Charles University

Faculty of Mathematics and Physics

Partial Differential Equations II

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Contents

Introduction

These notes are based on the course "Partial Differential Equations II" taught by doc. Kaplický during the summer semester of 2025. They were written and compiled by *a mere mortal student*, who certainly *makes many mistakes*, especially given the complexity of the topic. Readers are strongly advised that this is not a polished mathematical text, but rather the author's personal notes and interpretation of the lectures.

During exam preparation, some proofs were supplemented by other sources beyond doc. Kaplický's lectures. At the beginning of each such proof, the original source is indicated. Also, about three proofs are missing or are incomplete.

1 Winter semester addendum

1.1 Weak* convergence

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is reflexive, i.e.

$$\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*,$$

which means that balls in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ are weakly* compact. Moreover, $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is *separable*, from which it follows $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ with the weak* topology is metrizable and thus there exists a weakly * converging subsequence (from the balls).

Example (For people without Functional Analysis I)**.** Let X be a linear normed space, $\{x_n\} \subset X$ a sequence in X. We say x_n converges weakly to $x \in X$ whenever

$$f(x_n) \to f(x), \forall f \in X^*.$$

Let X^* be the topological dual to X, $\{f_n\} \subset X^*$ a sequence in X^*. We say f_n converges weakly* to $f \in X^*$ whenever

$$f_n(x) \to f(x), \forall x \in X^*,\ i.e.\, x(f_n) \to x(f),$$

where by $x(y), x \in X, y \in X^*$ we understand

$$\varepsilon_x : X^* \to \mathbb{K}, y \mapsto y(x).$$

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, every point $x \in \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ can be interpreted as a linear functional on $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, so given $\{x_n\} \subset \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, we can interpret is as a $\{x_n\} \subset \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, meaning given a weakly converging sequence in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, it is actually a weakly* converging sequence in $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$.

1.2 Regularity of parabolic problems

We are solving

$$\begin{aligned}\partial_t u - \nabla \cdot \mathbb{A}\nabla u + bu + \mathbf{c}\cdot\nabla u + \nabla\cdot(u\mathbf{d}) = f, &\text{ in } (0,T)\times\Omega,\\ u = 0, &\text{ on } (0,T)\times\partial\Omega,\\ u = u_0, &\text{ on } \{0\}\times\Omega,\end{aligned}$$

where $u:(0,T)\times\Omega\to\mathbb{R}$, $\Omega\in \mathrm{C}^{0,1}$, $\mathbb{A}$ uniformly elliptic on $(0,T)\times\Omega$, $\mathbb{A}, b, \mathbf{c}, \mathbf{d} \in \mathrm{L}_\infty\left((0,T)\times\Omega\right)$. The data are f, u_0, with minimal sensible regularity of

$$u_0 \in \mathrm{L}_2(\Omega)\,, f \in \mathrm{L}_2\Big((0,T);\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*\Big), \Omega\in \mathrm{C}^{0,1}.$$

We have shown that under these assumptions, there exists an unique weak solution

$$u \in \mathrm{L}_2\Big((0,T);\mathrm{W}_0^{1,2}(\Omega)\Big), \partial_t u \in \mathrm{L}_2\Big((0,T);\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*\Big)$$

We will now show that as in the elliptic case, we can hope for more regularity of the solution provided we provide more regularity of the data and the domain.

Theorem 1. *Let the assumptions of the previous theorem hold and let moreover* $\nabla\cdot\mathbf{d} \in L_\infty\left((0,T)\times\Omega\right), f \in L_2\Big((0,T);L_2(\Omega)\Big), \nabla\mathbb{A}, \partial_t\mathbb{A} \in L_\infty\left((0,T)\times\Omega\right)$. *Then the unique weak solution* u *satisfies for all* $\delta\in(0,1)$

$$\partial_t u \in L_2\Big((\delta,T);L_2(\Omega)\Big), u\in L_\infty\Big((\delta,T);\,W_0^{1,2}(\Omega)\Big)$$

and there exists $C>0$ *such that*

$$\|\partial_t u\|_{L_2((\delta,T);L_2(\Omega))} + \|u\|_{L_\infty\left((\delta,T);\,W_0^{1,2}(\Omega)\right)} \le \frac{C}{\delta}\Big(\|f\|_{L_2((0,T);L_2(\Omega))} + \|u_0\|_{L_2(\Omega)}\Big)$$

If moreover $u_0\in W_0^{1,2}(\Omega)$, *then*

$$\partial_t u \in L_2\Big((0,T);L_2(\Omega)\Big), u\in L_\infty\Big((0,T);\,W_0^{1,2}(\Omega)\Big),$$

and

$$\|\partial_t u\|_{L_2((0,T);L_2(\Omega))} + \|u\|_{L_\infty\left((0,T);\,W_0^{1,2}(\Omega)\right)} \le C\Big(\|f\|_{L_2((0,T);L_2(\Omega))} + \|u_0\|_{W_0^{1,2}(\Omega)}\Big).$$

Proof. (*Missing*) This proof is missing, but can be found in Bulíček, 2019a. One has to work with Galerkin approximations. □

Theorem 2. *Let* $\Omega\in C^{1,1}$. *If the assumptions of the above theorem hold*

- *with* $\delta\in(0,1)$, *then*
$$u\in L_2\Big((\delta,T);\,W_0^{2,2}(\Omega)\Big),$$
- *if moreover* $u_0\in W_0^{1,2}(\Omega)$, *then*
$$u\in L_2\Big((0,T);\,W_0^{2,2}(\Omega)\Big).$$

Proof. (*From: the lectures*) Take the weak formulation in $t\in(\delta,T)$. WLOG further assume $d=0$.

Then

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi - bu\varphi - \mathbf{c}\cdot\nabla u\varphi - \int_\Omega \partial_t u\varphi = \int_\Omega (f - bu - \mathbf{c}\cdot\nabla u - \partial_t u)\varphi,$$

and the integrand of the last integral is in $\mathrm{L}_2(\Omega)$ for a.e. $t \in (\delta, T)$. We can thus use the elliptic regularity results and write:

$$\|u\|^2_{\mathrm{W}^{2,2}(\Omega)} \leq C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\partial_t u\|^2_{\mathrm{L}_2(\Omega)}),$$

integrating both sides $\int_\delta^T \mathrm{d}t$ yields

$$\|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))} \leq C(\|f\|^2_{\mathrm{L}_2\left((\delta,T);\mathrm{L}_2(\Omega)\right)} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{W}^{1,2}(\Omega))} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))})$$

□

Theorem 3. *If data are smooth and satisfy the compatibility condiitons, then the weak solutions to the parabolic equation are smooth.*

Proof. (*From: the lectures*) No proof has been given. □

Remark (Compatibility condition). : Take the heat equation : $\partial_t u - \triangle u = f$ at time zero: $\triangle u(0) + f(0) = \partial_t u(0) \in \mathrm{W}^{1,2}_0(\Omega)$, so we need that $f(0) + \triangle u(0)$ has zero trace $\Rightarrow$ compatibility conditions.

1.3 Uniqueness of solutions to hyperbolic problems

Theorem 4 (Uniqueness of the solution to a hyperbolic equation)**.** *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that* $\mathbf{c} \in W^{1,\infty}(\Omega)$*. Then the weak solution to the hyperbolic equation is unique.*

Proof. (*From: Evans, 2010*) It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the equation, multiply it by $\varphi \in V$ fixed and integrate over Ω for $t \in (0,T)$ fixed:

$$< \partial_{tt} u(t), \varphi > + \int_\Omega \mathbb{A}(t)\nabla u(t)\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\varphi\,\mathrm{d}x - \int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\varphi\,\mathrm{d}x = 0.$$

Now, take a special test function

$$\psi(t) = \left(\int_t^s u(\tau)\,\mathrm{d}\tau\right)\chi_{(0,s)}(t),$$

for some $s \in (0,T)$. Then $\partial_t\psi(t) = -u(t)$ on $t \in (0,s)$. Next, integrate the equation in time over $(0,s)$.

$$\int_0^s < \partial_{tt} u(t), \psi > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t)\nabla u(t)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^s \int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t =$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s < \partial_{tt} u(t), \varphi > \mathrm{d}t =< \partial_t u(s), \psi(s) > - < \partial_t u(0), \psi(0) > - \int_0^s < \partial_t u(t), \partial_t\psi(t) > \mathrm{d}t\,,$$

and realize $\psi(s) = 0, \partial_t u(0) = 0$, so

$$-\int_0^s < \partial_t u(t), \partial_t \psi(t) > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t)\nabla u(t) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\psi \,\mathrm{d}x\,\mathrm{d}t - \int_0^s \int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\psi \,\mathrm{d}$$

but since $\partial_t\psi(t) = -u(t)$, we can actually write (time dependencies are omitted for brevity)

$$\int_0^s < \partial_t u, u > \mathrm{d}t + \int_0^s \int_\Omega -\mathbb{A}\nabla\partial_t\psi\cdot\nabla\psi - b\psi\partial_t\psi - \psi\mathbf{c}\cdot\nabla\partial_t\psi + \partial_t\psi\mathbf{d}\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2_{\mathrm{L}_2(\Omega)} - \int_\Omega \mathbb{A}\nabla\psi\cdot\nabla\psi + b\psi^2 + \psi\mathbf{c}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla\psi \,\mathrm{d}x\Big)\,\mathrm{d}t =$$

$$= \int_0^s \int_\Omega (\partial_t\mathbb{A})\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 + \psi\partial_t\mathbf{c}\cdot\nabla\psi + \underbrace{\partial_t\psi}_{=-u(t)}\,\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi - \psi\mathbf{d}\cdot\nabla\underbrace{\partial_t\psi}_{=-u(t)}\,)\,\mathrm{d}t\,\mathrm{d}x\,,$$

and upon integration (recall $\psi(s) = 0$, from the definition of ψ it follows $\nabla\psi(0) = 0$, and $u(0) = 0$,)

$$\frac{1}{2}\Big(\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathbb{A}(0)\nabla\psi(0)\cdot\nabla\psi(0) + b(0)\psi(0)^2 + \psi(0)\mathbf{c}(0)\cdot\nabla\psi(0) + \psi(0)\mathbf{d}(0)\nabla\psi(0)\,\mathrm{d}x\Big) =$$

$$= \int_0^s \int_\Omega \partial_t\mathbb{A}\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 - u\partial_t\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\,.$$

From this we obtain the following estimate:

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\psi(0)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C\Big(\int_0^s \|\psi\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big)\,\mathrm{d}t + \|\psi(0)\|^2_{\mathrm{L}_2(\Omega)},$$

where $C = C\Big(\|\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|b\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t b\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}\Big)$.
Define now the test function $\chi(t) = \int_0^t u(\tau)\,\mathrm{d}\tau$, and realize that in fact $\psi(t) = \chi(s) - \chi(t), \chi(0) = 0$.
Plugging this in the above inequalty yields

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s \|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big) + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)},$$

and using

$$\|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} = \|\chi(t) - \chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le 2\Big(\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)}\Big),$$

and the definition of $\chi(t)$, from which it follows

$$\|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le \int_0^s \|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\,,$$

we are allowed to write

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s 2\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big),$$

and so

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + (1 - 2sC)\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_1\Big(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big).$$

If we now choose $T_1 \in (0,T]$ small enough *s.t.* $1-2sC>0$ for $s \in (0,T_1]$, we finally obtain

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_2\Big(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big), \forall s \in (0,T_1],$$

which implies $u=0$ on $(0,T_1]$ by the Gronwall lemma: we have

$$\xi(t) \le \int_0^t \xi(s)\,\mathrm{d}s\,, \text{ for } a.a.\, t \in (0,T) \Rightarrow \xi(t) = 0\, a.e..$$

for $\xi \in \mathrm{L}_1((0,T))$ nonnegative[1]. If we now boostrap on $[T_1, 2T_1], [2T_1, 3T_1]$ etc., we obtain $u=0$ on $(0,T]$.

□

2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d$ open$\,, p \in [1,+\infty], k \in N$. We define

$$\mathrm{W}^{\mathrm{k,p}}(\Omega) = \Big\{ f \in \mathrm{L}_\mathrm{p}(\Omega)\,; D^\alpha f \in \mathrm{L}_\mathrm{p}(\Omega)\,, \forall |\alpha| \le k \Big\},$$

with the norm

$$\|f\|^p_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \|f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)} + \sum_{0<|\alpha|\le k} \|D^\alpha f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)}.$$

Recall that:

- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is Banach $\forall p$ and Hilbert for $p=2$.
- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is separable if $p<\infty$ and reflexive if $p>1, p<\infty$.

Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.

2.1 Tools from functional analysis

Definition 1 (Regularization kernel)**.** The function η is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\operatorname{supp}\eta \subset \mathrm{U}(0,1)$
- $\eta \ge 0$
- η is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$

Definition 2 (Regularization of a function)**.** Let η be a regularization kernel. Set[2]

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of $f \in \mathrm{L}_1(\Omega)_{\mathrm{loc}}$ by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

[1] In our case $\xi = \|u\|^2_{\mathrm{L}_2(\Omega)} + \|\chi\|^2_{\mathrm{W}^{1,2}(\Omega)}$.

[2] Another common choice is $\eta_k = k^d\eta(kx), k \in \mathbb{N}$.

Remark (Propertios of regularization). The regularization has the following properties:

- $f \in \mathrm{L_p}(\Omega) \Rightarrow f_\varepsilon \to f \,\mathrm{in}\, \mathrm{L_p}(\Omega)$ and also a.e
- $f \in \mathrm{L}_\infty(\Omega) \Rightarrow f_\varepsilon \to f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathrm{U}(x,\varepsilon)} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y$
- $\operatorname{supp} f_\varepsilon \subset \overline{U(\Omega,\varepsilon)}, f = 0 \,\mathrm{on}\, \mathrm{U}(x,\varepsilon) \Rightarrow f_\varepsilon(x) = 0$

Definition 3 ($\Omega' \subset\subset \Omega$)**.** $O \subset\subset \Omega$ means $\overline{O}$ is compact and $\overline{O} \subset \Omega$.

Definition 4 (Shift operator)**.** For $u \in \mathrm{L_p}(\Omega)\,, k \in \{1,\dots,d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

Lemma 1 (Approximation property of the shift operator)**.** *For $u \in L_p(\Omega)$, it holds $\tau_h u \to u$ in $L_p(\Omega)\,, h \to 0^+$.*

Lemma 2 (Partition of unity)**.** *Let $E \subset \mathbb{R}^d, \mathcal{G}$ be an open covering of E (possibly uncountable.) Then there exists a countable system $\mathcal{F}$ of nonnegative functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$ and*

1. *$\mathcal{F}$ is subordinate to $\mathcal{G}: \forall \varphi \in \mathcal{F} \exists U \in \mathcal{G} : \operatorname{supp}\varphi \subset U$*
2. *$\mathcal{F}$ is locally finite[3]: $\forall K \subset E$ compact$, \operatorname{supp}\varphi \cap K \neq \emptyset$ for at most finitely many $\varphi \in \mathcal{F}$.*
3. *$\sum_{\varphi\in\mathcal{F}} \varphi(x) = 1, \forall x \in E$.*

Proof. (*From: the lectures*) (Sketch) *Step 1 (If E is compact)*:

E compact $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}\, s.t.\, E \subset \bigcup_{j=1}^m U_j$. Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \cup_{j=1}^m K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set:

$$K_m = \Big\{ x \in U |\operatorname{dist}(x,\partial\Omega) \ge \frac{1}{m}, \|x\| \le m \Big\}.$$

Then clearly $K_1 \subset K_2 \dots$, and they "converge monotonously to U. Next, find $\phi_j \in C_c(U_j), \phi_j > 0 \,\mathrm{on}\, K_j$, e.g. $\phi_j = \theta(\operatorname{dist}(x,\partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

Step 2 (If E is open):

Approximate E by $K \subset E$ compact by the exhaustion argument, then the covering will enlarge from finite → countable (nontrivial reasoning). □

2.2 Density of smooth functions

Lemma 3 (Local approximation by smooth functions (using regularization))**.** *Assume $p \in [1,\infty), \Omega \subset \mathbb{R}^d$ open$, k \in \mathbb{N}, u \in W^{k,p}(\Omega)\,, \Omega_\varepsilon = \{x \in \Omega |\operatorname{dist}(x,\partial\Omega) > \varepsilon\}$. Then it holds*

1. *$D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ a.e. in $\Omega_\varepsilon, \forall |\alpha| \le k$*
2. *$u_\varepsilon \to u$ in $W^{k,p}(\Omega)_{\,loc}\,, \varepsilon \to 0^+$*

[3] In other words, φ_K is nonzero for at most finitely many $\varphi \in \mathcal{F} \Leftrightarrow$ points in K can be represented by finitely many functions $\varphi \in \mathcal{F}$.

Proof. (*From: the lectures*) First of all: (those are classical derivatives at the moment!)

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\Big(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y\Big) = \int_{\mathbb{R}^d} u(y)D^\alpha_x\eta_\varepsilon(x-y)\,\mathrm{d}y\,,$$

the integrable majorants are *e.g.* $\|\eta_\varepsilon\|_\infty |u|\chi_{\mathrm{U}(0,\varepsilon)}(x) \in \mathrm{L}_1(\Omega)$. Now picking $x \in \Omega_\varepsilon$ we realize $\forall y \in \mathbb{R}^d/\overline{\Omega} : x-y \geq \operatorname{dist}(x,\partial\Omega) \geq \varepsilon$, and so $\eta_\varepsilon(x-y) = 0$. Meaning the integrand is zero on the complement of Ω, and since η_ε has a compact support in Ω, we can integrate over $\Omega \supset \Omega_\varepsilon$ instead. Exchanging derivatives and using the definition of the weak derivative then yields

$$\begin{aligned}\int_\Omega u(y)D^\alpha_x\eta_\varepsilon(x-y)\,\mathrm{d}y = (-1)^{|\alpha|}\int_\Omega u(y)D^\alpha_y\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_\Omega D^\alpha_y u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \\ = \int_{\mathbb{R}^d} D^\alpha_y u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = (D^\alpha u)_\varepsilon.\end{aligned}$$

Take $V \subset\subset \Omega$ open, then

$$\|u-u_\varepsilon\|_{\mathrm{W}^{\mathrm{k,p}}(V)} = \sum_{|\alpha|\leq k}\|D^\alpha u - D^\alpha u_\varepsilon\|_{\mathrm{L_p}(V)} \to 0,$$

because $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \to D^\alpha u \,\text{in}\, \mathrm{L_p}(V)$, from the properties of regularization. □

Theorem 5 (Global approximation by smooth functions)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}, p \in [1,\infty)$. Then $C = \Big\{f \in C^\infty(\Omega), \operatorname{supp} f \text{ bounded}\Big\} \bigcap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, i.e.*

$$\overline{C \bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

If moreover Ω is bounded, it holds:

$$\overline{C^\infty(\Omega)\bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

Proof. (*From: the lectures*) Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)\,, \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega)\bigcap \mathrm{W}^{\mathrm{k,p}}(\Omega)$ *s.t.* $\|u-v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$. For every $j \in \mathbb{N}$ define an open set

$$\Omega_j = \Big\{x \in \Omega, \operatorname{dist}(x,\partial\Omega) > \frac{1}{j}\Big\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$. Next, set

$$U_j = \Omega_{j+1}/\overline{\Omega_{j-1}}, j = 1,2,\ldots,$$

where $\Omega_0 = \Omega_{-1} = \varnothing$. Since Ω_j are open, U_j are also open and $\Omega \subset \bigcup_{j\in\mathbb{N}} U_j \Rightarrow \exists\{\varphi_j\}_{j\in\mathbb{N}}$ partition of unity subordinate to $\{U_j\}_{j\in\mathbb{N}}$. We can write $u = \sum_{j\in\mathbb{N}} u\varphi_j$, where $u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega)\,, \operatorname{supp} u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$: set $v_j = (u\varphi_j)_{\varepsilon_j}$ and fix an arbitrary $\delta > 0$. By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(U)} < \frac{\delta}{2^{j-1}},$$

for any $U \subset\subset \Omega$, for $\varepsilon_j > 0$ sufficiently smalls. In fact, if needed, make the ε_j smaller so that

$$\operatorname{supp} v_j \subset \Omega_{j+2}/\overline{\Omega_{j-2}}.$$

That is completely possible: from the properties of regularization, we know

$$\operatorname{supp} v_j \subset \mathrm{B}(0,\varepsilon) + \overline{U_j},$$

and since

$$\overline{U_j} = \overline{\Omega_{j+1}}/\Omega_{j-1} \subset \overline{\Omega_{j+2}}/\Omega_{j-2} \subset \overline{\overline{\Omega_{j+2}}/\Omega_{j-2}} = \Omega_{j+2}/\overline{\Omega_{j-2}},$$

the compact set $\overline{U_j}$ is contained in the open set $\Omega_{j+2}/\overline{\Omega_{j-2}}$, meaning with ε_j small, the set $\overline{U_j} + \mathrm{B}(0,\varepsilon_j)$ will still be in $\Omega_{j+2}/\overline{\Omega_{j-2}}$. Also, let us take possibly ε_j even smaller to have a nice inequality: for fixed $N \in \mathbb{N}$:

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(U)} < \frac{1}{2 - \left(\frac{1}{2}\right)^{N+1}} \frac{\delta}{2^{j-1}},$$

meaning of $N \in \mathbb{N}$ will be evident later.

Set

$$v = \sum_{j\in\mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms ($\mathcal{F}$ is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u - v\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega_N)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u - v)(x) = \sum_{j=1}^{N+2} (u\varphi_j - v_j),$$

because for $m > N + 2$, *i.e.*, $m - 2 > N$ it holds the functions u_m, v_m, have their supports in

- $\operatorname{supp} u_m \subset U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}$,
- $\sup v_m \subset \Omega_{m+2}/\overline{\Omega_{m-2}}$,

but $\Omega_{m-1} \subset \Omega_{m-2} \subset \Omega_N$, for $N < m - 2$, meaning that the set Ω_N does not lie in the supports of those functions. The norm of sum is (recall $\Omega_N \subset\subset \Omega$, so the above estimate holds)

$$\|u - v\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega_N)} \le \sum_{j=1}^{N+2} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega_N)} < \delta \frac{1}{2 - \left(\frac{1}{2}\right)^{N+1}} \sum_{j=1}^{N+2} \frac{1}{2^j} = \delta.$$

It only remains to let $N \to \infty$ and realize

$$\|u - v\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N\in\mathbb{N}} \int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x = \sup_{N\in\mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \sup_{N\in\mathbb{N}} |D^\alpha f| \chi_{\Omega_N} \,\mathrm{d}x \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega}(x) \,\mathrm{d}x = \int_{\Omega} |D^\alpha f| \,\mathrm{d}x\,,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Alltogether,

$$\|u - v\|_{\mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega)} \le \delta, \forall \delta > 0$$

from which it follows $v \in \mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega)$ (this was not totally evident) and thus $v \in \mathrm{W}^{\mathrm{k},\mathrm{p}}(\Omega) \cap C^\infty(\Omega)$

so indeed we have showed the desired density. □

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Remark ($\mathrm{C}^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $\mathrm{C}^{k,\lambda}$ if: Ω is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0,1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : \mathrm{B}(0,\alpha) \subset \mathbb{R}^{d-1} :\to \mathbb{R}$ *s.t.* $a_j \in \mathrm{C}^{k,\lambda}(\mathrm{B}(0,\alpha)), \exists \mathbb{A}_j \mathbb{R}^d \to \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \le m : \varnothing \ne \partial\Omega \cap U_j = \mathbb{A}_j\left(\left\{(x', a_j(x') \in \mathbb{R}^d | x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}\right\}\right)$,
3. $\forall j \le m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \Omega$,
4. $\forall j \le m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$.

If $\lambda = 0$ we sometimes drop it and write $\Omega \in \mathrm{C}^{k,0} \Leftrightarrow \Omega \in \mathrm{C}^k$, if $k = 0, \lambda = 1$ we call $\Omega \in \mathrm{C}^{0,1}$ to be a Lipschitz domain. *Remember that $\lambda(\Omega) < \infty$ is a part of the definition.*

Theorem 6 (Global approximation by smooth functions up to the boundary)**.** *Let $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1,\infty)$. Then $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. (*From: Bulíček et al., 2018*) Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in \mathrm{C}^\infty(\overline{\Omega})$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\overline{\Omega}$,
2. partition of unity,
3. approximation of u on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^m$ be from the definition of $\mathrm{C}^{0,0}$ boundary. Then[4]

$$\overline{\Omega} \subset \bigcup_{j=0}^m U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to $\{U_j\}_{j=0}^m$. Since

$$u = \sum_{j=0}^m u\varphi_j, \text{ on } \Omega$$

observe that $u_j := u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, $\operatorname{supp} u_j \subset \operatorname{supp} \varphi_j \subset\subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d / \Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \dots, m\}$.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of Ω, it holds[5] $u\varphi_0 \in \mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)$. Since $\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d) = \mathrm{W}_0^{\mathrm{k,p}}(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)}}$, we can find $v_0 \in \mathcal{D}(\mathbb{R}^d)$ *s.t.*

$$\|v_0 - u_0\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} = \|v_0 - u\varphi_0\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

Notice that naturally

$$\mathcal{D}(\mathbb{R}^d) \subset \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d),$$

[4] Our choice $U_0 = \Omega$ is important, as without it the definition of $\mathrm{C}^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^m U_j$.
[5] This would not hold if the support were not compactly contained in Ω.

and so $v_0 \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$.

Case $j \in \{1, \dots, m\}$. We have a problem now: $\{U_j\}_{j=1}^m$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of Ω, where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate u_j on a larger *open* domain containing $\overline{\Omega}$ and then show this is also a good aproximation of u_j on $\Omega \subset \overline{\Omega}$.

Set $u_j = u\varphi_j$, and denote

$$\begin{aligned} S_j &= \mathbb{A}_j\Big(\Big\{(x', x_d) | a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in \mathrm{U}(0,\alpha)\Big\}\Big), \\ \Omega_j &= \mathbb{R}^d / \overline{S_j}, \end{aligned}$$

Realize that since $u = 0$ outside of Ω, also u_j is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega_j)$. Now pick $\delta \in \left(0, \frac{\beta}{2}\right)$, where β is from the definition of $\mathrm{C}^{0,0}$ and set

$$\begin{aligned} S_j^\delta &= \mathbb{A}_j\Big(\Big\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\Big\}\Big), \\ \Omega_j^\delta &= \mathbb{R}^d / \overline{S_j^\delta}, \end{aligned}$$

The trick is to shift the (support of) function u_j "into" Ω_j^δ

$$\tau_\delta u_j\big(\mathbb{A}_j\big(x', a_j(x')\big)\big) = u_j\big(\mathbb{A}_j\big(x', a_j(x') + \delta\big)\big), x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\operatorname{supp}(\tau_\delta u_j) = \operatorname{supp}(u_j) - \mathrm{B}(0,\delta),$$

from which it follows $\tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\big(\Omega_j^\delta\big)$; we have only shifted the function u_j, but since we have also shifted S_j, qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta$ and $\Omega \subset \Omega_j^\delta \cap \Omega_j$, and also $\Omega \subset \Omega_j$, $\Omega \subset \Omega_j^\delta \cap \Omega_j$, and the fact τ_δ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\big(\Omega_j \cap \Omega_j^\delta\big)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ *s.t.*

$$\|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to u_j, we are done once we approximate $\tau_\delta u_j$ by a function from $\mathrm{C}^\infty\left(\overline{\Omega}\right)$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $\mathrm{C}^\infty\left(\Omega_j^\delta\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right)$.

Show $\Omega \subset\subset \Omega_j^\delta$*:* We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k\big(\big\{(x', x_d) | x_d = a_k(x'), x' \in \mathrm{U}(0,\alpha)\big\}\big),$$

and the set Ω_j^δ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j\Big(\Big\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\Big\}\Big).$$

Realize it suffices to show $\partial\Omega \notin \overline{S_j}$, as then it wont be excluded from $\mathbb{R}^d$ and thus will end up in

Ω_j^δ. *Thanks to continuity of* a_j, we may write

$$\overline{S_j} = \mathbb{A}_j\bigg(\bigg\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\bigg\}\bigg),$$

i.e., the " $<$ " have changed to " $\le$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j\big(\big\{(x', x_d)|x_d = a_j(x'), x' \in \mathrm{U}(0,\alpha)\big\}`\big) \notin \mathbb{A}_j\bigg(\bigg\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\bigg\}\bigg),$$

which is equivalent to

$$\bigg((a_j \le a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta)\bigg) \vee \bigg((a_j > a_j - \delta) \wedge (a_j \ge a_j - \frac{\beta}{2} - \delta)\bigg).$$

Our choice has been $\delta \in \left(0, \frac{\beta}{2}\right)$, and $\beta > 0$ from the definition of $\Omega \in \mathrm{C}^{0,0}$, so the second statement is clearly true $\forall j \in 1, \ldots, m$. Consequently $\partial\Omega \notin \overline{S}_j$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

Approximation of $\tau_\delta u_j$. Since Ω_j^δ is open, by the local approximation theorem there $\exists v_j \in \mathrm{C}^\infty\left(\mathbb{R}^d\right)$ such that for any $U \subset\subset \Omega_j^\delta$:

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{k,p}(U)} \frac{\varepsilon}{2(m+1)},$$

and so in particular, (as we have shown above $\Omega \subset \overline{\Omega} \subset \Omega_j^\delta$,)

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{2(m+1)}.$$

Realize that the function v_j is smooth on all $\mathbb{R}^d$, so in particular $v_j \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$.

Approximation of u.

Finally, let us set

$$v = \sum_{j=0}^m v_j.$$

Then $v \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ and it holds

$$\begin{aligned}
\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} &= \left\|\sum_{j=0}^m u_j - \sum_{j=0}^m v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \left\|\sum_{j=0}^m u_j - v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \sum_{j=0}^m \|u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \\
&\le \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} + \sum_{j=1}^m \|\tau_\delta u_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \\
&< \frac{\varepsilon}{m+1} + 2\sum_{j=1}^m \frac{\varepsilon}{2(m+1)} = \varepsilon
\end{aligned}$$

This proof may still have some flaws, but the author has decided to move on. □

Remark (What is $\mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$). Recall

$$C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \left\{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\right\}.$$

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, does there exist

$$\overline{u} \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right),\ s.t.\, \overline{u}|_\Omega = u,\ \|\overline{u}\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

?

The answer is **yes**, if Ω is nice enough. Notice however this is not as simple as in the case of Lebesgue spaces, where we could just extend the function by zero trivailly. We are dealing with derivatives, that are somehow regular, and if we extend a nonzero function by zero, it might mess up the regularity of the derivatives.

We will be using C^1 diffeomorphisms heavily, so we investigate some of their properties first.

Lemma 4 (Properties of C^1 diffeomorphisms)**.** *Let $U, V \subset \mathbb{R}^d$ be open, $\Phi : U \to V$ be a C^1 diffeomorphism and let $\tilde{U} \subset \mathbb{R}^d$ s.t. $\tilde{U} \subset\subset U$. Then*

1. $\Phi(\tilde{U}) \subset\subset V$,
2. *if moreover $\tilde{U}$ is compact, then* [6]

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \leq C\|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

Proof. (*From: the lectures*)

Ad 1.: No proof has been given.

Ad 2.: Just a change of variables formula:

$$\|u \circ \Phi\|^p_{\mathrm{L_p}(\tilde{U})} = \int_{\tilde{U}} |u \circ \Phi|^p \,\mathrm{d}x = \int_{\Phi(\tilde{U})} |u|^p |\det \nabla\Phi|^{-1} \,\mathrm{d}x\,.$$

Since Φ is one-to-one, we know $|\det \nabla\Phi| > 0$ on U, and since $\Phi \in \mathrm{C}^1(U)$ and $\tilde{U} \subset U \Rightarrow \Phi \in \mathrm{C}^1\left(\tilde{U}\right) \Rightarrow \det\nabla\Phi \in \mathrm{C}^0\left(\tilde{U}\right)$, and since $\tilde{U}$ is compact, $|\det \nabla\Phi| \geq C_1 > 0 \Leftrightarrow |\det\nabla\Phi|^{-1} \leq \frac{1}{C_1}$. In total

$$\|u \circ \Phi\|^p_{\mathrm{L_p}(\tilde{U})} \leq \frac{1}{C_1}\int_{\Phi(\tilde{U})} |u|^p \,\mathrm{d}x = C\|u\|^p_{\mathrm{L_p}(\Phi(\tilde{U}))}.$$

As for the derivative, we have $\forall i \in \{1, \dots, d\}$:

$$\begin{aligned}\int_{\tilde{U}} |\partial_i(u \circ \Phi)|^p \,\mathrm{d}x \leq \int_{\tilde{U}} |\nabla(u\circ\Phi)|^p \,\mathrm{d}x = \int_{\tilde{U}} \left|\nabla\Phi\Big((\nabla u)\circ\Phi\Big)\right|^p \mathrm{d}x \leq \\ \leq \|\nabla\Phi\| \int_{\tilde{U}} |(\nabla u)\circ\Phi|^p \,\mathrm{d}x = \|\nabla\Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p |\det\nabla\Phi|^{-1}\,\mathrm{d}x \leq C\|\nabla\Phi\|\int_{\Phi(\tilde{U})} |\nabla u|^p \,\mathrm{d}x \leq \\ \leq C\|\nabla u\|^p_{\mathrm{L}_p(\Phi(\tilde{U}))},\end{aligned}$$

where $\|\nabla\Phi\|$ is *e.g.* the operator norm of the matrix $\nabla\Phi$. □

Lemma 5 (Flat extension)**.** *Let $\alpha, \beta > 0, K \subset\subset U(0,\alpha) \times [0,\beta)$ be compact. Then there $\exists C > 0$, a linear operator*

$$E : C^1\left((B(0,\alpha) \times [0,\beta])\right) \to C^1\left((B(0,\alpha)\times[-\beta,\beta])\right),$$

and the set $\tilde{K} \subset\subset U(0,\alpha)\times[-\beta,\beta)$ such that $\forall u \in C^1\left(B(0,\alpha)\times[0,\beta]\right)$ it holds

1. *$Eu = u$ on $B(0,\alpha)\times[0,\beta]$,*

[6] For $\tilde{U}$ compact: $\tilde{U} \subset\subset V \Leftrightarrow \tilde{U} \subset V$.

