Charles University

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Continuum mechanics

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6.1 Stability of fluid flows

Let us investigate the following PDE:

$$\begin{cases} \partial_t u = \partial_{xx} u + au, & \text{in}\, \Omega = (0,1) \\ u(t,x) = 0, & \text{on}\, x = 0, x = 1 \end{cases}.$$

Clearly, $\hat{u}(t,x) = 0$ is a solution, moreover it is a *steady solution*. Our interest is whether, given some initial condition $u_0(x)$the solution converges to the steady one; in other words, whether

$$" \lim_{t\to\infty} u(t,x) = 0",$$

in some sense of convergence.

6.1.1 Energy theory

One way would be to linearize and then deploy some linearized stability analysis techniques (as in the case of ODE's). However, we do not want to lose any information (*i.e.*, any phenomena) described by the full system; a different technique is needed. Let us measure the convergence in the $\mathrm{L}_2(\Omega)$ norm, "the energy norm", *i.e.*

$$\|u\|_{\mathrm{L}_2(\Omega)} = \left(\int_\Omega |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Meaning we are interested in the conditions under which

$$u \to 0 \,\text{in}\, \mathrm{L}_2(\Omega) \Leftrightarrow \|u\|_{\mathrm{L}_2(\Omega)} \to 0.$$

Let us investigate the following quantity:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)u(t,x)\,\mathrm{d}x = \int_\Omega \partial_t u(t,x)u(t,x)\,\mathrm{d}x = \\ &= \int_\Omega \left(u\partial_{xx}u + au^2\right)\mathrm{d}x = -\int_\Omega (\partial_x u)^2\,\mathrm{d}x + a\int_\Omega u^2\,\mathrm{d}x = \\ &= -\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} \le -\frac{1}{C_p^2}\|u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} = \\ &= -\left(\frac{1}{C_p^2} - a\right)\|u\|^2_{\mathrm{L}_2(\Omega)},\end{aligned}$$

where we used the Poincare inequality in the form $\|u\|_{\mathrm{L}_2(\Omega)} \le C_p\|\partial_x u\|_{\mathrm{L}_2(\Omega)}$ (we have zero trace). So we have :

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} \le -2\left(\frac{1}{C_p^2} - a\right)\|u\|^2_{\mathrm{L}_2(\Omega)},$$

so if

$$\frac{1}{C_p^2} - a > 0,$$

the norm satisfies the following differential inequality

$$\|u\|_{\mathrm{L}_2(\Omega)} \le \exp\left(-t\sqrt{\frac{1}{C_p^2} - a}\right)\|u_0\|_{\mathrm{L}_2(\Omega)},$$

and so

$$\|u(t,x)\|_{\mathrm{L}_2(\Omega)} \to 0, \text{ as } t \to \infty.$$

We are not happy yet. The Poincare constant is undetermined, so let us get an estimate for it. The equation has the form

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} &= -\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)} + a\|u\|^2_{\mathrm{L}_2(\Omega)} = -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}}\right) \le \\ &\le -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \max_{u\in \mathrm{W}_0^{1,2}(\Omega)}\frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|_{\mathrm{L}_2(\Omega)}}\right),\end{aligned}$$

let us define

$$\frac{1}{a_{\text{crit}}} = \max_{u\in \mathrm{W}_0^{1,2}(\Omega)}\frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}},$$

and so we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}_2(\Omega)} \le -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\left(\frac{1}{a} - \frac{1}{a_{\text{crit}}}\right) = -a\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}\frac{a_{\text{crit}} - a}{a a_{\text{crit}}}.$$

We see that if $a < a_{\text{crit}} \Leftrightarrow \frac{1}{a} > \frac{1}{a_{\text{crit}}}$ the $\mathrm{L}_2(\Omega)$ norm vanishes exponentially. But *how much is it?* That depends on a_{crit}, so let us define the functional

$$F : \mathrm{W}_0^{1,2}(\Omega) \to \mathbb{R}^+, F : u \mapsto \frac{\|u\|^2_{\mathrm{L}_2(\Omega)}}{\|\partial_x u\|^2_{\mathrm{L}_2(\Omega)}},$$

and find its extrema. The Gateux derivative at the extrema is

$$
\begin{aligned}
0 = &\delta F(u_{\text{ext}})[v] = \frac{\mathrm{d}}{\mathrm{d}t} F(u_{\text{ext}} + tv)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\int_\Omega (u_{\text{ext}} + tv)(u_{\text{ext}} + tv)\,\mathrm{d}x}{\int_\Omega (\partial_x u_{\text{ext}} + t\partial_x v)(\partial_x u_{\text{ext}} + t\partial_x v)\,\mathrm{d}x}\Big|_{t=0} = \\
&= 2\frac{\int_\Omega u_{\text{ext}} v\,\mathrm{d}x \int_\Omega (\partial_x u_{\text{ext}})^2\,\mathrm{d}x - \int_\Omega u_{\text{ext}}^2\,\mathrm{d}x \int_\Omega \partial_x u_{\text{ext}} \partial_x v\,\mathrm{d}x}{\left(\int_\Omega (\partial_x u_{\text{ext}})^2\,\mathrm{d}x\right)^2} = \\
&= \frac{1}{\left(\int_\Omega (\partial_x u_{\text{ext}})^2\,\mathrm{d}x\right)^2}\left(\int_\Omega u_{\text{ext}} v\,\mathrm{d}x - \frac{\int_\Omega u_{\text{ext}}^2\,\mathrm{d}x}{\int_\Omega (\partial_x u_{\text{ext}})^2\,\mathrm{d}x}\int_\Omega \partial_x u_{\text{ext}} \partial_x v\,\mathrm{d}x\right) = \\
&= \frac{1}{\int_\Omega (\partial_x u_{\text{ext}})^2\,\mathrm{d}x}\frac{1}{a_{\text{crit}}}\left(\int_\Omega (a_{\text{crit}} u_{\text{ext}} v - \partial_x u_{\text{ext}} \partial_x v)\,\mathrm{d}x\right),
\end{aligned}
$$

and so we see

$$
\delta F(u_{\text{ext}})[v] = 0 \Leftrightarrow \int_\Omega (a_{\text{crit}} u_{\text{ext}} v - \partial_x u_{\text{ext}} \partial_x v)\,\mathrm{d}x = 0,
$$

This is a weak formulation of the problem

$$
\int_\Omega (a_{\text{crit}} u_{\text{ext}} + \partial_{xx} u_{\text{ext}}) v\,\mathrm{d}x\,, \forall v \in \mathrm{W}_0^{1,2}(\Omega) \Leftrightarrow \begin{cases} \partial_{xx} u_{\text{ext}} = a_{\text{crit}} u_{\text{ext}}\,, & \text{in}\,\Omega \\ u_{\text{ext}} = 0, & \text{on}\,\partial\Omega \end{cases}.
$$

This is an *eigenproblem for the elliptic operator* (from the original parabolic operator). The solution is the following:

$$
\begin{aligned}
u_{\text{ext}}^n &= C \sin\left(\sqrt{a_{\text{crit}}^n}\, x\right), \\
a_{\text{crit}}^n &= n^2\pi^2, n \in \mathbb{N}
\end{aligned}
$$

The *smallest eigenvalue* is

$$
a_{\text{crit}} = \pi^2.
$$

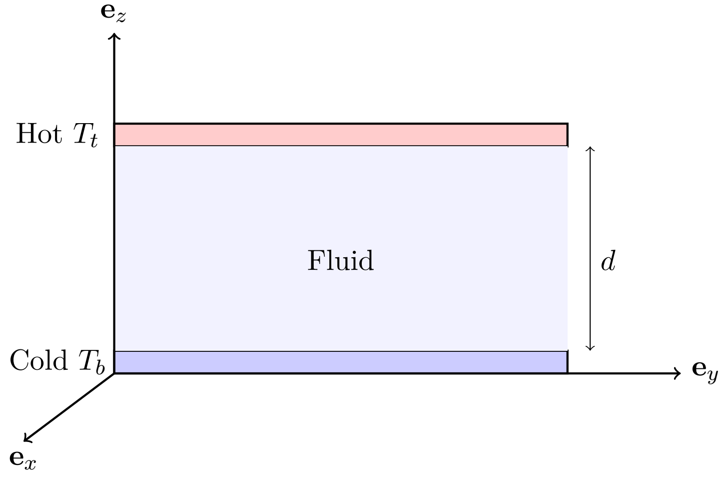
This means that

$$
\forall a < a_{\text{crit}} = \pi^2,
$$

the perturbations in the initial condition decay exponentially.

6.1.2 Rayleigh-Bénard convection

Let us use the developed theory on the problem of Rayleigh-Bénard convection.



There are two parallel infinite plates, the top with the temperature T_t and the bottom one with the temperature T_b in the gravitational field $\mathbf{g} = -g\mathbf{e}_z$. The space is filled with a compressible Navier Stokes fluid, so the governing equations are

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\nabla \cdot \mathbf{v}) &= 0, \\
\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \rho\mathbf{g} + \nabla \cdot (-p_{\mathrm{th}}(\rho, \theta)\mathbb{I} + \lambda(\nabla \cdot \mathbf{v})\mathbb{I} + 2\mu\mathbb{D}) \\
\rho c_V \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \nabla \cdot (\kappa \nabla \theta) + \left(p_{\mathrm{th}}(\theta, \rho) - \frac{\partial p_{\mathrm{th}}(\theta, \rho)}{\partial \theta}\right)(\nabla \cdot \mathbf{v}) + \mathbb{T} : \mathbb{D},
\end{aligned}
$$

Why is anything happening at all?

- gravitational field is crucial, as without it, buyoancy oscillations won't work
- dependence of density on temperature is also essential

Boussinesq-Oberbeck approximation

Doing a stability analysis of a system of nonlinear PDEs is difficult. We adopt the following assumptions, known as the *Oberbeck-Boussinesq approximation*

- the density depends only on the temperature, not on pressure, and only linearly: $\rho(\theta) = \rho_{\mathrm{ref}}(1 - \alpha(\theta - \theta_{\mathrm{ref}}))$
- working with compressible fluids is a nightmare, lets make it incompressible: $\nabla \cdot \mathbf{v} = 0$.
- the density in the momentum equation is constant in the first term and the same in the temperature equation
- ignore all the nonlinear terms in the thermal equation

Doing all this produces the following system of equations:

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0, && (6.1) \\
\rho_{\mathrm{ref}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \rho_{\mathrm{ref}}(1 - \alpha(\theta - \theta_{\mathrm{ref}}))\mathbf{g} + \nabla \cdot (-p_{\mathrm{th}}(\rho, \theta)\mathbb{I} + 2\mu\mathbb{D}) && (6.2) \\
\rho_{\mathrm{ref}} c_V \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \nabla \cdot (\kappa \nabla \theta), && (6.3)
\end{aligned}
$$

with the boundary conditions

$$
\begin{cases}
\theta &= T_t, \text{ on } \{z = d\} \\
\theta &= T_b, \text{ on } \{z = 0\}
\end{cases}
$$

Note that, physically, this *approximation makes no sense.* Since $\mathbb{T} : \mathbb{D} = 0$, there is no viscous dissipation, but also $\mathbb{T} \neq \mathbb{O}$, so we are just losing energy but the temperature does not increase. Nevertheless, this approximation is popular and often used.

Steady state, pure conduction

With the system of our interest 6.1 defined, let us investigate the stability of the steady state. What does it look like?

Clearly, the simplest case would be

$$\hat{\mathbf{v}} = \mathbf{0},$$

and the remaning qualities need to solve the stationary versions of the present equations; the equation for the temperature reads

$$0 = \nabla \cdot \left(\kappa \nabla \hat{\theta}\right),$$

with the solution

$$\hat{\theta} = \hat{\theta}(z) = -\frac{T_b - T_t}{d} z + T_{bot} = -\beta z + T_{bot}, \tag{6.4}$$

where we have denoted

$$\beta = \frac{T_b - T_t}{d},$$

as the temperature gradient. The pressure has to solve the rest of the momentum equation, that is

$$0 = -\nabla \hat{p} - \rho_{\text{ref}} (1 - \alpha(\theta(z) - \theta_{\text{ref}})) g \mathbf{e}_z,$$

which has the solution

$$\hat{p}(z) = -\rho_{\text{ref}} g \int_0^z (1 - \alpha(\theta(s) - \theta_{\text{ref}}))\, \mathrm{d}s\,.$$

The triple $\left(\hat{\mathbf{v}}, \hat{\theta}, \hat{p}\right)$ will be our steady solution from now on.

