# Thermodynamics and mechanics of solids

Kamil Belan

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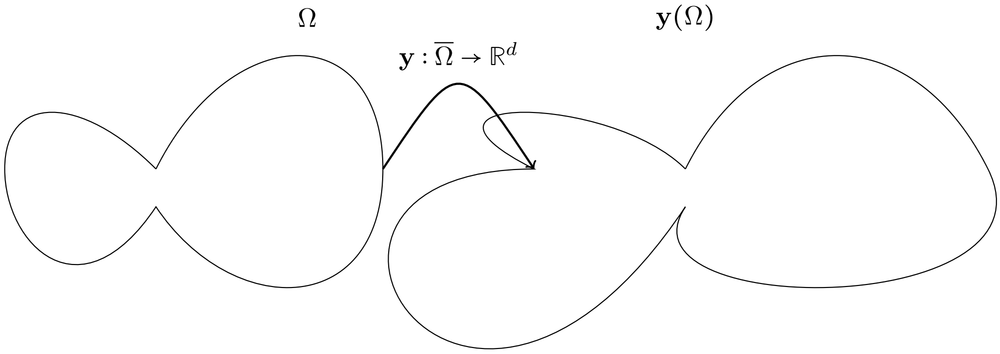
# 1 TODO

- include missing lecture about potential forces
- include missing lecture about rank one convexity
- include weak convergence symbol
- fix bold greek letters

# 2 Geometry

# 3 Deformation

Suppose we are given an abstract body $\Omega \subset \mathbb{R}^d, d = 2, 3$. Choosing a particular state, we denote it the **reference configuration**. After the acting of forces, the body deforms into the **current, deformed configuration**.



The mapping that produces the deformation is called the **deformation**, denoted $\mathbf{y}$, i.e.

$$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^d.$$

Of large interest will be the **deformation gradient**

$$\mathbb{F}(\mathbf{x}) = \nabla y(\mathbf{x}), (\nabla y)_{ij} = \frac{\partial y^i}{\partial x^j},$$

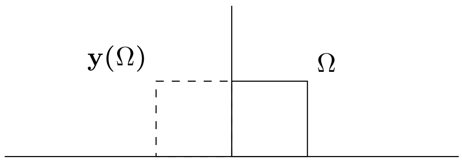
on which we put some physically sound restrictions, such asdet $\mathbb{F} > 0$. This means in particular that the determinant is nonzero, but also that preserves orientations of bases:

$$(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0 \Rightarrow (\mathbb{F}\mathbf{e}_1 \times \mathbb{F}\mathbf{e}_2) \cdot \mathbb{F}\mathbf{e}_3 > 0.$$

**Example.** Suppose the deformation is given as

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x},$$

i.e., $\mathbb{F} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, det $\mathbb{F} = -1$. This is an example of a deformation that is *forbidden*.




Imagine it is a sheet of paper in a plane - you cannot reflect it without lifting it from the plane.

## 3.1 Displacement

Another useful way of describing the deformation is by using the **displacement vector u**:

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x},$$

so taking the gradient gives

$$\nabla \mathbf{u}(\mathbf{x}) = \mathbb{F}(\mathbf{x}) - \mathbb{I}.$$

*Remark.* It is interesting that we do not place any restrictions on the determinant of the displacement gradient.

## 3.2 Changes of measures

We need to examine how the volume, area and lengths change under deformation. In what follows, for a set $\omega \subset \mathbb{R}^d$ in the reference configuration we denote $\omega^y \subset \mathbb{R}^d$ to be the deformed body in the current configuration, i.e.

$$\omega^y = \mathbf{y}(\omega).$$

### 3.2.1 Change of volume

Using the change of variable theorem we obtain

$$\lambda(\omega^y) = \int_{\omega^y} 1 \,\mathrm{d}\mathbf{x}^y = \int_\omega \det \mathbb{F}(\mathbf{x}) \,\mathrm{d}\mathbf{x}\,,$$

so we write $\mathrm{d}\mathbf{x}^y = \det \mathbb{F} \,\mathrm{d}\mathbf{x}$. This motivates "our" definition of the determinant of the deformation gradient:

$$\det \mathbb{F}(\mathbf{x}) = \lim_{r\to 0} \frac{\lambda(\mathbf{y}(B(\mathbf{x}, r))}{\lambda(B(\mathbf{x}, r))}, \tag{1}$$

where $B(\mathbf{x}, r)$ is a (closed) ball centered at $\mathbf{x}$ of radius $r$.

### 3.2.2 Change of lengths

Suppose the line segment $\mathbf{x} + \Delta\mathbf{x}$ undergoes deformation. How does its length change? Taylor expansion yields:

$$\mathbf{y}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \mathbb{F}(\mathbf{x})\Delta\mathbf{x} + \text{h.o.t.},$$

where h.o.t. stands for higher order terms. Using the expansion, we can write

$$\|\mathbf{y}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{y}(\mathbf{x})\|^2 = (\Delta\mathbf{x})^\top \mathbb{F}^\top \mathbb{F}\, \Delta\mathbf{x} = (\Delta\mathbf{x})^\top \mathbb{C}(\mathbf{x})\Delta\mathbf{x},$$

where we have denoted

$$\mathbb{C}(\mathbf{x}) = \mathbb{F}^\top(\mathbf{x})\mathbb{F}(\mathbf{x}),$$

as the **Right Cauchy Green tensor**.

**Example.** Let the deformation $\mathbf{y}$ be given as $\mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v}, \mathbf{v} \in \mathbb{R}^d, \mathbb{R} \in \mathrm{SO(d)} = \left\{\mathbb{A} \in \mathbb{R}^{d\times d}, \mathbb{A}^\top\mathbb{A} = \mathbb{A}\mathbb{A}^\top = \mathbb{I}, \det \mathbb{A} = 1^1 \det \mathbb{A} > 0\right\}$. Then $\mathbb{F}= \mathbb{R}$, $\mathbb{C}= \mathbb{I}$.

### 3.2.3 Change of surfaces

For $\mathbb{A} \in \mathbb{R}^{d\times d}$ regular we define the **cofactor matrix** $\operatorname{cof} \mathbb{A}$ as

$$\operatorname{cof} \mathbb{A} = (\det \mathbb{A})\mathbb{A}^{-\top},$$

which is an interesting quantity whatsoever; we will use the following theorem

**Theorem 1** (Piola's identity)**.** *Let* $\mathbf{y} \in C^2(\Omega; \mathbb{R}^d)$*, then* $\forall \mathbf{x} \in \Omega$ :

$$\nabla \cdot (\operatorname{cof} \nabla\mathbf{y}(\mathbf{x})) = \mathbf{0}.$$

For a regular matrix $\mathbb{A}$, we also have the identity

$$\mathbb{A}^{-1} = \frac{1}{\det \mathbb{A}}(\operatorname{cof} \mathbb{A})^\top, \tag{2}$$

What about the determinant of the cofactor? Clearly

$$\det \operatorname{cof} \mathbb{A} = (\det \mathbb{A})^d \det \mathbb{A}^{-\top} = (\det \mathbb{A})^{d-1},$$

[1] From the fact $\mathbb{A}$ is orthogonal automatically follows $\det \mathbb{A} = \pm 1$.

so we can also express equation 2 in a different way

$$\mathbb{A}^{-1} = \frac{(\operatorname{cof}\mathbb{A})^{\top}}{(\det\operatorname{cof}\mathbb{A})^{1/d-1}}. \tag{3}$$

From geometry, recall the change of variables for surface integration:

$$\int_{\partial\omega^{y}} \mathbf{n}^{y}\,\mathrm{d}S^{y} = \int_{\partial\omega} \operatorname{cof}\mathbb{F}\mathbf{n}\,\mathrm{d}S\,,$$

where $\mathbf{n}^y$ is the outward unit normal to the deformed boundary $\omega^y$. Informally, we write $\mathbf{n}^y\,\mathrm{d}S^y = \operatorname{cof}\mathbb{F}\mathbf{n}\,\mathrm{d}S$. We can also explicitely express the normal to the deformed boundary as

$$\mathbf{n}^{y}(\mathbf{x}^{y}) = \frac{\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})}{\|\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})\|}, \mathbf{x}\in\partial\omega, \mathbf{y}(\mathbf{x})\in\partial\omega^{y}. \tag{4}$$

## 3.3 Affine transformations

An example of deformation is the so called **affine transformation**.

**Example.** Suppose the folowing deformation:

$$\mathbf{y}(\mathbf{x}) = \mathbb{A}\mathbf{x} + \mathbf{v}, \mathbb{A}\in\mathbb{R}^{d\times d}, \mathbf{v}\in\mathbb{R}^{d}, \det\,\mathbb{F} > 0.$$

Clearly then $\mathbb{F}(\mathbf{x}) = \mathbb{A}$.

It is crucial to realize how $\mathbb{F}, \mathbb{F}^{\top}, \mathbb{F}^{-\top}$ work.

- $\mathbb{F}$ takes a vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration* and maps it to the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ in the *current configuration*
- $\mathbb{F}^{-1}$ takes the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ from the *current configuration* and maps it to the vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration*
- $\mathbb{F}^{\top}$ is defined through: $\mathbb{F}\mathbf{u}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbb{F}^{\top}\mathbf{w}$, and since $\mathbb{F}$ is defined on the reference configuration, $\mathbb{F}^{\top}$ must take something from the *current configuration* and return something from the *reference configuration.*
- $\mathbb{F}^{-\top}$ consequently takes something from the *referrence configuration* and maps it to something from the *current configuration.*

**Example.** What when $\mathbb{C} = \mathbb{I}$? Can we say something about $\mathbb{F}$? Write $\mathbb{C} = \mathbb{F}^{\top}\mathbb{F} = \mathbb{I}$, so $\mathbb{F}^{\top} = \mathbb{F}^{-1}, \det\mathbb{F} > 0$. From this we have $\mathbb{F}(\mathbf{x}) = \mathbb{R}(\mathbf{x}), \mathbf{x}\in\Omega$, where $\mathbb{R}$ is a rotation tensor. Investigate the cofactor of the deformation gradient:

$$\operatorname{cof}\mathbb{F} = \det\mathbb{F}\mathbb{F}^{-\top} = \operatorname{cof}\mathbb{R} = 1\mathbb{R}(\mathbf{x}) = \mathbb{F}(\mathbf{x}).$$

This implies $\operatorname{cof}\mathbb{F} = \mathbb{F}$. Recall Piola's identity:

$$\mathbf{0} = \nabla\cdot\operatorname{cof}\mathbb{F} = \nabla\cdot\mathbb{F}(\mathbf{x}) = \nabla^{2}\mathbf{y}(\mathbf{x}).$$

We have the identity: and since the LHS is zero, we also have $\|\nabla\nabla\mathbf{y}\| = 0 \Rightarrow \mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v}$. Let $\mathbb{R}$ be piecewise affine. Then $\mathbb{R}_1(\mathbb{I} - \mathbf{n}\otimes\mathbf{n}) = \mathbb{R}_2(\mathbb{I} - \mathbf{n}\otimes\mathbf{n})$, so $\mathbb{R}_1 - \mathbb{R}_2 = (\mathbb{R}_1 = \mathbb{R}_2) = (\mathbb{R}_1 - \mathbb{R}_2)\mathbf{n}\otimes\mathbf{n} = \mathbf{a}\otimes\mathbf{b}$, but that is not possible for two rotations; the rank of the RHS is one, whereas the LHS is not.

