{jm}}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}\right) = \frac{\partial \varepsilon^l_n}{\partial X^j} + \frac{\partial \varepsilon^l_j}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}, \end{aligned}$$

the Riemann curvature tensor is (linear approximation)

$$
\begin{aligned}
0 = R^i{}_{jkm} &\approx \frac{\partial \Gamma^i{}_{jm}}{\partial X^k} - \frac{\partial \Gamma^i{}_{km}}{\partial X^j} \\
&= \frac{\partial}{\partial X^k}\left(\frac{\partial \varepsilon^i_j}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^j} - \frac{\partial \varepsilon_{mj}}{\partial X^i}\right) - \frac{\partial}{\partial X^j}\left(\frac{\partial \varepsilon^i_k}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^k} - \frac{\partial \varepsilon_{km}}{\partial X^i}\right) = \\
&= \frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i},
\end{aligned}
$$

so the compatibility conditions are

$$
\frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i} = 0.
$$

1.3 Surface geometry

In this part, we will work with surfaces embedded in $\mathbb{R}^3$.

Let $G = \{\mathbf{u}\} \subset \mathbb{R}^2$ be the parametrization space and $\mathbf{\Phi} : G \subset \mathbb{R}^2 \to \mathbb{R}^3$ is the parametrization, so the points of the surface are

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{u}), \mathbf{x} \in \mathbb{R}^3.$$

Definition 7. The indices $i, j, k, \dots \in \{1, 2, 3\}$ will denote objects from $\mathbb{R}^3$ and indices $\alpha, \beta, \gamma, \dots \in \{1, 2\}$ will denote indices of objects from $\mathbb{R}^2$.

1.3.1 Tangent and normal vectors

As in the previous story, we can define (basis) tangent vectors:

$$\mathbf{t}_1 = \frac{\partial \mathbf{\Phi}}{\partial u^1}, \mathbf{t}_2 = \frac{\partial \mathbf{\Phi}}{\partial u^2},$$

and on surfaces, of importance is also the normal vector

$$\mathbf{n} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|}.$$

1.3.2 Distances and angles

The metric tensor on the surface is given by

$$\mathbb{g}_s = \begin{bmatrix} \mathbf{t}_1 \cdot \mathbf{t}_1 & \mathbf{t}_1 \cdot \mathbf{t}_2 \\ \mathbf{t}_1 \cdot \mathbf{t}_2 & \mathbf{t}_2 \cdot \mathbf{t}_2 \end{bmatrix},$$

or in context of diff. geo. it is called **the first fundamental form**.

1.3.3 Derivatives

In $\mathbb{R}^3$, we know how to differentiate tangent vectors (using Christoffel symbols). The metric tensor in $\mathbb{R}^3$ is given by

$$\mathbb{g} = \begin{bmatrix} \mathbb{g}_s & 0 \\ 0 & 1 \end{bmatrix}$$

Realize that, veiwed in $\mathbb{R}^3$ the relation can be written as

$$\frac{\partial \mathbf{t}_\alpha}{\partial u^\beta} = \Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + b_{\alpha\beta}\mathbf{n},$$

because viewed from $\mathbb{R}^3$, the basis vectors are $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ and we have just denoted $b_{\alpha\beta} = \Gamma^3{}_{\alpha\beta}$.

What about the derivative of the normal vector? From the length of $\mathbf{n}$ we know

$$\mathbf{n} \cdot \mathbf{n} = 1 \Rightarrow \frac{\partial \mathbf{n}}{\partial u^\alpha} \cdot \mathbf{n} = 0,$$

so that means the derivative is perpendicular to the normal direction, so

$$\frac{\partial \mathbf{n}}{\partial u^\alpha} = A^\gamma{}_\alpha \mathbf{t}_\gamma.$$

Next trick is to realize

$$0 = \frac{\partial}{\partial u^\alpha}(\mathbf{n} \cdot \mathbf{t}_\beta) = \frac{\partial \mathbf{n}}{\partial u^\alpha} \cdot \mathbf{t}_\beta + \mathbf{n} \cdot \frac{\partial \mathbf{t}_\beta}{\partial u^\alpha} = A^\gamma{}_\alpha \mathbf{t}_\gamma \cdot \mathbf{t}_\beta + \mathbf{n} \cdot \left(\Gamma^\delta{}_{\alpha\beta}\mathbf{t}_\delta + b_{\alpha\beta}\mathbf{n}\right) = A^\gamma{}_\alpha g_{s,\gamma\beta} + b_{\alpha\beta},$$

from which it follows

$$A^\gamma{}_\alpha = -g^{\gamma\beta} b_{\beta\alpha}.$$

1.3.4 Commutation of derivatives

What are the *implications* of

$$\frac{\partial^2 \mathbf{t}_\alpha}{\partial u^\beta \partial u^\gamma} = \frac{\partial^2 \mathbf{t}_\alpha}{\partial u^\beta \partial u^\alpha}?$$

Write

$$0 = \frac{\partial^2 \mathbf{t}_\alpha}{\partial u^\beta \partial u^\gamma} - \frac{\partial^2 \mathbf{t}_\alpha}{\partial u^\beta \partial u^\alpha} = (\text{ something })\mathbf{t}_\delta + (\text{ something different })\mathbf{n},$$

so we see the whole thing splits into two parts. It can be shown

Theorem 1 (Gauss relation)**.**

$$R_{\psi\beta\delta\alpha} = b_{\alpha\beta} b_{\psi\delta} - b_{\alpha\delta} b_{\psi\beta}.$$

Theorem 2 (Codazzi-Mainardi relation)**.**

$$b_{\alpha\beta}|_\delta - b_{\alpha\delta}|_\beta = 0$$

1.3.5 Surfaces evolving in time

Now the points of the surface are given by

$$\mathbf{x} = \mathbf{\Phi}(t, \mathbf{u}), \text{ where } \mathbf{\Phi} : \mathbb{R} \times G \to \mathbb{R}^3.$$

We can define the **velocity of the surface**:

$$\mathbf{v}_s = \frac{\partial \mathbf{\Phi}}{\partial t}(t, \mathbf{u}).$$

The basis of everything has always been Gauss theorem; we will be interested in the quantity of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S\,,$$

where $S(t)$ is a time-dependent surface. Let us try the approach from Reynolds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{\Phi}(t,\mathbf{x}(t))^{-1}}\psi(t,\mathbf{\Phi}(t,\mathbf{u}))\sqrt{\det\mathbb{g}_s}\,\mathrm{d}u^1\,\mathrm{d}u^2 =,$$

and now we need to calculate the derivatives. Start slow:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{t}_\alpha}{\mathrm{d}t} &= \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{\Phi}}{\partial u^\alpha}(t,\mathbf{u})\right) = \frac{\partial}{\partial u^\alpha}\underbrace{\left(\frac{\partial\mathbf{\Phi}}{\partial t}(t,\mathbf{u})\right)}_{=\mathbf{v}_s(t,\mathbf{u})} = \frac{\partial}{\partial u^\alpha}\left(\mathbf{v}_\parallel + v_\perp\mathbf{n}\right) = \\
&= \frac{\partial\mathbf{v}_\parallel}{\partial u^\alpha} + \frac{\partial(v_\perp\mathbf{n})}{\partial u^\alpha} = \frac{\partial\left(v_\parallel^\beta\mathbf{t}_\beta\right)}{\partial u^\alpha} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} + v_\perp\frac{\partial\mathbf{n}}{\partial u^\alpha} = \\
&= \frac{\partial v_\perp^\beta}{\partial u^\alpha}\mathbf{t}_\beta + \frac{\partial\mathbf{t}_\beta}{\partial u^\alpha}v_\parallel^\beta + \frac{\partial v_{per}}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta}b_{\beta\alpha}\mathbf{t}_\gamma = \\
&= \frac{\partial v_\parallel^\beta}{\partial u^\alpha}\mathbf{t}_\beta + v_\parallel^\beta\Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + v_\parallel^\beta b_{\alpha\beta}\mathbf{n} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta}b_{\alpha\beta}\mathbf{t}_\gamma = \\
&= v_\parallel^\beta|_\alpha\mathbf{t}_\beta - v_\perp g^{\gamma\beta}b_{\alpha\beta}\mathbf{t}_\gamma + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n} = \\
&= \left(v_\parallel^\beta|_\alpha - v_\perp g^{\beta\gamma}b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n}.
\end{aligned}$$

So all in all

$$\frac{\mathrm{d}\mathbf{t}_\alpha}{\mathrm{d}t} = \left(v_\parallel^\beta|_\alpha - v_\perp g^{\beta\gamma}b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n}.$$

Next ingredient is the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{g}_s$, so in components:

$$\frac{\mathrm{d}g_{\alpha\beta}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{t}_\alpha\cdot\mathbf{g}_\beta) = \cdots = v_\parallel^\delta|_\alpha g_{\delta\beta} + v_\parallel^\delta|_\beta g_{\delta\alpha} - 2v_\perp b_{\alpha\beta}.$$

After some further manipulation, the final formula becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S = \int_{S(t)}\frac{\mathrm{d}\psi}{\mathrm{d}t}(t,\mathbf{x}) + \psi(t,\mathbf{x})\left(\nabla\cdot\mathbf{v}_{\parallel S} - 2v_\perp(t,\mathbf{x})K(t,\mathbf{x})\right)\mathrm{d}S\,, \tag{16}$$

where

$$\nabla\cdot\mathbf{v}_{\parallel S} - 2v_\perp K := v^\beta(t,\mathbf{u})_\parallel|_\beta - 2v_\perp(t,\mathbf{u})K(t,\mathbf{u})\Big|_{\mathbf{u}=\mathbf{\Phi}(t,\mathbf{x})^{-1}}\,,$$

$$K = \frac{1}{2}g^{\beta\alpha}b_{\alpha\beta}$$

is the mean curvature.