Perturbation of the steady state, convection

What happens if now perturb the steady state? We are solving the following system

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, \\ \rho_{\text{ref}} (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) = -\rho_{\text{ref}} (1 - \alpha(\theta - \theta_{\text{ref}})) g \mathbf{e}_z - \nabla p + \mu \bigtriangleup \mathbf{v} \\ \rho_{\text{ref}} c_V (\partial_t \theta + (\mathbf{v} \cdot \nabla) \theta) = \kappa \bigtriangleup \theta \end{cases},$$

with the initial conditions given as

$$\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}} = \tilde{\mathbf{v}}, \theta = \hat{\theta} + \tilde{\theta}, p = \hat{p} + \tilde{p}$$

all at $t = 0$. The the hatted variables are the steady state, the tildas are perturbations. Incompressibility yields

$$\nabla \cdot (\hat{\mathbf{v}} + \tilde{\mathbf{v}}) = 0 = \nabla \cdot \tilde{\mathbf{v}},$$

because $\hat{\mathbf{v}} = \mathbf{0}$. The momentum equation is

$$\rho_{\text{ref}} (\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}) = -\rho_{\text{ref}} \left(1 - \alpha\left(\hat{\theta} + \tilde{\theta} - \theta_{\text{ref}}\right)\right) g \mathbf{e}_z - \nabla(\hat{p} + \tilde{p}) + \mu \bigtriangleup \tilde{\mathbf{v}},$$

so upon plugging in $\nabla \hat{p} = -\rho_{\text{ref}} \left(1 - \alpha\left(\hat{\theta} - \theta_{\text{ref}}\right)\right) g \mathbf{e}_z$, the equation becomes

$$\rho_{\text{ref}} (\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}) = \rho_{\text{ref}} \alpha g \mathbf{e}_z \tilde{\theta} - \nabla \tilde{p} + \mu \bigtriangleup \tilde{\mathbf{v}},$$

and finally, the heat equation is

$$\rho_{\text{ref}} c_V \left(\partial_t \hat{\theta} + \partial_t \tilde{\theta} + (\tilde{\mathbf{v}} \cdot \nabla)\left(\hat{\theta} + \tilde{\theta}\right)\right) = \kappa \bigtriangleup \left(\hat{\theta} + \tilde{\theta}\right),$$

so upon using the fact $\partial_t \hat{\theta} = 0, \bigtriangleup \hat{\theta} = 0$, we obtain

$$\rho_{\text{ref}} c_V \left(\partial_t \hat{\theta} + (\tilde{\mathbf{v}} \cdot \nabla)\left(\hat{\theta} + \tilde{\theta}\right)\right) = \kappa \bigtriangleup \tilde{\theta}.$$

Altogether, the perturbation $\left(\tilde{\mathbf{v}}, \tilde{\theta}, \tilde{p}\right)$ solves the following system

$$\nabla \cdot \tilde{\mathbf{v}} = 0, \tag{6.5}$$

$$\rho_{\text{ref}} (\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}) = +\rho_{\text{ref}} \alpha g \mathbf{e}_z \tilde{\theta} - \nabla \tilde{p} + \mu \bigtriangleup \tilde{\mathbf{v}} \tag{6.6}$$

$$\rho_{\text{ref}} c_V \left(\partial_t \tilde{\theta} + (\tilde{\mathbf{v}} \cdot \nabla)\left(\hat{\theta} + \tilde{\theta}\right)\right) = \kappa \bigtriangleup \tilde{\theta}. \tag{6.7}$$

Non-dimensonalisation

We seek some quality that would characterize the stability of the flow. It would be most convenient for this quantity to be dimensionless, accounting for all the geometry, choice of units and other things. For that, let us choose

- a characteristic length: $l_{\text{char}} := d$
- a characteristic density $\rho_{\text{char}} := \rho_{\text{ref}}$
- a characteristic temperature $\theta_{\text{char}} = T_b - T_t$
- a characteristic time $t_{\text{char}} = ?$

There is however a problem: *how to choose the characteristic time?* We have no characteristic velocity [1], because

$$\hat{\mathbf{v}} = \mathbf{0}.$$

There are some candidates whose units include seconds:

$$[\mu] = \text{Pa} \cdot \text{s}, [g] = {}^{\text{m}}/_{\text{s}^2}, [\kappa] = {}^{\text{W}}/_{\text{m}\cdot\text{K}},$$

so we can in theory choose one and using that define t_{char}. Let us continue: with the characteristic time being chosen, we can also set the characteristic velocity as

$$v_{\text{char}} = \frac{l_{\text{char}}}{t_{\text{char}}}.$$

so we can rewrite the qualities [2]

$$\tilde{\mathbf{v}} = v_{\text{char}}\mathbf{v}^*, t = t_{\text{char}}t^*, \mathbf{x} = l_{\text{char}}\mathbf{x}^*, \tilde{\theta} = \theta_{\text{char}}\theta^*, \tilde{p} = \frac{\rho_{\text{ref}}l_{\text{char}}^2}{t_{\text{char}}^2}p^*$$

Plugging this into the equations for the perturbation 6.5 and denoting (for the time being) the dimensionless variables with stars, we obtian

$$\begin{aligned}
\nabla^* \cdot \mathbf{v}^* &= 0 \\
\frac{\rho_{\text{ref}}d^2}{\mu t_{\text{char}}}(\partial_{t^*}\mathbf{v}^* + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^*) &= -\nabla^* p^* + \triangle^* \mathbf{v}^* + \frac{\alpha g \theta_{\text{char}} d \rho_{\text{ref}} t_{\text{char}}}{\mu}\theta^* \mathbf{e}_z \\
\partial_{t^*}\theta^* + (\mathbf{v}^* \cdot \nabla^*)\theta^* &= \nabla \cdot \left(\frac{\kappa t_{\text{char}}}{\rho_{\text{ref}} c_V d^2}\nabla^* \theta^*\right) - v_z^*,
\end{aligned}$$

where we have denoted

$$v_z^* = \rho_{\text{ref}} c_V \tilde{\mathbf{v}} \cdot \nabla \hat{\theta} = \rho_{\text{ref}} c_V \tilde{\mathbf{v}} \cdot \nabla\left(\frac{-\theta_{\text{char}}}{d}z + T_b\right) = -\rho_{\text{ref}} c_V \frac{\theta_{\text{char}}}{d}\mathbf{e}_z \cdot \tilde{\mathbf{v}}$$

And now we see how the choice of t_{char} influences the equations. I can require one of the following

$$\begin{aligned}
\frac{\rho_{\text{ref}}d^2}{\mu t_{\text{char}}} &= 1, \\
\frac{\alpha g \theta_{\text{char}} d \rho_{\text{ref}} t_{\text{char}}}{\mu} &= 1, \\
\frac{\kappa t_{\text{char}}}{\rho_{\text{ref}} c_V d^2} &= 1.
\end{aligned}$$

[1] If we would, that would suffice, as we have characteristic length.

[2]
$$\text{Pa} = {}^{\text{N}}/_{\text{m}} = {}^{\text{kg}\cdot\text{m}}/_{\text{s}^2\text{m}^2} = {}^{\text{kg}/\text{m}^3\text{m}^3}/_{\text{s}^2\text{m}} = {}^{\text{kg}/\text{m}^3\text{m}^2}/_{\text{s}^2}$$

Each of these choices are sensible. In our case, we are interested in the thermal conduction mainly, so let us choose

$$t_{\text{char}} = \frac{\rho_{\text{ref}} c_V d^2}{\kappa}.$$

Finally, we arrive to the following system of equations (we omit the stars and tildas)

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= 0, \\
\frac{1}{\mathrm{Pr}}\left(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) &= -\nabla p + \triangle\, \mathbf{v} + \mathrm{Ra}\,\theta \mathbf{e}_z, \\
\partial_t \theta + (\mathbf{v} \cdot \nabla)\theta &= \triangle\, \theta + v_z,
\end{aligned}$$

where

$$\mathrm{Pr} = \frac{\nu}{k} = \frac{\rho_{\text{ref}} d^2}{\mu t_{\text{char}}}, \tag{6.8}$$

is the Prandtl number and

$$\mathrm{Ra} = \frac{\alpha g \theta_{\text{char}} d^3}{\nu k}, \nu = \frac{\mu}{\rho_{\text{ref}}}, k = \frac{\kappa}{\rho_{\text{ref}} c_V}. \tag{6.9}$$

is the Rayleigh number.

Another form of the equations can be derived when rescaling the temperateure (choosing a different characteristic temperature)[3]

$$\theta = \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ra}}}\theta^*, \tag{6.10}$$

and this leads (of course, other quantities will have to be rescaled as well)

$$\begin{aligned}
\nabla^* \cdot \mathbf{v}^* &= 0 \\
\partial_{t^*} \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* &= -\nabla^* p^* + \triangle^*\, \mathbf{v}^* + \sqrt{\mathrm{Ra}}\,\theta^* \mathbf{e}_z \\
\mathrm{Pr}\left(\partial_{t^*} \theta^* + (\mathbf{v}^* \cdot \nabla^*)\theta^*\right) &= \triangle^*\, \theta^* - \sqrt{\mathrm{Ra}}\, v_z^*.
\end{aligned}$$

This scaling is popular in mathematical literature and *we will stick to it.* It is also common to denote

$$\mathrm{R} := \sqrt{\mathrm{Ra}}.$$

So finally finally, we are solving the rescaled system 6.5 in the nondimensionalised version (all the functions are the tilded functions, but we do not write it anymore)

$$\nabla \cdot \mathbf{v} = 0, \tag{6.11}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \triangle\, \mathbf{v} + \mathrm{R}\,\theta \mathbf{e}_z \tag{6.12}$$

$$\mathrm{Pr}\left(\partial_t \theta + (\mathbf{v} \cdot \nabla)\theta\right) = \triangle\, \theta - \mathrm{R}\, v^z, \tag{6.13}$$

with the boundary conditions

To add another issue, realise that we are working on unbounded domains (the plates are infinite), so integrals over the domain are problematic. This can be solved by assuming periodic boundary conditions on lateral faces. The original boundary conditions read as

[3] Of course θ^* is totally different than the previous one

$$\mathbf{v} = \mathbf{v}, \text{ on } \{z = 0, z = d\},$$
$$\theta = T_t, \text{ on } \{z = d\},$$
$$\theta = T_b, \text{ on } \{z = 0\},$$

and since the steady state satisfies them, the perturbation must be compatible, and so the boundary conditions for the *perturbation read as*

$$\mathbf{v} = \mathbf{0}, \text{ on } \{z = 0, z = d\}, \tag{6.14}$$
$$\theta = 0, \text{ on } \{z = d\}, \tag{6.15}$$
$$\theta = 0, \text{ on } \{z = 0\}. \tag{6.16}$$

Stability analysis

Let us take now the momentum equation, multiply $\cdot\mathbf{v}$ and integrate $\int_\Omega \mathrm{d}x$.This yields:

$$\int_\Omega (\text{ equation })\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x = -\int_\Omega \nabla p\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega (\triangle\,\mathbf{v})\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega \mathrm{R}\,\theta\mathbf{e}_z\cdot\mathbf{v}\,\mathrm{d}x\,,$$

realize that

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{v}\cdot\nabla\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)\mathrm{d}x = \int_\Omega (\nabla\cdot\mathbf{v})\frac{\mathbf{v}\cdot\mathbf{v}}{2}\,\mathrm{d}x = 0,$$

and

$$\int_\Omega \nabla p\cdot\mathbf{v}\,\mathrm{d}x = -\int_\Omega p(\nabla\cdot\mathbf{v})\,\mathrm{d}x = 0,$$

and also

$$\int_\Omega \triangle\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x = -\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x\,,$$

where we have used the periodicity of the boundary conditions (and zero trace of the perturbation) and incompressibility. This means

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} = -\|\nabla\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathrm{R}\,\theta\mathbf{e}_z\cdot\mathbf{v}\,\mathrm{d}x\,,$$

which is exactly the similiar expression to the one derived at the beginning of our studies of the stability analysis. [4]. It is evident that when

$$\mathrm{R} = 0 = \mathrm{Re}\,,$$

the norm decays exponentially. Is there a chance this happens *even for nonzero Rayleigh number?*.

Let us repeat the previous manipulation; only now we wish to capture the evolution of the temperature perturbation aswell, so investigate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x + \mathrm{Pr}\int_\Omega \theta^2\,\mathrm{d}x\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \mathrm{Pr}\|\theta\|^2_{\mathrm{L}_2(\Omega)}\right).$$

Above we have shown

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} = -\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x + \int_\Omega \mathrm{R}\theta\mathbf{e}_z\cdot\mathbf{v}\,\mathrm{d}x\,,$$

[4] We could again use Poincare to obtain the estimate for $\|\nabla\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} \leq \frac{1}{C_p}\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)}$ and stuff.

and the norm of the temperature perturbation is in fact

$$
\begin{aligned}
\mathrm{Pr}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2_{\mathrm{L}_2(\Omega)} &= \mathrm{Pr}\int_\Omega \theta\partial_t\theta\,\mathrm{d}x = \int_\Omega \theta\bigtriangleup\theta - \theta\,\mathrm{R}\,v^z - \theta\mathrm{Pr}(\mathbf{v}\cdot\nabla)\theta\,\mathrm{d}x = \\
&= \int_{\partial\Omega}\theta\nabla\theta\cdot\mathbf{n}\,\mathrm{d}S - \int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - \int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x - 2\mathrm{Pr}\int_\Omega \theta\mathbf{v}\cdot\nabla\theta\,\mathrm{d}x = \\
&= -\int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - \int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x - 2\mathrm{Pr}\left(\int_{\partial\Omega}\theta^2\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S - \int_\Omega \theta(\nabla\cdot(\theta\mathbf{v}))\,\mathrm{d}x\right) = \\
&= -\int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - \int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x + 2\mathrm{Pr}\int_\Omega \theta(\nabla\theta\cdot\mathbf{v} + \theta(\nabla\cdot\mathbf{v}))\,\mathrm{d}x = \\
&= -\int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - \int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x - 2\int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x = -\int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - 3\int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x
\end{aligned}
$$

because of the boundary conditions and the form the stationary solution 6.11, 6.14, 6.10 [5]. All in all

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \mathrm{Pr}\|\theta\|^2_{\mathrm{L}_2(\Omega)}\right) = -\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{v}\,\mathrm{d}x - \int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x - 2\int_\Omega \mathrm{R}\,\theta v^z\,\mathrm{d}x\,.
$$

Introduce yet a different notation:

$$
\mathcal{D}(\mathbf{v},\theta) := \int_\Omega \nabla\mathbf{v}:\nabla\mathbf{v}\,\mathrm{d}x + \int_\Omega \nabla\theta\cdot\nabla\theta\,\mathrm{d}x\,, \mathcal{J}(\mathbf{v},\theta) = -2\int_\Omega \theta v^z\,\mathrm{d}x\,,
$$

so we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \mathrm{Pr}\|\theta\|^2_{\mathrm{L}_2(\Omega)}\right) = -\mathcal{D} + \mathrm{R}\,\mathcal{J} = -\mathcal{D}\,\mathrm{R}\left(\frac{1}{\mathrm{R}} - \frac{\mathcal{J}}{\mathcal{D}}\right).
$$