# 4 Forces

## 4.1 Forces in the deformed configuration

Recall $\mathbf{y} : \overline{\Omega} \to \overline{\Omega}^y$.We can define the **volume density of applied forces** $\mathbf{f}^y : \overline{\Omega}^y \to \mathbb{R}^3$ (in newtons per cubic meters, e.g. gravity). The same on the boundary $\mathbf{g}^y : \Gamma_N^y \to \mathbb{R}^3$ (**surface density of applied forces** (in newtons per square meters = Pascals, e.g. hydrostatic pressure.)

### 4.1.1 Cauchy stress tensor

**Lemma 1** (Stress principle of Euler and Cauchy)**.** *There exists a (Cauchy) stress vector function* $\mathbf{t}^y : \overline{\Omega}^y \times \mathcal{S}^{d-1} \to \mathbb{R}^d$ *with the following properties.*

1. *If* $\mathbf{x}^y \in \Gamma_N^y$*, then* $\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbf{g}^y(\mathbf{x}^y)$*, where* $\mathbf{n}^y$ *is the unit outer normal vector to* $\partial\Omega^y$ *at* $\mathbf{x}^y$*.*
2. $\forall \omega^y \subset \Omega^y$ *it holds that* $\int_{\omega^y} \mathbf{f}(\mathbf{x^y})\, \mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)\, \mathrm{d}S^y = 0$*. (Balance of forces in static equilibrium.*
3. $\forall \omega^y \subset \Omega^y$ *it holds that* $\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y)\, \mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y)\, \mathrm{d}S^y = \mathbf{0}$*. (Balance of moment of forces in static equilibrium.)*

*Euler says that the direct consequence of this is the existence of* $\mathbb{T}^y(\mathbf{x}^y)$ *such that*

$$\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y, \tag{5}$$

*where the tensorial quantity* $\mathbb{T}$ *is called the* ***Cauchy stress tensor****.*

### 4.1.2 Balance equations in the deformed configuration

Classical physics gives us 2 fundamental relations: Newtons second law for momenta and for angular momenta. We examine these in the continuum mechanics setting.

From second property it follows:

$$\int_{\omega^y} \mathbf{f}^y(\mathbf{x}^y)\, \mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\, \mathrm{d}S^y = \int_{\omega^y} \mathbf{f}^y(\mathbf{x})^y\, \mathrm{d}\mathbf{x}^y + \int_{\omega^y} \nabla \cdot \mathbb{T}^y(\mathbf{x}^y)\, \mathrm{d}\mathbf{x}^y = 0, \tag{6}$$

so using the localization theorem we get

$$\mathbf{f}^y(\mathbf{x}^y) + \nabla \cdot \mathbf{T}^y(\mathbf{x}^y) = \mathbf{0}, \forall \mathbf{x}^y \in \Omega^y.$$

From the third property it follows

$$\begin{aligned}
&\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y)\, \mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\, \mathrm{d}S^y = \\
&= \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y\, \mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} x_j^y (T_{km}^y n_m^y)\, \mathrm{d}S^y = \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y\, \mathrm{d}\mathbf{x}^y = \int_{\omega^y} \varepsilon_{ijk} \frac{\partial(x_j^y T_{km}^y)}{\partial x_m^y}\, \mathrm{d}\mathbf{x}^y = \\
&= \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y\, \mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} x_j^y \frac{\partial T_k m}{\partial x_m^y}\, \mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} \delta_{jm} T_{km}^y\, \mathrm{d}\mathbf{x}^y = \mathbf{0}.
\end{aligned}$$

The last term implies

$$\int_{\omega^y} \varepsilon_{ijk} T^y_{kj} = 0,$$

and using the localization theorem, we obtain

$$T^y_{ij}(\mathbf{x}^y) = T^y_{ji}(\mathbf{x}^y), \quad i.e. \mathbb{T}^y(\mathbf{x}^y) = (\mathbb{T}^y(\mathbf{x}^y))^\top. \tag{7}$$

The **Cauchy stress tensor is symmetric.**

## 4.2 Forces in the undeformed configuration

We have obtained the equations in the deformed configuration. That is however unconvenient - we solve the equations to find the deformed configuration. This brings us to find a new way to write the equations - in the reference configuration. The construction is a bit synthetic, our intuition will be guided by the requirement to obtain similair equations as in the current configuration.

### 4.2.1 Piola-Kirchhoff stresses

**Definition 1** (First Piola-Kirchhoff stress tensor)**.** Given the Cauchy stress tensor $\mathbb{T}^y(\mathbf{x}^\mathbf{y})$, we define the **First Piola Kirchhoff stress tensor**

$$\mathbb{T} : \overline{\Omega} \to \mathbb{R}^{3\times 3}, \mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{x}^y) \operatorname{cof} \mathbb{F}(\mathbf{x}) = \det \mathbb{F}(\mathbf{x}) \mathbb{T}^y(\mathbf{x}^y) \mathbb{F}^{-\top}(\mathbf{x}).$$

**Definition 2** (Second Piola-Kirchhoff stress tensor)**.** The quantity

$$\mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1}\mathbb{T}(\mathbf{x}) = \mathbb{S}(\mathbf{x})^\top,$$

is called the **second Piola-Kirchhoff stress tensor.**

*Remark.* The first PK tensor $\mathbb{T}$ is *not symmetric in general.*, but the second $\mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1} = \det \mathbb{F}(\mathbf{x})\mathbb{F}^{-1}\mathbb{T}^y(\mathbf{x}^y)\mathbb{F}^{-\top}(\mathbf{x})$ *is.* Also, we see that not every matrix can serve as $\mathbb{T}$; it must hold $\mathbb{T}(\mathbf{x})\big(\operatorname{cof} \mathbb{F}^{-1}\big)$ is symmetric.

*Remark.* We have the following identity (using Piola's identity):

$$\nabla \cdot \mathbb{T}(\mathbf{x}) = \det \mathbb{F}(\mathbf{x}) \nabla \cdot \mathbb{T}^y(\mathbf{x}^y)^y. \tag{8}$$

### 4.2.2 Balance equations in the deformed configuration

Recall the balance of momentum

$$-\nabla \cdot (\mathbb{T}^y(\mathbf{x}^y)) = \mathbf{f}^y(\mathbf{x}^y) \,\mathrm{in}\, \Omega^y,$$

multiplying by $\det \mathbb{F} > 0$ yields

$$\det \mathbb{F} \nabla \cdot (\mathbb{T}^y(\mathbf{x}^y)) = \det \mathbb{F}(\mathbf{x}) \mathbf{f}^y(\mathbf{x}^y), \tag{9}$$

which *begs* for the definition

$$\mathbf{f}(\mathbf{x}) = \det \mathbb{F}(\mathbf{x}) \mathbf{f}^y(\mathbf{y}(\mathbf{x})),$$

as the force in the *referential configuration.*

In total, the total acting body force on the body can be written as

$$\int_{\mathbf{y}(\omega)} \mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y = \int_\omega \mathbf{f}^y(\mathbf{y}(\mathbf{x}))\det\mathbb{F}(\mathbf{x})\,\mathrm{d}x = \int_\omega \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

We can do the same for the balance of surface forces; the total contact force acting on the body is

$$\begin{aligned}\int_{\Gamma_N^y} \mathbf{g}^y(\mathbf{x}^y)\,\mathrm{d}S^y &= \int_{\partial\omega^y} \mathbf{t}^y(\mathbf{x}^y,\mathbf{n}^y)\,\mathrm{d}S^y = \int_{\partial\mathbf{y}(\omega)} \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\,\mathrm{d}S^y = \\ &= \int_{\partial\omega} \mathbb{T}^y(\mathbf{y}(\mathbf{x}))\operatorname{cof}\mathbb{F}(t,\mathbf{x})\mathbf{n}\,\mathrm{d}S = \int_{\partial\omega} \mathbb{T}(\mathbf{x})\mathbf{n}\,\mathrm{d}S\,,\end{aligned}$$

so if we define

$$\mathbf{g}(\mathbf{x}) = \mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x}),$$

as the contact force in the *referential configuration*, we formally have a similiar expression.

# 5 Elasticity

**Definition 3** (Elasticity)**.** We say that a material is **elastic (or Cauchy elastic)** if there is a response function $\tilde{\mathbb{T}}^D : \Omega \times \mathbb{R}^{3\times3}_+ \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ such that

$$\mathbb{T}^y(\mathbf{x}^y) = \tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F}).$$

The response function is also called the **constitutive law**.

*Remark.* If we know the material is elastic, we also have the information about the First Piola-Kirchhoff stress, as $\mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{x}^y)\operatorname{cof}\mathbb{F}$, so

$$\mathbb{T}(\mathbf{x}) = \tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F})\operatorname{cof}\mathbb{F} = \tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}). \tag{10}$$

## 5.1 Frame invariance principle

The frame invariance principle states:

$$\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{R}\mathbf{x}) = \mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F})\mathbb{R}^\top, \forall\mathbb{R}\in\mathrm{SO}(3)\,, \forall\mathbf{x}\in\overline{\Omega},$$

from which it follows ($\tilde{\mathbb{T}}$ is defined in 10)

$$\tilde{\mathbb{T}}(\mathbf{x},\mathbb{R}\mathbb{F}) = \det(\mathbb{R}\mathbb{F})\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{R}\mathbb{F})(\mathbb{R}\mathbb{F})^{-\top} = \det(\mathbb{R}\mathbb{F})\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F})\mathbb{R}^\top\mathbb{R}\mathbb{F}^{-\top} = \det\mathbb{F}\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F})\mathbb{F}^{-\top} = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}),$$

thus

$$\tilde{\mathbb{T}}(\mathbf{x},\mathbb{R}\mathbb{F}) = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}), \text{ i.e. } \mathbb{R}^\top\tilde{\mathbb{T}}(\mathbf{x},\mathbb{R}\mathbb{F}) = \tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}), \forall\mathbb{R}\in\mathrm{SO}(3)\,, \forall\mathbb{F}\in\mathbb{R}^{3\times3}_+.$$

