(although this is a bit inaccurate). Realize that since $u = 0$ outside of Ω, also u_j is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega_j)$. Now pick $\delta \in \left(0, \frac{\beta}{2}\right)$, where β is from the definition of $\mathrm{C}^{0,0}$ and set

$$S_j^\delta = \mathbb{A}_j\left(\left\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$
$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

i.e.,

$$"\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d)|a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j\left(\left\{(x', x_d)|x_d < a_j(x') - \frac{\beta}{2} - \delta\right\}\right)."$$

The trick is to shift the (support of) function u_j "into" Ω_j^δ

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\mathrm{supp}\,(\tau_\delta u_j) = \mathrm{supp}\,(u_j) - \delta,$$

from which it follows $\tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)$; we have only shifted the function u_j, but since we have also shifted S_j, qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j, \Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$, and the fact τ_δ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ *s.t.*

$$\|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to u_j, we are done once we approximate $\tau_\delta u_j$ by a function from $\mathrm{C}^\infty\left(\overline{\Omega}\right)$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $\mathrm{C}^\infty\left(\Omega_j^\delta\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right)$.

Show $\Omega \subset\subset \Omega_j^\delta$*:* We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^{m} \mathbb{A}_k(\{(x', x_d)|x_d = a_k(x'), x' \in \mathrm{U}(0,\alpha)\}),$$

and the set Ω_j^δ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j\left(\left\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right).$$

Realize it suffices to show $\partial\Omega \notin \overline{S_j}$, as then it wont be excluded from $\mathbb{R}^d$ and thus will end up in Ω_j^δ. *Thanks to continuity of* a_j, we may write

$$\overline{S_j} = \mathbb{A}_j\left(\left\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

i.e., the " $<$ " have changed to " $\le$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d)|x_d = a_j(x'), x' \in \mathrm{U}(0,\alpha)\}`) \notin \mathbb{A}_j\left(\left\{(x', x_d)|a_j(x') - \frac{\beta}{2} - \delta \le x_d \le a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

which is equivalent to

$$\Big((a_j \le a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta)\Big) \vee \Big((a_j > a_j - \delta) \wedge (a_j \ge a_j - \frac{\beta}{2} - \delta)\Big).$$

Our choice has been $\delta \in \left(0, \frac{\beta}{2}\right)$, and $\beta > 0$ from the definition of $\Omega \in \mathrm{C}^{0,0}$, so the second statement is clearly true $\forall j \in 1, \dots, m$. Consequently $\partial\Omega \notin \overline{S}_j$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

Approximation of $\tau_\delta u_j$. Since Ω_j^δ is open there $\exists v_j \in \mathrm{C}^\infty\left(\Omega_j^\delta\right)$ such that

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \|\tau_\delta w_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

What is more, since $\overline{\Omega} \subset \Omega_j^\delta$, we see $v_j \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ in fact.

Approximation of u.

Finally, let us set

$$v = \sum_{j=0}^m v_j.$$

Then $v \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ and it holds

$$\begin{aligned}
\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} &= \left\|\sum_{j=0}^m u_j - \sum_{j=0}^m v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \left\|\sum_{j=0}^m u_j - v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \sum_{j=0}^m \|u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \\
&\le \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} + \sum_{j=1}^m \|\tau_\delta u_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \\
&< \frac{\varepsilon}{m+1} + 2\sum_{j=1}^m \frac{\varepsilon}{2(m+1)} = \varepsilon
\end{aligned}$$

□

Remark (What is $\mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$). Recall

$$C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \left\{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\right\}.$$

In other literature, it is stated that also $\mathrm{C}^\infty\left(\overline{\Omega}\right)$ is dense in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ if $\Omega \in \mathrm{C}^{0,0}$. This probably means

$$\mathrm{C}^\infty\left(\overline{\Omega}\right) \subset \mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d).$$

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, does there exist $\overline{u} \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)$, $\quad s.t. \overline{u}|_\Omega = u, \|\overline{u}\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} \le C(\Omega)\|u\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$?

The answer is **yes**, if Ω is nice enough.

Lemma 4. *Let* $\alpha, \beta > 0, K \subset U(0,\alpha) \times [\alpha, \beta]$ *be compact. Then*

$$\exists C > 0, \exists E : C^1(\overline{U(0,\alpha)} \times [0,\beta]) \to C^1(\overline{U(0,\alpha)} \times [-\beta,\beta]), \exists \tilde{K} \subset U(0,\alpha) \times [-\beta, b)\ compact$$

such that:

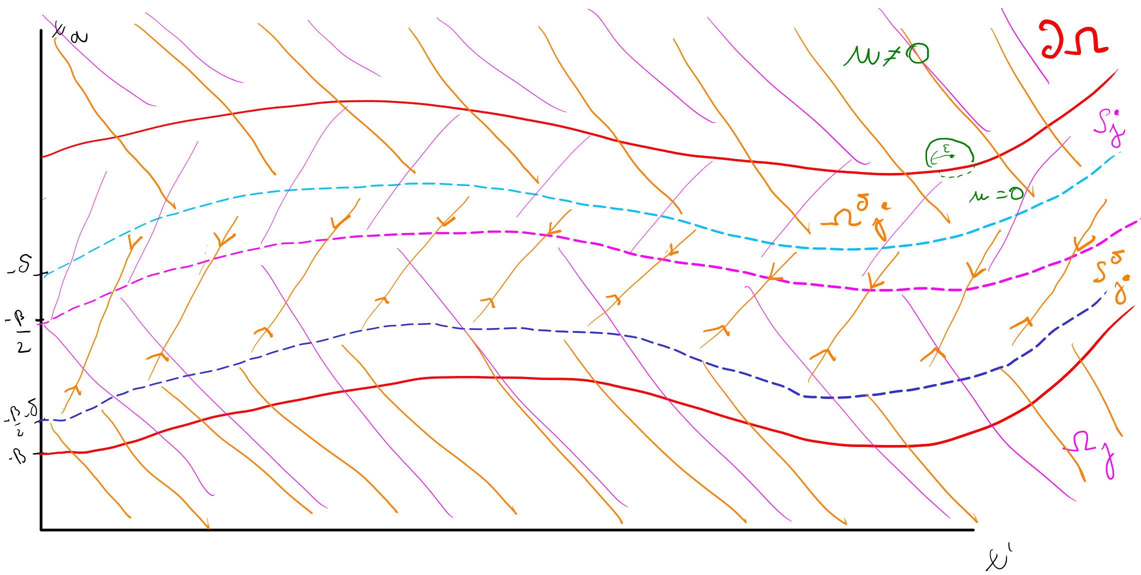

Figure 1: A cumbersome sketch of $\Omega_j, S_j, \Omega_j^\delta, S_j^\delta$

Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms ($\mathcal{F}$ is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for $m > N + 1$, *i.e.*, $m - 1 > N$ it holds $U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}, \Omega_N \subset \Omega_{m-1}$ meaning $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \varnothing \Rightarrow \operatorname{supp} u\varphi_j \cap \Omega_N = \operatorname{supp} v_j \cap \Omega_N = \varnothing$, since $\operatorname{supp} u\varphi_j \subset U_j, \operatorname{supp} v_j \subset \operatorname{supp} u\varphi_j \subset U_j, \forall j \geq m$. The norm of sum is

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let $N \to \infty$ and realize

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} \,\mathrm{d}x \int_{\mathbb{R}^d} |D^\alpha f| \chi_\Omega(x) \,\mathrm{d}x = \int_\Omega |D^\alpha f| \,\mathrm{d}x\,,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Alltogether,

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$ (this was not totally evident) and thus $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega) \cap C^\infty(\Omega)$ so indeed we have showed the desired density. □

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Remark ($\mathrm{C}^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $\mathrm{C}^{k,\lambda}$ if: Ω is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0,1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : \mathrm{B}(0,\alpha) \subset \mathbb{R}^{d-1} :\to \mathbb{R}\, s.t.\, a_j \in \mathrm{C}^{k,\lambda}(\mathrm{B}(0,\alpha)), \exists \mathbb{A}_j \mathbb{R}^d \to \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \leq m : \varnothing \neq \partial\Omega \cap U_j = \mathbb{A}_j\big(\{(x', a_j(x') \in \mathbb{R}^d | x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}\}\big)$,
3. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \Omega$,
4. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \mathbb{R}^d/\overline{\Omega}$.

If $\lambda = 0$ we sometimes drop it and write $\Omega \in \mathrm{C}^{k,0} \Leftrightarrow \Omega \in \mathrm{C}^{k}$, if $k = 0, \lambda = 1$ we call $\Omega \in \mathrm{C}^{0,1}$ to be a Lipschitz domain. *Remember that $\lambda(\Omega) < \infty$ is a part of the definition.*

Theorem 5 (Global approximation by smooth functions up to the boundary)**.** *Let $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C^{\infty}_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in \mathrm{C}^{\infty}\left(\overline{\Omega}\right)$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\overline{\Omega}$,
2. partition of unity,
3. approximation of u on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^m$ be from the definition of $\mathrm{C}^{0,0}$ boundary. Then[4]

$$\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to $\{U_j\}_{j=0}^m$. Since

$$u = \sum_{j=0}^{m} u\varphi_j, \text{ on } \Omega$$

observe that $u_j := u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \operatorname{supp} u_j \subset \operatorname{supp} \varphi_j \subset\subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d / \Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \dots, m\}$.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of Ω, it holds $u\varphi_0 \in \mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)$. Since $\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d) = \mathrm{W}^{\mathrm{k,p}}_0(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)}}$, we can find $v_0 \in \mathcal{D}(\mathbb{R}^d)$ *s.t.*

$$\|v_0 - u\varphi_0\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

Case $j \in \{1, \dots, m\}$. We have a problem now: $\{U_j\}_{j=1}^m$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of Ω, where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate u_j on a larger *open* domain containing $\overline{\Omega}$ and then show this is also a good aproximation of u_j on $\Omega \subset \overline{\Omega}$.

Set $w_j = u\varphi_j$, and denote

$$\begin{aligned} S_j &= \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in \mathrm{U}(0, \alpha)\right\}\right), \\ \Omega_j &= \mathbb{R}^d / \overline{S_j}, \end{aligned}$$

i.e.,

$$\text{"}\Omega_j = \Omega \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d \leq a_j(x') - \frac{\beta}{2}\right\}\right),\text{"}$$

[4]Our choice $U_0 = \Omega$ is important, as without it the definition of $\mathrm{C}^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^m U_j$.