Denote

$$
\frac{1}{\mathrm{R}_{\mathrm{crit}}} := \max_{\theta\in \mathrm{W}^{1,2}_0(\Omega),\mathbf{v}\in \mathrm{W}^{1,2}_0(\Omega)_{\mathrm{div}}} \frac{\mathcal{J}}{\mathcal{D}},
$$

and so

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{v}\|^2_{\mathrm{L}_2(\Omega)} + \mathrm{Pr}\|\theta\|^2_{\mathrm{L}_2(\Omega)}\right) = -\mathcal{D}\,\mathrm{R}\left(\frac{1}{\mathrm{R}} - \frac{\mathcal{J}}{\mathcal{D}}\right) \le -\mathcal{D}\,\mathrm{R}\left(\frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R}_{\mathrm{crit}}}\right).
$$

In the case

$$
\frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R}_{\mathrm{crit}}} = \frac{\mathrm{R}_{\mathrm{crit}} - \mathrm{R}}{\mathrm{R}_{\mathrm{crit}}\,\mathrm{R}} > 0.
$$

we see the perturbation decays exponentially. To calculate the critical value of the Rayleigh number, we need to minimize the functional

$$
(\mathbf{v},\theta) \mapsto \frac{\mathcal{J}(\mathbf{v},\theta)}{\mathcal{D}(\mathbf{v},\theta)}
$$

over the space

$$
X = \left\{(\boldsymbol{\varphi},\zeta)|\boldsymbol{\varphi}\in \mathrm{W}^{1,2}_0\left(\Omega;\mathbb{R}^d\right), \nabla\cdot\boldsymbol{\varphi} = 0, \zeta\in \mathrm{W}^{1,2}_0(\Omega)\right\}.
$$

That can be done for example by the method of Lagrange multipliers - we will minimize the functional

$$
\mathcal{F}(\mathbf{v},\theta) := \frac{\mathcal{J}(\mathbf{v},\theta)}{\mathcal{D}(\mathbf{v},\theta)} - \int_\Omega \lambda(\mathbf{x})(\nabla\cdot\mathbf{v})\,\mathrm{d}x\,,
$$

over $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)\times \mathrm{W}^{1,2}_0(\Omega)$. Evaluate the Gateaux derivative

$$
\delta\mathcal{F}(\mathbf{v},\theta)[\mathbf{u},\varphi] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{J}(\mathbf{v}+t\mathbf{u},\theta+t\varphi)}{\mathcal{D}(\mathbf{v}+t\mathbf{u},\theta+t\varphi)} - \int_\Omega \lambda\nabla\cdot(\mathbf{v}+t\mathbf{u})\,\mathrm{d}x\right)\Bigg|_{t=0}.
$$

[5] $\nabla\theta = -\frac{\mathrm{R}}{\mathrm{Pr}}\mathbf{e}_z$

The derivative of the numerator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(\mathbf{v}+t\mathbf{u},\theta+t\varphi)\Big|_{t=0} = -2\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(\theta+t\varphi)(v^z+tu^z)\Big|_{t=0}\,\mathrm{d}x = -2\int_\Omega \varphi v^z + \theta u^z\,\mathrm{d}x\,,$$

and the denominator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(\mathbf{v}+t\mathbf{u},\theta+t\varphi)\Big|_{t=0} = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(\nabla(\mathbf{v}+t\mathbf{u}):\nabla(\mathbf{v}+t\mathbf{u})+\nabla(\theta+t\varphi)\cdot\nabla(\theta+t\varphi))\Big|_{t=0}\,\mathrm{d}x = 2\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{u}+\nabla\varphi\cdot\nabla\theta\,\mathrm{d}x\,,$$

so using the Leibniz rule we obtain

$$\delta\mathcal{F}(\mathbf{u},\theta)[\mathbf{u},\varphi] = \frac{-2\left(\int_\Omega \varphi v^z+\theta u^z\,\mathrm{d}x\right)\mathcal{D}(\mathbf{v},\theta) - 2\mathcal{J}(\mathbf{v},\theta)\left(\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{u}+\nabla\varphi\cdot\nabla\theta\,\mathrm{d}x\right)}{\mathcal{D}(\mathbf{v},\theta)^2} - \int_\Omega \lambda(\mathbf{x})(\nabla\cdot\mathbf{u})\,\mathrm{d}x\,,$$

and this is zero provided (realize $\mathcal{D}(\mathbf{v},\theta)>0$.)

$$\left(\int_\Omega \varphi v^z+\theta u^z\,\mathrm{d}x\right)\mathcal{D}(\mathbf{v},\theta)+\mathcal{J}(\mathbf{v},\theta)\left(\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{u}+\nabla\varphi\cdot\nabla\theta\,\mathrm{d}x\right)+\frac{\mathcal{D}(\mathbf{v},\theta)^2}{2}\int_\Omega \lambda(\nabla\cdot\mathbf{u})\,\mathrm{d}x = 0 \Leftrightarrow,$$
$$\int_\Omega \varphi v^z+\theta u^z\,\mathrm{d}x + \frac{\mathcal{J}(\mathbf{v},\theta)}{\mathcal{D}(\mathbf{v},\theta)}\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{u}+\nabla\varphi\cdot\nabla\theta\,\mathrm{d}x + \frac{\mathcal{D}(\mathbf{v},\theta)}{2}\int_\Omega \lambda(\nabla\cdot\mathbf{u})\,\mathrm{d}x = 0,$$

if we recover that in the critical case the fraction containing $\mathcal{J},\mathcal{D}$ is in fact related to the critical Rayleigh number, we can write

$$0 = \int_\Omega \varphi v^z+\theta u^z\,\mathrm{d}x + \frac{1}{\mathrm{R}_{\mathrm{crit}}}\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{u}+\nabla\varphi\cdot\nabla\theta\,\mathrm{d}x + \int_\Omega \tilde{\lambda}(\nabla\cdot\mathbf{u})\,\mathrm{d}x\,,$$

where we have just rescaled the Lagrange multiplier $\tilde{\lambda} = (\mathcal{D}(\mathbf{v},\theta)/2)\lambda$. This seems familiar - let us move all the derivatives from $(\mathbf{u},\varphi)$:

$$0 = \int_\Omega v^z\varphi+\theta u^z\,\mathrm{d}x - \frac{1}{\mathrm{R}_{\mathrm{crit}}}\int_\Omega \triangle\,\mathbf{v}\cdot\mathbf{u}+\triangle\,\theta\varphi\,\mathrm{d}x + \int_\Omega \nabla\tilde{\lambda}\cdot\mathbf{u}\,\mathrm{d}x\,,\forall\varphi\in \mathrm{W}^{1,2}_0(\Omega)\,,\forall\mathbf{u}\in\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)\,.$$

This is exactly the weak formulation of the problem

$$v^z+\theta\mathbf{e}_z - \frac{1}{\mathrm{R}_{\mathrm{crit}}}\triangle\,\mathbf{v} - \frac{1}{\mathrm{R}_{\mathrm{crit}}}\triangle\,\theta + \nabla\tilde{\lambda} = 0,\ \text{in}\,\Omega,$$
$$\mathbf{v} = \mathbf{0},\ \text{on}\,\{z=d, z=0\},$$
$$\theta = 0,\ \text{on}\,\{z=d, z=0\},$$

which can be separated into

$$\triangle\,\theta = \mathrm{R}_{\mathrm{crit}} v^z, \tag{6.17}$$
$$\triangle\,\mathbf{v} = \mathrm{R}_{\mathrm{crit}}\theta\mathbf{e}_z + \nabla\lambda. \tag{6.18}$$

Which is remarkable; if we realize the Lagrange multiplier only should enforce $\nabla\cdot\mathbf{v}=0$, we can write

$$\triangle\,\theta = \mathrm{R}_{\mathrm{crit}} v^z, \tag{6.19}$$
$$\triangle\,\mathbf{v} = \mathrm{R}_{\mathrm{crit}}\theta\mathbf{e}_z + \nabla\lambda, \tag{6.20}$$
$$\nabla\cdot\mathbf{v} = 0. \tag{6.21}$$

This is once again a *(generalized) eigenproblem for the "linearized"*[6]*. version of the elliptic operator.*

It can be rewritten in this "suggestive notation":

$$\begin{bmatrix} \triangle & -\nabla & 0 \\ \nabla\cdot & 0 & 0 \\ 0 & 0 & \triangle \end{bmatrix}\begin{bmatrix} \mathbf{v}^* \\ \lambda \\ \theta^* \end{bmatrix} = \mathrm{R}_{\text{crit}}\begin{bmatrix} 0 & 0 & \mathbf{e}_z \\ 0 & 0 & 0 \\ \mathbf{e}_z\cdot & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v}^* \\ \lambda \\ \theta^* \end{bmatrix}$$

which is a generalized eigenvalue problem

$$\mathbb{A}\mathbf{x} = \mu\mathbb{B}\mathbf{x},$$

and so we see that we are interested in the (generalized) spectrum of some operators.

Solution to the (generalized) eigenproblem

Our examinition boils down to finding the *smallest possible eigenvalue* of a certain linear operator. We utilize some tricks along the way. First, take the divergence of the velocity equation from 6.19 and write

$$\nabla\cdot\triangle\,\mathbf{v} - \mathrm{R}_{\text{crit}}\partial_z\theta - \triangle\,\lambda = 0,$$

realize $\nabla\cdot\triangle\,\mathbf{v} = \triangle\,(\nabla\cdot\mathbf{v}) = 0$, so

$$\triangle\,\lambda = -\mathrm{R}_{\text{crit}}\partial_z\theta.$$

If we take laplacian of 6.19 instead, we obtain

$$\triangle\,\triangle\,\mathbf{v} - \mathrm{R}_{\text{crit}}\triangle\,(\theta\mathbf{e}_z) - \triangle\,\nabla\lambda = 0,$$

realize now $\triangle\,\nabla\lambda = \nabla\,\triangle\,\lambda = -\mathrm{R}_{\text{crit}}\nabla\partial_z\theta$, so we can eliminate λ whatsoever and arrive at the system

$$\begin{aligned} \triangle\,\triangle\,\mathbf{v} &= \mathrm{R}_{\text{crit}}(\triangle\,(\theta\mathbf{e}_z) - \nabla\partial_z\theta), \\ \triangle\,\theta &= \mathrm{R}_{\text{crit}}v^z, \\ \nabla\cdot\mathbf{v} &= 0. \end{aligned}$$

Now let us take the z component of the first equation, so we are solving

$$\begin{aligned} \triangle\,\triangle\,v^z &= \mathrm{R}_{\text{crit}}(\triangle\,\theta - \partial_{zz}\theta), \\ \triangle\,\theta &= \mathrm{R}_{\text{crit}}v^z, \\ \nabla\cdot\mathbf{v} &= 0. \end{aligned}$$

Next, we make the following ansatz [7]:

$$\mathbf{v} = \hat{\mathbf{v}}(z)\exp\left(i(k_x x + k_y y)\right), \theta = \hat{\theta}(z)\exp\left(i(k_x x + k_y y)\right),$$

and so the laplacian transformes as

$$\triangle\,\mathbf{v} = \triangle\,(\hat{\mathbf{v}}(z)\exp\left(i(k_x x + k_y y)\right)) = -\hat{\mathbf{v}}(z)\exp\left(i(k_x x + k_y y)\right)(k_x^2 + k_y^2) + \frac{\mathrm{d}^2\hat{\mathbf{v}}}{\mathrm{d}z^2}\exp\left(i(k_x x + k_y y)\right),$$

[6] The only linearization present is getting rid of the advective term, but that is kind of a "fake nonlinearity", arising from our Eulerian description

[7] This is technically the same as doing the Fourier transform of the system

so formally

$$\triangle \to \frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2,$$

where $k^2 = k_x^2 + k_y^2$. The first equation now becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2 \hat{v}^z = \mathrm{R}_{\mathrm{crit}}\left(\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\hat{\theta} - \frac{\mathrm{d}^2\hat{\theta}}{\mathrm{d}z^2}\right) = -\mathrm{R}_{\mathrm{crit}} k^2 \hat{\theta}, \tag{6.22}$$

and the second one

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\hat{\theta} = \mathrm{R}_{\mathrm{crit}}\hat{v}^z, \tag{6.23}$$

and we do not care about the last one for now. Apply now transformed laplacian to the first one 6.22 and write

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 \hat{v}^z = -\mathrm{R}_{\mathrm{crit}} k^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\hat{\theta},$$

plug this into the second equation 6.23 to obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 \hat{v}^z = -\mathrm{R}_{\mathrm{crit}}^2 k^2 \hat{v}^z,$$

and since $\sqrt{\mathrm{Ra}} = \mathrm{R}$, we have arrived to the equation[8]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 \hat{v}^z = -\mathrm{Ra}_{\mathrm{crit}} k^2 \hat{v}^z, z \in [0,1]. \tag{6.24}$$

This is fact an *eigenvalue problem for the (linear unbounded) sixth order differential operator*. To solve it, we *require boundary conditions*. The simplest assumption would be the so called *free-free* boundary conditions, see 6.1.2 for other choices.

$$\hat{v}^z = 0, \tag{6.25}$$

$$\frac{\mathrm{d}^2\hat{v}^z}{\mathrm{d}z^2} = 0, \tag{6.26}$$

$$\frac{\mathrm{d}^4\hat{v}^z}{\mathrm{d}z^4} = 0, \tag{6.27}$$

all on $\{z = 0, 1\}$. With these boundary conditions, one can show [9]

$$\hat{v}^z = \sum_{n=1}^{\infty} \hat{v}_n^z \sin(n\pi z),$$

for some coefficients $\hat{v}_n^z$. This representation really satisfies the above 6.25 boundary conditions. Using this representaion, 6.24 becomes

$$\sum_{n=1}^{\infty} \left(-n^2\pi^2 - k^2\right)^3 \hat{v}_n^z \sin(n\pi z) = -\mathrm{Ra}_{\mathrm{crit}} k^2 \sum_{n=1}^{\infty} \hat{v}_n^z \sin(n\pi z),$$

[8] The domain is transformed by the nondimensionalisation, $z = z^* d$.