## 5.2 Isotropic material

Recall $\mathbb{T}^y(\mathbf{x}^y) = \tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F}), \mathbf{y} : \overline{\Omega} \to \Omega^y = \mathbf{y}(\Omega)$. Take $\mathbf{x}_0 \in \overline{\Omega}$ general but fixed, take $\mathbf{v}(\mathbf{z}) = \mathbf{x}_0 + R^\top(\mathbf{z}-\mathbf{x}_0)$ for some $\mathbb{R}\in\mathrm{SO}(3)$ and define a *new deformation*

$$\tilde{\mathbf{y}} = \mathbf{y}\circ\mathbf{v}^{-1} : \mathbf{v}(\overline{\Omega}) \to \mathbf{y}(\overline{\Omega}), \tilde{\mathbf{y}}(\tilde{\mathbf{x}}) = \mathbf{y}(\mathbf{x}_0 + \mathbb{R}(\tilde{\mathbf{x}}-\mathbf{x}_0)).$$

This implies

$$\mathbf{x}_0^y = \mathbf{x}_0^{\tilde{y}}, \mathbb{T}^y(\mathbf{x}_0^y) = \tilde{\mathbb{T}}^D(\mathbf{x}_0,\mathbb{F}) = \mathbb{T}^{\tilde{y}}(\mathbf{x}_0^{\tilde{y}}) = \tilde{\mathbb{T}}^D(\mathbf{x}_0,\tilde{\mathbb{F}}(\mathbf{x}_0)) = \tilde{\mathbb{T}}^D(\mathbf{x}_0,\mathbb{F}(\mathbf{x}_0)\mathbb{R}).$$

**Definition 4** (Isotropic material)**.** We cal the material **isotropic** if it holds

$$\tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F}) = \tilde{\mathbb{T}}^D(\mathbf{x},\mathbb{F}\mathbb{R}), \forall\mathbb{R} \in \mathrm{SO}(3)\,, \forall\mathbb{F} \in \mathbb{R}_+^{3\times3}.$$

*Remark.* For the first Piola-Kirchhoff we obtain: $\mathbb{T}^D(\mathbf{x},\mathbb{F}\mathbb{R}) = \mathbb{T}^D(\mathbf{x},\mathbb{F})\mathbb{R}$, which means

$$\mathbb{T}^D(\mathbf{x},\mathbb{Q}\mathbb{F}\mathbb{R}) = \mathbb{Q}\tilde{\mathbb{T}}^D\mathbb{R}, \forall\mathbb{R},\mathbb{Q} \in \mathrm{SO}(3)\,, \forall\mathbb{F} \in \mathbb{R}_+^{3\times3}.$$

## 5.3 Hyperelastic materials

**Definition 5.** We say that a material is hyperelastic if there is a function $W : \overline{\Omega} \times \mathbb{R}_+^{3\times3} \to \mathbb{R}$ such that

$$\mathbb{T}(\mathbf{x}) = \tilde{\mathbb{T}}(\mathbf{x},\mathbb{F}) = \frac{\partial W(\mathbf{x},\mathbb{F})}{\partial\mathbb{F}}, \mathbb{F} = \nabla\mathbf{y}(\mathbf{x}).$$

The function $W, [W] = \frac{J}{m^3} = \frac{Nm}{m^3}$ = Pa is called **stored energy density.**

*Remark.* Evidently, $W$ has a potential.

## 5.4 Properties of W

It is physical to assume

1. $W \geq 0$ (energy is nonnegative)
2. $W(\mathbf{x},\mathbb{F}) = W(\mathbf{x},\mathbb{R}\mathbb{F}), \forall\mathbb{R} \in \mathrm{SO}(3)\,, \forall\mathbf{x} \in \overline{\Omega}, \forall\mathbb{F} \in \mathbb{R}_+^{3\times3}$. (energy does not change under rotations [2]
3. $W\big(\mathbf{x},\tilde{\mathbb{R}}\mathbb{U}\big) = W(\mathbf{x},\mathbb{U}), \mathbb{U} = \sqrt{\mathbb{C}}$. (matrices are from the polar decomposition)
4. $W(\mathbf{x},\mathbb{F}) \to \infty$ if $\det\mathbb{F} \to 0_+$ (it takes infinite energy to deform the body to a point)
5. $W(\mathbf{x},\mathbb{F}) \geq \alpha\big(\|\mathbb{F}\|^p + \|\mathrm{cof}\,\mathbb{F}\|^q + (\det\,\mathbb{F})^r\big) - d, \forall\alpha > 0, \forall p,q,r \geq 1, \forall d \in \mathbb{R}, \forall\mathbf{x} \in \overline{\Omega}, \forall\mathbb{F} \in \mathbb{R}_+^{3\times3}$.

**Definition 6** (Natural state of the body)**.** The natural state of the body is the state in which

$$W(\mathbf{x},\mathbb{F}) = 0 \wedge \frac{\partial W}{\partial\mathbb{F}}(\mathbf{x},\mathbb{F}) = \mathbb{0}. \tag{11}$$

*Remark* (Unnatural vegetable)*.* Not all materials (bodies) have its natural states. Zum Beispiel, carrot does not have a natural state.

From the previous work, we can write $\mathbb{R}^\top\frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}} = \frac{\partial W(\mathbf{x},\mathbb{F})}{\partial\mathbb{F}}$, and for brevity denote $W(\mathbf{x},\mathbb{R}\mathbb{F}) = W_R(\mathbf{x},\mathbb{F})$. Next, we *suppose we can Taylor expand*:

$$\begin{aligned} W_R\big(\mathbf{x},\mathbb{F}+\tilde{\mathbb{F}}\big) &= W\big(\mathbf{x},\mathbb{R}\mathbb{F}+\mathbb{R}\tilde{\mathbb{F}}\big) = W(\mathbf{x},\mathbb{R}\mathbb{F}) + \frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}} : \big(\mathbb{R}\tilde{\mathbb{F}}\big) + \text{h.o.t.} \\ &= W_R(\mathbf{x},\mathbb{F}) + \mathbb{R}^\top\frac{\partial W(\mathbf{x},\mathbb{R}\mathbb{F})}{\partial\mathbb{F}} : \tilde{\mathbb{F}} + \text{h.o.t.}. \end{aligned}$$

[2]If this was not true, you could create infinite energy by just spinning a rubber.

Moreover

$$W_R\big(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\big) = W_R(\mathbf{x}, \mathbb{F}) + \frac{\partial W_R(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}} : \tilde{\mathbb{F}}.$$

Altogether

$$\frac{\partial}{\partial \mathbb{F}} (W_R(\mathbf{x}, \mathbb{F}) - W_R(\mathbf{x}, \mathbb{F})) = \mathbb{0},$$

from which it follows [3]

$$W(\mathbf{x}, \mathbb{R}\mathbb{F}) = W(\mathbf{x}, \mathbb{F}) + k(\mathbb{R}).$$

Take $\mathbb{F} = \mathbb{I}$, then

$$W\big(\mathbf{x}, \mathbb{R}^2\big) = W(\mathbf{x}, \mathbb{R}\mathbb{R}) = W(\mathbf{x}, \mathbb{R}) + k(\mathbb{R}) = W(\mathbf{x}, \mathbb{I}) + 2k(\mathbb{R}),$$

so

$$W(\mathbf{x}, \mathbb{R}^n) = W(\mathbf{x}, \mathbb{I}) + nk(\mathbb{R}).$$

Since the set of rotations is closed and bounded, it is compact, so there exists a convergent subsequence of $\{\mathbb{R}^n\}$. Moreover, we assume $W$ to be continuous (we took the derivative...), so $\lim_{n\to\infty} W(\mathbf{x}, \mathbb{R}^n)$ exists and from the properties of $W$ we get it is finite. But then $k(\mathbb{R}) = 0$, as otherwise $nk(\mathbb{R}) \to \infty$. All in all, we have shown

$$W(\mathbf{x}, \mathbb{R}\mathbb{F}) = W(\mathbf{x}, \mathbb{F}). \tag{12}$$

**Definition 7** (Energy functional)**.** Let us have $\partial\Omega = \Gamma_N \cup \Gamma_D, \Gamma_N \cap \Gamma_D = \emptyset$, where the parts of the boundary are those when Neumann/Dirichlet boudary conditions are prescribed. The energy functional of the material is the functional

$$I(\mathbf{y}) = \int_\Omega W(\mathbf{x}, \mathbb{F}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}S\,,$$

where the first part corresponds to the stored energy and the remaining terms are the work done by the external loads.

*Remark.* If $\mathbf{y}$ is the minimizer of $I$, then $I(t\boldsymbol{\varphi} + \mathbf{y}) \geq I(\mathbf{y}), \forall t, \boldsymbol{\varphi}$. If we denote

$$a(t) := I(t\boldsymbol{\varphi} + \mathbf{y}),$$

then it most hold

$$0 = a'(0) = \frac{\mathrm{d}}{\mathrm{d}t} \left( \int_\Omega W(\mathbb{F} + t\nabla\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}x \right)\Bigg|_{t=0},$$

calculating the derivatives yields

$$\begin{aligned}
0 &= \int_\Omega \frac{\partial W(\mathbb{F})}{\partial \mathbb{F}} : \nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_\Omega \frac{\partial}{\partial x_j} \left( \frac{\partial W(\mathbb{F})}{\partial F_{ij}} \varphi_i \right) \mathrm{d}\mathbf{x} - \int_\Omega \frac{\partial}{\partial x_j} \left( \frac{\partial W(\mathbb{F})}{\partial F_{ij}} \right) \varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_{\Gamma_N} \frac{\partial W(\mathbb{F})}{\partial F_{ij}} \varphi_i n_j\,\mathrm{d}S - \int_\Omega \frac{\partial}{\partial x_j} \left( \frac{\partial W(\mathbb{F})}{\partial F_{ij}} \right) \varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S\,,
\end{aligned}$$

[3] The set of matrices with positive determinant is connected.

so it must hold

$$-\frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right) = f_i \text{ in}\,\Omega, \frac{\partial W(\mathbb{F})}{\partial F_{ij}} n_j = g_i \,\text{on}\,\Gamma_N.$$

This is exactly

$$-\nabla\cdot\mathbb{T} = \mathbf{f}\,\text{in}\,\Omega, \mathbb{T}\mathbf{n} = \mathbf{g},\ \text{on}\,\Gamma_N.$$

This implies that $\mathbf{y}$ minimizes energy $\Leftrightarrow$ $\mathbf{y}$ is governed by the equations of classical mechanis.

