If $\lambda = 0$ we sometimes drop it and write $\Omega \in C^{k,0} \Leftrightarrow \Omega \in C^{k}$, if $k = 0, \lambda = 1$ we call $\Omega \in C^{0,1}$ to be a Lipschitz domain. *Remember that* $\lambda(\Omega) < \infty$ *is a part of the definition.*

Theorem 5 (Global approximation by smooth functions up to the boundary)**.** *Let* $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1, \infty)$. *Then* $C^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$ *is dense in* $W^{k,p}(\Omega)$.

Proof. Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in \mathrm{C}^{\infty}\left(\overline{\Omega}\right)$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\overline{\Omega}$,
2. partition of unity,
3. approximation of u on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^{m}$ be from the definition of $\mathrm{C}^{0,0}$ boundary. Then[4]

$$\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to $\{U_j\}_{j=0}^{m}$. Since

$$u = \sum_{j=0}^{m} u\varphi_j, \text{ on } \Omega$$

observe that $u_j := u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \operatorname{supp} u_j \subset \operatorname{supp}\varphi_j \subset\subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d/\Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \dots, m\}$.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of Ω, it holds $u\varphi_0 \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)$. Since $\mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right) = \mathrm{W}^{\mathrm{k,p}}_0\left(\mathbb{R}^d\right) = \overline{\mathcal{D}\left(\mathbb{R}^d\right)}^{\|\cdot\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)}}$, we can find $v_0 \in \mathcal{D}\left(\mathbb{R}^d\right)$ *s.t.*

$$\|v_0 - u\varphi_0\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

Case $j \in \{1, \dots, m\}$. We have a problem now: $\{U_j\}_{j=1}^{m}$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of Ω, where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate u_j on a larger *open* domain containing $\overline{\Omega}$ and then show this is also a good aproximation of u_j on $\Omega \subset \overline{\Omega}$.

Set $w_j = u\varphi_j$, and denote

$$\begin{aligned} S_j &= \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in \mathrm{U}(0, \alpha)\right\}\right), \\ \Omega_j &= \mathbb{R}^d / \overline{S_j}, \end{aligned}$$

i.e.,

$$\text{"}\Omega_j = \Omega \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d \le a_j(x') - \frac{\beta}{2}\right\}\right),\text{"}$$

[4]Our choice $U_0 = \Omega$ is important, as without it the definition of $\mathrm{C}^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^{m} U_j$.