[9] This would in fact be the Fourier transformation of the original problem. Or, we can say we are looking for the solution as some Fourier series.

and since $\{\sin(n\pi z)\}$ formes a complete ON system, this implies

$$\mathrm{Ra}^n_{\mathrm{crit}} = \frac{\left(\pi^2 n^2 + k^2\right)^3}{k^2}, n \in \mathbb{N},$$

and as we are looking for the smallest one, our value is:

$$\mathrm{Ra}_{\mathrm{crit}} = \frac{\left(\pi^2 + k^2\right)^3}{k^2}.$$

But this still depends on $k_n = \frac{2\pi}{L} n$ the choice of L, *i.e.*, the choice of the periodicity of the boundary. So in fact we want to minimize this

$$\frac{\partial \mathrm{Ra}_{\mathrm{crit}}}{\partial k} = \frac{\left(k^2 + \pi^2\right)^2 \left(3k^2 - \left(k^2 + \pi^2\right)\right)}{k^2} = 0 \Rightarrow k_{\mathrm{crit}} = \frac{\pi^2}{2}.$$

Finally, plugging this in yields

$$\mathrm{Ra}_{\mathrm{crit}} = \frac{27}{4}\pi^4 \approx 657,51 \tag{6.28}$$

Our problem is finally solved. In the case

$$\mathrm{Ra} < \frac{27}{4}\pi^4,$$

every perturbation of the steady state *decays* exponentially - the steady state is stable. In the case

$$\mathrm{Ra} > \frac{27}{4}\pi^4,$$

every perturbation of the steady state *grows* exponentially - the steady state is unstable.

The case

$$\mathrm{Ra} = \frac{27}{4}\pi^4,$$

is the *bifurcation point*; the only thing that follows from our examination is that the L_2 norm of the perturbation does not change. In order to provide a more detailed description, we would require some advanced techniques from nonlinear bifurcation analysis...

Free-free boundary conditions

Let us return a bit to the boundary conditions. We have discussed 6.25 the free-free boundary conditions. Let us show these BC for the velocity correspond to the following BC for the velocity and stress

$$\begin{aligned} v^z &= 0 \,\text{on}\, \{z = 0, 1\}, \\ \mathbb{T}\mathbf{n} &= -p_{\mathrm{ambient}}\,\mathbf{n},\, \text{on}\, \{z = 0, 1\}, \end{aligned}$$

with p_{ambient} being the "ambient pressure". [10] The first condition is simply a no-penetration condition. Notice those are boundary conditions for the total quantites, so assuming the form

$$\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}}, p = \hat{p} + \tilde{p},$$

[10] I am not sure what that means, but we will se soon that it must be the steady pressure.

with the hatted variables being the steady solution and the variables with tildas are perturbations. Denote also

$$\begin{aligned}\hat{\mathbb{T}} &= -\hat{p} + 2\mu\hat{\mathbb{D}},\\ \tilde{\mathbb{T}} &= -\tilde{p} + 2\mu\tilde{\mathbb{D}},\\ \mathbb{T} &= \hat{\mathbb{T}} + \tilde{\mathbb{T}}.\end{aligned}$$

Since $(\hat{\mathbf{v}}, \hat{p}) = (\mathbf{0}, \hat{p})$ solves the equations, it must hold on $\{z = 0, z = d\}$

$$\hat{\mathbb{T}}\mathbf{n} = \left(-\hat{p} + 2\mu\hat{\mathbb{D}}\right)\mathbf{n} = -\hat{p}\mathbf{n} = -p_{\text{ambient}}\,\mathbf{n},$$

and so

$$p_{\text{ambient}} = \hat{p}, \text{ on } \{z = 0, z = d\}$$

If now the perturbed function ought to solve the problem with the boundary conditions, it must hold

$$\mathbb{T}\mathbf{n} = \left(-(\hat{p} + \tilde{p}) + 2\mu\tilde{\mathbb{D}}\right)\mathbf{n} = -\hat{p}\mathbf{n}, \text{ on } \{z = 0, z = d\},$$

which means

$$\left(-\tilde{p} + 2\mu\tilde{\mathbb{D}}\right)\mathbf{n} = \tilde{\mathbb{T}}\mathbf{n} = \mathbf{0}, \text{ on } \{z = 0, z = d\}.$$

This translates to[11]

$$\pm[T_{zx}, T_{yz}, T_{zz}]^\top = \mathbf{0},$$

which implies

$$\frac{\partial \tilde{v}^x}{\partial z} + \frac{\partial \tilde{v}^z}{\partial x} = 0, \frac{\partial \tilde{v}^y}{\partial z} + \frac{\partial \tilde{v}^z}{\partial y} = 0,$$

again on $\{z = 0, 1\}$. The last component yields trivial information, since $\tilde{v}^z = 0$ there. But if it is a sensibly regular solution, also its derivative (assuming continuity and whatever) is zero there, and so those conditions really mean

$$\frac{\partial \tilde{v}^x}{\partial z} = 0, \frac{\partial \tilde{v}^y}{\partial z} = 0.$$

Recall now that

$$\nabla \cdot \mathbf{v} = \frac{\partial \tilde{v}^x}{\partial x} + \frac{\partial \tilde{v}^y}{\partial y} + \frac{\partial \tilde{v}^z}{\partial z} = 0,$$

inside of Ω. Let us however suppose that it holds also *on the boundary*[12]. Differentiate w.r.t z, swap the derivatives and obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \tilde{v}^x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v^y}{\partial z}\right) + \frac{\partial^2 v^z}{\partial z^2} = 0,$$

on $\{z = 0, 1\}$. Since the first two terms are zero, we read

$$\frac{\partial^2 \tilde{v}^z}{\partial z^2} = 0 \,\text{on}\, \{z = 0, z = 1\}.$$

Finally, let us deal with the BC for the forth derivative. For that, recall that we have not yet discussed the boundary conditions for the temperature, which are:

$$\tilde{\theta} = 0 \,\text{on}\, \{z = 0, z = 1\}.$$

[11] on $\{z = 0, 1\}$ the outer unit normal $\mathbf{n}$ equals to $\mathbf{e}_z$.

[12] We are on $\{z = 0, 1\}$.

Take a look at 6.22 now: if we suppose *that the equation holds also on the boundary*, we can restrict to $\{z = 0, z = 1\}$, where the temperature perturbation vanishes. We thus see

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2 v^{*z} = 0 \,\mathrm{on}\, \{z = 0, z = 1\},$$

and since $\tilde{v}^z = \frac{\mathrm{d}^2 \hat{v}^z}{\mathrm{d}z^2} = 0$, we obtain

$$\frac{\mathrm{d}^4 v^{*z}}{\mathrm{d}z^4} = 0 \,\mathrm{on}\, \{z = 0, z = 1\}.$$

The case $\mathrm{Ra} > \mathrm{Ra}_{\mathbf{crit}}$ "slightly"

What happens when we perturb the system with

$$\mathrm{Ra} > \mathrm{Ra}_{\mathrm{crit}}\,,$$

meaning *slightly larger*? That would mean

$$\frac{\left(\pi^2 + k^2\right)^3}{k^2} > \frac{\left(\pi^2 + k^2_{\mathrm{crit}}\right)^3}{k^2_{\mathrm{crit}}},$$

As a toy problem, let us suppose the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\begin{bmatrix} \gamma_1(\mathrm{Ra}) & 0 \\ 0 & \gamma_2(\mathrm{Ra}) \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} -a q_1 q_2 \\ b q_1^2 \end{bmatrix},$$

where $\gamma_1(\mathrm{Ra}), \gamma_2(\mathrm{Ra})$ are some functions of the Rayleigh number. There are some regimes:

- $\gamma_1 > 0, \mathrm{Ra} < \mathrm{Ra}_{\mathrm{crit}}$: then q_1 is *damped exponentially* and the nonlinearity does not play a role,
- $\gamma_1 < 0, \mathrm{Ra} > \mathrm{Ra}_{\mathrm{crit}}$: then q_1 *grows exponentially* and therefore the nonlinearity cannot be ignored.
- $\gamma_2 >> 1$ means that the second equation is (almost) only a algebraic one, which we can solve, substitue back into the first one and obtain

  $$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -\gamma_1 q_1 - \frac{ab}{\gamma_2} q_1^3 = -\gamma_1 q_1 \left(1 + \frac{ab}{\gamma_1 \gamma_2} q_1^2\right),$$

  which is really interesting; it is only a cubic correction to a linear system (*i.e.*, a *quadratic* nonlinearity.) This model might serve as a precursor to the *Ginzburg - Landau* equations.

Try a similiar thing: suppose this system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} u^2 \\ -uv \end{bmatrix},$$

and investigate its stability.

First, linearize around the steady state $\mathbf{0}$:

$$\nabla \begin{bmatrix} -\frac{1}{\mathrm{Re}} u + v + u^2 \\ -\frac{1}{\mathrm{Re}} v - uv \end{bmatrix}(\mathbf{0}) = \begin{bmatrix} -\frac{1}{\mathrm{Re}} + 2u & 1 \\ -v & -\frac{1}{\mathrm{Re}} - u \end{bmatrix}(\mathbf{0}) = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}.$$

The eigenvalues are $-\frac{1}{\mathrm{Re}}$ with the degeneracy 2. So for Re not too large, this is negative and the steady state is stable, but for $\mathrm{Re} \to \infty$, this goes to zero and we can not really say anything using this theorem.

It is crucial that the matrix is symmetric. Let us investigate

$$\dot{\mathbf{u}} = \mathbb{A}\mathbf{u}, \mathbb{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix},$$

where λ, μ are eigenvalues. The solution of course is

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0.$$

Calculate the exponential, so start with Jordan decomposition of $\mathbb{A}$)

$$\mathbb{A} = \begin{bmatrix} 1 & \frac{1}{\mu-\lambda} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\mu-\lambda} 0 & 1 \end{bmatrix},$$

and the eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{\sqrt{1 + \frac{1}{(\mu-\lambda)^2}}} \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix}.$$

Notice that the eigenvectors are *not orthogonal*. The exponential thus is

$$\exp(t\mathbb{A}) = \begin{bmatrix} e^{\lambda t} & \frac{e^{\mu t} - e^{\lambda t}}{\mu-\lambda} \\ 0 & e^{\mu t} \end{bmatrix}$$

If we Taylor expand this, we obtain

$$\exp(t\mathbb{A}) \approx \mathbb{I} + \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix} t,$$

and so the general solution thus is

$$\mathbf{u} = \exp(t\mathbb{A})\mathbf{u}_0 \approx \begin{bmatrix} (1+\lambda t)u_{0,1} + t u_{0,1} \\ (1+\mu t)u_{0,2} \end{bmatrix}$$

This means that even though (supposing the eigenvalues are negative)

$$\lim_{t \to \infty} \mathbf{u}(t) = \mathbf{0},$$

initially, the solution grows linearly with time.

But anyways, the linearized stability tells us that the origin is stable.

Let us try the energy method. Investigate $\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|$, so calculate

$$\tilde{\mathbf{u}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 0 & -\frac{1}{\mathrm{Ra}} \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \cdot \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \underbrace{u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}}_{=0},$$

so rewriting this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 = -\frac{1}{\mathrm{Ra}}|\tilde{\mathbf{u}}|^2 + \tilde{u}\tilde{v}.$$

This is remarkable analogy; the first one is the "laplacian", the missing term is the convective nonlinear term, which does not add to the energy balance and the last one is the is the interaction term.

Using Younge:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 \le -\frac{1}{\mathrm{Re}}|\tilde{\mathbf{u}}|^2 + \frac{1}{2}\left(|\tilde{u}|^2 + |\tilde{v}|^2\right) = \left(-\frac{1}{\mathrm{Re}} + \frac{1}{2}\right)|\tilde{\mathbf{u}}|^2,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\mathbf{u}}|^2 \le \left(1 - \frac{2}{\mathrm{Re}}\right)|\tilde{\mathbf{u}}|^2,$$

which means that for $\mathrm{Re} > 2$, the perturbations do not decay, and for $\mathrm{Re} \le 2$, the perturbations decay (exponentially).

Do the complete analysis of the problem: solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\mathrm{Re}} & 0\\ 0 & -\frac{1}{\mathrm{Re}}\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} + \tilde{u}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}.$$

The stready solutions are (after some computations)

$$\tilde{u}^* = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right),$$
$$\tilde{v}^* = \mathrm{Re}\,\tilde{u}^{*2},$$

and also the origin $\mathbf{0}$. Realize however that the first steady state exists only if $\mathrm{Re} \ge 2$.

So if when $\mathrm{Re} < 2$, there is only one stationary point, the origin

$$\tilde{\mathbf{u}} = \mathbf{0},$$

and all trajectories are attracted towards that point. If $\mathrm{Re} > 2$, there are 3 stationary points, lying on the parabola $\tilde{v} = \mathrm{Re}\,\tilde{u}^{*2}$ with $\tilde{u} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right)$. We see that

$$\lim_{\mathrm{Re}\to\infty} \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{\mathrm{Re}^2}}\right) = 0,$$

which corresponds to the linearization theory. This can also be used to quantify how large of neighbourhood can be obtained in the linerized stability theory, as the "minus" stationary point is unstable.

6.1.3 Orr-Sommerfeld system

This is a system describing the stability of shear flows with a parabolic inflow profile in a channel. The velocity has the profile

$$\mathbf{v} = \hat{\mathbf{v}}(y)\exp\left(i(\beta x + \alpha z - \omega t)\right).$$

The equation for the perturbation of the vertical component $\tilde{v}^y$ of the velocity is (in the Fourier space)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}-^2}k^2\right)(-i\omega + i\alpha v^z)\tilde{v}^y - i\alpha\frac{\mathrm{d}^2 v^z}{\mathrm{d}y^2}\tilde{v}^y = \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\right)^2\tilde{v}^y,$$

where $v^z(y) = 1 - y^2$.