Are there some other qualities of $W$? It is natural to assume

$$W(\mathbb{I}) = 0 \Rightarrow W(\mathbb{R}) = 0, \forall \mathbb{R}\in \mathrm{SO}(3)$$

and $W(\mathbb{F}) > 0$ whenever $\mathbb{F} \notin \mathrm{SO}(3)$ This however implies $W$ is not convex! Assume

$$\mathbb{R}_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, \mathbb{R}_2 = \mathbb{I},$$

then

$$W\left(\frac{1}{4}\mathbb{R}_1 + \frac{3}{4}\mathbb{R}_2\right) > \frac{1}{4}W(\mathbb{R}_1) + \frac{3}{4}W(\mathbb{R}_2) = 0.$$

**Example** (Minimizer does not exist)**.** Assume $J(u) = \int_0^1 \left(1-(u'(x))^2\right)^2 + u(x)^2\,\mathrm{d}x\,, u\in \mathrm{W}^{1,4}(0,1), u(0) = u(1) = 0$, and find the minimum of $J$. First of all, $J > 0$, so the minimum also. I can take $u_k$ such that $u_k'(x) = 1$ on $(0,1/2)$ and $u_k'(x) = -1$ on $(1/2,1)$. Then $J(u_k)\to 0 \Rightarrow \inf J = 0$ but there is no minimizer.

Not everything is lost...

**Definition 8** (Polyconvexity, 1977 J.M. Ball)**.** $W:\mathbb{R}^{3\times3}\to\mathbb{R}\cup\{\infty\}$ is polyconvex provided there exists convex and lower-semicontinuous function $h:\mathbb{R}^{19}\to\mathbb{R}\cup\{\infty\}$:

$$W(\mathbb{A}) = h(\mathbb{A}, \operatorname{cof}\mathbb{A}, \det\mathbb{A}).$$

**Example.**

- If $W$ is convex and lower-semicontinuous then $W$ is polyconvex.
- $W(\mathbb{A}) = \det\mathbb{A}$ is polyconvex but not convex.

*Remark* (Weak convergence in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$). Let $1 < p < \infty$ and $\{\mathbf{u_k}\}\subset \mathrm{L_p}(\Omega;\mathbb{R}^3)$. We say $\{\mathbf{u_k}\}$ converges weakly to $\mathbf{u}$ in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$ provided

$$\int_\Omega \mathbf{u_k}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,, \forall\boldsymbol{\varphi}\in \mathrm{L}_{p'}(\Omega;\mathbb{R}^3).$$

**Theorem 2** (Magic)**.** *Assume that* $\mathbf{y}^k$ *converges weakly to* $\mathbf{y}$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$, $\Omega\subset\mathbb{R}^3\in C^{0,1}, p > 3$. *Then* $\det\nabla\mathbf{y}^k$ *converges weakly to* $\det\nabla\mathbf{y}$ *in* $L_{\frac{p}{3}}(\Omega)$. *Moreover* $\operatorname{cof}\nabla\mathbf{y}^l$ *converges weakly to* $\operatorname{cof}\nabla\mathbf{y}$ *in* $L_{\frac{p}{2}}(\Omega;\mathbb{R}^{3\times3})$.

*Proof.* Only in 2 dimensions. The determinant can be written as:

$$\det \nabla \mathbf{y} = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(y_1 \frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1 \frac{\partial y_2}{\partial x_1}\right) = \nabla \cdot \left(y_1, -\frac{\partial y_2}{\partial x_1}\right),$$

so then

$$\int_\Omega \det \nabla \mathbf{y}^k \varphi \,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1^k \frac{\partial y_2^k}{\partial x_2}\right)\varphi \,\mathrm{d}x - \int_\Omega \frac{\partial}{\partial x_2}\left(y_1^k \frac{\partial y_2^k}{\partial x_1}\right)\varphi \,\mathrm{d}x = -\int_\Omega y_1^k \frac{\partial y_2^k}{\partial x_2}\frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x + \int_\Omega \frac{\partial \varphi}{\partial x_2} y_1^k \frac{\partial y_2^k}{\partial x_1}\,\mathrm{d}x\,,$$

and the result follows from the embedding theorems and strong convergence (strong times weak gives weak convergence). □

## 5.5 Rank-one convexity

Assume the following domain: $\Omega = (1,2) \times (0,4\pi) \times (1,2)$ and the deformation $\mathbf{y} : \overline{\Omega} \to \mathbb{R}^3, \mathbf{y}(x_1,x_2,x_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3), \mathbb{F} = \begin{bmatrix} \cos x_2 & -x_1 \sin x_2 & 0 \\ \sin x_2 & x_1 \cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
We can calculate $\det \mathbb{F} = x_1 \cos^2 x_2 + x_1 \sin^2 x_2 = x_1$. But even though the deformation has positive determinant, we still face self-penetration issues, i.e., $\mathbf{y}$ *is not injective.*

**Theorem 3** (Ciarlet-Nečas condition)**.** *Let* $p > 3$ *and let* $\det \mathbb{F} > 0$ *a.e. in* $\Omega \subset \mathbb{R}^3, \mathbf{y} \in W^{1,p}(\Omega; \mathbb{R}^3)$. *If*

$$\int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \lambda(\mathbf{y}(\Omega))$$

*then* $\mathbf{y}$ *is injective almost everywhere in* $\Omega$*, i.e.,* $\exists \omega \subset \Omega : \lambda(\omega) = 0, \mathbf{y}|_{\Omega/\omega}$ *is injective.*

Is the determinant condition of any use? Let us compute, assuming $\mathbf{y} = \mathbf{0}$ on $\partial\Omega$.

$$\int_\Omega \det \mathbb{F}\,\mathrm{d}\mathbf{x} = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1 \frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1 \frac{\partial y_2}{\partial x_1}\right)\mathrm{d}\mathbf{x} = \int_{\partial\Omega} y_1 \frac{\partial y_2}{\partial x_2} n_1 - y_1 \frac{\partial y_2}{\partial x_1} n_2 \,\mathrm{d}S \underbrace{\Rightarrow}_{y=0 \text{ on } \partial\Omega} \int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

This is powerful! Assume that $\mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}$ on $\partial\Omega$, then

$$\int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \lambda(\Omega) \det \mathbb{F}.$$

Now let

$$I(\mathbf{y}) = \int_\Omega \det \mathbb{F}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \mathbf{y} \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^3)\,, \mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}.$$

Then $I$ is constant[4] and it holds

$$I(\mathbf{y}) = \lambda(\Omega) \det \mathbb{F}.$$

[4] All constant functionals are convex.

# 6 Linearized elasticity

Recall the Right Cauchy-Green tensor: $\mathbb{C} = \mathbb{F}^\top\mathbb{F}$. Using it, we can define

**Theorem 4** (Green-Lagrange strain tensor/Green-St.-Venaint strain tensor)**.** *Let $\mathbb{C}$ be the Right Cauchy-Green tensor. We define the Green-Lagrange strain tensor/Green-St.-Venaint strain tensor as*

$$\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I}).$$

*Remark.* The Green-St.-Venaint strain tensor can be rewritten as:

$$\mathbb{E} = \frac{1}{2}\big((\mathbb{I} + \nabla\mathbf{u})^\top(\mathbb{I} + \nabla\mathbf{u}) - \mathbb{I}\big) = \frac{1}{2}\big(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\big) + \frac{1}{2}(\nabla\mathbf{u})^\top\nabla\mathbf{u} = \mathbb{e}(\mathbf{u}) + \frac{1}{2}\mathbb{C}(\nabla\mathbf{u}).$$

For the stored energy density, we can write

$$W(\mathbb{F}) = W(\mathbb{R}\mathbb{F}) = \overline{W}(\mathbb{C}(\mathbb{F})) = \hat{W}(\mathbb{E}(\mathbb{F})).$$

and also

$$W(\mathbb{F}) = \hat{W}(\mathbb{e}(\mathbf{u}) + \mathbb{C}(\nabla\mathbf{u})).$$

It is our assumption that

$$\hat{W}(\mathbb{0}) = 0, \hat{W}(\mathbb{E}) > 0 \,\text{if}\, \mathbb{E} \neq \mathbb{0},$$

and also that

$$\mathbb{C}(\nabla\mathbf{u}) = \mathbf{0}.$$

Using Taylor expansion, we can write

$$\hat{W}(\mathbb{e}(\mathbf{u})) = \hat{W}(\mathbb{0}) + \frac{\partial\hat{W}}{\partial\mathbb{e}}(\mathbb{0})\mathbb{e}(\mathbf{u}) + \frac{1}{2}\frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{0})\mathbb{e}(\mathbf{u})\mathbf{e}(u) + \text{h.o.t.}.$$

Since $\hat{W}(\mathbb{0}) = \frac{\partial\hat{W}}{\partial\mathbb{0}}(\mathbb{0}) = 0$ the above (formal) manipulation leads us to the definition

**Definition 9** (Tensor of elastic constants)**.**

$$\mathcal{C} = \frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{0}), C_{ijkl} = \frac{\partial^2\hat{W}}{\partial e_{ij}\partial e_{kl}}.$$

*Remark.* Since we assume $\hat{W}$ is smooth, we have some symmetries, and from the general 81 components of $C_{ijkl}$ only 21 are unique.

With the notion of a tensor of elastic constants, we can then write the stored energy density as

$$w(\mathbb{e}) = \frac{1}{2}(\mathcal{C}\mathbb{e}) : \mathbb{e}.$$

Following our definition $\mathbb{T} = \frac{\partial\hat{W}}{\partial\mathbb{F}}$ we see

$$\sigma = \frac{\partial w(\mathbb{e})}{\partial\mathbb{e}} = \mathbb{C}\mathbb{e}, \sigma_{ij} = C_{ijkl}e_{kl}.$$

Is a useful notion of stress. It is denoted as the *Cauchy stress.* or in components

$$\sigma_{ij} = C_{ijkl}e_{kl}.$$

## 6.1 Equations

Rewritting the equations in the linearized elasticity setting we obtain the system

$$-\nabla\cdot\sigma = -\nabla\cdot(\mathcal{C}\mathbb{e}) = \mathbf{f}\,\mathrm{in}\,\Omega$$
$$\sigma\mathbf{n} = \mathbf{g}\,\mathrm{on}\,\Gamma_N,$$
$$\mathbf{u} = \mathbf{0}\,\mathrm{on}\,\Gamma_D.$$

The weak formulation can be obtained as

$$\int_\Omega \frac{\partial}{\partial x_j}(C_{ijkl}e_{kl})v_i\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,,\,\forall\mathbf{v}\in\mathrm{W}^{1,2}\big(\Omega;\mathbb{R}^3\big)\,,u=0\,\mathrm{on}\,\Gamma_D,$$

so

$$\int_\Omega C_{ijkl}e_{kl}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} C_{ijkl}e_{kl}v_in_j\,\mathrm{d}S = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,,$$

which can be rewritten as

$$\underbrace{\int_\Omega \mathbb{C}\mathbb{e}(\mathbf{u})\cdot\mathbb{e}(\mathbf{v})\,\mathrm{d}\mathbf{x}}_{:=B(u,v)} = \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}S,}_{:=L(v)}$$

where we have denoted

$$\mathbb{e}(\mathbf{v}) = \mathrm{sym}\,(\nabla\mathbf{v}).$$

We are looking for

$$u\in V = \{u\in\mathrm{W}^{1,2}\big(\Omega;\mathbb{R}^3\big)\,,\mathrm{tr}\,u=0\,\mathrm{on}\,\Gamma_D\}:B(u,v)=L(v)\forall v\in V,$$

and to prove the existence, we will use the Lax-Milgram lemma. Show that

- $L\in V^*$
- $B: V\times V\to\mathbb{R}$ is V-bounded and V-coercive

Realize that in order to show the properties, we would have to be able to control $\nabla\mathbf{u}$ by $\mathrm{sym}\,(\nabla\mathbf{u})$. Is that even possible?