2 Linearised elasticity

The static setting of the linearised elasticity theory is

$$\nabla \cdot \tau + \mathbf{f} = \mathbf{0}, \tag{17}$$

and for now we will want to solve for the stress, that is

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ & \tau_{yy} & \tau_{yz} \\ & & \tau_{zz} \end{bmatrix},$$

since τ is symmetric. Recall the compatibility conditions

$$\tau = \lambda(\operatorname{tr}\varepsilon)\mathbb{I} + 2\mu\varepsilon, \tag{18}$$

$$\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top}\right), \tag{19}$$

$$\nabla \times \left((\nabla \times \varepsilon)^{\top}\right) = \mathbb{O}, \tag{20}$$

$$\triangle\,\tau + \frac{1}{1+\nu}\nabla\nabla\operatorname{tr}\tau = -\left(\nabla\mathbf{f}(\nabla\mathbf{f})^{\top}\right) - \frac{\nu}{1-\nu}(\nabla\cdot\mathbf{f})\mathbb{I} \tag{21}$$

2.1 Plane stress/strain problems

In each of the cases, the stress/strain tensors have a *special structure*:

$$\tau(x,y) = \begin{bmatrix} \tau_{xx}(x,y) & \tau_{xy}(x,y) & 0 \\ \tau_{xy}(x,y) & \tau_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

and the same for the strain tensor. Inverting the stress-strain relation yields

$$\varepsilon = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau)\mathbb{I}\right),$$

but since $I_{zz} = 1$, in general we have

$$\varepsilon(x,y) = \begin{bmatrix} \varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0 \\ \varepsilon_{xy}(x,y) & \varepsilon_{yy}(x,y) & 0 \\ 0 & 0 & \varepsilon_{zz}(x,y) \end{bmatrix},$$

for $\varepsilon_{zz}(x,y) \neq 0$.

Remark (Notation). Note that in the following, operators acting on tensors will always respect the dimensionality of the tensor (so i will write $\operatorname{tr}\tau_{2D}$ instead of $\operatorname{tr}_{2D}\tau_{2D}$. And the same for the laplacian, divergence and so on

2.2 Plane stress problem

The stress is given as

$$\tau = 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)\mathbb{I},$$

where τ has the structure 22. It must hold

$$0 = \tau_{zz} = 2\mu\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}),$$

so

$$0 = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + (2\mu + \lambda)\varepsilon_{zz},$$

and that yields a condition on ε_{zz}:

$$\varepsilon_{zz} = -\frac{\lambda}{2\mu + \lambda} \operatorname{tr}_{2D} \varepsilon_{2D}.$$

The constitutive relation can than be rewritten as

$$\tau_{2D} = 2\mu\varepsilon_{2D} + \lambda(\operatorname{tr}\varepsilon_{2D} + \varepsilon_{zz})\mathbb{I}_{2D} = 2\mu\left(\varepsilon_{2D} + \frac{\lambda}{2\mu + \lambda}(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right).$$

The "2D Beltrami-Michel equations" can be derived from:

$$\triangle\,\tau + \frac{1}{1+\nu}\nabla\nabla \operatorname{tr}\tau = -\left(\nabla\mathbf{f} + (\nabla\mathbf{f})^\top\right) - \frac{\nu}{1-\nu}(\nabla\cdot\mathbf{f})\mathbb{I},$$

but there is a problem: the zz equation yields:

$$0 + \frac{1}{1+\nu}\frac{\partial^2}{\partial z^2}\left(\operatorname{tr}\tau_{2D} = 0 - \frac{\nu}{1-\nu}(\nabla\cdot\mathbf{f}_{2D})\right),$$

but in our case τ_{2D} is not a function of z, so of course we would have

$$\nabla\cdot\mathbf{f}_{2D} = 0,$$

which is not generally true! The forces are given to us. Try something different: take the trace of the Beltrami-Michell equation and obtain (after some calculation)

$$\triangle \operatorname{tr}\tau = -\frac{1+\nu}{1-\nu}\nabla\cdot\mathbf{f},$$

so rewritting in "2D" view:

$$\left(\triangle_{2D} + \frac{\partial^2}{\partial z^2}\right)\operatorname{tr}\tau_{2D} = -\frac{1+\nu}{1-\nu}\nabla\cdot\mathbf{f}_{2D}.$$

That maybe did not help much, because the z derivative is still zero, but here comes the time for some handwaving: what about we use the above equation to replace the troublemaking term? We would obtain

$$\triangle_{2D}\operatorname{tr}\tau_{2D} - \nu\frac{1+\nu}{1-\nu}\nabla\cdot\mathbf{f}_{2D} = -\frac{1+\nu}{1-\nu}(\nabla\cdot\mathbf{f}_{2D}),$$

so after some manipulation

$$\triangle\operatorname{tr}\tau_{2D} = -(1+\nu)\nabla\cdot\mathbf{f}_{2D}.$$

In total, the problem is described as

$$\mathbf{0}_{2D} = \nabla\cdot\tau_{2D} + \mathbf{f}_{2D}, \tag{23}$$

$$\tau_{2D} = 2\mu\left(\varepsilon_{2D} + \frac{\lambda}{2\mu+\lambda}(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right) \tag{24}$$

$$\triangle\operatorname{tr}\tau_{2D} = -(1+\nu)\nabla\cdot\mathbf{f}_{2D}. \tag{25}$$

2.3 Plain strain problem

This time, the structure of the stress and strain are:

$$\varepsilon(x,y) = \begin{bmatrix} \varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0 \\ \varepsilon_{xy}(x,y) & \varepsilon_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tau(x,y) = \begin{bmatrix} \tau_{xx}(x,y) & \tau_{xy}(x,y) & 0 \\ \tau_{xy}(x,y) & \tau_{yy}(x,y) & 0 \\ 0 & 0 & \tau_{zz}(x,y) \end{bmatrix}.$$

Using a similiar approach, we can calculate, using $\varepsilon = \frac{1}{2\mu}\Big(\tau - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau)\mathbb{I}\Big)$,

$$\tau_{zz} = \lambda \operatorname{tr}\varepsilon_{2D},$$

so the constituive relation is

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau_{2D} + \tau_{zz})\mathbb{I}_{2D}\right) = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D} - \frac{\lambda}{3\lambda+2\mu}\lambda(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right),$$

so taking the trace we can obtain: $\tau_{zz} = \frac{\lambda}{\lambda(\lambda+\mu)}\operatorname{tr}\tau_{2D}$ and plugging it into the original equation yields

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{2(\lambda+\mu)}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D}\right).$$

As for the Beltami-Michell equations, taking the trace gives us again

$$\triangle \operatorname{tr}\tau = -\frac{1+\nu}{1-\nu}\nabla\cdot\mathbf{f},$$

and in *plain strain*, we are able to simply do

$$\triangle \operatorname{tr}\tau_{2D} = -\frac{1}{1-\nu}\nabla\cdot\mathbf{f}_{2D},$$

without any magic. In total, the equations we are solving are

$$\mathbf{0}_{2D} = \nabla\cdot\tau_{2D} + \mathbf{f}, \tag{26}$$

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{2(\lambda+\mu)}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D}\right), \tag{27}$$

$$\triangle \operatorname{tr}\tau_{2D} = -\frac{1}{1-\nu}\nabla\cdot\mathbf{f}_{2D}. \tag{28}$$

2.4 Airy stress function

Let us assume that the force is given as

$$\mathbf{f}_{2D} = -\nabla\varphi,$$

i.e. the force is conservative. Moreover, let us use the following ansatz:

$$\tau_{2D} = \begin{bmatrix} \frac{\partial^2\Phi}{\partial y^2} + \varphi & \frac{\partial^2\Phi}{\partial x\partial y} \\ \frac{\partial^2\Phi}{\partial x\partial y} & \frac{\partial^2\Phi}{\partial x^2} + \varphi \end{bmatrix},$$

for some function $\Phi(x,y)$ called the *Airy stress function*. Why that would be useful? Calculate the divergence of the stress:

$$\nabla \cdot \tau_{2D} = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial y^2} + \varphi\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x \partial y}\right) \\ -\frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x^2} + \varphi\right) \end{bmatrix} = \nabla \varphi,$$

so identically we have

$$\mathbf{0}_{2D} = \nabla \cdot \tau_{2D} - \nabla \varphi,$$

and one of our equations is solved. What about the remaining ones? Beltrami-Michell:

$$\triangle \operatorname{tr} \tau_{2D} = \triangle \left(\triangle \Phi + 2\varphi\right) = \triangle \triangle \Phi + \triangle \varphi.$$

Using this in plain strain case:

$$\triangle \triangle \Phi + \frac{1 - 2\nu}{1 - \nu} \triangle \varphi = 0,$$

and in plain stress case:

$$\triangle \triangle \Phi + (1 - \nu) \triangle \varphi = 0.$$

Let us take a glimpse at the biharmonic equation.

