

# Partial differential equations II

Kamil Belan

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# 1 Winter semester addendum

## 1.1 Weak\* convergence

Since  $L_\infty((0, T); L_2(\Omega))$  is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of  $L_\infty((0, T); L_2(\Omega))$  is reflexive, i.e.

$$L_\infty((0, T); L_2(\Omega)) \cong \left( L_1((0, T); L_2(\Omega)) \right)^*,$$

which means that balls in  $L_\infty((0, T); L_2(\Omega))$  are weakly\* compact. Moreover,  $L_1((0, T); L_2(\Omega))$  is *separable*, from which it follows  $L_\infty((0, T); L_2(\Omega))$  with the weak\* topology is metrizable and thus there exists a weakly\* converging subsequence (from the balls).

**Example** (For people without Functional Analysis I). Let  $X$  be a linear normed space,  $\{x_n\} \subset X$  a sequence in  $X$ . We say  $x_n$  converges weakly to  $x \in X$  whenever

$$f(x_n) \rightarrow f(x), \forall f \in X^*.$$

Let  $X^*$  be the topological dual to  $X$ ,  $\{x_n\} \subset X^*$  a sequence in  $X^*$ . We say  $f_n$  converges weakly\* to  $f \in X^*$  whenever

$$f_n(x) \rightarrow f(x), \forall x \in X, \text{ i.e. } x(f_n) \rightarrow x(f),$$

where by  $x(y), x \in X, y \in X^*$  we understand

$$\varepsilon_x : X^* \rightarrow \mathbb{K}, y \mapsto y(x).$$

Since  $L_\infty((0, T); L_2(\Omega)) \cong \left( L_1((0, T); L_2(\Omega)) \right)^*$ , every point  $x \in L_\infty((0, T); L_2(\Omega))$  can be interpreted as a linear functional on  $L_1((0, T); L_2(\Omega))$ , so given  $\{x_n\} \subset L_\infty((0, T); L_2(\Omega))$ , we can interpret it as a  $\{x_n\} \subset \left( L_1((0, T); L_2(\Omega)) \right)^*$ , meaning given a weakly converging sequence in  $L_\infty((0, T); L_2(\Omega))$ , it is actually a weakly\* converging sequence in  $L_1((0, T); L_2(\Omega))$ .

## 1.2 Regularity of parabolic problems

**Theorem 1.** *Let the assumptions of the previous theorem hold and  $\Omega \in C^{1,1}, \delta \in (0, 1)$ . Then  $u \in L_2((\delta, T); W^{2,2}(\Omega))$ .*

*Proof.* Take the weak formulation in  $t \in (\delta, T)$ . WLOG further assume  $d = 0$ . Then

$$\int_{\Omega} \mathbb{A} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi - b u \varphi - \mathbf{c} \cdot \nabla u \varphi - \int_{\Omega} \partial_t u \varphi = \int_{\Omega} (f - b u - \mathbf{c} \cdot \nabla u - \partial_t u) \varphi,$$

and the integrand of the last integral is in  $L_2(\Omega)$  for a.e.  $t \in (\delta, T)$ . We can thus use the elliptic regularity results and write:

$$\|u\|_{W^{2,2}(\Omega)}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{W^{1,2}(\Omega)}^2 + \|\partial_t u\|_{L_2(\Omega)}^2),$$

integrating both sides  $\int_{\delta}^T dt$  yields

$$\|u\|_{L_2((\delta,T);L_2(\Omega))}^2 \leq C(\|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2((0,T);W^{1,2}(\Omega))}^2 + \|u\|_{L_2((\delta,T);L_2(\Omega))}^2)$$

□

**Theorem 2.** *If data are smooth and satisfy the compatibility conditions, then the weak solutions to the parabolic equation are smooth.*

*Proof.* no. □

*Remark* (Compatibility condition). : Take the heat equation :  $\partial_t u - \Delta u = f$  at time zero:  $\Delta u(0) + f(0) = \partial_t u(0) \in W_0^{1,2}(\Omega)$ , so we need that  $f(0) + \Delta u(0)$  has zero trace  $\Rightarrow$  compatibility conditions.

### 1.3 Uniqueness of solutions to hyperbolic problems

**Theorem 3** (Uniqueness of the solution to a hyperbolic equation). *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that  $\mathbf{c} \in W^{1,\infty}(\Omega)$ . Then the weak solution to the hyperbolic equation is unique.*

*Proof.* It is enough that if  $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$ . To do that, take the equation, multiply it by  $\varphi \in V$  fixed and integrate over  $\Omega$  for  $t \in (0, T)$  fixed:

$$\langle \partial_{tt} u(t), \varphi \rangle + \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \varphi \, dx + \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \varphi \, dx - \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \varphi \, dx = 0.$$

Now, take a special test function

$$\psi(t) = \left( \int_t^s u(\tau) \, d\tau \right) \chi_{(0,s)}(t),$$

for some  $s \in (0, T)$ . Then  $\partial_t \psi(t) = -u(t)$  on  $t \in (0, s)$ . Next, integrate the equation in time over  $(0, s)$ .

$$\int_0^s \langle \partial_{tt} u(t), \psi \rangle \, dt + \int_0^s \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s \langle \partial_{tt} u(t), \varphi \rangle \, dt = \langle \partial_t u(s), \psi(s) \rangle - \langle \partial_t u(0), \psi(0) \rangle - \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt,$$

and realize  $\psi(s) = 0, \partial_t u(0) = 0$ , so

$$- \int_0^s \langle \partial_t u(t), \partial_t \psi(t) \rangle \, dt + \int_0^s \int_{\Omega} \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, dx \, dt + \int_0^s \int_{\Omega} (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, dx \, dt - \int_0^s \int_{\Omega} u(t) \mathbf{d}(t) \cdot \nabla \psi \, dx \, dt = 0,$$

but since  $\partial_t \psi(t) = -u(t)$ , we can actually write (time dependencies are omitted for brevity)

$$\int_0^s \langle \partial_t u, u \rangle \, dt + \int_0^s \int_{\Omega} -\mathbb{A} \nabla \partial_t \psi \cdot \nabla \psi - b \psi \partial_t \psi - \psi \mathbf{c} \cdot \nabla \partial_t \psi + \partial_t \psi \mathbf{d} \cdot \nabla \psi \, dx \, dt = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\begin{aligned}
& \frac{1}{2} \int_0^s \frac{d}{dt} \left( \|u\|_{L_2(\Omega)}^2 - \int_{\Omega} \mathbb{A} \nabla \psi \cdot \nabla \psi + b \psi^2 + \psi \mathbf{c} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla \psi \, dx \right) dt = \\
& = \int_0^s \int_{\Omega} (\partial_t \mathbb{A}) \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 + \psi \partial_t \mathbf{c} \cdot \nabla \psi + \underbrace{\partial_t \psi \cdot \mathbf{c} \cdot \nabla \psi}_{=-u(t)} - \psi \partial_t \mathbf{d} \cdot \nabla \psi - \psi \mathbf{d} \cdot \nabla \underbrace{\partial_t \psi}_{=-u(t)} \, dx \, dt,
\end{aligned}$$

and upon integration (recall  $\psi(s) = 0$ , from the definition of  $\psi$  it follows  $\nabla \psi(0) = 0$ , and  $u(0) = 0$ ),

$$\begin{aligned}
& \frac{1}{2} \left( \|u(s)\|_{L_2(\Omega)}^2 + \int_{\Omega} \mathbb{A}(0) \nabla \psi(0) \cdot \nabla \psi(0) + b(0) \psi(0)^2 + \psi(0) \mathbf{c}(0) \cdot \nabla \psi(0) + \psi(0) \mathbf{d}(0) \cdot \nabla \psi(0) \, dx \right) = \\
& = \int_0^s \int_{\Omega} \partial_t \mathbb{A} \nabla \psi \cdot \nabla \psi + \partial_t b \psi^2 - u \partial_t \mathbf{c} \cdot \nabla \psi - \psi \partial_t \mathbf{d} \cdot \nabla \psi + \psi \mathbf{d} \cdot \nabla u \, dx \, dt.
\end{aligned}$$

From this we obtain the following estimate:

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\psi(0)\|_{W^{1,2}(\Omega)}^2 \leq C \left( \int_0^s \|\psi\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\psi(0)\|_{L_2(\Omega)}^2,$$

where  $C = C(\|\mathbb{A}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbb{A}\|_{L_{\infty}(\Omega)}, \|b\|_{L_{\infty}(\Omega)}, \|\partial_t b\|_{L_{\infty}(\Omega)}, \|\mathbf{c}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{c}\|_{L_{\infty}(\Omega)}, \|\mathbf{d}\|_{L_{\infty}(\Omega)}, \|\partial_t \mathbf{d}\|_{L_{\infty}(\Omega)})$ .

Define now the test function  $\chi(t) = \int_0^t u(\tau) \, d\tau$ , and realize that in fact  $\psi(t) = \chi(s) - \chi(t)$ ,  $\chi(0) = 0$ . Plugging this in the above inequality yields

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left( \int_0^s \|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2 \right) dt + \|\chi(s)\|_{L_2(\Omega)}^2,$$

and using

$$\|\chi(s) - \chi(t)\|_{W^{1,2}(\Omega)}^2 = \|\chi(t) - \chi(s)\|_{W^{1,2}(\Omega)}^2 \leq 2 \left( \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \right),$$

and the definition of  $\chi(t)$ , from which it follows

$$\|\chi(s)\|_{L_2(\Omega)}^2 \leq \int_0^s \|u\|_{L_2(\Omega)}^2 \, dt,$$

we are allowed to write

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{L_2(\Omega)}^2 \leq C \left( \int_0^s 2 \|\chi(s)\|_{W^{1,2}(\Omega)}^2 + 2 \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + 2 \|u\|_{L_2(\Omega)}^2 \, dt \right),$$

and so

$$\|u(s)\|_{L_2(\Omega)}^2 + (1 - 2sC) \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_1 \left( \int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right).$$

If we now choose  $T_1 \in (0, T]$  small enough s.t.  $1 - 2sC > 0$  for  $s \in (0, T_1]$ , we finally obtain

$$\|u(s)\|_{L_2(\Omega)}^2 + \|\chi(s)\|_{W^{1,2}(\Omega)}^2 \leq C_2 \left( \int_0^s \|\chi(t)\|_{W^{1,2}(\Omega)}^2 + \|u(t)\|_{L_2(\Omega)}^2 \, dt \right), \forall s \in (0, T_1],$$

which implies  $u = 0$  on  $(0, T_1]$  by the Gronwall lemma: we have

$$\xi(t) \leq \int_0^t \xi(s) \, ds, \text{ for } a.a. \, t \in (0, T) \Rightarrow \xi(t) = 0 \, a.e..$$

for  $\xi \in L_1((0, T))$  nonnegative<sup>1</sup>. If we now bootstrap on  $[T_1, 2T_1], [2T_1, 3T_1]$  etc., we obtain  $u = 0$  on  $(0, T]$ . □

## 2 Sobolev spaces revisited

Let  $\Omega \subset \mathbb{R}^d$  open,  $p \in [1, +\infty]$ ,  $k \in \mathbb{N}$ . We define

$$W^{k,p}(\Omega) = \left\{ f \in L_p(\Omega) ; D^\alpha f \in L_p(\Omega), \forall |\alpha| \leq k \right\},$$

with the norm

$$\|f\|_{W^{k,p}(\Omega)}^p = \|f\|_{L_p(\Omega)}^p + \sum_{0 < |\alpha| \leq k} \|D^\alpha f\|_{L_p(\Omega)}^p.$$

Recall that:

- $W^{k,p}(\Omega)$  is Banach  $\forall p$  and Hilbert for  $p = 2$ .
- $W^{k,p}(\Omega)$  is separable if  $p < \infty$  and reflexive if  $p > 1, p < \infty$ .

*Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.*

### 2.1 Tools from functional analysis

**Definition 1** (Regularization kernel). The function  $\eta$  is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\text{supp } \eta \subset U(0, 1)$
- $\eta \geq 0$
- $\eta$  is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x) dx = 1$

**Definition 2** (Regularization of a function). Let  $\eta$  be a regularization kernel. Set<sup>2</sup>

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of  $f \in L_1(\Omega)_{\text{loc}}$  by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

*Remark* (Properties of regularization). The regularization has the following properties:

- $f \in L_p(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  in  $L_p(\Omega)$  and also a.e

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<sup>1</sup>In our case  $\xi = \|u\|_{L_2(\Omega)}^2 + \|\chi\|_{W^{1,2}(\Omega)}^2$ .

<sup>2</sup>Another common choice is  $\eta_k = k^d \eta(kx)$ ,  $k \in \mathbb{N}$ .

- $f \in L_\infty(\Omega) \Rightarrow f_\varepsilon \rightarrow f$  a.e and \*-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) \eta_\varepsilon(x-y) dy = \int_{U(x,\varepsilon)} f(y) \eta_\varepsilon(x-y) dy$
- $\text{supp } f_\varepsilon \subset \overline{U(\Omega, \varepsilon)}, f = 0 \text{ on } U(x, \varepsilon) \Rightarrow f_\varepsilon(x) = 0$

**Definition 3** ( $\Omega' \subset\subset \Omega$ ).  $O \subset\subset \Omega$  means  $\overline{O}$  is compact and  $\overline{O} \subset \Omega$ .

**Definition 4** (Shift operator). For  $u \in L_p(\Omega), k \in \{1, \dots, d\}, h > 0$ , we introduce the shift operator

$$\tau_h u(x) = u(x + h \mathbf{e}_k)$$

**Lemma 1** (Approximation property of the shift operator). For  $u \in L_p(\Omega)$ , it holds  $\tau_h u \rightarrow u$  in  $L_p(\Omega), h \rightarrow 0^+$ .

**Lemma 2** (Partition of unity). Let  $E \subset \mathbb{R}^d, \mathcal{G}$  be an open covering of  $E$  (possibly uncountable.) Then there exists a countable system  $\mathcal{F}$  of nonnegative functions  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $0 \leq \varphi \leq 1$  and

1.  $\mathcal{F}$  is subordinate to  $\mathcal{G} : \forall \varphi \in \mathcal{F} \exists U \in \mathcal{G} : \text{supp } \varphi \subset U$
2.  $\mathcal{F}$  is locally finite<sup>3</sup>:  $\forall K \subset E$  compact,  $\text{supp } \varphi \cap K \neq \emptyset$  for at most finitely many  $\varphi \in \mathcal{F}$ .
3.  $\sum_{\varphi \in \mathcal{F}} \varphi(x) = 1, \forall x \in E$ .

*Proof.* (Sketch) *Step 1* (If  $E$  is compact):

$E$  compact  $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}$  s.t.  $E \subset \bigcup_{j=1}^m U_j$ . Moreover,  $\exists K_j \subset U_j$  compact such that  $E \subset \bigcup_{j=1}^m K_j$ . That follows from the exhaustion argument: for  $U \subset \mathbb{R}^d$  open, you can approximate it by a compact set:

$$K_m = \left\{ x \in U \mid \text{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m \right\}.$$

Then clearly  $K_1 \subset K_2 \dots$ , and they "converge monotonously to  $U$ ". Next, find  $\phi_j \in C_c(U_j), \phi_j > 0$  on  $K_j$ , e.g.  $\phi_j = \theta(\text{dist}(x, \partial U_j))$ . Then use convolution:  $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$  small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

*Step 2* (If  $E$  is open):

Approximate  $E$  by  $K \subset E$  compact by the exhaustion argument, then the covering will enlarge from finite  $\rightarrow$  countable (nontrivial reasoning).  $\square$

## 2.2 Density of smooth functions

**Lemma 3** (Local approximation by smooth functions (using regularization)). Assume  $p \in [1, \infty), \Omega \subset \mathbb{R}^d$  open,  $k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$ . Then it holds

1.  $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$  a.e. in  $\Omega_\varepsilon, \forall |\alpha| \leq k$
2.  $u_\varepsilon \rightarrow u$  in  $W^{k,p}(\Omega)_{loc}, \varepsilon \rightarrow 0^+$

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<sup>3</sup>In other words,  $\varphi_K$  is nonzero for at most finitely many  $\varphi \in \mathcal{F} \Leftrightarrow$  points in  $K$  can be represented by finitely many functions  $\varphi \in \mathcal{F}$ .