**Example.** Let $u=0\,\mathrm{on}\,\partial\Omega$. In particular, let us take $\mathbf{u}\in\mathcal{D}(\Omega;\mathbb{R}^n)$. Then

$$\exists C>0:\int_\Omega|\mathbb{e}(\mathbf{u})|^2\,\mathrm{d}\mathbf{x}\geq c\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,.$$

Can this hold? Make a quick test: Take $\mathbf{u}$ such that $\mathbb{e}(\mathbf{u})=\mathbb{0}$, so $\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)=0$, so of course:

$$\nabla\mathbf{u} = -(\nabla\mathbf{u})^\top,$$

and $\nabla\mathbf{u}$ must have the form

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2}\\ \frac{\partial u_2}{\partial x_1} & 0\end{bmatrix},$$

where $\frac{\partial u_1}{\partial x_1}=\frac{\partial u_2}{\partial x_2}=0$, but since $\mathbf{u}=\mathbf{0}$ at the boundary, it also holds that $\mathbf{u}=\mathbf{0}$ in $\Omega$. Okay, so that not disprove the above inequality.

Let us try something else (although unsure what this means):

$$\int_\Omega |\mathbb{e}(\mathbf{u})|^2 \, \mathrm{d}x = \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \mathrm{d}x$$
$$= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \left(\frac{\partial u_j}{\partial x_i}\right)^2 \mathrm{d}x = \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \, \mathrm{d}x\,,$$

where we used the symmetry property. Integrating by parts two times to obtain "$\partial_i u_i \partial_j u_j = (\partial_j u_j)^2$ [5]. All in all

$$\frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_i}{\partial x_i}\right)^2 \mathrm{d}x \geq 0.$$

**Theorem 5** (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain ($\Omega \in C^{0,1}$). Then there exists $C > 0$ such that $\forall \mathbf{u} \in W^{1,2}((\Omega;\mathbb{R}^n))$ it holds*

$$\left(\|\mathbb{e}(\mathbf{u})\|^2_{L_2((\Omega;\mathbb{R}^{n\times n}))} + \|\mathbf{u}\|^2_{L_2((\Omega;\mathbb{R}^n))}\right) \geq c\|\mathbf{u}\|_{W^{1,2}((\Omega;\mathbb{R}^n))}.$$

**Definition 10** (Axial vectors)**.** Let $\mathbb{A} = -\mathbb{A}^\top, \mathbb{A} \in \mathbb{R}^{n\times n}$. Then there is $b \in \mathbb{R}^n$ such that $\mathbb{A}\mathbf{v} = \mathbf{b} \times \mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{b}$ is called the axial vector of $\mathbb{A}$.

*Remark* ($\mathbb{R}^n$). This truly holds in $\mathbb{R}^n$, not only in $\mathbb{R}^3$. We only have to replace $\times$ by $\wedge$, the outter product.

Assume that $\mathbf{u} \in C^2(\Omega;\mathbb{R}^3)$. Then

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial e_{ik}}{\partial x_j}(\mathbf{u}) + \frac{\partial e_{ij}}{\partial x_k}(\mathbf{u}) - \frac{\partial e_{jk}}{\partial x_i}(\mathbf{u}).$$

If now $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, then $\mathbf{u}$ is an affine function, because $\frac{\partial^2 u_i}{\partial x_j \partial x_k}, \forall i,j,k \in \{1,2,3\}$. [6] It must thus hold

$$u_i(x) = a_i + b_{ij}x_j,$$

and $\frac{\partial u_i}{\partial x_j} = b_{ij} = -b_{ji}$, because $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, so it must be skew symmetric. The skew-symmetry also means it can be written

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d} \times \mathbf{x}.$$

If additionaly we assume that $\mathbf{u} = \mathbf{0}$ on some $\Gamma_D \subset \partial\Omega, \mathcal{H}(\Gamma_D) > 0$ and $\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d} \times \mathbf{x}$, then $\mathbf{u} = \mathbf{0}$ identically in $\Omega$ This moreover means that

$$\mathbf{u} \mapsto \|\mathbb{e}(\mathbf{u})\|_{\mathrm{L}_2((\Omega;\mathbb{R}^{n\times n}))}$$

is a norm on

$$V = \{\mathbf{w} \in \mathrm{W}^{1,2}((\Omega;\mathbb{R}^3)), \mathbf{w} = \mathbf{0} \,\mathrm{on}\, \Gamma_D\}$$

which is equivalent to the norm of $\mathrm{W}^{1,2}((\Omega;\mathbb{R}^3))$.

Coming back to our equation $B(u,v) = L(v), \forall v \in V$, we have showed everything to use Lax-Milgram $\Rightarrow \exists! u \in V$. This also means the functional

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega (\mathbb{C}\mathbb{e}(\mathbf{v}) : \mathbb{e}(\mathbf{v}) - L(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,, \forall v \in V.$$

has an unique minimizer.

[5] Sign does not change as we integrate 2 times. Also, we have homogenous Dirichlet

[6] Recall that $\Omega$ is simply connected.

## 6.2 Convex analysis

We will deal with the analysis of the functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, f$ is convex.

**Definition 11** (Epigraph of a set)**.** The epigraph of a function $f$ is the set

$$\operatorname{epi} f = \{(x, y) : y \geq f(x)\}$$

*Remark.* With the notion of $\operatorname{epi} f$ we can work with sets instead of functions. Moreover, it holds

- $\operatorname{epi} f$ is closed $\Leftrightarrow$ $f$ is lower-semicontinuous,
- $f$ is convex $\Leftrightarrow$ $\operatorname{epi} f$ is convex

From one of the consequences of Hahn-Banach theorem (oddělovací věty), we obtain the existence of such $\xi \in \mathbb{R}^n$ (dependent of $x$) that for fixed $x$ it yields

$$f(z) \geq f(x) + \xi \cdot (z - x), \forall z \in \mathbb{R}^n.$$

*If $f$ is differentiable* at $x$, then

$$\xi = \nabla f(x).$$

But in general it does not have to be differentiable. This motivates the following definition

**Definition 12** (Subgradient, subdifferential)**.** The function $\xi(x)$ such that

$$f(z) \geq f(x) + \xi(x) \cdot (z - x), \forall z \in \mathbb{R}^n,$$

is called the **subgradient** of $f$ at $x$. The set of all subgradients of $f$ at $x$ is called the **subdifferential** of $f$ at $x$ and it is denoted $\partial f(x)$.

Let $f : \mathbb{R}^{n \times m} \to \mathbb{R} \cup \{\infty\}$, convex and lower semicontinuous [7], $f \neq \infty$. The function $\xi(\mathbb{X})$ such that

$$f(\mathbb{Y}) \geq f(\mathbb{X}) + \xi(\mathbb{X}) \cdot (\mathbb{Y} - \mathbb{X}), \forall \mathbb{Y} \in \mathbb{R}^{n \times m},$$

is called the subgradient of $f$ at $\mathbb{X}$. The set of all subgradients of $f$ at $\mathbb{X}$ is called the subdifferential and denoted $\partial f(\mathbb{X})$.

*Remark.*
- If $\partial f(\mathbb{X})$ is a singleton, then $\nabla f(\mathbb{X})$ exist.
- $\partial f(\mathbb{X})$ is convex
- $0 \in \partial f(x) \forall x \in \mathbb{R}^n$ is a condition for the minimizer.

**Definition 13** (Indicator function)**.** Let $K \subset \mathbb{R}^{n \times m}$ be a closed convex nonempty set. The function $I_K(\mathbb{X})$ given as

$$I_K(\mathbb{X}) = \begin{cases} 0, & \text{if } \mathbb{X} \in K \\ +\infty, & \text{otherwise} \end{cases},$$

is called the indicator function of $K$

[7] $f(x) \leq \liminf_{k \to \infty} f(x_k), x_k \to x$

The indicator function is hepful for constraint minimization. If $f$ is reasonably (at least finitely valued on $K$) then it holds:

$$\min_K f = \min_{\mathbb{R}^{n\times m}} (f + I_K).$$

**Example** (Unit interval)**.** Let $K = [0,1]$. What is $\partial I_K(x)$?

If $x \in (0,1)$, then $I_K(x) = 0$ so the only $\xi$ such that $I_K(y) \geq 0 + \xi(y-x)$ holds is $\xi = 0$.

If $x = 0, x = 1$ then $\partial I_K(0) = (-\infty, 0], \partial I_K(1) = [0,\infty)$. This resembles a normal "vector", but in fact it is not a single vector and more a "cone" of vectors.

**Definition 14** (Normal cone to a set)**.** Let $K$ be closed convex nonempty set. The subdifferential of the indicator function $I_K$ is called the normal cone to the set $K$ and it is denoted by $N_K$.

**Example.** Minimize $x^2$ on $[1,2]$. We are looking for

$$\min_{[1,2]} x^2 = \min_{\mathbb{R}} \big(x^2 + I_{[1,2]}(x)\big).$$

It must hold at the minimum

$$0 \in \partial\big(x^2 + I_{[1,2]}(x)\big) \Leftrightarrow -\partial I_{[1,2]}(x) \subset \partial x^2 \Leftrightarrow \big(x^2\big)' \in -N_{[1,2]}(x)$$

**Example.** Take a square $K = [0,1] \times [0,1] \subset \mathbb{R}^2$. We know $K \in C^{0,1}$ so the outter normal exist at a.a. points on the boundary. The outter normal does not exist in the corners, but the normal cone does.