2.5 Bending of a narrow rectangular beam by uniform load

Assume we have a narrow rectangular beam of length L, height h and width b, subjected to the load $q\mathbf{e}_y$, which is constant in the x-direction. $[q] = \frac{\mathrm{N}}{\mathrm{m}}$.

Boundary conditions are *essential*: they specify the problem. In our case, the **front/back face** is traction free:

$$\pm \tau \mathbf{e}_z = \mathbf{0}, \text{ on } \{z = \pm \frac{b}{2}\},$$

the **bottom face** is also stress free:

$$\tau \mathbf{e}_y = \mathbf{0}, \text{ on } \{y = \frac{h}{2}\},$$

the **top face** is subjected to the load

$$\tau \mathbf{e}_y = \frac{-q}{b}\mathbf{e}_y, \text{ on } \{y = -\frac{h}{2}\}.$$

On the lateral faces, we would like *something like*

$$\pm \tau \mathbf{e}_x = \mathbf{f}(y, z), \text{ on } \{x = \pm \frac{L}{2}\},$$

however, in our analysis, we are only interested in the fact whether the force can support the beam - but we dont care about the exact distribution of it. Thus, we require the *balance of forces:*

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{f}(y, z)\, \mathrm{d}y\, \mathrm{d}z = \frac{qL}{2}\mathbf{e}_y, \tag{29}$$

and moreover we require the *balance of torques:*

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{r} \times \mathbf{f}(y,z)\, \mathrm{d}y\, \mathrm{d}z = \mathbf{0}, \tag{30}$$

So the roadplan is to find the stress τ and check whether 29 and 30 are satisfied.

From the symmetry of the load, we assume that

$$\tau^{zz} = 0,$$

so our problem is essentialy a *plane stress problem.* Let us sum up our analysis (this takes some work)

$$\begin{aligned}
t^{xy}\left(x, y = \frac{h}{2}\right) &= 0\\
t^{yy}\left(x, y = \frac{h}{2}\right) &= 0\\
t^{xy}\left(x, y = -\frac{h}{2}\right) &= 0\\
t^{yy}\left(x, y = -\frac{h}{2}\right) &= -\frac{q}{b}\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{xy}\left(x = \pm\frac{L}{2}, y\right) \mathrm{d}y &= \mp\frac{qL}{2},\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{xx}\left(x = \pm\frac{L}{2}, y\right) \mathrm{d}y &= 0,\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} y\tau^{xx}\left(x = \pm\frac{L}{2}, y\right) \mathrm{d}y &= 0^2
\end{aligned}$$

Remember, that on the lateral sides, x is fixed, so *the coordinates are y and z*; some manipulation with the cross product and stuff is needed, for example:

$$\mathbf{r} \times \tau\mathbf{e}_x = \pm\left(z\tau^{xx}\mathbf{e}_z \times \mathbf{e}_x + z\tau^{xy}\mathbf{e}_z \times \mathbf{e}_y + y\tau^{xx}\mathbf{e}_y \times \mathbf{e}_x + y\tau^{xy}\mathbf{e}_y \times \mathbf{e}_y\right)$$

Evidently, the system is complicated enough. We thus make the following assumptions:

- the material of interest is a homogenous isotropc elastic solid
- the beam is massless $\Leftrightarrow$ the predominant force is the external load (not the body force)

From our work on the plain-stress problem, we know the Airy-stress function will be helpful for us. It will be convenient to find Φ in the form

$$\Phi = \Phi(x,y) = Ay^3 + by^5 + Cyx^2 + Dx^2y^3 + Ex^2,$$

where A, B, C, D, E are some constants fitted so that Φ solves the homogenous biharmonic equation:

$$\triangle \triangle \Phi = 0,$$

(recall that since we have no body forces, $\varphi = 0$.) Once we solve for the stress field, we can obtain the strain field using the constitutive relation and then solve for the displacement (solve a linear PDE) using the definition of the linearised strain tensor; see 2.2.

It can be shown the deflection of the middle point is

$$\delta = \frac{5}{384}\frac{qL^4}{EI_{zz}}\left(1 + \frac{12}{5}\frac{h^2}{L^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right),$$

where I_{zz} is a component of the inertia tensor:

$$I_{zz} = \frac{bh^3}{12}$$

2.6 Biharmonic equation in $\mathbb{R}^2$

Let $\Phi(x, y)$ be the Airy stress function. In the previous, we have come up to the problem of solving

$$\begin{cases} \triangle \triangle \Phi = 0, & \text{in } \Omega \subset \mathbb{R}^2, \\ \text{some boundary conditions}, & \text{on } \partial\Omega. \end{cases}$$

We are in $\mathbb{R}^2$, so we immedietaly use complex analysis: $z = x+iy, x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$, and for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we make the following identification

$$f(x, y) \Leftrightarrow f(z, \overline{z}),$$

and the derivatives are

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial z}(z, \overline{z})\frac{\partial z}{\partial x} + \frac{\partial f}{\partial z}(z, \overline{z})\frac{\partial \overline{z}}{\partial x} = \frac{\partial f}{\partial z}(z, \overline{z}) + \frac{\partial f}{\partial z}(z, \overline{z}),$$

$$\frac{\partial f}{\partial y}(x, y) = i\left(\frac{\partial f}{\partial z}(z, \overline{z}) - \frac{\partial f}{\partial \overline{z}}(z, \overline{z})\right),$$

from which it follows

$$\frac{\partial f}{\partial z}(z, \overline{z}) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x, y) - i\frac{\partial f}{\partial y}(x, y)\right),$$

$$\frac{\partial f}{\partial \overline{z}}(z, \overline{z}) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x, y) + i\frac{\partial f}{\partial y}(x, y)\right).$$

If we now take a look at the laplacian of a function $f(x, y)$, we can formally manipulate:

$$\triangle f(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f(x, y) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f(x, y) = 4\frac{\partial^2 f(z, \overline{z})}{\partial \overline{z}\partial z},$$

so in total

$$\triangle f(x, y) = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}(z, \overline{z}).$$

Using this, we can rewrite the *Laplace equation* to the form

$$\frac{\partial^2 g(z, \overline{z})}{\partial z \partial \overline{z}} = 0.$$

Let us solve it. It must be:

$$\frac{\partial g(z,\overline{z})}{\partial \overline{z}} = C_1(\overline{z}), g(z,\overline{z}) = \underbrace{\int C_1(\overline{z})\,\mathrm{d}\overline{z}}_{:=d_1(\overline{z})} + d_2(z)$$

so

$$g(z,\overline{z}) = d_1(\overline{z}) + d_2(z).$$

Now for the biharmonic equation, we need to solve

$$\frac{\partial^2 \Phi}{\partial z \partial \overline{z}} = d_1(\overline{z}) + d_2(z),$$

so that gives $\frac{\partial \Phi}{\partial \overline{z}} = z d_1(\overline{z}) + D_2(z) + e_1(\overline{z})$, and

$$\Phi(z,\overline{z}) = z D_1(\overline{z}) + \overline{z} D_2(z) + E_1(\overline{z}) + E_2(z).$$

In total, we have been able to derive:

$$\Phi(x,y) = \Re((\overline{z}\gamma(z) + \chi(z)))\Big|_{z=x+iy} = \Re\Big(\overline{(x+iy)}\gamma(x+iy) + \chi(x+iy)\Big).$$

2.7 Elliptic hole in uniformly stressed infinite plane

Suppose an infinite plane with a elliptic hole Ω with the standard paramateres a, b. The boundary conditions are

$$\begin{aligned} \tau \mathbf{n} &= \mathbf{0}, \text{ on } \partial\Omega, \\ \lim_{x^2+y^2\to\infty} \tau(x,y) &= S\mathbb{I}, \end{aligned}$$

where $S \in \mathbb{R}$ is given. The problem can be reformulated as

$$\triangle \triangle \Phi(x,y) = 0, \tau = \begin{bmatrix} \frac{\partial^2\Phi}{\partial y^2} & -\frac{\partial^2\Phi}{\partial x \partial y} \\ -\frac{\partial^2\Phi}{\partial x \partial y} & \frac{\partial^2\Phi}{\partial x^2} \end{bmatrix}, \tag{31}$$

plus the boundary conditions. The general representation of the solution is $\Phi = \Re(\overline{z}\psi(z) + \chi(z))$, moreover, we adopt a sensible coordinate system: **elliptical coordinates**

$$\begin{aligned} z &= c\cosh\zeta, \\ z &= x + iy. \end{aligned}$$

Equivalently

$$\begin{aligned} x &= c\cosh\xi\cos\eta, \\ y &= c\sinh\xi\sin\eta, \\ \zeta &= \xi + i\eta. \end{aligned}$$

It follows immedietaly:

$$\begin{aligned} \left(\frac{x}{c\cosh\xi}\right)^2 + \left(\frac{y}{c\sinh\xi}\right)^2 &= 1, \\ \left(\frac{x}{c\cos\eta}\right)^2 - \left(\frac{y}{c\sin\eta}\right)^2 &= 1, \end{aligned}$$

so the lines ξ = const are *ellipses* and the lines η = const are hyperbolas. This will be useful, as we can represent the boundary of the ellipse $\partial\Omega$ as some coordinate line ξ = const.

Through some simple calculation, we can show

$$\mathbf{g}_\eta = \sqrt{J}(-\sin\alpha\mathbf{e}_1 + +\cos\alpha\mathbf{e}_2),$$
$$\mathbf{g}_\zeta = \sqrt{J}(\cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2),$$

where α is the angle between the x axis and $\mathbf{g}_\xi(\xi,\eta)$ and

$$J = c^2\left(\sinh^2\zeta\cos^2\eta + \cosh^2\zeta\sin^2\eta\right).$$

We are interested in the quantity [3]

$$\exp(i2\alpha) = \frac{\sinh\zeta}{\sinh\overline{\zeta}},$$

Formally, for the normalised vectors, we can write something like

$$\begin{bmatrix}\mathbf{g}_{\hat{\xi}}\\ \mathbf{g}_{\hat{\eta}}\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} =$$

To solve for the stress, we need to express the stress in the elliptical coordinates:

$$\tau = \begin{bmatrix}\tau^{\xi\xi} & \tau^{\xi\eta}\\ \tau^{\xi\eta} & \tau^{\eta\eta}\end{bmatrix} = \tau^{xx}\mathbf{e}_x\otimes\mathbf{e}_x + \cdots = t^{\hat{\xi}\hat{\xi}}\mathbf{g}_{\hat{\xi}}\otimes\mathbf{g}_{\hat{\xi}} + \ldots.$$

Thats just some similiarity transformation, we are representing the matrix in a different basis. The traces must be preserved:

$$\tau^{xx} + \tau^{yy} = \tau^{\hat{\xi}\hat{\xi}} + \tau^{\hat{\eta}\hat{\eta}}$$

and similiarly, it can be shown

$$\tau^{\hat{\eta}\hat{\eta}} - \tau^{\hat{\xi}\hat{\xi}} + 2i\tau^{\hat{\xi}\hat{\eta}} = \exp(i2\alpha)(\tau^{yy} - t^{xx} + 2i\tau^{xy}).$$

Combining **all of this** we obtain for the Airy stress function the following relations

$$\tau^{xx} + \tau^{yy} = 4\Re\frac{\mathrm{d}\psi}{\mathrm{d}z},$$
$$\tau^{yy} - \tau^{xx} + 2i\tau^{xy} = 2\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right).$$

[3] That describes the rotation of the coordinate lines, which could mean "the ripping of the ellipse" when pulling

Solving this system (heh) gives

$$\begin{aligned}\psi &= \frac{1}{2} S \sinh \zeta,\\ \chi &= \frac{1}{2} S c^2 \zeta \cosh(2\xi_0)\\ t^{\hat{\eta}\hat{\eta}} &= \frac{2S\sinh(2\xi_0)}{\cosh(2\xi_0) - \cos(2\eta)},\\ \max_{\eta\in(0,2\pi)} \tau^{\hat{\eta}\hat{\eta}} &= 2S\frac{a}{b},\\ \min_{\eta\in(0,2\pi)} \tau^{\hat{\eta}\hat{\eta}} &= 2S\frac{b}{a}.\end{aligned}$$

We see that if b is small (the ellipse is very flat), the maximum explodes; the quantity $2\frac{a}{b}$ is called the stress coefficient factor. Just note that even if the stress at infinity is controlled, the stress at the tips can be enormous.

3 Stability of fluid flows

Let us investigate the following PDE:

$$\begin{cases}\partial_t u = \frac{\partial^2 u}{\partial x^2} + au, & \text{in}\,\Omega = (0,1)\\ u(t,x) = 0, & \text{on}\, x = 0, x = 1\end{cases}.$$

Clearly, $\hat{u}(t,x) = 0$ is a solution, moreover it is a *steady solution.* Our interest is whether, given some $u(t=0,x) = u_0(x)$ initial condition, the solution converges to the steady one; in other words, whether

$$\text{"} \lim_{t\to\infty} u(t,x) = 0\text{"},$$

in some sense of convergence. We will be interested in certain questions:

1. In what sense is the convergence?

3.1 Energy theory

As opposed to the ODE stability theory, we do not want to linearize anything. Let use measure the convergence in the $\mathrm{L}_2(\Omega)$ norm, "the energy norm", *i.e.*

$$\|u\|_{\mathrm{L}_2(\Omega)} = \left(\int_\Omega |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Meaning we are interested in the conditions under which

$$u \to 0 \,\text{in}\, \mathrm{L}_2(\Omega) \Leftrightarrow \|u\|_{\mathrm{L}_2(\Omega)} \to 0.$$

Let us investigate the following quantity:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)u(t,x)\,\mathrm{d}x = \int_\Omega \partial_t u(t,x)u(t,x)\,\mathrm{d}x = \\
&= \int_\Omega \left(\partial_{xx}u + au^2\right)\mathrm{d}x = -\int_\Omega (\partial_x u)^2\,\mathrm{d}x + a\int_\Omega u^2\,\mathrm{d}x = \\
&= -\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} \le -\frac{1}{C_p^2}\|u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} = \\
&= -\left(\frac{1}{C_p^2} - a\right)\|u\|^2_{\mathrm{L}_2(\Omega)},
\end{aligned}
$$

where we used the Poincare inequality in the form $\|u\|_{\mathrm{L}_2(\Omega)} \le C_p\|\partial_x u\|_{\mathrm{L}_2(\Omega)}$ (we have zero trace). So we have :

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} \le -2\left(\frac{1}{C_p^2} - a\right)\|u\|^2_{\mathrm{L}_2(\Omega)},
$$

so if

$$
\frac{1}{C_p^2} - a > 0,
$$

the system has the following solution: **ADD IT**, and so

$$
\|u(t,x)\|_{\mathrm{L}_2(\Omega)} \to 0,\ \text{as}\, t \to \infty.
$$

We are not happy yet. The Poincare constant is undetermined, so let us get an estimate for it. The equation has the form

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} &= -\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} = -a\|\partial_x\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x\|^2_{\mathrm{L}_2(\Omega)}}\right) \le \\
&\le -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \max_{u\in \mathrm{W}_0^{1,2}(\Omega)}\frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|_{\mathrm{L}_2(\Omega)}}\right),
\end{aligned}
$$

let us define

$$
\frac{1}{a_{\text{crit}}} = \max_{u\in \mathrm{W}_0^{1,2}(\Omega)}\frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}},
$$

and so we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} \le -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \frac{1}{a_{\text{crit}}}\right) = -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\frac{a_{\text{crit}} - a}{a a_{\text{crit}}}.
$$

We see that if $a < a_{\text{crit}} \Leftrightarrow \frac{1}{a} > \frac{1}{a_{\text{crit}}}$ the $\mathrm{L}_2(\Omega)$ norm vanishes exponentially. But *how much is it?* That depends on a_{crit}, so let us define the functional

$$
F : \mathrm{L}_2(\Omega) \to \mathbb{R}^+, F : u \mapsto \frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}},
$$

and find its extrema. The Gateux derivative at the extrema is

$$0 = \delta F(u_{\mathrm{ext}})[v] = \frac{\mathrm{d}}{\mathrm{d}t} F(u_{\mathrm{ext}} + tv)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\int_\Omega (u_{\mathrm{ext}} + tv)(u_{\mathrm{ext}} + tv)\,\mathrm{d}x}{\int_\Omega (\partial_x u_{\mathrm{ext}} + t\partial_x v)(\partial_x u_{\mathrm{ext}} + t\partial_x v)\,\mathrm{d}x}\Big|_{t=0} =$$

$$= 2\frac{\int_\Omega u_{\mathrm{ext}} v\,\mathrm{d}x \int_\Omega (\partial_x u_{\mathrm{ext}})^2\,\mathrm{d}x - \int_\Omega u_{\mathrm{ext}}^2\,\mathrm{d}x \int_\Omega \partial_x u_{\mathrm{ext}} \partial_x v\,\mathrm{d}x}{\left(\int_\Omega (\partial_x u_{\mathrm{ext}})^2\,\mathrm{d}x\right)^2} =$$

$$= \frac{1}{\left(\int_\Omega (\partial_x u_{\mathrm{ext}})^2\,\mathrm{d}x\right)^2}\left(\int_\Omega u_{\mathrm{ext}} v\,\mathrm{d}x - \frac{\int_\Omega u_{\mathrm{ext}}^2\,\mathrm{d}x}{\int_\Omega (\partial_x u_{\mathrm{ext}})^2\,\mathrm{d}x}\int_\Omega \partial_x u_{\mathrm{ext}} \partial_x v\,\mathrm{d}x\right) = \frac{1}{\int_\Omega (\partial_x u_{\mathrm{ext}})^2\,\mathrm{d}x}\frac{1}{a_{\mathrm{crit}}}\left(\int_\Omega (a_{\mathrm{crit}} u\right.$$

and so we see

$$\delta F(u_{\mathrm{ext}})[v] = 0 \Leftrightarrow \int_\Omega (a_{\mathrm{crit}} u_{\mathrm{ext}} v - \partial_x u_{\mathrm{ext}} \partial_x v)\,\mathrm{d}x\,,\ \forall v \in \mathrm{W}_0^{1,2}(\Omega)\,.$$

This is a weak formulation of the problem

$$\int_\Omega (a_{\mathrm{crit}} u_{\mathrm{ext}} + \partial_{xx} u_{\mathrm{ext}}) v\,\mathrm{d}x\,,\ \forall v \in \mathrm{W}_0^{1,2}(\Omega) \Leftrightarrow \begin{cases} \partial_{xx} u_{\mathrm{ext}} = -a_{\mathrm{crit}} u_{\mathrm{ext}}\,, & \text{in}\,\Omega \\ u_{\mathrm{ext}} = 0, & \text{on}\,\partial\Omega \end{cases}.$$

This is an *eigenproblem for the elliptic operator*. The solution is the following:

$$\begin{aligned} u_{\mathrm{ext}}^n &= C\sin\left(\sqrt{a_{\mathrm{crit}}^n}\,x\right), \\ a_{\mathrm{crit}}^n &= n^2\pi^2, n \in \mathbb{N} \end{aligned}$$

The *smallest eigenvalue* is

$$a_{\mathrm{crit}} = \pi^2.$$

This means that $\forall a < a_{\mathrm{crit}} = \pi^2$ the perturbations in the initial condition decay exponentially.

3.2 Rayleigh-Bénard convection

Let us use the developed theory on the problem of Rayleigh-Bénard convection. There are two plates, the top with the temperature T_t and the bottom one with the temperature T_b in the gravitational field $\mathbf{g} = -g\mathbf{e}_z$. The governing equations are

$$\begin{cases} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\nabla\cdot\mathbf{v}) = 0, \\ \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} + \nabla\cdot(-p_{\mathrm{th}}(\rho,\theta)\mathbb{I} + \lambda(\nabla\cdot\mathbf{v})\mathbb{I} + 2\mu\mathbb{D}) \\ \rho c_V\frac{\mathrm{d}\theta}{\mathrm{d}t} = \nabla\cdot(\kappa\nabla\theta) + \left(p_{\mathrm{th}}(\theta,\rho) - \frac{\partial p_{\mathrm{th}}(\theta,\rho)}{\partial\theta}\right)(\nabla\cdot\mathbf{v}) + \mathbb{T}^y : \mathbb{D}, \end{cases}$$

Why something happens?

- gravitational field is crucial, as without it, buyoancy oscillations won't work
- dependence of density on temperature is also essential

3.2.1 Boussinesq approximation

Doing a stability analysis of a system of nonlinear PDEs is difficult. Make the following assumptions: Oberbeck-Boussinesq approximation

- the density depends only on the temperature, not on pressure, and only linearly: $\rho(\theta) = \rho_{\text{ref}}(1 - \alpha(\theta - \theta_{\text{ref}}))$
- working with compressible fluids is a nightmare, lets make it incompressible: $\nabla \cdot \mathbf{v} = 0$.
- the density in the momentum equation is constant in the first term and the same in the temperature equation
- ignore all the nonlinear terms in the thermal equation

Doing all this produces the following system of equations:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0, \\ \rho_{\text{ref}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \rho_{\text{ref}}(1 - \alpha(\theta - \theta_{\text{ref}}))\mathbf{g} + \nabla \cdot (-p_{\text{th}}(\rho, \theta)\mathbb{I} + 2\mu\mathbb{D}) \\ \rho_{\text{ref}} c_V \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \nabla \cdot (\kappa \nabla \theta), \end{aligned}$$

with the boundary conditions

$$\begin{cases} \theta = T_{top} & , \text{on } z = d \\ \theta = T_{bot} & , \text{on } z = 0 \end{cases}$$

Note that ,physically, this makes no sense. Since $\mathbb{T}^y : \mathbb{D} = 0$, there is no viscous dissipation, but since $\mathbb{T}^y \neq \mathbb{0}$, we are just losing energy but the temperature does not increase.

3.2.2 Steady state, pure conduction ($\mathbf{v} = \mathbf{0}$)

In the case $\mathbf{v} = \mathbf{0}$ the equation for the temperature θ becomes $0 = \nabla \cdot (\kappa \nabla \theta)$ that has the solution

$$\hat{\theta} = -\frac{T_{bot} - T_{top}}{d} z + T_{bot} = -\beta z + T_{bot},$$

and the equation for the pressure reads as

$$0 = -\nabla p - \rho_{\text{ref}}(1 - \alpha(\theta - \theta_{\text{ref}}))g\mathbf{e}_z.$$

3.2.3 Perturbation of the steady

What happens if now perturb the steady state? We are solving the following system

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, \\ \rho_{\text{ref}}(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\rho_{\text{ref}}(1 - \alpha(\theta - \theta_{\text{ref}}))g\mathbf{e}_z - \nabla p + \mu \triangle \mathbf{v} \\ \rho_{\text{ref}} c_V (\partial_t \theta + (\mathbf{v} \cdot \nabla)\theta) = \kappa \triangle \theta \end{cases},$$

with the initial condiitons $\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}} = \tilde{\mathbf{v}}, \theta = \hat{\theta} + \tilde{\theta}, p = \hat{p} + \tilde{p}$ all at $t = 0$ and the hatted variables being the steady state 3.2.2. The equations for the perturbations (after some manipulation) become

$$\begin{cases} \nabla \cdot \tilde{\mathbf{v}} = 0, \\ \rho_{\text{ref}} \left(\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} \right) = +\rho_{\text{ref}} \alpha \tilde{\theta} g \mathbf{e}_z - \nabla \tilde{p} + \mu \triangle \tilde{\mathbf{v}} \\ \rho_{\text{ref}} c_V \left(\partial_t \tilde{\theta} + (\tilde{\mathbf{v}} \cdot \nabla) \left(\hat{\theta} + \tilde{\theta} \right) \right) = \kappa \triangle \tilde{\theta} \end{cases},$$

3.2.4 Non-dimensonalisation

Next, we need to non-dimensionalize the equations. For that, we need to chose

- a characteristic length: $l_{char} := d$
- a characteristic density $\rho_{char} := \rho_{\text{ref}}$
- a characteristic temperature $\theta_{char} = T_{bot} - T_{top}$
- a characteristic time $t_{char} = ?$

There is however a problem: how to choose the characteristic time? We have no characteristic velocity [4], because $\hat{\mathbf{v}} = \mathbf{0}$. There are some candidates whose units include seconds: $[\mu] = \text{Pa s}, [g] = \frac{\text{m}}{\text{s}^2}, [\kappa] = \frac{\text{W}}{\text{m K}}$.

Whatever, let us continue:

$$\tilde{\mathbf{v}} = v_{char} \mathbf{v}^*, \tilde{\theta} = \theta_{char} \theta^*, \mathbf{x} = l_{char} \mathbf{x}^*,$$

where the starred variables denote dimensionless ones. Plugging all this into the equations yields

$$\nabla^* \cdot \mathbf{v}^* = 0$$

$$\frac{\rho_{ref} d^2}{\mu t_{char}} \left(\partial_{t^*} \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla^*) \mathbf{v}^* \right) = -\nabla^* p^* + \triangle^* \mathbf{v}^* + \frac{\alpha g \theta_{char} d \rho_{ref} t_{char}}{\mu} \theta^* \mathbf{e}_z$$

$$\partial_{t^*} \theta^* + (\mathbf{v}^* \cdot \nabla^*) \theta^* = \nabla \cdot \left(\frac{\kappa t_{char}}{\rho_{ref} c_V d^2} \nabla^* \theta^* \right) + \underbrace{v_z^*}_{\rho c_V \tilde{\mathbf{v}} \cdot \nabla \hat{\theta}},$$

And now we see how the choice of t_{char} influences the equations. I can require one of the following

$$\frac{\rho_{ref} d^2}{\mu t_{char}} = 1$$

$$\frac{\alpha g \theta_{char} d \rho_{ref} t_{char}}{\mu} = 1$$

$$\frac{\kappa t_{char}}{\rho_{ref} c_V d^2} = 1.$$

Each of these choices are sensible. In our case, we are inrested in the thermal conduction mainly, so let us choose

$$t_{char} = \frac{\rho_{ref} c_V d^2}{\kappa}.$$

[4] If we would, that would suffice, as we have characteristic length

Finally, we arrive to the following system of equations (we omit the stars and tildas)

$$\nabla \cdot \mathbf{v} = 0,$$
$$\frac{1}{\mathrm{Pr}}(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \triangle\,\theta + \mathrm{Ra}\,\theta \mathbf{e}_z,$$
$$\partial_t \theta + (\mathbf{v}\cdot\nabla)\theta = \triangle\,\theta + v_z,$$

where

$$\mathrm{Pr} = \frac{\nu}{k} = \frac{\rho_{ref} d^2}{\mu t_{char}}, \tag{32}$$

is the Prandtl number and

$$\mathrm{Ra} = \frac{\alpha g \theta_{char} d^3}{\nu k}, \nu = \frac{\mu}{\rho_{ref}}, k = \frac{\kappa}{\rho_{ref} c_V}. \tag{33}$$

is the Rayleigh number.