2. $\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \le \|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}$. *Actually,*

$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)),W^{1,p}(U(0,\alpha)\times(-\beta,\beta)))} = 2^{\frac{1}{p}}$$

3. *if* $\operatorname{supp} u \subset K$ *then* $\operatorname{supp} Eu \subset \tilde{K}$

Proof. (*From: the lectures*) (The set $\mathrm{U}(0,\alpha)\times[0,\beta)$ is a cylinder of radius α and height β)

The proof is constructive: for the assumed u we write ($x = (x', x_d)$, where $x' \in \mathrm{B}(0,\alpha) \subset \mathbb{R}^{d-1}, x_d \in [0,\beta] \subset \mathbb{R}$)

$$Eu(x', x_d) = \begin{cases} u(x', x_d), & x_d \ge 0 \\ -3u(x', -x_d) + 4u(x', -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Does Eu lie in $\mathrm{C}^1(\mathrm{B}(0,\alpha)\times[-\beta,\beta])$? Since $u \in \mathrm{C}^1(\mathrm{B}(0,\alpha)\times[0,\beta])$, it us continuous in the "lower cylinder", check only the transition through the origin plane: take some $a = (x', 0) \in \mathrm{B}(0,\alpha)\times\{0\}$. Then

$$\lim_{x\to a} Eu(x) = \begin{cases} u(a), & x_d \ge 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so Eu is continuous. The derivatives

$$\lim_{x\to a} \partial_d Eu(x', x_d) = \begin{cases} \partial_d u(x', 0), & x_d \ge 0 \\ \left(-3\partial_d u(x', -x_d)(-1) + 4\partial_d u(x', -\frac{x_d}{2})(-\frac{1}{2})\right)\Big|_{x_d=0} = \partial_d u(x', 0), & x_d < 0, \end{cases}$$

and also for any $i \in \{1, \dots, d-1\}$

$$\lim_{x\to a} \partial_i Eu(x_1, \dots, x_d) = \begin{cases} = \partial_i u(x_1, \dots, 0), & x_d \ge 0 \\ = -3\partial_i u(x_1, \dots, 0) + 4\partial_i u(x_1, \dots, 0) = \partial_i u(x_1, \dots, 0) \end{cases}$$

so the the derivative is also continuous. Thus, we have

$$Eu \in C^1 \subset \mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta)).$$

The first property is clear from the definition of Eu, the estimates of the norm: (all derivatives can in fact be assumed classical)

$$\begin{aligned}
\|Eu\|^p_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} &= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = \\
&= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} \\
&+ + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1} \|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} = \\
&= \|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|4u - 3u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} + \\
&+ \sum_{|\alpha|=1} \|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1} \|D^\alpha (4u - 3u)\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} \\
&= 2\|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + 2\sum_{|\alpha|=1} \|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} = 2\|u\|^p_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))}
\end{aligned}$$

and so

$$\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} = 2^{\frac{1}{p}}\|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}.$$

Where we have used the obvious fact

$$\int_{U(0,\alpha)\times(0,\beta)} |f|^p \, dx = \int_{U(0,\alpha)\times(-\beta,0)} |f|^p \, dx\,,$$

We will assume (although this is *with* a loss of generality), that

$$Eu\big(x', x_d\big) = 0 \Leftrightarrow (u(x', x_d) = 0 \vee 3u(x', -x_d) = 0 \vee 4u(x', -\frac{x_d}{2}) = 0),$$

i.e., that support lies in

$$\operatorname{supp} Eu \subset \big\{(x', x_d) \in U(0,\alpha) \times [0,\beta] | u(x', x_d) = 0\big\} \cup \big\{(x', x_d) \in U(0,\alpha) \times [-\beta, 0] | u(x', -x_d) \neq 0\big\} \cup$$
$$\cup \left\{(x', x_d) \in U(0,\alpha) \times [-\beta, 0] | u(x', -\frac{x_d}{2}) \neq 0\right\}.$$

Denote $\Phi_1, \Phi_2 : U(0,\alpha) \times (0,\beta) \to U(0,\alpha) \times (-\beta,\beta)$ to be the mappings

$$\Phi_1(x', x_d) = (x', -x_d),$$
$$\Phi_2(x', x_d) = (x', -\frac{x_d}{2}),$$

then clearly Φ_1, Φ_2 are C^1 diffeomorphisms and

$$u(x', -x_d) = (u \circ \Phi_1)(x', x_d), u\left(x', -\frac{x_d}{2}\right) = (u \circ \Phi_2)\big(x', x_d\big),$$

i.e.,

$$u\big(x', x_d\big) = \big(u \circ \Phi_1^{-1}\big)\big(x', -x_d\big) = \big(u \circ \Phi_2^{-1}\big)\left(x', -\frac{x_d}{2}\right),$$

and so

$$\operatorname{supp} Eu \subset \operatorname{supp} u \cup \Phi_1^{-1}(\operatorname{supp} u) \cup \Phi_2^{-1}(\operatorname{supp} u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as $\operatorname{supp} u \subset\subset K$. Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\operatorname{supp} Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have $K \subset\subset U(0,\alpha) \times [0,\beta) \Rightarrow K \subset\subset U(0,\alpha) \times (-\beta,\beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset\subset U(0,\alpha) \times (0,\beta)$, which really implies

$$\tilde{K} \subset\subset U(0,\alpha) \times [-\beta,\beta).$$

□

Let us prove the main result.

Theorem 7 (Extension of Sobolev functions). *Let $\Omega \in C^{k-1,1}, k \in \mathbb{N}, p \in [1,\infty], V \subset \mathbb{R}^d$ open such that $\Omega \subset\subset V$. Then there is $E : W^{k,p}(\Omega) \to W^{k,p}\big(\mathbb{R}^d\big)$ bounded linear operator such that*

1. $\forall u \in W^{k,p}(\Omega) : Eu = u$ *a.e. in* Ω,
2. $\forall u \in W^{k,p}(\Omega) : \operatorname{supp} Eu \subset V \supset\supset \Omega$,

3. $\|E\|_{\mathcal{L}(W^{k,p}(\Omega),W^{k,p}(\mathbb{R}^d))} \le C, C = C(p,\Omega,V)$.

Proof. (*From: the lectures, with some incompletions*) **Will only be presented for** $k = 1, \Omega \in C^1, p < \infty$**.** The strategy is:

1. covering of $\overline{\Omega}$ & partition of unity
2. obtain a diffeomorphism from the fact $\Omega \in \mathrm{C}^{1,0}$,
3. suitable composition & cut off,
4. flat extension,
5. show existence of $E : \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) \to \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)$ with the desired properties,
6. extend via density

Covering of Ω*:* In the following, $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ are as in the definition of a $\mathrm{C}^{1,0}$ domain. Set $U_0 = \Omega$ and realize

$$\overline{\Omega} \subset \bigcup_{j=0}^m U_j,$$

i.e., $\{U_j\}_{j=0}^m$ is an open covering of $\overline{\Omega}$. Denote $\{\varphi_j\}_{j=0}^m$ as the partition of unity subordinate to $\{U_j\}_{j=0}^m$.

Diffeomorphism & flat extension For $j \in \{1,\dots,m\}$ we define $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to U_j$ by

$$\Phi_j\left(y', y_d\right) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

(A bit confusingly, we will however be "interpreting" Φ_j as $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to \mathbb{R}^d$, with it being extended by zero on $\mathbb{R}^d / \overline{U_j}$, as we need Φ_j^{-1} to be defined on the whole $\mathbb{R}^d$.)

As $\Omega \in \mathrm{C}^{1,0}$, we know $a_j \in \mathrm{C}^1\left(\mathrm{B}(0,\alpha)\right) \subset \mathrm{C}^1\left(\mathrm{U}(0,\alpha)\right)$, and so ϕ_j is a C^1 diffemorphism. Let us denote by $\tilde{E} : \mathrm{C}^1\left(\mathrm{B}(0,\alpha) \times [-\beta,\beta]\right) \to \mathrm{C}^1\left(\mathrm{B}(0,\alpha) \times [-2\beta,\beta]\right)$ the extension operator from the Flat extension lemma. Then we for any $u \in C^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) : u = \sum_{j=1}^m \varphi_j u$ define

$$Eu = \varphi_0 u + \sum_{j=1}^m \left(\eta \tilde{E}((\varphi_j u) \circ \Phi_j)\right) \circ \Phi_j^{-1},$$

where $\eta \in \mathrm{C}^\infty\left(\mathrm{U}(0,\alpha) \times \mathbb{R}\right)$ is a cut-off function

$$\eta\left(y', y_d\right) \begin{cases} = 1 \text{ on} & y_d \ge 0, \\ = 0 \text{ on} & y_d \le -2h, , \\ \in (0,1) & \text{else}. \end{cases}$$

for some parameter $h > 0$ which will be defined later. With this definition, $\forall x \in \Omega$ it holds

$$(Eu)(x) = (\varphi_0 u)(x) + \sum_{j=1}^m \left(\eta \tilde{E}(u\varphi_j \circ \Phi_j)\right) \underbrace{\left(\Phi_j^{-1}(x)\right)}_{\in \mathrm{U}(0,\alpha)\times(-\beta,\beta)} =$$

$$= (\varphi_0 u)(x) + \sum_{j=1}^m (u\varphi_j \circ \Phi_j)\left(\Phi_j^{-1}(x)\right) = (\varphi_0 u)(x) + \sum_{j=1}^m (u\varphi_j)(x) = \sum_{j=0}^m (u\varphi_j)(x) = u(x) + \sum_{j=0}^m \varphi_j(x) = u(x)$$

since $\eta(y) = 1, \tilde{E}(\varphi_j u \circ \Phi_j)(y) = (\varphi_j u \circ \Phi_j)(y)$ for $y \in \mathrm{U}(0,\alpha) \times (-\beta,\beta)$, according to our definition of η and the properties of the extension $\tilde{E}$.

The motivation behind the cutoff is the following: we know $\operatorname{supp}\varphi_j \subset U_j$, so since $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to U_j$, we have $\operatorname{supp} u\varphi_j \circ \Phi_j \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta)$ and from the properties of the flat extension operator we also have $\operatorname{supp} \tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-2\beta,\beta)$. Since moreover $\operatorname{supp}\eta = \mathrm{U}(0,\alpha) \times (-2h,\infty)$, in total

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) = \operatorname{supp}\eta \cap \operatorname{supp}\tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h,\infty) \cap (-\beta,\beta),$$

and also [7]

$$\operatorname{supp} Eu \subset \Phi_j\big(\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j)\big).$$

We need to prove $\operatorname{supp} Eu \subset V$, where V is some neighbourhood of Ω. If it holds

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta),$$

the desired property holds, as then

$$\operatorname{supp} Eu \subset \operatorname{supp} u_0 \cup \bigcup_{j=1}^{m} \Phi_j\big(\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j)\big) \subset U_0 \cup \bigcup_{j=1}^{m} \Phi_j(\mathrm{U}(0,\alpha) \times (-\beta,\beta)) \subset \bigcup_{j=0}^{m} U_j,$$

and from the assumptions $\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j$, and the union is open. Meaning, the support is contained within an open set, in which Ω is compactly contained.[8] In total, if

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h,\infty) \cap (-\beta,\beta) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta).$$

So if h is such that $-2h > -\beta \Leftrightarrow h < \frac{\beta}{2}$, we can guarantee $\operatorname{supp} Eu \subset \bigcup_{j=0}^{m} U_j$ and that is what we want.

Finally, E is clearly linear, its norm: (we are using the lemma about flat extension and the properties of C^1 diffeomorphisms together with the facts $\eta \le 1$ on $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, $\Phi_j(\mathrm{U}(0,\alpha) \times (0,\beta)) \subset U_j, \Phi_j^{-1}(\mathbb{R}^d) \subset$ [9], $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, [10])

$$\begin{aligned}\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &= \big\|\big(\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big) \circ \Phi_j^{-1}\big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\big\|\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = \\ &= C\big\|\tilde{E}(u\varphi_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} \le C\|u\varphi_j \circ \Phi_j\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))} \le \\ &\le C\|u\varphi_j\|_{\mathrm{W}^{1,p}(U_j \cap \Omega)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},\end{aligned}$$

from which it clearly follows $\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))} \le C$.

So all the properties hold for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$, let us show them also for $u \in \mathrm{W}^{1,p}(\Omega)$. Pick an arbitrary $u \in \mathrm{W}^{1,p}(\Omega)$, find $\{u_k\} \subset C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u_k \to u \text{ in } \mathrm{W}^{1,p}(\Omega)$. Since E is continous, we know $\lim_{k\to\infty} Eu_k$ exists. Let us set

$$Eu := \lim_{k\to\infty} Eu_k,$$

where Eu_k is defined above for $u_k \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$.

Ad 3): Clearly $E(\alpha u) = \lim_{k\to\infty} E(\alpha u_k) = \alpha Eu, E(u+v) = \lim_{k\to\infty} E(u_k + v_k) = \lim_{k\to\infty} Eu_k + Ev_k = Eu + Ev$, since E on the continous functions is linear. Also

[7] Generally, $\operatorname{supp} v \circ \Psi^{-1} = \overline{\{x \in \mathbb{R}^d | u(\Psi^{-1}(x)) \ne 0\}} = \overline{\{x \in \mathbb{R}^d | \Psi^{-1}(x) \in \operatorname{supp} u\}} = \overline{\{x \in \mathbb{R}^d | x \in \Psi(\operatorname{supp} u)\}} = \Psi(\operatorname{supp} u)$.

[8] This might not exactly be the formulation of the theorem, but is pretty close. But come on, we are already proving something different then the original formulation...

[9] We can somehow extend Φ_j^{-1} by zero from $U_j \subset \Omega \subset \mathbb{R}^d$ to be defined on the whole $\mathbb{R}^d$ (i guess)

[10] ...even though the assumptions to use those are not totally valid... but doc. Kaplicky is okay with that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} = \left\|\lim_{k\to\infty} Eu_k\right\|_{W^{1,p}(\mathbb{R}^d)} = \lim_{k\to\infty} \|Eu_k\|_{W^{1,p}(\mathbb{R}^d)} \le \|E\| \lim_{k\to\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^d)} = \|E\|\|u\|_{W^{1,p}(\mathbb{R}^d)},$$

(we are using $\{Eu_k\}$ has a limit); we see our above definition truly yields a continuous linear operator.
Ad 1): $\forall\, a.a.\, x \in \Omega : Eu(x) = \lim_{k\to\infty} Eu_k(x) = \lim_{k\to\infty} u_k(x) = u(x)$,
Ad 2): $\operatorname{supp} Eu_k \subset \mathrm{U}(\Omega,\varepsilon) \Rightarrow \operatorname{supp} Eu \subset \overline{\mathrm{U}(\Omega,\varepsilon)} \subset V$.

$$\operatorname{supp} Eu = \left\{x \in \mathbb{R}^d \middle| \lim_{k\to\infty} Eu_k \neq 0\right\} \subset \bigcap_{k=1}^{\infty} \underbrace{\operatorname{supp} Eu_k}_{\subset V} \subset V.$$

We are done.

□

Remark ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $\Omega \in \mathrm{C}^{0,1}$ and $p \in (1,\infty), k \in \mathbb{N}$, but the construction of the extension must be different. "It seems the result is not known for $\Omega \in \mathrm{C}^{0,1}$ and $p = 1$, or $p = \infty$."

2.4 Embedding theorems

From the definition of $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ it immediately follows $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L}_{\mathrm{p}}(\Omega)$. Can we obtain $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L}_q(\Omega)$ for some $q > p$? The answer **yes**, if Ω is again nice enough (and there will also be some dependence on the dimension of $\mathbb{R}^d$.)

2.4.1 Theorems for $p \le d$

Example. Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1,x_2) = \int_{-\infty}^{x_1} \partial_1 u(s,x_2)\,\mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1,s)\,\mathrm{d}s\,,$$

so

$$\begin{aligned}
\|u\|^2_{\mathrm{L}_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \\
&\le \left(\int_{\mathbb{R}}\int_{-\infty}^{x_1} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{-\infty}^{x_2} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) = \\
&= \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right) = \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)^2 = \\
&= \|\nabla u\|^2_{\mathrm{L}_1(\mathbb{R}^2)},
\end{aligned}$$

so

$$\|u\|_{\mathrm{L}_2(\mathbb{R}^2)} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^2)}.$$

This can be generalized in two ways:

- $d > 2$,

- less smoothness.

Lemma 6 (Gagliardo)**.** *Let $d \geq 2$. Let $\hat{u}_i : \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative and measurable for $j \in \{1, \dots, d\}$. We define*

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), \hat{\mathrm{dx}}_j = \mathrm{d}x_1 \dots \mathrm{d}x_{j-1} \, \mathrm{d}x_{j+1} \dots \mathrm{d}x_d \, .$$

Consider the functions $u_j : \mathbb{R}^d \to \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x) \, \mathrm{d}x \leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} \hat{\mathrm{dx}}_j \right)^{\frac{1}{d-1}}. \tag{1}$$

(Both integrals can be infinity.)

Proof. (*From: the lectures*) Induction by dimension: *The case $d = 2$.*:

$$\int_{\mathbb{R}^2} u_1(x_1, x_2) u_2(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2) \hat{u}_2(x_1) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \left(\int_{\mathbb{R}} \hat{u}_1(x_2) \, \mathrm{d}x_2 \right) \left(\int_{\mathbb{R}} \hat{u}_2(x_1) \, \mathrm{d}x_1 \right) =$$
$$= \left(\int_{\mathbb{R}} \hat{u}_1(\hat{x}_1) \hat{\mathrm{dx}}_1 \right) \left(\int_{\mathbb{R}} \hat{u}_2(\hat{x}_2) \hat{\mathrm{dx}}_2 \right).$$

(an equality in fact.) We have used Fubini once, which is permitted, as we have measurability + nonnegativity.

The case $d \to d+1$ Before we procceed, recall the "generalized Holder", all functions are nonnegative

$$\int_\Omega \prod_{j=1}^d f_j \, \mathrm{d}x \leq \prod_{j=1}^d \left(\int_\Omega f_j^{p_j} \, \mathrm{d}x \right)^{\frac{1}{p_j}},$$

where $\sum_{j=1}^d \frac{1}{p_j} = 1$. See that if we take $p_j = d$, then $\sum_{j=1}^d \frac{1}{d} = \frac{1}{d} \sum_{j=1}^d 1 = 1$, so

$$\int_\Omega \prod_{j=1}^d f_j \, \mathrm{d}x \leq \prod_{j=1}^d \left(\int_\Omega f_j^d \, \mathrm{d}x \right)^{\frac{1}{d}}.$$

Let us compute:

$$\int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^d u_j \overbrace{u_{d+1}}^{=\hat{u}(\hat{x}_{d+1})} \overbrace{\mathrm{d}x_1 \dots \mathrm{d}x_d}^{=\hat{\mathrm{dx}}_{d+1}} \mathrm{d}x_{d+1} = \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}} \prod_{j=1}^d u_j \, \mathrm{d}x_{d+1} \right)}_{\text{Holder}} u_{d+1} \hat{\mathrm{dx}}_{d+1} \leq$$

$$\leq \int_{\mathbb{R}^d} \underbrace{\left(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d}} u_{d+1} \, \hat{\mathrm{dx}}_{d+1}}_{\text{Holder}} \leq \left(\int_{\mathbb{R}^d} u_{d+1}^d \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \left(\int_{\mathbb{R}^d} \left(\prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x) \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d}\frac{d}{d-1}} \mathrm{d}\hat{x}_{d+1} \right)$$

$$\leq \left(\int_{\mathbb{R}^d} u_{d+1}^d \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} u_j^d \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d-1}\frac{d-1}{1}} \mathrm{d}\hat{x}_j \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{d-1}{d}\frac{1}{d-1}} =$$

$$= \left(\int_{\mathbb{R}^d} u_{d+1}^d \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d \, \mathrm{d}x_{d+1} \, \mathrm{d}\hat{x}_j \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} = \left(\int_{\mathbb{R}^d} u_{d+1}^d \, \mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^d \left(\int_{\mathbb{R}^d} u_j^d \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d}} =$$

$$= \prod_{j=1}^{d+1} \left(\int_{\mathbb{R}^d} u_j^d \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d}},$$

where the induction step is taken for the function

$$v_j = \left(\int_{\mathbb{R}} u_j^d \, \mathrm{d}x_{d+1} \right)^{\frac{1}{d-1}},$$

that is clearly nonnegative and measurable. □

Remark. Sometimes, the lemma is stated as: $\hat{u}_i \in \mathrm{L}_\infty \left(\mathbb{R}^{d-1}\right), \operatorname{supp} \hat{u}_i$ is compact $\forall i \in \{1, \ldots, d\}$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d |u_j(x)| \, \mathrm{d}x \le \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_j(\hat{x}_j)|^{d-1} \mathrm{d}\hat{\mathrm{x}}_\mathrm{i} \right)^{\frac{1}{d-1}} = \prod_{j=1}^d \|\hat{u}_j\|_{\mathrm{L}_{d-1}(\mathbb{R}^{d-1})}.$$

The difference is that in our version, we have nonnegativity in the assumptions and do not requiry compact supports and essential boundedness, as we work with integrals that are possibly infinite.

Theorem 8 (Gagliardo-Nirenberg)**.** *Let* $p \in [1, d)$*. Then* $\forall u \in W^{1,p}(\mathbb{R}^d)$*:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where $p^* = \frac{dp}{d-p}$.

Proof. (*From: the lectures*) Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$, then use density, as $\mathrm{W}^{1,p}(\mathbb{R}^d) = \mathrm{W}_0^{1,p}(\mathbb{R}^d)$.

$$\forall j \in \{1, \ldots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_d) \, \mathrm{d}s\,.$$

This estimate is independent of $j \in \{1, \ldots, d\}$, so it holds

$$|u(x)| \le \int_{\mathbb{R}} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s\,.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \le \left(\int_{\mathbb{R}^d} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s \right)^{\frac{d}{d-1}} = \prod_{j=1}^d \left(\int_{\mathbb{R}} |\nabla u|(\ldots, s, \ldots) \, \mathrm{d}s \right)^{\frac{1}{d-1}}.$$

Denote

$$u_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) = \left(\int_{\mathbb{R}^d} |\nabla u| \overbrace{(\ldots, x_j, \ldots)}^{=x \text{ in fact}} \mathrm{d}x_j \right)^{\frac{1}{d-1}},$$

which is a function independent of $x_j, u_j \equiv u_j(\hat{x}_j)$. So the integral (the $\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)$ norm)

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} \, \mathrm{d}x \le \int_{\mathbb{R}^d} \prod_{j=1}^d u_j \, \mathrm{d}x \underbrace{\le}_{\text{Gagliardo lemma}} \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} u_j^{d-1}(\hat{x}_j) \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d-1}} =$$

$$= \left(\prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) \, \mathrm{d}x_j \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u| \, \mathrm{d}x \right)^{\frac{d}{d-1}},$$

and so

$$\|u\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)}^{\frac{d}{d-1}} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^d)}^{\frac{d}{d-1}},$$

meaning ($1^* = \frac{d}{d-1}$)

$$\|u\|_{L_{1^*}(\mathbb{R}^d)} \leq 1\|\nabla u\|_{L_1(\mathbb{R}^d)}.$$

If now $p \in (1,d)$, we investigate for what q can we obtain estimate of $\||u|^q\|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)}$:

$$\|u\|^q_{L_{\frac{qd}{d-1}}(\mathbb{R}^d)} = \||u|^q\|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1}|\nabla u|\,\mathrm{d}x \underbrace{\leq}_{\text{Holder}} q\|\nabla u\|_{L_p(\mathbb{R}^d)}\left\||u|^{q-1}\right\|_{L_{p'}(\mathbb{R}^d)} =$$
$$= q\|\nabla u\|_{L_p(\mathbb{R}^d)}\|u\|^{q-1}_{L_{(q-1)p'}(\mathbb{R}^d)}.$$

We want $(q-1)p' = \frac{qd}{d-1}$, so we can divide both sides:

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$\frac{q}{d-1} = \frac{p}{d-p} \Leftrightarrow \frac{qd}{d-1} = \frac{dp}{d-p} = p^*, q = \frac{p(d-1)}{d-p}$$

and thus

$$\|u\|^q_{L_{p^*}(\mathbb{R}^d)} \leq q\|\nabla u\|_{L_p(\mathbb{R}^d)}\|u\|^{q-1}_{L_{p^*}(\mathbb{R}^d)} \Leftrightarrow \|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

$\Rightarrow$ statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$: let $u \in W^{1,p}(\mathbb{R}^d)$, be arbitrary. Then $\exists\{u_k\} \subset \mathcal{D}(\mathbb{R}^d) : u_k \to u$ in $W^{1,p}(\mathbb{R}^d)$. Moreover, we have showed that $\forall k \in \mathbb{N}$:

$$\|u_k\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)},$$

so passing to the (strong) limit and using the continuity of the norm indeed yields

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)}.$$

We are done.

□

Remark.
- It is evident that nonzero constants are not in $W^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.
- the set $\mathbb{R}^d$ is of course unbounded, so we have no ordering of $L_p(\Omega)$ spaces.
- of course, we require no smoothness of the domain

Theorem 9. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W^{1,p}_0(\Omega), \forall p \in [1,d)$ the statement of the previous theorem holds.*

Proof. (*From: the lectures*) An immediate corollary of the previous theorem: we have showed the inequality for $u \in \mathcal{D}(\mathbb{R}^d)$, but WLOG it holds also for $u \in \mathcal{D}(\Omega)$ (i can keep the integrals over $\mathbb{R}^d$, but in the end only values from Ω count) and since $W^{1,p}_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$, i can again extend it on the whole $W^{1,p}_0(\Omega)$. □

Remark. In the proof of theorem we showed that $\forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\Omega)} \leq q\|\nabla u\|_{\mathrm{L_p}(\Omega)}\|u\|^{q-1}_{\mathrm{L}_{\frac{p(q-1)}{p-1}}(\Omega)},$$

for q such that $\frac{qd}{d-1} \leq p^*$.

Definition 5 (Continuous & compact embeddings)**.** Let X, Y be linear normed spaces. We say

- X is continuously embedded into Y, $X \hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and

$$\forall x \in X : \|x\|_Y \leq C\|x\|_X.$$

- X is compactly embedded into Y, $X \hookrightarrow\hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and

$$\forall A \subset X \text{ bounded} : \overline{A}^Y \text{ is compact in } Y.$$

This is the same as saying[11] $X \subset Y$ (is a subspace) and the identity $\mathrm{id} : X \to Y$ is

- a bounded linear operator, $\mathrm{id} \in \mathcal{L}(X, Y)$
- is a compact linear operator, $\mathrm{id} \in \mathcal{K}(X, Y)$

Theorem 10 (Embedding theorem for $p \leq d$)**.** *Let $\Omega \in C^{0,1}, p^* = \frac{dp}{d-p}$. Then*

- *if $p \in [1, d)$, then*
$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)\ \forall q \in [1, p^*],$$
- *if $p \in [1, d)$, then*
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega), \forall q \in [1, p^*),$$
- *if $p = d$, then*
$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega)\ \forall q \in [1, \infty),$$
- *if $p = d$, then*
$$W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)\ \forall q \in [1, \infty),$$
(the same as above, i.e., every continuous embedding is also compact.)

Proof. (*From: the lectures & Bulíček et al., 2018*) We would like to use the previous lemmas + extension.

Ad continuity for $p < d$:

Recall that the composition of continuous operators yields a continuous operator. In our case:

- the extension operator $E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ is continuous
- identity $I_1 : \mathrm{W}^{1,p}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\mathbb{R}^d)$ is continous (Gagliardo-Nirenberg: $\|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$.)
- restriction $I_2 : \mathrm{L}_{p^*}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\Omega)$ is continuous (monotonicity of the L. integral: $\Omega \subset \mathbb{R}^d \Rightarrow \|u\|_{\mathrm{L}_{p^*}(\Omega)} \leq \|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)}$.)

[11] Really, we have $\|x\|_Y = \|\,\mathrm{id}\,x\|_Y \leq \|\mathrm{id}\|_{\mathcal{L}(X,Y)}\|x\|_X$, and if $A \subset X$ is bounded, than from the definition of $\mathrm{id} \in \mathcal{K}(X, Y) : \mathrm{id}(A) = A \subset Y$ is relatively compact in Y.

- identity $I_3 : \mathrm{L}_{p^*}(\Omega) \to \mathrm{L}_q(\Omega)$ is continous (embedding of Lebesgue spaces: Ω is bounded $\Rightarrow \mathrm{L}_{p^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\, \forall q \in [1, p^*]$)

Together, the mapping
$$\mathrm{id} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\Omega)\,, \mathrm{id} = I_3 \circ I_2 \circ I_1 \circ E$$
is continuous, and so $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1, p^*]$.

Ad continuity for $p = d$:

If $p = d$, we have (this holds generally) $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega)\, \forall r \in [1, d)$, (embedding of Lebesgue spaces, $\mathrm{L}_d(\Omega) \hookrightarrow \mathrm{L}_r(\Omega)\,, \forall r \in [1, d)$). Notice $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d-$, which means we can for all $q \in [1, \infty)$ find $r \in [1, d)$ s.t. $r^* > q$. Consequently,
$$\forall q \in [1, \infty) \exists r \in [1, d)\; s.t.\, \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,.$$
Notice also that $\forall r \in [1, d)$ we always have
$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega)$$
(that's just renaming p with r in embedding for $p < d$). Then it holds
$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,,$$
and so $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1, \infty)$.

Ad compactness for $p < d$:

It suffices to show $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0, 1)$ is relatively compact in $\mathrm{L}_q(\Omega)$, which is, since $\mathrm{L}_q(\Omega)$ is complete, equivalent to U being totally bounded in $\mathrm{L}_q(\Omega)$.[12] Extend the functions to $\mathrm{W}^{1,p}(\mathbb{R}^d)$ using the extension operator E, so $EU \subset \mathrm{W}^{1,p}(\mathbb{R}^d)$. Take some yet undetermined $\delta > 0$ and denote by $(EU)_\delta$ the set of regularized functions from EU with some kernel η. Our next strategy is the following:

1. show $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0, R))$,
2. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_1(\Omega)$,
3. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)$.

Since the supports of the functions from EU are uniformly bounded[13], we know [14] $\exists R > 0$ *s.t.*
$$\forall v \in (EU)_\delta : \mathrm{supp}\, v \subsetneq \mathrm{B}(0, R) \subset \mathbb{R}^d.$$
(Also remember that $\Omega \subset \mathrm{B}(0, R)$.) Moreover, from the properties of mollification[15] it follows:
$$(EU)_\delta \subset \mathrm{C}^1\left((\mathrm{B}(0, R))\right).$$

Next up, calculate for $v \in (EU)_\delta$ the norm $\|v\|_{\mathrm{C}^1(\mathrm{B}(0,1))}$. Realize that in fact $v = (Eu)_\delta$ for some

[12] A metric space P is totally bounded if there exists a finite ε−net: a finite open covering of P by balls centered in P of radii smaller than ε.

[13] From the properties of extension, we know $\forall u \in \mathrm{W}^{1,p}(\Omega) : \mathrm{supp}\, Eu \subset V$ with V open *s.t.* $\Omega \subset\subset V$.

[14] Properties of mollification include $\mathrm{supp}(Eu)_\delta \subset \mathrm{B}(0, \delta) + \mathrm{supp}\, Eu$.

[15] We have $(EU)_\delta \subset \mathrm{C}^\infty(\mathbb{R}^d)$.

$u \in U$, and so

$$|\int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y\,| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,$$

as $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$. Also

$$\begin{aligned}|\nabla \int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y\,| = |\int_{\mathbb{R}^d} Eu(y)\nabla_x\eta_\delta(x-y)\,\mathrm{d}y\,| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\nabla\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le \\ \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,\end{aligned}$$

using the same arguments. In total

$$\forall v \in (EU)_\delta : \|v\|_{\mathrm{C}^1(\mathrm{B}(0,R))} \le C,$$

or in other words, all functions from $(EU)_\delta$ are uniformly bounded in $\mathrm{C}^1(\mathrm{B}(0,R)) \Rightarrow$ they are uniformly bounded and uniformly equicontinuous (that is implied by uniform boundedness of the derivatives). Thus we can use Arzela-Ascoli theorem and state

$$(EU)_\delta \subset\subset \mathrm{C}^0(\mathrm{B}(0,R)).$$

Since $\mathrm{C}^0(\mathrm{B}(0,R))$ is complete, this also means $(EU)_\delta$ is totally bounded in $\mathrm{C}^0(\mathrm{B}(0,R))$. Using the fact ($\mathrm{B}(0,R)$ is compact) [16]

$$\mathrm{C}^0(\mathrm{B}(0,R)) \hookrightarrow\hookrightarrow \mathrm{L}_1(\mathrm{U}(0,R)),$$

we also see that $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0,R))$.

Next, take an arbitrary $u \in U$ and compute (we are using the fact $Eu = u$ *a.e.* in $\Omega, \Omega \subset \mathrm{U}(0,R)$.)

$$\begin{aligned}\|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} &\le \Big\|\overbrace{Eu}^{:=v} - (Eu)_\delta\Big\|_{\mathrm{L}_1(\mathrm{U}(0,R))} = \int_{\mathrm{U}(0,R)}|v-v_\delta|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}|v(x)-\int_{\mathbb{R}^d}v(y)\eta_\delta(x-y)\,\mathrm{d}y\,|\,\mathrm{d}x = \\ &= \int_{\mathrm{U}(0,R)}|v(x)-\int_{\mathbb{R}^d}v(x-y)\eta_\delta(y)\,\mathrm{d}y\,|\,\mathrm{d}x = \big[x\mapsto x+y\big] = \int_{\mathrm{U}(0,R)}|\int_{\mathbb{R}^d}v(x+y)-v(x)\underbrace{\eta_\delta(y)}_{\le 1}\,\mathrm{d}y \\ &\le \int_{\mathrm{U}(0,R)}|\int_{\mathbb{R}^d}(v(x+y)-v(y))\eta_\delta(y)\,\mathrm{d}y\,|\,\mathrm{d}x \le \int_{\mathrm{U}(0,R)}\int_{\mathbb{R}^d}\frac{|v(x+y)-v(x)|}{|y|}|\eta_\delta(y)||y|\,\mathrm{d}y\,\mathrm{d}x \le \\ &\underbrace{\le}_{\text{Fubini}} \int_{\mathbb{R}^d}\int_{\mathrm{U}(0,R)}\frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y\,.\end{aligned}$$

Estimate the inner integral: assume v is smooth, $v \in \mathcal{D}(\mathrm{U}(0,R))$ and write

[16]Take some sequence $\{f_k\} \subset \mathrm{C}^0(\mathrm{B}(0,R))$ that is bounded, then

$$\lim_{k\to\infty}\int_{\mathrm{U}(0,R)}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}\lim_{k\to\infty}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}|f|\,\mathrm{d}x = \int_{\mathrm{B}(0,R)}|f|\,\mathrm{d}x \le \infty,$$

the majorant being *e.g.* $\max_{k\in\mathbb{N}}\|f_k\|_\infty$. Hence every bounded sequence in $\mathrm{C}^0(\mathrm{B}(0,R))$ has a converging (sub)sequence in $\mathrm{L}_1(\mathrm{U}(0,R))$.

$$\int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x = \int_{\mathrm{U}(0,R)} \frac{1}{|y|}\Big|\int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(v(x+sy))}_{\nabla v(x+sy)\cdot y}\,\mathrm{d}s\,\Big|\,\mathrm{d}x \underbrace{\le}_{\text{C-S}} \int_{\mathrm{U}(0,R)} \frac{1}{|y|}\int_0^1 |\nabla v(x+sy)||y|\,\mathrm{d}s\,\mathrm{d}x =$$

$$= \int_{\mathrm{U}(0,R)} \int_0^1 |\nabla v(x+sy)|\,\mathrm{d}s\,\mathrm{d}x = \int_0^1 \int_{\mathrm{U}(0,R)} |v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s\,.$$

The last integral can be further manipulated by using the change of variables $z := x + sy \in \{x+sy|x \in \mathrm{U}(0,R)\} = \mathrm{U}(0,R) + sy = \mathrm{U}(sy,R)$. Since $\nabla v \in \mathcal{D}\left(\mathrm{U}(0,R)\right)$, the integral is nonzero only for $z \in \mathrm{U}(sy,R) \cap \mathrm{U}(0,R) \subset \mathrm{U}(0,R)$ so we can write

$$\int_0^1 \int_{\mathrm{U}(0,R)} |\nabla v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s = \int_0^1 \int_{\mathrm{U}(sy,R)\cap\mathrm{U}(0,R)} |\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \le \int_0^1 \int_{\mathrm{U}(0,R)} |\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \le$$

$$\le \int_0^1 \|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))} (\lambda(\mathrm{U}(0,R)))^{\frac{1}{p'}}\,\mathrm{d}s \le C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))},$$

and so we have shown

$$\forall v \in \mathcal{D}\left(\mathrm{U}(0,1)\right) : \int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x \le C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

Now, take[17] $v \in \mathrm{W}^{1,p}_0(\mathrm{U}(0,R))$, then $\exists\{v_k\} \subset \mathcal{D}(\mathrm{U}(0,R)) : v_k \to v \,\mathrm{in}\, \mathrm{W}^{1,p}_0(\mathrm{U}(0,R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y)-v_k(x)|}{|y|}\,\mathrm{d}x \le C(R)\|\nabla v_k\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \to C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

Putting it all together:

$$\|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} \le \int_{\mathbb{R}^d}\int_{\mathrm{U}(0,R)} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y \le C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}\int_{\mathbb{R}^d} \underbrace{|y|}_{\le\delta}\eta_\delta(y)\,\mathrm{d}y \le$$

$$\le C(R)\delta\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}\int_{\mathbb{R}^d}\eta_\delta(y)\,\mathrm{d}y = C(R)\delta\|v\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} = C(R)\delta\|Eu\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} \le$$

$$\le C_1\delta\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C_1\delta.$$

where we have used the properties of the reg. kernel η_δ, the extension operator E and the fact $u \in U$.

Now fix $\varepsilon > 0$, find $\{(Eu_k)_\delta\}_{k=1}^m$ a finite $\frac{\varepsilon}{2}$-net in $(EB)_\delta$ in $\mathrm{L}_1(\mathrm{U}(0,R))$ (which is possible, since we have total boundedness in $\mathrm{L}_1(\mathrm{U}(0,R))$.) We will show $\{u_k\}_{k=1}^m$is a (finite) ε–net in $\mathrm{L}_1(\Omega)$.

Up to now, $\delta > 0$ has been undetermined; now comes the time - set

$$\delta > 0\,\mathrm{s.t.}\, C_1\delta < \frac{\varepsilon}{4}.$$

Fix an arbitrary $u \in U$, and find $j \in \{1,\dots,m\}$ *s.t.* $\|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L}_1(\mathrm{U}(0,R))} < \frac{\varepsilon}{2}$. Compute

$$\|u-u_j\|_{\mathrm{L}_1(\Omega)} \le \|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu)_\delta - (Eu_j)_\delta\right\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu_j)_\delta - u_j\right\|_{\mathrm{L}_1(\Omega)} \le C_1\delta + \frac{\varepsilon}{2}C_1\delta < \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2} < \varepsilon,$$

where we have used the above estimate and the fact $\Omega \subset \mathrm{B}(0,R)$. Thus, we have shown U is totally bounded in $\mathrm{L}_1(\Omega)$ and so

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_1(\Omega)\,.$$

[17]Recall we have $v \in (EU)_\delta$ and so $\operatorname{supp} v \subsetneq \mathrm{B}(0,R)$, meaning it is "zero on $\mathrm{S}(0,R)$" - in the sense of traces.

It remains to show the validity for a general $q \in [1, p^*)$. Using the interpolation theorem on Lebesgue spaces [18] we obtain

$$\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|_{\mathrm{L}_1(\Omega)}^{\theta} \|u\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta},$$

where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Let us now show U is totally bounded in $\mathrm{L}_q(\Omega)$, *i.e.*, $\forall \epsilon > 0$ there exists a finite $\epsilon-$ net in U in $\mathrm{L}_q(\Omega)$. Pick $\{u_j\}_{j=1}^m \subset U$ that is an $\beta > 0$ net in $\mathrm{L}_1(\Omega)$, where β will be determined later. Then it holds

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \|u - u_j\|_{\mathrm{L}_1(\Omega)}^{\theta} \|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta} \le \beta^{\theta} \|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta}.$$

Since we have already shown $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_{p^*}(\Omega)$, we know (again u, u_j are in U)

$$\|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta} \le C_2 \|u - u_j\|_{\mathrm{W}^{1,p}(\Omega)}^{1-\theta} \le C_2 2^{1-\theta},$$

and so

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \beta^{\theta} C_2 2^{1-\theta}.$$

We see that if we choose β *s.t.*

$$\beta < \left(\frac{\epsilon}{C_2 2^{1-\theta}} \right)^{\frac{1}{\theta}},$$

we obtain

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \epsilon,$$

i.e., $\{u_j\}_{j=1}^m$ is a ϵ–set in $\mathrm{L}_q(\Omega)$. Thus $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*)$.

Ad compactness for $p = d$

Finally, let us show the last result. It holds

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega), \forall r \in [1, d).$$

Moreover, we have just shown

$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_s(\Omega), \forall s \in [1, r^*).$$

Notice that $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d^-$, so $\forall q \in [1, \infty)$ fixed $\exists r \in [1, d) : r^* > q$, *i.e.*, $q \in [1, r^*)$. But then

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty).$$

And realize that implies $\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\ \forall q \in [1, \infty)$: if $\{u_n\}$ is bounded in $\mathrm{W}^{1,d}(\Omega)$, then it is bounded in $\mathrm{W}^{1,r}(\Omega)$, as the identity between those spaces is continuous, and the above compact embedding tells us $\{u_{n_k}\}$ is convergent in $\mathrm{L}_q(\Omega)$ for some $\{n_k\}$. In total, $\{u_n\} \subset \mathrm{W}^{1,d}(\Omega)$ has a subsequence $\{u_{n_k}\}$ convergent in $\mathrm{L}_q(\Omega)$. We are done. □

2.4.2 Theorems for $p > d$

We know we will encounter Holder spaces. Let us recall some of their properties in a remark.

Remark (Properties of Holder spaces). Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $k \in \mathbb{N}_0, \lambda \in [0, 1]$. The

[18] In the case $q \in [r, s)$ it holds $\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|_{\mathrm{L}_r(\Omega)}^{\theta} \|u\|_{\mathrm{L}_s(\Omega)}^{1-\theta}$, $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$.

norm on the space $\mathrm{C}^{0,1}\left(\overline{\Omega}\right)$ is defined as

$$\|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} = \|u\|_{\mathrm{C}^{k}(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x\neq y, x,y,\in\overline{\Omega}} \frac{|D^\alpha u(x) - D^\alpha(y)|}{|x-y|^\lambda},$$

and the space

$$\mathrm{C}^{k,\lambda}\left(\overline{\Omega}\right) := \left\{u \in \mathrm{C}^{k}\left(\overline{\Omega}\right) \big| \|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} \leq \infty\right\},$$

where we identify

$$\mathrm{C}^{k,0}\left(\overline{\Omega}\right) = \mathrm{C}^{k}\left(\overline{\Omega}\right).$$

Moreover, we have the following embeddings: $\forall \alpha \in [0,1]$ it holds

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\beta}\left(\Omega\right), \forall \beta \in [1,\alpha],$$

and

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [1,\alpha).$$

A fresh start of a new chapter calls for a fresh new lemma.

Lemma 7 (Morrey)**.** *Let* $u \in \mathcal{D}\left(\mathbb{R}^d\right)$*. Then* $\forall x_1, x_2 \in \mathbb{R}^d, \forall \mu \in (0,1]$ *it holds*

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu [\nabla u]_{L_{1,\mu}(\mathbb{R}^d)},$$

with

$$[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \sup_{\rho>0} \int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \,\mathrm{d}y\,.$$

Proof. (*From: Bulíček et al., 2018*) Pick arbitrary but fixed $x_1, x_2 \in \mathbb{R}^d$. Denote by C_ρ the closed cube with a side of length ρ *s.t.* x_1 and x_2 lie on opposite faces. Then $\rho \leq |x_1 - x_2| \leq \rho\sqrt{d}$ (it is not closer then the height and not further then the diagonal.)

Let us begin by first computing the deviation of $u(x_i)$ from the mean value of u on C_ρ:

$$\left|\frac{1}{\lambda(C_\rho)} \int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)\right| = \left|\int_{C_\rho} \frac{u(x) - u(x_i)}{\rho^d}\,\mathrm{d}x\right| \leq \int_{C_\rho} \frac{|u(x) - u(x_i)|}{\rho^d}\,\mathrm{d}x\,.$$

What other can we use when estimating differences than Newton - Leibniz, right? Since $u \in \mathcal{D}\left(\mathbb{R}^d\right) \subset \mathrm{C}^1\left(\mathbb{R}^d\right)$ it holds for $i \in \{1,2\}$ and $\forall x \in C_\rho$:

$$\begin{aligned}|u(x) - u(x_i)| &\leq \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} u(x_i + s(x - x_i))\,\mathrm{d}s\right| = \left|\int_0^1 \nabla u(x_i + s(x-x_i)) \cdot (x - x_i)\,\mathrm{d}s\right| \leq \\ &\leq \int_0^1 |\nabla u(x_i + s(x-x_i))||x - x_i|\,\mathrm{d}s \leq \rho\sqrt{d} \int_0^1 |\nabla u(x_i + s(x - x_i))|\,\mathrm{d}s\,.\end{aligned}$$

Notice that it is important $x \in C_\rho$ and x_1, x_2 are on the opposite sides. With this estimate for $|u(x) - u(x_i)|$ and Fubini we can write for the deviattion

$$\left|\frac{1}{\rho^d} \int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)\right| \leq \sqrt{d} \int_{C_\rho} \frac{1}{\rho^{d-1}} \int_0^1 |\nabla u(x_i + s(x - x_i)|\,\mathrm{d}s\,\mathrm{d}x = \sqrt{d} \int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x-x_i))|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s\,.$$

This calls for a sensible change of variables. Denote $z = x_i + s(x - x_i)$, then under this transformation the cube C_ρ becomes

$$z \in x_i + s(C_\rho - x_i) = x_i(1-s) + C_{s\rho} = x_i(1-s) + [0, s\rho]^d := C^i_{s\rho}$$

which, since x_i is taken from the faces and $s \le 1$, is a cube with its "origin" somewhere in C_ρ and a side of length $s\rho \le \rho$, which implies $C^i_{s\rho} \subset [0,R]^d$ for some R. The integral then becomes (clearly $|\det \nabla_x z| = s^d$),

$$\sqrt{d}\int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x-x_i))|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s = \sqrt{d}\int_0^1 \int_{C^i_{s\rho}} \frac{|\nabla u(z)|}{s^d\rho^{d-1}}\,\mathrm{d}z\,\mathrm{d}s = \sqrt{d}\rho^\mu \int_0^1 s^{\mu-1}\int_{C^i_{s\rho}} \frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z\,\mathrm{d}s\,,$$

where we are being a bit suggestively imaginative. The "deviation estimate" then becomes

$$|\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)| \le \sqrt{d}\rho^\mu \int_0^1 s^{\mu-1} \underbrace{\int_{C^i_{s\rho}} \frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z}_{\le [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}}\,\mathrm{d}s \le \sqrt{d}\rho^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \int_0^1 s^{\mu-1}\,\mathrm{d}s =$$

$$= \frac{\sqrt{d}}{\mu}\rho^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)},$$

where we used that $C^i_{s\rho} \subset [0,R]^d$ and that $0 \in \mathbb{R}^d$, and so we could estimate the integral over $C^i_{s\rho}$ by $[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}$. Triangle inequality and the fact $\rho \le |x_1 - x_2|$ concludes our proof:

$$|u(x_1) - u(x_2)| \le |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_1)| + |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_2)| \le \frac{2\sqrt{d}}{\mu}\rho^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \le$$

$$\le \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}.$$

□

Remark. It is sufficient when $u \in \mathrm{C}^1_0(\mathbb{R}^d)$.

Gagliardo had in fact two lemmas, so let us even the game for Morrey.

Lemma 8. *Let $p \in (d,\infty)$, and let $\mu = 1 - \frac{d}{p}$. Then $\forall u \in \mathcal{D}(\mathbb{R}^d)$ it holds*

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} \le \left(1 + \frac{4\sqrt{d}}{\mu}\right)\|u\|_{W^{1,p}(\mathbb{R}^d)},$$

where

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}|u(x)| + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu}.$$

Proof. (*From: Bulíček et al., 2018*) We prove the assertion by estimating both terms in the above norm. To obtain those specific constants, we will pay some more attention to our procceeding. Let us also state the trivial: $\mu = 1 - \frac{d}{p} \in (0,1)$ for $p \in (d,\infty)$.

Begin with the differences: choose an arbitrary $\rho > 0$ and compute

$$\int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\,\mathrm{d}y \le \left(\int_{[0,\rho]^d}\left(\frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\right)^p \mathrm{d}y\right)^{\frac{1}{p}} \left(\lambda([0,\rho]^d)\right)^{\frac{p-1}{p}} = \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \frac{\rho^{\frac{d(p-1)}{p}}}{\rho^{d-1+\mu}} = \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

because $\frac{d(p-1)}{p} - d + 1 - \mu = \frac{dp-d}{p} - d + 1 - 1 + \frac{d}{p} = \frac{dp}{p} - d = 0$. Taking the suprema yields

$$[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \leq \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

Going with this into the first Morrey lemma, we see

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

meaning

$$\sup_{x_1,x_2\in\mathbb{R}^d, x_1\neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{\mu}} \leq \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

To estimate the infinity norm, we can actually exploit the above result aswell: pick $x \neq y \in \mathbb{R}^d$ and write

$$|u(x)| - |u(y)| \leq |u(x) - u(y)| \leq \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

and so

$$|u(x)| \leq |u(y)| + \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)}.$$

Now fix $\rho \geq 2|x - y| > 0$, integrate both sides w.r.t y over $[0,\rho]^d$ and obtain

$$|u(x)|\rho^d \leq \int_{[0,\rho]^d} |u(y)|\,\mathrm{d}y + \frac{2\sqrt{d}}{\mu}\rho^d\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

which upon using Holder in the integral becomes

$$|u(x)|\rho^d \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)}\rho^{\frac{d(p-1)}{p}} + \frac{2\sqrt{d}}{\mu}\rho^d\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

Since we have lost y, we can in fact choose $\rho = 1$, and upon taking the supremum write

$$\sup_{x\in\mathbb{R}^d} |u(x)| \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

and so in total

$$\|u\|_{\mathrm{C}^{0,\mu}(\mathbb{R}^d)} \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{4\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)} = \left(1 + \frac{4\sqrt{d}}{\mu}\right)\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

$\square$

Remark. Since $\mathcal{D}\left(\mathbb{R}^d\right)$ is dense in $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$, the above lemma holds also for $u \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$. We have to be careful to pick a good representant though.

Theorem 11 (Embedding theorems for $p > d$)**.** *Let* $\Omega \in C^{0,1}, d \in \mathbb{N}, p > d$, *i.e.*, $p \in (d,\infty]$. *Denote* $\mu^* = 1 - \frac{d}{p} \in (0,1)$. *Then*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*],$$

and

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*).$$

Proof. (*From: Bulíček et al., 2018*) *Ad continuous:*

Since $\Omega \in C^{0,1}$, we are able to use the extension theorem; recall $\forall u \in W^{1,p}(\Omega) : \operatorname{supp} Eu \subset V$, where $\overline{\Omega} \subset V$. Let us deal with the case $p \in (d, \infty)$ first. We have shown

$$\|u\|_{C^{0,\mu^*}(\overline{\Omega})} \leq \|u\|_{C^{0,\mu^*}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|u\|_{W^{1,p}(\mathbb{R}^d)}, \forall u \in W^{1,p}\left(\mathbb{R}^d\right).$$

Realize that in fact

$$\|u\|_{C^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{C^{0,\mu^*}(\overline{\Omega})},$$

as $\partial\Omega$ is a set of zero Lebesgue measure and so in fact $Eu = u \text{ on } \overline{\Omega}$, which together with the obvious fact $Eu \in W^{1,p}\left(\mathbb{R}^d\right)$ gives

$$\|u\|_{C^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{C^{0,\mu^*}(\overline{\Omega})} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

This exactly means

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\mu^*}\left(\overline{\Omega}\right).$$

Realize also that

$$C^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$

and so

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*].$$

If now $p = \infty$, realize that (by embedding of Lebesgue spaces)

$$W^{1,\infty}(\Omega) \hookrightarrow W^{1,q}(\Omega), \forall q \in [1, \infty).$$

From the previous result it follows

$$W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in \left[0, 1 - \frac{d}{q}\right],$$

and notice that $1 - \frac{d}{q} \to 1$ as $q \to \infty$. This means that $\forall q \in [1, \infty)$ it holds

$$W^{1,\infty}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, 1 - \frac{d}{q}].$$

Since $q \in [1, \infty)$ was arbitrary, we conclude it must be

$$W^{1,\infty}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, 1].$$

[19]

[19] The norm

$$\|u\|_{C^{0,1-\frac{d}{q}}(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |u(x)| + \sup_{x_1 \neq x_2 \in \overline{\Omega}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{1-\frac{d}{q}}},$$

actually allows one to pass to the limit $q \to \infty$, since the supremum is independent of the exponent in the denonimanotor, the function on the RHS is continuous. So if we know $\|u\|_{C^{0,1-\frac{d}{q}}(\overline{\Omega})} \leq C\|u\|_{W^{1,\infty}(\Omega)}, \forall q \in [1, \infty)$, we can pass to the limit on the LHS and obtain $\|u\|_{C^{0,1}(\overline{\Omega})} \leq C\|u\|_{W^{1,\infty}(\Omega)}$, which is the only missing possibility.

Ad compactness This will be a bit of a cheating: we know

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [0, \mu^*],$$

and for Holder spaces it also holds

$$\mathrm{C}^{0,\beta}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \beta),$$

which means if we choose $\beta = \mu^*$, we in fact obtain

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*).$$

Using the same arguments as in the case of the compact embedding for $p < d$, we can conclude

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*).$$

We are done. □

Remark. Note that in the case of $p > d$, from

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$

and[20]

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{L}_\infty(\Omega), \forall \alpha \in [0, 1]$$

it follows[21]

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega), \forall p > d.$$

But that of course means (Ω is bounded) that

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty],$$

whenever $p > d$.

Remark (Summary). Let us summarize the obtained results. If $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, then upon denoting

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p},$$

it holds

- if $p < d$, then
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*],$$
and
$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*),$$
- if $p = d$, then
$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty)$$

[20] Clearly,

$$\|u\|_{\mathrm{L}_\infty(\Omega)} \le \|u\|_{\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right)} = \|u\|_{\mathrm{L}_\infty(\Omega)} + \sup_{x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}.$$

[21] Again, composition of a compact and continuous (linear) operators yields a compact operator independently of the order.

and

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1, \infty),$$

- if $p > d$, then

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$

and

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$

which also imply

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty(\Omega)\,,$$

and

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega)\,.$$

Remark (Summary - embedding into Lebesgue spaces). It can be guiding to look only at the embeddings into some Lebesgue spaces. Let $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, denote

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p}.$$

Then it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,,$$

where q is given:

- if $p < d$, then $q \in [1, p^*]$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

Also, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)\,,$$

where q is given as

- if $p < d$, then $q \in [1, p^*)$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

2.5 Trace theorems

There are many many troubles with the boundary. Another one we have yet not encountered arises with *e.g.* homogenous Dirichlet boundary coniditons: it should hold

$$u = 0 \,\mathrm{on}\, \partial\Omega,$$

but typically, u is an element of some Sobolev space on Ω, and $\lambda_d(\partial\Omega) = 0$. We cannot sensibly talk about pointwise values on a set of measure zero, they can be arbitrary. This can be dealt with provided - as expected - when Ω, *i.e.*, $\partial\Omega$ is benevolent enough.

Realize also that for Ω at least $\mathrm{C}^{0,0}$ we also have the density of $\mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\Omega)$, meaning the values on $\partial\Omega$ are well defined at least for a dense subset of $\mathrm{W}^{1,p}(\Omega)$. It should not be too difficult to extend to the whole $\mathrm{W}^{1,p}(\Omega)$, right...?

Moreover, in the case $p > d, \Omega \in \mathrm{C}^{0,1}$ we already know $\forall u \in \mathrm{W}^{1,p}(\Omega)$ there exists $u^* \in \mathrm{C}^0\left(\overline{\Omega}\right)$, such that $u = u^*$ *a.e.* in Ω, and so in these cases, the values $u^* \restriction_{\partial\Omega}$ are well defined. What if $p \leq d$?

Remark (The space $\mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega)$). In the winter semester, we have defined

$$\mathrm{L_p}(\partial\Omega) \equiv \mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega)\,,$$

i.e., the lebesgue spaces are taken w.r.t the $d-1$ dimensional (normalized complete) Hausdorff measure $\mathcal{H}_{d-1}$.

Theorem 12 (Continuous trace theorem)**.** *Let $\Omega \in C^{0,1}, p \in [1,\infty]$, denote $p^{\#} = \frac{dp-p}{d-p}$. Then there is a continuous linear operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L_q(\partial\Omega)$, *with q being*

- *if $p < d$, then $q \in [1, p^{\#}]$,*
- *if $p = d$, then $q \in [1,\infty)$,*
- *if $p > d$, then $q \in [1,\infty]$.*

Moreover, $\forall u \in C^\infty\left(\overline{\Omega}\right)$ it holds

$$\mathrm{tr}\, u = u \restriction_{\partial\Omega},$$

meaning $\mathrm{tr} : W^{1,p}(\Omega) \to L_p(\partial\Omega)$ *is an extension of* $\tilde{\mathrm{tr}} : C^\infty\left(\overline{\Omega}\right) \to C^0\left(\partial\Omega\right)$.

Proof. (*From: Bulíček et al., 2018*) The strategy is the following

1. define $\tilde{\mathrm{tr}}$ for smooth functions,
2. obtain estimates for $\tilde{\mathrm{tr}}$ using embedding theorems,
3. extend $\tilde{\mathrm{tr}}$ to the whole space, which defines tr.

Case $p < d$:

As we have mentioned, the case for functions smooth up to the boundary is evident. Let us so define $\tilde{\mathrm{tr}} : \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) \to \mathrm{C}^0\left(\partial\Omega\right)$, by

$$\tilde{\mathrm{tr}}\, u = u \restriction_{\partial\Omega}\,.$$

Then clearly $\tilde{\mathrm{tr}}$ is a well defined linear [22] operator.

(We are using the notation from the definition of a $\mathrm{C}^{0,1}$ domain). Let us for clarity define (and also recall)

$$\begin{aligned}
G_j &= \mathbb{A}_j\left(\left\{(x', a_j(x')| x' \in \mathrm{U}(0,\alpha)\right\}\right),\\
G_j^+ &= \mathbb{A}_j\left(\left\{x', a_j(x') + b | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\right\}\right),\\
G_j^- &= \mathbb{A}_j\left(\left\{x', a_j(x') - b | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\right\}\right).
\end{aligned}$$

Within this notation, $G_j \subset \partial\Omega, G_j^+ \subset \Omega, G_j^- \subset \mathbb{R}^d / \overline{\Omega}$ and $U_j = G_j \cup G_j^+ \cup G_j^-$. Moreover, $\{U_j\}_{j=1}^m$ are open sets *s.t.* $\partial\Omega \subset \bigcup_{j=1}^m U_j$. Denote $\{\varphi_j\}_{j=1}^m \subset \mathcal{D}\left(\mathbb{R}^d\right)$ to be the partition of unity subordinate to this (open) covering. Realize moreover that since $\mathcal{H}_{d-1}(\partial\Omega) < \infty$, it holds

$$\mathrm{L}_{p^{\#}}(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, p^{\#}].$$

[22] Be careful about the continuity of $\tilde{\mathrm{tr}}$ here. $\mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right)$ is not a normed linear space. We could state the continuity in a different setting, but we do not actually need it for our purposes.

So if we are able to show

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

we have the rest of the estimates for free.

Take $u \in \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$, and denote $u_j = u\varphi_j, j \in \{1,\dots,m\}$. Let for the moment $p > 1$, so $p^\# = \frac{dp-p}{d-p} > \frac{d-1}{d-1} = 1$. As $u_j \in \mathrm{C}^\infty\left(\mathbb{R}^d\right)$, it holds $u_j \in \mathrm{C}^1\left(\overline{G_j^+}\right)$. Moreover, since $\operatorname{supp} u_j \subset U_j$ and as $U_j \cap \mathbb{A}_j(\{(x', a_j(x') + b | x' \in \mathrm{U}(0,\alpha), b \in [0,\infty)\}) = \varnothing$, it holds

$$u_j\big(\mathbb{A}_j(x', a_j(x') + \beta)\big) = 0.$$

With those qualities, if we denote (for an arbitrary $j \in \{1,\dots,m\}$; this will become $\|u_j\|_{\mathrm{L}_{p^\#}(\partial\Omega)}$)

$$v(x') = \left|u_j\big(\mathbb{A}_j(x', a_j(x'))\big)\right|^{p^\#} = \left|u_j\big(\mathbb{A}_j(x', a_j(x'))\big)\right|^{\frac{dp-p}{d-p}},$$

we can write (recall $\mathbb{A}_j$ is orthogonal and $x' \in \mathrm{U}(0,\alpha)$)

$$\begin{aligned}
|v(x')| &= \left|\left|u_j\big(\mathbb{A}_j(x', a_j(x'))\big)\right|^{\frac{dp-p}{d-p}} - \left|u_j\big(\mathbb{A}_j(x', a_j(x') + \beta)\big)\right|^{\frac{dp-p}{d-p}}\right| \le \left|\int_{a_j(x')+\beta}^{a_j(x')} \frac{\partial}{\partial s}\left|u_j\big(\mathbb{A}_j(x', s)\big)\right| \mathrm{d}s\right| = \\
&= \left|\int_{a_j(x')+\beta}^{a_j(x')} p^\# \left|u_j\big(\mathbb{A}_j(x', s)\big)\right|^{\frac{dp-d}{d-p}} \operatorname{sign}\big(u_j(\mathbb{A}_j(x', s))\big) \nabla u_j\big(\mathbb{A}_j(x', s)\big) \cdot \mathbb{A}_j(x', 1)\, \mathrm{d}s\right| \le \\
&\le p^\# \int_{a_j(x')}^{a_j(x')+\beta} \left|u_j\big(\mathbb{A}_j(x', s)\big)\right|^{\frac{dp-d}{d-p}} \left|\nabla u_j\big(\mathbb{A}_j(x', s)\big)\right| \underbrace{\left|\mathbb{A}_j(x', 1)\right|}_{=|(x',1)|} \mathrm{d}s \le \\
&\le p^\# \sqrt{1+\alpha^2} \int_{a_j(x')}^{a_j(x')+\beta} \left|u_j\big(\mathbb{A}_j(x', s)\big)\right|^{\frac{dp-d}{d-p}} \left|\nabla u_j(x', s)\right| \mathrm{d}s\,.
\end{aligned}$$

Integrate this inequality over $\mathrm{U}(0,\alpha)$ and write (recall the definition of G_j^+, we are using Fubini, substitution theorem and the fact $|\det \mathbb{A}_j = 1|$)

$$\begin{aligned}
\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' &\le p^\# \sqrt{1+\alpha^2} \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \left|u_j\big(\mathbb{A}_j(x', s)\big)\right|^{\frac{dp-d}{d-p}} \left|\nabla u_j\big(\mathbb{A}_j(x', s)\big)\right| \mathrm{d}s\,\mathrm{d}x' \le \\
&\le p^\# \sqrt{1+\alpha^2} \int_{\mathbb{A}_j(\{(x', a_j(x')+b) | x' \in \mathrm{U}(0,\alpha), b \in [0,\beta]\})} |u_j|^{\frac{dp-d}{d-p}} |\nabla u_j(x)|\,\mathrm{d}x = \\
&= p^\# \sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{\frac{dp-d}{d-p}} |\nabla u(x)|\,\mathrm{d}x \le p^\# \sqrt{1+\alpha^2} \|\nabla u\|_{\mathrm{L}_p(G_j^+)} \left\| |u_j|^{\frac{dp-d}{d-p}} \right\|_{\mathrm{L}_{p'}(G_j^+)},
\end{aligned}$$

and since $\frac{dp-d}{d-p} p' = \frac{dp-d}{d-p}\frac{p}{p-1} = \frac{dp}{d-p} = p^*$, we have

$$\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' = p^\# \sqrt{1+\alpha^2} \|\nabla u\|_{\mathrm{L}_p(G_j^+)} \|u_j\|^{\frac{dp-d}{d-p}}_{\mathrm{L}_{p^*}(G_j^+)}.$$

Since $G_j^+ \in \mathrm{C}^{0,1}$, the last term can be estimated using the continuous embedding theorems:

$$\|u_j\|^{\frac{dp-d}{d-p}}_{\mathrm{L}_{p^*}(G_j^+)} \le C\|u_j\|^{\frac{dp-d}{d-p}}_{\mathrm{W}^{1,p}(G_j^+)},$$

whereas the integral on the LHS actually is[23]

$$\int_{\mathrm{U}(0,\alpha)} |v(x')|\,\mathrm{d}x' = \int_{\mathrm{U}(0,\alpha)} \left|u_j\big(\mathbb{A}_j(x', a_j(x'))\big)\right|^{p^\#} \mathrm{d}x = \int_{\mathbb{A}_j(\{(x',a_j(x'))|x'\in\mathrm{U}(0,\alpha)\})} |u_j(x)|^{p^\#}\,\mathrm{d}x = \|u_j\|^{p^\#}_{\mathrm{L}_{p^\#}(G_j)},$$

and so we write

$$\|u_j\|^{p^\#}_{\mathrm{L}_{p^\#}(G_j)} \le C\|\nabla u_j\|_{\mathrm{L}_p\left(G_j^+\right)} \|u_j\|^{\frac{dp-d}{d-p}}_{\mathrm{W}^{1,p}\left(G_j^+\right)} \ge C\|u_j\|^{\frac{dp-d+d-p}{d-p}}_{\mathrm{W}^{1,p}\left(G_j^+\right)} = C\|u_j\|^{p^\#}_{\mathrm{W}^{1,p}\left(G_j^+\right)},$$

and so

$$\|u_j\|_{\mathrm{L}_{p^\#}(G_j)} \le C\|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}.$$

This has been done for $p > 1$, but in fact taking the limit $p \to 1^+$ is allowed here (without a proof). Hence the above estimate holds $\forall p \in [1, d)$.

The estimates have so far been local - let us glue them together. Recall $\partial\Omega = \bigcup_{j=1}^m G_j, G_j^+ \subset \Omega$, and so

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} = \left\|\sum_{j=1}^m u_j\right\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_{p^\#}(\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(\operatorname{supp} u_j\cap\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(G_j)} \le$$
$$\le C\sum_{j=1}^m \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where we have used the fact $0 \le \varphi_j \le 1$ in the last inequality. And so we have shown

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le \|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall u \in \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right).$$

Now let $u \in \mathrm{W}^{1,p}(\Omega)$ be arbitrary. Since $\Omega \in \mathrm{C}^{0,1}$, there $\exists\{u_k\} \subset \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right)$ $s.t.$ $u_k \to u$ in $\mathrm{W}^{1,p}(\Omega)$. Set

$$\operatorname{tr} u = \lim_{k\to\infty} \tilde{\operatorname{tr}}\, u_k.$$

It is not totally evident that the definition is sensible, *i.e.*, that $\tilde{\operatorname{tr}} u_k$ converges. But using our derived estimate

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

it is easy to show that the sequence $\left\{\tilde{\operatorname{tr}} u_k\right\}$ is Cauchy in $\mathrm{L}_q(\partial\Omega)$, and so it is convergent. Also, from the arithmetic of the limits and linearity of $\tilde{\operatorname{tr}}$ we see tr is linear. Next, check

$$\|\operatorname{tr} u\|_{\mathrm{L}_q(\partial\Omega)} = \left\|\lim_{k\to\infty} \tilde{\operatorname{tr}} u_k\right\|_{\mathrm{L}_q(\partial\Omega)} = \lim_{k\to\infty} \|u_k \restriction_{\partial\Omega}\|_{\mathrm{L}_q(\partial\Omega)} \le C\lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\Omega)} = C\|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall q \in [1, p^\#].$$

and so inded $\operatorname{tr} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\partial\Omega), \forall q \in [1, p^\#]$ and it surely is bounded.

Case $p = d$

In this case, we have $\mathrm{id} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega), \mathrm{W}^{1,r}(\Omega)), \forall r \in [1, d)$ and the previous result tells us $\operatorname{tr} \in \mathcal{L}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1, r^\#]$. Observe that $r^\# = \frac{dr-r}{d-r} \to \infty$ as $r \to d^-$, meaning $\forall q \in [1, \infty)$

[23] Careful, this is a bit inaccurate (although not mentioned in the reference also); we should also include the volume form from the definition of the surface integral into our computation. However, that can be estimated by a constant(and included in C), since our $\Omega \in \mathrm{C}^{0,1}$.

there exists $r \in [1, d)$ $s.t.\, r^{\#} > q$, $i.e.\, q \in [1, r^{\#})$, which in fact means

$$\operatorname{tr} \in \mathcal{L}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall r \in [1, d), \forall q \in [1, \infty).$$

But then $\operatorname{tr} \circ \operatorname{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is a continous linear operator $\forall q \in [1, \infty)$, as it as a composition of continuous linear operators.

Case $p > d$

This is the easiest case: we know that $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*]$, so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right) \subset \mathrm{C}^0\left(\partial\Omega\right) \subset \mathrm{L}_\infty\left(\partial\Omega\right),$$

and so in total

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right),$$

which together with the fact $(\mathcal{H}_{d-1}(\partial\Omega) < \infty)$

$$\mathrm{L}_\infty\left(\partial\Omega\right) \hookrightarrow \mathrm{L}_q(\partial\Omega), \forall q \in [1, \infty),$$

concludes the proof. □

Remark. These results are very similiar to the results obtained in the case of embedding theorems - and it is not surprising, since it might seem we have in fact shown the embeddings of some Sobolev spaces into some Lebesgue spaces. One needs to be careful however, as it *is not an embedding* - we are taking Ω open, so $\partial\Omega \notin \Omega$. It only makes sense in the case of $p > d$.

The chapter will be concluded by stating the compact analogue to the embedding theorems.

Theorem 13 (Compact trace theorem)**.** *Let $d \in \mathbb{N}, \Omega \in C^{0,1}$, denote $p^{\#} = \frac{dp-p}{d-p}$, and let* tr *be the trace operator from the previous theorem. Then*

$$\operatorname{tr} \in \mathcal{K}\big(W^{1,p}(\Omega), L_q(\partial\Omega)\big),$$

where q is

- *if $p < d$, then $q \in [1, p^{\#})$,*
- *if $p = d$, then $q \in [1, \infty)$,*
- *if $p > d$, then $q \in [1, \infty]$,*

Proof. (*From: Bulíček et al., 2018*) *Case $p < d$:* Let us adopt the custom that when talking about the properties of $u \in \mathrm{W}^{1,p}(\Omega)$, in the space $\mathrm{L}_q(\partial\Omega)$, we are always talking about the properties of $\operatorname{tr} u$ in $\mathrm{L}_q(\partial\Omega)$...

It will be pretty similiar to the continuous case, so let us skip only to the key estimate. We are not apriori sure in which space we will be able to comfortably show the compactness, so let first

$q \in [1, \infty)$ and compute. Then we might use some interpolation estimates...

$$\int_{\mathrm{U}(0,\alpha)} \left|u_j\big(\mathbb{A}_j\big(x', a_j(x')\big)\big)\right|^q \mathrm{d}x' \le \left| \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \frac{\partial}{\partial s} \left|u_j\big(\mathbb{A}_j\big(x', s\big)\big)\right|^q \mathrm{d}s\, \mathrm{d}x' \right| \le$$
$$\le q\sqrt{1+\alpha^2} \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \left|u_j\big(\mathbb{A}_j\big(x', s\big)\big)\right| \left|\nabla u_j\big(\mathbb{A}_j\big(x', s\big)\big)\right| \mathrm{d}s\, \mathrm{d}x' \le$$
$$\le q\sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{q-1} |\nabla u_j(x)|\, \mathrm{d}x \le q\sqrt{1+\alpha^2} \|\nabla u_j\|_{\mathrm{L}_q\left(G_j^+\right)} \|u_j\|^{q-1}_{\mathrm{L}_{q'(q-1)}\left(G_j^+\right)},$$

where $q'(q-1) = \frac{q}{q-1}(q-1) = q$, so all in all

$$\|u_j\|_{\mathrm{L}_q(G_j)} \le C(q,\Omega) \|\nabla u_j\|^{\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \le C(q,\Omega) \|u_j\|^{\frac{1}{q}}_{\mathrm{W}^{1,q}\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)}$$

which leads to

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} = \left\| \sum_{j=1}^m u_j \right\|_{\mathrm{L}_q(\partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(\operatorname{supp} u_j \cap \partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(G_j)} \le$$
$$\le C \sum_{j=1} \|u_j\|^{\frac{1}{q}}_{\mathrm{W}^{1,q}\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \le C_1 \|u\|^{\frac{1}{q}}_{\mathrm{W}^{1,p}(\Omega)} \|u\|^{1-\frac{1}{q}}_{\mathrm{L}_q(\Omega)}.$$

Denote $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$; we will show $\operatorname{tr} U$ is totally bounded in $\mathrm{L}_q(\partial\Omega)$ for *some* $q \in [1,\infty)$. Let $\varepsilon > 0$ be given. Realize that $\forall p \in [1,d)$ it is $p < p^*$, and so it always holds $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\mathrm{p}(\Omega)$. For the moment, we pick $q = p$. Denote $\{u_k\}_{k=1}^m$ to be the δ–net from U in $\mathrm{L}_\mathrm{p}(\Omega)$, where δ will be chosen suitably later. Let now $u \in U$ be arbitrary and find $u_i \in U$ *s.t.* $\|u - u_i\|_{\mathrm{L}_\mathrm{p}(\Omega)} \le \delta$. Using the estimate from above we have

$$\|u - u_i\|_{\mathrm{L}_p(\partial\Omega)} \le C_1 \|u - u_i\|^{\frac{1}{q}}_{\mathrm{W}^{1,p}(\Omega)} \|u - u_i\|^{1-\frac{1}{q}}_{\mathrm{L}_p(\Omega)} \le C_1 2^{\frac{1}{q}} \delta^{1-\frac{1}{q}},$$

where we have used the fact $u, u_i \in U$. We see that upon choosing

$$\delta < \left(\frac{\varepsilon}{C_1 2^{\frac{1}{q}}} \right)^{\frac{1}{1-q}},$$

we in fact have

$$\|u - u_i\|_{\mathrm{L}_\mathrm{p}(\partial\Omega)} < \varepsilon,$$

and so $\{\operatorname{tr} u_i\}_{i=1}^m \subset U$ is a ε–net in $\mathrm{L}_\mathrm{p}(\partial\Omega)$, meaning

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_\mathrm{p}(\partial\Omega)\big).$$

Since now $\mathrm{L}_p(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega)$, $\forall q \in [1,p]$, we have also shown

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1,p].$$

The remaining case is when $q \in (p, p^{\#})$. As in the case of compact embedding of Sobolev spaces, a suitable interpolation theorem will do the job for us. It holds

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \le \|u\|^{\theta}_{\mathrm{L}_p(\partial\Omega)} \|u\|^{1-\theta}_{\mathrm{L}_{p^{\#}}(\partial\Omega)},$$

where $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^\#}$. Let now $\{u_i\}_{i=1}^m \subset U$ be such that $\{\operatorname{tr} u_i\}_{i=1}^m$ is the β–net in $\mathrm{L_p}(\partial\Omega)$ whose existence we have just proven $\forall \beta > 0$. Recall also that since U is bounded and $\operatorname{tr} \in \mathcal{L}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big) \forall q \in [1, p^\#]$, there exists $0 < C_2 < \infty$ *s.t.*

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \leq C_2, \forall u \in U.$$

and so $\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \leq C_2, \forall u \in U$ in particular.

Finally, $\forall \in U$ it holds

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \leq \|u - u_i\|^\theta_{\mathrm{L_p}(\partial\Omega)} \|u - u_i\|^{1-\theta}_{\mathrm{L}_{p^\#}(\partial\Omega)} \leq \beta^\theta (2C_2)^{1-\theta},$$

so if we choose

$$\beta < \left(\frac{\varepsilon}{(2C_2)^{1-\theta}}\right)^{\frac{1}{\theta}},$$

then

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \leq \varepsilon, \forall u \in U.$$

which concludes the proof $\{\operatorname{tr} u_i\}_{i=1}^m$ is an ε–net in U in $\mathrm{L}_q(\partial\Omega)$ also for $q \in (p, p^\#)$. In total, we have showed the compactness of the trace operator for all $q \in [1, p^\#)$.

Case $p = d$

In this case $\operatorname{tr} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega), \mathrm{L}_q(\partial\Omega)), \forall q \in [1, \infty)$. We also know $\operatorname{id} \in \mathcal{L}\big(\mathrm{W}^{1,d}(\Omega), \mathrm{W}^{1,r}(\Omega)\big), \forall r \in [1, d)$ and that

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1, r^\#).$$

Repating the same arguments, we see $r^\# \to \infty$ as $r \to d^-$, meaning $\forall q \in [1, \infty)$ there exists $r \in [1, d)$ *s.t.* $r^\# > q$, and consequently this implies

$$\forall q \in [1, \infty) \exists r \in [1, d)\ s.t.\ \operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big).$$

But then the operator $\operatorname{tr} \circ \operatorname{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is compact $\forall q \in [1, \infty)$, as it is a composition of a (linear) continuous and compact operator.

Case $p > d$

This case is again trivial: from the embedding theorems, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu),$$

so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right) \subset \mathrm{C}^0\left(\partial\Omega\right) \subset \mathrm{L}_\infty\left(\partial\Omega\right),$$

meaning

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right),$$

which together with the embedding

$$\mathrm{L}_q(\partial\Omega) \hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right), \forall q \in [1, \infty).$$

completes the proof for any $q \in [1, \infty]$. We are done once again.

$\square$

2.6 Fine properties of Sobolev spaces

In this last chapter, we present some other useful properties of Sobolev spaces.

2.6.1 Composition of Sobolev functions

In the chapter about extension of functions from Sobolev spaces we have seen: $U, V \subset \mathbb{R}^d$ open, $u \in \mathrm{W}^{1,p}(U)$, $\Phi : U \to V$ a C^1-diffemorphism $\Rightarrow u \circ \Phi \in \mathrm{W}^{1,p}(V)$. Is there any other class of mappings that guarantee the composition remains in some Sobolev spaces?

Theorem 14 (Derivative of superposition)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1, \infty]$, $u : \Omega \to \mathbb{R}^d$ be in $W^{1,p}(\Omega)$. Denote for an arbitrary $a \in \mathbb{R}$ the set*

$$\Omega_a = \{x \in \Omega | u(x) = a\}.$$

Then

1. *$\nabla u = 0$ on Ω_a,*
2. *if $f \in C^{0,1}\left(\mathbb{R}^d\right)$ s.t. $\|f\|_{L_\infty(\Omega)} < \infty$ (it is a Lipschitz continuous function), then $f \circ u - f(0) \in W^{1,p}(\Omega)$ and it holds*

$$\nabla(f \circ u) = \begin{cases} (f' \circ u)\nabla u, & a.e.\ in \{u \notin S\}, \\ 0, & a.e.\ in \{u \in S\} \end{cases},$$

where

$$S = \left\{s \in \mathbb{R} | f'(s) \text{ does not exist}\right\}$$

(in the strong sense).

Proof. (*From: the lectures & Bulíček et al., 2018*) The proof has been presented for the case $f \in \mathrm{C}^1(\mathbb{R}), \|f\|_{\mathrm{L}_\infty(\mathbb{R})} < \infty$. In the general setting, one can play with *a.e.* convergence and Rademacher theorem.

Denote $L = \|f\|_{\mathrm{L}_\infty(\mathbb{R})}$. Since f is continuous, $f \circ u$ is measurable. Moreover, from the Lipschitzity on the whole $\mathbb{R}$ it follows

$$|(f \circ u)(x) - f(0)| \le L|u(x) - 0| = L|u(x)|,$$

and since $u \in \mathrm{L}_\mathrm{p}(\Omega)$, also $f \circ u - f(0) \in \mathrm{L}_\mathrm{p}(\Omega)$. To obtain the wanted result, we need to show $\nabla(f \circ u)$ exists, so let us move on to the second claim. The strategy is to work with some regularization and then pass to the limit with the support size. In concrete terms, pick $\varphi \in \mathcal{D}(\Omega)$, and regularization u_ε *s.t.* $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > 2\varepsilon$. Then we can write

$$-\int_\Omega (f \circ u_\varepsilon)(x)\partial_i\varphi(x)\,\mathrm{d}x = \int_\Omega \left(f' \circ u_\varepsilon\right)(x)\partial_i u_\varepsilon(x)\varphi\,\mathrm{d}x,$$

using per partes and chain rules (those are strong derivatives). Now we would like to pass to the limit $\varepsilon \to 0^+$. Take the LHS:

$$\lim_{\varepsilon\to 0^+} \left|\int_\Omega (f \circ u_\varepsilon - f \circ u)\partial_i\varphi\,\mathrm{d}x\right| \le \lim_{\varepsilon\to 0^+}\int_\Omega |f \circ u_\varepsilon - f \circ u||\partial_i\varphi|\,\mathrm{d}x \le L\|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega)} \lim_{\varepsilon\to 0^+}\int_{\Omega\cap\operatorname{supp}\varphi} |u_\varepsilon - u|\,\mathrm{d}x = 0,$$

as $u_\varepsilon \to u$ in $\mathrm{L}_1(\Omega \cap \operatorname{supp}\varphi)$ (notice from the theorem on mollification we actually obtain this convergence in $\mathrm{L}_1(\Omega_\varepsilon)$, but we have made a good choice of φ) The RHS can be manipulated

$$\lim_{\varepsilon\to 0^+}\left|\int_\Omega \big(f'\circ u_\varepsilon\big)\partial_i u_\varepsilon\varphi-\big(f'\circ u\big)\partial_i u\varphi\,\mathrm{d}x\right|\le \lim_{\varepsilon\to 0^+}\int_\Omega\big|\big(f'\circ u_\varepsilon-f'\circ u\big)\big||\partial_i u||\varphi|\,\mathrm{d}x+$$
$$+\lim_{\varepsilon\to 0^+}\int_\Omega \big|f'\circ u_\varepsilon\big||\partial_i u_\varepsilon-\partial_i u||\varphi|\,\mathrm{d}x\le$$
$$\le\|\varphi\|_{\mathrm{L}_\infty(\Omega)}\lim_{\varepsilon\to 0^+}\int_{\Omega\cap\operatorname{supp}\varphi}\big|f'\circ u_\varepsilon-f'\circ u\big||\partial_i u|\,\mathrm{d}x+L\|\varphi\|_{\mathrm{L}_\infty(\Omega)}\lim_{\varepsilon\to 0^+}\int_{\Omega\cap\operatorname{supp}\varphi}|\partial_i u_\varepsilon-\partial_i u|\,\mathrm{d}x=$$
$$=\|\varphi\|_{\mathrm{L}_\infty(\Omega)}\lim_{\varepsilon\to 0^+}\int_\Omega\big|f'\circ u_\varepsilon-f'\circ u\big||\partial_i u|\,\mathrm{d}x\,,$$

where we have used $\nabla u_\varepsilon\to\nabla u$ in $\mathrm{L}_1(\Omega\cap\operatorname{supp}\varphi)$. As for the second integral, recall that $u_\varepsilon\to u$ *a.e.* in $\Omega\cap\operatorname{supp}\varphi$ and that f' is globally continous. Thus with the majorant

$$\big|f'\circ u_\varepsilon-f'\circ u\big||\partial_i u|\le L\|u\|_{\mathrm{W}^{1,p}(\Omega)}|u_\varepsilon-u|,$$

that is integrable on the bounded set $\operatorname{supp}\varphi\cap\Omega$ we see the second integral is zero aswell. All in all, we have shown

$$-\int_\Omega(f\circ u)\partial_i\varphi\,\mathrm{d}x=\int_\Omega\big(f'\circ u\big)\partial_i u\varphi\,\mathrm{d}x\,,\forall\varphi\in\mathcal{D}(\Omega)\,,$$

which is exactly what we need to state that the ∂_i-weak derivative of $f\circ u$ exists and is equal to $(f'\circ u)\partial_i u$. Finally, the following estimate holds:

$$\Bigg\|\underbrace{\nabla(f\circ u)}_{=\nabla(f\circ u-f(0))}\Bigg\|_{\mathrm{L_p}(\Omega)}=\big\|\big(f'\circ u\big)\nabla u\big\|_{\mathrm{L_p}(\Omega)}\le L\|\nabla u\|_{\mathrm{L_p}(\Omega)}\le L\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

and so

$$f\circ u-f(0)\in\mathrm{W}^{1,p}(\Omega)\,.$$

Let us now deal with the first assertion. We will first show $\nabla u=0$ *a.e.* on Ω_0. For that, choose a special function

$$f(x)=\begin{cases}x, & x>0,\\ 0, & x\le 0,\end{cases}$$

and so $f\circ u=u^+$. If we now set

$$f_\varepsilon(x)=\begin{cases}\big(x^2+\varepsilon^2\big)^{1/2}-\varepsilon, & x>0,\\ 0, & x\le 0,\end{cases}$$

with the derivative being

$$f_\varepsilon'(x)=\begin{cases}\frac{x}{(x^2+\varepsilon^2)^{1/2}}, & x>0,\\ 0, x\le 0\end{cases}$$

and so $\lim_{\varepsilon\to 0^+}f_\varepsilon(x)=f(x)$, and $\lim_{\varepsilon\to 0^+}f_\varepsilon'(x)=\chi_{\mathbb{R}^+}(x)$ *a.e.* in $\mathbb{R}$, meaning

$$\lim_{\varepsilon\to 0^+}f_\varepsilon\circ u=u^+,\ \lim_{\varepsilon\to 0^+}f_\varepsilon'\circ u=\chi_{x\in\Omega|u(x)>0},$$

a.e. in Ω. Using the preivously obtained general expression and passing to the limit yields

$$-\int_\Omega (f\circ u)\nabla\varphi\,\mathrm{d}x = \int_\Omega \nabla u\varphi\chi_{\{x\in\Omega|u(x)>0\}}\,\mathrm{d}x\,,$$

and realizing[24] $u^- = (-u)^+$, we also have

$$-\int_\Omega (f\circ u)\nabla\varphi\,\mathrm{d}x = -\int_\Omega \nabla u\varphi\chi_{\{x\in\Omega|u(x)<0\}}\,\mathrm{d}x\,,$$

meaning (in the weak sense)

$$\nabla u^+ = \nabla u\chi_{\{x\in\Omega|u(x)>0\}}, \nabla u^- = -\nabla u\chi_{\{x\in\Omega|u(x)<0\}}.$$

But since $u = u^+ - u^-$, it holds $\nabla u = \nabla u^+ - \nabla u^-$ *a.e.*. We see $\nabla u^- = \nabla u^+ = 0$, on Ω_0, and so $\nabla u = 0$ on *a.e.* on Ω_0. Taking $\tilde{u} = u + c$ for $c\in\mathbb{R}$ arbitrary, we see $\nabla\tilde{u} = \nabla u = 0$ *a.e.* on Ω_0, meaning $\nabla\tilde{u} = 0$ *a.e.* on $\{x\in\Omega|\tilde{u} - c = 0\} = \Omega_c$. Since $\tilde{u}, c$ were arbitrary, we are done with this proof.

If $u\notin S$, the above presentation works just fine. If $u(x)\in S$, we know $\nabla u = 0$ *a.e.* on Ω_u, and since $\lambda(S) = 0$ by Rademacher theorem, we have also the above conclusion. This argumentation is sloppy, but has not been presented in full detail.

□

2.6.2 Difference quotients

There is some interplay between the weak derivative and difference quotients, which can be useful in some proofs, *e.g.*, regularity results for elliptic problems. We have seen that if a function posseses classical derivatives, it also has weak derivatives and they conincide *a.e.*. Is something there something to be said about the opossite implication?

Definition 6 (Difference quotient)**.** Let $u:\mathbb{R}^d\to\mathbb{R}, h\in\mathbb{R}$. For each $i\in\{1,\ldots,d\}$ we define the difference quotient as

$$\Delta_h^i u(x) = \frac{u(x+he_i) - u(x)}{h},$$

where e_i is the canoncal base vector of $\mathbb{R}^d$.

Remark. If $\partial_i u(x)$ exists, then clearly

$$\partial_i u(x) = \lim_{h\to 0}\Delta_h^i u(x).$$

Let us jump straight to the main result.

Theorem 15. *(From: Bulíček et al., 2018) Let $\Omega\subset\mathbb{R}^d$ open, $p\in[1,\infty]$ and $u\in L_p(\Omega)$. Denote $\forall\delta>0$ the set*

$$\Omega_\delta = \{x\in\Omega|\operatorname{dist}(x,\partial\Omega)>\delta\}.$$

Then it holds

1. *If moreover $u\in W^{1,p}(\Omega)$, then $\forall i\in\{1,\ldots,d\}, \delta\in(0,1), h\in\left(0,\frac{\delta}{2}\right)$*

$$\left\|\Delta_h^i u\right\|_{L_p(\Omega_\delta)} \le \|\partial_i u\|_{L_p(\Omega)}.$$

2. *Let $p\in(1,\infty]$ and let there exist $\{C_i\}_{i=1}^d$ constants s.t. $\forall i\in\{1,\ldots,d\}, \delta\in(0,1), h\in\left(0,\frac{\delta}{2}\right)$*

[24] Recall $u^+ = \max(0,u), u^- = -\min(0,u)$ are both nonnegative functions, $u = u^+ - u^-, |u| = u^+ + u^-$. It holds trivially $u^- = -\min(0,u) = \max(0,-u) = (-u)^+$.

it holds

$$\left\|\Delta_h^i u\right\|_{L_p(\Omega_\delta)} \leq C_i.$$

Then $u \in W^{1,p}(\Omega)$ *and moreover* $\forall i \in \{1, \ldots, d\}$ *we have*

$$\left\|\partial_i u\right\|_{L_p(\Omega)} \leq C_i.$$

Proof. Ad 1. Let first $p \in [1, \infty)$. *What else do we use to estimate differences then..., right?* If needed, extend u by zero outside of Ω and set $u_\varepsilon = u \star \omega_\varepsilon$. Recall that $u_\varepsilon \to u$ in $\mathrm{W}^{1,p}\Big(\Omega_{\frac{\delta}{2}}\Big)$. Realize that since $u_\varepsilon \in \mathrm{C}^1\left(\mathbb{R}^d\right)$, we have

$$\Delta_h^i u_\varepsilon(x) = \frac{u_\varepsilon(x + he_i) - u_\varepsilon(x)}{h} = \frac{1}{h}\int_0^h \partial_t u_\varepsilon(x + te_i)\,\mathrm{d}t\,.$$

Let us compute $\left\|\Delta_h^i\right\|_{\mathrm{L}_p(\Omega_\delta)}$. We have

$$\begin{aligned}\left|\Delta_h^i u_\varepsilon\right|^p &= \left|\frac{1}{h}\int_0^h \partial_t u_\varepsilon(x + te_i)\,\mathrm{d}t\right|^p \leq \frac{1}{h^p}\bigg(\int_0^h \left|\partial_t u_\varepsilon(x + te_i)\right|\mathrm{d}t\bigg)^p \leq \frac{1}{h^p}\Bigg(\bigg(\int_0^h \left|\partial_t u_\varepsilon(x + te_i)\right|^p\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\int_0^h 1\,\mathrm{d}t\bigg)^{\frac{p-1}{p}}\Bigg)^p = \\ &= \frac{1}{h^p}\int_0^h \left|\partial_t u_\varepsilon(x + te_i)\right|^p\mathrm{d}t\,h^{p-1} = \frac{1}{h}\int_0^h \left|\partial_t u_\varepsilon(x + te_i)\right|^p\mathrm{d}t\,,\end{aligned}$$

and so integrating over Ω_δ yields

$$\left\|\Delta_h^i u_\varepsilon\right\|_{\mathrm{L_p}(\Omega_\delta)}^p \leq \frac{1}{h}\int_{\Omega_\delta}\int_0^h \left|\partial_t u_\varepsilon(x + te_i)\right|^p\mathrm{d}t\,\mathrm{d}x = \frac{1}{h}\int_0^h\int_{\Omega_\delta} \left|\partial_t u_\varepsilon(x + te_i)\right|^p\mathrm{d}x\,\mathrm{d}t\,,$$

denote $z = x + te_i$, then $z \in \Omega_\delta + te_i$, meaning the points have shifted towards Ω in one direction by at most $t \leq h < \frac{\delta}{2}$. If we integrate over $\Omega_{\frac{\delta}{2}}$ instead, we make the domain larger and so

$$\left\|\Delta_h^i u_\varepsilon\right\|_{\mathrm{L_p}(\Omega_\delta)}^p \leq \frac{1}{h}\int_0^h\int_{\Omega_{\frac{\delta}{2}}} \left|\partial_t u_\varepsilon(z)\right|^p\mathrm{d}z\,\mathrm{d}t \leq \frac{1}{h}h\left\|\partial_t u_\varepsilon\right\|_{\mathrm{L}_p\left(\Omega_{\frac{\delta}{2}}\right)}^p,$$

so if we pass to the limit $\varepsilon \to 0^+$ (we are in Ω_δ), we see

$$\left\|\Delta_h^i u\right\|_{\mathrm{L_p}(\Omega_\delta)} \leq \left\|\partial_t u\right\|_{\mathrm{L_p}\left(\Omega_{\frac{\delta}{2}}\right)} \leq \left\|\partial_t u\right\|_{\mathrm{L_p}(\Omega)}.$$

If now $p = \infty$, the things are not that simple - Ω is not bounded apriori, so we have no ordering of Lebesgue spaces. Let us fix that: take some $R > 0$ and denote $\Omega^R = \Omega \cap \mathrm{U}(0, R)$. If $u \in \mathrm{W}^{1,\infty}(\Omega)$, it must be $u \in \mathrm{W}^{1,p}\big(\Omega^R\big)$ for all $p \in [1, \infty)$. On Ω_δ^R we have from the above

$$\left\|\Delta_h^i u\right\|_{\mathrm{L_p}\left(\Omega_\delta^R\right)} \leq \left\|\partial_t u\right\|_{\mathrm{L_p}(\Omega^R)},$$

and since Ω_δ^R is bounded, we can pass to the limit $p \to \infty$ and write

$$\left\|\Delta_h^i u\right\|_{\mathrm{L}_\infty\left(\Omega_\delta^R\right)} \leq \left\|\partial_t u\right\|_{\mathrm{L}_\infty(\Omega^R)},$$

and passing to the limit $R \to \infty$ we obtain

$$\left\|\Delta_h^i u\right\|_{\mathrm{L}_\infty(\Omega_\delta)} \leq \left\|\partial_t u\right\|_{\mathrm{L}_\infty(\Omega)}.$$

Ad 2. Once again extend u by zero outside of Ω and let for the moment $p \in (1, \infty)$. Fix

$\{h_n\} \subset \left(0, \frac{\delta}{2}\right) s.t. h_n \to 0$ and consider the functions

$$v_n^i = \Delta_{h_n}^i u \chi_{\Omega_{2h_n}}.$$

Clearly (meaning from the assumptions)

$$\|v_n\|_{\mathrm{L_p}(\Omega)}^i = \left\|\Delta_{h_n}^i u\right\|_{\mathrm{L_p}(\Omega_{2h_n})} \le C_i,$$

and from the reflexivity of $\mathrm{L_p}(\Omega), p \in (1, \infty)$, we know $\exists v_i \in \mathrm{L_p}(\Omega)$ such that

$$v_n^i \rightharpoonup v^i,$$

as $n \to \infty$. Since the norm is weak lower semicontinuous, we also have the estimate

$$\|v_i\|_{\mathrm{L_p}(\Omega)} \le C_i.$$

It remains to show now that v_i are weak derivatives of u. Before we procced, let us mention the following identity: let $\varphi \in \mathcal{D}(\Omega)$, then

$$\int_{\mathbb{R}^d} \Delta_h^i u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} u\Delta_{-h}^i \varphi \,\mathrm{d}x\,.$$

Really, it holds **THIS NEEDS TO BE FINISHED**

$$\int_{\mathbb{R}^d} \Delta_h^i u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{u(x+he_i) - u(x)}{h} \varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{\varphi(x)u(x+he_i) - \varphi(x)u(x)}{h} \,\mathrm{d}x\,,$$

change the variables $y = x + he_i$, then the above integral is equal to

$$\int_{\mathbb{R}^d} \frac{\varphi(y-he_i)u(y) - \varphi(y-he_i)u(y-he_i)}{h} \,\mathrm{d}y =$$

Using this, we can obtain

$$\begin{aligned}
\int_\Omega v_i\varphi \,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega v_n^i \varphi \,\mathrm{d}x = \lim_{n\to\infty} \int_{\Omega_{2h_n}} \Delta_{h_n}^i u\varphi \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} \Delta_{h_n}^i u\varphi \,\mathrm{d}x = \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} u\Delta_{-h_n}^i \varphi \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} u \frac{\varphi(x - h_n e_i) - \varphi(x)}{h_n} u(x) \,\mathrm{d}x = -\int_{\mathbb{R}^d} u\partial_i\varphi \,\mathrm{d}x = \\
&= -\int_\Omega u\partial_i \varphi \,\mathrm{d}x
\end{aligned}$$

which really means $v_i = \partial_i u$ in the weak sense. The fact we can integrate over $\mathbb{R}^d$ instead of Ω_{2h_n} comes from the fact $\operatorname{supp}\varphi \subset \Omega_{2h_n}$ for some n large enough, and since the whole integrand is zero on any larger set then Ω_{2h_n}, meaning we without a doubt expand the integration domain to the whole space; the interchange of differentiation and integration is simple, because φ has a compact support. All in all, we have shown $v_i = \partial_i u$ weakly and $\|v_i\|_{\mathrm{L_p}(\Omega)} \le C_i$, so we are done with the case $p \in (1, \infty)$.

If $p = \infty$, we can repeat the same arguments as in the above case: create a bounded domain $\Omega_R = \Omega \cap \mathrm{U}(0, R)$, realize that

$$\|v_i\|_{\mathrm{L_p}(\Omega^R)} \le C_i, \forall p \in (1, \infty),$$

and then pass to the limits $R \to \infty, p \to \infty$.

$\square$

Remark. We have thus shown that if $p \in (1, \infty)$, then

$$\Delta_h^i u \rightharpoonup \partial_i u, h \to 0^+.$$

2.6.3 Representation of duals

The last stop within our adventure through Sobolev spaces will be the dual. We present a representation theorem for dual spaces of certain Sobolev spaces. The proof will use techniques developed later, in the chapter 4.

Theorem 16. *Let $\Omega \in C^{0,1}$. Let $X = W_0^{1,r}(\Omega)\,, r \in (1,\infty)$ with equivalent norm $|||u||| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$. Then $\forall \Phi \in X^* \exists \mathbf{F} \in L_{r'}(\Omega)$ such that*

$$\forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_\Omega \mathbf{F} \cdot \nabla \varphi \, \mathrm{d}x\,,$$

and moreover it holds $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}$.

Proof. (*From: the lectures*) We solve the problem

$$\begin{cases} -\nabla \cdot \left(|\nabla u|^{r-2} \nabla u\right) & = \Phi, \text{ in } \Omega, \\ u & = 0, \text{ on } \partial\Omega \end{cases}. \tag{2}$$

Let us check the solution exists: denote

$$a_0 = 0, \mathbf{a}(x, z, \mathbf{p}) = |\mathbf{p}|^{r-2}\mathbf{p}.$$

Then clearly

$$|a_i(x, z, \mathbf{p})| \le |\mathbf{p}|^{r-1},$$

so we have the right grow conditions, next:

$$\mathbf{a}(x, z, p) \cdot \mathbf{p} = |\mathbf{p}|^{r-2}\mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^r,$$

so we have coercivity, and finally monotonicity; realize that actually[25]

$$\mathbf{a}(x, z, p) = \frac{1}{r} \nabla |\mathbf{p}|^r,$$

and for $r > 1$ is the function $\mathbf{p} \mapsto |\mathbf{p}|^r$ strictly convex. So from the classification of convex functions we know

$$\frac{1}{r}|\mathbf{p}|^r \text{ is strictly convex } \Leftrightarrow \nabla \frac{1}{r}|\mathbf{p}|^r = \mathbf{a} \text{ is strictly monotone}.$$

By the existence & uniqueness theorem on problems with monotone operator, we conclude such $u \in \mathrm{W}_0^{1,r}(\Omega)$ exists and is unique by. The weak formulation of the above problem is:

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \Phi(\varphi).$$

So it seems proimising to set $\mathbf{F} = |\nabla u|^{r-2} \nabla u$. If we test now the weak formulation with u itself (recall what norm are we using on X)

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^r = \Phi(u) \le \|\Phi\|_{X^*} |||u||| = \|\Phi\|_{X^*} \|\nabla u\|_{\mathrm{L}_r(\Omega)}.$$

[25] $\nabla |\mathbf{p}|^r = r|\mathbf{p}|^{r-1} |\mathbf{p}|^{-1}\mathbf{p} = r|\mathbf{p}|^{r-2}$

If now $\|\nabla u\|_{\mathrm{L}_r(\Omega)} = 0$, then $\Phi = 0$ and we are finished; if it is nonzero, then we can divide and write

$$\|\nabla u\|^{r-1}_{\mathrm{L}_r(\Omega)} \le \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|^{r-1}_{\mathrm{L}_r(\Omega)} = \left\||\nabla u|^{r-1}\right\|_{\mathrm{L}_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \le \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{\mathrm{B}_X(0,1)} |\Phi(\varphi)| = \sup_{\mathrm{B}_X(0,1)} \left|\int_\Omega \mathbf{F}\cdot\nabla\varphi \,\mathrm{d}x\right| \le \sup_{\mathrm{B}_X(0,1)} \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}\|\nabla\varphi\|_{\mathrm{L}_r(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)},$$

which concludes $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}$. □

3 Nonlinear elliptic equations - compactness methods

So far we have dealt with only linear equations, *i.e.*, linear operators. When dealing with nonlinear operators, things get much more involved. This chapter will be dedicated to the study of nonlinear problems using compactness methods, which will especially allow us to solve problems with nonlinearities in the non-leading terms. **Be especially careful about the assumptions on the operators - they do not need to be linear.**

Let us start with some technical, mostly known lemmas.

3.1 Nemytskii operators

Lemma 9 (Fatou)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable, let $\{f_n\}$ be a sequence of measurable nonnegative functions. Then it holds*

$$\int_\Omega \liminf_{n\to\infty} f_n \,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x\,.$$

(Both integrals can be infinity.)

Theorem 17 (Iegorov)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable and of finite Lebesgue measure, let $\{f_n\}$ and f be measurable and finite a.e. in Ω. Then the following statements are equivalent*

1. $f_n(x) \to f(x), \forall$ *a.a.* $x \in \Omega$,
2. $\forall\varepsilon > 0 \exists G \subset \Omega$ *open s.t.* $\lambda(G) < \varepsilon$ *and* $f_n \rightrightarrows f$ *on* Ω/G.

Theorem 18 (Vitali)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable and of finite Lebesgue measure, let $\{f_n\}$ and f be measurable s.t. $f_n(x) \to f(x)$for a.a. $x \in \Omega$. Let moreover be true*

$$\forall\varepsilon > 0 \exists\delta > 0 : \forall n \in \mathbb{N}, \forall H \subset \Omega \, s.t.\, \lambda(H) < \delta \Rightarrow \int_H |f_n| \,\mathrm{d}x \le \varepsilon.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x\,.$$

Remark (What is $f(\cdot, x)$?)*.* In the following definitions and throughout the chapter in general, we use the "$\cdot$" to denote variables in the following sense. Just to make it clear, this means the following. If U, V, W are some sets and $f : U \times V \to W$, we write

- $f(u, \cdot) : V \to W$, as the mapping $V \ni v \mapsto f(u, v) \in W$ for some $u \in U$ fixed,
- $f(\cdot, v) : U \to W$, as the mapping $U \ni u \mapsto f(u, v) \in W$ for some $v \in V$ fixed.

Definition 7 (Caratheodory function)**.** Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}$. We say the function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is Caratheodory if

1. $f(x, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is continuous for almost all $x \in \Omega$,
2. $f(\cdot, y) : \Omega \to \mathbb{R}$ is measurable for all $y \in \mathbb{R}^N$.

Definition 8 (Nemytskii operator)**.** Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Caratheodory. For $\mathbf{u} : \Omega \to \mathbb{R}^N, x \in \Omega$ we define the Nemytskii operator N_f: as

$$N_f : \mathbf{u}(x) \mapsto f(x, \mathbf{u}(x)),$$

meaning N_f is an operator between function spaces, whose image is the function $N_f \mathbf{u} : \Omega \to \mathbb{R}$ *s.t.*

$$N_f \mathbf{u}(x) = f(x, \mathbf{u}(x)), x \in \Omega.$$

Remark. We write **u** in bold to stress $u : \Omega \to \mathbb{R}^N$ is a vector valued mapping, even though we might drop the notation later on.

Remark. This definition is similiar to the one used in the Caratheodory theory of ordinary differential equations, but not the same.

Theorem 19 (On Nemytskii operators)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable, $N \in \mathbb{N}, f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Caratheodory. Then*

1. *if $\mathbf{u} : \Omega \to \mathbb{R}^N$ is measurable, then $N_f \mathbf{u} : \Omega \to \mathbb{R}$ is also measurable,*
2. *if there are numbers $p_i \in [1, \infty), i \in \{1, \ldots, N\}$, an exponent $p \in [1, \infty)$, a function g s.t. $g \in L_p(\Omega)$, and a constant $C \geq 0$ such that for almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$ it holds*

$$|f(x, y)| \leq g(x) + C \sum_{i=1}^{N} |y_i|^{\frac{p_i}{p}},$$

then the Nemytskii operator $N_f : \mathbf{u} \mapsto N_f \mathbf{u}$ is continuous from $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \ldots L_{p_N}(\Omega)$ to $L_p(\Omega)$ and moreover it maps sets bounded in $L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \ldots L_{p_N}(\Omega)$ to sets bounded in $L_p(\Omega)$. We write

$$N_f \in \mathcal{C}(L_{p_1}(\Omega), \ldots, L_{p_N}(\Omega); L_p(\Omega)).$$

Proof. (*From: the lectures*) No proof □

Remark. The information about boundedness of images of bounded sets is not trivial - *the Nemytskii operator is not linear in general.*

3.2 Fixed point theorems

The majority of the problems we will be dealing with will be solved by using some fixed points theorem. Here we state two of them (without proofs).

Definition 9 (Compact operator)**.** Let X, Y be normed linear spaces, $M \subset X$. The mapping $F : M \to Y$ is called a compact operator on M into Y provided

1. F is continuous,
2. $F(M \cap K) \subset Y$ is relatively compact in Y for any bounded $K \subset X$.

Remark.

- We have no linearity of F! So continuity cannot follow from compactness and we have to assume it explicitely. In general, compactness $\Rightarrow$ boundedness $\neq$ continuity for nonlinear operators.
- one might wonder why just not state $F(K) \subset Y$ is relatively compact in Y for $K \subset M$ bounded. First of all, this is pretty much the same, but in the applications, this formulation will be more suitable for us. Secondly, in general M need not be a subspace of X, let alone a normed linear space;in such cases the notion of boundedness of $K \subset M$ is not defined at all. On the other hand, boundedness of $K \subset X$ can be measured in the metric of X.
- The definition is from *Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2*

Theorem 20 (Brouwer fixed point theorem)**.** *Let $N \in \mathbb{N}$ and $K \subset \mathbb{R}^N$, be a nonempty convex closed bounded subset of $\mathbb{R}^N$. Assume further that $F : K \to K$ is continuous. Then F has a fixed point in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. (*From: the lectures*) No proof. □

Theorem 21 (Schauder fixed point theorem)**.** *Let X be a linear normed space and $K \subset X$ be a nonempty convex closed bonded subset of X. Assume further that $F : K \to K$ is compact on K into K and $F(K) \subset K$. Then there is fixed point of F in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. (*From: the lectures*) No proof. □

Remark.

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bouded, it is automatically compact, and since $F : K \to K$ is continuous, F is compact[26]. For Schauder, we have to assume this extra.
- realize again that in Schauder, the set K does not have to be a subspace of X.
- proof of Brouwer with N=1 is easy, based on Darboux property.

3.3 Problem protypes

In this chapter some nonlinear elliptic equations are discussed. The strategy of solving them (*i.e.*, proving a solution exists) will be always the same - define a suitable operator and show it has a fixed point.

Example. Suppose the following problem:

$$\begin{cases} -\triangle u + g(u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

where $f \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*, g : \mathbb{R} \to \mathbb{R}$ is continuous and has a controlled grow *s.t.* $\exists C > 0, \exists \alpha \in (0,1]$:

$$\forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

Even though g might be nonlinear, we assume only (sub)linear growth of it - so this is kind of cheating.

[26] Image of a compact set under continuous mapping is a compact set

Theorem 22 (Existence)**.** *Let $\Omega \in C^{1,1}, f \in \left(W_0^{1,2}(\Omega)\right)^*, g$ is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi + g(u)\varphi \,\mathrm{d}x =< f, \varphi >_{(W_0^{1,2}(\Omega))^*} .$$

If $f \in L_2(\Omega)$, then the solution $u \in W^{2,2}(\Omega)$.

Proof. (*From: the lectures*) We define $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ such that[27]

$$Sw = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x =< f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x .$$

Clearly, when $Sw = w$, *i.e.*, S has a fixed point and then

$$Sw = w \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla w \cdot \nabla \varphi \,\mathrm{d}x =< f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x ,$$

and so the solution w to our problem exists. S is defined in an implicit way, but it might be guiding to think of it, written in strong formulation, as

$$Sw = -(\triangle)^{-1}(f - g(w),$$

meaning Sw is the solution u to (modulo boundary conditions)

$$-\triangle u = f - g(w),$$

also meaning Sw is the solution u to the Poisson equation with the RHS "build from w."

First, let us show S is well defined:

$$\left| \int_\Omega g(w)\varphi \,\mathrm{d}x + < f, \varphi > \right| \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} \|\varphi\|_{\mathrm{W}^{1,2}(\Omega)} + \|\varphi\|_{\mathrm{L}_2(\Omega)} \|g(w)\|_{\mathrm{L}_2(\Omega)},$$

and

$$\int_\Omega |g(w)|^2 \,\mathrm{d}x \le \int_\Omega C^2(1 + |w|^\alpha)^2 \,\mathrm{d}x \le \int_\Omega 2C^2\left(1 + |w|^{2\alpha}\right) \mathrm{d}x \le \int_\Omega 2C^2\left(1 + |w|^2\right) \mathrm{d}x < \infty,$$

where we used the Young inequality[28], $\alpha \le 1$ and the fact S is defined on $\mathrm{L}_2(\Omega)$, meaning $w \in \mathrm{L}_2(\Omega)$. With that information, we see the last integral is finite and so truly $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ is well defined.

Also, the estimate $|g(w)| \le C(1 + |w|^\alpha), \alpha \in (0, 1]$ together with the fact $C(1 + w^\alpha) \in \mathrm{L}_2(\Omega)$ for $w \in \mathrm{L}_2(\Omega)$ means the Nemytskii operator

$$w \mapsto g(w) \in \mathcal{C}(\mathrm{L}_2(\Omega); \mathrm{L}_2(\Omega)),$$

is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$.

Let us now show that S is continuous as a whole. Define $I_g \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$ as

$$I_g(\varphi) = \int_\Omega g(w)\varphi \,\mathrm{d}x - < f, \varphi > .$$

We are interested in the continuity of the (nonlinear!) mapping $g \mapsto I_g$ as a mapping from $\mathrm{L}_2(\Omega)$

[27] Meaning the image Sw of w is the function u such that the integral equality holds. We could also assume $S : \mathrm{L}_2(\Omega) \to \mathrm{W}^{1,2}(\Omega)$, but for compactness we need the origin and target spaces to be the same.

[28] In the form $(a + b)^2 \le 2\left(a^2 + b^2\right)$.

to $\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$. Let $\{g_k\} \subset \mathrm{L}_2(\Omega)$ be a sequence *s.t.* $g_k \to g$ in $\mathrm{L}_2(\Omega)$. Then

$$\|I_{g_k} - I_g\|_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*} = \sup_{\varphi \in \mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} |(I_{g_k} - I_g)(\varphi)| = \sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \left|\left(\int_\Omega g_k \varphi \,\mathrm{d}x - < f, \varphi > - \int_\Omega g\varphi \,\mathrm{d}x + < f, \varphi >\right)\right| \le$$
$$\sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \int_\Omega |g - g_k||\varphi| \,\mathrm{d}x \le \|g - g_k\|_{\mathrm{L}_2(\Omega)} \to 0,$$

so we have shown $g \to g_k$ in $\mathrm{L}_2(\Omega) \Rightarrow I_g \to I_{g_k}$ in $\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$, and so $(g \mapsto I_g) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*\Big)$. Finally, define the operator $L : \left(\mathrm{W}_0^{1,2}(\Omega)\right)^* \to \mathrm{W}_0^{1,2}(\Omega)$, as

$$Lh = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x =< h, \varphi >,$$

or in other words, L maps $h \mapsto u$, where u solves

$$\begin{cases} -\triangle u = h, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega \end{cases}.$$

or in other other, informal, words

$$Lh = -(\triangle)^{-1} h.$$

Realize that this approach been studiud in the chapter about Fredholm theory extensively. In this simple case however, recall the apriori estimates that the solution satisfies (that is guaranteed from the properties of the Poisson equation)

$$\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} \le C\|h\|_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*},$$

which together with the linearity of L checks $h \mapsto u$ is continuous. Putting it all together:

- $(w \mapsto g(w)) \in \mathcal{C}(\mathrm{L}_2(\Omega)\,;\mathrm{L}_2(\Omega))$,
- $(g(w) \mapsto f - g(w)) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*\Big)$, [29]
- $(f - g(w) \mapsto u) \in \mathcal{C}\Big(\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*; \mathrm{W}_0^{1,2}(\Omega)\Big)$,

In total, the composition is continuous and yields S.

Next, we would like to show S is a compact (nonlinear) operator. We start with showing S maps bounded sets in $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_2(\Omega)$; for that we need apriori estimates: test the weak formulation with u: (we are doing this a bit more careful that might

[29] In the sense we interpret the RHS as a point in the dual space. Precisely, we should write something like $\left(g(w) \mapsto \left(\varphi \in \mathrm{W}_0^{1,2}(\Omega) \mapsto < f, \varphi > - \int_\Omega g(w)\varphi \,\mathrm{d}x\right)\right) \in \mathcal{C}\Big(\mathrm{L}_2(\Omega)\,;\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*\Big)$.

seem needed, but come on, it is the first example...)

$$\begin{aligned}\underbrace{\|\nabla u\|^2_{\mathrm{L}_2(\Omega)}}_{C_p^2\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)}\leq} &\leq \|f\|_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \|g(w)\|_{\mathrm{L}_2(\Omega)}\|u\|_{\mathrm{L}_2(\Omega)} \leq \\ &\leq \frac{\|f\|_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*}}{\sqrt{2\varepsilon_1}}\sqrt{2\varepsilon_1}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{\|g(w)\|_{\mathrm{L}_2(\Omega)}}{\sqrt{2\varepsilon_2}}\sqrt{2\varepsilon_2}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} \leq \\ &\leq (\varepsilon_1+\varepsilon_2)\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{1}{4\varepsilon_1}\|f\|^2_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*} + \frac{1}{4\varepsilon_2}\|g(w)\|^2_{\mathrm{L}_2(\Omega)} \leq \\ &\leq (\varepsilon_1+\varepsilon_2)\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} + \frac{1}{4\min(\varepsilon_1,\varepsilon_2)}\left(\|f\|^2_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*} + \|g(w)\|^2_{\mathrm{L}_2(\Omega)}\right),\end{aligned}$$

and since $\|g(w)\|^2_{\mathrm{L}_2(\Omega)} \leq \|C(1+|w|^\alpha)\|^2_{\mathrm{L}_2(\Omega)} \leq 4C^2\Big((\lambda(\Omega))^2 + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big)$ and Poincare inequality, we can write

$$\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq \frac{1}{4\big(C_p^2-(\varepsilon_1+\varepsilon_2)\big)\min(\varepsilon_1,\varepsilon_2)}\left(\|f\|^2_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*} + 4\Big((\lambda(\Omega))^2 + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big)\right),$$

which, for $\varepsilon_1, \varepsilon_2$ sufficiently small, just an estimate of the type

$$\|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq C\left(1 + \|f\|^2_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*} + \|w\|^2_{\mathrm{L}_2(\Omega)}\right).$$

As we can see, if $w \in \mathrm{L}_2(\Omega)$ is bounded, also $u \in \mathrm{W}_0^{1,2}(\Omega)$ is bounded, so S maps bounded sets from $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega)$. Since moreover $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_2(\Omega)$, this means S is compact from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$.

To conclude the proof, we need that there exists $K \subset \mathrm{L}_2(\Omega)$ closed convex bounded nonempty *s.t.* $S(K) \subset K$, so we can use Schauder. That clearly will be the case when we show $S\big(\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)\big) \subset \mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)$ for some $R > 0$. But this is simple - the estimate

$$\|u\|^2_{\mathrm{L}_2(\Omega)} \leq \|u\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq C\left(\underbrace{1 + \|f\|^2_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*}}_{:=C_1} + \|w\|^2_{\mathrm{L}_2(\Omega)}\right),$$

tells us if the R were to exist, it must hold

$$R^2 \leq CC_1 + CR^2 \Leftrightarrow R^2(1-C) \leq CC_1.$$

If $1-C>0$, then just take $R \leq \sqrt{\frac{CC_1}{1-C}}$, if $1-C<0$, any $R>0$ will do.

And so we see such an R exists in all cases $\Rightarrow$ the image of a ball is in a ball for some $R \Rightarrow S$ is compact on $\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R)$ into $\mathrm{B}_{\mathrm{L}_2(\Omega)}(0,R) \Rightarrow$ it has a fixed point by Schauder $\Leftrightarrow$ the solution exists.

For the regularity part of the assertion, realize that u_0 solves

$$\begin{cases} -\triangle u_0 = f - g(u_0), & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases},$$

and if $f \in \mathrm{L}_2(\Omega)$, then $f - g(u_0) \in \mathrm{L}_2(\Omega)$ and so from the regularity theory for elliptic equations we get

$$u \in \mathrm{W}^{2,2}(\Omega)\,.$$

□

Theorem 23 (Uniqueness). *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be weak solutions to the above problem. Let $f \in \left(W_0^{1,2}(\Omega)\right)^*$ and let g be continuous. Let either extra*

1. *g is nondecreasing, or*
2. *$g \in C^1(\mathbb{R}), \|g'\|_{L_\infty(\mathbb{R})}$ small.*

Then $u_1 = u_2$.

Proof. (*From: the lectures*) In the linear case, $u_1 - u_2$ would be a solution to the problem with zero data. In the nonlinear case, we have now certainity $u_1 - u_2$ solves anything, so we will have to be more careful.

Subtract the equations for u_1, u_2 and test the weak formulation with $u_1 - u_2$.:

$$\int_\Omega |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2)\,\mathrm{d}x = 0.$$

In the first case, the second term is nonnegative, so for the whole thing to be zero it has to hold

$$0 = \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \geq C_p \|u_1 - u_2\|^2_{\mathrm{W}^{1,2}(\Omega)} \Rightarrow u_1 - u_2 = 0 \in \mathrm{W}_0^{1,2}(\Omega)\,.$$

In the second case, the second does not have a sign, but we can estimate it

$$\left|\int_\Omega (g(u_1) - g(u_2)(u_1 - u_2))\,\mathrm{d}x\right| \leq \int_\Omega \left\|g'\right\|_{\mathrm{L}_\infty(\mathbb{R})} \left|u_1 - u_2\right|^2 \Big|\,\mathrm{d}x \leq \left\|g'\right\|_{\mathrm{L}_\infty(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)},$$

meaning

$$\|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \pm \|g\|_{\mathrm{L}_\infty(\mathbb{R})} C_p \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} = (1 \pm \|g\|_{\mathrm{L}_\infty(\mathbb{R})} C_p) \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} \leq 0.$$

If the term is positive we again obtain $u_1 = u_2$ in $\mathrm{W}_0^{1,2}(\Omega)$ from the Poincare inequality, if it is negative, we require

$$\left\|g'\right\|_{\mathrm{L}_\infty(\mathbb{R})} C_p < 1,$$

and then we have the same result. □

Example. Suppose the following problem

$$\begin{cases} -\triangle u + b(\nabla u) = f, & \text{in}\,\Omega \\ u = 0, & \text{on}\,\partial\Omega. \end{cases}$$

where $f \in (\mathrm{W}_0^{1,2}(\Omega))^*$, and the function $b : \mathbb{R}^d \to \mathbb{R}$ is continuous and (essentialy) bounded. The weak formulation is:

$$\text{find}\, u \in \mathrm{W}_0^{1,2}(\Omega)\ s.t. : \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi + b(\nabla u)\varphi\,\mathrm{d}x = < f, \varphi >_{\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*},$$

Theorem 24. *Let $f \in \left(W_0^{1,2}(\Omega)\right)^*, \Omega \in C^{0,1}, b : \mathbb{R}^d \to \mathbb{R}$ continuous and essentialy bounded. Then there is a weak solution to the above problem.*

Proof. (*From: the lectures*) As in the previous: define the operator $S : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$, such as

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \, dx = < f, \varphi > - \int_\Omega b(\nabla w) \varphi \, dx \,,$$

or equivalently

$$Sw = u \Leftrightarrow u \text{ is the solution to } \begin{cases} -\triangle u = f - b(\nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases},$$

or informally

$$Sw = -(\triangle)^{-1}(f - b(\nabla w)),$$

modulo boundary conditions. Once again, if we are able to show S has a fixed point, this information is equivalent to the considered problem having a solution.

Let us first show S is well defined. Holder yields

$$\|\nabla u\|_{L_2(\Omega)} \|\nabla \varphi\|_{L_2(\Omega)} \le \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W_0^{1,2}(\Omega)} + \|b\|_{L_\infty(\mathbb{R}^d)} \|\varphi\|_{L_1(\Omega)},$$

use Poincare inequality and $W_0^{1,2}(\Omega) \hookrightarrow L_1(\Omega)$, to obtain

$$\|u\|_{W_0^{1,2}(\Omega)} \le \underbrace{C\left(\|f\|_{\left(W_0^{1,2}(\Omega)\right)^*} + \|b\|_{L_\infty(\mathbb{R}^d)} \right)}_{:=R},$$

and so we see that the target space really is $W_0^{1,2}(\Omega)$. Also, the above estimate is extra nice, as it does not depend on w whatsoever. If we denote the RHS by $R > 0$, we immediately see $\|u\|_{W_0^{1,2}(\Omega)} = \|Sw\|_{W_0^{1,2}(\Omega)} \le R$, meaning that for sure $S\left(B_{W_0^{1,2}(\Omega)}(0,R)\right) \subset B_{W_0^{1,2}(\Omega)}(0,R)$. Let us show other properties needed for compactness.

First, $\nabla : W_0^{1,2}(\Omega) \to L_2\left(\Omega;\mathbb{R}^d\right), w \mapsto \nabla w$ is linear and bounded,

$$\|\nabla w\|_{L_2(\Omega)} \le \|w\|_{W_0^{1,2}(\Omega)},$$

and so continuous, the map $\mathbf{y} \mapsto b(\mathbf{y})$ satisfies the growth

$$|b(\mathbf{y})| \le \underbrace{\|b\|_{L_\infty(\mathbb{R}^d)}}_{\text{"}=g(x)\text{"}} + 0 \cdot \sum_{i=1}^d |y_i|^{2/2},$$

and since $\lambda(\Omega) < \infty$, we have (in particular) $\|b\|_{L_\infty(\mathbb{R}^d)} \in L_2(\Omega)$. Putting this together means $N_b : \nabla w \mapsto b(\nabla w)$ is continuous from[30] $L_2\left(\Omega;\mathbb{R}^d\right)$ to $L_2(\Omega;\mathbb{R})$,

$$(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}\left(L_2\left(\Omega;\mathbb{R}^d\right); L_2(\Omega;\mathbb{R})\right)$$

by Nemytskii. Next, the mapping

$$b(\nabla w) \mapsto b(\nabla w) + f$$

from $L_2(\Omega)$ to $\left(W_0^{1,2}(\Omega)\right)^*$ in the sense

$$b(\nabla w) \mapsto \left(W_0^{1,2}(\Omega) \ni \varphi \mapsto < f, \varphi >_{\left(W_0^{1,2}(\Omega)\right)^*} + \int_\Omega b(\nabla w) \varphi \, dx \right),$$

[30]We should be more precise and write it is continuous from $L_2\left(\Omega;\mathbb{R}^d\right)$ to $L_2(\Omega;\mathbb{R})$, as $b : \mathbb{R}^d \to \mathbb{R}$.

is also continous: let $\{b_k\} \subset \mathrm{L}_2(\Omega)$ *s.t.* $b_k \to b$, in $\mathrm{L}_2(\Omega)$, then

$$\sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \left| < f,\varphi > - \int_\Omega b_k(\nabla)\varphi \,\mathrm{d}x - < f,\varphi > + \int_\Omega b(\nabla w)\varphi \,\mathrm{d}x \right| \leq \sup_{\mathrm{U}_{\mathrm{W}_0^{1,2}(\Omega)}(0,1)} \int_\Omega |b(\nabla w) - b_k(\nabla w)||\varphi| \,\mathrm{d}x \leq$$
$$\leq \|b_k - b\|_{\mathrm{L}_2(\Omega)} \to 0,$$

and so the mapping $b(\nabla w) \mapsto b(\nabla w) + f$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$. Finally, the mapping $L : \left(\mathrm{W}_0^{1,2}(\Omega)\right)^* \to \mathrm{W}_0^{1,2}(\Omega)$,

$$Lh = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x =< h, \varphi >,$$

i.e.,

$$Lh = u \Leftrightarrow u \,\text{solves} \begin{cases} - \triangle u = h, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega, \end{cases}$$

is also continuous between the spaces - it is just again the solution operator to the Poisson equation with a RHS $h \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$. Altogether

- $(w \mapsto \nabla w) \in \mathcal{C}\left(\mathrm{W}_0^{1,2}(\Omega)\,; \mathrm{L}_2\left(\Omega; \mathbb{R}^d\right)\right)$,
- $(\nabla w \mapsto b(\nabla w)) \in \mathcal{C}\left(\mathrm{L}_2\left(\Omega; \mathbb{R}^d\right); \mathrm{L}_2(\Omega; \mathbb{R})\right)$,
- $(b(\nabla w) \mapsto f - b(\nabla w)) \in \mathcal{C}\left(\mathrm{L}_2(\Omega)\,; \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*\right)$,
- $(f - b(\nabla w) \mapsto u) \in \mathcal{C}\left(\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*; \mathrm{W}_0^{1,2}(\Omega)\right)$,

and so S as a composition of those mappings is continuous from $\mathrm{W}_0^{1,2}(\Omega)$ to $\mathrm{W}_0^{1,2}(\Omega)$. It remains to show S is compact: we already have continuity, consider now $\{w_k\}_{k\in\mathbb{N}} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded. Then, by the compact embedding $\mathrm{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_1(\Omega)$, there $\exists\{u_k\} \subset \{w_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$ (also bounded) *s.t.* $u_k \to u$ in $\mathrm{L}_1(\Omega)$. To conclude, use the following trick (as when showing uniqueness to the previous problem): subtract equation for u_k from equation for u_l and test with $u_l - u_k$

$$\int_\Omega \nabla u_l \cdot \nabla(u_l - u_k) \,\mathrm{d}x + \int_\Omega b(\nabla u_l)(u_l - u_k) \,\mathrm{d}x - < f, u_l - u_k >$$
$$- \left(\int_\Omega \nabla u_k \cdot \nabla(u_l - u_k) \,\mathrm{d}x + \int_\Omega b(\nabla u_k)(u_l - u_k) \,\mathrm{d}x - < f, u_l - u_k \right),$$

meaning

$$C\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq \|\nabla(u_l - u_k)\|^2_{\mathrm{L}_2(\Omega)} \leq \int_\Omega |b(\nabla u_l) - b(\nabla u_k)\,||u_l - u_k| \,\mathrm{d}x \leq 2\|b\|_{\mathrm{L}_\infty(\Omega)}\|u_l - u_k\|_{\mathrm{L}_1(\Omega)},$$

so

$$\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \leq C\|u_l - u_k\|_{\mathrm{L}_q(\Omega)}.$$

And we are finished - the sequence $\{u_n\}$ converges in $\mathrm{L}_1(\Omega)$, so is Cauchy in $\mathrm{L}_1(\Omega)$, and the above inequality proves that also $\{u_n\}$ is Cauchy in a complete space $\mathrm{W}_0^{1,2}(\Omega)$, meaning $\{u_n\}$ converges in $\mathrm{W}_0^{1,2}(\Omega)$. All in all, $\{u_n\} \subset \mathrm{W}_0^{1,2}(\Omega)$ is a convergent subsequence of a bounded sequence $\{w_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$, meaning S is compact on $\mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)$ into $\mathrm{B}_{\mathrm{W}_0^{1,2}(\Omega)}(0,R)$ and so must possess a fixed point by the Schauder fixed point theorem. □

4 Nonlinear elliptic equations - monotone operator theory

The method of compact perturbations allowed us to solve some nonlinear problems. This relied on the fact that nonlinearities were present mainly in the nonleading terms and so we were able to use our theory we developed for linear equations (continuity of the solution operators, regularity results, etc.) In this chapter we examine a different approach that will allow us to solve a particular class of problems even with nonlinearities in the leading term.

4.1 Coercivity and monotonicity

Consider the following problem:

$$
\begin{aligned}
-\sum_{i=1}^{d} \partial_i(a_i(x,u(x),\nabla u(x))) + a_0(x,u(x),\nabla u(x)) = f(x), \text{ in } \Omega \\
u = u_0, \text{ on } \partial\Omega,
\end{aligned}
$$

or written more concisely

$$
-\nabla \cdot (\mathbf{a}(x,u(x),\nabla u(x))) + a_0(x,u(x),\nabla u(x)) = f(x), \text{ in } \Omega, \tag{3}
$$

$$
u = u_0, \text{ on } \Omega, \tag{4}
$$

where $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$,

$$
\mathbf{a}(x,u(x),\nabla u(x)) = (a_1(x,u(x),\nabla u(x)),\dots,a_d(x,u(x),\nabla u(x)))),
$$

is some *nonlinear* vector function. The data of the problem are

- $\Omega \in C^{0,1}$,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, i \in \{0,\dots,d\}$ are Caratheodory in $x \in \Omega$ and $(z,\mathbf{p}) \in \mathbb{R} \times \mathbb{R}^d$,
- *growth condition*: $\exists C > 0, r \in (1,\infty), h \in \mathrm{L}_{r'}(\Omega) : \forall i \in \{0,\dots,d\}, \forall\ a.a.\ x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d :$
$$
|a_i(x,z,\mathbf{p})| \le C\big(|z|^{r-1} + |\mathbf{p}|^{r-1}\big) + h(x),
$$
- $f \in \left(\mathrm{W}_0^{1,r}(\Omega)\right)^*$,
- $u_0 \in \mathrm{W}^{1,r}(\Omega)$,

and the unknown is $u : \Omega \to \mathbb{R}$.

Remark. The Nemytskii operator[31] $(u,\mathbf{p}) \mapsto a_i(\cdot,u,\mathbf{p})$ is continuous from $(\mathrm{L}_r(\Omega))^{d+1}$ to $\mathrm{L}_{r'}(\Omega)$. by Nemystkii theorem. Truly, generally, the estimate

$$
|f(x,\mathbf{y})| \le g(x) + C\sum_{j=1}^{d} |y_i|^{\frac{p_i}{p}}, g \in \mathrm{L}_\mathrm{p}(\Omega),
$$

guarantess continuity from $\mathrm{L}_{p_1}(\Omega) \times \dots \times \mathrm{L}_{p_d}(\Omega)$ to $\mathrm{L}_\mathrm{p}(\Omega)$. So in our case, the target space is $\mathrm{L}_{r'}(\Omega)$, since $h \in \mathrm{L}_{r'}(\Omega)$, and the exponent from the origin space can be found by

$$
\text{"}p_i = \frac{p_i}{p}p,\text{"}
$$

[31]Again meaning $N_{a_i}(u,\mathbf{p})(x) = a_i(x,u(x),\mathbf{p}(x))$.

i.e., taking the exponents we in fact have the estimates for and multiplying them by the target space exponent. In our case $p_i = (r-1)r' = r - 1\frac{r}{r-1} = r$.

Definition 10 (Coercivity)**.** We say that $\{a_i\}_{i=0}^d$ is coercive if $\exists C_1 > 0, \exists C_2(x) \in \mathrm{L}_1(\Omega) : \forall\, a.e.\, x \in \Omega, \forall (z,p) \in \mathbb{R}^{d+1}$:

$$\sum_{i=1}^{d} a_i(x,z,\mathbf{p})p_i + a_0(x,z,\mathbf{p})z \geq C_1|\mathbf{p}|^r - C_2(x),$$

i.e.,

$$\mathbf{a}(x,z,\mathbf{p}) \cdot \mathbf{p} + a_0(x,z,\mathbf{p})z \geq C_1|\mathbf{p}|^r - C_2(x).$$

Definition 11 (Monotonicity)**.** We say that $\{a_i\}_{i=0}^d$ is monotone if $\forall\, a.a.\, x \in \Omega, \forall (z_1,\mathbf{p}_1),(z_2,\mathbf{p}_2) \in \mathbb{R}^{d+1}$:

$$(\mathbf{a}(x,z_1,\mathbf{p}_1) - \mathbf{a}(x,z_2,\mathbf{p}_2)) \cdot (\mathbf{p}_1 - \mathbf{p}_2) + (a_0(x,z_1,\mathbf{p}_1) - a_0(x,z_2,\mathbf{p}_2))(z_1 - z_2) \geq 0.$$

We say $\{a_i\}_{i=0}^d$ is strictly monotone if $\forall\, a.a.\, x \in \Omega, \forall (z_1,\mathbf{p}_1,),(z_2,\mathbf{p}_2) \in \mathbb{R}^{d+1}\ s.t.\ (z_1,p_1) \neq (z_2,p_2)$ it holds

$$(\mathbf{a}(x,z_1,\mathbf{p}_1) - \mathbf{a}(x,z_2,\mathbf{p}_2)) \cdot (\mathbf{p}_1 - \mathbf{p}_2) + (a_0(x,z_1,\mathbf{p}_1,) - a_0(x,z_2,\mathbf{p}_2))(z_1 - z_2) > 0.$$

4.2 Existence and uniqueness of the weak solution

Definition 12 (Weak solution)**.** We say that $u \in \mathrm{W}^{1,r}(\Omega)$ is a weak solution to the above problem 3 provided

$$\operatorname{tr} u = \operatorname{tr} u_0 \text{ on } \partial\Omega,$$

$u - u_0 \in \mathrm{W}_0^{1,r}(\Omega)$ and it holds

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega \mathbf{a}(\cdot,u,\nabla u)\cdot\nabla\varphi + a_0(\cdot,u,\nabla u)\varphi \,\mathrm{d}x =< f,\varphi >_{(\mathrm{W}^{1,r}(\Omega))^*},$$

Now comes the proof of existence and uniqueness – let us prepare for that with a useful lemma.

Lemma 10. *Let $g : B(0,R) \subset \mathbb{R}^N \to \mathbb{R}^N$ be continuous, $N \in \mathbb{N}, R > 0$, and $\forall c \in S(0,R) : g(c)\cdot c \geq 0$. Then, there is $c_0 \in B(0,R) : g(c_0) = 0$.*

Proof. (*From: the lectures*) By contradiction. Let $g \neq 0$ in $\mathrm{U}(0,R)$. Let us define

$$h(x) = -R\frac{g(x)}{|g(x)|}.$$

Then $h \in C(\mathrm{B}(0,R)), h(\mathrm{B}(0,R)) \subset \mathrm{S}(0,R) \subset \mathrm{B}(0,R)$, so by Brouwer there

$$\exists x_0 \in \mathrm{B}(0,R) : h(x_0) = x_0 \Leftrightarrow -R\frac{g(x_0)}{|g(x_0)|} = x_0.$$

Take the dot product with $x_0 \in \mathrm{S}(0,R)$ and write

$$\underbrace{-R\frac{\overbrace{g(x_0)\cdot x_0}^{\geq 0}}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in \mathrm{S}(0,R),$$

so that is a contradiction. ☐

Remark. We have in fact shown that this c_0 is from $\mathrm{S}(0,R)$.

Theorem 25 (Existence and uniqueness)**.** *Let $\Omega \in C^{0,1}, r \in (1,\infty), f \in \left(W^{1,r}(\Omega)\right)^*, u_0 \in W^{1,r}(\Omega)$, $\{a_i\}_{i=0}^d$ be Caratheodory, coercive and monotone and let them also satisfy the growth condition. Then, there is a weak solution to the problem 3. If, moreover, $\{a_i\}_{i=0}^d$ is strictly monotone, then the weak solution is unique.*

Proof. (*From: the lectures*) The strategy is the following:

1. Galerkin approximation,
2. uniform estimates,
3. limit passage,
4. identification of limits,

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

Galerkin:

Since $\mathrm{W}_0^{1,r}(\Omega)$ is separable there $\exists\{w_i\}_{i=1}^\infty \subset \mathrm{W}_0^{1,r}(\Omega)$ that is a dense[32] linearly independent subset. We search for $\left\{\alpha_n^j\right\} \subset \mathbb{R}, n \in \mathbb{N}, j \in \{1,\dots,n\}$ such that

$$u_n(x) := u_0(x) + \sum_{j=1}^n \alpha_n^j w_j(x),$$

satisfy $\forall j \in \{1,\dots,n\}$

$$\int_\Omega \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla w_j + a_0(x,u_n,\nabla u_n) w_j \,\mathrm{d}x =< f, w_j > .$$

We claim such $\left\{\alpha_n^j\right\}_{j=1}^n \subset \mathbb{R}^n$ exist $\forall n \in \mathbb{N}$ by the previous lemma. We define a vector function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ (the dependence on α_n^j is through u_n)

$$\mathbf{F}\left(\alpha_n^1,\dots,\alpha_n^n\right) := \left\{\int_\Omega \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla w_j + a_0(x,u_n,\nabla u_n) w_j \,\mathrm{d}x - < f, w_j >\right\}_{j=1}^n .$$

Let us show $\mathbf{F}$ is continuous (maybe more carefully then needed, but just to make it clear)

- $\left(\alpha_n^1,\dots,\alpha_n^n\right) \mapsto \left(u_n = u_0 + \sum_{j=1}^n \alpha_n^j w_j\right)$, is continuous from $\mathbb{R}^n$ to $\mathrm{W}^{1,r}(\Omega)$, it is just a (finite) linear combination[33]
- $u_n \mapsto \nabla u_n$ is continuous from $\mathrm{W}^{1,r}(\Omega)$ to $(\mathrm{L}_r(\Omega))^n$, as it is linear and $\|\nabla u_n\|_{\mathrm{L}_r(\Omega)} \le \|u_n\|_{\mathrm{W}^{1,r}(\Omega)}$,
- $(u_n,\nabla u_n) \mapsto \mathbf{a}(\cdot,u_n,\nabla u_n)$ is continuous from $\mathrm{L}_r(\Omega)\times(\mathrm{L}_r(\Omega))^n$ to $(\mathrm{L}_{r'}(\Omega))^d$ by Nemytskii and $(u_n,\nabla u_n) \mapsto a_0(\cdot,u_n,\nabla u_n)$ is also continuous from $\mathrm{L}_r(\Omega)\times(\mathrm{L}_r(\Omega))^n$ to $\mathrm{L}_{r'}(\Omega)$ by Nemytskii,
- $I_{\nabla w_j} : \mathbf{a}(\cdot,u_n,\nabla u_n) \mapsto \mathbb{R}$, defined by $I_{\nabla w_j}(\mathbf{a}(\cdot,u_n,\nabla u_n)) = \int_\Omega \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla w_j \,\mathrm{d}x$, is continuos from $(\mathrm{L}_{r'}(\Omega))^n$ to $\mathbb{R}$, because it is linear and $\left|\int_\Omega \mathbf{a}\cdot\nabla w_j \,\mathrm{d}x\right| \le \|\mathbf{a}\|_{(\mathrm{L}_{r'}(\Omega))^n}\|\nabla w_j\|_{(\mathrm{L}_r(\Omega))^n}$,

[32]It can be chosen such that it is itself dense, not only its span.
[33]And $+,\cdot$ is continuous on normed spaces...

- $I_{w_j} : a_0(\cdot, u_n, \nabla u_n) \mapsto \mathbb{R}$, defined by $I_{w_j}(a_0(\cdot, u_n, \nabla u_n)) = \int_\Omega a_0(x, u_n, \nabla u_n) w_j \,\mathrm{d}x$, is continuous from $\mathrm{L}_{r'}(\Omega)$ to $\mathbb{R}$ because it is linear and $\left|\int_\Omega a_0 w_j \,\mathrm{d}x\right| \le \|a_0\|_{\mathrm{L}_{r'}(\Omega)} \|w_j\|_{\mathrm{L}_r(\Omega)}$,
- (trivially) (*e.g.*) $a_0(\cdot, u_n, \nabla u_n) \mapsto < f, w_j >$ is continuous from $\mathrm{L}_{r'}(\Omega)$ to $\mathbb{R}$, because it is just a constant,
- and finally (this notation is extra embarassing)

$$\left(\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla w_n \,\mathrm{d}x\,, \int_\Omega a_0(x, u_n, \nabla u_n) \,\mathrm{d}x\,, < f, w_j >\right) \mapsto \left(\sum_{i=1}^n \left(\int_\Omega \mathbf{a} \cdot \nabla w_j + a_0 w_j \,\mathrm{d}x - < f, w_j >\right) \mathbf{e}_i\right) =$$
$$= \left\{\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot w_j + a_0(x, u_n, \nabla u_n) w_j \,\mathrm{d}x - < f, w_j >\right\}_{j=1}^n$$

is continuous from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^n$, because it is just a finite linear combination ("assembly of the vector").

On total, we get $\left(\alpha_n^1, \dots, \alpha_n^n\right) \mapsto \mathbf{F}\left(\alpha_n^1, \dots, \alpha_n^n\right)$ is continuous from $\mathbb{R}^n$ to $\mathbb{R}^n$, as it is just a composition of continuous mappings (listed above in the right order). Next, compute (realize $u_n - u_0 = \sum_{i=1}^n \alpha_n^i w_n$)

$$\mathbf{F}(\boldsymbol{\alpha}_n) \cdot \boldsymbol{\alpha}_n = \sum_{i=1}^n \alpha_n^i \left(\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot w_i + a_0(x, u_n, \nabla u_n) \,\mathrm{d}x - < f, w_i >\right) =$$
$$= \int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u_0) + a_0(x, u_n, \nabla u_n)(u_n - u_0) \,\mathrm{d}x - < f, u_n - u_0 > .$$

This can be further manipulated by using coercivity, one has (also with the help of Cauchy and Schwarz)

$$\begin{aligned}
\mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u_0) + a_0(x, u_n, \nabla u_n)(u_n - u_0) \ge{}& \\
&\ge C_1 |\nabla u_n|^r - C_2(x) - \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla u_0 - a_0(x, u_n, \nabla u_n) u_0 \\
&\ge C_1 |\nabla u_n|^r - (C_2(x) + |\mathbf{a}(\dots)||\nabla u_0| + |a_0(\dots)||u_0|),
\end{aligned}$$

so we can write (notice we do not split $u_n - u_0$)

$$\begin{aligned}
&\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u_0) + a_0(x, u_n, \nabla u_n)(u_n - u_0) \,\mathrm{d}x - < f, u_n - u_0 > \ge \\
&\ge \int_\Omega (C_1 |\nabla u_n|^r - C_2(x) - |\mathbf{a}(\dots)||\nabla u_0| - |a_0(\dots)||u_0|) \,\mathrm{d}x - \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*} \|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)} \ge \\
&\ge C_1 \|\nabla u_n\|_{\mathrm{L}_r(\Omega)}^r - \|C_2\|_{\mathrm{L}_1(\Omega)} - \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*} \|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)} - \int_\Omega |\mathbf{a}(\dots)||\nabla u_0| + |a_0(\dots)||u_0| \,\mathrm{d}x\,.
\end{aligned}$$

Let us now estimate. The first term can be rewritten using triangle inequality and Poincare as

$$\begin{aligned}
\|\nabla u_n\|_{\mathrm{L}_r(\Omega)}^r &= \left|\|\nabla u_n\|_{\mathrm{L}_r(\Omega)}\right|^r \ge \left|\|\nabla(u_n - u_0)\|_{\mathrm{L}_r(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\right|^r \ge \left|\|\nabla(u_n - u_0)\|_{\mathrm{L}_r(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r\right| \ge \\
&\ge \left|C\|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r\right| \text{"} \ge C\|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r, \text{"}
\end{aligned}$$

where we made the constant C larger without renaming it to obtain a positive quantity. Next, we can estimate the integral with $a_0, \mathbf{a}$ from growth conditions:

$$\begin{aligned}
|a_0(x, u_n, \nabla u_n)| &\le h(x) + C\left(|u_n|^{r-1} + |\nabla u_n|^{r-1}\right), \\
|\mathbf{a}(x, u_n, \nabla u_n)| &\le C(d) \sum_{i=1}^d |a_i(x, u_n, \nabla u_n)| \le C(d) d\left(h(x) + C\left(|u_n|^{r-1} + |\nabla u_n|^{r-1}\right)\right),
\end{aligned}$$

so the integral becomes

$$\begin{aligned}
&\int_\Omega |\mathbf{a}||\nabla u_0| + |a_0||u_0| \,\mathrm{d}x \le \\
&\le \int_\Omega |h||u_0| + C|u_0||u_n|^{r-1} + C|u_0||\nabla u_n|^{r-1} \,\mathrm{d}x + C(d)d \int_\Omega |h||\nabla u_0| + C|u_n|^{r-1}|\nabla u_0| + C|\nabla u_n|^{r-1}|\nabla u_0| \,\mathrm{d}x \le \\
&\le \|h\|_{\mathrm{L}_{r'}(\Omega)}\|u_0\|_{\mathrm{L}_r(\Omega)} + C\|u_0\|_{\mathrm{L}_r(\Omega)}\|u_n\|^{r-1}_{\mathrm{L}_r(\Omega)} + C\|u_0\|_{\mathrm{L}_r(\Omega)}\|\nabla u_n\|^{r-1}_{\mathrm{L}_r(\Omega)} + \\
&+ C(d)d\Big(\|h\|_{\mathrm{L}_{r'}(\Omega)}\|\nabla u_0\|^{r-1}_{\mathrm{L}_r(\Omega)} + C\|u_n\|^{r-1}_{\mathrm{L}_r(\Omega)}\|\nabla u_0\|_{\mathrm{L}_r(\Omega)} + \|\nabla u_n\|^{r-1}_{\mathrm{L}_r(\Omega)}\|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big),
\end{aligned}$$

this is a lot of terms, but it gets easier when i use Poincare and estimate everything w.r.t. $\|\nabla u_n\|_{\mathrm{L}_q(\Omega)}$:

$$\begin{aligned}
\le&\|h\|_{\mathrm{L}_{r'}(\Omega)}\|u_0\|_{\mathrm{L}_r(\Omega)} + \|h\|_{\mathrm{L}_{r'}(\Omega)}\|\nabla u_0\|^{r-1}_{\mathrm{L}_r(\Omega)} + \\
&+ \|\nabla u_n\|^{r-1}_{\mathrm{L}_r(\Omega)}\Big(C\|u_0\|_{\mathrm{L}_r(\Omega)} + C\|u_0\|_{\mathrm{L}_r(\Omega)} + C(d)d\Big(\|\nabla u_0\|_{\mathrm{L}_r(\Omega)} + \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big)\Big),
\end{aligned}$$

now use Young inequality:

$$|ab| \le \frac{a^p}{p} + \frac{b^q}{q}, \frac{1}{p} + \frac{1}{q} = 1,$$

so in particular

$$|ab| \le \frac{a^r}{r} + \frac{b^{\frac{r}{r-1}}}{\frac{r}{r-1}}.$$

Using this and going for the maximum over all constants gives (realize that also h is given)

$$\le C\Big(\|u_0\|^r_{\mathrm{L}_r(\Omega)} + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|\nabla u_n\|^r_{\mathrm{L}_r(\Omega)}\Big).$$

Upon combining the above estimations, we obtain (constants are changing from line to line and do not really make sense)

$$\begin{aligned}
\mathbf{F}(\boldsymbol{\alpha}_n)\cdot\boldsymbol{\alpha}_n &\ge C_1\|\nabla u_n\|^r_{\mathrm{L}_r(\Omega)} - \|C_2\|_{\mathrm{L}_1(\Omega)} - \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)} + \\
&- C\Big(\|u_0\|^r_{\mathrm{L}_r(\Omega)} + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|\nabla u_n\|^r_{\mathrm{L}_r(\Omega)}\Big) \ge \\
&\ge C_1\Big(C\|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)}\Big) - \|C_2\|_{\mathrm{L}_1(\Omega)} - \underbrace{\|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}}_{\text{Younge}} + \\
&- C\Big(\|u_0\|^r_{\mathrm{L}_r(\Omega)} + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + C\|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)}\Big) \ge \\
&\ge C_1\|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - C_2\left(\underbrace{1}_{\|C_2\|_{\mathrm{L}_1(\Omega)}\cdots} + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|u_0\|^r_{\mathrm{L}_r(\Omega)}\|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\right),
\end{aligned}$$

so we have

$$\mathbf{F}(\boldsymbol{\alpha}_n)\cdot\boldsymbol{\alpha}_n \ge C_1\|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - C_2\Big(1 + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|u_0\|^r_{\mathrm{L}_r(\Omega)}\|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

which is very nice, as it depends on data mostly. As for the first term, realize

$$u_n - u_0 = \sum_{j=1}^n \alpha_n^j w_n \Rightarrow \|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)} = \|\boldsymbol{\alpha}_n\cdot(w_1,\dots,w_n)\|_{\mathrm{W}^{1,r}(\Omega)},$$

which seems trivial, but from the properties of the norm and the fact $\{w_n\}$ are linearly independent, the mapping

$$\mathbb{R}^n \ni \boldsymbol{\alpha}_n \mapsto \|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}, \|u_n - u_0\|_{\mathrm{W}^{1,r}(\Omega)} = \|\boldsymbol{\alpha}_n \cdot (w_1, \dots, w_n)\|_{\mathrm{W}^{1,r}(\Omega)},$$

is a norm on $\mathbb{R}^n$. And since all norms on finite dimensional spaces are equivalent, there exists $K_1(n) > 0\, s.t.$

$$\forall \alpha \in \mathbb{R}^n : K_1(n)|\boldsymbol{\alpha}_n| \le \|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}.$$

But this means

$$\begin{aligned}\mathbf{F}(\boldsymbol{\alpha}_n) \cdot \boldsymbol{\alpha}_n &\ge C_1 \|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} - C_2\Big(1 + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|u_0\|^r_{\mathrm{L}_r(\Omega)} \|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),\\ &\ge K_1(n) C_1 |\boldsymbol{\alpha}_n|^r - C_2\Big(1 + \|\nabla u_0\|^r_{\mathrm{L}_r(\Omega)} + \|u_0\|^r_{\mathrm{L}_r(\Omega)} \|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),\end{aligned}$$

i.e., if we choose $\boldsymbol{\alpha}_n$ so that its norm is large enough, the above product stays always positive. So, in other words:

$$\exists R > 0, \forall \boldsymbol{\alpha}_n \in \mathrm{S}(0, R) \subset \mathbb{R}^n : \mathbf{F}(\boldsymbol{\alpha}_n) \cdot \boldsymbol{\alpha}_n > 0,$$

and hence by the previous lemma

$$\exists \boldsymbol{\alpha}_n \in \mathrm{S}(0, R) : \mathbf{F}(\boldsymbol{\alpha}_n) = 0,$$

and we fix these $\boldsymbol{\alpha}_n = \big(\alpha_n^1, \dots, \alpha_n^n\big)$ from now on.

Uniform estimates

They follow from the previous manipulation: since we have $\mathbf{F}(\boldsymbol{\alpha}_n) \cdot \boldsymbol{\alpha}_n = 0$

$$\|u_n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} \le C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

and so

$$\|u_n\|_{\mathrm{W}^{1,r}(\Omega)} \le C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

together with (just take the $\|\cdot\|^{r'}_{\mathrm{L}_{r'}(\Omega)}$ norm of the growth estimates and use the above two)

$$\forall j \in \{0, \dots, d\} : \|a_j(x, u_n, \nabla u_n)\|^{r'}_{\mathrm{L}_{r'}(\Omega)} \le C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|^{r'}_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

Limit passage

Since the sequences $\{u_n\}, \{a_i(\cdot, u_n, \nabla u_n)\}$ are uniformly bounded, from the separability of the spaces we can extract sequences (not renamed):

$$u^n \rightharpoonup u \,\mathrm{in}\, \mathrm{W}^{1,r}(\Omega)\,, a_j \rightharpoonup b_j \,\mathrm{in}\, \mathrm{L}_{r'}(\Omega)\,, j \in \{0, \dots, d\}.$$

The weak convergences allow us to write

$$\lim_{n\to\infty} \int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot w_j + a_0(x, u_n, \nabla u_n) w_j \,\mathrm{d}x - < f, w_j > = \int_\Omega \mathbf{b} \cdot \nabla w_j + b_0 \cdot w_j \,\mathrm{d}x - < f, w_j >, \forall j \in \mathbb{N}$$

and so

$$\forall j \in \mathbb{N} \int_\Omega \mathbf{b} \cdot \nabla w_j + b_0 w_j \,\mathrm{d}x = < f, w_j > .$$

And since $\{w_j\}_{j\in\mathbb{N}}$ is dense in $\mathrm{W}_0^{1,r}(\Omega)$ (plus the convergences and linearity of the integral) we have

$$\forall\varphi\in\mathrm{W}_0^{1,r}(\Omega):\int_\Omega \mathbf{b}\cdot\nabla\varphi+b_0\varphi\,\mathrm{d}x=<f,\varphi>.$$

Identification of limits

To solve the problem, we need to show that actually $b_j(x)=a_j(x,u,\nabla u),\forall j\in\{0,\dots,d\}$. For that, we use monotonicity & the *Minty trick*: take $u_n,\nabla u_n,v,\mathbf{V}$ together with monotonicity

$$\begin{aligned}0\le&\int_\Omega(\mathbf{a}(x,u_n,\nabla u_n)-\mathbf{a}(x,v,\mathbf{V}))\cdot(\nabla u_n-\mathbf{V})+(a_0(x,u_n,\nabla u_n)-a_0(x,v,\mathbf{V}))(u_n-v)\,\mathrm{d}x=\\=&\underbrace{\int_\Omega\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_n+a_0(x,u_n,\nabla u_n)u_n\,\mathrm{d}x}_{=I_n}+\\-&\int_\Omega\mathbf{a}(x,u_n,\nabla u_n)\cdot\mathbf{V}+\mathbf{a}(x,v,\mathbf{V})\cdot\nabla u_n+a_0(x,u_n,\nabla u_n)v+a_0(x,v,\mathbf{V})u_n\,\mathrm{d}x+\\+&\int_\Omega\mathbf{a}(x,v,\mathbf{V})\cdot\mathbf{V}+a_0(x,v,\mathbf{V})v\,\mathrm{d}x\end{aligned}$$

If we now denote

$$I_n=\underbrace{\int_\Omega\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla(u_n-u_0)+a_0(x,u_n,\nabla u_n)\cdot(u_n-u_0))\,\mathrm{d}x}_{=<f,u_n-u_0>}+\int_\Omega\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_0+a_0(x,u_n,\nabla u_n)u_0\,\mathrm{d}x,$$

we may write (realize $u_n-u_0=\sum_{j=1}^n\alpha_n^jw_j$, and as the equation is valid with testing by w_j, it is valid when testing with u_n-u_o) by using the equation

$$I_n=<f,u_n-u_0>+\int_\Omega\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_0+a_0(x,u_n,\nabla u_n)u_0\,\mathrm{d}x,$$

but the limit of that is

$$\lim_{n\to\infty}I_n=<f,u-u_0>+\int_\Omega\mathbf{b}\cdot\nabla u_0+b_0u_0\,\mathrm{d}x=\int_\Omega\mathbf{b}\cdot\nabla(u-u_0)+b_0(u-u_0)\,\mathrm{d}x+\int_\Omega\mathbf{b}\cdot\nabla u_0+b_0u_0\,\mathrm{d}x=\int_\Omega\mathbf{b}\cdot\nabla u+b_0u\,\mathrm{d}x,$$

as the rest has subtracted. Taking this back to the original estimate and passing to the limit yields

$$\begin{aligned}0\le&\int_\Omega\mathbf{b}\cdot\nabla u+b_0u-\mathbf{b}\cdot\mathbf{V}-\mathbf{a}(x,v,\mathbf{V})\cdot\nabla u-b_0v-a_0(x,v,\mathbf{V})u+\mathbf{a}(x,v,\mathbf{V})\cdot\mathbf{V}+a_0(x,v,\mathbf{V})v\,\mathrm{d}x=\\=&\int_\Omega\mathbf{b}\cdot(\nabla u-\mathbf{V})+b_0(u-v)-\mathbf{a}(x,v,\mathbf{V})\cdot(\nabla u-\mathbf{V})-a_0(x,v,\mathbf{V})(u-v)\,\mathrm{d}x=\\=&\int_\Omega(\mathbf{b}-\mathbf{a}(x,v,\mathbf{V}))\cdot(\nabla u-\mathbf{V})+(b_0-a_0(x,v,\mathbf{V})(u-v))\,\mathrm{d}x\end{aligned}$$

So far, $v,\mathbf{V}$ have been arbitrary. If we take them now as

$$V=\nabla u-\lambda\mathbf{g},\mathbf{g}\in\mathrm{L}_r(\Omega),v=u,$$

then

$$0\le\int_\Omega\lambda\mathbf{g}\cdot(\mathbf{b}-\mathbf{a}(x,u,\nabla u-\lambda\mathbf{g}))\,\mathrm{d}x.$$

Also λ is of our choosing - if it is $\lambda>0$, we can divide the inequality by it and pass to the limit

$\lambda \to 0_+$ (using Nemytski and continuity in the variables of $\mathbf{a}$); this yields

$$0 \leq \int_\Omega \mathbf{g} \cdot (\mathbf{b} - \mathbf{a}(x,u,\nabla u))\, \mathrm{d}x\,.$$

Since $\mathbf{g}$ was arbitrary, we could have taken $\mathbf{g} \to -\mathbf{g}$, which would lead to

$$0 \geq \int_\Omega \mathbf{g} \cdot (\mathbf{b} - \mathbf{a}(x,u,\nabla u))\, \mathrm{d}x\,,$$

so

$$\int_\Omega \mathbf{g} \cdot (\mathbf{b} - \mathbf{a}(x,u,\nabla u))\, \mathrm{d}x = 0, \forall \mathbf{g} \in \mathrm{L}_r(\Omega)\,.$$

Using the Fundamental lemma of the calculus of variation, we see this must really mean

$$\mathbf{b} = \mathbf{a}(x,u,\nabla u),\ a.e. \text{ in}\, \Omega.$$

Differently, we could argue that $\mathbf{b} = \mathbf{a}(x,u,\nabla u)$ in $\mathrm{L}_{r'}(\Omega) = (\mathrm{L}_r(\Omega))^*$. almost everywhere. So from the previous results, we obtain

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega): \int_\Omega \mathbf{a}(x,u,\nabla u)\nabla\varphi + a_0(x,u,\nabla u)\varphi\, \mathrm{d}x =< f, \varphi >,$$

and we are almost done. It only remains to show the traces are correct, but since $u_n \rightharpoonup u$ in $\mathrm{W}^{1,r}(\Omega)$ and the trace operator is continuous, we immediately have

$$\operatorname{tr} u = \operatorname{tr} u_0.$$

Uniqueness: Let u_1, u_2 be two solutions. Then Use strict monotonicity, subtract the weak formulation and test with $u_2 - u_1$:

$$\int_\Omega \mathbf{a}(x,u_1,\nabla u_1) \cdot \varphi + a_0(x,u_1,\nabla u_1)\, \mathrm{d}x =< f, \varphi >,$$

and

$$\int_\Omega \mathbf{a}(x,u_2,\nabla u_2) \cdot \varphi + a_0(x,u_2,\nabla u_2)\, \mathrm{d}x =< f, \varphi >,$$

so subtracting the two and taking $\varphi = u_1 - u_2 = (u_1 - u_0) - (u_2 - u_0) \in \mathrm{W}_0^{1,r}(\Omega)$ yields

$$\int_\Omega (\mathbf{a}(x,u_2,\nabla u_2) - \mathbf{a}(x,u_1,\nabla u_1)) \cdot \nabla(u_2 - u_1) + (a_0(x,u_2,\nabla u_2) - a_0(x,u_1,\nabla u_1))(u_2 - u_1)\, \mathrm{d}x = 0,$$

with the integrand being nonnegative from monotonicity. But if we also have strict monotonocity, this would have to be strictly positive if $u_1 \neq u_2$ - and since it is zero, then it must mean $(u_1, \nabla u_1) = (u_2, \nabla u_2)$ $a.e.$ in Ω, and so really $u_1 = u_2$ in $\mathrm{W}^{1,r}(\Omega)$. □

Example (Nonlinearities vs weak convergence)**.** Let $g_n(x) = \sin(nx)$, then $g \rightharpoonup 0$ in $\mathrm{L}_2((0,4))$ (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx)\varphi\, \mathrm{d}x \geq \int_2^4 \sin^2(nx)\, \mathrm{d}x \to \frac{1}{2} \neq 0, \forall \varphi \in \mathrm{L}_2((0,4))\,,$$

so $\{u_n^2\} = \{\sin^2(nx)\}$ **does not converge weakly to** $0 = 0^2$.

Remark. It might seem that when dealing with nonlinear elliptic equation with a monotone coercive operator, it might seem one only needs to check the assumptions and then almost blindly use this theorem. However, the class of problems we can solve with the above procedure is quite broader than just the ones we discussed, but one needs to be a bit creative. So, it if benificial to think of the "proof" we have given above as not a typical proof of a mathematical statement, but rather as a procedure of solving similiar kinds of problems.

5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping $I : X \mapsto \mathbb{R}$

$$I(u) = \int_\Omega F(x, u, \nabla u) \, \mathrm{d}x \,,$$

with some basic assumptions:

- $\Omega \in \mathrm{C}^{0,1}$,
- $r \in (1, \infty)$,
- $X = u_0 + \mathrm{W}_0^{1,r}(\Omega)$, for some $u_0 \in \mathrm{W}^{1,r}(\Omega)$,
- $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is Caratheodory in x and $(z, \mathbf{p})$ with the following coercivity condition:

$$\exists C_1 > 0, C_2 \in \mathrm{L}_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x, z, p) \geq C_1 |p|^r - C_2(x).$$

Remark.

- from the assumptions it follows $\int_\Omega F(\cdot, u, \nabla u) \, \mathrm{d}x$ is defined $\forall u \in \mathrm{W}^{1,r}(\Omega)$,
- notice the setting is very similiar to the nonlinear equations previously,
- the fact $u \in u_0 + \mathrm{W}_0^{1,r}(\Omega), u_0 \in \mathrm{W}^{1,r}(\Omega)$, is nothing new - even it the previous chapter we had $u_0 \in \mathrm{W}^{1,r}(\Omega), u \in \mathrm{W}^{1,r}(\Omega)$ $s.t.\, u - u_0 \in \mathrm{W}_0^{1,r}(\Omega)$. Also, since we know $\mathrm{W}_0^{1,r}(\Omega)$ is a closed subspace of $\mathrm{W}^{1,r}(\Omega)$, the structure $u_0 + \mathrm{W}_0^{1,r}(\Omega)$ is exactly that of a factorspace.

5.1 Euler-Lagrange equations

The connection between PDE's and calculus of variations is presented in the following lemma.

Lemma 11. *Let $\Omega \in C^{0,1}, r \in (1, \infty), X = u_0 + W_0^{1,r}(\Omega), u_0 \in W^{1,r}(\Omega), F$ Caratheodory. Moreover, let the following condition hold: $\exists C > 0, h \in L_1(\Omega)$ such that*

$$\forall \; a.a\, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x, z, p)| + |\partial_z F(x, z, p)| \leq h(x) + C(|z|^r + |p|^r),$$

and let moreover $F(x, \cdot, \cdot) \in C^1\big(\mathbb{R}^{d+1}\big)$ for a.a. $x \in \Omega$.

Let now $u \in X$ be a local minimizer of I over X, i.e., $I(u) < \infty$ and

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho \Rightarrow I(u) \leq I(v).$$

*Then u is the weak solution to the **Euler-Lagrange equations**:*

$$\begin{aligned} -\nabla \cdot (\nabla_p F(x, u, \nabla u) + \partial_z F(x, u, \nabla u)) = 0, \; in \, \Omega \\ u = u_0, \; on \, \partial\Omega, \end{aligned}$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \nabla_p F(x, u, \nabla u) \cdot \nabla \varphi + \partial_z F(x, u, \nabla u) \varphi \, \mathrm{d}x = 0,$$

and $\operatorname{tr} u = \operatorname{tr} u_0$ *on* $\partial\Omega$.

Proof. (*From: the lectures*) First, if $u \in u_0 + \mathrm{W}_0^{1,r}(\Omega)$, then $\operatorname{tr} u = \operatorname{tr} u_0$ and that is true. Now fix some $\varphi \in \mathcal{D}(\Omega)$ and define

$$\iota : \mathbb{R} \to \mathbb{R}^* : \iota(t) = \int_\Omega F(x, u + t\varphi, \nabla(u + t\varphi))\,\mathrm{d}x\,.$$

Since now u minimizes I, we see ι has a (local) minimum at 0. We will show $\iota'(0)$ exists and is equal to the Euler-Lagrange equations. Denote now

$$l(\cdot, t) = F(\cdot, u + t\varphi, \nabla(u + t\varphi),$$

for $t \in \mathbb{R}$. To do that, we would like to swap the derivative and the integral, namely to write

$$\partial_t \iota(t) = \partial_t \int_\Omega l(\cdot, t)\,\mathrm{d}x = \int_\Omega \partial_t l(\cdot, t)\,\mathrm{d}x\,.$$

To do that, we have to check the assumptions:

- measurability of $l(\cdot, t)$,
- differentiability of $l(\cdot, t)$,
- majorant for the derivative,
- convergence of the integral on some neighbourhood of t.

Recall that from our assumptions (F is Caratheodory + smoothness)

- $F(\cdot, z, \mathbf{p})$ is continuous $\forall(z, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^d \Rightarrow l(\cdot, t)$ is measurable. so in particular it is measurable on some neighbourhood of $t = 0$,
- $F(x, \cdot, \cdot) \in \mathrm{C}^1\left(\mathbb{R}^{d+1}\right) \Rightarrow l(\cdot, t)$ is differentable w.r.t t.

It is hence valid to now compute $\partial_t l(\cdot, t)$

$$\partial_t l(\cdot, t) = \partial_z F(\cdot, u + t\varphi, \nabla(u + t\varphi))\varphi + \nabla_p F(\cdot, u + t\varphi, \nabla(u + t\varphi)) \cdot \nabla\varphi.$$

We have a nice majorant for this expression straight from the assumptions:

$$\begin{aligned} |\partial_t l(\cdot, t)| &= |\partial_z F(\cdot, u + t\varphi, \nabla(u + t\varphi))\varphi + \nabla_p F(\cdot, u + t\varphi, \nabla(u + t\varphi)) \cdot \nabla\varphi| \le \\ &\le 2(|h(\cdot)| + C(|u + t\varphi|^r + |\nabla(u + t\varphi)|^r))(|\varphi| + |\nabla\varphi|), \end{aligned}$$

which is in $\mathrm{L}_1(\Omega)$ for $h \in \mathrm{L}_1(\Omega)$, $u \in \mathrm{W}^{1,r}(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$. Also from the assumption u is a local minimizer we have

$$i(0) = \int_\Omega F(\cdot, u, \nabla u)\,\mathrm{d}x < \infty$$

Altogether, we have checked the necessary assumption, and we see that $\iota'(0)$ exists and is equal to

$$\iota'(0) = \int_\Omega \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi\,\mathrm{d}x = 0.$$

□

Example. Let

$$F(x, z, p) = \frac{1}{r}(1) + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x,z,p) = \left(\frac{r}{2}\frac{1}{r}2\left(1+|p|^2\right)^{\frac{r-2}{2}}\right)p - G = \left(1+|p|^2\right)^{\frac{r-2}{2}}p - G, \partial_z F(x,z,p) = -g.$$

We have

$$|\left(1+|p|^{2\frac{r-2}{2}}\right)p| \le \left(1+|p|^2\right)^{\frac{r-2}{2}}\left(1+|p|^2\right)^{\frac{1}{2}} = \left(1+|p|^2\right)^{\frac{r-1}{2}} \le C(1+|p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla\cdot\left(\left(1+|\nabla u|^2\right)^{\frac{r-2}{2}}\nabla u\right) = -\nabla\cdot G + g, & \text{in}\,\Omega \\ u = u_0, & \text{on}\,\partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall\varphi\in\mathcal{D}(\Omega): \int_\Omega \left(1+|\nabla u|^2\right)^{\frac{r-2}{2}}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega (G\cdot\nabla\varphi + g\varphi)\,\mathrm{d}x\,.$$

5.2 Minimization of (convex) functionals

We have seen that finding a minimizer is equivalent to solving the Euler-Lagrange equations. But how to find the minimizer? The approach is the following:

Remark (General approach).

- find some minimizing sequence :$\{u_n\}\subset X$ such that

$$\lim_{n\to\infty} I(u_n) = \inf_X I.$$

- we would then like to find the minimizer as some sort of a limit of $\{u_n\}$ - we need some kind of convergence or compactness. In our case, we will use weak convergence, thus from some compactness result, we will want to show

$$u_n \rightharpoonup u$$

  for some u,
- to check that actually $I(u)$ is the minimum of I, we need some kind of continuity of I - in our case, weak sequential lower semicontinuity will be enough:

$$I(u) \le \liminf_{n\to\infty} I(u_n),\ \text{as}\, u_n \rightharpoonup u.$$

In the following, we try to derive some sufficient conditions for the functional $I = \int_\Omega F(x,u,\nabla u)\,\mathrm{d}x$ to be weakly sequentially lower semicontinuous, *i.e.*, some conditions for $F(x,u,\nabla u)$.

Lemma 12. *Let $N\in\mathbb{N}, F:\mathbb{R}^N\to\mathbb{R}, F\in C^1\left(\mathbb{R}^N\right)$. Then*

1. *F is (strictly) convex $\Leftrightarrow$ ∇F is (strictly) monotone*
2. *If F is (strictly) convex, then*

$$\forall\xi_1,\xi_2\in\mathbb{R}^N, \xi_1\neq\xi_2 : F(\xi_1) - F(\xi_2) \ge \nabla F(\xi_2)\cdot(\xi_1-\xi_2).$$

Proof. (*From: the lectures*) Fix $\xi_1,\xi_2,\xi_1\neq\xi_2$, define $\varphi(t) = F(\xi_2 + t(\xi_1-\xi_2))$. Then $\varphi\in C^1(\mathbb{R})$

and it is also (strictly) convex: $\forall s, t \in \mathbb{R}:$

$$\begin{aligned}\varphi(\lambda s+(1-\lambda)t) &= F(\xi_2+(\lambda t+(1-\lambda)s)(\xi_1-\xi_2)) = F(\lambda(\xi_2+t(\xi_1-\xi_2))-\lambda\xi_2+\xi_2+(1-\lambda)s(\xi_1-\xi_2)) = \\ &= F(\lambda(\xi_2+t(\xi_1-\xi_2))+(1-\lambda)(\xi_2+s(\xi_1-\xi_2))) \leq \\ &\leq \lambda F(\xi_2+t(\xi_1-\xi_2))+(1-\lambda)F(\xi_2+s(\xi_1-\xi_2)) = \lambda\varphi(s)+(1-\lambda)\varphi(s).\end{aligned}$$

The derivative of φ is

$$\varphi'(t) = \nabla F(\xi_2+t(\xi_1-\xi_2))\cdot(\xi_1-\xi_2),$$

and so

$$" \Rightarrow ": (\nabla F(\xi_1)-\nabla F(\xi_2))\cdot(\xi_1-\xi_2) = \varphi'(1)-\varphi'(0) \geq 0,$$

if φ is (strictly) convex.

And $" \Leftarrow "$: Fix $t_1 > t_2$ and compute

$$(t_1-t_2)(\varphi'(t_1)-\varphi'(t_2)) = (\nabla F(\xi_2+t_1(\xi_1-\xi_2))-\nabla F(\xi_2+t_2(\xi_1-\xi_2)))\cdot(\xi_1-\xi_2)(t_1-t_2),$$

define

$$\eta_1-\eta_2 = (\xi_1-\xi_2)(t_1-t_2)$$

and we obtain

$$(t_1-t_2)(\varphi'(t_1)-\varphi'(t_2)) = (\nabla F(\eta_1)-\nabla F(\eta_2))\cdot(\eta_1-\eta_2)$$

and so if the ∇F is (strictly) monotonous, the LHS is (strictly) positive, meaning (strict) convexity of φ.

For 2) we already know F (strictly) convex $\Rightarrow \varphi$ (strictly) convex, and moreover realize

$$\varphi(t) = \varphi(t\cdot 1+(1-t)\cdot 0) \leq t\varphi(1)+(1-t)\varphi(0),$$

meaning

$$\Rightarrow \forall t \in \left(0, \frac{1}{2}\right): \frac{\varphi(1)-\varphi(0)}{1} \geq \frac{\varphi(t)-\varphi(0)}{t} \to \varphi'(0),$$

as $t \to 0_+$. And so $\varphi(1)-\varphi(0) \geq \varphi'(0)$, which is exactly the same as

$$F(\xi_1)-F(\xi_2) \geq \nabla F(\xi_2)\cdot(\xi_1-\xi_2).$$

□

Theorem 26 (Sequential weak lower semicontinuity)**.** *Let $N, M \in \mathbb{N}, \Omega$ be open and $F(x, \mathbf{z}, \mathbf{p})$: $\Omega \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ be Caratheodory and convex in $\mathbf{p}$. Let moreover $\exists C(x) \in L_1(\Omega)$ such that*

$$\forall\, a.a.\, x \in \Omega, \forall(\mathbf{z}, \mathbf{p}) \in \mathbb{R}^{N+M}: F(x, \mathbf{z}, \mathbf{p}) \geq C(x).$$

Let $\{\mathbf{u_n}\} \subset L_1(\Omega)$ converge to $\mathbf{u}$ in $L_1(\Omega)$, $\mathbf{u}_n \to \mathbf{u}$ and let $\{\mathbf{U}_n\} \subset L_1(\Omega)$ converge weakly to $\mathbf{U}$ in $L_1(\Omega)$, $\mathbf{U}_n \rightharpoonup \mathbf{U}$. Then it holds

$$\int_\Omega F(x, \mathbf{u}, \mathbf{U})\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega F(x, \mathbf{u}_n, \mathbf{U}_n)\,\mathrm{d}x$$

Proof. (*From: Bulíček, 2019b*) The proof will be given only if moreover $\forall\, a.e.\, x \in \Omega, \forall z \in \mathbb{R}^M$: $F(x, z, \cdot) \in C^1(\mathbb{R}^N)$. The main idea is the following: by the previous lemma:

$$\int_\Omega F(\cdot, \mathbf{u}_n, \mathbf{U}_n)\,\mathrm{d}x \geq \int_\Omega (F(\cdot, \mathbf{u}_n, \mathbf{U}) + \nabla_p F(\cdot, \mathbf{u}_n, \mathbf{U})\cdot(\mathbf{U}_n - \mathbf{U}))\,\mathrm{d}x\,,$$

Since $\{\mathbf{u}_n\}$ converges strongly in $\mathrm{L}_1(\Omega)$, there is a (not renamed) subsequence converging *a.e.* in Ω. By Egorov theorem that means

$$\forall\varepsilon > 0 \exists \Omega_\varepsilon \subset \Omega : \mathbf{u}_n \rightarrow\rightarrow \mathbf{u}_n, \lambda(\Omega/\Omega_\varepsilon) < \varepsilon.$$

Let us then write (we are using the fact $F(x,\mathbf{z},\mathbf{p}) - C(x) \geq 0$.)

$$\begin{aligned}\int_\Omega F(x,\mathbf{u}_n,\mathbf{U}_n)\,\mathrm{d}x = \int_\Omega (F(x,\mathbf{u}_n,\mathbf{U}_n) - C(x))\,\mathrm{d}x + \int_\Omega C(x)\,\mathrm{d}x \geq \\ \geq \int_{\Omega_\varepsilon} F(x,\mathbf{u}_n,\mathbf{U}) - C(x)\,\mathrm{d}x + \int_{\Omega_\varepsilon} F(x,\mathbf{u}_n,\mathbf{U}_n) - F(x,\mathbf{u}_n,\mathbf{U})\,\mathrm{d}x + \int_\Omega C(x)\,\mathrm{d}x \geq \\ \geq \int_{\Omega_\varepsilon} F(x,\mathbf{u}_n,\mathbf{U}) - C(x)\,\mathrm{d}x + \int_{\Omega_\varepsilon} \nabla_p F(x,\mathbf{u}_n,\mathbf{U}) \cdot (\mathbf{U}_n - \mathbf{U})\,\mathrm{d}x + \int_\Omega C(x)\,\mathrm{d}x\,.\end{aligned}$$

Now we take the limes inferior of both sides and write

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega F(x,\mathbf{u}_n,\mathbf{U}_n)\,\mathrm{d}x \geq \liminf_{n\to\infty} \Big(\int_{\Omega_\varepsilon} F(x,\mathbf{u}_n,\mathbf{U}) - C(x)\,\mathrm{d}x + \int_{\Omega_\varepsilon} \nabla_p F(x,\mathbf{u}_n,\mathbf{U}) \cdot (\mathbf{U}_n - \mathbf{U})\,\mathrm{d}x + \\ + \int_\Omega C(x)\,\mathrm{d}x\, Big) \geq \\ \geq \liminf_{n\to\infty} \int_{\Omega_\varepsilon} F(x,\mathbf{u}_n,\mathbf{U}) - C(x)\,\mathrm{d}x + \liminf_{n\to\infty} \left(\int_{\Omega_\varepsilon} \nabla_p F(x,\mathbf{u}_n,\mathbf{U}) \cdot (\mathbf{u_n} - \mathbf{U})\,\mathrm{d}x \right) + \\ + \int_\Omega C(x)\,\mathrm{d}x\,.\end{aligned}$$

The second integral actually has a limit, as $\mathbf{U}_n \rightharpoonup \mathbf{U}$ in Ω_ε, so $\mathbf{U}$ is bounded and $\nabla_p F(x,\mathbf{u}_n,\mathbf{U}_n) \in \mathrm{L}_\infty(\Omega_\varepsilon)$, because $\mathbf{u}_n \rightarrow\rightarrow \mathbf{u}$ on Ω_ε and $\nabla_p F(x,\mathbf{u}_n,\mathbf{U})$ is continuous in $\mathbf{u}_n$ from Caratheodory. As the first integral is nonnegative, we can use Fatou to estimate from below. But since $F(x,\cdot,\mathbf{p})$ is continuous from Caratheodory property and $\mathbf{u}_n \rightarrow\rightarrow \mathbf{u}$ on Ω_ε, the integrand has a limit and it actually holds:

$$\liminf_{n\to\infty} \int_\Omega F(x,\mathbf{u}_n,\mathbf{U}_n)\,\mathrm{d}x \geq \int_{\Omega_\varepsilon} F(x,\mathbf{u},\mathbf{U}) - C(x)\,\mathrm{d}x + \int_\Omega C(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon} F(x,\mathbf{u},\mathbf{U})\,\mathrm{d}x + \int_{\Omega/\Omega_\varepsilon} C(x)\,\mathrm{d}x\,,$$

and by taking the limit $\varepsilon \to 0^+$ and using the monotone convergence theorem we actually have

$$\liminf_{n\to\infty} \int_\Omega F(x,\mathbf{u}_n,\mathbf{U}_n)\,\mathrm{d}x \geq \int_\Omega F(x,\mathbf{u},\mathbf{U})\,\mathrm{d}x\,.$$

□

Remark.
- if $U_n \to U$ strongly $\Rightarrow u_n \to u, U_n \to U$ *a.e.* (up to a subsequence) and the claim follows from the Fatou lemma. [34]
- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \,\mathrm{in}\, \mathrm{L_p}(\Omega) \Rightarrow \int_\Omega |\nabla u|^p\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p\,\mathrm{d}x\,.$$

Theorem 27 (Existence of a minimizer). *Let $d \in \mathbb{N}, \Omega \in C^{0,1}, r \in (1,\infty), u_0 \in W^{1,r}(\Omega)$, denote $X = u_0 + W_0^{1,r}(\Omega)$. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be Caratheodory with the coercivity condition:*

[34] For Fatou, we need nonnegativity of the integrand, but that can be again met from the assumptions $F - c_2 \geq 0, F - c_2 \in \mathrm{L}_1(\Omega)$

$\exists C > 0, h \in L_1(\Omega)$ s.t.:

$$\forall \; a.a. \; x \in \Omega, \forall (z, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^d : F(x, z, \mathbf{p}) \geq C|\mathbf{p}|^r - h.$$

Let moreover F be convex in the last variable, i.e., let $\mathbf{p} \mapsto F(x, z, \mathbf{p})$ be convex $\forall$ a.a. $x \in \Omega, \forall z \in \mathbb{R}$. Then there exists $u \in X$ that minimizes I on X, i.e., $\operatorname{tr} u = \operatorname{tr} u_0$ on $\partial\Omega$ and

$$\forall v \in X : I(u) \leq I(v).$$

Proof. (*From: Bulíček, 2019b*) First of all, let us show

$$\Lambda := \inf_{u \in X} I(u) = \inf_{u \in X} \int_\Omega F(x, u, \nabla u) \, \mathrm{d}x \,,$$

is bounded from below. Coercivity states

$$F(x, u, \nabla u) \geq C|\nabla u|^r - h(x),$$

so

$$\int_\Omega F(x, u, \nabla u) \, \mathrm{d}x \geq \int_\Omega C|\nabla u|^r - h(x) \, \mathrm{d}x \geq C\|\nabla u\|^r_{\mathrm{L}_r(\Omega)} - \|h\|_{\mathrm{L}_1(\Omega)} \geq -\|h\|_{\mathrm{L}_1(\Omega)},$$

and since $\|h\|_{\mathrm{L}_1(\Omega)}$ is of course bounded, this means $\Lambda > -\infty$. Next, we show there exists a minimizing sequence $\{u_n\} \subset X$, *i.e.*, a sequence *s.t.*

$$\lim_{n \to \infty} I(u_n) = \Lambda.$$

Realize that from the definition of infimum, $\forall n \in \mathbb{N} \exists u_n \in X$ such that

$$I(u_n) \leq \Lambda + \frac{1}{n}.$$

We do not know the limit of $I(u_n)$ actually exists, so we have to be a bit more careful. On the other hand, limes superior is defined always, so we on one hand have

$$\limsup_{n \to \infty} I(u_n) \leq \Lambda,$$

and from the definition of the infimum

$$\Lambda \leq I(u_n),$$

we read

$$\Lambda \leq \liminf_{n \to \infty} I(u_n),$$

and so in total

$$\Lambda \leq \liminf_{n \to \infty} I(u_n) \leq \limsup_{n \to \infty} \leq \Lambda.$$

But this means all the inequalities must in fact be equalities, and the fact $\liminf_{n \to \infty} I(u_n) = \limsup_{n \to \infty} I(u_n)$ is equivalent to the fact $\lim_{n \to \infty} I(u_n)$ exists and is equal to Λ. And so we have obtained our minimizing ("infimizing") sequence.

Now, also from the property of the infimum we may write

$$I(u_n) = \int_\Omega F(x, u_n, \nabla u_n) \, \mathrm{d}x \leq \Lambda + 1,$$

and so upon using coercivity and estimating the integral from below we may write

$$\int_\Omega F(x, u_n, \nabla u_n)\,\mathrm{d}x \geq \int_\Omega C|\nabla u_n|^r - h\,\mathrm{d}x \geq C\|\nabla u_n\|^r_{\mathrm{L}_r(\Omega)} - \|h\|_{\mathrm{L}_1(\Omega)},$$

and so

$$\|\nabla u_n\|^r_{\mathrm{L}_r(\Omega)} \leq \frac{1}{C}\Big(\Lambda + 1 + \|h\|_{\mathrm{L}_1(\Omega)}\Big),\ i.e.,\ \|\nabla u_n\|_{\mathrm{L}_r(\Omega)} \leq C_1$$

Realize also that

$$\|\nabla u_n\|_{\mathrm{L}_r(\Omega)} = \|\nabla(u_n - u_0) + \nabla u_0\|_{\mathrm{L}_r(\Omega)} \geq \|\nabla(u_n - u_0)\|_{\mathrm{L}_r(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)} \geq C_p\|u_n - u_0\|_{\mathrm{W}^{1,r}_0(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)},$$

meaning

$$\|u_n - u_0\|_{\mathrm{W}^{1,r}_0(\Omega)} \leq \frac{1}{C_p}\Big(\|\nabla u_n\|_{\mathrm{L}_r(\Omega)} + \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big) \leq \frac{1}{C_p}\Big(C_1 + \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big) = C_2,$$

where we used Poincare and the fact $\{u_n\} \subset X$, so $\operatorname{tr} v_n = \operatorname{tr} u_0$. Finally, using

$$\|u_n - u_0\|_{\mathrm{W}^{1,r}_0(\Omega)} \geq \|u_n\|_{\mathrm{W}^{1,r}(\Omega)} - \|u_0\|_{\mathrm{W}^{1,r}(\Omega)},$$

we have obtained

$$\|u_n\|_{\mathrm{W}^{1,r}(\Omega)} \leq C_2 + \|u_0\|_{\mathrm{W}^{1,r}(\Omega)} = C_3,$$

which is an uniform bound of the functions from the sequence. $\{u_n\}$ is a bounded sequence in a reflexive space[35], so

$$u_n \rightharpoonup u,\ \text{in}\,\mathrm{W}^{1,r}(\Omega)\,,$$

in particular

$$\nabla u_n \rightharpoonup \nabla u\,\text{in}\,\mathrm{L}_r(\Omega)\,,$$

and also

$$u_n \to u,\ \text{in}\,\mathrm{L}_r(\Omega)\,.$$

Finally, this means

$$\Lambda = \lim_{n\to\infty}\int_\Omega F(x, u_n, \nabla u_n)\,\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega F(x, u_n, \nabla u_n)\,\mathrm{d}x \geq \int_\Omega F(x, u, \nabla u)\,\mathrm{d}x \geq \Lambda,$$

where we used the weak lower sequential semicontinuity of I and the fact $u \in X$, so $I(u) \geq \Lambda$. But the above manipulation shows that in fact all the inequalities must be equalities, and thus

$$I(u) = \int_\Omega F(x, u, \nabla u)\,\mathrm{d}x = \Lambda,$$

i.e., u is a minimizer. □

6 Evolutionary equations

6.1 Embedding theorems for Sobolev-Bochner spaces

In the whole section, we take X as a Banach space and $T > 0$. Let us recall just a few definitions

[35] X is the space $X = u_0 + \mathrm{W}^{1,r}_0(\Omega)$, and since $\mathrm{W}^{1,r}_0(\Omega)$ is a closed subspace of $\mathrm{W}^{1,r}(\Omega)$, the space X is actually a factorspace and its reflexivity is implied by the reflexivity of $\mathrm{W}^{1,r}(\Omega)$.

Definition 13 (Simple function, strong measurability, Bochner integral)**.** We call $s : I \to X$ a simple function, provided $\exists n \in \mathbb{N}$ such that $\exists \{x_j\}_{j=1}^n, \exists \{E_j\}_{j=1}^n \subset I$ that are pairwise disjoint and $\lambda_1\left(\bigcup_{j=1}^n E_j\right) < \infty$ such that

$$\forall t \in I : s(t) = \sum_{j=1}^n x_j \chi_{E_j}(t).$$

For a simple function s we define its Bochner integral over I as

$$\int_I s(t)\,\mathrm{d}t = \sum_{j=1}^n x_j \lambda_1(E_j).$$

A function $f : I \to X$ is called (strongly) measurable, provided $\exists \{s_n\}_{n\in\mathbb{N}}$ simple functions *s.t.*

$$s_n(t) \to f(t), \text{ in } X, \lambda_1 - \ a.a.\, t \in I,$$

meaning

$$\lim_{n\to\infty} \|s_n(t) - f(t)\|_X = 0, \forall \lambda_1 - \ a.a.\, t \in I.$$

We say a strongly measurable function $f : I \to X$ is Bochner integrable, provided $\exists \{s_n\}_{n\in\mathbb{N}}$ simple functions *s.t.* $s_n(t) \to f(t)$ in X and λ_1- *a.e.* in I and also

$$\lim_{n\to\infty} \int_I \|s_n(t) - f(t)\|_X \,\mathrm{d}\lambda_1 = 0.$$

Then we define the Bochner integral of f over I as

$$\int_I f(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_I s_n(t)\,\mathrm{d}t\,.$$

Definition 14 (Lebesgue-Bochner & Sobolev-Bochner spaces)**.** For $p \in [1, \infty)$ we define

$$\mathrm{L_p}\Big((0,T);X\Big) = \left\{ f : I \to X \text{ strongly measurable} \,|\, \int_I \|f(t)\|_X^p \,\mathrm{d}\lambda_1 < \infty \right\},$$

and for $p = \infty$

$$\mathrm{L}_\infty\Big((0,T);X\Big) = \left\{ f : I \to X \text{ strongly measurable} \,|\, \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X < \infty \right\},$$

together with the norms

$$\|f\|_{\mathrm{L_p}((0,T);X)} = \left(\int_I \|f(t)\|_X^p \,\mathrm{d}\lambda_1 \right)^{\frac{1}{p}}, p \in [1,\infty)$$

$$\|f\|_{\mathrm{L}_\infty((0,T);X)} = \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X.$$

We say a function $g \in \mathrm{L}_1\Big((0,T);X\Big)_{\mathrm{loc}}$ is the weak (time) derivative of the function $u \in \mathrm{L}_1\Big((0,T);X\Big)_{\mathrm{loc}}$, provided it holds

$$\forall \varphi \in \mathcal{D}(I) : \int_I g(t)\varphi(t)\,\mathrm{d}t = -\int_I u(t)\varphi'(t).$$

We write $g = \partial_t u$. Next, we define the set

$$\mathrm{W}^{1,\mathrm{p}}\Big((0,T);X\Big) = \left\{ f : I \to X \text{ strongly measurable} \,|\, f \in \mathrm{L_p}\Big((0,T);X\Big), \partial_t f \in \mathrm{L_p}\Big((0,T);X\Big) \right\},$$

with the norms

$$\|f\|_{\mathrm{W}^{1,\mathrm{p}}((0,T);X)} = \left(\|f\|^p_{\mathrm{L_p}((0,T);X)} + \|\partial_t f\|^p_{\mathrm{L_p}((0,T);X)}\right)^{\frac{1}{p}}, p \in [1,\infty)$$

$$\|f\|_{\mathrm{W}^{1,\infty}((0,T);X)} = \operatorname*{ess\,sup}_{t\in I}\left(\|f\|_{\mathrm{L}_\infty((0,T);X)} + \|f\|_{\mathrm{L}_\infty((0,T);X)}\right), p = \infty.$$

Definition 15 (Continuous functions, strong derivatives)**.** The standard definition of limits in normed spaces work of course: function $f : I \to X$ is said to be continuous on I provided $\forall \{t_n\}_{n\in\mathbb{N}} \subset I$ such that $t_n \to t$ in I it holds $f(t_n) \to f(t)$ in X. The space $\mathrm{C}^0([0,T];X)$ is then defined as usual

$$\mathrm{C}^0([0,T];X) = \left\{u : \overline{I} \to X \text{ continuous} \,|\, \max_{t\in[0,T]} \|u(t)\|_X < \infty\right\},$$

with the norm

$$\|u\|_{\mathrm{C}^0([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X.$$

We say f has a derivative at $t_0 \in I$, denoted $f'(t_0) \in X$, if

$$f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0+h) - f(t_0)}{h}$$

the limit exists. The spaces $\mathrm{C}^k(I;X), \mathrm{C}^\infty(I;X), \mathcal{D}(I;X)$ are then defined as expected.

Lemma 13 (Lebesgue theorem for vector valued functions)**.** *Let $\{f_n\}_{n\in\mathbb{N}} \subset L_1\big((0,T);X\big)$ such that $f_n(t) \to f(t)$ in X for a.a. $t \in I$. Then $f \in L_1\big((0,T);X\big)$ and it holds*

$$\lim_{n\to\infty} \int_I f_n(t)\,\mathrm{d}t = \int_I f(t)\,\mathrm{d}t\,.$$

Proof. (*From: the lectures*) No proof. □

Lemma 14. *Let $p \in [1,\infty)$. Then*

1. *The set*
$$\left\{u : I \to X | u(t) = \sum_{j=1}^N \varphi_j(t)x_j, N \in \mathbb{N}, \varphi_j \in \mathcal{D}(I), x_j \in X\right\},$$
is dense in $L_p\big((0,T);X\big)$. In particular, $\mathcal{D}((0,T);X)$ is dense in $L_p\big((0,T);X\big)$.

2. *Let $\omega \in \mathcal{D}(\mathbb{R})$ be a regularization kernel, and extend $u \in L_p\big((0,T);X\big)$ by zero outside of $(0,T)$. Then $u \star \omega_\varepsilon \to u$ as $\varepsilon \to 0^+$, in $L_p\big((0,T);X\big)$ and a.e. in $(0,T)$.*

Proof. (*From: Bulíček et al., 2018*) Ad 1.: We know already that simple functions are dense in $\mathrm{L_p}\big((0,T);X\big)$, *i.e.*, the functions in the form

$$u(t) = \sum_{j=1}^N x_j \chi_{E_j}(t),$$

for some $\{E_j\} \subset I$ (pairwise disjoint, etc.) From the theory of Lebesgue spaces, we know $\chi_{E_j}(t) \in \mathrm{L_p}(I), \forall p \in [1,\infty]$ can be approximated by functions from $\mathcal{D}(I)$; the assertion follows from this argumentation (although not very detailed).

Ad 2.: The proof is consequence of the similiar assertion valid for Lebesgue spaces. □

Theorem 28. *Let X be a Banach space, $p \in [1, \infty]$. Then*

$$W^{1,p}(I; X) \hookrightarrow C^0(I; X),$$

and moreover $\forall u \in W^{1,p}\Big((0,T); X\Big)$ and $\forall t, s$ such that $0 \le s \le t \le T$ it holds

$$u(t) = u(s) + \int_s^t \partial_\tau u(\tau)\, \mathrm{d}\tau,$$

with $\partial_t u$ being the weak time derivative.

Proof. (*From: Evans, 2010*) Extend u to $(-\infty, 0) \cup (T, \infty)$ by zero and mollify with some regularizator ω_ε,

$$u_\varepsilon = \omega_\varepsilon \star u.$$

Then it holds (we are taking classical derivatives!):

$$\partial_t u_\varepsilon(t) = \partial_t \int_{\mathbb{R}} u(y)\omega_\varepsilon(t-y)\, \mathrm{d}y = \int_{\mathbb{R}} u(y)\partial_t\omega_\varepsilon(t-y)\, \mathrm{d}y = -\int_{\mathbb{R}} u(y)\partial_y\omega_\varepsilon(t-y)\, \mathrm{d}y,$$

and if now take $t \in I_\varepsilon = \{t \in I \,|\, \mathrm{dist}\,(t, \partial I) > \varepsilon\} = (\varepsilon, T-\varepsilon)$, we see that $\forall y \in \mathbb{R}/\overline{I}$ it holds $t - y \ge \mathrm{dist}\,(t, \partial I) > \varepsilon$, but then $\omega_\varepsilon(t-y) = 0$ there, as $\mathrm{supp}\,\omega_\varepsilon \subset \mathrm{B}(0, \varepsilon)$. This means that for $t \in I_\varepsilon$ it actually holds

$$\partial_t u_\varepsilon(t) = -\int_{I_\varepsilon} u(y)\partial_y\omega_\varepsilon(t-y)\, \mathrm{d}y = -\int_I u(y)\omega_\varepsilon(t-y)\, \mathrm{d}y = \int_I \partial_y u(y)\omega_\varepsilon(t-y)\, \mathrm{d}y,$$

where we have just added zeros when integrating over I instead and used the definition of the weak derivative. The last line means

$$\partial_t u_\varepsilon = \omega_\varepsilon \star \partial_t u = (\partial_t u)_\varepsilon,\ a.e.\ \text{on}\ I_\varepsilon,$$

and so from the properties of mollification it follows

$$\partial_t u_\varepsilon \to \partial_t u\,\text{in}\,\mathrm{L_p}\Big((0,T); X\Big)_{\text{loc}},$$

and we also have

$$u_\varepsilon \to u\,\text{in}\,\mathrm{L_p}\Big((0,T); X\Big).$$

Using the standard Newton-Leibniz formula (for a continuously differentiable function in this case) we can write for all $0 < s < t < T$

$$u_\varepsilon(t) = u_\varepsilon(s) + \int_s^t \partial_\tau u_\varepsilon(\tau)\, \mathrm{d}\tau,$$

and using the above convergences, we obtain upon passing to the limit $\varepsilon \to 0^+$

$$u(t) = u(s) + \int_s^t \partial_\tau u(\tau)\, \mathrm{d}\tau,$$

for $a.a.\, s, t \in \mathbb{R}\, s.t.\, 0 < s < t < T$, and if we write $\forall\, a.a.\, s, t \in \mathbb{R}\, s.t.\, 0 \le s \le t \le T$, we have added only finitely many points, thus a set of measure zero. The second point is hence proved. It now only remains to realize the mapping

$$t \mapsto \int_0^t \partial_\tau u(\tau)\, \mathrm{d}\tau,$$

is continuous for $u \in \mathrm{W}^{1,\mathrm{p}}\Big((0,T);X\Big)\Big(\Rightarrow \partial_t u \in \mathrm{L}_\mathrm{p}\Big((0,T);X\Big)\Big)$ from the continuous dependence of the Lebesgue integral on the integration domain. So from the above it already follows

$$u \in \mathrm{C}^0\left([0,T];X\right).$$

Finally, to show the embedding is continuous, estimate the norm

$$\|u\|_{\mathrm{C}^0([0,T];X)} = \max_{t\in[0,T]}\|u(t)\|_X = \max_{t\in[0,T]}\left\|u(s) + \int_s^t \partial_\tau u(\tau)\,\mathrm{d}\tau\right\|_X \le \max_{t\in[0,T]}\|u(s)\|_X + \max_{t\in[0,T]}\int_s^t \|\partial_\tau u(\tau)\|_X\,\mathrm{d}\tau \le$$
$$\le \max_{t\in[0,T]}\|u(s)\|_X + \max_{t\in[0,T]}\|\partial_\tau u\|_{\mathrm{L}_\mathrm{p}((s,t);X)}(t-s)^{\frac{1}{p'}}.$$

Realize now that if $p = \infty$ we are essentialy done, as the above term is equivalent to the norm $\|u\|_{\mathrm{W}^{1,\infty}((0,T);X)}$. If $p < \infty$, we know $a.a.\, s \in (0,T)$ are Lebesgue points, so using the Lebesgue differentation theorem one might write

$$u(s) = \lim_{h\to 0^+}\frac{1}{h}\int_s^{s+h} u(\tau)\,\mathrm{d}\tau,$$

so

$$\max_{t\in[0,T]}\|u(s)\|_X = \max_{t\in[0,T]}\left\|\lim_{h\to 0^+}\frac{1}{h}\int_s^{s+h} u(\tau)\,\mathrm{d}\tau\right\|_X \le \max_{t\in[0,T]}\lim_{h\to 0^+}\frac{1}{h}\int_s^{s+h}\|u(\tau)\|_X\,\mathrm{d}\tau \le \|u\|_{\mathrm{L}_\mathrm{p}((0,T);X)}T^{\frac{1}{p'}},$$

so we finally can write

$$\|u\|_{\mathrm{C}^0([0,T];X)} \le \max_{t\in[0,T]}\|u(s)\|_X + \|\partial_\tau u\|_{\mathrm{L}_\mathrm{p}((0,T);X)}T^{\frac{1}{p'}} \le \|u\|_{\mathrm{L}_\mathrm{p}((0,T);X)}T^{\frac{1}{p'}} + \|\partial_\tau u\|_{\mathrm{L}_\mathrm{p}((0,T);X)}T^{\frac{1}{p'}} \le C\|u\|_{\mathrm{W}^{1,\mathrm{p}}((0,T);X)}$$

□

Remark. Recall that in the setting of Lebesgue spaces, we have seen

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right),$$

if $p > d$, where $\Omega \subset \mathbb{R}^d$. In this case, $I \subset \mathbb{R}$, so we might say "$d = 1$" and the result seems intuitive (other than the case $p = 1 = d$.)

Lemma 15 (Arzela-Ascoli)**.** *Let X, Y be Banach spaces such that $X \hookrightarrow\hookrightarrow Y$. Then*

$$C^1([0,T];X) \hookrightarrow\hookrightarrow C([0,T];Y).$$

Proof. (*From: the lectures*) No proof. □

Lemma 16 (Ehrling)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$. Then*

$$\forall\varepsilon > 0\,\exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \le \varepsilon\|u\|_{V_1} + C\|u\|_{V_3}.$$

Proof. (*From: the lectures*) By contradicition, assume

$$\exists\varepsilon > 0\, s.t.\, \forall n \in N\,\exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon\|u_n\|_{V_1} + n\|u_n\|_{V_3}.$$

WLOG we can assume $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$: truly, the inequaility is 1-homogenous and (the original) holds if $u_n = 0$. So we have

$$1 > \varepsilon\|u_n\|_{V_1} + n\|u_n\|_{V_3}.$$

In particular,

$$1 > n\|u_n\|_{V_3} \Leftrightarrow \|u_n\|_{V_3} < \frac{1}{n},$$

and so $u_n \to 0$ in V_3. Moreover,

$$\|u_n\|_{V_1} < \frac{1}{\varepsilon},$$

meaning $\{u_n\}$ is bounded in V_1 and since $V_1 \hookrightarrow\hookrightarrow V_2$ there exists $\{u_{n_k}\} \subset \{u_n\}$ *s.t.*: $u_{n_k} \to u$ in V_2 strongly. Since $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$, also $\|u\|_{V_2} = 1$ and because $V_2 \hookrightarrow V_3$, and $\{u_{n_k}\}$ converges in V_2, it also converges in V_3. But we have already shown $u_n \to 0$ in V_3, so it must be $u = 0$ in V_3 and also in V_2 from the continuous embedding; we have thus arrived to the contradiction $\|u\|_{V_2} = 1 \wedge u = 0 \,\text{in}\, V_2$. □

Theorem 29 (Aubin-Lions)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3, p \in [1,\infty)$. Then the space*

$$\mathcal{U} = \Big\{u \in L_p\Big((0,T);V_1\Big), \partial_t u \in L_1\Big((0,T);V_3\Big)\Big\},$$

with the norm

$$\|u\|_{\mathcal{U}} = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

is compactly embedded into $L_p\Big((0,T);V_2\Big)$,

$$\mathcal{U} \hookrightarrow\hookrightarrow L_p\Big((0,T);V_2\Big).$$

Proof. (*From: the lectures*) Strategy: I want to fix $M \subset \mathcal{U}$ bounded and show that it is precompact in $\mathrm{L_p}\Big((0,T);V_2\Big)$. That will be done in the following way:

1. mollify M by convolution,
2. use Arzela-Ascoli,
3. show compactness in $\mathrm{L_p}\Big((0,T);V_3\Big)$,
4. apply Ehrling lemma and show compactness in $\mathrm{L_p}\Big((0,T);V_2\Big)$.

Fix $M \subset \mathcal{U}$ bounded. Then $\exists C^* > 0 : \forall u \in M : \|u\|_{\mathcal{U}} \le C^*$.

Next, take

$$\varphi : \mathbb{R} \to [0,\infty), \varphi \in C^\infty(\mathbb{R}), \operatorname{supp}\varphi \subset (-1,0), \int_{\mathbb{R}} \varphi \,\mathrm{d}x = 1,$$

a regularization kernel, and $\forall \delta > 0$ set

$$\varphi_\delta(t) = \frac{1}{\delta}\varphi\Big(\frac{t}{\delta}\Big).$$

Now, extend functions from M to $(0,2T)$ in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0,T) \\ u(2T-t), & t \in (T,2T) \end{cases}.$$

Now mollify: for $\delta > 0, \delta < T$ fixed define

$$M_\delta = \big\{(\tilde{u} \star \varphi_\delta) \restriction_{(0,T)} | u \in M\big\}.$$

From the properties of regularization it follows

$$M_\delta \subset C^1([0,T];V_1),$$

and from Arzela-Ascoli and the fact $V_1 \hookrightarrow\hookrightarrow V_2$, one has

$$M_\delta \subset \mathrm{C}^1\left([0,T];V_1\right) \hookrightarrow\hookrightarrow \mathrm{C}([0,T];V_2) \hookrightarrow \mathrm{L_p}\Big((0,T);V_2\Big),$$

where we have also used the simple fact continuous functions are uniformly bounded on the compact set $[0,T]$, and thus integrable ($\lambda([0,T]) < \infty$.)

Our next goal is to estimate the distance of M and M_δ in $\mathrm{L_p}\Big((0,T);V_3\Big)$. As in the previous proofs, we will use some interpolation theorems for that, so let us first compute estimates for $p=1$ and for $p=\infty$. Before we procceed, let us first deploy our favourite trick; let $u \in M, t \in (0,T)$ be arbitrary. Then (recall $\operatorname{supp}\varphi_\delta \subset (0,-\delta)$)

$$\begin{aligned}\tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^0 \tilde{u}(t-s)\varphi_\delta(s)\,\mathrm{d}s = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s))\varphi_\delta(s)\,\mathrm{d}s = \\ = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s))\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\delta}^s \varphi_\delta(h)\,\mathrm{d}h\,\mathrm{d}s\,,\end{aligned}$$

by using the fact $\varphi_\delta(\delta) = 0$. Per partes then yields

$$\left((\tilde{u}(t) - \tilde{u}(t-s))\int_{-\delta}^s \varphi_\delta(h)\,\mathrm{d}h\right)\Bigg|_{s=-\delta}^0 - \int_{-\delta}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\int_{-\delta}^s \varphi_\delta(h)\,\mathrm{d}h\,\mathrm{d}s\,.$$

Realize the first term is zero, whereas the second term can be rewritten using Fubini

$$-\int_{-\delta}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\int_{-\delta}^s \varphi_\delta(h)\,\mathrm{d}h\,\mathrm{d}s = -\int_{-\delta}^0\int_h^0 \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\,\mathrm{d}s\,\varphi_\delta(h)\,\mathrm{d}h\,.$$

Let us now estimate the distance in $\mathrm{L}_1\Big((0,T);V_3\Big)$ norm (recall $\varphi_\delta(h)$ is just a nonnegative number)

$$\begin{aligned}\int_0^T \|\tilde{u}(t) - \tilde{u}_\delta(t)\|_{V_3}\,\mathrm{d}t \le \int_0^T\int_{-\delta}^0\int_h^0 \left\|\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\right\|_{V_3}\mathrm{d}s\,\varphi_\delta(h)\,\mathrm{d}h\,\mathrm{d}t \le \\ \le \delta\left(\int_0^T \|2\partial_t\tilde{u}(t)\|_{V_3}\,\mathrm{d}t\right) = 2\delta\|\partial_t\tilde{u}\|_{\mathrm{L}_1((0,T);V_3)} = 2\delta\|\partial_t u\|_{\mathrm{L}_1((0,T);V_3)} \le 2\delta\|u\|_{\mathcal{U}} \le 2\delta C^*\end{aligned}$$

because $\tilde{u} = u$ on $(0,T)$, and all functions from M have their norm $\|u\|_{\mathcal{U}} = \|u\|_{\mathrm{L_p}((0,T);V_1)} + \|\partial_t u\|_{\mathrm{L}_1((0,T);V_3)}$ bounded by C^*.

Now, esstimate the distance in $\mathrm{L}_\infty\Big((0,T);V_3\Big)$ norm

$$\begin{aligned}\operatorname*{ess\,sup}_{(0,T)} \|\tilde{u}(t) - \tilde{u}_\delta(t)\|_{V_3} \le \operatorname*{ess\,sup}_{(0,T)} \left\|\int_{-\delta}^0\int_h^0 \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\,\mathrm{d}s\,\varphi_\delta(h)\,\mathrm{d}h\right\|_{V_3} \le \\ \le \operatorname*{ess\,sup}_{(0,T)}\left(\int_T^0 \left\|\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s))\right\|_{V_3}\mathrm{d}s\right)\left(\int_{-\delta}^0 \varphi_\delta(h)\,\mathrm{d}h\right) = 2\|\partial_t\tilde{u}\|_{\mathrm{L}_1((0,T);V_3)} \le 2C^*.\end{aligned}$$

Using these estimates, we are now ready to show $M_\delta \subset\subset \mathrm{L_p}\Big((0,T);V_3\Big)$. Recall we have

$$u \in \mathrm{L}_q(\Omega)\,, \forall q \in [1,\infty] \Rightarrow \|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|^{\theta}_{\mathrm{L}_1(\Omega)}\|u\|^{1-\theta}_{\mathrm{L}_\infty(\Omega)},$$

where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{\infty}$, *i.e.*, $\theta = \frac{1}{q}$, so

$$\|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_3)} \le \|u-\tilde{u}_\delta\|^{\frac{1}{p}}_{\mathrm{L_1}((0,T);V_3)}\|u-\tilde{u}_\delta\|^{1-\frac{1}{p}}_{\mathrm{L}_\infty((O,T);V_3)} \le (2C^*\delta)^{\frac{1}{p}}(2C^*)^{1-\frac{1}{p}} = 2C^*\delta^{\frac{1}{p}},$$

And since $\delta > 0$ is under our control, this really means M_δ is totally bounded in $\mathrm{L_p}\Big((0,T);V_3\Big)$. Now we would like to use Ehrling. So far, we know

$$V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3,$$

but one has to believe this also implies

$$\mathrm{L_p}\Big((0,T);V_1\Big) \hookrightarrow\hookrightarrow \mathrm{L_p}\Big((0,T);V_2\Big) \hookrightarrow \mathrm{L_p}\Big((0,T);V_2\Big),$$

which is not evident at first sight. But if we believe, then we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le \mu\|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_1)} + C_\mu\|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_3)},$$

and plugging in our derived estimates, this means

$$\forall u \in M : \|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le \mu C^* + C_\mu 2C^*\delta^{\frac{1}{p}}.$$

Let now $\beta > 0$ be given. We are yet to choose $\mu > 0, \delta > 0$, so we can do it now: pick

$$\mu > 0 : \mu C^* < \frac{\beta}{2},$$

$$\delta > 0 : C_\mu 2C^*\delta^{1/p} < \frac{\beta}{2},$$

and after choosing these, one has

$$\forall u \in M : \|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_2)} < \beta.$$

Denote now $\{(\tilde{w}_k)_\delta\}_{k=1}^N$ the finite ε−net in M_δ in $\mathrm{L_p}\Big((0,T);V_2\Big)$ (which exists since above using Arzela-Ascoli we have showed $M_\delta \subset\subset \mathrm{L_p}\Big((0,T);V_2\Big)$. Let us show $\{w_k\}_{k=1}^N$ is a finite ε−net in M in $\mathrm{L_p}\Big((0,T);V_2\Big)$: let $\varepsilon > 0$ be given and choose $u \in M$. Arbitrary. Then one can find $k \in \{1,\dots,N\}$ such that $\|\tilde{u}_\delta - (\tilde{w}_k)_\delta\|_{\mathrm{L_p}((0,T);V_2)} < \frac{\varepsilon}{2}$ and so

$$\|u-w_k\|_{\mathrm{L_p}((0,T);V_2)} \le \|u-\tilde{u}_\delta\|_{\mathrm{L_p}((0,T);V_2)} + \|\tilde{u}_\delta - (\tilde{w}_k)_\delta\|_{\mathrm{L_p}((0,T);V_2)} + \|(\tilde{w}_k)_\delta - w_k\|_{\mathrm{L_p}((0,T);V_2)} < \beta + \frac{\varepsilon}{2} + \beta,$$

where we used the above estimate. If we now choose $\beta < \frac{\varepsilon}{4}$, we really see

$$\|u-w_k\|_{\mathrm{L_p}((0,T);V_2)} < \varepsilon,$$

meaning $\{w_k\} \subset M$ is a finite ε−net in M in $\mathrm{L_p}\Big((0,T);V_2\Big)$, and since M was arbitrary (but bounded), we are done. □

Remark. The pair $(\mathcal{U}, |||\cdot|||)$ is a Banach space.

6.2 Nonlinear parabolic equations

We will be dealing with the following problem:

$$\partial_t u - \nabla \cdot \mathbf{a}(x, u, \nabla u) + a_0(x, u, \nabla u) = f \text{ in } (0,T) \times \Omega, \tag{5}$$

$$u = u_0, \text{ on } \{0\} \times \Omega, \tag{6}$$

$$u = 0, \text{ on } (0,T) \times \partial\Omega. \tag{7}$$

The unknown is the function $u(t,x) : (0,T) \times \Omega \to \mathbb{R}$, and we are given

- $\Omega \in \mathrm{C}^{0,1}, T > 0, Q_T = (0,T) \times \Omega$,
- $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d, a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ are Caratheodory in $x \in \Omega$ and in $(z,p) \in \mathbb{R} \times \mathbb{R}^d$, with the following growth condition: $\exists C > 0, \exists r \in (1, \infty)$ *s.t.*

$$\forall\, a.a.\, x \in \Omega, \forall (z, \mathbf{p}) \in \mathbb{R}^{d+1} : |a_i(x, z, \mathbf{p})| \le C\left(1 + |z|^{r-1} + |\mathbf{p}|^{r-1}\right),$$

- and also with the coercivity condition: $\exists C_1, C_2 > 0, \exists q \in (1, \max(2, r))$:

$$\forall\, a.a.\, x \in \Omega, \forall (z, \mathbf{p}) \in \mathbb{R}^{d+1} : \mathbf{a}(x, z, \mathbf{p}) \cdot \mathbf{p} + a_0(x, z, \mathbf{p}) z \ge C_1 |\mathbf{p}|^r - C_2(1 + |z|^q),$$

- $f : (0,T) \times \Omega \to \mathbb{R}, u_0 : \Omega \to \mathbb{R}$ (and will also be in some Banach spaces)

Theorem 30 (Existence and uniqueness). *Let $\Omega \in C^{0,1}$, and let $\{a_i\}_{i=0}^d$ satisfy the above growth conditions and coercivity and let them moreover be monotone. Denote*

$$V = W_0^{1,r}(\Omega) \cap L_2(\Omega)\,.$$

Then $\forall f \in L_{r'}\Big((0,T); V^*\Big), \forall u_0 \in L_2(\Omega)$ exists a solution $u \in L_r\Big((0,T); V\Big)$ s.t. $\partial_t u \in L_{r'}\Big((0,T); V^*\Big), u \in C([0,T]; L_2(\Omega)), u(0) = u_0$ and moreover

$$\forall\, a.e.\, t \in (0,T), \forall \varphi \in V :< \partial_t u, \varphi > + \int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla \varphi + a_0(x, u, \nabla u) \varphi \,\mathrm{d}x =< f, \varphi >\,.$$

Finally, the solution is unique.

Remark.

- the requirement $V = \mathrm{W}_0^{1,r}(\Omega) \cap \mathrm{L}_2(\Omega)$ becomes redundent whenever $r^* = \frac{rd}{d-r} > 2$,
- recall that $\Big(\mathrm{L_r}\Big((0,T); V\Big)\Big)^* = \mathrm{L_{r'}}\Big((0,T); V^*\Big)$,
- on V we can assume *e.g.* $\|u\|_V = \|u\|_{\mathrm{L}_2(\Omega)} + \|u\|_{\mathrm{W}_0^{1,r}(\Omega)}$.

Proof. (*From: Bulíček et al., 2018*) The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

Rothe method:] Fix $m \in \mathbb{N}$ and divide the time interval $(0,T)$ into subintervals of equal length: denote $h = \frac{T}{m}, t_0 = 0, t_m = T, t_k = kh, k \in 0, \dots, m$. The goal is to solve a stationary variant of the problem on each subinterval $(t_k, t_{k+1}]$, approximate the time derivative using difference scheme and then pass to the (certain) limit $m \to \infty$.

Denote the (Bochner integral) mean

$$f_k = \frac{1}{h} \int_{t_k}^{t_{k+1}} f(t) \,\mathrm{d}t \,.$$

Realize that since $f \in \mathrm{L}_{r'}\Big((0,T); V^*\Big)$, we have constructed $f_k \in V^*$, *i.e.*, time independent. For our fixed $m \in \mathbb{N}$ we will denote $u_m(t)$ the approximate solution on the discretization of I. Set $u_m(t_0 = 0) = u_0$, and $\forall k \in \{0, \dots, m\}$ we will be looking for the solution $u_m(t_{k+1})$ of the following stationary problem on $(t_k, t_{k+1}]$:

$$\int_\Omega \frac{u_m(t_{k+1}) - u_m(t_k)}{h} \varphi \,\mathrm{d}x + \int_\Omega \mathbf{a}(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) \cdot \nabla \varphi + a_0(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) \varphi \,\mathrm{d}x =< f_k, \varphi > .$$

The sought solution should lie in $V = \mathrm{W}_0^{1,r}(\Omega) \cap \mathrm{L}_2(\Omega)$, which we can check with following apriori estimate derived under. Realize that this way, $u_m(t_{k+1})$ is a stationary solution that is constant in time on $(t_k, t_{k+1}]$.

It is a question whether the solution $u_m(t_{k+1}) \in V$ for the above steady problem exists or not; since we have growth, monotonocity, coercivity and regularity of the data and the boundary, we will assume without a proof that the solution exists - it could in theory be proven using the tools from the application of monotone operator theory on nonlinear elliptic equations. We will not assume the solution to be unique however.

A-priori and uniform estimates As usual test the weak formulation with the solution itself, *i.e.*, test with $u_m(t_{k+1}) \in V$.

$$\int_\Omega \frac{|u_m(t_{k+1})|^2 - u_m(t_{k+1}) u_m(t_{k+1})}{h} \,\mathrm{d}x +$$
$$+ \int_\Omega \mathbf{a}(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) \cdot \nabla u_m(t_{k+1}) + a_0(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) u_m(t_{k+1}) \,\mathrm{d}x =< f_k, u_m(t_{k+1}) >,$$

use the coercivity conditions and Poincare (recall $V = \mathrm{W}_0^{1,r}(\Omega) \cap \mathrm{L}_2(\Omega)$)

$$\int_\Omega \mathbf{a}(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) \cdot \nabla u_m(t_{k+1}) + a_i(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) u_m(t_{k+1}) \,\mathrm{d}x \geq$$
$$\geq \int_\Omega C_1 |\nabla u_m(t_{k+1})|^r - C_2(1 + |u_m(t_{k+1})|^q) \,\mathrm{d}x = C_1 \|\nabla u_m(t_{k+1})\|^r_{\mathrm{L}_r(\Omega)} - C_2 \Big(\lambda(\Omega) + \|u_m(t_{k+1})\|^q_{\mathrm{L}_q(\Omega)} \Big) \geq$$
$$\geq C_1 \|\nabla u_m(t_{k+1})\|^r_{\mathrm{L}_r(\Omega)} - C_2 \Big(1 + \lambda(\Omega) + \|u_m(t_{k+1})\|^{\max(2,r)}_{\mathrm{L}_{\max(2,r)}(\Omega)} \Big).$$

where we used $q < \max(2,r)$, so $\mathrm{L}_{\max(2,r)}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)$. The RHS can be estimated as

$$< f_k, u_m(t_{k+1}) > \leq \|f_k\|_{V^*} \|u_m(t_{k+1})\|_{V^*} = \|f_k\|_{V^*} \Big(\|u_m(t_{k+1})\|_{\mathrm{L}_2(\Omega)} + \|u_m(t_{k+1})\|_{\mathrm{W}_0^{1,r}(\Omega)} \Big) \leq$$
$$\leq C \Big(\|f_k\|^2_{V^*} + \|f_k\|^{r'}_{V^*} \Big) + \varepsilon \Big(\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)} + \|u_m(t_{k+1})\|^r_{\mathrm{W}_0^{1,r}(\Omega)} \Big).$$

Using these, the apriori estimates become

$$\int_\Omega u_m(t_{k+1})^2 - u_m(t_{k+1})u_m(t_k)\,\mathrm{d}x + h\Big(C_1\|\nabla u_m(t_{k+1})\|^r_{\mathrm{L}_r(\Omega)} - C_2\Big(1+\lambda(\Omega)+\|u_m(t_{k+1})\|^{\max(2,r)}_{\mathrm{L}_{\max(2,r)}(\Omega)}\Big)\Big) \le$$
$$\le Ch\Big(\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}\Big)+\varepsilon h\Big(\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}+\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\Big),$$

and when choosing $\varepsilon > 0$ and renaming all the constants

$$\int_\Omega u_m(t_{k+1})^2 - u_m(t_{k+1})u_m(t_k)\,\mathrm{d}x + Ch\Big(\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\Big) \le Ch\Big(\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}+\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}\Big)$$

Notice that (completing the square):

$$\int_\Omega u_m(t_{k+1})^2 - u_m(t_{k+1})u_m(t_k)\,\mathrm{d}x = \int_\Omega \frac{1}{2}u_m(t_{k+1})^2+\frac{1}{2}(u_m(t_{k+1})-u_m(t_k))^2-\frac{1}{2}u_m(t_k)^2\,\mathrm{d}x\,,$$

so when we sum these terms from $k = 0$ to some $j < m$ we see that a lot of things subtract in fact:

$$\sum_{k=0}^{j-1}\int_\Omega u_m(t_{k+1})^2 - u_m(t_{k+1})u_m(t_k)\,\mathrm{d}x = \frac{1}{2}\Bigg(\|u_m(t_j)\|^2_{\mathrm{L}_2(\Omega)}-\|u_0\|^2_{\mathrm{L}_2(\Omega)}+\sum_{k=0}^{j-1}\|u_m(t_{k+1})-u_m(t_k)\|^2_{\mathrm{L}_2(\Omega)}\Bigg).$$

So for $j < m$ we can write the a-priori estimate as (we of course sum all the weak formulations)

$$\|u_m(t_j)\|^2_{\mathrm{L}_2(\Omega)}+\sum_{k=0}^{j-1}\Big(\|u_m(t_{k+1})-u_m(t_k)\|^2_{\mathrm{L}_2(\Omega)}+C_1h\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\Big)\le$$
$$\le C\Bigg(\|u_0\|^2_{\mathrm{L}_2(\Omega)}+h\sum_{k=0}^{j-1}\Big(\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}+\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}\Big)\Bigg).$$

It remains to notice $u_m(t_{k+1})$ is constant on $(t_k, t_{k+1}]$, so it actually holds

$$\sum_{k=0}^{j-1}h\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)} = \sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\,\mathrm{d}t =$$
$$= \int_0^{t_j}\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\,\mathrm{d}t\,,$$

and also

$$\sum_{k=0}^{j-1}h\Big(\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}+\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}\Big) = \int_0^{t_j}\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}+\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t \le$$
$$\le \int_0^T\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}\,\mathrm{d}t+\int_0^{t_j}\|u_m(t_{k+1})\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\,,$$

so we can finally write the apriori estimates as

$$\|u_m(t_j)\|^2_{\mathrm{L}_2(\Omega)}+\sum_{k=0}^{j-1}\|u_m(t_{k+1})-u_m(t_k)\|^2_{\mathrm{L}_2(\Omega)}+C\int_0^{t_j}\|u_m(t_{k+1})\|^r_{\mathrm{W}^{1,r}_0(\Omega)}\,\mathrm{d}t\le$$
$$\le C\Big(\|u_0\|^2_{\mathrm{L}_2(\Omega)}+\int_0^T\|f_k\|^2_{V^*}+\|f_k\|^{r'}_{V^*}\,\mathrm{d}t\Big).$$

The RHS contains only data, so these are uniform estimates. Because are the summands are nonnegative and $j < m$ was arbitrary, we conclude it must hold

$$\|u_m\|_{\mathrm{L}_\infty((0,T);\mathrm{L}_2(\Omega))} \leq C,$$
$$\|u_m\|_{\mathrm{L}_\mathrm{r}\left((0,T);\mathrm{W}_0^{1,r}(\Omega)\right)} \leq C,$$

and using the growth condition we also obtain

$$\|a_i(\cdot, u_m, \nabla u_m)\|_{\mathrm{L}_{\mathrm{r}'}((0,T);\mathrm{L}_{r'}(\Omega))} \leq C.$$

Limit passage Using the reflexivity of (some of) the spaces, we see that since the above sequences are uniformly bounded, it must be (we are not renaming the subsequences)

$$u_m \rightharpoonup u, \text{ in } \mathrm{L}_\mathrm{r}\Big((0,T); \mathrm{W}_0^{1,r}(\Omega)\Big),$$
$$u_m \rightharpoonup^* u \,\text{in}\, \mathrm{L}_\infty\Big((0,T); \mathrm{L}_2(\Omega)\Big),$$
$$a_i(\cdot, u_m, \nabla u_m) \rightharpoonup b_i, i \in \{0, \dots, d\} \,\text{in}\, \mathrm{L}_{\mathrm{r}'}\Big((0,T); \mathrm{L}_{r'}(\Omega)\Big).$$

To be totally valid, we should check that the first convergencies are actually to the same function. We would like to obtain a sequence for the time derivative also. For that reason, let us do the following. On (t_k, t_{k+1}), set

$$\tilde{u}_m(t) = u_m(t_k) + \frac{t - t_k}{h}(u_m(t_{k+1}) - u_m(t_k)),$$

and we immeditaly see

$$\partial_t \tilde{u}_m(t) = \frac{u_m(t_k) - u_m(t_{k+1})}{h}, t \in (t_k, t_{k+1}),$$

and from the previous esimates we also have

$$\|\tilde{u}_m\|_{\mathrm{L}_\infty((0,T);\mathrm{L}_2(\Omega))} \leq C, \|\tilde{u}_m\|_{\mathrm{L}_\mathrm{r}\left((0,T);\mathrm{W}_0^{1,r}(\Omega)\right)} \leq C.$$

Rewriting the partial derivative from the weak formulation $u_m(t_{k+1})$ solves we obtain

$$\int_\Omega \partial_t \tilde{u}_m \varphi \,\mathrm{d}x =< f_k, \varphi > - \int_\Omega \mathbf{a}(x, u_m(t_{k+1}), \nabla u_m(t_{k+1})) \cdot \varphi + a_0(x, u_m(t_{k+1}), \nabla u_m(t_{k+1}))\,\mathrm{d}x \leq$$
$$\leq$$

REST OF THE PROOF IS MISSING □

7 Semigroup theory

The lands of nonlinear equations are hostile, let us retreat back to our welcoming safeplace - *linear equations*. We will study an elegant framework; consider the following problem

$$u' = Au, A \,\text{is a linear operator}$$
$$u(0) = u_0,$$

where $u : [0,\infty) \to \mathbb{R}$. We know that for example if $Au = au, a \in \mathbb{R}$ then

$$u(t) = u_0 e^{at}.$$

If $\mathbf{u} : [0,\infty) \to \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d\times d}$, then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{A}^k t^k.$$

It is not surprising that this can be extended to functions $u : [0,\infty) \to X$, taking values in an arbitrary Banach space X. In particular, for $A \in \mathcal{L}(X)$, it is sensible to set

$$u(t) = \exp(tA)u_0,$$

with

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!},$$

This works well for *bounded* operators. But in the context of PDE's, most operators are differential and these are generally unbounded, take for example

$$X = \mathrm{L}_2(\Omega), Au = \triangle u.$$

Our divine inspiration tells us we can *guess* the solution should be

$$u(t) = \exp(\triangle t)u_0,$$

but what is

$$\exp(\triangle t)?$$

To explore this question, we will have to develop some theory first.

7.1 (Unbounded) linear operators and (c_0-)semigroups

Definition 16 (Linear operator and its domain)**.** Let X be a Banach space over $\mathbb{K}$. A linear operator on X is a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace of X and $A : \mathcal{D}(A) \to X$ is linear. We call $\mathcal{D}(A)$ the domain of A.

Remark. Evidently, it is important that $\mathcal{D}(A)$ is a subspace of X (and not *e.g.* a subset); when evaluating $A(\alpha x + \beta y), \alpha, \beta \in \mathbb{K}, x, y \in \mathcal{D}(A)$, we need to be sure $\alpha x + \beta y \in \mathcal{D}(A)$ also.

Definition 17 (Semigroup, c_0 semigroup)**.** A family of linear bounded operators

$$\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$$

is called a semigroup, provided

1. $S(0) = \mathrm{id}$
2. $\forall s, t \geq 0 : S(t)S(s) = S(t+s)$.

If moreover

$$\forall x \in X : \lim_{t\to 0^+} S(t)x = x,$$

we call $\{S(t)\}$ a strongly continuous semigroup, written as c_0- semigroup.

Remark. • even though we have talked about unbounded operators, we stress that always

$$\{S(t)\}_{t\ge 0} \subset \mathcal{L}(X),$$

- $\{S(t)\}_{t\in\mathbb{R}}$ with the two conditions, an inverse element

$$(S(t))^{-1} = S(-t),$$

is an Abelian group $(\{S(t)\}_{t\in\mathbb{R}}, \circ)$ with respect to composition $\circ$.

Remark (Notation-wise). • **in the following, X is always a Banach space,**

- the exponential function defined on numbers is written as e^t, whereas the operator exponential will be written as $\exp(tA)$.

Lemma 17 (Properties of strongly continuous semigroups)**.** *Let $\{S(t)\}_{t\ge 0}$ be a c_0-semigroup in X. Then*

1. $\exists M \ge 1, \omega \in \mathbb{R}\, s.t.\, \forall S \in \{S(t)\}_{t\ge 0}, \forall t \ge 0 : \|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$,
2. $\forall x \in X$ *the mapping*
$$t \mapsto S(t)x$$
is continuous from $[0,\infty)$ *to* X.

Proof. (*From: the lectures*) Ad 1.: No proof has been given.

Ad 2.: Fix $t > 0, x \in X$ and compute

$$\lim_{h\to 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t)(S(h)x - x)\|_X \le \lim_{h\to 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X = 0,$$
$$\lim_{h\to 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t-h)(x - S(h)x)\|_X \le \lim_{h\to 0^+} \underbrace{\|S(t-h)\|_{\mathcal{L}(X)}}_{\le Me^{\omega(t-h)}} \|x - S(h)x\|_X = 0,$$

so

$$\lim_{h\to 0} \|S(t+h)x - S(t)x\|_X = 0.$$

$\square$

Definition 18 (Infinitesimal generator)**.** A linear operator $(A, \mathcal{D}(A))$ is called an infinitesimal generator of the semigroup $\{S(t)\}_{t\ge 0}$, provided

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h\to 0^+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X |\, \lim_{h\to 0^+} \frac{S(h)x - x}{h} \text{ exists in } X \right\},$$

Remark. Realize that the limit from the definition actually is

$$Ax = \lim_{h\to 0^+} \frac{S(h)x - x}{h} = \lim_{h\to 0^+} \frac{S(h+0)x - S(0)x}{h} = \lim_{h\to 0^+} \frac{S(h+0) - S(0)}{h} x = \frac{\mathrm{d}}{\mathrm{d}t} S(0)x,$$

i.e.,

$$A = \frac{\mathrm{d}}{\mathrm{d}t} S(0).$$

Theorem 31. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ in X. Then*

1. *If $x \in \mathcal{D}(A)$ then $\forall t \geq 0 : S(t)x \in \mathcal{D}(A)$ and moreover it holds*

$$AS(t)x = S(t)Ax = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)x).$$

2. *For $x \in X, t \geq 0$ set*

$$x_t = \int_0^t S(s)x \, \mathrm{d}s \,.$$

Then $x_t \in \mathcal{D}(A)$ and moreover it holds

$$A(x_t) = S(t)x - x.$$

Proof. (*From: the lectures*) Fix $x \in \mathcal{D}(A), t \geq 0$. Calculate

$$\lim_{h\to 0_+} \frac{S(h)S(t)x - S(t)x}{h} \overset{36}{=} \lim_{h\to 0_+} S(t)\frac{S(h)x - x}{h} = S(t)Ax.$$

This means $S(t)x \in \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$, moreover, if $t > 0$:

$$\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right),$$

estimate,

$$\begin{aligned}\left\|\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax\right\| &= \left\|\lim_{h\to 0^+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right)\right\| \leq \\ &= \lim_{h\to 0^+} \|S(t-h)\|_{\mathcal{L}(X)} \left\|\left(\frac{x - S(h)x}{-h} - S(h)Ax\right)\right\|_X \leq \\ &\leq \lim_{h\to 0^+} Me^{\omega(t-h)}\left\|\frac{x - S(h)x}{-h} - S(h)Ax\right\| = \lim_{h\to 0^+} Me^{\omega(t-h)}(Ax - Ax) = 0,\end{aligned}$$

as $S(t)$ is continous and $S(0) = \mathrm{id}$. We have thus shown the derivative exists, is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)x) = AS(t)x,$$

and since the RHS is continuos [37]we have also showed $t \mapsto S(t)x$ is $C^1([0,\infty))$. This justifies our computation above, that concludes

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h\to 0_+} \frac{1}{h}(S(h)x_t - x_t) = \lim_{h\to 0_+} \frac{1}{h}\left(S(h)\int_0^t S(s)x \, \mathrm{d}s - \int_0^t S(s)x \, \mathrm{d}s\right),$$

[36] $S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$

[37] We can easily check

$$\lim_{h\to 0^+} \|AS(t+h)x - AS(t)x\|_X \leq \lim_{h\to 0^+} \|S(t)\|_{\mathcal{L}(X)}\|AS(h)x - Ax\|_X = 0,$$

and very similiarly for the other limit.

and realize that

$$S(h)\int_0^t S(s)x\,\mathrm{d}s=\int_0^t S(h)S(s)x\,\mathrm{d}s=\int_0^t S(h+s)x\,\mathrm{d}s=\int_h^{t+h} S(s)x\,\mathrm{d}s\,,$$

which follows from the properties of the Bochner integral with respect to composition with linear bounded operators. Hence the previous computation continues as follows

$$=\lim_{h\to 0_+}\frac{1}{h}\left(\int_t^{t+h} S(s)x\,\mathrm{d}s-\int_0^h S(s)x\,\mathrm{d}s\right)=S(t)x-S(0)x=S(t)x-x,$$

which means $x_t\in\mathcal{D}(A)$ and the formula is proven. We used some kind of Lebesgue differentiation theorem, that also works on Banach spaces (with our strognly continuous semigroups). ☐

Definition 19 (Closed operator)**.** We say that a linear operator $(A,\mathcal{D}(A))$ is closed if $\forall\{u_n\}\subset\mathcal{D}(A): u_n\to u\wedge Au_n\to v$, for some $u,v\in X$, then it most hold

$$u\in\mathcal{D}(A)\wedge Au=v.$$

This also means that

$$\operatorname{graph} A=\{(x,Ax)|x\in\mathcal{D}(A)\}\subset X\times X$$

is closed in $(X\times X,\|\cdot\|_1)$.

Example. Let $\Omega\in C^{1,1}, X=\mathrm{L}_2(\Omega)\,,\mathcal{D}(A)=\mathrm{W}^{2,2}(\Omega)\cap\mathrm{W}_0^{1,2}(\Omega)\,,Au=\triangle u$. Then $(A,\mathcal{D}(A)$ is closed. Really, take $\{u_n\}\subset\mathrm{L}_2(\Omega): u_n\to u$ in $\mathrm{L}_2(\Omega)$ for some $u\in\mathrm{L}_2(\Omega)$. Suppose $Au_n\to v$ in $\mathrm{L}_2(\Omega)\,,v\in\mathrm{L}_2(\Omega)$. Suppose the following equation: find u_n such that

$$\begin{aligned}-\triangle u_n&=Au_n,\ \text{in}\,\Omega\\ u_n&=0\,\text{on}\,\partial\Omega.\end{aligned}$$

From the regularity theory for elliptic problems, we know that $\|u_n\|_{\mathrm{W}^{2,2}(\Omega)}\le C\|Au_n\|_{\mathrm{L}_2(\Omega)}\le C$, so we can extract $u_{n_k}\rightharpoonup u$ in $\mathrm{W}^{2,2}(\Omega)$. Realize moreover

$$\int_\Omega\triangle u_n\varphi\,\mathrm{d}x=\int_\Omega u_n\triangle\varphi\,\mathrm{d}x\,,\forall\varphi\in\mathcal{D}(\Omega),$$

and the limit of this is

$$\int_\Omega v\varphi\,\mathrm{d}x=\int_\Omega u\triangle\varphi\,\mathrm{d}x=\int_\Omega\varphi\triangle u\,\mathrm{d}x\,,$$

which means $\triangle u=v$ *a.e.* in Ω and that $u\in\mathcal{D}(A), Au=v$.

Theorem 32 (Closedness of infinitesimal generators)**.** *Let $(A,\mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\ge0}\subset\mathcal{L}(X)$. Then*

1. *$\mathcal{D}(A)$ is dense in X,*
2. *$(A,\mathcal{D}(A))$ is closed.*

Proof. (*From: the lectures*) Ad 1.: From the previous theorem it follows $\forall x\in X,\forall t\ge0: x_t\in\mathcal{D}(A)$, so

$$\frac{1}{t}x_t=\frac{1}{t}\int_0^t S(s)x\,\mathrm{d}s\in\mathcal{D}(A),$$

and moreover

$$\lim_{t\to0^+}\frac{x_t}{t}\lim_{t\to0^+}\int_0^t S(s)\,\mathrm{d}s=S(0)x\in X$$

, and so we have showed (again with the help of Lebesgue differentiation theorem)

$$\forall x \in X \exists \left\{ \frac{x_t}{t} \right\} \subset \mathcal{D}(A) : \frac{x_t}{t} \to x,$$

i.e., $\mathcal{D}(A)$ is dense in X.

Ad 2.: Take $\{x_n\} \subset \mathcal{D}(A) : x_n \to x$ in $X, Ax \to v \,\text{in}\, X$ and compute first [38]

$$\frac{(S(h)-\mathrm{id})x}{h} = \lim_{n\to\infty} \frac{(S(h)-\mathrm{id})x_n}{h} = \frac{1}{h}\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}(S(s)x_n)\,\mathrm{d}s = \frac{1}{h}\int_0^h AS(s)x_n\,\mathrm{d}s = \frac{1}{h}\int_0^h S(s)Ax_n,$$

so taking the limit yields

$$Ax = \lim_{h\to 0^+} \frac{(S(h)-\mathrm{id})x}{h} = \lim_{h\to 0^+} \frac{1}{h}\int_0^h S(s)v\,\mathrm{d}s = v,$$

Altogether, $x \in \mathcal{D}(A), Ax = v$. □

Lemma 18. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of c_0-semigroups $\{S(t)\}_{t\geq 0}, \{\tilde{S}(t)\}_{t\geq 0}$. Then*

$$\{S(t)\}_{t\geq 0} = \{\tilde{S}(t)\}_{t\geq 0}.$$

Proof. (*From: the lectures*) We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix $x \in \mathcal{D}(A), t > 0$. Then $g(s) := S(s)\tilde{S}(t-x)x$ satisfies $g \in C^1([0,t];X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$, as A, S commute. This means $g(0) = g(1)$ and from this it follows $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X, S$ continous $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$, and since $t \geq 0$ was arbitrary, we are done. □

7.2 Resolvent set & operator

Definition 20 (Resolvent of a linear operator)**.** Let $(A, \mathcal{D}(A))$ be a linear (possibly unbounded) operator on X. We define

1. resolvent set

$$\rho(A) = \left\{\lambda \in \mathbb{K} | \lambda\,\mathrm{id} - A \,\text{is invertible and}\, (\lambda\mathrm{id} - A)^{-1} \in \mathcal{L}(X)\right\},$$

2. resolvent operator $R(\lambda, A) : X \to \mathcal{D}(A) : R(\lambda, A) = (\lambda\mathrm{id} - A)^{-1}$, for $\lambda \in \rho(A)$.

Remark. If $(A, \mathcal{D}(A))$ is a closed linear operator: $\lambda \in \rho(A) \Leftrightarrow \lambda\mathrm{id} - A$ is a bijection of $\mathcal{D}(A)$ onto X.

Lemma 19. *Let $(A, \mathcal{D}(A))$ be a linear operator on X. Then it holds*

1. $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$
2. $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x,$
3. $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$, *and* $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A),$
4. *If moreover $(A, \mathcal{D}(A))$ is the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ s.t. $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, then $\forall \lambda > \omega$ it holds*

[38] This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that X is a Banach space and work with some functionals and Bochner integrals or whatever

- $\lambda \in \rho(A)$,
- $$R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t)\,\mathrm{d}t\,,$$
- $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda-\omega}$.

Remark. The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

Proof. (*From: the lectures*) Ad 1.:

$$AR(\lambda, A)x = (A - \lambda\mathrm{id})\underbrace{R(\lambda, A)}_{=(\lambda\mathrm{id}-A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$\begin{aligned} R(\lambda, A) - R(\eta, A) &= R(\lambda, A)(\mathrm{id} - (\lambda\mathrm{id} - A))R(\eta, A) = R(\lambda, A)(\eta\mathrm{id} - A - \lambda\mathrm{id} + A)R(\eta, A) = \\ &= (\eta - \lambda)R(\lambda, A)R(\eta, A) \end{aligned}$$

For $\lambda \neq \eta$ we also have

$$R(\lambda, A)R(\eta A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG asume $\omega = 0$, meaning $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$. Denote $\tilde{S}(t) = e^{-\omega t}S(t)$.

Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x\,\mathrm{d}t\,.$$

First of all, this is well defined as

$$\left\|\tilde{R}x\right\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X \,\mathrm{d}t = \frac{M}{\lambda}\|x\|_X,$$

and so $\left\|\tilde{R}\right\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}$, $\tilde{R} \in \mathcal{L}(X)$. Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Leftrightarrow \mathrm{id} = (\lambda\mathrm{id} - A)\tilde{R}.$$

For $x \in X, h > 0$ fixed compute

$$\begin{aligned} \frac{1}{h}\left(S(h)\tilde{R}x - \tilde{R}x\right) &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t}S(t+h)x - e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \\ &= \frac{1}{h}\left(\int_h^\infty e^{-\lambda(t-h)}S(t)x\,\mathrm{d}t - \int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \\ &= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t\,, \\ &= \int_h^\infty e^{-\lambda t}\frac{e^{\lambda h} - 1}{h} S(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t = \\ &= \frac{e^{\lambda h} - 1}{h}\int_h^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t = \end{aligned}$$

and so

$$\lim_{h\to 0^+}\frac{1}{h}\big(S(h)\tilde{R}x-\tilde{R}x\big)=\lim_{h\to 0^+}\frac{e^{\lambda h}-1}{h}\int_h^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t-\frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t=\lambda\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t-x=\lambda\tilde{R}x-x,$$

meaning truly

$$\tilde{R}x\in\mathcal{D}(A),A\tilde{R}x=\lambda\tilde{R}x-x\Rightarrow \mathrm{id}=(\lambda\mathrm{id}-A)\tilde{R}.$$

i.e.

To show that $\tilde{R}$ is also the second inverse, we need the following theorem:

$$x\in\mathcal{D}(A),A\,\text{closed}:A\tilde{R}x=A\Big(\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\Big)=\int_0^\infty e^{-\lambda t}\underbrace{AS(t)}_{=S(t)A}x\,\mathrm{d}t=\tilde{R}Ax,$$

which has been stated but not proved [39]. Using this however, one has

$$\mathrm{id}=\lambda\tilde{R}-A\tilde{R}=\lambda\tilde{R}-\tilde{R}A=\tilde{R}(\lambda\mathrm{id}-A),$$

and so in total

$$\mathrm{id}=\tilde{R}(\lambda\mathrm{id}-A)=(\lambda\mathrm{id}-A)\tilde{R}.$$

And so $\lambda\mathrm{id}-A$ has an inverse, which together with the fact $\tilde{R}$ is bounded concludes, $\lambda\in\rho(A)$ and $\tilde{R}=R(\lambda,A)$. The fact that $R(\lambda,A)$ follows from the remark above. □

Definition 21 (Contraction semigroup)**.** We say that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup if

$$\forall t\geq:\|S(t)\|_{\mathcal{L}(X)}\leq 1.$$

Theorem 33 (Hille-Yosida)**.** *Let $M\geq 1,\omega\in\mathbb{R}$. A linear $(A,\mathcal{D}(A))$ on a Banach space X generates a c_0-semigroup (meaning it is its infinitesimal generator) satysfing $\forall t\geq 0:\|S(t)\|_{\mathcal{L}(X)}\leq Me^{\omega t}$* ***if and only if***

1. *$(A,\mathcal{D}(A))$ is closed,*
2. *$\mathcal{D}(A)$ is dense in X,*
3. *$\forall\lambda>\omega,n\in\mathbb{N}:\lambda\in\rho(A)$ and it holds $\|R^n(\lambda,A)\|_{\mathcal{L}(X)}\leq\frac{M}{(\lambda-\omega)^n}$.*

Proof. (*From: the lectures*) *Proof from: lectures* If $M=1,\omega=0$, then $\|R(\lambda,A)\|_{\mathcal{L}(X)}\leq\frac{1}{\lambda}\Rightarrow\|R^n(\lambda,A)\|_{\mathcal{L}(X)}\leq\frac{1}{\lambda^n}$. " $\Rightarrow$ " has been proven, now show the other direction. The plan is to

1. approximate A by $\{A_n\}\subset\mathcal{L}(X)$,
2. construct S_n for A_n as previously,
3. estimate and limit passage.

Approximation: See the analogy: $a\in\mathbb{R}:\frac{n}{n-a}\to 1$, we would like $nR(n,A)=n(n\mathrm{id}-A)^{-1}\to\mathrm{id}$, as then our obvious candidate for the aproximation of A would be

$$A_nx=A(nR(n,A))x,$$

and thanks to the property $AR(\lambda,A)=\lambda R(\lambda,A)-\mathrm{id}$, we actually see $AnR(n,A)=n(nR(n,A)-\mathrm{id})$, which clearly is a linear bounded operator.

[39] It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

Let us so calculate the norm of $nAR(n,A) = n(nR(n,A) - \mathrm{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$, (This approx. is called the Yosida approximation.)

Fix some $x \in \mathcal{D}(A)$ and write

$$\|nR(n,A)x - x\|_X = \|R(n,A)Ax\|_X \le \|R(n,A)\|_{\mathcal{L}(X)}\|Ax\|_X \le \frac{1}{n}\|Ax\|_X \to 0 \,\mathrm{as}\, n \to \infty.$$

If

$$y \in X : \|nR(n,A)y - y\|_X \le \|nR(n,A)(y-x)\|_X + \|nR(n,A)x - x\|_X + \|x-y\|_X \le 2\|y-x\| + \underbrace{\|nR(n,a)x - x\|_X}_{\to 0},$$

but $\|y - x\|_X$ can be made arbitrarily small from density of $\mathcal{D}(A)$ in X, so in fact

$$nR(n,A)y \to y \,\mathrm{in}\, X, \forall y \in X.$$

And so $nR(n,A)$ really approximates id.

Using this gives us

$$\forall x \in \mathcal{D}(A) : A_n x = nAR(n,A)x = n\,\overbrace{R(n,A)A}^{=AR(n,A)}\,x \to Ax \,\mathrm{in}\, X$$

pointwisely. Define now (this is only possible since $\{A_n\} \subset \mathcal{L}(X)$)

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} = \left\|\sum_{k=0}^{\infty} \frac{1}{k!}(tA_n)^k\right\|_{\mathcal{L}(X)} = \left\|\sum_{k=0}^{\infty} \frac{1}{k!}\big(-nt\mathrm{id} + n^2 tR(n,A)\big)^k\right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\|\sum_{k=0}^{\infty} \frac{1}{k!}(-nt\mathrm{id})^k \sum_{k=0}^{\infty} \frac{\big(n^2 tR(n,A)\big)^k}{k!}\right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives

$$\|S_n(t)\|_{\mathcal{L}(X)} \le e^{-nt}\,\mathrm{id} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!}\|nR(n,A)\|_X^k \le e^{-nt}e^n t = 1,$$

as $\|nR(n,A)\|^k \le 1$. This means

$$\{S_n(t)\}_{\mathcal{L}(X)} \le 1 = Me^{\omega t},$$

for our case $M = 1, \omega = 0$. Now show that this converges: fix $x \in \mathcal{D}(A)$, compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_n(s)S_m(t-s)x)\,\mathrm{d}s\right\|_X = \left\|\int_0^t S_n(s)(A_n - A_m)S_m(t-s)x\,\mathrm{d}s\right\|_X \le$$
$$\underbrace{\le}_{\|S_l\|_{\mathcal{L}(X)} \le 1} \quad t\|(A_n - A_m)x\|_X,$$

and since $\{A_n x\}_{n\in\mathbb{N}}$ converges, it is Cauchy and thus $S_n(t)x \subset X$ is Cauchy, but since X is Banach, it also converges. Finally, for $y \in X$, we have

$$\|S_n(t)y - S_m(t)y\|_X \le \|S_n(t)(y-x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x-y)\|_X \le 2\|x-y\|_X + t\|(A_n - A_m)x\|_X,$$

and since again $\mathcal{D}(A)$ is dense in X, the norm $\|x-y\|$ can be made aribtrarily small $\Rightarrow \{S_n(y)\}$ converges $\forall y \in X$. By Banach-Steinhaus we can now argue that since S_n are linear continuous operators on a Banach space, that are uniformly bounded by 1 and coverge pointwisely $\forall y \in X$, the operators must also converge in $\mathcal{L}(X)$, meaning

$$\exists S(t) \in \mathcal{L}(X)\; s.t.\, S_n(t)y \to S(t)y, \forall t \ge 0, \forall y \in X,$$

and so $S(t)_{t\ge 0}$ really is a c_0−semigroup.

It remains to answer this question. Is $(A, \mathcal{D}(A))$ the infinitesimal generator of $\{S(t)\}_{t\ge0}$? Let $(\tilde{A}, \mathcal{D}(\tilde{A}))$ be the infinitesimal generator of $\{S(t)\}_{t\ge0}$. Compute

$$S_n(t)x - x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} S_n(s)x \,\mathrm{d}s = \int_0^t S_n(x)A_n x \,\mathrm{d}s\,,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \le \|S_n(s)(A_n - A)x\|_X + \|(S_n(s) - S(s))Ax\|_X \to 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \,\mathrm{d}s\,.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t\to0_+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A} \,\mathrm{on}\, \mathcal{D}(A).$$

The opposite inclusion is simple: fix $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$, and so $\lambda\mathrm{id} - A : \mathcal{D}(A) \to X$ is onto, but also $\lambda\mathrm{id} - A = \lambda\mathrm{id} - \tilde{A}$ on $\mathcal{D}(A)$, and so $\lambda\mathrm{id} - \tilde{A} : \mathcal{D}(A) \to X$ is onto. From the previous theorem, we know $\lambda\mathrm{id} - \tilde{A}$ is one-to-one, so $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Altogether, $A = \tilde{A}, \mathcal{D}(A) = \mathcal{D}(\tilde{A})$.

$\square$

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