**Definition 15** (Fenchel (convex) conjugate — Legendre transformation)**.** Let $x^*$ be a slope i have chosen (it is given). I require

$$f(x) \geq x^* \cdot x - k, \forall x \in \mathbb{R}^{n\times m},$$

which means $k \geq x^* \cdot x - f(x), \forall x \mathbb{R}^{n\times m}$, and so we can define

$$f^*(x^*) \coloneqq \sup_{x\in\mathbb{R}^{n\times m}} (x^* \cdot x - f(x)).$$

*Remark.* $f^*$ is always convex even if $f$ is not. But when $f$ is convex and lower-semicontinuous, then

$$f^{**} = f, \text{ (biconjugate)}.$$

**Theorem 6** (Fenchel identity)**.** *Let* $x^* \in \partial f(x)$*. Then*

$$x^* \cdot x = f(x) + f^*(x^*).$$

*Proof.* Let us assume that $x^* \in \partial f(x)$. Then it must hold

$$f(y) \geq f(x) + x^* \cdot (y - x), \forall y,$$

so

$$x^* \cdot x - f(x) \geq x^* \cdot y - f(y),$$

and taking the supremum over $y$ yields[8]

$$x^* \cdot x - f(x) = \sup_y (x^* \cdot y - f(y)) = f^*(x^*).$$

We have thus obtained

$$x^* \cdot x = f(x) + f^*(x^*).$$

$\square$

*Remark* (Minimization of $f \Leftrightarrow$ minimization of $f^*$). We see that it holds:

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*).$$

## 6.3 Problem of a man...

Assume a person is pulling a box of weight $m$ of weight $m$ of weight $m$ of weight $m$ by a spring. If he is pulling just a little, the box does not move, only the spring is deformed - but in a reversible, elastic way. To move the box, the man needs to pull at least with the force $\sigma_0 = mgc$, where c is some friction coefficient. When he us pulling with force greater than $\sigma_0$, the box is moving and does not require any extra force to be moved (the system to be deformed). The deformation can be decomposed as

$$\mathbb{e} = \tilde{\mathbb{e}} + \mathbb{p},$$

where $\mathbb{e}$ is the total strain, $\tilde{\mathbb{e}}$ is the elastic strain and $\mathbb{p}$ is the plastic strain.

## 6.4 von Mieses elatoplasticity

The elasticity part is described as

$$\begin{cases} -\nabla \cdot \sigma = \mathbf{f}, & \text{in the bulk} \\ \sigma \mathbf{n} = \mathbf{g}, & \text{on the boundary} \end{cases},$$

with some constituive relation $\sigma = \mathcal{C}\tilde{\mathbb{e}} = \mathcal{C}(\mathbb{e} - \mathbb{p})$. What about the plastic part?

$$\begin{cases} \dot{\mathbb{p}}(t) \in N_K(\sigma), \\ \mathbb{p}(0) = \mathbb{p}_0, \end{cases}$$

where $K$ is a convex closed subset such that $0 \in K$. This means that the plastic deformation is zero inside $K$, i.e. for some stresses.

*Remark.* Very often, the deformation is considered "incompressible", i.e.,

$$\det \mathbb{F} = 1,$$

which in linear case translates into

$$\operatorname{tr} \varepsilon = 0.$$

[8] The inequality becomes equality, as it can be reached by taking $y = x$.

In most cases, the set $K$ is given as

$$K = \{\sigma : \varphi(\sigma) \leq 0\},$$

where $\varphi$ is the **yield function.** The set

$$\{\sigma | \varphi(\sigma) = 0\}$$

is called the **yield surface.** Very often we have

$$\varphi(\sigma) = |\sigma^D| - c_0,$$

where $|\cdot|$ denotes the Frobenius norm and

$$\sigma^D = \sigma - \frac{1}{3}(\operatorname{tr}\sigma)\mathbb{I},$$

is the *deviatoric part of the stress tensor.*

### 6.4.1 Plastic evolution

From the previous we have

$$\dot{\mathbb{p}} = \begin{cases} \mathbb{0}, & \text{if } \varphi(\sigma) < 0, \\ \frac{\lambda}{|\sigma^D|}\sigma^D, & \text{if } \varphi(\sigma) = 0, \lambda \geq 0 \end{cases}.$$

Also $\dot{\mathbb{p}} \in N_K(\sigma) = \partial I_K(\sigma)$ so

$$\sigma \in \partial I_K^*(\dot{\mathbb{p}}),$$

where

$$I_K^*(\dot{\mathbb{p}}) = \sup_{\mathbb{q} \in \mathbb{R}^{3\times 3}} (\dot{\mathbb{p}} : \mathbb{q} - I_K(\mathbb{q})) = \sup_{\mathbb{q} \in K} \dot{\mathbb{p}} : \mathbb{q},$$

is the Fenckel transformation of $I_K$, also called the **supporting function** of $\dot{\mathbb{p}}$. We are able to rewrite the supremum to take the form[9]

$$I_K^*(\dot{\mathbb{p}}) = \dot{\mathbb{p}} : \frac{c_0}{|\dot{\mathbb{p}}|}\dot{\mathbb{p}},$$

if however the second term lies in $K$. Realize now that if $\operatorname{tr}\dot{\mathbb{p}} = 0$ then

$$I_K^*(\dot{\mathbb{p}}) = c_0|\dot{\mathbb{p}}|,$$

and if $\operatorname{tr}\dot{\mathbb{p}} \neq 0$, then $I_K^*(\dot{\mathbb{p}}) = +\infty$. If we now define the **dissipation potential** $D$ as

$$D(\dot{\mathbb{p}}) = \begin{cases} c_0|\dot{\mathbb{p}}|, & \text{if } \operatorname{tr}\dot{\mathbb{p}} = 0 \\ +\infty, & \text{otherwise} \end{cases},$$

we get the following condition

$$\sigma \in \partial D(\dot{\mathbb{p}}).$$

[9] To utilize Cauchy-Schwarz later.

Let us summarise a bit. For the stress tensor we have $\sigma = \mathcal{C}(\mathbb{e} - \mathbb{p}) \in D(\dot{\mathbb{p}})$. The general relation also yields $\sigma = \frac{\partial w(\tilde{\mathbb{e}})}{\partial \tilde{\mathbb{e}}} = \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \tilde{\mathbb{e}}}$, where $w(\tilde{\mathbb{e}}) = \frac{1}{2} C\tilde{\mathbb{e}} : \tilde{\mathbb{e}}$ is the free energy density. Using the chain rule we obtain the condition

$$\frac{\partial w(\mathbb{e} - \mathbb{p})}{\partial \mathbb{p}} \in \partial D(\dot{\mathbb{p}}).$$

In total, we are solving the following system

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D(\dot{\mathbb{p}}), & \text{in}\, \Omega (\textit{flow rule}) \\ \mathbb{p}(0) = \mathbb{p}_0, & \text{in}\, \Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e} - \mathbb{p})) = \mathbf{f}, & \text{in}\, \Omega \\ \text{boundary condiitons}\,, & \text{on}\, \partial\Omega \end{cases}.$$

How to solve the system?

### 6.4.2 Discrete time setting

Let us take $t \in [0, T]$ and fix $\tau = \frac{T}{N}, N \in \mathbb{N}$ for some $N >> 1$. Assume that using some discrete scheme, we are able to calculate $\mathbb{p}$ at a certain time. Then we must solve

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D\Big(\frac{\mathbb{p}-\dot{\mathbb{p}}_{\mathbb{k}-\mathbb{1}}}{\tau}\Big), & \text{in}\, \Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k, & \text{in}\, \Omega \end{cases}.$$

Which are the E-L equations of the functional [10]

$$I(\mathbf{u}, \mathbb{p}) = \int_\Omega w(\mathbb{e}(\mathbf{u}) - \mathbb{p})\, \mathrm{d}x + \tau \int_\Omega D\Big(\frac{1}{\tau}(\mathbb{p} - \mathbb{p}_{k-1})\Big)\, \mathrm{d}x - \int_\Omega \mathbf{f}_k \cdot \mathbf{u}\, \mathrm{d}x - \int_{\Gamma_N} \mathbf{g}_k \cdot u\, \mathrm{d}S\,.$$

Really, taking the variaton with respect to $\mathbf{u}$ gives us

$$-\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k,$$

and the variation with respect to $\mathbb{p}$ gives us

$$0 \in -\sigma + \partial D\Big(\frac{1}{\tau}(\mathbb{p} - \mathbb{p}_{k-1})\Big).$$

If we want to minimize this functional, *i.e.*, solve the equations, it must hold [11] $D(\mathbb{q}) \neq +\infty$ (for $\mathbb{q}$ being the argument). From our assumptions on the dissipation potential this however implies.

$$D(\mathbb{q}) = c_0 |\mathbb{q}|, \operatorname{tr} \mathbb{q} = 0,$$

and we say the evolution is **rate-independent**. We see that $D$ is 1-homogenous:

$$D(\alpha \mathbb{q}) = \alpha D(\mathbb{q}).$$

Rewritting the functional now yields:

$$I(\mathbf{u}, \mathbb{p}) = \frac{1}{2} \int_\Omega \mathcal{C}(\mathbb{e}(\mathbf{u}) - \mathbb{p}) : (\mathbb{e}(\mathbf{u}) - \mathbb{p})\, \mathrm{d}x + \int_\Omega c_0 |\mathbb{p} - \mathbb{p}_{k-1}|\, \mathrm{d}x - L_k(\mathbf{u}), \mathbb{p}(0) = \mathbb{p}_0,$$

[10] We have guessed it.

[11] If not, we have no chance of minimizing it.

where $L_k(\mathbf{u})$ is the loading (at the $k$-th time step.) The sought solution is the pair $(\mathbf{u}_k, \mathbb{p}_k)$ which satisfies

$$I(\mathbf{u}_k, \mathbb{p}_k) = \min_{\mathbf{u},\mathbb{p}} I(\mathbf{u}, \mathbb{p}).$$

## 6.5 Rheological models

### 6.5.1 Dashpots

Or *tlumič* in Czech. The stress is assumed to take the form

$$\sigma = \mathcal{D}\dot{\mathbb{e}}(\nabla\mathbf{u}), \sigma_{ij} = D_{ijkl}\dot{e}_{kl}(\nabla\mathbf{u}),$$

where $\mathcal{D}$ is the **tensor of viscosity constants.** [12]

### 6.5.2 Kelvin-Voigt mateial

The response of some materials can be modelled as a "parallel composition of a spring and a dashpot." Then, the total stress is

$$\sigma = \sigma_p + \sigma_e,$$

that is the sum of the plastic and the elastic stresses. The strain is of course the same:

$$\mathbb{e} = \mathbb{e}_p = \mathbb{e}_e.$$

The governing equations thus are

$$\begin{aligned} -\nabla\cdot(\mathcal{C}\mathbb{e}(\mathbf{u}) + \mathcal{D}\dot{\mathbb{e}}(\mathbf{u})) &= \mathbf{f}, \text{ in } \Omega \\ (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}})\mathbf{n} &= \mathbf{0}, \text{ on } \Gamma_N \\ \mathbf{u} &= \mathbf{0}, \text{ on } \Gamma_D \\ \mathbb{e}(t=0) &= \mathbb{e}_0, \text{ in } \Omega. \end{aligned}$$

Let us obtain the energy *formally* balance. As usual, multiply the first equation by $\dot{\mathbf{u}}$ and integrate $\int_\Omega \mathrm{d}\mathbf{x}$.

$$\int_\Omega -\nabla\cdot(\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}})\cdot\dot{\mathbf{u}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

using Gauss

$$\int_\Omega (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}}) : \nabla\dot{\mathbf{u}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x = {}^{13} \int_\Omega \mathcal{C}\mathbb{e} : \dot{\mathbb{e}}\,\mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}} : \dot{\mathbb{e}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,,$$

and now we rewrite

$$= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\right)\mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

[12] People say viscosity stresses or viscous stress. This is used, but nonetheless it is wrong.
[13] It holds $\dot{\mathbb{e}}(\mathbf{u}) = \mathbb{e}(\dot{\mathbf{u}})$.

and integrate in time:

$$\int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

Remember that

$$w(\mathbb{e}(\mathbf{u})) = \frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u}),$$

so we have obtained

$$\int_\Omega w(\mathbb{e}(\mathbf{u}(T)))\,\mathrm{d}x - \int_\Omega w(\mathbb{e}(\mathbf{u}(0)))\,\mathrm{d}x + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

### 6.5.3 Maxwell material

This is the case when we "put the spring and the dashpot in serial composition". The total stress is

$$\sigma = \sigma_p = \sigma_e,$$

and the total strain is

$$\varepsilon = \mathbb{e}_p + \mathbb{e}_e.$$

## 6.6 Internal parameters

A lot of materials can be described using some internal parameters $\mathbb{z}$ (scalars, vectos, tensors; we take the tensor case for generality); for example, plastic strain, fatique, damage, length of a crack, delamination.