Another form of the equations can be derived when rescaling the temperateure (choosing a different characteristic temperature)[5]

$$\theta = \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ra}}}\theta^*,$$

and this leads (of course, other quantities will have to be rescaled as well)

$$\nabla^* \cdot \mathbf{v}^* = 0$$
$$\partial_{t^*} \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* = -\nabla^* p^* + \triangle^*\,\mathbf{v}^* + \sqrt{\mathrm{Ra}}\,\theta^* \mathbf{e}_z$$
$$\mathrm{Pr}\,(\partial_{t^*} \theta^* + (\mathbf{v}^* \cdot \nabla^*)\theta^*) = \triangle^*\,\theta^* + \sqrt{\mathrm{Ra}}\,v_z^*.$$

This scaling is popular in mathematical literature and *we will stick to it.* It is also common to denote

$$R := \sqrt{\mathrm{Ra}}\,.$$

So finally finally, we are solving

$$\nabla \cdot \mathbf{v} = 0,$$
$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \triangle\,\mathbf{v} + \mathrm{R}\,\theta \mathbf{e}_z$$
$$\mathrm{Pr}\,(\partial_t \theta + (\mathbf{v}\cdot\nabla)\theta) = \triangle\,\theta + \mathrm{R}\,v_z.$$

To add another issue, realise that we are working on unbounded domains, so integrals over the domain are problematic. This can be solved using periodic boundary conditions on lateral faces.

Let us take now the velocity equation, multiply $\cdot\mathbf{v}$ and integrate $\int_\Omega \mathrm{d}x$.This yields:

$$\int_\Omega (\,\text{equation}\,)\cdot\mathbf{v}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega (\mathbf{v}\cdot\nabla)\cdot\mathbf{v}\,\mathrm{d}x\Big) = -\int_\Omega \nabla p\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega \triangle\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega \mathrm{R}\,\theta \mathbf{e}_z \mathbf{v}\,\mathrm{d}x\,,$$

[5]Of course θ^* is totally different than the previous one

realize that

$$\int_\Omega (\mathbf{v}\cdot\nabla)\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{v}\cdot\nabla\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)\mathrm{d}x = \int_\Omega (\nabla\cdot\mathbf{v})\frac{\mathbf{v}\cdot\mathbf{v}}{2}\,\mathrm{d}x = 0,$$

and

$$\int_\Omega \nabla p\cdot\mathbf{v}\,\mathrm{d}x = -\int_\Omega p(\nabla\cdot\mathbf{v})\,\mathrm{d}x = 0,$$

and also

$$\int_\Omega \triangle\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \nabla\mathbf{v}:\nabla\mathbf{v}\,\mathrm{d}x\,,$$

where we have used the periodicity of the boundary conditions and incompressibility. This means

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} = -\|\nabla\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathrm{R}\,\theta\mathbf{e}_z\cdot\mathbf{v}\,\mathrm{d}x\,,$$

which is exactly the similiar expression to the one derived at the beginning of our studies of the stability analysis. [6]. It is evident that when

$$\mathrm{R} = 0 = \mathrm{Re}\,,$$

the norm decays exponentially.

Let us repeat the previous maniupulation. Define

$$\psi = \frac{1}{2}\int_\Omega \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x + \mathrm{Pr}\,\frac{1}{2}\int_\Omega \theta^2\,\mathrm{d}x\,,$$

and investigate (we are using the equations extensively)

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{v}\,\mathrm{d}x - \int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x + 2\int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x\,.$$

Introduce yet a different notation:

$$\mathcal{D} := \int_\Omega \nabla\mathbf{v}:\nabla\mathbf{v}\,\mathrm{d}x + \int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x\,, \mathcal{I} = 2\int_\Omega \theta v^z\,\mathrm{d}x\,,$$

so we have

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mathcal{D} + \sqrt{\mathrm{Ra}}\,\mathcal{J} = -\mathcal{D}\sqrt{\mathrm{Ra}}\left(\frac{1}{\sqrt{\mathrm{Ra}}} - \frac{\mathcal{J}}{\mathcal{D}}\right).$$

Denote

$$\sqrt{\mathrm{Ra}_{\mathrm{crit}}} := \max_{\theta\in\mathrm{W}^{1,2}_0(\Omega),\mathbf{v}\in\mathrm{W}^{1,2}_0(\Omega)_{\mathrm{xiv}}} \frac{\mathcal{J}}{\mathcal{D}},$$

and we are interested when

$$\frac{1}{\sqrt{\mathrm{Ra}}} - \frac{1}{\sqrt{\mathrm{Ra}_{\mathrm{crit}}}} = \frac{\sqrt{\mathrm{Ra}_{\mathrm{crit}}} - \sqrt{\mathrm{Ra}}}{\sqrt{\mathrm{Ra}_{\mathrm{crit}}}\sqrt{\mathrm{Ra}}} > 0.$$

Let us define the functional

$$\mathcal{F}(\mathbf{v},\theta) := \frac{\mathcal{I}(\mathbf{v},\theta)}{\mathcal{D}(\mathbf{v},\theta)} - \int_\Omega \lambda(\mathbf{x})(\nabla\cdot\mathbf{v})\,\mathrm{d}x\,,$$

[6] We could again use Poincare to obtain the estimate for $\|\nabla\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} \le \frac{1}{C_p}\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)}$ and stuff.

for some Lagrange multiplier $\lambda(\mathbf{x})$. We now seek:

$$\max_{\mathbf{v}\in W_0^{1,2}(\Omega),\theta\in W_0^{1,2}(\Omega)} \mathcal{F}(\mathbf{v},\theta),$$

so evaluate the Gateaux derivative

$$\delta\mathcal{F}(\mathbf{v}^*,\theta^*)[\mathbf{v},\theta] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{I}(\mathbf{v}^*+t\mathbf{v},\theta^*+t\theta)}{\mathcal{D}(\mathbf{v}^*+t\mathbf{v},\theta^*+t\theta)} - \int_\Omega \lambda\nabla\cdot(\mathbf{v}^*+t\mathbf{v})\,\mathrm{d}x\right)\Bigg|_{t=0},$$

it is easy to realize the numerator is

$$2\int_\Omega \theta v^{*z} + \theta v^z \,\mathrm{d}x\,,$$

and the denominator is

$$2\int_\Omega \nabla\mathbf{v}^* : \nabla\mathbf{v} + \nabla\theta^*\cdot\nabla\theta\,\mathrm{d}x\,,$$

so using the Leibniz rule we obtain

$$\begin{aligned} 0 &= \frac{1}{\mathcal{D}(\mathbf{v}^*,\theta^*)}\left(\frac{\mathrm{d}\mathcal{I}(\mathbf{v}^*,\theta^*)}{\mathrm{d}t} - \frac{\mathcal{I}}{\mathcal{D}}(\mathbf{v}^*,\theta^*)\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}(\mathbf{v}^*,\theta^*) - \int_\Omega \lambda(\nabla\cdot\mathbf{v}^*)\,\mathrm{d}x\right) \\ &\equiv \frac{1}{\mathcal{D}^*}\left(\frac{\mathrm{d}\mathcal{I}^*}{\mathrm{d}t} - \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\frac{\mathrm{d}\mathcal{D}^*}{\mathrm{d}t} - \int_\Omega \lambda(\nabla\cdot\mathbf{v}^*)\,\mathrm{d}x\right), \end{aligned}$$

where we have just rescaled the Lagrange multiplier. This becomes

$$= \frac{1}{\mathcal{D}^*}\left(\int_\Omega \theta v^{*z} + \theta^* v^z\,\mathrm{d}x - \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\left(\int_\Omega \nabla\mathbf{v}^* : \mathbf{v}\,\mathrm{d}x + \int_\Omega \nabla\theta^*\cdot\nabla\theta\,\mathrm{d}x - \int_\Omega \lambda(\nabla\cdot\mathbf{v}^*)\,\mathrm{d}x\right)\right).$$

Hmm, this is familiar - it is a weak formulation of some problem! (If we take $\mathbf{v},\theta$ to be test functions...). The problem then becomes

$$\begin{aligned} -\int_\Omega \left(-\sqrt{\mathrm{Ra}_{\mathrm{crot}}}\,\triangle\,\theta^* + v^{*z}\right)\theta\,\mathrm{d}x = 0 &\Leftrightarrow -\frac{1}{\sqrt{\mathrm{Ra}_{\mathrm{crit}}}}\triangle\,\theta^* + v^{*z} = 0, \\ \int_\Omega \left(\theta^*\mathbf{e}_z - \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\triangle\,\mathbf{v}^* - \nabla\lambda\right)\cdot\mathbf{v}\,\mathrm{d}x = 0 &\Leftrightarrow -\frac{1}{\sqrt{\mathrm{Ra}_{\mathrm{crit}}}}\triangle\,\mathbf{v}^* + \theta^*\mathbf{e}_z - \nabla\lambda = 0. \end{aligned}$$

So the pair $(\mathbf{v}^*,\theta^*)$ maximizes $\mathcal{F}(\mathbf{v},\theta)$ iff it solves the system above. We must thus solve now

$$\begin{aligned} -\triangle\,\mathbf{v}^* - \nabla\lambda + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\theta^*\mathbf{e}_z &= 0, \\ \triangle\,\theta^* + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,v^{*z} &= 0,, \\ \nabla\cdot\mathbf{v}^* &= 0, \end{aligned}$$

which is *exactly the system for the perturbtion, only linearized nd stationary.* It can be rewritten in this "suggestive notation":

$$\begin{bmatrix} \triangle & -\nabla & 0 \\ \nabla\cdot & 0 & 0 \\ 0 & 0 & \triangle \end{bmatrix} \begin{bmatrix} \mathbf{v}^* \\ \lambda \\ \theta^* \end{bmatrix} = -\sqrt{\mathrm{Ra}_{\mathrm{crit}}} \begin{bmatrix} 0 & 0 & \mathbf{e}_z \\ 0 & 0 & 0 \\ \mathbf{e}_z\cdot & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}^* \\ \lambda \\ \theta^* \end{bmatrix}$$

which is a generalized eigenvalue problem

$$\mathbb{A}\mathbf{x} = \mu\mathbb{B}\mathbf{x},$$

and so we see that we are interested in the (generalized) spectrum of some operators. Coming back to our conditions for maximizing, taking the divergence of the first equation:

$$0 = \underbrace{-\nabla\cdot\triangle\,\mathbf{v}^*}_{=\triangle\,(\nabla\cdot\mathbf{v})=0} - \triangle\,\lambda + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,(\nabla\cdot(\theta\mathbf{e}_z)),$$

taking the laplacian of the first one yields

$$\triangle\,\triangle\,\mathbf{v}^* - \underbrace{\triangle\,\nabla\lambda}_{=\nabla\,\triangle\,\lambda} + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\triangle\,(\theta^*\mathbf{e}_z),$$

so combining the two yields (eliminating λ):

$$\begin{aligned} \triangle\,\triangle\,\mathbf{v}^* + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\nabla(\nabla\cdot(\theta^*\mathbf{e}_z)) + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\triangle\,(\theta^*\mathbf{e}_z) = 0, \\ \triangle\,\theta^* + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,v^{*z} = 0. \end{aligned}$$

Now let us take the z component of the first equation, so we are solving

$$\begin{aligned} \triangle\,\triangle\,\mathbf{v}^{*z} - \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\frac{\partial^2\theta^*}{\partial z^2} + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,\triangle\,()\theta^* = 0, \\ \triangle\,\theta^* + \sqrt{\mathrm{Ra}_{\mathrm{crit}}}\,v^{*z} = 0, \end{aligned}$$

and Fourier transform it:

$$\mathbf{v}^* = \hat{\mathbf{v}}(z)\exp\left(i(k_x x + k_y y)\right), \theta^* = \hat{\theta}(z)\exp\left(i(k_x x + k_y y)\right),$$

so formally

$$\triangle \to \frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2,$$

and the equations become **(now we are solving for the hatted variables, the amplitudes, without renaming anything.)**

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2 v^{*z} - \mathrm{Ra}_{\mathrm{crit}} \tag{34}$$

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\theta^* + \mathrm{Ra}_{\mathrm{crit}}\,v^{*z}. \tag{35}$$

Now we apply the Fourier transformed laplacian to the first one and write

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 v^{*z} - \mathrm{Ra}_{\mathrm{crit}}\, k^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\theta^*,$$

and plug this into the remaining equation gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 v^{*z} = -\mathrm{Ra}_{\mathrm{crit}}\, k^2 v^{*z}, z \in [0,1].$$

We have thus derived a sixth order ODE for the velocity, which is in fact an *eigenvalue problem for the (linear unbounded) operator.* The problem is about the boundary condition: it makes *some* sense to assume:

$$\begin{aligned} v^{*z} &= 0, \\ \frac{\mathrm{d}^2 v^{*z}}{\mathrm{d}z^2} &= 0, \\ \frac{\mathrm{d}^4 v^{*z}}{\mathrm{d}z^4} &= 0, \end{aligned}$$

all on $\{z = 0, 1\}$. Really, it can be shown

$$v^{*z} = \sum_{n=1}^{\infty} v_n^{*z} \sin(n\pi z),$$

for some numbers v_n^{*z}. This representation really *splnuje* the above boundary conditions. Plug this an write:

$$\left(-n^2\pi^2 - k^2\right)^3 v_n^{*z} = -\mathrm{Ra}_{\mathrm{crit}}\, v_n^{*z},$$

and so

$$\mathrm{Ra}^n_{\mathrm{crit}} = \frac{\left(\pi^2 n^2 + k^2\right)^3}{k^2}, n \in \mathbb{N}.$$

As we are looking for the smallest one, our value is:

$$\mathrm{Ra}_{\mathrm{crit}} = \frac{\left(\pi^2 + k^2\right)^3}{k^2}.$$

Notice that this still depends on $k_n = \frac{2\pi}{L} n$ the choice of L, *i.e.*, the choice of the periodicity of the boundary. So in fact we want to minimize this

$$\frac{\partial \mathrm{Ra}_{\mathrm{crit}}}{\partial k} = \frac{\left(k^2 + \pi^2\right)^2 \left(3k^2 - \left(k^2 + \pi^2\right)\right)}{k^2} = 0 \Rightarrow k_{\mathrm{crit}} = \frac{\pi^2}{2},$$

plugging this yields

$$\mathrm{Ra}_{\mathrm{crit}} = \frac{27}{4}\pi^4 \tag{36}$$

3.2.5 Free-free boundary conditions

But hold on, how did we get the boundary conditions? Those are called the *free-free* boundary conditions.

$$v^z = 0 \,\text{on}\, \{z = 0, 1\}, \mathbb{T}^y \mathbf{n} = -p_{\text{ambient}}\,\mathbf{n}, \text{ on } \{z = 0, 1\},$$

and

$$\mathbf{v} = \mathbf{0} + \tilde{\mathbf{v}}, \mathbb{T}^y = -(\hat{p} + \tilde{p})\mathbb{I} + \dots .$$

The pressure in the steady case also satisfies the boundary conditions, namely

$$-\hat{p}\mathbb{I}\mathbf{n} = -p_{\text{ambient}}\,\mathbf{n}.$$

This means

$$\tilde{\mathbb{T}}^y = -\tilde{p}\mathbb{I} + 2\mu\left(\frac{1}{2}(\nabla\tilde{\mathbf{v}}) + (\nabla\tilde{\mathbf{v}})^\top\right),$$

with the following boundary conditions

$$\tilde{\mathbb{T}}^y\mathbf{n} = \mathbf{0}, \tilde{v}^z = 0 \,\text{on}\, \{z = 0, 1\}.$$

This translates to[7]

$$[T_{zx}, T_{yz}, T_{zz}]^\top = \mathbf{0},$$

which implies

$$\frac{\partial \tilde{v}^x}{\partial z} + \frac{\partial \tilde{v}^z}{\partial x} = 0, \frac{\partial \tilde{v}^y}{\partial z} + \frac{\partial \tilde{v}^z}{\partial y} = 0,$$

again on $\{z = 0, 1\}$. Since $\tilde{v}^z = 0$ there, also its derivative (assuming continuity...) is zero there, and so those conditions really mean

$$\frac{\partial \tilde{v}^x}{\partial z} = 0, \frac{\partial \tilde{v}^y}{\partial z} = 0.$$

Recall that

$$\nabla \cdot \mathbf{v} = \frac{\partial \tilde{v}^x}{\partial x} + \frac{\partial \tilde{v}^y}{\partial y} + \frac{\partial \tilde{v}^z}{\partial z} = 0,$$

inside of Ω. Let us however suppose that it holds also *on the boundary*[8]. Differentiate w.r.t z, swap the derivatives and obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \tilde{v}^x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v^y}{\partial z}\right) + \frac{\partial^2 v^z}{\partial z^2} = 0,$$

on $\{z = 0, 1\}$. Since the first two terms are zero, we read

$$\frac{\partial^2 \tilde{v}^z}{\partial z^2} = 0 \,\text{on}\, \{z = 0, 1\}.$$

Finally, let us deal with the BC for the forth derivative. For that, recall that we have not yet discussed the boundary conditions for the temperature, which are: $\tilde{\theta} = 0 \,\text{on}\, \{z = 0, 1\}$. Take a look at 34 now, using the boundary conditions and the definition of $\triangle$, we obtain [9]

[7] on $\{z = 0, 1\}$ the outer unit normal $\mathbf{n}$ equals to $\mathbf{e}_z$.

[8] We are on $\{z = 0, 1\}$.

[9] Again, we are also assuming the equations hold on the boundary aswell.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2 v^{*z} = 0 \,\mathrm{on}\, \{z = 0, 1\},$$

so in particular

$$\frac{\mathrm{d}^4 v^{*z}}{\mathrm{d}z^4} = 0 \,\mathrm{on}\, \{z = 0, 1\}.$$

3.2.6 The case $\mathrm{Ra} > \mathrm{Ra}_{\mathbf{crit}}$ "slightly"

What happens when we perturb the system with

$$\mathrm{Ra} > \mathrm{Ra}_{\mathrm{crit}}\,,$$

meaning *slightly larger*? That would mean

$$\frac{\left(\pi^2 + k^2\right)^3}{k^2} > \frac{\left(\pi^2 + k^2_{\mathrm{crit}}\right)^3}{k^2_{\mathrm{crit}}},$$

As a toy problem, let us suppose the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\begin{bmatrix} \gamma_1(\mathrm{Ra}) & 0 \\ 0 & \gamma_2(\mathrm{Ra}) \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} -a q_1 q_2 \\ b q_1^2 \end{bmatrix},$$

where $\gamma_1(\mathrm{Ra}), \gamma_2(\mathrm{Ra})$ are some functions of the Rayleigh number. THere are some regimes:

- $\gamma_1 > 0, \mathrm{Ra} < \mathrm{Ra}_{\mathrm{crit}}$: then q_1 is *damped exponentially* and the nonlinearity does not play a role,
- $\gamma_1 < 0, \mathrm{Ra} > \mathrm{Ra}_{\mathrm{crit}}$: then q_1 *grows exponentially* and therefore the nonlinearity cannot be ignored.
- $\gamma_2 >> 1$ means that the second equation is (almost) only a algebraic one, which we can solve, substitue back into the first one and obtain

  $$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -\gamma_1 q_1 - \frac{ab}{\gamma_2} q_1^3 = -\gamma_1 q_1\left(1 + \frac{ab}{\gamma_1\gamma_2} q_1^2\right),$$

  which is really interesting; it is only a cubic correction to a linear system (*i.e.*, a *quadratic* nonlinearity.) This model might serve as a precursor to the *Ginzburg - Landau* equations.

Try a similiar thing: suppose this system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + u\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} u^2 \\ -uv \end{bmatrix},$$

and investigate its stability.

First, linearize around the steady state $\mathbf{0}$:

$$\nabla\begin{bmatrix} -\frac{1}{\mathrm{Re}} u + v + u^2 \\ -\frac{1}{\mathrm{Re}} v - uv \end{bmatrix}(\mathbf{0}) = \begin{bmatrix} -\frac{1}{\mathrm{Re}} + 2u & 1 \\ -v & -\frac{1}{\mathrm{Re}} - u \end{bmatrix}(\mathbf{0}) = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}.$$

The eigenvalues are $-\frac{1}{\text{Re}}$ with the degeneracy 2. So for Re not too large, this is negative and the steady state is stable, but for $\text{Re} \to \infty$, this goes to zero and we can not really say anything using this theorem.

It is crucial that the matrix is symmetric. Let us investigate

$$\dot{\mathbf{u}} = \mathbb{A}\mathbf{u}, \mathbb{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix},$$

where λ, μ are eigenvalues. The solution of course is

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0.$$

Calculate the exponential, so start with Jordan decomposition of $\mathbb{A}$)

$$\mathbb{A} = \begin{bmatrix} 1 & \frac{1}{\mu-\lambda} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\mu-\lambda} 0 & 1 \end{bmatrix},$$

and the eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{\sqrt{1 + \frac{1}{(\mu-\lambda)^2}}} \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix}.$$

Notice that the eigenvectors are *not orthogonal*. The exponential thus is

$$\exp(t\mathbb{A}) = \begin{bmatrix} e^{\lambda t} & \frac{e^{\mu t} - e^{\lambda t}}{\mu-\lambda} \\ 0 & e^{\mu t} \end{bmatrix}$$

If we Taylor expand this, we obtain

$$\exp(t\mathbb{A}) \approx \mathbb{I} + \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix} t,$$

and so the general solution thus is

$$\mathbf{u} = \exp(t\mathbb{A})\mathbf{u}_0 \approx \begin{bmatrix} (1+\lambda t)u_{0,1} + t u_{0,1} \\ (1+\mu t)u_{0,2} \end{bmatrix}$$

This means that even though (supposing the eigenvalues are negative)

$$\lim_{t \to \infty} \mathbf{u}(t) = \mathbf{0},$$

initially, the solution grows linearly with time.

But anyways, the linearized stability tells us that the origin is stable.

Let us try the energy method. Investigate $\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|$, so calculate

$$\tilde{\mathbf{u}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 = \begin{bmatrix} -\frac{1}{\text{Re}} & 0 \\ 0 & -\frac{1}{\text{Ra}} \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \cdot \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \underbrace{u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}}_{=0},$$

so rewriting this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 = -\frac{1}{\text{Ra}}|\tilde{\mathbf{u}}|^2 + \tilde{u}\tilde{v}.$$

This is remarkable analogy; the first one is the "laplacian", the missing term is the convective nonlinear term, which does not add to the energy balance and the last one is the is the interaction term.

Using Younge:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 \le -\frac{1}{\mathrm{Re}}|\tilde{\mathbf{u}}|^2 + \frac{1}{2}\left(|\tilde{u}|^2 + |\tilde{v}|^2\right) = \left(-\frac{1}{\mathrm{Re}} + \frac{1}{2}\right)|\tilde{\mathbf{u}}|^2,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 \le \left(1 - \frac{2}{\mathrm{Re}}\right)|\tilde{\mathbf{u}}|^2,$$

which means that for $\mathrm{Re} > 2$, the perturbations do not decay, and for $\mathrm{Re} \le 2$, the perturbations decay (exponentially).

Do the complete analysis of the problem: solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\mathrm{Re}} & 0\\ 0 & -\frac{1}{\mathrm{Re}}\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} + \tilde{u}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}.$$

The stready solutions are (after some computations)

$$\tilde{u}^* = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right),$$
$$\tilde{v}^* = \mathrm{Re}\,\tilde{u}^{*2},$$

and also the origin $\mathbf{0}$. Realize however that the first steady state exists only if $\mathrm{Re} \ge 2$.

So if when $\mathrm{Re} < 2$, there is only one stationary point, the origin

$$\tilde{\mathbf{u}} = \mathbf{0},$$

and all trajectories are attracted towards that point. If $\mathrm{Re} > 2$, there are 3 stationary points, lying on the parabola $\tilde{v} = \mathrm{Re}\,\tilde{u}^{*2}$ with $\tilde{u} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right)$. We see that

$$\lim_{\mathrm{Re}\to\infty} \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right) = 0,$$

which corresponds to the linearization theory. This can also be used to quantify how large of neighbourhood can be obtained in the linerized stability theory, as the "minus" stationary point is unstable.

3.3 Orr-Sommerfeld system

This is a system describing the stability of shear flows with a parabolic inflow profile in a channel. The velocity has the profile

$$\mathbf{v} = \hat{\mathbf{v}}(y)\exp\left(i(\beta x + \alpha z - \omega t)\right).$$

The equation for the perturbation of the vertical component $\tilde{v}^y$ of the velocity is (in the Fourier space)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}-^2}k^2\right)(-i\omega + i\alpha v^z)\tilde{v}^y - i\alpha\frac{\mathrm{d}^2 v^z}{\mathrm{d}y^2}\tilde{v}^y = \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\right)^2\tilde{v}^y,$$

where $v^z(y) = 1 - y^2$.

3.4 Eigenvalue problem

How to solve the eigenvalue problem? We are interested in only the *smallest* eiegenvalue. Techniques will be presented on the toy problem:

$$\begin{aligned}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} &= \lambda u,\\ u &= 0, \text{ on } \{x = 0, 1\}\\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= 0, \text{ on } \{x = 0, 1\}\end{aligned}$$

3.4.1 Analytical approach

Imagine we are using finite differences to solve the equation with the RHS f:

$$\begin{bmatrix} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0, \text{ on } \{x = 0\} \\ \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0, \text{ on } \{x = 1\} \end{bmatrix} \begin{bmatrix} u_1 \\ \dots \\ u_N \end{bmatrix} = \begin{bmatrix} 0 \\ f_k \\ 0 \end{bmatrix},$$

but when we are trying to solve the eigenproblem, we encounter a problem with the weird boundary conditions (try to think about the RHS).

Since the operator is linear, we can write the solution analytically

$$u(x) = c_1 \cos\left(x\lambda^{1/4}\right) + c_2 \sin\left(x\lambda^{1/4}\right) + c_3 e^{-x\lambda^{1/4}} + c_4 e^{x\lambda^{1/4}},$$

where c_1, c_2, c_3, c_4 are the coefficients to be determined from the boundary conditions. To have $\mathbf{c} \neq \mathbf{0}$, we require :

$$\mathbb{A}\mathbf{c} = \mathbf{0}, \det \mathbb{A} = 0,$$

which yields a condition for λ, in particular

$$4\left(e^{-\lambda^{1/4}} - e^{\lambda^{1/4}}\right)\lambda \sin\left(\lambda^{1/4}\right) = 0.$$

This is a non-linear algebraic equation, that can be solved using *e.g.* numerics.

3.4.2 Numerical approach

Some existing software packages are able to solve eigenvalue problems of the type

$$\mathbf{L}u = \lambda u,$$

where $\mathbf{L}$ is a second order operator. We are able to rewrite this equation in the form

$$\mathbf{D}_k \mathbf{D}_k u = \lambda u,$$

where

$$\mathbf{D}_k = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2,$$

and we are solving:

$$\mathbf{D}_k v = \lambda u, \mathbf{D}_k u - v = \lambda u, \ i.e., \begin{bmatrix} 0 & \mathbf{D}_k \\ \mathbf{D}_k & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is equivalent to

$$\mathbf{D}_k = (\lambda + 1)v, \mathbf{D}_k\mathbf{D}_k u = (\lambda + 1)\mathbf{D}_k v, \Rightarrow \mathbf{D}_k\mathbf{D}_k u = \lambda(\lambda + 1)u$$

so if we solve the above eigenvalue problem for λ, we have found the eigenvalue $\mu = \lambda(\lambda + 1)$ of the original problem. What about the boundary conditions? Again assuming the equations hold also on the boundary, we are able to show

$$u = v = 0, \text{ on } \{x = 0, 1\}.$$