6.1.4 Eigenvalue problem for differential operators

We have seen that many times, our examination has led us to the study of an eigenproblem of a differential operator. Let us discuss briefly how to deal with this problem. It is crucial that we are interested in only the *smallest* eiegenvalue. Techniques will be presented on the toy problem:

$$
\begin{aligned}
\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} &= \lambda u, \\
u &= 0, \text{ on } \{x = 0, 1\} \\
\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= 0, \text{ on } \{x = 0, 1\}
\end{aligned}
$$

Analytical approach

Imagine we are using finite differences to solve the equation with the RHS f:

$$
\begin{bmatrix} \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0, \text{ on } \{x = 0\} \\ \frac{\mathrm{d}^4u}{\mathrm{d}x^4} \\ \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0, \text{ on } \{x = 1\} \end{bmatrix} \begin{bmatrix} u_1 \\ \dots \\ u_N \end{bmatrix} = \begin{bmatrix} 0 \\ f_k \\ 0 \end{bmatrix},
$$

but when we are trying to solve the eigenproblem, we encounter a problem with the weird boundary conditions (try to think about the RHS).

Anyways, since the operator is linear, we can write the solution analytically

$$
u(x) = c_1 \cos\left(x\lambda^{1/4}\right) + c_2 \sin\left(x\lambda^{1/4}\right) + c_3 e^{-x\lambda^{1/4}} + c_4 e^{x\lambda^{1/4}},
$$

where c_1, c_2, c_3, c_4 are the coefficients to be determined from the boundary conditions. To have $\mathbf{c} \neq \mathbf{0}$, we require :

$$
\mathbb{A}\mathbf{c} = \mathbf{0}, \det \mathbb{A} = 0,
$$

which yields a condition for λ, in particular

$$
4\left(e^{-\lambda^{1/4}} - e^{\lambda^{1/4}}\right)\lambda \sin\left(\lambda^{1/4}\right) = 0.
$$

This is a non-linear algebraic equation, that can be solved using *e.g.* numerically. So, even though we at first coud tackle the problem ananalytically, we have come up to the embarassing restriction of not being able to solve nonlinear *algebraic* equations by hand.

Numerical approach

Some existing software packages are able to solve eigenvalue problems of the type

$$
\mathbf{L}u = \lambda u,
$$

where $\mathbf{L}$ is a second order operator. For our problem with a fourth order (or sixth order) operator, we are able to rewrite this equation in the form

$$
\mathbf{D}_k \mathbf{D}_k u = \lambda u,
$$

where

$$
\mathbf{D}_k = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2,
$$

and we are solving:

$$\mathbf{D}_k v = \lambda u, \mathbf{D}_k u - v = \lambda u,\ i.e.,\begin{bmatrix} 0 & \mathbf{D}_k \\ \mathbf{D}_k & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is equivalent to

$$\mathbf{D}_k = (\lambda+1)v, \mathbf{D}_k\mathbf{D}_k u = (\lambda+1)\mathbf{D}_k v, \Rightarrow \mathbf{D}_k\mathbf{D}_k u = \lambda(\lambda+1)u$$

so if we solve the above eigenvalue problem for λ, we have found the eigenvalue $\mu = \lambda(\lambda+1)$ of the original problem. What about the boundary conditions? Again assuming the equations hold also on the boundary, we are able to show

$$u = v = 0, \text{ on } \{x = 0, 1\}.$$

CHAPTER 7

Classical problems

CHAPTER 8

Mixtures

CHAPTER 9

Addenda

9.1 Differential geometry

9.2 Differential geometry, tensor calculus

How to write $\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u}$ etc. in polar, cylindrical and other coordinates? Can the deformation $\boldsymbol{\gamma}$ be viewed as a change of *something* rather than the body itself? Notice that there are similiarities between change of coordinates $\mathbf{x} = \mathbf{x}(\boldsymbol{\gamma})$ and the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{x})$, the Left Cauchy-Green tensor $\mathbb{B}$ and the metric tensor $\mathbb{g}$.

9.2.1 Curvilinear coordinates

Let us for now *pretend* we are still in our flat Euclidian space $\mathbb{E}^n$, and we are just changing our coordinates... consider $\mathbf{x} = \left(x^1, x^2, \ldots, x^n\right)$ as the cartesian coordinates of the point $\mathbf{x}$ and a different set $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, *e.g.*, the polar coordinates

$$x^1(r,\varphi) = r\cos\varphi, x^2(r,\varphi) = r\sin\varphi, [x,y] = [x^1, x^2], [r,\varphi] = [\xi^1, \xi^2].$$

That means every point $\mathbf{x}$ in a plane can be described by using $[x^1, x^2]$ or $[r,\varphi]$. We are used to analysis in cartesian coordinates - how can i do it in a more general setting?

Viewed from a different perspective, we do not need to pretend everything takes place in $\mathbb{E}^n$. Let $G \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ ("space of coordinates") and let $\boldsymbol{\Theta} : G \to \mathcal{M}$ be a C^1 diffemorphism between G and a $\mathcal{M} \subset \mathbb{E}^n$ an open subset of the Euclidian space (it can in fact be a manifold). In this view, all points $\mathbf{x} \in \mathcal{M}$ can be uniquely represented as $\mathbf{x} = \boldsymbol{\Theta}(\boldsymbol{\xi})$, for some $\boldsymbol{\xi} \in G$. Consider some examples

Example (Polar, spherical and cylindrical coordinates).

- polar coordinates: $G = (0,\infty) \times (0, 2\psi) \subset \mathbb{R}^2, \mathcal{M} = \mathcal{S}^2, \boldsymbol{\xi} = (r,\varphi), \boldsymbol{\Theta} : (r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi), \mathbf{x} = (r\cos\varphi, r\sin\varphi)$,
- spherical coordinates: $G = (0,\infty) \times (0,\psi) \times (0, 2\psi) \subset \mathbb{R}^3, \mathcal{M} = \mathcal{S}^3, \boldsymbol{\xi} = (r,\theta,\varphi), \boldsymbol{\Theta} : (r,\theta,\varphi) \mapsto (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \mathbf{x} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$,
- cylindrical coordinates: $G = (0,\infty) \times (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^3, \mathcal{M} = \mathcal{S}^2 \times \mathbb{R}, \boldsymbol{\xi} = (r,\varphi,z), \boldsymbol{\Theta} : (r,\varphi,z) \mapsto (r\cos\varphi, r\sin\varphi, z), \mathbf{x} = (r\cos\varphi, r\sin\varphi, z)$.

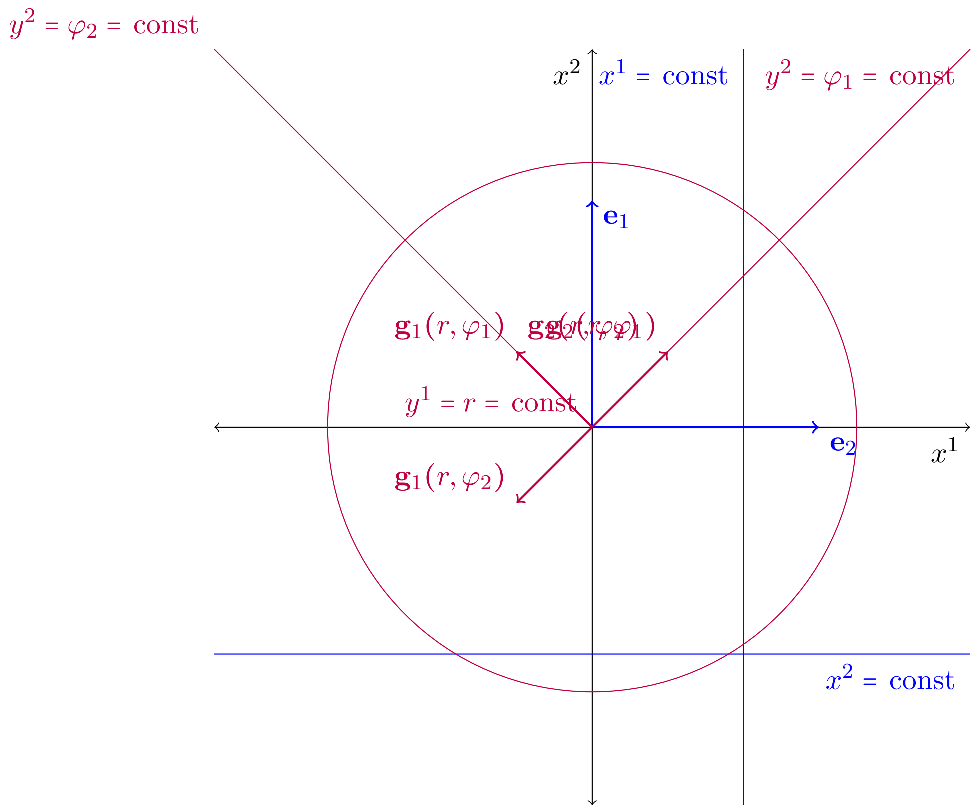


Figure 9.1: Coordinate lines and basis vectors in cartesian and polar coordinates (*the length of the vectors is the same...*)

It is customary, however imprecise, to write $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ instead of $\mathbf{x} = \boldsymbol{\Theta}(\boldsymbol{\xi})$. By the first one we mean $\mathbf{x}$ can be obtained through some function of the argument $\boldsymbol{\xi}$, but it it is not true that $\mathbf{x}$ itself is a function. In fact, $\mathbf{x}$ is a point in $\mathcal{M}$, that is the image of $\boldsymbol{\xi}$ under $\boldsymbol{\Theta}$.

This approach is elegant, as from the very start we are connecting the properties of $\mathbf{x} \in \mathcal{M}$, *i.e.*, the properties of (the manifold) $\mathcal{M} \subset \mathbb{E}^n$ with the propertis of $\boldsymbol{\xi} \in G$, *i.e.*, with the properties of $G \subset \mathbb{R}^n$. Also, it immedietaly allows for the comparison with continuum mechanics: instead of $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ we are just writing $\mathbf{x} = \boldsymbol{\Theta}(\boldsymbol{\xi})$.

Remark. The name curvilinear coordinates come from the fact that the image $\boldsymbol{\Theta}\left(\left\{\xi^k = \text{const}\right\}\right)$ are not "straight lines"

Definition 1 (Coordinate lines)**.** For $j \in \{1, \ldots, n\}$ and $\xi^j \in \mathbb{R}$ such that $\left(\xi^1, \ldots, \xi^j, \ldots, \xi^n\right) \in G$ we define the $j-$th coordinate lines/curve $\boldsymbol{\gamma}_j$ as the curve

$$\gamma_j(\xi^j) = \mathbf{x}(\xi^1, \ldots, \xi^j, \ldots, \xi^n),\ i.e., \gamma_j\left(\xi^j\right) = \boldsymbol{\Theta}\left(\xi^1, \ldots, \xi^j, \ldots, \xi^n\right),$$

and the rest of ξ^i remain arbitrary.

Basis of a vector space

In cartesian coordinates: $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where the vectors are *tangent to the coordinate lines*, that is

$$\mathbf{e}_i = \frac{\mathrm{d}\boldsymbol{\gamma}^i}{\mathrm{d}x^i}, \tag{9.1}$$

In a curvilinear coordinate system, we can repeat the same construction. We can *define a vector tangent to the coordinate line*

$$\mathbf{g}_i(\boldsymbol{\xi}) = \frac{\mathrm{d}\boldsymbol{\gamma}_i}{\mathrm{d}\xi^i}\left(\xi^i\right) = \frac{\partial \boldsymbol{\Theta}}{\partial \xi^i}(\boldsymbol{\xi}). \tag{9.2}$$

The problem is that the vectors $\mathbf{g}_i$ are not constant in space! It is a vector field!.

Remark. If it is evident what variable we are differentiating with respect to (which is not always), we write

$$\frac{\partial}{\partial \xi^i} = \partial_i,$$

so

$$\mathbf{g}_i(\boldsymbol{\xi}) = \partial_i \boldsymbol{\Theta}(\boldsymbol{\xi}).$$

This makes the manipulation with indices a bit easier, but sometimes covers the true meaning...

Vector fields

A vector $\mathbf{v}$ is independent of a basis; i can express it w.r.t $\{\mathbf{e}_j\}$ and $\{\mathbf{g}_j\}$ also:

$$\mathbf{v} = v^i \mathbf{e}_i = \nu^i(\boldsymbol{\xi})\mathbf{g}_i(\boldsymbol{\xi}).$$

(Note that in general $v^i \neq \nu^i$.) What about its derivatives? We already sense trouble, as the "curvilinear basis" is not constant!

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial\left(v^j \mathbf{e}_j\right)}{\partial x^i} = \frac{\partial v^j}{\partial x^i}\mathbf{e}_j,$$

works perfectly fine in cartesian coordinates, as $\mathbf{e}_j = \text{const}$. In curvilinear setting

$$\frac{\partial \mathbf{v}}{\partial \xi^i} = \frac{\partial\left(v^j \mathbf{g}_j\right)}{\partial \xi^i} = \frac{\partial v^j}{\partial \xi^i}\mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial \xi^i}, \tag{9.3}$$

as generally

$$\frac{\partial \mathbf{g}_j}{\partial \xi^i} \neq \mathbf{0}.$$

We can identify the last term, as differentiating a vector *should give* a vector, so in particular it can be expressed w.r.t to the basis $\{\mathbf{g}_k\}$:

$$\frac{\partial \mathbf{g}_j}{\partial \xi^i} = \Gamma^k{}_{ji}\mathbf{g}_k,$$

where $\Gamma^k{}_{ji}$ are the coefficients of the linear combinations. Thanks to the *commutation of the partial derivatives* [1], it holds

$$\Gamma^k{}_{ji} = \Gamma^k{}_{ij}, \tag{9.4}$$

i.e., $\Gamma^k{}_{ij}$ is symmetric in ij. Well, that did not help *very* much, as we don't know $\Gamma^k{}_{ij}$, but at least we have the symmetry property. Going back to 9.3:

$$\frac{\partial \mathbf{v}}{\partial \xi^i} = \frac{\partial\left(v^j \mathbf{g}_j\right)}{\partial \xi^i} = \frac{\partial v^j}{\partial \xi^i}\mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial \xi^i} = \frac{\partial v^k}{\partial \xi^i}\mathbf{g}_k + v^j \Gamma^k{}_{ij}\mathbf{g}_k = \left(\frac{\partial v^k}{\partial \xi^i} + \Gamma^k_{ij} v^j\right)\mathbf{g}_k. \tag{9.5}$$

[1] We are still in flat $\mathbb{R}^d$, i.e. euclidian space. No curvature, torsion, that would obstruct the commutation properties.

In short

$$\frac{\partial \mathbf{v}}{\partial \xi^i} = \left(\frac{\partial v^k}{\partial \xi^i} + \Gamma^k_{ij} v^j\right)\mathbf{g}_k.$$

Compare it to

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial v^k}{\partial x^i}\mathbf{e}_k.$$

This leads us to the definition

Definition 2 (Covariant derivative of a vector field)**.** The quantity:

$$\nabla_i v^j = \frac{\partial v^j}{\partial \xi^i} + \Gamma^j{}_{ik} v^k = \partial_i v^j + \Gamma^j{}_{ik} v^k \tag{9.6}$$

is called the **covariant derivative of the components vector field v**. We have also shown

$$\partial_i \mathbf{v} = \nabla_i v^j \mathbf{g}_j.$$

Dot product

The number $\mathbf{v} \cdot \mathbf{u}$ is obtained in a special manner:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{e}_i \cdot u^j \mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{e}_j) v^i u^j = \delta_{ij} v^i u^j.$$

I can of course write the vectors in a different basis:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{g}_i \cdot u^j \mathbf{g}_j = (\mathbf{g}_i \cdot \mathbf{g}_j) v^i u^j = g_{ij} v^i u^j.$$

Definition 3 (Metric tensor)**.** The tensor $\mathbb{g}$ such that $\forall \mathbf{v} = v^i \mathbf{g}_i, \mathbf{u} = u^j \mathbf{g}_j$ it holds:

$$\mathbf{v} \cdot \mathbf{u} = g_{ij} v^i u^j, g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

is called the **metric tensor**.

Dual space

The (vector) dual space is the space of all linear forms on the underlying vector space. In particular it is a vector space itself, so

$$\forall \mathbf{l} \in V^* : \mathbf{l} = l_i \mathbf{e}^i,$$

where $\mathbf{e}^i$ is the i-th basis vector. The action of the forms can be described as

$$\mathbf{l}(\mathbf{v}) = l_i \mathbf{e}^i (v^j \mathbf{e}_j) = l_i v^j \mathbf{e}^i(\mathbf{e}_j), \forall \mathbf{v} \in V.$$

If it holds $\mathbf{e}^{\mathbf{i}}(\mathbf{e}_j) = \delta^i_j$, we call the basis $\mathbf{e}^i$ dual to $\mathbf{e}_j$. What about curvilinear setting? We can adopt the same definition

Definition 4. We call the basis $\mathbf{g}^j$ of V^* the dual basis to $\mathbf{g}_i$ iff

$$\mathbf{g}^j(\mathbf{g}_i) = \delta^j_i.$$

For the original basis we had $\mathbf{g}_i = \frac{\partial \boldsymbol{\Theta}}{\partial \xi^i} = \frac{\partial \Theta^j}{\partial \xi^i}\mathbf{e}_j$, in the dual case (using the chain rule):

$$\mathbf{g}^i(\mathbf{g}_j) = \delta^i_j = \frac{\partial \xi^i}{\partial \xi^j} = \frac{\partial \xi^i}{\partial \Theta^k}\frac{\partial \Theta^k}{\partial \xi^j}\mathbf{e}^k(\mathbf{e}_k) = \left(\frac{\partial \xi^i}{\partial \Theta^k}\mathbf{e}^k\right)\left(\frac{\partial \Theta^k}{\partial \xi^j}\mathbf{e}_k\right) = \left(\frac{\partial \xi^i}{\partial \Theta^k}\mathbf{e}^k\right)(\mathbf{g}_j)$$

so i can conclude

$$\mathbf{g}^i = \frac{\partial \xi^i}{\partial \Theta^k}\mathbf{e}^k.$$

Recall that we have the *Riesz representation theorem*:

$$\forall \mathbf{l} \in V^* \exists! \mathbf{u} \in V : \mathbf{l}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{u}, \forall \mathbf{v} \in V.$$

This implies

$$l_i \mathbf{g}^i(v^j \mathbf{g}_j) = v^m u^n g_{mn},\ i.e.,\ l_i v^j \delta^i_j = u^m v^n g_{mn},\ i.e.,\ l^i v_i = u^m v^i g_{mi},\ i.e.,\ l_i = g_{im} u^m.$$

So $l_i = g_{im}u^m$, where $\mathbf{u}$ represents $\mathbf{l}$. It is common to write

$$l_i = g_{im} l^m.$$

Covector fields

How to compute $\frac{\partial \mathbf{l}}{\partial y^i}$? Just change the location of the index :)

$$\frac{\partial \mathbf{l}}{\partial \xi^i} = \frac{\partial (l_j \mathbf{g}^j)}{\partial \xi^i} = \frac{\partial l^j}{\partial \xi^i}\mathbf{g}^j + \frac{\partial \mathbf{g}^j}{\partial \xi^i} l_j.$$

Again, the last term must be expressable in the dual basis, so

$$\frac{\partial \mathbf{l}}{\partial \xi^i} = \frac{\partial l_j}{\partial \xi^i}\mathbf{g}^j + \tilde{\Gamma}^j{}_{im} l_j \mathbf{g}^m, \tag{9.7}$$

where again $\tilde{\Gamma}^j{}_{im} = \tilde{\Gamma}^j{}_{mi}$ are the coefficients of the linear combinations, "that are symmetric".

What is the relation between $\Gamma^k{}_{im}$ and $\tilde{\Gamma}^k{}_{im}$? Recall $\delta^i_j = \mathbf{g}^i(\mathbf{g}_j)$, so differentiating can lead us to

$$\Gamma^j_{im} = -\Gamma^j_{im}. \tag{9.8}$$

Definition 5. Let $\mathbf{l}$ be a covector field. The quantity

$$\nabla_i l_j = \partial_i l_j - \Gamma^k{}_{ij} l_k \tag{9.9}$$

is called **the covariant derivative of the components covector field l**. We have shown

$$\partial_i \mathbf{l} = \nabla_i l_j \mathbf{g}^j.$$

Finally, for the tensor field of type $(0,2)$, *i.e.*, for bilinear forms, one can obtain

$$
\begin{aligned}
\frac{\partial \mathbb{A}}{\partial \xi^i} &= \frac{\partial}{\partial \xi^i}(A_{mn}\mathbf{g}^m \otimes \mathbf{g}^n) = \frac{\partial A_{mn}}{\partial \xi^i}\mathbf{g}^m \otimes \mathbf{g}^n + A_{mn}\frac{\partial \mathbf{g}^m}{\partial \xi^i} \otimes \mathbf{g}^n + A_{mn}\mathbf{g}^m \otimes \frac{\partial \mathbf{g}^n}{\partial \xi^i} = \\
&= \frac{\partial A_{mn}}{\partial \xi^i}\mathbf{g}^m \otimes \mathbf{g}^n - A_{mn}\Gamma^m{}_{ik}\mathbf{g}^k \otimes \mathbf{g}^n - A_{mn}\mathbf{g}^m \otimes \Gamma^n{}_{ik}\mathbf{g}^k = \\
&= \frac{\partial A_{mn}}{\partial \xi^i}\mathbf{g}^m \otimes \mathbf{g}^n - \Gamma^m{}_{ik}A_{mn}\mathbf{g}^k \otimes \mathbf{g}^n - \Gamma^n{}_{ik}A_{mn}\mathbf{g}^m \otimes \mathbf{g}^k = \\
&= \frac{\partial A_{mn}}{\partial \xi^i}\mathbf{g}^m \otimes \mathbf{g}^n - \Gamma^l{}_{im}A_{ln}\mathbf{g}^m \otimes \mathbf{g}^n - \Gamma^l{}_{in}A_{ml}\mathbf{g}^m \otimes \mathbf{g}^n = \\
&= \left(\frac{\partial A_{mn}}{\partial \xi^i} - \Gamma^l{}_{im}A_{ln} - \Gamma^l{}_{in}A_{ml}\right)\mathbf{g}^m \otimes \mathbf{g}^n,
\end{aligned}
$$

so if we denote

$$\nabla_i A_{mn} = \partial_i A_{mn} - \Gamma^l{}_{im}A_{ln} - \Gamma^l{}_{in}A_{ml},$$

we can write

$$\partial_i \mathbb{A} = \nabla_i A_{mn}\mathbf{g}^m \otimes \mathbf{g}^n.$$

Direct expression of the Christoffel symbols

All our formulas depend on the Christoffel symbols. How to compute them? With the above relation, we can express $\nabla_j g_{mn}$:

$$\nabla_j g_{mn} = \partial_j g_{mn} - \Gamma^k{}_{jm}g_{kn} - \Gamma^{k_{jn}}g_{mk}.$$

Moreover, we can directly differentiate.

$$\frac{\partial g_{mn}}{\partial \xi^j} = \frac{\partial(\mathbf{g}_m \cdot \mathbf{g}_n)}{\partial \xi^j} = \frac{\partial \mathbf{g}_m}{\partial \xi^j} \cdot \mathbf{g}_n + \mathbf{g}_m \frac{\partial \mathbf{g}_n}{\partial \xi^j} = \Gamma^k{}_{mj}\mathbf{g}_k \cdot \mathbf{g}_n + \mathbf{g}_m \cdot \Gamma^k{}_{nj}\mathbf{g}_k = \Gamma^k{}_{mj}g_{kn} + \Gamma^k{}_{nj}g_{mk},$$

but realize that if we rearrange, we exactly obtain the covariant derivative $\nabla_j g_{mn}$ on the LHS. We have shown a remarkable identity

$$\nabla_j g_{mn} = 0, \tag{9.10}$$

which also implies

$$\partial_i \mathbb{g} = \mathbb{0}. \tag{9.11}$$

This property is particularly useful, as it allows us to express the Christoffel symbols. Using cyclic permuation, we can write

$$
\begin{aligned}
A &= \frac{\partial g_{mn}}{\partial \xi^j} = \Gamma^k{}_{mj}g_{kn} + \Gamma^k{}_{nj}g_{mk}, \\
B &= \frac{\partial g_{jm}}{\partial \xi^j} = \Gamma^k{}_{jn}g_{kn} + \Gamma^k{}_{mn}g_{jk}, \\
C &= \frac{\partial g_{nj}}{\partial \xi^n} = \Gamma^k{}_{nm}g_{kj} + \Gamma^k{}_{jm}g_{nk}.
\end{aligned}
$$

Taking $A - B - C$ yields

$$\frac{\partial g_{mn}}{\partial y^j} - \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial \xi^m} = -2\Gamma^k{}_{nm}g_{jk},$$

multiplying by g^{jl} gives

$$-2\Gamma^k{}_{nm}\delta_k^l = g^{jl}\left(\frac{\partial g_{mn}}{\partial \xi^j} - \frac{\partial g_{jm}}{\partial \xi^n} - \frac{\partial g_{nj}}{\partial \xi^m}\right),$$

from which it follows

$$\Gamma^l{}_{nj} = \frac{1}{2}g^{lm}\left(\frac{\partial g_{mn}}{\partial \xi^j} + \frac{\partial g_{jm}}{\partial \xi^n} - \frac{\partial g_{nj}}{\partial \xi^m}\right). \tag{9.12}$$

Interchangability of the derivatives

In euclidian space:

$$\frac{\partial^2 \mathbf{v}}{\partial x^j \partial x^i} = \frac{\partial^2 \left(v^k \mathbf{e}_k\right)}{\partial x^j \partial x^i} = \left(\frac{\partial^2 v^k}{\partial x^j \partial x^i}\right)\mathbf{e}_k = \left(\frac{\partial^2 v^k}{\partial x^i \partial x^j}\right)\mathbf{e}_k = \frac{\partial^2 \mathbf{v}}{\partial x^i \partial x^j},$$

where $\mathbf{e}_k$ are the *cartesian basis vectors*. Will it hold even in curvilinear coordinate systems?

$$\begin{aligned}\frac{\partial^2 \mathbf{v}}{\partial \xi^j \partial \xi^k} - \frac{\partial^2 \mathbf{v}}{\partial \xi^j \partial \xi^k} &= \partial_j\partial_k \mathbf{v} - \partial_k\partial_j \mathbf{v} = \partial_j\left(\nabla_k v^i\right)\mathbf{g}_i - \partial_k\left(\nabla_j v^i\right)\mathbf{g}_i = \left(\nabla_j\nabla_k v^i\right)\mathbf{g}_i - \left(\nabla_k\nabla_j v^i\right)\mathbf{g}_i = \\ &= \left(\frac{\partial \Gamma^i{}_{jm}}{\partial \xi^k} - \frac{\partial \Gamma^i{}_{km}}{\partial \xi^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}\right)v^m \mathbf{g}_i.\end{aligned}$$

We have skipped the calculation, although it it not trivial whatsoever. Let us just state the quantity $\nabla_k v^i$ are coordinates of a tensor field of type $(1,1)$, and the covariant derivative of such coordinates would be

$$\nabla_j A_k{}^i = \partial_j A_k{}^i - \Gamma^l{}_{jk} A_l{}^i + \Gamma^i{}_{jl} A_k{}^l,$$

so one would have to manipulate

$$\nabla_j \nabla_k v^i = \partial_j \nabla_k v^i - \Gamma^l{}_{jk}\nabla_l v^i + \Gamma^i{}_{jl}\nabla_k v^l.$$

Definition 6 (Riemann curvature tensor)**.** The *tensor*[2]

$$R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial y^k} - \frac{\partial \Gamma^i{}_{km}}{\partial y^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}, \tag{9.13}$$

is called the **Riemann curvature tensor**. We have shown that it holds

$$\nabla_i\nabla_j v^k - \nabla_j\nabla_i v^k = R^k{}_{ijl}v^l.$$

We see that if the Riemann curvature tensor is zero, then effectively we are in the case of a flat euclidian space, as the derivatives commute. In other words, in flat euclidian space, the Riemann curvature tensor is always zero. If we flip this, we see that if we have a space with zero Riemann curvature tensor, *we have a chance* that the derivatives commute, i.e. that the structure is euclidian.

[2] This truly is a tensor, even though the Christoffel symbols are not.

Example (Interpretation in continuum mechanics).

$$\mathbf{x} = \boldsymbol{\Theta}(\boldsymbol{\xi}) \text{ vs } \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}),$$
$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i} \text{ vs } \mathbf{g}_I = \frac{\partial \boldsymbol{\chi}}{\partial X^I}, \text{ i.e. } (\mathbf{g}_M)^i = F^i{}_M = \frac{\partial \chi^i}{\partial X^M},$$
$$g_{IJ} = \mathbf{g}_I \cdot \mathbf{g}_J = (\mathbb{F}^\top \mathbb{F})_{IJ} = (\mathbb{C})_{IJ},$$

So
$$\mathbb{g} = \mathbb{C} = \mathbb{F}^\top \mathbb{F} = (\nabla \boldsymbol{\chi})^\top \boldsymbol{\chi}. \tag{9.14}$$

The way our metric tensor $\mathbb{g}$ is not insignificant, not all positive definite symmetric bilinear forms can be factorized as in our case. In particular, our vectors $\mathbf{g}_M = \frac{\partial \boldsymbol{\chi}}{\partial X^M}$ solve the following system of PDE's:
$$\frac{\partial \mathbf{g}_M}{\partial X^J} = \Gamma^K{}_{MJ} \mathbf{g}_K.$$

It turns out **ciarletIntroductionDifferentialGeometry2005** this system has *an unique solution* if and only if
$$\frac{\partial^2 \mathbf{g}_m}{\partial X^I \partial X^J} = \frac{\partial^2 \mathbf{g}_m}{\partial X^J \partial X^I},$$
i.e., when the second partial derivatives commute. But we have shown in the previous that this means
$$\partial_{JI} \mathbf{g}_M = \partial_{IJ} \mathbf{g}_M = \frac{\partial \left(\Gamma^K{}_{IM} \mathbf{g}_K \right)}{\partial X^J} = \frac{\partial \left(\Gamma^K{}_{MJ} \mathbf{g}_K \right)}{\partial X^I},$$
this is equivalent to
$$\partial_{JI} \mathbf{g}_m - \partial_{IJ} \mathbf{g}_n = 0 \Leftrightarrow \ldots R^I{}_{JKM} = 0.$$

So this implies that all physically admissable deformations produce a deformed space with zero Riemann curvature tensor, *i.e.*, a flat space.

9.2.2 Calculus

Gradient

For a scalar function, we set
$$(\nabla \varphi)_i = \frac{\partial \varphi}{\partial x^i},$$
so the summation convention forces us to write
$$\nabla \varphi = \frac{\partial \varphi}{\partial x^i} \mathbf{e}^i,$$
meaning the gradient is in fact a covector. This causes a lot of misconseption. In fact, the functions $\frac{\partial \varphi}{\partial x^i}$ are coordinates of the differential covector
$$\mathbf{d}\varphi = \frac{\partial \varphi}{\partial x^i} \mathbf{e}^i = \frac{\partial \varphi}{\partial x^i} \mathbf{d}x^i,$$
but in physics, it is common to put the coordinates of a *covector* into a *vector*, and so $\nabla \varphi$ is born.

Expressed in curvilinear coordinates
$$\nabla \varphi = \frac{\partial \varphi}{\partial x^i} \mathbf{e}^i = \frac{\partial \varphi}{\partial \xi^j} \underbrace{\frac{\partial \xi^j}{\partial x^i} \mathbf{e}^i}_{=\mathbf{g}^j} = \frac{\partial \varphi}{\partial \xi^j} \mathbf{g}^j.$$

Realize that for a scalar field, it holds[3] $\partial_j = \nabla_j$, so in fact

$$\nabla\varphi = \nabla_j f \mathbf{g}^j.$$

What about the gradient of a vector field? In the cartesian coordinate system:

$$\nabla\mathbf{v} = \nabla\left(v^i\mathbf{e}_i\right) = \frac{\partial v^i}{\partial x^j}\mathbf{e}_i \otimes \mathbf{e}^j,$$

and in curvilinear coordinates:

$$\begin{aligned}
\nabla\left(v^i\mathbf{g}_i\right) &= \nabla\left(v^i\frac{\partial\Theta^m}{\partial\xi^i}\mathbf{e}_m\right) = \frac{\partial}{\partial\Theta^j}\left(v^i\frac{\partial\Theta^m}{\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \left(\frac{\partial v^i}{\partial\Theta^j}\frac{\partial\Theta^m}{\partial\xi^i} + v^i\frac{\partial^2\Theta^m}{\partial\Theta^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \left(\frac{\partial v^i}{\partial\xi^n}\frac{\partial\xi^n}{\partial\Theta^j}\frac{\partial\Theta^m}{\partial\xi^i} + v^i\frac{\partial^2\Theta^m}{\partial\Theta^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial\Theta^m}{\partial\xi^u}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial\Theta^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial\Theta^m}{\partial\xi^i}\right)\frac{\partial\xi^l}{\partial\Theta^j}\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial\Theta^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial\Theta^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial\Theta^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^l}{\partial\Theta^j}\mathbf{e}^j\right) = \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\frac{\partial\mathbf{g}^i}{\partial\xi^l}\otimes\mathbf{g}^l = \\
&= \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\Gamma^s{}_{il}\mathbf{g}_s\otimes\mathbf{g}^l = \left(\frac{\partial v^s}{\partial\xi^l} + \Gamma^s{}_{il}v^i\right)\mathbf{g}_s\otimes\mathbf{g}^l = \\
&= \nabla_l v^s\mathbf{g}_s\otimes\mathbf{g}^l,
\end{aligned}$$

and so

$$\nabla\mathbf{v} = \nabla_i v^j\mathbf{g}_j\otimes\mathbf{g}^i.$$

Until now, we have not discussed the fact $|\mathbf{g}_i| \neq 1$, which is a kind of a problem. Let us define

$$\mathbf{v} = v^i\mathbf{g}_i = v^i|\mathbf{g}_i|\frac{\mathbf{g}_i}{|\mathbf{g}_i|} = v^{\hat{i}}\mathbf{g}_{\hat{i}},$$

where we have defined

$$v^{\hat{i}} = |\mathbf{g}^i|v^i, \mathbf{g}_{\hat{i}} = \frac{\mathbf{g}_i}{|\mathbf{g}_i|}.$$

But! the differential formulas work for $v^i, \mathbf{g}_i$, **not for** $v^{\hat{i}}, \mathbf{g}_{\hat{i}}$! We have to be careful

$$\nabla\varphi = \nabla_j\varphi\mathbf{g}^j = |\mathbf{g}^i|\nabla_j\varphi\mathbf{g}^{\hat{i}}, \tag{9.15}$$

$$\nabla\mathbf{v} = \nabla_i v^j\mathbf{g}_j\otimes\mathbf{g}^i = |\mathbf{g}_j|\left|\mathbf{g}^i\right|\nabla_i v^j\mathbf{g}_{\hat{j}}\otimes\mathbf{g}^{\hat{i}}. \tag{9.16}$$

For the divergence of a vector field, we know: $\nabla\cdot\mathbf{v} = \operatorname{tr}(\nabla\mathbf{v})$, so

$$\nabla\cdot\mathbf{v} = \operatorname{tr}(\nabla\mathbf{v}) = \operatorname{tr}\left(\nabla_i v^j\mathbf{g}_j\otimes\mathbf{g}^i\right) = \nabla_j v^j.$$

The divergence of a tensor field a can be tricky, but be guided by the summation convention; for the tensor of type $(2,0)$ (a bivector)

$$\mathbb{A} = A^{ij}\mathbf{g}_i\otimes\mathbf{g}_j$$

we can define

$$\nabla\cdot\mathbb{A} = \nabla_j A^{ij}\mathbf{g}_i.$$

For the tensors of a different type, we need to change the position of the indices to obtain a bivector. Notice that the result is a vector, not a bivector.

[3] We can formally suppose φ is a vector field with all components being φ, then $\nabla_j\varphi = \partial_j\varphi + \Gamma^k{}_{jl}\varphi^l$, but $\Gamma^k{}_{jl}\varphi^l = \varphi\Gamma^k{}_{jk} = 0$.

Laplace-Beltrami operator

$$\triangle\,\varphi = \frac{1}{\sqrt{\det\mathbb{g}}}\frac{\partial}{\partial\xi^i}\left(\sqrt{\det\mathbb{g}}g^{ij}\frac{\partial\varphi}{\partial\xi^j}\right),$$

on one hand:

$$\triangle\,\varphi = \nabla\cdot\nabla\varphi = \nabla_i(\nabla\varphi)^i = \nabla_i\left(g^{ij}\frac{\partial\varphi}{\partial\xi^j}\right) = \nabla_i g^{ij}\partial_j\varphi,$$

where we have rised the index

$$(\nabla\varphi)^i = g^{ij}(\nabla\varphi)_j = g^{ij}\frac{\partial\varphi}{\partial\xi^j},$$

so using the covariant derivative definition

$$\nabla\cdot\nabla\varphi = \frac{\partial}{\partial\xi^i}\left(g^{ij}\frac{\partial\varphi}{\partial\xi^j}\right) + \Gamma^i{}_{il}g^{lj}\frac{\partial\varphi}{\partial\xi^j},$$

on the other

$$\begin{aligned}
\triangle\,\varphi &= \frac{1}{\sqrt{\det\mathbb{g}}}\left(\frac{\partial}{\partial\xi^i}\left(\sqrt{\det\mathbb{g}}\right)g^{ij}\frac{\partial\varphi}{\partial\xi^j} + \sqrt{\det\mathbb{g}}\frac{\partial g^{ij}}{\partial\xi^i}\frac{\partial\varphi}{\partial\xi^j} + \sqrt{\det\mathbb{g}}g^{ij}\frac{\partial^2\varphi}{\partial\xi^i\partial\xi^j}\right) = \\
&= \frac{1}{\sqrt{\det\mathbb{g}}}\left(\frac{1}{2\sqrt{\det\mathbb{g}}}\frac{\partial}{\partial\xi^i}(\det\mathbb{g})g^{ij}\frac{\partial\varphi}{\partial\xi^j} + \sqrt{\det\mathbb{g}}\frac{\partial g^{ij}}{\partial\xi^i} + \sqrt{\det\mathbb{g}}g^{ij}\frac{\partial^2\varphi}{\partial\xi^i\partial\xi^j}\right) = \\
&= \frac{1}{\sqrt{\det\mathbb{g}}}\left(\frac{1}{2}\operatorname{tr}\left(\mathbb{g}^{-1}\frac{\partial\mathbb{g}}{\partial\xi^i}\right)g^{ij}\frac{\partial\varphi}{\partial\xi^j} - \sqrt{\det\mathbb{g}}\left(\Gamma^j{}_{kn}g^{in} - \Gamma^i{}_{km}g^{mj}\right) + \sqrt{\det\mathbb{g}}g^{ij}\frac{\partial^2\varphi}{\partial\xi^i\partial\xi^j}\right) = \\
&= \frac{1}{\sqrt{\det\mathbb{g}}}\left(\frac{1}{2}\left(g^{mn}\frac{\partial g_{mn}}{\partial\xi^i}\right)g^{ij}\frac{\partial\varphi}{\partial\xi^j} - \right)
\end{aligned}$$

Bipolar coordinates

Define $\boldsymbol{\xi} = [\alpha,\beta]$, where

$$\alpha + i\beta = \log\frac{y + i(x+a)}{y + i(x-a)}.$$

This can be inversed and write

$$\begin{aligned}
x &= \frac{a\sinh\alpha}{\cosh\alpha - \cos\beta}, \\
y &= \frac{a\sin\beta}{\cosh\alpha - \cos\beta},
\end{aligned}$$

moreover,

$$\begin{aligned}
(x - a\coth\alpha)^2 + y^2 &= \frac{a^2}{\sinh^2\alpha}, \\
x^2 + (y - a\cot\beta)^2 &= \frac{a^2}{\sin^2\beta}.
\end{aligned}$$

Calculate *everything* for this coordinate system.

In general $\mathbf{g}_i = \frac{\partial x^j}{\partial \xi^i}\mathbf{e}_j$, so in our case

$$\begin{aligned}
\mathbf{g}_\alpha &= \frac{\partial x}{\partial \alpha}\mathbf{e}_x + \frac{\partial y}{\partial \alpha}\mathbf{e}_y \\
&= \left(\frac{(a\cosh\alpha)(\cosh\alpha - \cos\beta) - a\sinh\alpha\sinh\alpha}{(\cosh\alpha-\cos\beta)^2}\right)\mathbf{e}_x + \left(\frac{a\cos\beta(\cosh\alpha-\cos\beta) - a\sin\beta\sinh\alpha}{(\cosh\alpha-\cos\beta)^2}\right)\mathbf{e}_y = \\
&= \frac{a}{(\cosh\alpha-\cos\beta)^2}((1-\cosh\alpha\cos\beta)\mathbf{e}_x - (\sin\beta\sinh\alpha)\mathbf{e}_y), \\
\mathbf{g}_\beta &= \frac{\partial x}{\partial \beta}\mathbf{e}_x + \frac{\partial y}{\partial \beta}\mathbf{e}_y \\
&= \cdots = \\
&= \frac{a}{(\cosh\alpha-\cos\beta)^2}(-(\sin\beta\sinh\alpha)\mathbf{e}_x + (-1+\cosh\alpha\cos\beta)\mathbf{e}_y).
\end{aligned}$$

We can see that $\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = 0$ and so

$$\mathbb{g} = \left(\frac{a}{\cosh\alpha-\cos\beta}\right)^2\mathbb{I}, \mathbb{g}^{-1} = \left(\frac{\cosh\alpha-\cos\beta}{a}\right)^2\mathbb{I}.$$

Coming back to Laplace-Beltrami operator, we can calculate

$$\left(\sqrt{\det\mathbb{g}}\mathbb{g}^{-1}\right) = \left(\frac{a}{\cosh\alpha-\cos\beta}\right)^2\left(\frac{\cosh\alpha-\cos\beta}{\alpha}\right)^2\mathbb{I} = \cdots = \mathbb{I},$$

and calculating a bit more yields

$$\triangle\varphi \to \left(\frac{\cosh\alpha-\cos\beta}{a}\right)^2\triangle_{\alpha\beta}\varphi.$$

Remark (Relation to complex analysis)*.* This can be seen as a conformal transformation

$$\gamma = f(z),$$

where

$$\begin{aligned}
\gamma &= \alpha + i\beta, \\
z &= x + iy,
\end{aligned}$$

Let us write

$$f(z) = f^x(x,y) + if^y(x,y) \Leftrightarrow \mathbf{f}(\mathbf{x}) = [f^x(\mathbf{x}), f^y(\mathbf{x})], z = x+iy,$$

and compute

$$\frac{\partial\mathbf{f}}{\partial\mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ \frac{\partial f^y}{\partial x} & \frac{\partial f^y}{\partial y} \end{bmatrix}.$$

Recall Cauchy-Riemmann conditions:

$$\begin{aligned}
\frac{\partial f^x}{\partial x} &= \frac{\partial f^y}{\partial y}, \\
\frac{\partial f^x}{\partial y} &= -\frac{\partial f^y}{\partial x},
\end{aligned}$$

using which the gradient becomes:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ -\frac{\partial f^x}{\partial y} & \frac{\partial f^y}{\partial y} \end{bmatrix},$$

which is an **orthogonal matrix:**

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top = \left(\left(\frac{\partial f^x}{\partial x}\right)^2 + \left(\frac{\partial f^y}{\partial y}\right)^2\right)\mathbb{I}.$$

Realize that all this structure comes just from the fact that the transformation is given through a holomorfic function.

Compatibility conditions in linearised elasticity

$$R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial \xi^k} - \frac{\partial \Gamma^i{}_{km}}{\partial \xi^j} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km},$$

and we know

$$R^i{}_{jkm} = 0 \Leftrightarrow \mathbb{C} = \mathbb{F}^\top\mathbb{F}, \mathbb{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}.$$

All this works in fully *nonlinear setting!*. In the classical lecture, we have been able to obtain compatibility condition in *linearised elasticity*: $\nabla \times \mathbb{e} = \mathbb{0}, \mathbb{e} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right)$.

Consider the following setting:

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}), \\ \mathbf{u} &= \chi(\mathbf{X}) - \mathbf{X}, \\ \nabla \mathbf{u} &= \mathbb{F} - \mathbb{I}, \\ \mathbb{F} &= \mathbb{I} + \nabla \mathbf{u}, \end{aligned}$$

then

$$\mathbb{C} = \mathbb{F}^\top\mathbb{F} = \left(\mathbb{I} + (\nabla \mathbf{u})^\top\right)(\mathbb{I} + \nabla \mathbf{u}) = \mathbb{I} + 2\mathbb{e} + \text{ h.o.t.},$$

and so

$$\mathbb{g}^{-1} = \mathbb{I} - 2\mathbb{e}.$$

The Christoffel symbols are

$$\begin{aligned} \Gamma^l{}_{nj} &= \frac{1}{2} g^{lm}\left(\frac{\partial g_{mn}}{\partial X^j} + \frac{\partial g_{jm}}{\partial X^n} - \frac{\partial g_{nj}}{\partial X^m}\right) \\ &\approx \frac{1}{2}(\mathbb{I} - 2\mathbb{e})^{lm}\left(\frac{\partial}{\partial X^j}(\mathbb{I} + 2\mathbb{e})_{mn} + \frac{\partial}{\partial X^n}(\mathbb{I} + 2\mathbb{e})_{jm} - \frac{\partial}{\partial X^m}(\mathbb{I} + 2\mathbb{e})_{nj}\right), \\ &\approx \delta^{lm}\left(\frac{\partial \varepsilon_{mn}}{\partial X^j} + \frac{\partial \varepsilon_{jm}}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}\right) = \frac{\partial \varepsilon^l_n}{\partial X^j} + \frac{\partial \varepsilon^l_j}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}, \end{aligned}$$

the Riemann curvature tensor is (linear approximation)

$$
\begin{aligned}
0 = R^i{}_{jkm} &\approx \frac{\partial \Gamma^i{}_{jm}}{\partial X^k} - \frac{\partial \Gamma^i{}_{km}}{\partial X^j} \\
&= \frac{\partial}{\partial X^k}\left(\frac{\partial \varepsilon^i_j}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^j} - \frac{\partial \varepsilon_{mj}}{\partial X^i}\right) - \frac{\partial}{\partial X^j}\left(\frac{\partial \varepsilon^i_k}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^k} - \frac{\partial \varepsilon_{km}}{\partial X^i}\right) = \\
&= \frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i},
\end{aligned}
$$

so the compatibility conditions are

$$
\frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i} = 0.
$$

9.2.3 Surface geometry

In this part, we will work with surfaces embedded in $\mathbb{R}^3$.

Let $G = \{\mathbf{u}\} \subset \mathbb{R}^2$ be the parametrization space and $\mathbf{\Phi} : G \subset \mathbb{R}^2 \to \mathbb{R}^3$ is the parametrization, so the points of the surface are

$$
\mathbf{x} = \mathbf{\Phi}(\mathbf{u}), \mathbf{x} \in \mathbb{R}^3.
$$

Definition 7. The indices $i, j, k, \dots \in \{1,2,3\}$ will denote objects from $\mathbb{R}^3$ and indices $\alpha, \beta, \gamma, \dots \in \{1,2\}$ will denote indices of objects from $\mathbb{R}^2$.

Tangent and normal vectors

As in the previous story, we can define (basis) tangent vectors:

$$
\mathbf{t}_1 = \frac{\partial \mathbf{\Phi}}{\partial u^1} = \partial_1 \mathbf{\Phi}, \mathbf{t}_2 = \frac{\partial \mathbf{\Phi}}{\partial u^2} = \partial_2 \mathbf{\Phi},
$$

and on surfaces, of importance is also the normal vector

$$
\mathbf{n} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|}.
$$

Distances and angles

The metric tensor on the surface is given by

$$
\mathbb{g}_s = \begin{bmatrix} \mathbf{t}_1 \cdot \mathbf{t}_1 & \mathbf{t}_1 \cdot \mathbf{t}_2 \\ \mathbf{t}_1 \cdot \mathbf{t}_2 & \mathbf{t}_2 \cdot \mathbf{t}_2 \end{bmatrix},
$$

or in context of differential geometry, this object is called **the first fundamental form**.

Derivatives

In $\mathbb{R}^3$, we know how to differentiate tangent vectors (using Christoffel symbols). Can this be helpful to us? Looking at our surface from $\mathbb{R}^3$, we see the (local) orthogonal basis on it is formed by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$, so thhe metric tensor in $\mathbb{R}^3$ is given by

$$
\mathbb{g} = \begin{bmatrix} \mathbb{g}_s & 0 \\ 0 & 1 \end{bmatrix}.
$$

In $\mathbb{R}^3$, we have no problem writing the derivatives of the tangent vectors

$$\partial_\alpha \mathbf{t}_\beta = \Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + b_{\alpha\beta}\mathbf{n},$$

where we have just denoted $b_{\alpha\beta} = \Gamma^3{}_{\alpha\beta}$.

What about the derivative of the normal vector? From the length of $\mathbf{n}$ we know

$$\mathbf{n}\cdot\mathbf{n} = 1 \Rightarrow \partial_\alpha \mathbf{n}\cdot\mathbf{n} = 0,$$

so that means the derivative is perpendicular to the normal direction, so

$$\partial_\alpha \mathbf{n} = A^\gamma{}_\alpha \mathbf{t}_\gamma.$$

Next trick is to realize

$$0 = \partial_\alpha\big(\mathbf{n}\cdot\mathbf{t}_\beta\big) = \partial_\alpha\mathbf{n}\cdot\mathbf{t}_\beta + \mathbf{n}\cdot\partial_\alpha\mathbf{t}_\beta = A^\gamma{}_\alpha\mathbf{t}_\gamma\cdot\mathbf{t}_\beta + \mathbf{n}\cdot\big(\Gamma^\delta{}_{\alpha\beta}\mathbf{t}_\delta + b_{\alpha\beta}\mathbf{n}\big) = A^\gamma{}_\alpha g_{\gamma\beta} + b_{\alpha\beta},$$

from which it follows

$$A^\gamma{}_\alpha = -g^{\gamma\beta}b_{\beta\alpha}.$$

(those are in fact the components of $\mathbb{g}_s$.)

Commutation of derivatives

What are the *implications* of

$$\partial_{\beta\gamma}\mathbf{t}_\alpha = \partial_{\gamma\beta}\mathbf{t}_\alpha?$$

Write

$$0 = \partial_{\beta\gamma}\mathbf{t}_\alpha - \partial_{\gamma\beta}\mathbf{t}_\alpha = (\,\text{something}\,)\mathbf{t}_\delta + (\,\text{something different}\,)\mathbf{n},$$

so we see the whole thing splits into two parts, that are perpendicular to each other; this must however mean that both of them are zero, which can be shown to be equivalent to

Theorem 1 (Gauss relation)**.**

$$R_{\gamma\beta\delta\alpha} = b_{\alpha\beta}b_{\gamma\delta} - b_{\alpha\delta}b_{\gamma\beta}.$$

Theorem 2 (Codazzi-Mainardi relation)**.**

$$\nabla_\alpha b_{\beta\gamma} - \nabla_\gamma b_{\beta\alpha} = 0$$

Surfaces evolving in time

Now the points of the surface are given by

$$\mathbf{x} = \mathbf{\Phi}(t,\mathbf{u}), \text{ where } \mathbf{\Phi} : \mathbb{R}\times G \to \mathbb{R}^3.$$

We can define the **velocity of the surface**:

$$\mathbf{v}_s = \frac{\partial\mathbf{\Phi}}{\partial t}(t,\mathbf{u}) = \partial_t\mathbf{\Phi}(t,\mathbf{u}).$$

The basis of everything has always been Gauss theorem; we will be interested in the quantity of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S\,,$$

where $S(t)$ is a time-dependent surface. Let us try the approach from Reynolds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{\Phi}^{-1}(t,\mathbf{x})}\psi(t,\mathbf{\Phi}(t,\mathbf{u}))\sqrt{\det \mathbb{g}_s}\,\mathrm{d}u^1\,\mathrm{d}u^2 =,$$

which is now a time-indepdent integral, meaning we can differentiate through. Let us first calculate the derivatives. Start slow:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{t}_\alpha}{\mathrm{d}t} &= \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{\Phi}}{\partial u^\alpha}(t,\mathbf{u})\right) = \frac{\partial}{\partial u^\alpha}\underbrace{\left(\frac{\partial\mathbf{\Phi}}{\partial t}(t,\mathbf{u})\right)}_{=\mathbf{v}_s(t,\mathbf{u})} = \frac{\partial}{\partial u^\alpha}\left(\mathbf{v}_\parallel + v_\perp\mathbf{n}\right) = \\
&= \frac{\partial\mathbf{v}_\parallel}{\partial u^\alpha} + \frac{\partial(v_\perp\mathbf{n})}{\partial u^\alpha} = \frac{\partial\left(v_\parallel^\beta\mathbf{t}_\beta\right)}{\partial u^\alpha} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} + v_\perp\frac{\partial\mathbf{n}}{\partial u^\alpha} = \\
&= \frac{\partial v_\perp^\beta}{\partial u^\alpha}\mathbf{t}_\beta + \frac{\partial\mathbf{t}_\beta}{\partial u^\alpha}v_\parallel^\beta + \frac{\partial v_{per}}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta}b_{\beta\alpha}\mathbf{t}_\gamma = \\
&= \frac{\partial v_\parallel^\beta}{\partial u^\alpha}\mathbf{t}_\beta + v_\parallel^\beta\Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + v_\parallel^\beta b_{\alpha\beta}\mathbf{n} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta}b_{\alpha\beta}\mathbf{t}_\gamma = \\
&= v_\parallel^\beta|_\alpha\mathbf{t}_\beta - v_\perp g^{\gamma\beta}b_{\alpha\beta}\mathbf{t}_\gamma + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n} = \\
&= \left(v_\parallel^\beta|_\alpha - v_\perp g^{\beta\gamma}b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n}.
\end{aligned}$$

So all in all

$$\partial_t\mathbf{t}_\alpha = \left(\nabla_\alpha v_\parallel^\beta - v_\perp g^{\beta\gamma}b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \partial_\alpha v_\perp\right)\mathbf{n}.$$

Next ingredient is the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{g}_s$, so in components:

$$\partial_t g_{\alpha\beta} = \partial_t\left(\mathbf{t}_\alpha\cdot\mathbf{g}_\beta\right) = \cdots = \nabla_\alpha v_\parallel^\delta g_{\delta\beta} + \nabla_\beta v_\parallel^\delta g_{\delta\alpha} - 2v_\perp b_{\alpha\beta}.$$

After some further manipulation, the final formula becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\psi(t,\mathbf{x})\,\mathrm{d}S = \int_{S(t)}\partial_t\psi(t,\mathbf{x}) + \psi(t,\mathbf{x})\left(\nabla\cdot\mathbf{v}_{\parallel S} - 2v_\perp(t,\mathbf{x})K(t,\mathbf{x})\right)\mathrm{d}S\,, \tag{9.17}$$

where

$$\left(\nabla\cdot\mathbf{v}_{\parallel S}\right)(t,\mathbf{x}) - 2v_\perp K \coloneqq \nabla_\beta v^\beta(t,\mathbf{u})_\parallel - 2v_\perp(t,\mathbf{u})K(t,\mathbf{u})\Big|_{\mathbf{u}=\mathbf{\Phi}^{-1}(t,\mathbf{x})}\,,$$

and

$$K = \frac{1}{2}g^{\beta\alpha}b_{\alpha\beta}$$

is the mean curvature.

9.3 Biharmonic equation

9.4 Convex analysis