The model

$$\sigma = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{e}} w(\mathbb{e}, \mathbb{z}),$$

with the flow rule

$$0 \in \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}).$$

is called the **generalized Kelvin-Voigt** model/material. From now on, we will be using $\varphi$ for the stored energy density. There is some analogy:

- $\varphi$ is the stored energy density = potential of stress
- $\zeta$ is the (pseudo)potential of dissipative forces.

To do anything, we need to obtain some energy balance, so test by $\dot{\mathbf{u}}$. Investigate the terms:

$$\sigma : \dot{\mathbb{e}} = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{e}},$$

realize now that from the flow rule it follows

$$(\partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{z}) + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z})) : \dot{\mathbb{z}} = 0,$$

so i can add it to the previous term and obtain

$$\sigma : \dot{\mathbb{e}} = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) : \dot{\mathbb{z}} + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}) : \dot{\mathbb{z}} = 0,$$

Realize now that we have obtained

$$\partial_{\mathbb{e}}\varphi(\mathbb{e}, \mathbb{z})\dot{\mathbb{e}} + \partial_{\mathbb{z}}\varphi(\mathbb{e}, \mathbb{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e}, \mathbb{z}),$$

and denoting the quantity

$$\xi(\dot{\mathbb{e}},\dot{\mathbb{z}}) := \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : \dot{\mathbb{z}},$$

as the *rate of the dissipation* we obtain

$$\sigma : \dot{\mathbb{e}} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e},\mathbb{z}) + \xi(\dot{\mathbb{e}},\dot{\mathbb{z}}).$$

What are the properties of $\xi$? First of all, we require

$$\xi \geq 0.$$

Assume $\zeta$ is a covex function:

$$\zeta(0,0) \geq \zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) + \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : (-\dot{\mathbb{e}}) + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : (-\dot{\mathbb{z}}).$$

Moreover, assume now $\zeta(0,0) = 0$. We have

$$\xi(\dot{\mathbb{e}},\dot{\mathbb{z}}) = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : \dot{\mathbb{e}} + \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) : \dot{\mathbb{z}} \geq \zeta(\dot{\mathbb{e}},\dot{\mathbb{z}}) \geq 0.$$

Finally, the total power balance becomes

$$\int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\rho|\dot{\mathbf{u}}|^2\,\mathrm{d}x + \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbb{e},z)\,\mathrm{d}x + \int_\Omega \xi(\dot{\mathbb{e}},\dot{z})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

and the total energy balance becomes

$$\int_\Omega \frac{1}{2}\rho\big(|\dot{\mathbf{u}}(T)|^2 - |\dot{\mathbf{u}}(0)|^2\big)\,\mathrm{d}x + \int_\Omega \big(\varphi(\mathbb{e}(T),z(T)) - \varphi(\mathbb{e}(0),z(0))\big)\,\mathrm{d}x + \int_0^T\int_\Omega \xi(\dot{\mathbb{e}},\dot{z})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

# 7 Thermodynamics in the framework of GSM (generalized standard materials)

*Having obtained some knowledge of thermodynamical quantites, we are ready to generalize the theory. We will see that the evolution of a specimen can be acquired by the knowledge of the stored energy density $\psi$ and the dissipation "potential" $\zeta$*

Denote

$$\psi = \psi(\mathbb{e},\mathbb{z},\theta),\ \zeta = \zeta(\dot{\mathbb{e}},\dot{\mathbb{z}})$$

to be *the stored energy* and the *dissipation potential.* Here $\theta > 0$ denotes the absolute thermodynamic temperature. Let us denote

$$\sigma_{el} = \frac{\partial\psi}{\partial\mathbb{e}},\ \sigma_{in} = \frac{\partial\psi}{\partial\mathbb{z}},\ s = -\frac{\partial\psi}{\partial\theta},$$

as the elastic and inelastic stress and the entropy density. Moreover, define

$$w(\mathbb{e},\mathbb{z},\theta,s) = \psi(\mathbb{e},\mathbb{z},\theta) + \theta s$$

as the **internal energy density**. If we calculate the time derivative of the internal energy density we obtain:

$$\dot{w} = \frac{\partial}{\partial t}(\psi(\mathbb{e},z,\theta) + \theta s) = \frac{\partial\psi}{\partial\mathbb{e}} : \dot{\mathbb{e}} + \frac{\partial\psi}{\partial\mathbb{z}} : \dot{\mathbb{z}} + \underbrace{\frac{\partial\psi}{\partial\theta}\dot{\theta} + \dot{\theta}s}_{=-s\dot{\theta}+\dot{\theta}s=0} + \theta\dot{s}.$$

We *postulate*:

$$\dot{w} = \sigma_{el} : \dot{\mathbb{e}} + \sigma_{in} : \dot{\mathbb{z}} + \xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) - \nabla \cdot \mathbf{j},$$

where $\mathbf{j}$ is the heat flux. From this postulate, we obtain

$$\xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) = \theta \dot{s} + \nabla \cdot \mathbf{j}. \tag{13}$$

A common modelling choice is the dependency

$$\mathbf{j} = \mathbf{j}(\theta, \mathbb{e}, \mathbb{z}, \nabla\theta) = -\mathbb{K}(\mathbb{e}, \mathbb{z}, \theta)\nabla\theta,$$

known as the *Fourier law.* Here

$$\mathbb{K} \in \{\mathbb{A} \in \mathbb{R}^{3\times 3} | \mathbb{A} > 0\},$$

is the *matrix of heat flux coefficients.* This is a classical example of a constitutive law.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s(t,\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{1}{\theta}(\xi - \nabla\cdot\mathbf{j})\,\mathrm{d}\mathbf{x} = \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}\mathbf{x} + \int_\Omega \frac{\nabla\cdot(\mathbb{K}\nabla\theta)}{\theta}\,\mathrm{d}\mathbf{x} = \\
&= \int_{\partial\Omega} \frac{\mathbb{K}\nabla\theta}{\theta}\cdot\mathbf{n}\,\mathrm{d}S - \int_\Omega \mathbb{K}\nabla\theta\cdot\nabla\left(\frac{1}{\theta}\right)\mathrm{d}\mathbf{x} + \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}\mathbf{x} = \\
&= \int_\Omega \left(\frac{\xi}{\theta} + \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2}\right)\mathrm{d}\mathbf{x} - \int_{\partial\Omega}\frac{\mathbf{j}}{\theta}\cdot\mathbf{n}\,\mathrm{d}S\,.
\end{aligned}$$

This relation is known as the *Clausius-Duhem inequality.*[14]

From the definition of $s$

$$s = -\frac{\partial\psi}{\partial\theta}(\theta, \mathbb{e}, \mathbb{z}),$$

it follows

$$\dot{s} = -\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}},$$

and so

$$\theta\dot{s} = \underbrace{-\frac{\partial^2\psi}{\partial\theta^2}\theta}_{:=C_V}\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : (\dot{\mathbb{e}}\theta) - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : (\dot{\mathbb{z}}\theta) = C_V\dot{\theta} - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : (\dot{\mathbb{e}}\theta) - \frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : (\dot{\mathbb{z}}\theta),$$

where we have identified

$$C_V = -\theta\frac{\partial^2\psi}{\partial\theta^2},$$

as the *heat capacity at the constant volume.* Coming back to 13, we read

$$C_V\dot{\theta} + \nabla\cdot\mathbf{j} = \xi(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}}.$$

This is our *heat equation,* the right hand side are the sources. We could identify the derivatives of the potential with lets say some derivative of $\sigma_{el}$, but let us keep the "thermodynamics and mechanics separated."; although it does not really make sense. In total

[14] Although inequalty, there appears only the equality sign "=". I do not actually know what that means.

$$
\begin{aligned}
C_V\dot{\theta} - \nabla\cdot(\mathbb{K}\nabla\theta) &= \xi + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{e}} : \dot{\mathbb{e}} + \theta\frac{\partial^2\psi}{\partial\theta\partial\mathbb{z}} : \dot{\mathbb{z}},\\
\rho\ddot{\mathbf{u}} - \nabla\cdot(\sigma_{el} + \sigma_{in}) &= \mathbf{f},\\
0 &\in \partial_{\dot{\mathbb{z}}}\zeta(\dot{\mathbb{e}}, \dot{\mathbb{z}}) + \partial_{\mathbb{z}}\psi(\mathbb{e}, \mathbb{z}, \theta),
\end{aligned}
$$

plus of course some initial and boundary conditions.

# 8 Summary

*At the end, the lecture is summarized.*

It began with deformation:

$$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^3, \nabla\mathbf{y} = \mathbb{F}, \mathbb{C} = \mathbb{F}^\top\mathbb{F}, \det\mathbb{F} > 0.$$

and some quantities associated with these. A little excursion allowed as to define

$$W = W(\nabla\mathbf{y}) = W(\mathbb{F}),$$

to be the stored energy density. Note that later on, we have called it $\psi$. Coming back to deformation, we have defined various stress measures:

$$\mathbb{T}^y, \mathbb{T} = \mathbb{T}^y \operatorname{cof}\mathbb{F}, \mathbb{S} = \mathbb{F}^{-1}\mathbb{T}.$$

Wanting to show existence of solutions, we needeed the convexity of some functionals. A problem with rotations however meant we needed to lower our expectations and we had to discover polyconvexity and rank-1 convexity. This included *e.g.* Legendre-Hadamard condition.