*Proof.* First of all:

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\left(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y) dy\right) = \int_{\mathbb{R}^d} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy,$$

the integrable majorants are *e.g.*  $\|\eta_\varepsilon\|_\infty |u| \chi_{U(0,\varepsilon)}(x) \in L_1(\Omega)$ . Now picking  $x \in \Omega_\varepsilon$  we realize  $\forall y \in \mathbb{R}^d/\Omega : x-y \geq \text{dist}(x, \partial\Omega) \geq \varepsilon$ , and so  $\eta_\varepsilon(x-y) = 0$ . Exchanging derivatives and using the definition of the weak derivative

$$\int_{\Omega} u(y)D_x^\alpha\eta_\varepsilon(x-y) dy = (-1)^{|\alpha|} \int_{\Omega} u(y)D_y^\alpha\eta_\varepsilon(x-y) dy = \int_{\Omega} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} D_y^\alpha u(y)\eta_\varepsilon(x-y) dy = (D^\alpha u)_\varepsilon.$$

Take  $V \subset\subset \Omega$  open, then

$$\|u - u_\varepsilon\|_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u - D^\alpha u_\varepsilon\|_{L_p(V)} \rightarrow 0,$$

because  $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \rightarrow D^\alpha u$  in  $L_p(V)$ , from the properties of regularization.  $\square$

**Theorem 4** (Global approximation by smooth functions). *Let  $\Omega \subset \mathbb{R}^d$  be open,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $C = \{f \in C^\infty(\Omega), \text{supp } f \text{ bounded}\} \cap W^{k,p}(\Omega)$  is dense in  $W^{k,p}(\Omega)$ , i.e.*

$$\overline{C \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

If moreover  $\Omega$  is bounded, it holds:

$$\overline{C^\infty \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega).$$

*Proof.* Let  $u \in W^{k,p}(\Omega), \varepsilon > 0$ . I want to show  $\exists v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$  s.t.  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ . For every  $j \in \mathbb{N}$  define an open set

$$\Omega_j = \left\{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{j}\right\}.$$

Clearly,  $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$ . Next, set

$$U_j = \Omega_{j+1} / \overline{\Omega_{j-1}}, j = 1, 2, \dots,$$

where  $\Omega_0 = \Omega_{-1} = \emptyset$ . Since  $\Omega_j$  are open,  $U_j$  are also open and  $\Omega \subset \bigcup_{j \in \mathbb{N}} U_j \Rightarrow \exists \{\varphi_j\}_{j \in \mathbb{N}}$  partition of unity subordinate to  $\{U_j\}_{j \in \mathbb{N}}$ . We can write  $u = \sum_{j \in \mathbb{N}} u\varphi_j$ , where  $u\varphi_j \in W^{k,p}(\Omega), \text{supp } u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$ . This is ready for convolution with  $\varepsilon_j > 0$ : set  $v_j = (u\varphi_j)_{\varepsilon_j}$  and fix an arbitrary  $\delta > 0$ . By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{\delta}{2^{j-1}},$$

for  $\varepsilon_j > 0$  sufficiently small, which we now fix so the above inequality holds. To have a nice inequality, we actually want:

$$\|v_j - u\varphi_j\|_{W^{k,p}(\Omega)} < \frac{2^N}{2^{N+1} - 1} \frac{\delta}{2^{j-1}},$$

meaning of  $N \in \mathbb{N}$  will be evident later.

Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then  $v \in C^\infty(\Omega)$ , (not clearly in  $W^{k,p}(\Omega)$  however) as  $\forall x \in \Omega$  the sum contains at most finitely many terms ( $\mathcal{F}$  is locally finite.)

Take the  $N \in \mathbb{N}$  and estimate the norm  $\|u - v\|_{W^{k,p}(\Omega)}$ . Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking  $x \in \Omega_N$  i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for  $m > N + 1$ , i.e.,  $m - 1 > N$  it holds  $U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}$ ,  $\Omega_N \subset \Omega_{m-1}$  meaning  $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \emptyset \Rightarrow \text{supp } u\varphi_j \cap \Omega_N = \text{supp } v_j \cap \Omega_N = \emptyset$ , since  $\text{supp } u\varphi_j \subset U_j$ ,  $\text{supp } v_j \subset \text{supp } u\varphi_j \subset U_j$ ,  $\forall j \geq m$ . The norm of sum is

$$\|u - v\|_{W^{k,p}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{W^{k,p}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let  $N \rightarrow \infty$  and realize

$$\|u - v\|_{W^{k,p}(\Omega_N)} \rightarrow \|u - v\|_{W^{k,p}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| dx = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) dx = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} dx = \int_{\mathbb{R}^d} |D^\alpha f| \chi_\Omega(x) dx = \int_\Omega |D^\alpha f| dx,$$

since  $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$ , and  $|D^\alpha f|$  is nonnegative, so the sequence under the integral is nondecreasing. Altogether,

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows  $v \in W^{k,p}(\Omega)$  (this was not totally evident) and thus  $v \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$  so indeed we have showed the desired density.  $\square$

*Remark.* It is nice that we only require  $\Omega$  to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

*Remark* ( $C^{k,\lambda}$  domain). Recall we call  $\Omega \subset \mathbb{R}^d$  to be of class  $C^{k,\lambda}$  if:  $\Omega$  is open and bounded,  $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0, 1], \alpha, \beta \in \mathbb{R}^+, \exists$  open sets  $U_j \subset \mathbb{R}^d, \exists a_j : B(0, \alpha) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R} \text{ s.t. } a_j \in C^{k,\lambda}(B(0, \alpha)), \exists \mathbb{A}_j \mathbb{R}^d \rightarrow \mathbb{R}^d$  affine orthogonal matrices such that

1.  $\partial\Omega \subset \bigcup_{j=1}^m U_j$ ,
2.  $\forall j \leq m : \partial\Omega \cap U_j = \mathbb{A}_j(\{(x', a_j(x')) \in \mathbb{R}^d | x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}\})$ ,
3.  $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \Omega$ ,
4.  $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in U(0, \alpha), b \in (0, \beta)\}) \subset \mathbb{R}^d / \overline{\Omega}$ .



If  $\lambda = 0$  we sometimes drop it and write  $\Omega \in C^{k,0} \Leftrightarrow \Omega \in C^k$ , if  $k = 0, \lambda = 1$  we call  $\Omega \in C^{0,1}$  to be a Lipschitz domain. *Remember that  $\lambda(\Omega) < \infty$  is a part of the definition.*

**Theorem 5** (Global approximation by smooth functions up to the boundary). *Let  $\Omega \in C^{0,0}$ ,  $k \in \mathbb{N}, p \in [1, \infty)$ . Then  $C_{\bar{\Omega}}^\infty(\mathbb{R}^d)$  is dense in  $W^{k,p}(\Omega)$ .*

*Proof.* Let  $u \in W^{k,p}(\Omega)$ , and  $\varepsilon > 0$ , be given. We wish to find  $v \in C^\infty(\bar{\Omega})$  s.t.  $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$ .

The sketch is simple:

1. covering of  $\bar{\Omega}$ ,
2. partition of unity,
3. approximation of  $u$  on the covering sets,
4. glue it together.

Set  $U_0 = \Omega$ , and let  $\{U_j\}_{j=1}^m$  be from the definition of  $C^{0,0}$  boundary. Then<sup>4</sup>

$$\bar{\Omega} \subset \bigcup_{j=0}^m U_j,$$

Take  $\{\varphi_j\}$  to be the partition of unity on  $\bar{\Omega}$ , subordinate to  $\{U_j\}_{j=0}^m$ . Since

$$u = \sum_{j=0}^m u\varphi_j, \text{ on } \Omega$$

observe that  $u_j := u\varphi_j \in W^{k,p}(\Omega)$ ,  $\text{supp } u_j \subset \text{supp } \varphi_j \subset U_j$ . **Also, we define**  $u(x) = 0, \forall x \in \mathbb{R}^d/\Omega$ . The proofs differs in the cases  $j = 0$  and  $j \in \{1, \dots, m\}$ .

*Case  $j = 0$ .* We have  $\text{supp } u\varphi_0 \subset U_0 = \Omega$ . That means that after the extension of  $u\varphi_0$  by zero outside of  $\Omega$ , it holds  $u\varphi_0 \in W^{k,p}(\mathbb{R}^d)$ . Since  $W^{k,p}(\mathbb{R}^d) = W_0^{k,p}(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{W^{k,p}(\mathbb{R}^d)}}$ , we can find  $v_0 \in \mathcal{D}(\mathbb{R}^d)$  s.t.

$$\|v_0 - u\varphi_0\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{m+1}.$$

*Case  $j \in \{1, \dots, m\}$ .* We have a problem now:  $\{U_j\}_{j=1}^m$  covers  $\partial\Omega$ , which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of  $\partial\Omega$ , the kernel would pick up values from outside of  $\Omega$ , where  $u = 0$  and the mollification would not be a good approximation. Instead, we approximate  $u_j$  on a larger *open* domain containing  $\bar{\Omega}$  and then show this is also a good approximation of  $u_j$  on  $\Omega \subset \bar{\Omega}$ .

Set  $w_j = u\varphi_j$ , and denote

$$S_j = \mathbb{A}_j \left( \left\{ (x', x_d) \mid a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j = \mathbb{R}^d / \overline{S_j},$$

i.e.,

$${}^{\text{''}}\Omega_j = \Omega \cup \mathbb{A}_j \left( \left\{ (x', x_d) \mid x_d \leq a_j(x') - \frac{\beta}{2} \right\} \right),{}^{\text{''}}$$

---

<sup>4</sup>Our choice  $U_0 = \Omega$  is important, as without it the definition of  $C^{0,0}$  boundary only means  $\partial\Omega \subset \bigcup_{j=1}^m U_j$ .



(although this is a bit inaccurate). Realize that since  $u = 0$  outside of  $\Omega$ , also  $u_j$  is zero there and in particular it is zero on that "lower strip". Clearly then  $u_j \in W^{k,p}(\Omega_j)$ . Now pick  $\delta \in (0, \frac{\beta}{2})$ , where  $\beta$  is from the definition of  $C^{0,0}$  and set

$$S_j^\delta = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

i.e.,

$${}^{\prime\prime}\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j \left( \left\{ (x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta \right\} \right).{}^{\prime\prime}$$

The trick is to shift the (support of) function  $u_j$  "into"  $\Omega_j^\delta$

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in U(0, \alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\text{supp}(\tau_\delta u_j) = \text{supp}(u_j) - \delta,$$

from which it follows  $\tau_\delta u_j \in W^{k,p}(\Omega_j^\delta)$ ; we have only shifted the function  $u_j$ , but since we have also shifted  $S_j$ , qualitatively there is no difference. Since  $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j$ ,  $\Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$ , and the fact  $\tau_\delta$  is an isometry between Sobolev spaces, we also have  $u_j, \tau_\delta u_j \in W^{k,p}(\Omega_j \cap \Omega_j^\delta)$ . Moreover, from the properties of the shift operator it follows  $\exists \delta > 0$  s.t.

$$\|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} \leq \|u_j - \tau_\delta u_j\|_{W^{k,p}(\Omega_j \cap \Omega_j^\delta)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know  $\tau_\delta u_j$  is already close to  $u_j$ , we are done once we approximate  $\tau_\delta u_j$  by a function from  $C^\infty(\overline{\Omega})$ . Notice that if we show  $\overline{\Omega} \subset \Omega_j^\delta$ , then clearly  $C^\infty(\Omega_j^\delta) \subset C^\infty(\overline{\Omega})$ .

*Show  $\Omega \subset \Omega_j^\delta$ :* We already know  $\Omega \subset \Omega_j^\delta$ , so it suffices to show  $\partial\Omega \subset \Omega_j^\delta$ . Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k(\{(x', x_d) | x_d = a_k(x'), x' \in U(0, \alpha)\}),$$

and the set  $\Omega_j^\delta$  is given as  $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$ , where

$$S_j = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right).$$

Realize it suffices to show  $\partial\Omega \not\subset \overline{S_j}$ , as then it wont be excluded from  $\mathbb{R}^d$  and thus will end up in  $\Omega_j^\delta$ . *Thanks to continuity of  $a_j$* , we may write

$$\overline{S_j} = \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

i.e., the " $<$ " have changed to " $\leq$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d) | x_d = a_j(x'), x' \in U(0, \alpha)\}) \not\subset \mathbb{A}_j \left( \left\{ (x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in U(0, \alpha) \right\} \right),$$

which is equivalent to

$$\left( (a_j \leq a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta) \right) \vee \left( (a_j > a_j - \delta) \wedge (a_j \geq a_j - \frac{\beta}{2} - \delta) \right).$$

Our choice has been  $\delta \in (0, \frac{\beta}{2})$ , and  $\beta > 0$  from the definition of  $\Omega \in C^{0,0}$ , so the second statement is clearly true  $\forall j \in 1, \dots, m$ . Consequently  $\partial\Omega \notin \overline{S_j}$  which leads to  $\partial\Omega \subset \Omega_j^\delta$ , and since also  $\Omega \subset \Omega_j^\delta$ , we have  $\overline{\Omega} \subset \Omega_j^\delta$ .

*Approximation of  $\tau_\delta u_j$ .* Since  $\Omega_j^\delta$  is open there  $\exists w_j \in C^\infty(\Omega_j^\delta)$  such that

$$\|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega)} \leq \|\tau_\delta u_j - w_j\|_{W^{k,p}(\Omega_j^\delta)} < \frac{\varepsilon}{4(m+1)}.$$

What is more, since  $\overline{\Omega} \subset \Omega_j^\delta$ , we see  $w_j \in C^\infty(\overline{\Omega})$  in fact. What is even more, we see  $\text{supp } w_j \subset \text{supp } u_j - \delta$  so  $\text{supp } \tau_{-\delta} w_j \subset \text{supp } u_j \subset \Omega$ , and since  $w_j$  is defined on the whole  $\mathbb{R}^d$ , also  $v_j := \tau_{-\delta} w_j$  is defined on the whole  $\mathbb{R}^d$ . Altogether,  $v_j = \tau_{-\delta} w_j \in \mathcal{D}(\mathbb{R}^d)$ , and from the properties of the shift operator, there  $\exists \delta > 0$  s.t.

$$\|v_j - w_j\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{4(m+1)}.$$

*Approximation of  $u$ .*

Finally, let us set

$$v = \sum_{j=0}^m v_j.$$

Then  $v \in \mathcal{D}(\mathbb{R}^d)$  and it holds

$$\begin{aligned} \|u - v\|_{W^{k,p}(\Omega)} &= \left\| \sum_{j=0}^m u_j - \sum_{j=0}^m v_j \right\|_{W^{k,p}(\Omega)} = \left\| \sum_{j=0}^m u_j - v_j \right\|_{W^{k,p}(\Omega)} \leq \sum_{j=0}^m \|u_j - v_j\|_{W^{k,p}(\Omega)} \leq \\ &\leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - u_j\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m+1} + \sum_{j=1}^m \|v_j - w_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|w_j - \tau_\delta u_j\|_{W^{k,p}(\Omega)} + \sum_{j=1}^m \|\tau_\delta u_j - u_j\|_{W^{k,p}(\Omega)} \\ &< \frac{\varepsilon}{m+1} + 2 \sum_{j=1}^m \frac{\varepsilon}{4(m+1)} + \sum_{j=1}^m \frac{\varepsilon}{2(m+1)} = \varepsilon \end{aligned}$$

Finally,

$$\mathcal{D}(\mathbb{R}^d) \subset C_\Omega^\infty(\mathbb{R}^d) \subset C^\infty(\overline{\Omega}).$$

*This proof may still have some flaws, but the author has decided to move on.* □

*Remark* (What is  $C_\Omega^\infty(\mathbb{R}^d)$ ). Recall

$$C_\Omega^\infty(\mathbb{R}^d) = \left\{ u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d) \right\}.$$

### 2.3 Extension of Sobolev functions

*Problem of extension:* For  $u \in W^{k,p}(\Omega)$ , does there exist

$$\bar{u} \in W^{k,p}(\mathbb{R}^d), \text{ s.t. } \bar{u}|_{\Omega} = u, \|\bar{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{W^{k,p}(\Omega)}$$

?

The answer is **yes**, if  $\Omega$  is nice enough.

We will be using  $C^1$  diffeomorphisms heavily, so we investigate some of their properties first.

**Lemma 4** (Properties of  $C^1$  diffeomorphisms). *Let  $U, V \subset \mathbb{R}^d$  be open,  $\Phi : U \rightarrow V$  be a  $C^1$  diffeomorphism and let  $\tilde{U} \subset \mathbb{R}^d$  s.t.  $\tilde{U} \subset\subset U$ . Then*

1.  $\Phi(\tilde{U}) \subset\subset V$ ,
2. if moreover  $\tilde{U}$  is compact, then <sup>5</sup>

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \leq C \|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

*Proof.* Ad 1.: No proof has been given.

Ad 2.: Just a change of variables formula:

$$\|u \circ \Phi\|_{L_p(\tilde{U})}^p = \int_{\tilde{U}} |u \circ \Phi|^p dx = \int_{\Phi(\tilde{U})} |u|^p |\det \nabla \Phi|^{-1} dx.$$

Since  $\Phi$  is one-to-one, we know  $|\det \nabla \Phi| > 0$  on  $U$ , and since  $\Phi \in C^1(U)$  and  $\tilde{U} \subset U \Rightarrow \Phi \in C^1(\tilde{U}) \Rightarrow \det \nabla \Phi \in C^0(\tilde{U})$ , and since  $\tilde{U}$  is compact,  $|\det \nabla \Phi| \geq C_1 > 0 \Leftrightarrow |\det \nabla \Phi|^{-1} \leq \frac{1}{C_1}$ . In total

$$\|u \circ \Phi\|_{L_p(\tilde{U})}^p \leq \frac{1}{C_1} \int_{\Phi(\tilde{U})} |u|^p dx = C \|u\|_{L_p(\Phi(\tilde{U}))}^p.$$

As for the derivative, we have  $\forall i \in \{1, \dots, d\}$ :

$$\begin{aligned} \int_{\tilde{U}} |\partial_i(u \circ \Phi)|^p dx &\leq \int_{\tilde{U}} |\nabla(u \circ \Phi)|^p dx = \int_{\tilde{U}} |\nabla \Phi((\nabla u) \circ \Phi)|^p dx \leq \\ &\leq \|\nabla \Phi\| \int_{\tilde{U}} |(\nabla u) \circ \Phi|^p dx = \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p |\det \nabla \Phi|^{-1} dx \leq C \|\nabla \Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p dx \leq \\ &\leq C \|\nabla u\|_{L_p(\Phi(\tilde{U}))}^p, \end{aligned}$$

where  $\|\nabla \Phi\|$  is e.g. the operator norm of the matrix  $\nabla \Phi$ . □

**Lemma 5** (Flat extension). *Let  $\alpha, \beta > 0, K \subset\subset U(0, \alpha) \times [0, \beta]$  be compact. Then there  $\exists C > 0$ , a linear operator*

$$E : C^1((B(0, \alpha) \times [0, \beta])) \rightarrow C^1((B(0, \alpha) \times [-\beta, \beta])),$$

*and the set  $\tilde{K} \subset\subset U(0, \alpha) \times [-\beta, \beta]$  such that  $\forall u \in C^1(B(0, \alpha) \times [0, \beta])$  it holds*

1.  $Eu = u$  on  $B(0, \alpha) \times [0, \beta]$ ,

---

<sup>5</sup>For  $\tilde{U}$  compact:  $\tilde{U} \subset\subset V \Leftrightarrow \tilde{U} \subset V$ .

2.  $\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \leq \|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}$ . *Actually,*

$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)), W^{1,p}(U(0,\alpha)\times(-\beta,\beta)))} = 2^{\frac{1}{p}}$$

3. if  $\text{supp } u \subset K$  then  $\text{supp } Eu \subset \tilde{K}$

*Proof.* (The set  $U(0,\alpha) \times [0,\beta]$  is a cylinder of radius  $\alpha$  and height  $\beta$ )

The proof is constructive: for the assumed  $u$  we write  $(x = (x', x_d)$ , where  $x' \in B(0,\alpha) \subset \mathbb{R}^{d-1}$ ,  $x_d \in [0,\beta] \subset \mathbb{R}$ )

$$Eu(x', x_d) = \begin{cases} u(x', x_d), & x_d \geq 0 \\ -3u(x', -x_d) + 4u(x', -\frac{x_d}{2}), & x_d < 0. \end{cases}$$

Does  $Eu$  lie in  $C^1(B(0,\alpha) \times [-\beta,\beta])$ ? Since  $u \in C^1(B(0,\alpha) \times [0,\beta])$ , it is continuous in the "lower cylinder", check only the transition through the origin plane: take some  $a = (x', 0) \in B(0,\alpha) \times \{0\}$ . Then

$$\lim_{x \rightarrow a} Eu(x) = \begin{cases} u(a), & x_d \geq 0 \\ -3u(a) + 4u(a) = u(a), & x_d < 0, \end{cases}$$

so  $Eu$  is continuous. The derivatives

$$\lim_{x \rightarrow a} \partial_d Eu(x', x_d) = \begin{cases} \partial_d u(x', 0), & x_d \geq 0 \\ (-3\partial_d u(x', -x_d)(-1) + 4\partial_d u(x', -\frac{x_d}{2})(-\frac{1}{2}))|_{x_d=0} = \partial_d u(x', 0), & x_d < 0, \end{cases}$$

and also for any  $i \in \{1, \dots, d-1\}$

$$\lim_{x \rightarrow a} \partial_i Eu(x_1, \dots, x_d) = \begin{cases} = \partial_i u(x_1, \dots, 0), & x_d \geq 0 \\ = -3\partial_i u(x_1, \dots, 0) + 4\partial_i u(x_1, \dots, 0) = \partial_i u(x_1, \dots, 0) & x_d < 0 \end{cases}$$

so the derivative is also continuous. Thus, we have

$$Eu \in C^1 \subset W^{1,p}(U(0,\alpha) \times (-\beta,\beta)).$$

The first property is clear from the definition of  $Eu$ , the estimates of the norm: (all derivatives can in fact be assumed classical)

$$\begin{aligned} \|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))}^p &= \|Eu\|_{L_p(U(0,\alpha)\times(-\beta,\beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L_p(U(0,\alpha)\times(-\beta,\beta))}^p = \\ &= \|Eu\|_{L_p(U(0,\alpha)\times(0,\beta))}^p + \|Eu\|_{L_p(U(0,\alpha)\times(-\beta,0))}^p \\ &+ + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L_p(U(0,\alpha)\times(0,\beta))}^p + \sum_{|\alpha|=1} \|D^\alpha Eu\|_{L_p(U(0,\alpha)\times(-\beta,0))}^p = \\ &= \|u\|_{L_p(U(0,\alpha)\times(0,\beta))}^p + \|4u - 3u\|_{L_p(U(0,\alpha)\times(-\beta,0))}^p + \\ &+ \sum_{|\alpha|=1} \|D^\alpha u\|_{L_p(U(0,\alpha)\times(0,\beta))}^p + \sum_{|\alpha|=1} \|D^\alpha (4u - 3u)\|_{L_p(U(0,\alpha)\times(-\beta,0))}^p \\ &= 2\|u\|_{L_p(U(0,\alpha)\times(0,\beta))}^p + 2 \sum_{|\alpha|=1} \|D^\alpha u\|_{L_p(U(0,\alpha)\times(0,\beta))}^p = 2\|u\|_{W^{k,p}(U(0,\alpha)\times(0,\beta))}^p \end{aligned}$$

and so

$$\|Eu\|_{W^{1,p}(U(0,\alpha) \times (-\beta,\beta))} = 2^{\frac{1}{p}} \|u\|_{W^{1,p}(U(0,\alpha) \times (0,\beta))}.$$

Where we have used the obvious fact

$$\int_{U(0,\alpha) \times (0,\beta)} |f|^p dx = \int_{U(0,\alpha) \times (-\beta,0)} |f|^p dx,$$

We will assume (although this is *with* a loss of generality), that

$$Eu(x', x_d) = 0 \Leftrightarrow (u(x', x_d) = 0 \vee 3u(x', -x_d) = 0 \vee 4u(x', -\frac{x_d}{2}) = 0),$$

i.e., that support lies in

$$\begin{aligned} \text{supp } Eu \subset & \{(x', x_d) \in U(0, \alpha) \times [0, \beta] | u(x', x_d) = 0\} \cup \{(x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -x_d) \neq 0\} \cup \\ & \cup \left\{ (x', x_d) \in U(0, \alpha) \times [-\beta, 0] | u(x', -\frac{x_d}{2}) \neq 0 \right\}. \end{aligned}$$

Denote  $\Phi_1, \Phi_2 : U(0, \alpha) \times (0, \beta) \rightarrow U(0, \alpha) \times (-\beta, \beta)$  to be the mappings

$$\begin{aligned} \Phi_1(x', x_d) &= (x', -x_d), \\ \Phi_2(x', x_d) &= (x', -\frac{x_d}{2}), \end{aligned}$$

then clearly  $\Phi_1, \Phi_2$  are  $C^1$  diffeomorphisms and

$$u(x', -x_d) = (u \circ \Phi_1)(x', x_d), u\left(x', -\frac{x_d}{2}\right) = (u \circ \Phi_2)(x', x_d),$$

i.e.,

$$u(x', x_d) = (u \circ \Phi_1^{-1})(x', -x_d) = (u \circ \Phi_2^{-1})\left(x', -\frac{x_d}{2}\right),$$

and so

$$\text{supp } Eu \subset \text{supp } u \cup \Phi_1^{-1}(\text{supp } u) \cup \Phi_2^{-1}(\text{supp } u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as  $\text{supp } u \subset K$ . Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\text{supp } Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have  $K \subset U(0, \alpha) \times [0, \beta] \Rightarrow K \subset U(0, \alpha) \times (-\beta, \beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset U(0, \alpha) \times (0, \beta)$ , which really implies

$$\tilde{K} \subset U(0, \alpha) \times [-\beta, \beta].$$

□

Let us prove the main result.

**Theorem 6** (Extension of Sobolev functions). *Let  $\Omega \in C^{k-1,1}$ ,  $k \in \mathbb{N}$ ,  $p \in [1, \infty]$ ,  $V \subset \mathbb{R}^d$  open such that  $\Omega \subset V$ . Then there is  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$  bounded linear operator such that*

1.  $\forall u \in W^{k,p}(\Omega) : Eu = u \text{ a.e. in } \Omega,$
2.  $\forall u \in W^{k,p}(\Omega) : \text{supp } Eu \subset V \supset \supset \Omega,$
3.  $\|E\|_{\mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^d))} \leq C, C = C(p, \Omega, V).$

*Proof.* **Will only be presented for  $k = 1, \Omega \in C^1, p < \infty$ .** The strategy is:

1. covering of  $\overline{\Omega}$  & partition of unity
2. obtain a diffeomorphism from the fact  $\Omega \in C^{1,0}$ ,
3. suitable composition & cut off,
4. flat extension,
5. show existence of  $E : C_{\overline{\Omega}}^{\infty}(\mathbb{R}^d) \rightarrow W^{k,p}(\mathbb{R}^d)$  with the desired properties,
6. extend via density

*Covering of  $\Omega$ :*

$$\overline{\Omega} \subset \Omega \cup \bigcup_{j=1}^m U_j$$

with  $U_j, a_j, \mathbb{A}_j, \alpha, \beta$  as in the definition of a  $C^1$  domain. In particular,  $a_j \in C^1(U(0, \alpha))$ .

*Construction of  $E$ :* We denote  $\{\varphi_j\}_{j=0}^m$  partition of unity subordinate to  $\{U_j\}_{j=1}^m$ . For  $j \in \{1, \dots, n\}$  we define  $\phi_j : U(0, \alpha) \times (-\beta, \beta) \rightarrow U_j$  by

$$\phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

Trivially  $\phi_j$  is  $C^1$  diffeomorphism. Let us denote by  $\tilde{E}$  the extension operator from the previous lemma. Then we have for  $u \in C_{\overline{\Omega}}^{\infty}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$ . We define

$$Eu = \varphi_0 u + \sum_{j=1}^m \left( \eta \tilde{E}((\varphi_j u) \circ \phi_j) \right) \circ \phi_j^{-1},$$

where  $\eta$  is a cut-off function  $\eta = 1$  on  $y_d \geq 0, \in (0, 1)$  else,  $= 0$  on  $y_d \leq -h$ , for some parameter  $h > 0$  which will be defined later. We also take  $\eta \in C^{\infty}$ . Due to our construction,

$$\phi_j^{-1}(U(0, \alpha) \times [-2h, \beta)) \subset U(\Omega, \varepsilon) \subset U(\Omega, 2\varepsilon) \subset V,$$

for some  $\varepsilon > 0$ .

*Properties of  $E$ :* It is clear that

- $E$  is linear from its definition
- 1) holds, as  $\phi_j$  and  $\phi_j^{-1}$  cancel *somewhere*
- 2) holds for  $h < \frac{\beta}{2}$



- 3) we use the previous lemma:

$$\begin{aligned}
\left\| \underbrace{\left( \eta \tilde{E}(\varphi_j u \circ \phi_j) \right)}_{\text{supp}(\cdot) \subset U(0, \alpha) \times (-\beta, \beta)} \circ \phi_j^{-1} \right\|_{W^{1,p}(\mathbb{R}^d)} &\leq C \left\| \eta \tilde{E}(\varphi_j u \circ \phi_j) \right\|_{W^{1,p}(U(0, \alpha) \times (-\beta, \beta))} \\
&\stackrel{\text{previous lemma}}{\leq} C \left\| \varphi_j u \circ \phi_j \right\|_{W^{1,p}(U(0, \alpha) \times (0, \beta))} \\
&\stackrel{\text{previous lemma}}{\leq} C \left\| \varphi_j u \right\|_{W^{1,p}(U_j \cap \Omega)} \leq \|u\|_{W^{1,p}(\Omega)} \Rightarrow \|E\| \leq C.
\end{aligned}$$

So all the properties hold for  $u \in C_{\Omega}^{\infty}(\mathbb{R}^d)$ . We need to show them also for  $u \in W^{1,p}(\Omega)$ . Pick an arbitrary  $u \in W^{1,p}(\Omega)$ , find  $\{u_k\} \subset C_{\Omega}^{\infty}(\mathbb{R}^d) : u_k \rightarrow u$  in  $W^{1,p}(\Omega)$ .

Ad 1): Since  $E$  is continuous, then  $E u_k \rightarrow E u$  in  $W^{1,p}(\mathbb{R}^d)$ . Since  $\Omega \subset \mathbb{R}^d \Rightarrow E u = u$  in  $W^{1,p}(\Omega)$ .

Ad 2):  $\text{supp } E u_k \subset U(\Omega, \varepsilon) \Rightarrow \text{supp } E u \subset \overline{U(\Omega, \varepsilon)} \subset V$ .

□

*Remark* ( $\Omega \in C^{0,1}$  suffices). The theorem is still valid if we assume only  $C^{0,1}$  and  $p \in (1, \infty), k > 1$ .

## 2.4 Embedding theorems

**Example.** Let  $u \in \mathcal{D}(\mathbb{R}^2)$ . Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2) ds = \int_{-\infty}^{x_2} \partial_2 u(x_1, s) ds,$$

so

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u(x_1, x_2)|^2 dx_1 dx_2 \leq \int_{\mathbb{R}} |\partial_1 u(s, x_2)| ds \int_{\mathbb{R}} |\partial_2 u(x_1, s)| ds dx_1 dx_2 \leq \left( \int_{\mathbb{R}^2} |\nabla u|^2 d\lambda^2 \right)^2,$$

so

$$\|u\|_{L_2(\mathbb{R}^2)} \leq \|\nabla u\|_{L_1(\mathbb{R}^2)}.$$

**Lemma 6.** Let  $d \geq 2$ . Let  $\hat{u}_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be nonnegative and measurable for  $j \in \{1, \dots, d\}$ . We define

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), d\hat{x}_j = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_d.$$

Consider the functions  $u_j : \mathbb{R}^d \rightarrow \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$ . Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x) dx \leq \prod_{j=1}^d \left( \int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} d\hat{x}_j \right)^{\frac{1}{d-1}}. \quad (1)$$

*Proof.* Induction by d.

$$1. \ d = 2 : \int_{\mathbb{R}^d} u_1(x_1, x_2) u_2(x_1, x_2) dx_1 dx_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2) \hat{u}_2(x_1) dx_1 dx_2 \stackrel{\text{Fubini}}{=} \int_{\mathbb{R}} \hat{u}_1(x_2) dx_2 \int_{\mathbb{R}} \hat{u}_2 dx_1.$$

2.

$$\begin{aligned}
d \rightarrow d+1 : \int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^d u_j(x) dx_{d+1} u_{d+1} dx d\hat{x}_{d+1} \\
&\stackrel{\text{Holder}}{\leq} \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} (u_j(x))^d dx_{d+1} \right)^{\frac{1}{d}} u_{d+1}(x) d\hat{x}_{d+1} \\
&\stackrel{\text{Holder}}{\leq} \left( \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \int_{\mathbb{R}} u_j^d(x) dx_{d+1} \right)^{\frac{1}{d-1}} d\hat{x}_{d+1} \right)^{\frac{d-1}{d}} \left( \int_{\mathbb{R}^d} u_{d+1}^d dx_{d+1} \right)^{\frac{1}{d}} \\
&\stackrel{\text{induction step}}{\leq} \left( \int_{\mathbb{R}^d} u_{d+1}^d d\hat{x}_{d+1} \right)^{\frac{1}{d}} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d(x) dx_{d+1} d\hat{x}_j d\hat{x}_{d+1} \right)^{\frac{d-1}{d} \frac{1}{d-1}}.
\end{aligned}$$

□

**Theorem 7** (Gagliardo-Nirenberg). *Let  $p \in [1, d)$ . Then  $\forall u \in W^{1,p}(\mathbb{R}^d)$ :*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where  $p^* = \frac{dp}{d-p}$ .

*Proof.* Estimate for  $u \in \mathcal{D}(\mathbb{R}^d)$ :

$$\forall j \in \{1, \dots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d) ds$$

independet of  $x_j$ , so

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds.$$

Next, consider  $p = 1, p^* = \frac{d}{d-1}$  and estimate:

$$|u|^{\frac{d}{d-1}} \leq \prod_{j=1}^d \underbrace{\left( \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) ds \right)^{\frac{1}{d-1}}}_{u_j \text{ independent of } x_j},$$

so the integral

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} dx \leq \int_{\mathbb{R}^d} \prod_{j=1}^d u_j dx \stackrel{\text{previous lemma}}{\leq} \left( \prod_{j=1}^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x) dx_j d\hat{x}_j \right)^{\frac{1}{d-1}} = \left( \int_{\mathbb{R}^d} |\nabla u| dx \right)^{\frac{d}{d-1}}.$$

If  $p \in (1, d)$ , compute

$$\|u\|_{L_{\frac{dp}{d-1}}^d(\mathbb{R}^d)}^q = \| |u|^q \|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1} |\nabla u| dx \stackrel{\text{Holder}}{\leq} \|\nabla u\|_{L_p(\mathbb{R}^d)} \|u\|_{L_{(q-1)p'}(\mathbb{R}^d)}^{q-1}.$$

We want  $\frac{(q-1)p'}{p-1} = \frac{qd}{d-1}$ , so

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$q\frac{d}{d-1} = p^*.$$

$\Rightarrow$  statement holds for  $u \in \mathcal{D}(\mathbb{R}^d)$ . To finish, use density of  $\mathcal{D}(\mathbb{R}^d)$  in  $W^{1,p}(\mathbb{R}^d)$ .  $\square$

*Remark.* • It is evident that nonzero constants are not in  $W^{1,p}(\mathbb{R}^d)$  and that also the inequality does not hold for them.

- the set  $\mathbb{R}^d$  is of course unbounded, so we have no ordering of  $L_p(\Omega)$  spaces.
- of course, we require no smoothness of the domain

**Theorem 8.** *Let  $\Omega \subset \mathbb{R}^d$  be open. Then  $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1, d)$  the statement of the previous theorem holds.*

*Proof.* An immediate corollary of the previous theorem.  $\square$

*Remark.* In the proof of theorem we showed that  $\forall u \in W^{1,p}(\mathbb{R}^d)$  it holds

$$\|u\|_{L_{\frac{qd}{d-1}}(\Omega)}^q \leq q \|\nabla u\|_{L_p(\Omega)} \|u\|_{L_{\frac{p(q-1)}{p-1}}(\Omega)}^{q-1},$$

for  $q$  such that  $\frac{qd}{d-1} \leq p^*$ .

**Theorem 9** (Embedding theorem). *Let  $\Omega \subset C^{0,1}, p^* = \frac{dp}{1-p}$ . If  $p \in [1, d)$  then*

$$W^{1,p}(\Omega) \subset L_q(\Omega) \forall q \in [1, p^*].$$

Moreover, if  $q < p^*$ , then

$$W^{1,p}(\Omega) \subset\subset L_q(\Omega).$$

If  $p = d$ , then

$$W^{1,p}(\Omega) \subset L_q(\Omega) \forall q < \infty, W^{1,p}(\Omega) \subset\subset L_q(\Omega) \forall 1 \leq q < \infty.$$

*Proof.* We would like to use the previous theorem + extension.

Ad continuity for  $p < d : E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$  the extension is continuous. We also know

- identity  $I_1 : W^{1,p}(\mathbb{R}^d) \rightarrow L_{p^*}(\mathbb{R}^d)$  is continuous,
- restriction  $I_2 : L_{p^*}(\mathbb{R}^d) \rightarrow L_{p^*}(\Omega)$  is continuous,
- identity  $I_3 : L_{p^*}(\Omega) \rightarrow L_q(\Omega)$  is continuous.

Together, the mapping  $id : W^{1,p}(\Omega) : L_q(\Omega)$ ,  $id = I_3 \circ I_2 \circ I_1 \circ E$  identity is continuous. If  $p=d$ , then  $W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \forall r \in [1, d)$ , and  $r^* \rightarrow \infty$  as  $r \rightarrow d^-$ . For  $q \in [1, \infty)$  find  $r \in [1, d)$  s.t.  $r^* > q$ . Then

$$W^{1,d}(\Omega) \subset W^{1,r}(\Omega) \subset L_{r^*}(\Omega) \subset L_q(\Omega),$$

using the previous results.

Ad compactness: We show  $W^{1,p}(\Omega) \subset L_q(\Omega)$  using Arzela-Ascoli and then it will get technical: show compactness in smooth functions, then show compactness in  $L_1(\Omega)$ , then approximate the norm of  $L_q(\Omega)$  using the obtained quantities.

Consider  $B = U_{W^{1,p}(\Omega)}(0, 1)$  and extend it to  $EB$ . Fix  $\delta > 0$  and let  $\eta$  be a regularization kernel. Then  $\exists R > 0 : \text{supp}(EB)_\delta \subset \overline{U(0, R)} \subset \mathbb{R}^d$  (i.e. all the functions from  $EB$  have the support contained in the ball). Moreover,  $(EB)_\delta \subset C^1(\overline{U(0, R)})$ . Actually, it is bounded in  $C^1(\overline{U(0, R)})$ .  $\underbrace{\subset}_{\text{Arzela-Ascoli}} C(\overline{U(0, R)})$  (uniform equicontinuity comes from uniform boundedness of the gradients,  $\nabla(u * \eta_\delta) = u * \nabla \eta_\delta$ .) Altogether  $(EB)_\delta$  is relatively compact in

$$C(\overline{U(0, R)}) \xRightarrow[\text{the space } C(\overline{U(0, R)}) \text{ is complete}]{\text{Arzela-Ascoli}} \text{bounded in } C(\overline{U(0, R)}) \xRightarrow[\text{bounded domain}]{\text{}} \text{bounded in } L_1(U(0, R)).$$

Next, take

$$\begin{aligned} u \in B : \|u - (Eu)_\delta\|_{L_q(\Omega)} &\leq \|Eu - (Eu)_\delta\|_{L_q(U(0, R))} = \int_{U(0, R)} |v - v_\delta| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} v(x+y) - v(x) \eta_\delta(y) dy \right| dx \leq \\ &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} |\eta_\delta(y)| |y| dy \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy}_{\text{Fubini}}. \end{aligned}$$

Estimate the inner integral: assume  $v$  is smooth and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|} \left| \int_0^1 \frac{d}{ds} (v(x+sy)) ds \right| dx \leq \underbrace{\int_{\mathbb{R}^d} \int_0^1 |\nabla v|(x+sy) ds dx}_{\text{Cauchy Schwartz}} \leq \underbrace{C(R) \left( \int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}}_{\text{Holder}}.$$

Now, take  $v \in W_0^{1,p}(U(0, R))$ , then  $\exists \{v_k\} \subset \mathcal{D}(U(0, R)) : v_k \rightarrow v$  in  $W^{1,p}(U(0, R))$ . So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} dx \leq C(R) \left( \int_{\mathbb{R}^d} |\nabla v_k|^p dx \right)^{\frac{1}{p}} \rightarrow C(R) \left( \int_{\mathbb{R}^d} |\nabla v|^p dx \right)^{\frac{1}{p}}.$$

So finally

$$\|u - (Eu)_\delta\|_{L_q(\Omega)} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x+y) - v(x)|}{|y|} dx |y| \eta_\delta(y) dy \leq \underbrace{C(R) \delta \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla u|^p dx \right)^{\frac{1}{p}} dx}_{|y| \leq \delta} \leq C_1 \delta.$$

Fix  $\varepsilon > 0$ , find finite  $\frac{\varepsilon}{2}$ -net in  $(EB)_\delta$  in  $L_1(U(0, R))$  (that is possible since we have total boundedness in  $L_1(U(0, R))$ .) Set  $\delta > 0$  s.t.  $C_1 \delta \frac{\varepsilon}{4}$ .<sup>7</sup> Denote the  $\frac{\varepsilon}{2}$ -net as  $\{Eu_k\}_{k=1}^m$ ,  $m \in \mathbb{N}$ . We show  $\{u_k\}_{k=1}^m$  is

<sup>7</sup>The order of the choices is not precise...

a  $\varepsilon$ -net in  $B$ . Fix  $u \in B$ , find  $j \in \{1, \dots, m\} : \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(U(0,R))}$ . Compute

$$\|u - u_j\|_{L_1(\Omega)} \leq \|u - (Eu)_\delta\|_{L_1(\Omega)} + \|(Eu)_\delta - (Eu_j)_\delta\|_{L_1(\Omega)} + \|(Eu_j)_\delta - u_j\|_{L_1(\Omega)} \leq 2C_1\delta + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus, we have shown

$$W^{1,p}(\Omega) \subset L_1(\Omega).$$

It remains to show the validity for a general  $q$ . Let  $q \in [1, p^*) : \|v\|_{L_q(\Omega)} \leq \|v\|_{L_1(\Omega)}^\alpha \|v\|_{L_{p^*}(\Omega)}^{1-\alpha}$ , for  $\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}$ ,  $\alpha \in (0, 1]$ . Is  $B$  totally bounded in  $L_q(\Omega)$ ? Let us compute

$$\|u - u_j\|_{L_q(\Omega)} \leq \|u - u_j\|_{L_1(\Omega)}^\alpha \underbrace{\|u - u_j\|_{L_{p^*}(\Omega)}^{1-\alpha}}_{\leq C, W^{1,p}(\Omega) \subset L_{p^*}(\Omega)} \leq C\varepsilon^\alpha.$$

□

## 2.5 Trace theorems

## 2.6 Composition of sobolev functions

## 2.7 Difference quotients

# 3 Nonlinear elliptic equations as compact perturbations

**Theorem 10** (Nemytskii). *Let  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $N \in \mathbb{N}$ ,  $\Omega \subset \mathbb{R}^d$  measurable,  $f$  Caratheodory. Then*

1. *if  $u : \Omega \rightarrow \mathbb{R}^N$  is measurable then  $f(\cdot, u)$  is also measurable*
2. *If there is  $p_i \in [1, +\infty)$ ,  $i \in \{1, \dots, N\}$ ,  $q \in [1, \infty)$ ,  $g \in L_q(\Omega)$ ,  $C > 0$  such that for almost all*

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x, y)| \leq g(x) + c \sum_{i=1}^N |y_i|^{p_i/q}$$

*, then  $u \mapsto f(\cdot, u)$  is continuous from  $L_{p_1}(\Omega) \times \dots \times L_{p_N}(\Omega)$  to  $L_q(\Omega)$ . Moreover, it maps bounded sets to bounded sets.*

*Proof.* No proof

□

**Definition 5** (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2). Let  $X, Y$  be normed linear spaces,  $M \subset X$ . The mapping  $F : M \rightarrow Y$  is called a compact operator on  $M$  into  $Y$  if  $F$  is continuous and  $F(M \cap K)$  is relatively compact in  $Y$  for any bounded  $K \subset X$ .

*Remark.* We have no linearity of  $F$ ! So continuity cannot follow from compactness (we have compactness  $\Rightarrow$  boundedness  $\neq$  continuity for nonlinear operators)

**Theorem 11** (Brouwer fixed point theorem). *Let  $K \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$  be a nonempty convex closed bounded. Assume that  $F : K \rightarrow K$  is continuous. Then  $F$  has a fixed point in  $K$ , i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

*Proof.* No proof □

**Theorem 12** (Schauder fixed point theorem). *Let  $K \subset X$  be a nonempty convex closed bounded subset of a linear normed space  $X$ . Assume that  $F$  is compact on  $K$  into  $K$  and  $F(K) \subset K$ . Then there is fixed point of  $F$  in  $K$ .*

*Proof.* No proof □

- for Brouwer,  $K \subset \mathbb{R}^N$  so since it is closed and bounded, it is automatically compact, and since  $F : K \rightarrow K$  is continuous,  $F$  is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with  $N=1$  is easy, based on Darboux property.

### 3.0.1 Problem prototypes

In this chapter some nonlinear elliptic equations are discussed.

**Example.** Suppose the following problem:

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$g : \mathbb{R} \rightarrow \mathbb{R}, f \in (W_0^{1,2}(\Omega))^*, \text{ continuous, } \exists \alpha \in [0, 1) : \forall s \in \mathbb{R} : |g(s)| \leq C(1 + |s|^\alpha).$$

**Theorem 13** (Existence). *Let  $\Omega \in C^{1,1}$ ,  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g$  is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle_{(W_0^{1,2}(\Omega))^*}.$$

If  $f \in L_2(\Omega)$ , then the solution  $u \in W^{2,2}(\Omega)$ .

*Proof.* We define  $S : L_2(\Omega) \rightarrow L_2(\Omega)$  such that

$$Sw = u \Leftrightarrow \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx.$$

$S$  is well defined:

$$|\text{RHS}| \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|\varphi\|_{W^{1,2}(\Omega)} + \|\varphi\|_{L_2(\Omega)} \|g(w)\|_{L_2(\Omega)},$$

and

$$\int_{\Omega} |g(w)|^2 \, dx \leq \int_{\Omega} C(1 + |w|^\alpha)^2 \, dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) \, dx \leq \int_{\Omega} C(1 + |w|^2) \, dx \leq \infty,$$

where we used the Young inequality and  $\alpha \leq 1$ . We have thus shown the mapping  $w \mapsto g(w)$  is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$  by Nemytskii. Next,  $S$  is continuous:

- $w \mapsto g(w)$  is continuous from  $L_2(\Omega)$  to  $L_2(\Omega)$
- $w \mapsto (\varphi W_0^{1,2}(\Omega) \rightarrow \langle f, \varphi \rangle - \int_{\Omega} g(w) \varphi \, dx)$  is continuous from  $L_2(\Omega)$  to  $(W_0^{1,2}(\Omega))^*$

- $F \rightarrow u$ , where  $u$  is the weak solution of

$$\begin{cases} -\Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

, is linear and continuous from  $(W_0^{1,2}(\Omega))^*$  to  $W_0^{1,2}(\Omega)$ .

In total, the composition is continuous and yields  $S$ . Next, we would like to show  $S$  is compact. We start with showing  $S$  maps bounded sets in  $L_2(\Omega)$  to bounded sets in  $W_0^{1,2}(\Omega)$ ; for that we need apriori estimates: test the weak formulation with  $u$ :

$$\|\nabla u\|_{L_2(\Omega)}^2 \leq \varepsilon \|u\|_{W^{1,2}(\Omega)}^2 + C \left( \|f\|_{(W^{1,2}(\Omega))^*}^2 + \|g(w)\|_{L_2(\Omega)}^2 \right) \underset{\text{Young}}{\leq} C \left( \|f\|_{(W_0^{1,2}(\Omega))^*} + 1 + \|w\|_{L_2(\Omega)}^2 \right),$$

from which follows  $S$  is compact from  $L_2(\Omega)$  to  $L_2(\Omega)$  by compact embedding. Now we need to show  $S(U(0, R)) \subset U(0, R)$  for some  $R > 0$ . From the previous we know:

$$\frac{C}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \left( \|f\|_{(W_0^{1,2}(\Omega))^*} + \|g\|_{L_2(\Omega)}^2 \right),$$

so since

$$\tilde{C} \int_{\Omega} |g(w)|^2 dx \leq \int_{\Omega} C(1 + |w|^{2\alpha}) dx \underset{\text{Young}}{\leq} \int_{\Omega} \left( C + \frac{c}{4} |w|^2 \right) dx$$

we know

$$\frac{c}{2} \|u\|_{L_2(\Omega)}^2 \leq \frac{c}{2} \|u\|_{W^{1,2}(\Omega)}^2 \leq \tilde{C} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + C\lambda(\Omega) + \frac{c}{4} \|w\|_{L_2(\Omega)}^2,$$

and thus

$$\|u\|_{L_2(\Omega)}^2 \leq \underbrace{\frac{2\tilde{C}}{c} \|f\|_{(W_0^{1,2}(\Omega))^*}^2 + 2\frac{C}{c}}_{=\overline{C}} + \frac{1}{2} \|w\|_{L_2(\Omega)}^2.$$

so if  $\overline{C} + \frac{1}{2}R^2 < R^2$ , we are done<sup>8</sup>. But such an  $R$  of course exists (says doc. Kaplicky)  $\Rightarrow$  the image of a ball is in a ball for some  $R \Rightarrow S$  is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that  $u_0$  solves  $\begin{cases} -\Delta u_0 = f - g(u_0) \in L_2(\Omega) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$

So from the regularity theory for elliptic equations we get

$$u \in W^{2,2}(\Omega).$$

□

**Theorem 14** (Uniqueness). *Let  $u_1, u_2 \in W_0^{1,2}(\Omega)$  be weak solutions to the above problem. Let  $f \in (W_0^{1,2}(\Omega))^*$ ,  $g$  be continuous. Let either*

1.  $g$  is nondecreasing
2.  $g \in C^1(\mathbb{R})$ ,  $\|g'\|_{\infty}$  small.

---

<sup>8</sup>The constants are most probably messed up.

Then  $u_1 = u_2$ .

*Proof.* We subtract the equations for  $u_1, u_2$  and test with  $u_1 - u_2$ :

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2) \, dx = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1 - u_2)\|_{L_2(\Omega)} \geq C \|u_1 - u_2\|_{W^{1,2}(\Omega)}^2 \Rightarrow u_1 - u_2 = 0.$$

$$|\int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2) \, dx| \leq \int_{\Omega} \|g'\|_{\infty} |u_1 - u_2|^2 \, dx \leq \|g'\|_{\infty} C_P \|\nabla(u_1 - u_2)\|_{L_2(\Omega)}^2 = 0 \Rightarrow u_1 = u_2,$$

whenever  $C\|g'\|_{\infty} < 1$ .  $\square$

**Example.** Suppose the following problem

$$\begin{cases} -\Delta u + b(\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where  $f \in (W_0^{1,2}(\Omega))^*$ ,  $b$  is continuous and bounded. The weak formulation is

$$u \in W_0^{1,2}(\Omega) \wedge \forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi + b(\nabla u) \varphi \, dx = \langle f, \varphi \rangle,$$

and the first apriori estimates (test with  $u$ )

$$\|\nabla u\|_{L_2(\Omega)} \leq \|f\|_{(W_0^{1,2}(\Omega))^*} \|u\|_{W_0^{1,2}(\Omega)} + \int_{\Omega} |u| \, dx \|b\|_{L_{\infty}(\Omega)}.$$

**Theorem 15.** Let  $f \in (W_0^{1,2}(\Omega))^*$ ,  $\Omega \in C^{0,1}$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}$  continuous and bounded. Then there is a weak solution to the above problem.

*Proof.*  $S : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$ ,  $Sw = u$  iff  $u$  solves

$$\begin{cases} -\Delta u = f - b(\nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle - \int_{\Omega} b(\nabla w) \varphi \, dx.$$

Clearly,  $S$  is well defined and

$$\|Sw\|_{W_0^{1,2}(\Omega)} \leq C \underbrace{(\|f\|_{(W_0^{1,2}(\Omega))^*} + \|b\|_{L_{\infty}(\Omega)})}_{:=R},$$

meaning  $S(\overline{U(0, R)}) \subset \overline{U(0, R)}$ . Moreover,  $S$  is continuous, as  $S$  is the composition of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show  $S$  is compact: we



already have continuity, consider  $\{w_k\}_{k \in \mathbb{N}} \subset W_0^{1,2}(\Omega)$  bounded. Then  $\exists \{u_k\} \subset W_0^{1,2}(\Omega)$  bounded:  $u_k \rightarrow u$  in  $L_1(\Omega)$  by embedding up to a subsequence. Next, use the following trick: substitute equation for  $u_k$  from equation for  $u_l$  and test with  $u_l - u_k$

$$C \|u_l - u_k\|_{W_0^{1,2}(\Omega)}^2 \leq \|\nabla(u_l - u_k)\|_{L_2(\Omega)}^2 \leq \int_{\Omega} |b(\nabla u_l) - b(\nabla u_k)| |u_l - u_k| dx \leq 2 \|b\|_{L_{\infty}(\Omega)} \|u_l - u_k\|_{L_1(\Omega)}.$$

All in all,  $S$  has a fixed point by Schauder, which is of course the weak solution.  $\square$

But this says  $\{u_k\}$  is Cauchy in  $W_0^{1,2}(\Omega)$ .

## 4 Nonlinear elliptic equations - monotone operator theory

**Lemma 7.** Let  $g : B(0, R) \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$  be continuous,  $N \in \mathbb{N}, R > 0$ , and  $\forall c \in S(0, R) : g(c) \cdot c \geq 0$ . Then, there is  $c_0 \in B(0, R) : g(c_0) = 0$ .

*Proof.* By contradiction. Let  $g \neq 0$  in  $U(0, R)$ . Let us define

$$h(x) = -R \frac{g(x)}{|g(x)|}.$$

Then  $h \in C(\overline{B(0, R)}), h(\overline{B(0, R)}) \subset S(0, R)$ , so by Brouwer there  $\exists x_0 \in \overline{B(0, R)} : h(x_0) = x_0 \Rightarrow -R \frac{g(x_0)}{|g(x_0)|} = x_0$ . Take the dot product with  $x_0$  and write

$$\underbrace{-R \frac{g(x_0) \cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in S(0, R),$$

so that is a contradiction.  $\square$

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i (a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x))) = f(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

The data are

- $\Omega \in C^{0,1}$ ,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, i \in \{1, \dots, d\}$  are Caratheodory in  $x$  and  $(u, \nabla u)$ .
- $f \in (W_0^{1,r}(\Omega))^*$ ,

and the unknown is  $u : \Omega \rightarrow \mathbb{R}$ .

*Remark.* The function  $(u, p) \mapsto a_i(\cdot, u, p)$  is continuous from  $(L_r(\Omega))^{d+1}$  to  $L_{r'}(\Omega)$ . by Nemystkii theorem.

**Definition 6** (Coercivity). We say that  $\{a_i\}_{i=0}^d$  are coercive if  $\exists C_1 > 0, C_2 \in L_1(\Omega) : \text{a.e. } x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} :$

$$\sum_{i=1}^d a_i(x, z, p) p_i + a_0(x, z, p) \geq C_1 |p|^r - C_2(x), \text{ i.e. } a(x, z, p) \cdot p \geq C_1 |p|^r - C_2(x)$$

**Definition 7** (Monotonicity). We say that  $\{a_i\}_{i=0}^d = a$  is monotone if for almost all

$$x \in \Omega, \forall (z_1, p_1), (z_2, p_2) \in \mathbb{R}^{d+1} : (a(x, z_1, p_1) - a(x, z_2, p_2)) \cdot (p_1 - p_2) + (a_0(x, z_1, p_1) - a_0(x, z_2, p_2)) \cdot (z_1 - z_2) \geq 0.$$

Very similarly we define strict monotonicity.

**Definition 8** (Weak solution). We say that  $u \in W^{1,r}(\Omega)$  is a weak solution to the above problem if

- $u = u_0$  in the sense of traces on  $\partial\Omega$ ,

•

$$\int_{\Omega} a(\cdot, u, \nabla u) \cdot \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle, \forall \varphi \in W_0^{1,r}(\Omega).$$

**Theorem 16** (Existence and uniqueness). *Let  $\Omega \in C^{0,1}$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $r \in (1, \infty)$ ,  $\{a_i\}_{i=1}^d$  be Caratheodory, coercive and  $m$  and let them also satisfy the growth conditions. Finally, let  $f \in (W^{1,r}(\Omega))^*$ . Then, there is a weak solution to the problem. If, moreover,  $\{a_i\}_{i=1}^d$  is strictly monotone, then the weak solution is unique.*

*Proof.* The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

**Galerkin:** Since  $W_0^{1,r}(\Omega)$  is separable  $\Rightarrow \exists \{w_i\}_{i=1}^{\infty}$  that is a dense<sup>9</sup> linearly independent subset of  $W_0^{1,r}(\Omega)$ . We search for  $n \in \mathbb{N}$  such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where  $\alpha_j \in \mathbb{R}$  and  $u^n$  satisfy

$$\forall j \in \{1, \dots, n\} : \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla w_j + a_0(\cdot, u^n, \nabla u^n) w_j \, dx = \langle f, w_j \rangle.$$

We claim such  $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$  exist  $\forall n \in \mathbb{N}$  by the previous lemma. We define a vector function

$$F(\alpha^n) := \left\{ \int_{\Omega} a \cdot \nabla w_j + a_0 w_j \, dx - \langle f, w_j \rangle \right\}_{j=1}^n,$$

---

<sup>9</sup>It can be chosen such that it is itself dense, not only its span

from Nemystkii  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F$  is continuous on  $\mathbb{R}^n$ . Moreover, it holds

$$\begin{aligned} F(\alpha^n) \cdot \alpha^n &\geq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \nabla(u^n - u_0) + a_0(u^n - u_0) \, dx - \langle f, u^n - u_0 \rangle \\ &\stackrel{\text{coercivity}}{\geq} \int_{\Omega} C_1 |\nabla u^n|^r - (C_2(\cdot) + |a| |\nabla u_0| + |a_0| |u_0|) \, dx - \|u^n\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*} - \|u_0\|_{W^{1,r}(\Omega)} \|f\|_{(W_0^{1,r}(\Omega))^*}, \end{aligned}$$

together with the fact

$$\|\nabla u^n\|_{L_r(\Omega)}^r \geq \left( \|\nabla(u - u_0)\|_{L_r(\Omega)} - \|\nabla u_0\|_{L_r(\Omega)} \right)^r \geq \|\nabla(u^n - u_0)\|_{L_r(\Omega)}^r - \|\nabla u_0\|_{L_r(\Omega)}^r \geq C \|u^n - u_0\|_{W^{1,r}(\Omega)}^r - \|\nabla u_0\|_{L_r(\Omega)}^r,$$

Next, realize that  $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{W^{1,r}(\Omega)}$  is a norm equivalent to  $|\alpha^n|$  (Euclidian norm). So that means  $\exists K_1(n) > 0 : \forall \alpha \in \mathbb{R}^n : K_1(n) |\alpha^n| \leq \|u^n - u_0\|_{W^{1,r}(\Omega)}$ . For  $|\alpha^n| = R, R > 0$  determined later estimate  $F(\alpha^n) \cdot \alpha^n \geq c \|u^n - u_0\|_{W^{1,r}(\Omega)} - \tilde{c} \left( \|\nabla u_0\|_{L_r(\Omega)}^r + 1 + \|u_0\|_{L_r(\Omega)}^r + \|f\|_{(W_0^{1,r}(\Omega))^*}^r \right)$  (which is not a trivial computation). And so  $\exists R > 0, \forall \alpha^n \in S(0, R) \subset \mathbb{R}^n : F(\alpha^n) \cdot \alpha^n > 0$ , so from the previous lemma  $\exists \alpha^n \in S(0, R) : F(\alpha^n) = 0$ , and we fix these  $\alpha^n$ . **Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|_{W^{1,r}(\Omega)}^r \leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}^r \right),$$

and

$$\begin{aligned} \|u^n\|_{W^{1,r}(\Omega)} &\leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}^r \right), \\ \forall j \in \{0, \dots, d\} : \|a_j(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}^{r'} &\leq C \left( 1 + \|u_0\|_{W^{1,r}(\Omega)}^r + \|f\|_{(W^{1,r}(\Omega))^*}^r \right), \end{aligned}$$

**Limit passage** From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \text{ in } W^{1,r}(\Omega), a_j \rightharpoonup \alpha_j \text{ in } L_{r'}(\Omega).$$

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_{\Omega} \alpha \cdot \nabla w_j + \alpha_0 w_j \, dx = \langle f, w_j \rangle,$$

and from density of  $\{w_j\}_{j \in \mathbb{N}}$  in  $W^{1,r}(\Omega)$  we have

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} \alpha \cdot \nabla \varphi + \alpha_0 \varphi \, dx = \langle f, \varphi \rangle.$$

**Identification of  $\alpha$ 's** We want to show  $\alpha_j = a_j(\cdot, u, \nabla u), j \in \{0, \dots, d\}$ . For that, we use the *Minty trick*:

$$\begin{aligned} 0 &\leq \int_{\Omega} (a(\cdot, u^n, \nabla u^n) - a(\cdot, v, V)) \cdot (\nabla u^n - V) + (a_0(\cdot, u^n, \nabla u^n) - a_0(\cdot, v, V)) \cdot (u^n - v) \\ &\leq \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla u^n + a_0(\cdot, u^n, \nabla u^n) \cdot u^n \, dx + \\ &\quad - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) V + a_0(\cdot, u^n, \nabla u^n) v - a(\cdot, v, V) + a_0(\cdot, v, V) \cdot (u^n - v)) \, dx. \end{aligned}$$

Denote

$$I^n = \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot \nabla (u^n - u_0) + a_0(\cdot, u^n, \nabla u^n) \cdot (u^n - u_0) \, dx + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx,$$

by using the equation we obtain

$$I^n = \langle f, u^n - u_0 \rangle + \int_{\Omega} a(\cdot, u^n, \nabla u^n) \cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \, dx \rightarrow \langle f, u - u_0 \rangle + \int_{\Omega} \alpha \nabla u_0 + \alpha_0 u_0 \, dx = \int_{\Omega} \alpha \nabla u + \alpha_0 u \, dx,$$

as the rest has subtracted. In total, we have

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, v, V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot, v, V))(u - v) \, dx.$$

So far,  $v, V$  have been arbitrary. If we take

$$V = \nabla u - \lambda \psi, \psi \in L_r(\Omega), v = u,$$

then  $0 \leq \int_{\Omega} (\alpha - a(\cdot, \nabla u + \lambda \psi)) \lambda \psi \, dx$ , so if we take  $\lambda > 0$  and pass to the limit  $\lambda \rightarrow 0_+$  (using Nemytskii theorem) we can write

$$0 \leq \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx.$$

Since  $\psi$  was arbitrary, we could have taken  $\psi \rightarrow -\psi$ , which in total means

$$0 = \int_{\Omega} (\alpha - a(\cdot, u, \nabla u)) \psi \, dx$$

Finally, from the previous results, we obtain

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle,$$

and we are almost done. It only remains to show the traces are correct, bt since  $u^n \rightharpoonup u$  in  $W^{1,r}(\Omega)$  and from the continuity of the traces, we obtain

$$\text{tr } u = \text{tr } u_0.$$

**Uniqueness:** Let  $u_1, u_2$  be two solutions. Use strict monotonicity, subtract the weak formulation and test with  $u_2 - u_1$ :

$$\int_{\Omega} \underbrace{(a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla (u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)}_{:=T} \, dx = 0,$$

where  $T \geq 0$ , so from strict monotonicity we obtain  $T = 0$  a.e. in  $\Omega$  but that means  $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$ , a.e. in  $\Omega \Rightarrow u_1 = u_2$  in  $W^{1,r}(\Omega)$ .  $\square$

**Example** (Nonlinearities vs weak convergence). Let  $g_n(x) = \sin(nx)$ , then  $g \rightharpoonup 0$  in  $L_2((0,4))$  (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx) \varphi \, dx \geq \int_0^4 \sin^2(nx) \, dx \rightarrow \frac{1}{2} \neq 0, \forall \varphi \in L_2((0,4)),$$

so  $\{u_n^2\} = \{\sin^2(nx)\}$  **does not converge weakly to**  $0 = 0^2$ .

*Remark.* The method of the presented proof is **very important**.

**Theorem 17.** Let  $\Omega \in C^{0,1}$ . Let  $X = W_0^{1,r}(\Omega)$ ,  $r \in (1, \infty)$  with equivalent norm  $\|u\| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$ . Then

$$\forall \varphi \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t. : } \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

*Proof.* We solve the problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{r-2} \nabla u) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Such  $u \in W_0^{1,r}(\Omega)$  exists and is unique by the above theorem. In this case:  $a(x, z, p) = |p|^{r-2}p$ ,  $a_0() = 0$ . Coercivity is clear, monotonicity will be shown in the tutorials<sup>10</sup>. Write the weak formulation of the above problem:

$$\forall \varphi \in W_0^{1,r}(\Omega) : \int_{\Omega} |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \, dx = \Phi(\varphi).$$

Set  $\mathbf{F} = |\nabla u|^{r-2} \nabla u$ , and test the weak formulation with  $u$  itself:

$$\|\nabla u\|_{L_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{L_r(\Omega)}.$$

If now  $\|\nabla u\|_{L_r(\Omega)} = 0$ , then  $\Phi = 0$  and we are finished, if it is nonzero, then

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|_{L_r(\Omega)}^{r-1} = \| |\nabla u|^{r-1} \|_{L_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{L_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{L_{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{B_X(0,1)} |\Phi(\varphi)| = \sup_{B_X(0,1)} \left| \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dx \right| \leq \sup_{B_X(0,1)} \|\mathbf{F}\|_{L_{r'}(\Omega)} \|\nabla \varphi\|_{L_r(\Omega)} = \|\mathbf{F}\|_{L_{r'}(\Omega)},$$

so  $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}$ . □

## 5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset W^{1,r}(\Omega) \rightarrow \mathbb{R}, u \mapsto \int_{\Omega} F(\cdot, u, \nabla u) \, dx,$$

with the basic assumptions  $\Omega \in C^{0,1}$ ,  $r \in (1, \infty)$ ,  $X = u_0 + W_0^{1,r}(\Omega)$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in L_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x, z, p) \geq C_1 |p|^r - c_2(x).$$

---

<sup>10</sup>This was a lie

*Remark.* From the assumptions it follows  $\int_{\Omega} F(\cdot, u, \nabla u) dx$  is defined  $\forall u \in W^{1,r}(\Omega)$ .

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

**Lemma 8.** *Let  $\Omega \in C^{0,1}$ ,  $r \in (1, \infty)$ ,  $X = u_0 + W_0^{1,r}(\Omega)$ ,  $u_0 \in W^{1,r}(\Omega)$ ,  $F$  Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall a. a \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x, z, p)| + |\partial_z F(x, z, p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x, \cdot, \cdot) \in C^1(\mathbb{R}^{d+1}).$$

*Let now  $u \in u_0 + W_0^{1,r}(\Omega)$  be a local minimizer of  $I$  over  $X$ , i.e.,*

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_{\Omega} F(\cdot, u, \nabla u) dx \leq \int_{\Omega} F(\cdot, u+v, \nabla(u+v)) dx, F(\cdot, u, \nabla u) \in L_1(\Omega).$$

*Then  $u$  is the weak solution to the **Euler-Lagrange equations**:*

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u) + \partial_z F(\cdot, u, \nabla u)) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega \end{cases},$$

*i.e.,*

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi + \partial_z F(\cdot, u, \nabla u) \varphi dx = 0, \text{tr } u = \text{tr } u_0 \text{ on } \partial\Omega.$$

*Proof.* First  $\text{tr } u = \text{tr } u_0$  holds, so we are fine. Now fix  $\varphi \in \mathcal{D}(\Omega)$  and define

$$\iota : \mathbb{R} \rightarrow \mathbb{R}^*, \iota(\tau) = \int_{\Omega} \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:= l(\tau, \cdot)} dx.$$

Now  $\iota$  has a local minimum in 0. We show that  $\iota'(0)$  exists and is equal to the of Euler-Lagrange equations.

- $l(\tau, \cdot)$  is measurable for  $\tau$  from some neighbourhood of 0.
- $l(\tau, \cdot)$  is differentiable

Moreover

$$\begin{aligned} \partial_{\tau} l(\tau, \cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla \varphi = \\ &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla \varphi. \end{aligned}$$

Also

$$i(0) = \int_{\Omega} F(\cdot, u, \nabla u) dx < \infty$$

and

$$|\partial_{\tau} l(\tau, \cdot)| \leq (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla \varphi|^r) + |h(x)|)(|\varphi| + |\nabla \varphi|) \in L_1(\Omega).$$

Altogether,  $\iota(\tau)$  is finite on  $(-1, 1)$ ,  $\iota'(\tau)$  exists and

$$\iota'(0) = \int_{\Omega} \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla \varphi dx.$$

□

**Example.** Let

$$F(x, z, p) = \frac{1}{r}(1 + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x, z, p) = \left( \frac{r}{2} \frac{1}{r} 2(1 + |p|^2)^{\frac{r-2}{2}} \right) p - G = (1 + |p|^2)^{\frac{r-2}{2}} p - G, \partial_z F(x, z, p) = -g.$$

We have

$$|(1 + |p|^2)^{\frac{r-2}{2}} p| \leq (1 + |p|^2)^{\frac{r-2}{2}} (1 + |p|^2)^{\frac{1}{2}} = (1 + |p|^2)^{\frac{r-1}{2}} \leq C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left( (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \right) = -\nabla \cdot G + g, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} (1 + |\nabla u|^2)^{\frac{r-2}{2}} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} (G \cdot \nabla \varphi + g\varphi) \, dx.$$

*Remark.* We have  $\{u_n\} \subset X$  s.t.  $\lim_{n \rightarrow \infty} I(u_n) = \inf_X I$ . Then use

- compactness:  $u_n \rightarrow u$  in some sense (weak convergence)
- weak lower semicontinuity  $I(u) \leq \liminf_{n \rightarrow \infty} I(u_n)$

**Lemma 9.** Let  $F : \mathbb{R}^N \rightarrow \mathbb{R}, F \in C^1(\mathbb{R}^N), N \in \mathbb{N}$ . Then

1.  $F$  is (strictly) convex  $\Leftrightarrow \nabla F$  is (strictly) monotone
2. If  $F$  is (strictly) convex, then

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2 : F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

*Proof.* Fix  $\xi_1, \xi_2, \xi_1 \neq \xi_2$ , define  $\varphi(t) = F(\xi_2 + t(\xi_1 - \xi_2))$ . Then  $\varphi \in C^1(\mathbb{R})$  and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1 - \xi_2)) \cdot (\xi_1 - \xi_2).$$

So

$$" \Rightarrow " : (\nabla F(\xi_1) - \nabla F(\xi_2)) \cdot (\xi_1 - \xi_2) = \varphi'(1) - \varphi'(0) \underset{\varphi \text{ convex or strictly convex}}{\geq} 0.$$

And  $" \Leftarrow " : \text{Fix } t_1 > t_2 \text{ and compute}$

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1 - \xi_2)) - \nabla F(\xi_2 + t_2(\xi_1 - \xi_2))) \cdot (\xi_1 - \xi_2)(t_1 - t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1 - \xi_2)(t_1 - t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2)) \cdot (\eta_1 - \eta_2)$$

and we are in the same situation as before. For 2) we already know  $F$  (strictly) convex  $\Rightarrow \varphi$  (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \rightarrow \varphi'(0),$$

as  $t \rightarrow 0_+$ . And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

**Theorem 18.** Let  $M, N \in \mathbb{N}, \Omega$  open,  $F : \Omega \times \mathbb{R}^{N+M} \rightarrow \mathbb{R}$  Caratheodory,  $F$  convex in  $p \in \mathbb{R}^n$ , i.e.  $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot)$  is convex and  $\exists c_2 \in L_1(\Omega), \forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M, \forall p \in \mathbb{R}^N : F(x, z, p) \geq c_2(x)$ . Let  $u_n \rightarrow u$  in  $L_1(\Omega), U_n \rightarrow U$  in  $L_1(\Omega)$ . Then

$$\int_{\Omega} F(\cdot, u, U) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} F(\cdot, u_n, U_n) dx.$$

*Proof.* The proof will be given only if moreover  $\forall a.e. x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot) \in C^1(\mathbb{R}^N)$ . Idea: by the previous lemma:

$$\int_{\Omega} F(\cdot, u_n, U_n) dx \geq \int_{\Omega} (F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U) \cdot (U_n - U)) dx,$$

and we have uniform convergence in the first term and second term and weak convergence in  $L_1(\Omega)$  in the last term. If  $\Omega$  is bounded, we can find  $K_k \subset K_{k+1} \subset \Omega$  s.t.  $\lambda(\Omega \setminus \bigcup_{k \in \mathbb{N}} K_k) = 0$ , and moreover  $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$  are compact,  $u_n \rightarrow u$  on  $K_k, \|u\|_{L^\infty(K_k)} + \|U\|_{L^\infty(K_k)} \leq k$  up to a subsequence. We can now extract a subsequence  $u_n \rightarrow u$  a.e. and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E}_k \text{ s.t. } u_n \rightarrow u \text{ on } \tilde{E}_k \wedge \lambda(\Omega \setminus \tilde{E}_k) < \frac{1}{k}.$$

Now define

$$\hat{E}_k = \bigcup_{j=1}^k \tilde{E}_j, E_k = \hat{E}_k \cap \{x \in \Omega, \text{dist}(x, \partial\Omega) > \frac{1}{k}\},$$

and  $E_k$  satisfy <sup>11</sup>

$$\lambda\left(\Omega \setminus \bigcup_k E_k\right) = 0.$$

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \leq k \wedge |U(x)| \leq k\}$$

and we also have  $\lambda(\Omega \setminus \bigcup_k F_k) = 0$ . FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\left(\Omega \setminus \bigcup_k K_k\right) = 0.$$

□

*Remark.* • if  $U_n \rightarrow U$  strongly  $\Rightarrow u_n \rightarrow u, U_n \rightarrow U$  a.e. (up to a subsequence) and the claim follows from the Fatou lemma. <sup>12</sup>

<sup>11</sup>"This is homework", says doc. Kaplicky

<sup>12</sup>For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions  $F - c_2 \geq 0, F - c_2 \in L_1(\Omega)$



- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L_p(\Omega) \Rightarrow \int_{\Omega} |\nabla u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p dx.$$

**Lemma 10** (Arzela-Ascoli). *Let  $X, Y$  be Banach spaces,  $X \subset\subset Y$ . Then*

$$C^1([0, T]; X) \subset\subset C([0, T]; Y).$$

**Lemma 11** (Ehrling). *Let  $V_1, V_2, V_3$  be Banach spaces s.t.  $V_1 \subset\subset V_2 \subset V_3$ . Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + C \|u\|_{V_3}.$$

*Proof.* By contradiction, assume

$$\exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{N} \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n \|u_n\|_{V_3}.$$

WLOG we can assume  $\{u_n\} \subset S_{V_2}(0, 1)$ : truly, the inequality is 1-homogenous and holds if  $u_n = 0$ . In particular we see  $\|u_n\|_{V_3} < \frac{1}{n}$ , so  $u_n \rightarrow 0$  in  $V_3$ . Moreover,  $\{u_n\}$  is bounded in  $V_1$  and since  $V_1 \subset\subset V_2$  there exists  $\{u_{n_k}\} \subset \{u_n\}$  s.t.:  $u_{n_k} \rightarrow u$  in  $V_2$  strongly. Since  $\{u_n\} \subset S_{V_2}(0, 1)$ , also  $\|u\|_{V_2} = 1$ . Finally, taking the limit passage yields  $0 \geq \|u\|_{V_3}$  and so  $u = 0$  in  $V_3$  and also in  $V_2$ . But that is a contradiction with the fact  $\{u_n\} \subset S_{V_2}(0, 1)$ .  $\square$

**Theorem 19** (Aubin-Lions). *Let  $V_1, V_2, V_3$  be Banach spaces s.t.  $V_1 \subset\subset V_2 \subset V_3, p \in [1, \infty)$ . Then the space*

$$\mathcal{U} = \{u \in L_p((0, T); V_1), \partial_t u \in L_1((0, T); V_3)\},$$

*with the norm*

$$\|u\| = \|u\|_{L_p((0, T); V_1)} + \|\partial_t u\|_{L_1((0, T); V_3)},$$

*satisfies*

$$\mathcal{U} \subset\subset L_p((0, T); V_2).$$

*Proof.* Strategy: I want to fix  $M \subset \mathcal{U}$  bounded and show that it is precompact in  $L_p((0, T); V_2)$ . That will be done in the following way:

1. Mollify  $M$  by convolution
2. use Arzela-Ascoli
3. show compactness in  $L_p((0, T); V_3)$
4. apply Ehrling lemma and show compactness in  $L_p((0, T); V_2)$ .

Fix  $M \subset \mathcal{U}$  bounded. Then  $\exists C^* > 0 : \forall u \in M : \|u\| \geq C^*$ .

Next, take

$$\varphi : \mathbb{R} \rightarrow [0, \infty), \varphi \in C^\infty(\mathbb{R}), \text{supp } \varphi \subset (-1, 0), \int_{\mathbb{R}} \varphi dx = 1,$$

a regularization kernel, then  $\forall \delta > 0$  define  $\varphi_\delta(t) := \frac{1}{\delta} \varphi(\frac{t}{\delta})$ .

Now, extend functions from  $M$  to  $(0, 2T)$  in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0, T) \\ u(2T - t), & t \in (T, 2T) \end{cases}.$$

Now mollify: for  $\delta > 0, \delta < T$  fixed define

$$M_\delta = \{ (\tilde{u} \star \varphi_\delta) \Big|_{(0, T)} \mid u \in M \}.$$

From the properties of regularization it follows  $M_\delta \subset C^1([0, T]; V_1) \underset{\text{A.A.}}{\subset\subset} C([0, T]; V_2) \subset L_p((0, T); V_2)$ .

Now estimate the distance of  $M$  and  $M_\delta$  in  $L_p((0, T); V_3)$ : for

$$u \in M, t \in (0, T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^0 \tilde{u}(t-s) \varphi_\delta(s) ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \varphi_\delta(s) ds = \int_{-\delta}^0 (\tilde{u}(t) - \tilde{u}(t-s)) \frac{d}{ds} \int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) d\sigma \Big|_{s=-\delta}^0 - \int_{-\delta}^0 \frac{d}{ds} (\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^s \varphi_\delta(\sigma) d\sigma ds,$$

since the first bracket is 0 and by denoting the first term in the second integrand by  $\tilde{u}'(t-s)$  this becomes (using Fubini)

$$= - \int_{-\delta}^0 \int_{\sigma}^0 \tilde{u}'(t-s) ds \varphi_\delta(\sigma) d\sigma,$$

and we see

$$\|\tilde{u}(t) - \tilde{u}_\delta(t)\|_{V_3} \leq \int_{-\delta}^0 \int_{\sigma}^0 \|\tilde{u}'(t-s)\|_{V_3} ds \varphi_\delta(\sigma) d\sigma.$$

$L_1((0, T); V_3)$  estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3} dt \leq \int_0^T \int_{-\delta}^0 \int_{\sigma}^0 \|u'(t-s)\|_{V_3} ds \varphi_\delta(\sigma) d\sigma dt \leq 2\delta \|u'\|_{L_1((0, T); V_3)} \leq 2\delta C^*$$

$L_\infty((0, T); V_3)$  estimate:

$$\|u - u_\delta\|_{L_\infty((0, T); V_3)} \leq 2\|u'\|_{L_1((0, T); V_3)} \leq 2C^*$$

It remains to show  $M_\delta \subset\subset L_p((0, T); V_2)$ :

$$\|u - u_\delta\|_{L_p((0, T); V_3)} \leq \|u - u_\delta\|_{L_1((0, T); V_3)}^{1/p} \|u - u_\delta\|_{L_\infty((0, T); V_3)}^{1-1/p} \leq 2C^* \delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u - u_\delta\|_{L_p((0, T); V_2)} \leq \mu \|u - u_\delta\|_{L_p((0, T); V_1)} + C_\mu \|u - u_\delta\|_{L_p((0, T); V_3)}.$$

This means

$$\forall u \in M : \|u - u_\delta\|_{L_p((0,T);V_2)} \leq C^* + C\mu 2C^* \delta^{1/p}.$$

Now fix  $\varepsilon > 0$  and find

$$\mu > 0 : \mu C^* < \frac{\varepsilon}{2}, \delta > 0, C\mu 2C^* \delta^{1/p} < \frac{\varepsilon}{2} \Rightarrow \forall u \in M : \|u - u_\delta\|_{L_p((0,T);V_2)} < \varepsilon.$$

This means  $\exists \{w_k\}_{k=1}^N \subset M : \{(w_k)_\delta\}_{k=1}^N$  is  $\varepsilon$ -net in  $M$  in  $L_p((0,T);V_2)$ . If we now fix  $u \in M$ , then

$$\exists K \in \{1, \dots, N\} : \|u_{\delta-(w_K)_\delta}\|_{L_p((0,T);V_2)} < \varepsilon.$$

□

*Remark.* The pair  $(\mathcal{U}, \|\cdot\|)$  is a Banach space.

We will be dealing with the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot a(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = f & \text{in } (0, T) \times \Omega, \\ u = u_0, & \text{on } \{0\} \times \Omega, \\ u = 0, & \text{on } (0, T) \times \partial\Omega \end{cases}.$$

The unknown is the function  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$ , and we are given  $\Omega \in C^{0,1}$ ,  $T > 0$ ,  $Q_T = (0, T) \times \Omega$ ,  $f : Q \rightarrow \mathbb{R}$  or  $f : (0, T) \rightarrow X$  a Banach space,  $u_0 : \Omega \rightarrow \mathbb{R}$ ,  $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions:  $\exists r > 1, \exists C > 0 : a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : |a_0(x, z, p)| + |a(x, z, p)| \leq C(1 + |z|^{r-1} + |p|^{r-1})$  and  $\exists C_1, C_2 > 0, q \in (1, \max(2, r)) a.e. x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1} : a(x, z, p)p + a_0(\dots)z \geq C_1|p|^r - C_2(1 + |z|q)$ .

**Theorem 20.** *Let  $\Omega \in C^{0,1}$ ,  $a, a_0$  satisfy growth conditions and coercivity, let  $\{a_i\}_{i=0}^d$  be monotone. Denote  $V = W_0^{1,r}(\Omega) \cap L_2(\Omega)$ . Then  $\forall f \in L_r((0, T); V^*), u_0 \in L_2(\Omega) \exists u \in L_r((0, T); V)$  s.t.  $\partial_t u \in L_r((0, T); V^*), u \in C([0, T]; L_2(\Omega)), u(0) = u_0$  and moreover*

$$a.e. t \in (0, T), \forall \varphi \in V : \langle \partial_t u, \varphi \rangle + \int_\Omega a(\cdot, u, \nabla u) \nabla \varphi + a_0(\cdot, u, \nabla u) \varphi \, dx = \langle f, \varphi \rangle.$$

Finally, the solution is unique.

*Proof.* The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

*Rothe method:* Fix  $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$  and approximate the derivative with

$$\partial_t u(t, x) \approx \frac{1}{h}(u(t, x) - u(t - h, x)).$$

Define  $u_0 = u_0, u_{k+1} \in V$  as a solution of

$$\frac{1}{h}(u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \text{ in } \Omega, u_{k+1} = 0 \text{ on } \partial\Omega.$$

Define

$$f_{k+1} := \int_{kh}^{(k+1)h} f \, dt,$$

then the weak formulation becomes

$$\int_{\Omega} \frac{u_{k+1} - u_k}{h} \varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla \varphi + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) \varphi \, dx = \langle f_{k+1}, \varphi \rangle.$$

We *claim without a proof* that the solutions  $\{u_k\}_{k=0}^n \subset V$  exist.

To obtain a-priori estimates, test the equation with  $u_{k+1}$ . This yields:

$$\int_{\Omega} |u_{k+1}|^2 - u_k u_{k+1} \, dx = \int_{\Omega} \frac{1}{2} |u_{k+1}|^2 + \frac{1}{2} (u_{k+1} - u_k) - \frac{|u_k|^2}{2} \, dx \Rightarrow \sum_{k=0}^{j-1} \int_{\Omega} |u_{k+1}|^2 - u_{k+1} u_k \, dx = \frac{1}{2} \|u_j\|_{L_2(\Omega)}^2 - \frac{1}{2} \|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} (u_k +$$

so

$$\int_{\Omega} a(\dots) \nabla \cdot u_{k+1} + a_0(\dots) u_{k+1} \, dx \geq C_1 \int_{\Omega} |\nabla u_{k+1}|^r \, dx - C_2 \int_{\Omega} (1 + |u_{k+1}|^q) \, dx,$$

$$\langle f_{k+1}, u_{k+1} \rangle \leq \|f_{k+1}\|_{V^*} \left( \|u_{k+1}\|_{W_0^{1,r}(\Omega)} + \|u_{k+1}\|_{L_2(\Omega)} \right) \leq \varepsilon \left( \|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r + \|u_{k+1}\|_{L_2(\Omega)}^2 \right) + C \left( \|f_{k+1}\|_{V^*}^{r'} + \|f_{k+1}\|_{V^*}^2 \right).$$

$$\text{So together } \|u_j\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} \left[ (u_{k+1} - u_k)^2 + h \|u_{k+1}\|_{W_0^{1,r}(\Omega)}^r \right] \leq C \left( \|u_0\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \|u_{k+1}\|_{L_2(\Omega)}^2 + \sum_{k=0}^{j-1} h \left( \|f\|_{V^*}^{r'} \|f\|_{V^*}^2 \right) \right) \quad \square$$

Let us now define  $u^n(t) = u_k$  for  $t \in (h(k-1), hk)$ , then

$$\|u^n\|_{L_{\infty}((0,T);L_2(\Omega))}^2 + \|u^n\|_{L_2((0,T);W_0^{1,r}(\Omega))}^2 < C(\text{data}).$$

Now set  $\tilde{u}^n(t) = u_{k-1} + \frac{t-t_{k-1}}{h}(u_k - u_{k-1})$  for  $t \in (t_{k-1}, t_k)$  and

$$k \in \{1, \dots, n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the equation to the form

$$\int_{\Omega} \partial_t \tilde{u}^n \varphi + a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi \, dx = \langle f^n, \varphi \rangle,$$

where  $f^n(t) := f_k$  in in

$$(t_{k-1}, t_k), k \in \{1, \dots, n\}.$$

We are now ready to use growth and apriori estimates:

$$\|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(Q_T)} \leq C(\text{data}).$$

For the norm of the time derivative:

$$\sup_{\varphi \in S_V(0,1)} \langle \partial_t \tilde{u}^n(t), \varphi \rangle = \sup_{\varphi \in S_V(0,1)} \langle f^n, \varphi \rangle - \int_{\Omega} (a(\cdot, u^n, \nabla u^n) \cdot \nabla f + a_0(\cdot, u^n, \nabla u^n) \varphi) dx,$$

at any  $t \in (0, T)$ . So using Holder:

$$\|\partial_t \tilde{u}^n(t)\|_{V^*} \leq \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)},$$

and integrating

$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{r'} dt \leq C \left( \int_0^T \|f^n\|_{V^*}^{r'} + \|a(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{L_{r'}(\Omega)} dt \right) \leq TC(\text{data})$$

## 6 Semigroup theory

We consider the equation

$$\begin{aligned} u' &= Au, A \text{ is linear} \\ u(0) &= u_0, \end{aligned}$$

where  $u : [0, \infty) \rightarrow \mathbb{R}$ . We know that for example if  $Au = au, a \in \mathbb{R}$  then

$$u(t) = u_0 e^{at}.$$

If  $\mathbf{u} : [0, \infty) \rightarrow \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d \times d}$ , then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{A}^k t^k.$$

This can be extended to  $u : [0, \infty) \rightarrow X, X$  a banach space,  $A \in \mathcal{L}(X)$ , then

$$u(t) = \exp(tA)u_0,$$

where the operator exponential is the same. This works well for unbounded operators, but suppose now

$$X = L_2(\Omega), Au = \Delta u.$$

We *guess* the solution should be

$$u(t) = \exp(\Delta t)u_0,$$

but what is

$$\exp(\Delta t)?$$

**Definition 9** (Linear operator and its domain). Let  $X$  be a Banach space over  $\mathbb{K}$ . Linear operator on  $X$  is a couple  $(A, \mathcal{D}(A))$ , where  $\mathcal{D}(A)$  is a subspace of  $X$  and  $A : \mathcal{D}(A) \rightarrow X$  is linear.

**Definition 10.** A family  $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$  is called a semigroup if

1.  $S(0) = \text{id}$
2.  $\forall s, t \geq 0 : S(t)S(s) = S(t+s).$

If moreover  $\forall x \in X : S(t)x \rightarrow x$ , as  $t \rightarrow 0_+$ , we call  $\{S(t)\}$  a  $c_0$ -semigroup (strongly continuous).

*Remark.*  $\{s(t)\}_{t \in \mathbb{R}}$  with the two conditions is an Abelian group  $(\{S(t)\}_{t \in \mathbb{R}}, \circ)$  with

$$(S(t))^{-1} = S(-t). \quad (3)$$

*Remark* ( $X = \text{Banach}$ ). **In the following,  $X$  is always a Banach space.**

**Lemma 12.** Let  $\{S(t)\}_{t \geq 0}$  be a  $c_0$ -semigroup in  $X$ . Then

1.  $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t},$
2.  $\forall x \in X, t \mapsto S(t)x \in C([0, \infty); X).$

*Proof.* 1  $\Rightarrow$  2. Fix  $t > 0, x \in X$  compute

$$\lim_{h \rightarrow 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t)(S(h)x - x)\|_X \leq \lim_{h \rightarrow 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \rightarrow 0.$$

now compute  $\lim_{h \rightarrow 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h \rightarrow 0_+} \|S(t-h)(x - S(h)x)\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \rightarrow 0.$   $\square$

**Definition 11** (Infinitesimal generator). A linear operator  $(A, \mathcal{D}(A))$  is called a infinitesimal generator of the semigroup  $\{S(t)\}_{t \geq 0}$ , if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X \mid \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\},$$

**Theorem 21.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  in  $X$ . Then

1.  $x \in \mathcal{D}(A) \Rightarrow \forall t \geq 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x),$
2.  $x \in X \wedge t \geq 0 \Rightarrow x_t = \int_0^t S(s)x \, ds \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x.$

*Proof.* Fix  $x \in \mathcal{D}(A), t \geq 0$ . Calculate

$$\lim_{h \rightarrow 0_+} \frac{S(h)S(t)x - S(t)x}{h} \stackrel{13}{=} \lim_{h \rightarrow 0_+} S(t) \frac{S(h)x - x}{h} = S(t)Ax,$$

---


$$\stackrel{13}{=} S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$$

(convergence is in the norm of the Banach space  $X$ ). This means  $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$ , moreover, if  $t > 0$ :

$$\lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \rightarrow 0_+} S(t-h) \left( \frac{x - S(h)x}{-h} - S(h)Ax \right),$$

estimate,

$$\left\| \lim_{h \rightarrow 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \rightarrow 0_+} S(t-h) \left( \frac{x - S(h)x}{-h} - S(h)Ax \right) \right\| = \lim_{h \rightarrow 0_+} \left\| \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \rightarrow 0_+} S(t-h) \right\|$$

as  $S(t)$  is continuous and  $S(0) = \text{id}$ . Clearly,  $t \mapsto S(t)x$  is  $C^1([0, \infty))$  and

$$\frac{d}{dt}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h \rightarrow 0_+} \frac{1}{h}(S(h)x_t - x_t) = \lim_{h \rightarrow 0_+} \frac{1}{h} \left( \int_h^{t+h} S(s)x \, ds - \int_0^t S(s)x \, ds \right),$$

realize that

$$S(h)x_t = \int_0^t S(s+h)x \, ds = \int_h^{t+h} S(s)x \, ds,$$

so the previous computation continues as follows

$$= \lim_{h \rightarrow 0_+} \frac{1}{h} \left( \int_t^{t+h} S(s)x \, ds - \int_0^h S(s)x \, ds \right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

**Definition 12** (Closed operator). We say that a linear operator  $(A, \mathcal{D}(A))$  is closed if  $\forall \{u_n\} \subset \mathcal{D}(A) : u_n \rightarrow u \wedge Au_n \rightarrow v$ , for some  $u, v \in X$ , then it must hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that  $\{(x, Ax) | x \in \mathcal{D}(A)\} \subset X \times X$  is closed in  $(X \times X, \|\cdot\|_1)$ .

**Example.** Let  $\Omega \in C^{1,1}$ ,  $X = L_2(\Omega)$ ,  $\mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ ,  $Au = \Delta u$ . Then  $(A, \mathcal{D}(A))$  is closed. Really, take  $\{u_n\} \subset L_2(\Omega) : u_n \rightarrow u$  in  $L_2(\Omega)$  for some  $u \in L_2(\Omega)$ . Suppose  $Au_n \rightarrow v$  in  $L_2(\Omega)$ ,  $v \in L_2(\Omega)$ . Suppose the following equation: find

$$u_n \text{ s.t. } -\Delta u_n = Au_n, u_n \text{ on } \partial\Omega.$$

From the regularity theory for elliptic problems, we know that  $\|u_n\|_{W^{2,2}(\Omega)} \leq C\|Au_n\|_{L_2(\Omega)} \leq C$ , so we can extract  $u_{n_k} \rightharpoonup u$  in  $W^{2,2}(\Omega)$ . Realize moreover

$$\int_{\Omega} \Delta u_n \varphi \, dx = \int_{\Omega} u_n \Delta \varphi \, dx, \forall \varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_{\Omega} v \varphi \, dx = \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} \Delta u \varphi \, dx,$$

which means  $\Delta u = v$  a.e. in  $\Omega$  and that  $u \in \mathcal{D}(A)$ ,  $Au = v$ .

**Theorem 22.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0} \subset X$ . Then

1.  $\mathcal{D}(A)$  is dense in  $X$ ,
2.  $(A, \mathcal{D}(A))$  is closed.

*Proof.* Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t} \int_0^t S(s)x \, ds \underbrace{\in \mathcal{D}(A)}_{\text{prev. thm}}, \quad \frac{x_t}{t} \rightarrow x \text{ in } X,$$

Ad 2.: Take  $\{x_n\} \subset \mathcal{D}(A) : x_n \rightarrow x \text{ in } X, Ax \rightarrow v \text{ in } X$ . Compute<sup>14</sup>

$$\frac{(S(h) - \text{id})x_n}{h} = \frac{1}{h} \int_0^h \frac{d}{ds}(S(s)x_n) \, ds = \frac{1}{h} \int_0^h AS(s)x_n \, ds = \frac{1}{h} \int_0^h S(s) \underbrace{Ax_n}_{\rightarrow v}, \text{ so taking the limit yields } \frac{(S(h) - \text{id})x}{h} =$$

Altogether,  $x \in \mathcal{D}(A), Ax = v$ . □

**Lemma 13.** Let  $(A, \mathcal{D}(A))$  be the infinitesimal generator of  $c_0$ -semigroups  $\{S(t)\}_{t \geq 0}, \{\tilde{S}(t)\}_{t \geq 0}$ . Then

$$\{S(t)\}_{t \geq 0} = \{\tilde{S}(t)\}_{t \geq 0}.$$

*Proof.* We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix  $x \in \mathcal{D}(A), t > 0$ . Then  $g(s) := S(s)\tilde{S}(t-s)x$  satisfies  $g \in C^1([0, t]; X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$ , as  $A, S$  commute. This means  $g(0) = g(t)$  and from this it follows  $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$ . Since  $\overline{\mathcal{D}(A)} = X, S$  continuous  $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$ , and since  $t \geq 0$  was arbitrary, we are done. □

**Definition 13** (Resolvent of a linear operator). Let  $(A, \mathcal{D}(A))$  be a linear (possibly unbounded) operator on  $X$ . We define

1. resolvent set

$$\rho(A) = \left\{ \lambda \in \mathbb{K} \mid \lambda \text{id} - A \text{ is invertible and } (\lambda \text{id} - A)^{-1} \in \mathcal{L}(X) \right\},$$

2. resolvent operator  $R(\lambda, A) : X \rightarrow \mathcal{D}(A) : R(\lambda, A) = (\lambda \text{id} - A)^{-1}$ , for

$$\lambda \in \rho(A).$$

*Remark.* If  $(A, \mathcal{D}(A))$  is a closed linear operator:  $\lambda \in \rho(A) \Leftrightarrow \lambda \text{id} - A$  is a bijection of  $\mathcal{D}(A)$  onto  $X$ .

**Lemma 14.** Let  $(A, \mathcal{D}(A))$  be a linear operator on  $X$ . It holds

1.  $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$
2.  $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x,$

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<sup>14</sup>This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that  $X$  is a Banach space and work with some functionals and Bochner integrals or whatever



3.  $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$ , and  $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A)$ ,
4. If moreover  $(A, \mathcal{D}(A))$  is the infinitesimal generator of a  $c_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  s.t.  $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ , then

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t) dt \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

*Remark.* The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

*Proof.* Ad 1.:

$$AR(\lambda, A)x = (A - \lambda \text{id}) \underbrace{R(\lambda, A)}_{=(\lambda \text{id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\text{id} - (\lambda \text{id} - A))R(\eta, A) = R(\lambda, A)(\eta \text{id} - A - \lambda \text{id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For  $\lambda \neq \eta$  we also have

$$R(\lambda, A)R(\eta, A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG assume  $\omega = 0$ , meaning  $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$ . Denote  $\tilde{S}(t) = e^{-\omega t} S(t)$ .

Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x dt.$$

First of all, this is well defined as

$$\|\tilde{R}x\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X dt = \frac{M}{\lambda} \|x\|_X,$$

and so  $\|\tilde{R}\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}$ ,  $\tilde{R} \in \mathcal{L}(X)$ . Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda \tilde{R}x - x \Leftrightarrow \text{id} = (\lambda \text{id} - A)\tilde{R}.$$

For  $x \in X, h > 0$  fixed compute

$$\begin{aligned} \frac{1}{h}(S(h)\tilde{R}x - \tilde{R}x) &= \frac{1}{h} \left( \int_0^\infty e^{-\lambda t} S(t+h)x - e^{-\lambda t} S(t)x dt \right) = \\ &= \frac{1}{h} \left( \int_h^\infty e^{-\lambda(t-h)} S(t)x dt - \int_0^\infty e^{-\lambda t} S(t)x dt \right) = \\ &= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x dt - \frac{1}{h} \int_0^h e^{-\lambda t} S(t)x dt = \\ &= e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} \rightarrow \lambda e^{-\lambda t}, \text{ as } h \rightarrow 0_+ \end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t)e^{-\lambda t}\frac{e^{h\lambda}-1}{h}S(t)x \rightarrow \lambda e^{-\lambda t}S(t)x \text{ on } (0,\infty) \text{ as } h \rightarrow 0_+.$$

The norm of this can be estimated as  $\|\lambda e^{-\lambda t}S(t)x\| \leq Ce^{-\lambda t}M\|x\|_X \in L_1((0,\infty))$ . Altogether, we obtain  $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda \text{id} - A)\tilde{R}x = x$ .

To proceed further, we need the following theorem:

$$x \in \mathcal{D}(A), A \text{ closed} : A\tilde{R}x = A\left(\int_0^\infty e^{-\lambda t}S(t)x \, dt\right) = \int_0^\infty e^{-\lambda t} \underbrace{AS(t)}_{=S(t)A} x \, dt = \tilde{R}Ax,$$

which has been stated but not proved<sup>15</sup>. Finally, we can write:  $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda \text{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$ . Moreover, we have also shown the mapping is a bijection.  $\square$

**Definition 14** (Contraction semigroup). We say that  $\{S(t)\}_{t \geq 0}$  is a contraction semigroup if

$$\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

**Theorem 23** (Hille-Yosida). Let  $M \geq 1, \omega \in \mathbb{R}$ . A linear  $(A, \mathcal{D}(A))$  on a Banach space  $X$  generates a  $c_0$ -semigroup (meaning it is its infinitesimal generator) satisfying  $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$  **if and only if**

1.  $(A, \mathcal{D}(A))$  is closed,
2.  $\mathcal{D}(A)$  is dense in  $X$ ,
3.  $\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}$ .

*Proof.* If  $M = 1, \omega = 0$ , then  $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}$ . "  $\Rightarrow$  " has been proven, now show the other direction. The plan is to

1. approximate  $A$  by  $\{A_n\} \subset \mathcal{L}(X)$ ,
2. construct  $S_n$  for  $A_n$  as previously,
3. estimate and limit passage.

*Approximation:* See the analogy:  $a \in \mathbb{R} : \frac{n}{n-a} \rightarrow 1$ , we would like  $nR(n, A) \rightarrow \text{id}$ . Calculate the norm of  $nAR(n, A) = n(nR(n, A) - \text{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$ , (This approx. is called the Yosida approximation.) For  $x \in \mathcal{D}(A)$  fixed:

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leq \|R(n, A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \frac{1}{n} \|Ax\|_X \rightarrow 0 \text{ as } n \rightarrow \infty.$$

If

$$y \in X : \|nR(n, A)y - y\|_X \leq \|nR(n, A)(y - x)\|_X + \|nR(n, A)x - x\|_X + \|x - y\|_X \leq 2\|y - x\|_X + \underbrace{\|nR(n, A)x - x\|_X}_{\rightarrow 0},$$

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<sup>15</sup>It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

but  $\|y - x\|_X$  can be made arbitrarily small from density of  $\mathcal{D}(A)$  in  $X$ , so in fact

$$nR(n, A)y \rightarrow y \text{ in } X, \forall y \in X.$$

And so  $nR(n, A)$  really approximates  $\text{id}$ .

Using this gives us

$$\forall x \mathcal{D}(A) : A_n x = n \overbrace{AR(n, A)}^{=AR(n, A)} x = n R(n, A) A x \rightarrow A x \text{ in } X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \leq \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (t A_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id} + n^2 t R(n, A))^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt \text{id})^k \sum_{k=0}^{\infty} \frac{(n^2 t R(n, A))^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives  $\leq e^{-nt} \text{id} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n, A)\|_X^k \leq e^{-nt} e^{nt} = 1$ , as  $\|nR(n, A)\|_X^k \leq 1$ . This means  $\{S_n(t)\}_{\mathcal{L}(X)} \leq 1$ .

Now show that this converges: fix  $x \in \mathcal{D}(A)$ , compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{d}{ds} (S_n(s)S_m(t-s)x) ds \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x ds \right\|_X \leq \underbrace{t}_{\|S_t\|_{\mathcal{L}(X)} \leq 1} \|A_n - A_m\|_X$$

and since  $X$  is Banach, it is convergent also. Finally, for  $y \in X$ , we have

$$\|S_n(t)y - S_m(t)y\|_X \leq \|S_n(t)(y - x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(t)(x - y)\|_X \leq 2\|x - y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that  $\{S_n(t)y\}$  is Cauchy uniformly on  $[0, T], T > 0 \Rightarrow \exists S(t) : S_n(t)y \rightarrow S(t)y \forall y \in X, t > 0$ . And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain  $\{S(t)\}_{t \geq 0}$  is a  $c_0$ -semigroup.

It remains to answer this question. Is  $(A, \mathcal{D}(A))$  the infinitesimal generator of  $\{S(t)\}_{t \geq 0}$ ? Let  $(\tilde{A}, \mathcal{D}(\tilde{A}))$  be the infinitesimal generator of  $\{S(t)\}_{t \geq 0}$ . Compute

$$S_n(t)x - x = \int_0^t \frac{d}{ds} S_n(s)x ds = \int_0^t S_n(s)A_n x ds,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \leq \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \rightarrow 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \, ds.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t \rightarrow 0_+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A} \text{ on } \mathcal{D}(A)).$$

The opposite inclusion is simple: fix  $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$ , and so  $\lambda \text{id} - A : \mathcal{D}(A) \rightarrow X$  is onto, but also  $\lambda \text{id} - A = \lambda \text{id} - \tilde{A}$  on  $\mathcal{D}(A)$ , and so  $\lambda \text{id} - \tilde{A} : \mathcal{D}(A) \rightarrow X$  is onto. From the previous theorem, we know  $\lambda \text{id} - \tilde{A}$  is one-to-one, so  $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$ . Altogether,  $A = \tilde{A}$ ,  $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$ .  $\square$

## 7 (Some) exercises

### 7.1 4.3.2025

**Example** (Coefficients for smooth extension). Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for  $u \in \mathcal{D}(\mathbb{R}^d)$ . Find  $\{c_j\}_{j=1}^{k+1}$  in such a way that  $Eu \in C^k(\mathbb{R}^d)$ . Moreover, take  $d = 1$ .

*Proof.* For  $k = 0, j = 1$  we take  $c_1 = 1, c_j = 0, j \neq 1$ . For  $k = 1$ , compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k+1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take  $x_d = 0$  in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \dots, k\}.$$

That is a linear system of  $k + 1$  equations. Is it solvable?  $\square$

### 7.2 8.4.2025

**Example** (Laplace). Let  $a_0 = 0, a(\cdot, z, p) = p$ . Then  $|a(\dots)| \leq |p|$ , growth can be accomplished for  $r = 2, a(\dots) \cdot p \geq |p|^2$ . We can thus apply the theorem to our laplace equation

**Example.** Let  $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}(1 + |p|^2)$ . Then it is clearly Caratheodory, it is bounded  $|a(\dots)| \leq |p| \frac{\pi}{2}$ , so the growth conditions yield, it is coercive as  $\operatorname{atan}(1 + |p|^2) \geq \frac{\pi}{4} |p|^2$ , and it is monotone

$$(\operatorname{atan}(1 + |p_1|^2)p_1 - \operatorname{atan}(1 + |p_2|^2)p_2)(p_1 - p_2) = \int_0^1 \sum_{j=1}^d \frac{d}{ds} \operatorname{atan}(1 + |p_2 + s(p_1 - p_2)|^2) (p_2 + s(p_1 - p_2)) \, ds (p_1 - p_2)_j$$