Realizing we are stuck in full theory, we began exploring linearized elasticity. To show existence, we refreshed the Korn's inequality. And because that all seemed easy, a question about time dependence has been asked: is everything truly stationary?

No, it is not; that lead us to von Mises elastoplasticity and to a class of materials, such as Kelvin-Voigt or Maxwell materials. Generalizing this framework and also including some internal variables, we have given the foundations of (the thermodynamics of)gave generalized standard materials: this was especially elegant, as from the Helmholtz free energy and the dissipation potential, we were able to derive evolution equations for the important thermodynamical quantites. This included some energy/power estimates, balances and the notion of entropy and its rate.

# 9 (Some) tutorials

## 9.1 Change of observer

The requirement of material frame indifference yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}), \forall\mathbb{Q} \in \operatorname{orth}.$$

## 9.2 Change of reference configuration

The requiremenent of material symmetry yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{P}), \forall \mathbb{P} \in \mathcal{G},$$

where $\mathcal{G}$ is the symmetry group of the material.

## 9.3 Consequences of isotropic hyperelastic solid

*Remark* (Groups unim, orth). The "biggest sensible" symmetry group is the unimodular group:

$$\text{unim} = \{\mathbb{P}, \det \mathbb{P} = \pm 1\}.$$

There exists another common group:

$$\text{orth}\,\{\mathbb{Q}, \mathbb{Q}\mathbb{Q}^\top = \mathbb{Q}^\top\mathbb{Q} = \mathbb{I}\} \subset \text{unim}\,.$$

We thus have $W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}), \forall \mathbb{Q} \in \text{orth}\,, \forall \mathbb{F}.$

Use *polar decomposition*: $\mathbb{F} = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R}, \mathbb{R} \in \text{orth}\,, \mathbb{U}, \mathbb{V}\,\text{positively definite}\,, \mathbb{U} = \sqrt{\mathbb{C}}, \mathbb{V} = \sqrt{\mathbb{B}}.$

To make use of it, we first use m.f.i. and then isotropy and then first use isotropy and then m.f.i. So from material frame indifference

$$W = \hat{F}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}\left(\mathbb{R}^\top\mathbb{R}\mathbb{U}\right) = \hat{W}(\mathbb{U}) = \overline{W}(\mathbb{C}),$$

where we have taken $\mathbb{Q} = \mathbb{R}^\top$. Note that this works universaly (without the need of isotropy), as it comes from the objectivity consideration.

From isotropy

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \overline{W}(\mathbb{C}) = \overline{W}\left((\mathbb{F}\mathbb{Q})^\top(\mathbb{F}\mathbb{Q})\right) = \overline{W}\left(\mathbb{Q}^\top\mathbb{F}^\top\mathbb{F}\mathbb{Q}\right) = \overline{W}\left(\mathbb{Q}^\top\mathbb{C}\mathbb{Q}\right), \forall Q \in \text{orth}\,, \forall \mathbb{C}\,\text{admissable}\,.$$

Now backwards using the second polar decomposition:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \hat{W}(\mathbb{V}\mathbb{R}\mathbb{Q}) = \hat{W}\left(\mathbb{V}\mathbb{R}\mathbb{R}^\top\right) = \hat{W}\left(\sqrt{\mathbb{B}}\right) = \tilde{W}(\mathbb{B}),$$

$$W = \tilde{W}(\mathbb{B}) = \tilde{W}\left(\mathbb{Q}\mathbb{F}(\mathbb{Q}\mathbb{F})^\top\right) = \tilde{W}\left(\mathbb{Q}\mathbb{B}\mathbb{Q}^\top\right).$$

So far, we have shown

$$W(t, \mathbf{X}) = \overline{W}(\mathbb{C}(t, \mathbf{X}), \mathbf{X}) = \overline{W}\left(\mathbb{Q}\mathbb{C}\mathbb{Q}^\top\right),$$
$$W(t, \mathbf{X}) = \tilde{W}(\mathbb{B}(t, \mathbf{X}), \mathbf{X}) = \tilde{W}\left(\mathbb{Q}\mathbb{B}\mathbb{Q}^\top\right),$$

In HW, we will know

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}}, T_{iJ} = 2\frac{\partial \hat{W}}{\partial F_{iJ}}$$

and we can show

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}} = 2\mathbb{F}\frac{\partial \tilde{W}(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}, \mathbb{T}^y = \ldots, \mathbb{S} = 2\frac{\partial \overline{W}(\mathbb{C})}{\partial \mathbb{C}}.$$

**Definition 16** (Isotropic functions)**.** We say the functions $\hat{a}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \hat{\mathbf{a}}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \hat{\mathbb{A}}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha), \alpha = 1, \dots, N$ are isotropic functions (of their respective arguments) if it holds

$$
\begin{aligned}
\hat{a}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha) &= \hat{a}\big(y_\alpha, \mathbb{Q}\mathbf{y}_\alpha, \mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),\\
\mathbb{Q}\hat{\mathbf{a}}(y_\alpha, \mathbf{y}_\alpha, \mathbb{Y}_\alpha) &= \hat{\mathbf{a}}\big(y_\alpha, \mathbb{Q}\mathbf{y}_\alpha, \mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),\\
\mathbb{Q}\hat{\mathbb{A}}\mathbb{Q}^\top &= \hat{\mathbb{A}}\big(y_\alpha, \mathbb{Q}\mathbf{y}_\alpha, \mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),
\end{aligned}
$$

.

So we see that $\overline{W}(\mathbb{C}), \tilde{W}(\mathbb{B})$ are **scalar isotropic functions of 1 tensorial (symmetric) argument.**

**Theorem 7** (Representation theorem for scalar isotropic functions)**.** *Let* $\psi = \hat{\psi}(\mathbb{A}) = \hat{\psi}(\mathbb{Q}\mathbb{A}\mathbb{Q}^\top)$ *be a scalar isotropic function of a single symmetric tensorial variable. Then it must hold*

$$\hat{\psi}(\mathbb{A}) \equiv \hat{\psi}(\mathrm{I}_1(\mathbb{A}), \mathrm{I}_2(\mathbb{A}), \mathrm{I}_3(\mathbb{A})),$$

*where*

$$
\begin{aligned}
\mathrm{I}_1(\mathbb{A}) &= \operatorname{tr}\mathbb{A},\\
\mathrm{I}_2(\mathbb{A}) &= \frac{1}{2}\Big((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}\mathbb{A}^2\Big),\\
\mathrm{I}_3(\mathbb{A}) &= \det\mathbb{A},
\end{aligned}
$$

*are the invariants of* $\mathbb{A}$*.*

*Proof.* $\det(\mathbb{A} - \lambda\mathbb{I}) = -\lambda^3 + \lambda^2 \mathrm{I}_1 - \lambda \mathrm{I}_2 + \mathrm{I}_3 = p_\lambda(\mathbb{A})$ We will prove a different assertion:

$\mathbb{A}, \mathbb{B}$ are symmetric with the same invariants $\Leftrightarrow \exists \mathbb{Q} : \mathbb{A} = \mathbb{Q}\mathbb{B}\mathbb{Q}^\top$ " $\Leftarrow$ " is trivial, as then the matrices are simILLiar, so they have the same char. polynomial, so they have the same invariants. $\Rightarrow$ have same eigenvalues, so if i write the spectral decomposiiton, i can write

$$\mathbb{A} = \mathbb{Q}\Lambda\mathbb{Q}^\top, \mathbb{B} = \mathbb{Q}\Lambda\mathbb{R}^\top = \mathbb{R}\mathbb{Q}^\top\mathbb{A}\mathbb{Q}\mathbb{R}^\top.$$

Now suppose that the function is not a function of the invariants: $\hat{\psi} \neq \tilde{\psi}(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$. That means $\exists \mathbb{A}_, \mathbb{A}_2$ such that $\mathrm{I}_1(\mathbb{A}_1) = \mathrm{I}_1(\mathbb{A}_2)$ and the same for the remaining invariants. Using the previous assertion, we have

$$\exists \mathbb{Q} : \mathbb{A}_1 = \mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top \Rightarrow \hat{\psi}(\mathbb{A}_1) = \hat{\psi}\big(\mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top\big) = \hat{\psi}(\mathbb{A}_2), \text{ but } \hat{\psi}(\mathbb{A}_1) \neq \hat{\psi}(\mathbb{A}_2).$$

□

Since using polar decomposition it can be shown the invariants of $\mathbb{B}, \mathbb{C}$ are the same we recieve

$$W = \tilde{W}(\mathrm{I}_1(\mathbb{B}), \mathrm{I}_2(\mathbb{B}), \mathrm{I}_3(\mathbb{B})) = \overline{W}(\mathrm{I}_1(\mathbb{C}), \mathrm{I}_2(\mathbb{C}), \mathrm{I}_3(\mathbb{C})).$$

## 9.4 Representation in terms of principial stresses

... in terms of the eigenvalues $\mathbb{U}, \mathbb{V}$. The invariants can be expressed as

$$\begin{aligned}
\mathrm{I}_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\
\mathrm{I}_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \\
\mathrm{I}_3 &= \lambda_1\lambda_2\lambda_3.
\end{aligned}$$

Often in materials science the quantites can be expressed in these variables:

**Example** (Ogden materials)**.**

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{n} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\lambda_k} + \lambda_3^{\alpha_k} - 3\right)$$

How to calculate e.g. $\mathbb{T}$ in this representation?

$$\mathbb{T} = 2\frac{\partial W(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)}{\partial \mathbb{B}}\mathbb{F} = 2\frac{\partial \hat{W}}{\partial \mathbb{B}}(\lambda_1(\mathbb{B}), \lambda_2(\mathbb{B}), \lambda_3(\mathbb{B}))2\frac{\partial \hat{W}}{\partial \lambda_i}\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}.$$

What (in the hell) is $\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}$? [15]

$$\mathbb{B}(s) = \sum_{\alpha=1}^{3} \omega_\alpha(s)\mathbf{g}_\alpha(s) \otimes \mathbf{g}_\alpha(s), \forall s \in I$$

where $I$ is some open interval and $\{\mathbf{g}_\alpha\}$ is an ON eigenbasis of $\mathbb{B}$. Next, realize

$$\omega_1(s) = \mathbf{g}_1(s) \cdot \mathbb{B}(s)\mathbf{g}_1(s),$$

and differentiate this:

$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{g}_1}{\mathrm{d}s} \cdot \mathbb{B}\mathbf{g}_1 + \mathbf{g}_1\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}s}\mathbf{g}_1 + \mathbf{g}_1 \cdot \mathbb{B}\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}s} = \frac{1}{2} + +0.$$

[15] Recall the Daleckii-Krein theorem: