# Partial differential equations II

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## Contents

# 1 Winter semester addendum

## 1.1 Weak* convergence

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is reflexive, i.e.

$$\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*,$$

which means that balls in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ are weakly* compact. Moreover, $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is *separable*, from which it follows $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ with the weak* topology is metrizable and thus there exists s weakly * converging subsequence (from the balls).

**Example** (For people without Functional Analysis I)**.** Let $X$ be a linear normed space, $\{x_n\} \subset X$ a sequence in $X$. We say $x_n$ converges weakly to $x \in X$ whenever

$$f(x_n) \to f(x), \forall f \in X^*.$$

Let $X*$ be the topological dual to $X$, $\{x_n\} \subset X^*$ a sequence in $X$. We say $f_n$ converges weakly* to $f \in X^*$ whenever

$$f_n(x) \to f(x), \forall x \in X^*,\ i.e.\, x(f_n) \to x(f),$$

where by $x(y), x \in X, y \in X.*$ we understand

$$\varepsilon_x : X^* \to \mathbb{K}, y \mapsto y(x).$$

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, every point $x \in \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ can be interpreted as a linear functional on $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, so given $\{x_n\} \subset \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, we can interpret is as a $\{x_n\} \subset \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, meaning given a weakly converging sequence in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, it is actually a weakly* converging sequence in $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$.

## 1.2 Regularity of parabolic problems

**Theorem 1.** *Let the assumptions of the previous theorem hold and $\Omega \in C^{1,1}, \delta \in (0,1)$. Then $u \in L_2\Big((\delta,T);\, W^{2,2}(\Omega)\Big)$.*

*Proof.* Take the weak formulation in $t \in (\delta,T)$. WLOG further assume $d = 0$. Then

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi - bu\varphi - \mathbf{c}\cdot\nabla u\varphi - \int_\Omega \partial_t u\varphi = \int_\Omega (f - bu - \mathbf{c}\cdot\nabla u - \partial_t u)\varphi,$$

and the integrand of the last integral is in $\mathrm{L}_2(\Omega)$ for a.e. $t \in (\delta,T)$. We can thus use the elliptic regularity results and write:u

$$\|u\|^2_{\mathrm{W}^{2,2}(\Omega)} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\partial_t u\|^2_{\mathrm{L}_2(\Omega)}),$$

integrating both sides $\int_\delta^T \mathrm{d}t$ yields

$$\|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{L}_2((0,T);\mathrm{W}^{1,2}(\Omega))} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))})$$

□

**Theorem 2.** *If data are smooth and satisfy the compatibility condiitons, then the weak solutions to the parabolic equation are smooth.*

*Proof.* no. □

*Remark* (Compatibility condition). : Take the heat equation : $\partial_t u - \triangle u = f$ at time zero: $\triangle u(0) + f(0) = \partial_t u(0) \in \mathrm{W}^{1,2}_0(\Omega)$, so we need that $f(0) + \triangle u(0)$ has zero trace $\Rightarrow$ compatibility conditions.

### 1.3 Uniqueness of solutions to hyperbolic problems

**Theorem 3** (Uniqueness of the solution to a hyperbolic equation)**.** *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that $\mathbf{c} \in W^{1,\infty}(\Omega)$. Then the weak solution to the hyperbolic equation is unique.*

*Proof.* It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the equation, multiply it by $\varphi \in V$ fixed and integrate over $\Omega$ for $t \in (0,T)$ fixed:

$$< \partial_{tt} u(t), \varphi > + \int_\Omega \mathbb{A}(t)\nabla u(t) \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\varphi \,\mathrm{d}x - \int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\varphi\,\mathrm{d}x = 0.$$

Now, take a special test function

$$\psi(t) = \left(\int_t^s u(\tau)\,\mathrm{d}\tau\right)\chi_{(0,s)}(t),$$

for some $s \in (0,T)$. Then $\partial_t\psi(t) = -u(t)$ on $t \in (0,s)$. Next, integrate the equation in time over $(0,s)$.

$$\int_0^s < \partial_{tt} u(t), \psi > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t)\nabla u(t)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^s\int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^s\int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s < \partial_{tt} u(t), \varphi > \mathrm{d}t =< \partial_t u(s), \psi(s) > - < \partial_t u(0), \psi(0) > - \int_0^s < \partial_t u(t), \partial_t\psi(t) > \mathrm{d}t\,,$$

and realize $\psi(s) = 0, \partial_t u(0) = 0$, so

$$-\int_0^s < \partial_t u(t), \partial_t\psi(t) > \mathrm{d}t + \int_0^s\int_\Omega \mathbb{A}(t)\nabla u(t)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^s\int_\Omega (bu(t) + \mathbf{c}\cdot\nabla u(t))\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^s\int_\Omega u(t)\mathbf{d}(t)\cdot\nabla\psi\,\mathrm{d}x\,\text{ţ} = 0,$$

but since $\partial_t\psi(t) = -u(t)$, we can actually write (time dependencies are omitted for brevity)

$$\int_0^s < \partial_t u, u > \mathrm{d}t + \int_0^s\int_\Omega -\mathbb{A}\nabla\partial_t\psi\cdot\nabla\psi - b\psi\partial_t\psi - \psi\mathbf{c}\cdot\nabla\partial_t\psi + \partial_t\psi\mathbf{d}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0,$$

rewriting the LHS as a time derivative of something, we obtain

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2_{\mathrm{L}_2(\Omega)} - \int_\Omega \mathbb{A}\nabla\psi\cdot\nabla\psi + b\psi^2 + \psi\mathbf{c}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla\psi\,\mathrm{d}x\Big)\,\mathrm{d}t =$$
$$= \int_0^s\int_\Omega (\partial_t\mathbb{A})\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 + \psi\partial_t\mathbf{c}\cdot\nabla\psi + \underbrace{\partial_t\psi}_{=-u(t)}\,\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi - \psi\mathbf{d}\cdot\nabla\underbrace{\partial_t\psi}_{=-u(t)}\,)\,\mathrm{d}t\,\mathrm{d}x\,,$$

and upon integration (recall $\psi(s) = 0$, from the definition of $\psi$ it follows $\nabla\psi(0) = 0$, and $u(0) = 0$,)

$$\frac{1}{2}\Big(\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathbb{A}(0)\nabla\psi(0)\cdot\nabla\psi(0) + b(0)\psi(0)^2 + \psi(0)\mathbf{c}(0)\cdot\nabla\psi(0) + \psi(0)\mathbf{d}(0)\nabla\psi(0)\,\mathrm{d}x\Big) =$$
$$= \int_0^s\int_\Omega \partial_t\mathbb{A}\nabla\psi\cdot\nabla\psi + \partial_t b\psi^2 - u\partial_t\mathbf{c}\cdot\nabla\psi - \psi\partial_t\mathbf{d}\cdot\nabla\psi + \psi\mathbf{d}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\,.$$

From this we obtain the following estimate:

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\psi(0)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C\Big(\int_0^s \|\psi\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big)\mathrm{d}t + \|\psi(0)\|^2_{\mathrm{L}_2(\Omega)},$$

where $C = C\Big(\|\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbb{A}\|_{\mathrm{L}_\infty(\Omega)}, \|b\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t b\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{c}\|_{\mathrm{L}_\infty(\Omega)}, \|\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}, \|\partial_t\mathbf{d}\|_{\mathrm{L}_\infty(\Omega)}\Big)$.
Define now the test function $\chi(t) = \int_0^t u(\tau)\,\mathrm{d}\tau$, and realize that in fact $\psi(t) = \chi(s) - \chi(t), \chi(0) = 0$. Plugging this in the above inequalty yields

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s \|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big) + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)},$$

and using

$$\|\chi(s) - \chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} = \|\chi(t) - \chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le 2\Big(\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)}\Big),$$

and the definition of $\chi(t)$, from which it follows

$$\|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le \int_0^s \|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\,,$$

we are allowed to write

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)} \le C\Big(\int_0^s 2\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + 2\|u\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big),$$

and so

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + (1 - 2sC)\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_1\Big(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big).$$

If we now choose $T_1 \in (0, T]$ small enough *s.t.* $1 - 2sC > 0$ for $s \in (0, T_1]$, we finally obtain

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_2\Big(\int_0^s \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)}\,\mathrm{d}t\Big), \forall s \in (0, T_1],$$

which implies $u = 0$ on $(0, T_1]$ by the Gronwall lemma: we have

$$\xi(t) \le \int_0^t \xi(s)\,\mathrm{d}s\,, \text{ for } a.a.\, t \in (0, T) \Rightarrow \xi(t) = 0\; a.e..$$

for $\xi \in \mathrm{L}_1((0,T))$ nonnegative[1]. If we now boostrap on $[T_1, 2T_1], [2T_1, 3T_1]$ etc., we obtain $u = 0$ on $(0,T]$.

$\square$

# 2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1, +\infty]$, $k \in N$. We define

$$\mathrm{W^{k,p}}(\Omega) = \left\{ f \in \mathrm{L_p}(\Omega)\,;\, D^\alpha f \in \mathrm{L_p}(\Omega)\,,\, \forall |\alpha| \le k \right\},$$

with the norm

$$\|f\|^p_{\mathrm{W^{k,p}}(\Omega)} = \|f\|^p_{\mathrm{L_p}(\Omega)} + \sum_{0<|\alpha|\le k} \|D^\alpha f\|^p_{\mathrm{L_p}(\Omega)}.$$

Recall that:

- $\mathrm{W^{k,p}}(\Omega)$ is Banach $\forall p$ and Hilbert for $p = 2$.
- $\mathrm{W^{k,p}}(\Omega)$ is separable if $p < \infty$ and reflexive if $p > 1, p < \infty$.

*Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.*

## 2.1 Tools from functional analysis

**Definition 1** (Regularization kernel)**.** The function $\eta$ is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\operatorname{supp} \eta \subset \mathrm{U}(0,1)$
- $\eta \ge 0$
- $\eta$ is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$

**Definition 2** (Regularization of a function)**.** Let $\eta$ be a regularization kernel. Set[2]

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(x/\varepsilon), \varepsilon > 0.$$

We define the smoothing of $f \in \mathrm{L}_1(\Omega)_{\mathrm{loc}}$ by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

*Remark* (Propertios of regularization). The regularization has the following properties:

- $f \in \mathrm{L_p}(\Omega) \Rightarrow f_\varepsilon \to f \,\mathrm{in}\, \mathrm{L_p}(\Omega)$ and also a.e

[1]In our case $\xi = \|u\|^2_{\mathrm{L_2}(\Omega)} + \|\chi\|^2_{\mathrm{W}^{1,2}(\Omega)}$.
[2]Another common choice is $\eta_k = k^d\eta(kx), k \in \mathbb{N}$.

- $f \in \mathrm{L}_\infty(\Omega) \Rightarrow f_\varepsilon \to f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathrm{U}(x,\varepsilon)} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y$
- $\operatorname{supp} f_\varepsilon \subset \overline{U(\Omega,\varepsilon)}, f = 0 \,\text{on}\, \mathrm{U}(x,\varepsilon) \Rightarrow f_\varepsilon(x) = 0$

**Definition 3** ($\Omega' \subset\subset \Omega$). $O \subset\subset \Omega$ means $\overline{O}$ is compact and $\overline{O} \subset \Omega$.

**Definition 4** (Shift operator). For $u \in \mathrm{L_p}(\Omega), k \in \{1,\dots,d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

**Lemma 1** (Approximation property of the shift operator). *For $u \in L_p(\Omega)$, it holds $\tau_h u \to u$ in $L_p(\Omega), h \to 0^+$.*

**Lemma 2** (Partition of unity). *Let $E \subset \mathbb{R}^d, \mathcal{G}$ be an open covering of $E$ (possibly uncountable.) Then there exists a countable system $\mathcal{F}$ of nonnegative functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$ and*

1. *$\mathcal{F}$ is subordinate to $\mathcal{G} : \forall \varphi \in \mathcal{F} \exists U \in \mathcal{G} : \operatorname{supp}\varphi \subset U$*
2. *$\mathcal{F}$ is locally finite[3]: $\forall K \subset E$ compact$, \operatorname{supp}\varphi \cap K \neq \emptyset$ for at most finitely many $\varphi \in \mathcal{F}$.*
3. *$\sum_{\varphi\in\mathcal{F}} \varphi(x) = 1, \forall x \in E$ .*

*Proof.* (Sketch) *Step 1 (If $E$ is compact)*:

$E$ compact $\Rightarrow \exists m \in \mathbb{N} : \exists U_j \in \mathcal{G}\, s.t.\, E \subset \bigcup_{j=1}^m U_j$ . Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \cup_{j=1}^m K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set:

$$K_m = \left\{ x \in U \middle| \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{m}, \|x\| \le m \right\}.$$

Then clearly $K_1 \subset K_2 \dots$, and they "converge monotonously to $U$. Next, find $\phi_j \in C_c(U_j), \phi_j > 0 \,\text{on}\, K_j$, e.g. $\phi_j = \theta(\operatorname{dist}(x, \partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally

$$\varphi_j = \frac{\psi_j}{\sum_k \psi_k}.$$

*Step 2 (If $E$ is open)*:

Approximate $E$ by $K \subset E$ compact by the exhaustion argument, then the covering will enlarge from finite $\to$ countable (nontrivial reasoning). □

## 2.2 Density of smooth functions

**Lemma 3** (Local approximation by smooth functions (using regularization)). *Assume $p \in [1,\infty), \Omega \subset \mathbb{R}^d$ open$, k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega_\varepsilon = \{x \in \Omega | \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$. Then it holds*

1. *$D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ a.e. in $\Omega_\varepsilon, \forall |\alpha| \le k$*
2. *$u_\varepsilon \to u$ in $W^{k,p}(\Omega)_{loc}, \varepsilon \to 0^+$*

[3]In other words, $\varphi_K$ is nonzero for at most finitely many $\varphi \in \mathcal{F} \Leftrightarrow$ points in $K$ can be represented by finitely many functions $\varphi \in \mathcal{F}$.

*Proof.* First of all:

$$\forall x \in \Omega : D^\alpha(u_\varepsilon(x)) = D^\alpha\Big(\int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y\Big) = \int_{\mathbb{R}^d} u(y) D^\alpha_x \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_\Omega u(y) D^\alpha_x \eta_\varepsilon(x-y)\,\mathrm{d}y\,,$$

the integrable majorants are *e.g.* $\|\eta_\varepsilon\|_\infty |u| \chi_{\mathrm{U}(0,\varepsilon)}(x) \in \mathrm{L}_1(\Omega)$. Now picking $x \in \Omega_\varepsilon$ we realize $\forall y \in \mathbb{R}^d/\Omega : x - y \geq \operatorname{dist}(x, \partial\Omega) \geq \varepsilon$, and so $\eta_\varepsilon(x-y) = 0$. Exchanging derivatives and using the definition of the weak derivative

$$\int_\Omega u(y) D^\alpha_x \eta_\varepsilon(x-y)\,\mathrm{d}y = (-1)^{|\alpha|} \int_\Omega u(y) D^\alpha_y \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_\Omega D^\alpha_y u(y) \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} D^\alpha_y u(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = (D^\alpha u)_\varepsilon.$$

Take $V \subset\subset \Omega$ open, then

$$\|u - u_\varepsilon\|_{\mathrm{W}^{\mathrm{k,p}}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u - D^\alpha u_\varepsilon\|_{\mathrm{L}_\mathrm{p}(V)} \to 0,$$

because $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \to D^\alpha u \,\mathrm{in}\, \mathrm{L}_\mathrm{p}(V)$, from the properties of regularization. □

**Theorem 4** (Global approximation by smooth functions)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C = \Big\{f \in C^\infty(\Omega), \operatorname{supp} f$ bounded$\Big\} \bigcap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, i.e.*

$$\overline{C \bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

*If moreover $\Omega$ is bounded, it holds:*

$$\overline{C^\infty \bigcap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega)\,.$$

*Proof.* Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)\,, \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega) \bigcap \mathrm{W}^{\mathrm{k,p}}(\Omega)$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$. For every $j \in \mathbb{N}$ define an open set

$$\Omega_j = \Big\{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\Big\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1} \forall j \in \mathbb{N}, \bigcup_{j=1}^\infty \Omega_j = \Omega$. Next, set

$$U_j = \Omega_{j+1}/\overline{\Omega_{j-1}}, j = 1, 2, \ldots,$$

where $\Omega_0 = \Omega_{-1} = \varnothing$. Since $\Omega_j$ are open, $U_j$ are also open and $\Omega \subset \bigcup_{j\in\mathbb{N}} U_j \Rightarrow \exists \{\varphi_j\}_{j\in\mathbb{N}}$ partition of unity subordinate to $\{U_j\}_{j\in\mathbb{N}}$. We can write $u = \sum_{j\in\mathbb{N}} u\varphi_j$, where $u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega)\,, \operatorname{supp} u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$: set $v_j = (u\varphi_j)_{\varepsilon_j}$ and fix an arbitrary $\delta > 0$. By the properties of regularization, we have

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\delta}{2^{j-1}},$$

for $\varepsilon_j > 0$ sufficiently small, which we now fix so the above inequality holds. To have a nice inequality, we actually want:

$$\|v_j - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{2^N}{2^{N+1} - 1}\frac{\delta}{2^{j-1}},$$

meaning of $N \in \mathbb{N}$ will be evident later.

Set

$$v = \sum_{j \in \mathbb{N}} v_j,$$

then $v \in C^\infty(\Omega)$, (not clearly in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ however) as $\forall x \in \Omega$ the sum contains at most finitely many terms ($\mathcal{F}$ is locally finite.)

Take the $N \in \mathbb{N}$ and estimate the norm $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$. Observe (the sum again contains only finitely many terms)

$$u - v = \sum_{j=1}^{\infty} (u\varphi_j - v_j),$$

so taking $x \in \Omega_N$ i have

$$(u - v)(x) = \sum_{j=1}^{N+1} (u\varphi_j - v_j),$$

because for $m > N + 1$, *i.e.*, $m - 1 > N$ it holds $U_m = \Omega_{m+1}/\overline{\Omega_{m-1}}, \Omega_N \subset \Omega_{m-1}$ meaning $\forall j \geq m > N + 1 : U_m \cap \Omega_N = \varnothing \Rightarrow \operatorname{supp} u\varphi_j \cap \Omega_N = \operatorname{supp} v_j \cap \Omega_N = \varnothing$, since $\operatorname{supp} u\varphi_j \subset U_j, \operatorname{supp} v_j \subset \operatorname{supp} u\varphi_j \subset U_j, \forall j \geq m$. The norm of sum is

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \delta \frac{2^N}{2^{N+1} - 1} \sum_{j=1}^{N+1} \frac{1}{2^j} = \delta.$$

It only remains to let $N \to \infty$ and realize

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

by Lévi's theorem:

$$\sup_{N \in \mathbb{N}} \int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega_N}(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \sup_{N \in \mathbb{N}} |D^\alpha f| \chi_{\Omega_N} \,\mathrm{d}x \int_{\mathbb{R}^d} |D^\alpha f| \chi_{\Omega}(x) \,\mathrm{d}x = \int_{\Omega} |D^\alpha f| \,\mathrm{d}x \,,$$

since $\Omega_{N-1} \subset \Omega_N \forall N \in \mathbb{N}$, and $|D^\alpha f|$ is nonnegative, so the sequence under the integral is nondecreasing. Alltogether,

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \leq \delta, \forall \delta > 0$$

from which it follows $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$ (this was not totally evident) and thus $v \in \mathrm{W}^{\mathrm{k,p}}(\Omega) \cap C^\infty(\Omega)$ so indeed we have showed the desired density. □

*Remark.* It is nice that we only require $\Omega$ to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

*Remark* ($\mathrm{C}^{k,\lambda}$ domain). Recall we call $\Omega \subset \mathbb{R}^d$ to be of class $\mathrm{C}^{k,\lambda}$ if: $\Omega$ is open and bounded, $\exists m \in \mathbb{N}, k \in \mathbb{N}_0, \lambda \in [0,1], \alpha, \beta \in \mathbb{R}^+, \exists$ open sets $U_j \subset \mathbb{R}^d, \exists a_j : \mathrm{B}(0,\alpha) \subset \mathbb{R}^{d-1} :\to \mathbb{R}\, s.t.\, a_j \in \mathrm{C}^{k,\lambda}(\mathrm{B}(0,\alpha)), \exists \mathbb{A}_j \mathbb{R}^d \to \mathbb{R}^d$ affine orthogonal matrices such that

1. $\partial\Omega \subset \bigcup_{j=1}^m U_j$,
2. $\forall j \leq m : \varnothing \neq \partial\Omega \cap U_j = \mathbb{A}_j\big(\{(x', a_j(x') \in \mathbb{R}^d | x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}\}\big)$,
3. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') + b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \Omega$,
4. $\forall j \leq m : \mathbb{A}_j(\{(x', a_j(x') - b) | x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}) \subset \mathbb{R}^d/\overline{\Omega}$.

If $\lambda = 0$ we sometimes drop it and write $\Omega \in C^{k,0} \Leftrightarrow \Omega \in C^{k}$, if $k = 0, \lambda = 1$ we call $\Omega \in C^{0,1}$ to be a Lipschitz domain. *Remember that $\lambda(\Omega) < \infty$ is a part of the definition.*

**Theorem 5** (Global approximation by smooth functions up to the boundary)**.** *Let $\Omega \in C^{0,0}$, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C^{\infty}_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

*Proof.* Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, and $\varepsilon > 0$, be given. We wish to find $v \in \mathrm{C}^{\infty}\left(\overline{\Omega}\right)$ *s.t.* $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple:

1. covering of $\overline{\Omega}$,
2. partition of unity,
3. approximation of $u$ on the covering sets,
4. glue it together.

Set $U_0 = \Omega$, and let $\{U_j\}_{j=1}^m$ be from the definition of $\mathrm{C}^{0,0}$ boundary. Then[4]

$$\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j,$$

Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to $\{U_j\}_{j=0}^m$. Since

$$u = \sum_{j=0}^{m} u\varphi_j, \text{ on } \Omega$$

observe that $u_j := u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \operatorname{supp} u_j \subset \operatorname{supp} \varphi_j \subset\subset U_j$. **Also, we define** $u(x) = 0, \forall x \in \mathbb{R}^d / \Omega$. The proofs differs in the cases $j = 0$ and $j \in \{1, \dots, m\}$.

*Case* $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset U_0 = \Omega$. That means that after the extension of $u\varphi_0$ by zero outside of $\Omega$, it holds $u\varphi_0 \in \mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)$. Since $\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d) = \mathrm{W}^{\mathrm{k,p}}_0(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)}}$, we can find $v_0 \in \mathcal{D}(\mathbb{R}^d)$ *s.t.*

$$\|v_0 - u\varphi_0\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

*Case* $j \in \{1, \dots, m\}$. We have a problem now: $\{U_j\}_{j=1}^m$ covers $\partial\Omega$, which is a *closed* set and we cannot simply use local approximation theorem. One could imagine if we were to mollify in the neighbourhood of $\partial\Omega$, the kernel would pick up values from outside of $\Omega$, where $u = 0$ and the mollification would not be a good approximation. Instead, we approximate $u_j$ on a larger *open* domain containing $\overline{\Omega}$ and then show this is also a good aproximation of $u_j$ on $\Omega \subset \overline{\Omega}$.

Set $w_j = u\varphi_j$, and denote

$$\begin{aligned} S_j &= \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} < x_d < a_j(x'), x' \in \mathrm{U}(0, \alpha)\right\}\right), \\ \Omega_j &= \mathbb{R}^d / \overline{S_j}, \end{aligned}$$

*i.e.*,

$$\text{"}\Omega_j = \Omega \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d \le a_j(x') - \frac{\beta}{2}\right\}\right),\text{"}$$

[4] Our choice $U_0 = \Omega$ is important, as without it the definition of $\mathrm{C}^{0,0}$ boundary only means $\partial\Omega \subset \bigcup_{j=1}^m U_j$.

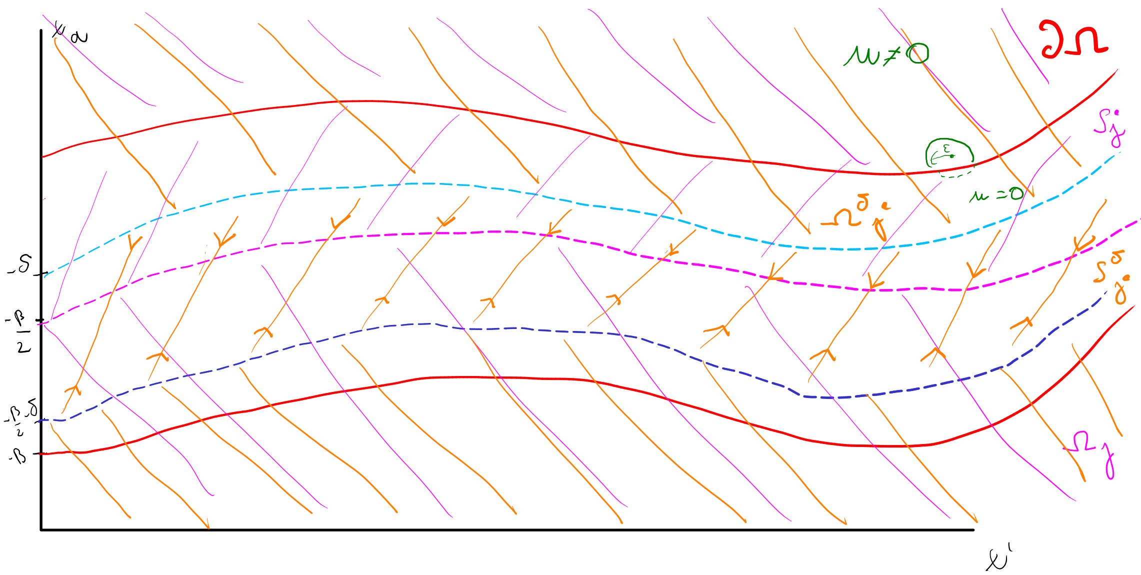


Figure 1: A cumbersome sketch of $\Omega_j, S_j, \Omega_j^\delta, S_j^\delta$

(although this is a bit inaccurate). Realize that since $u = 0$ outside of $\Omega$, also $u_j$ is zero there and in particular it is zero on that "lower strip". Clearly then $u_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega_j)$. Now pick $\delta \in \left(0, \frac{\beta}{2}\right)$, where $\beta$ is from the definition of $\mathrm{C}^{0,0}$ and set

$$S_j^\delta = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$
$$\Omega_j^\delta = \mathbb{R}^d / \overline{S_j^\delta},$$

*i.e.*,

$$"\Omega_j^\delta = \Omega \cup \mathbb{A}_j(\{(x', x_d) | a_j(x') - \delta < x_d < a_j(x')\}) \cup \mathbb{A}_j\left(\left\{(x', x_d) | x_d < a_j(x') - \frac{\beta}{2} - \delta\right\}\right)."$$

The trick is to shift the (support of) function $u_j$ "into" $\Omega_j^\delta$

$$\tau_\delta u_j(\mathbb{A}_j(x', a_j(x'))) = u_j(\mathbb{A}_j(x', a_j(x') + \delta)), x' \in \mathrm{U}(0,\alpha) \subset \mathbb{R}^{d-1}.$$

Realize that in fact

$$\mathrm{supp}\,(\tau_\delta u_j) = \mathrm{supp}\,(u_j) - \delta,$$

from which it follows $\tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)$; we have only shifted the function $u_j$, but since we have also shifted $S_j$, qualitatively there is no difference. Since $\Omega \subset \Omega_j^\delta \subset \Omega_j^\delta \cap \Omega_j, \Omega \subset \Omega_j \subset \Omega_j^\delta \cap \Omega_j$, and the fact $\tau_\delta$ is an isometry between Sobolev spaces, we also have $u_j, \tau_\delta u_j \in \mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)$. Moreover, from the properties of the shift operator it follows $\exists \delta > 0$ *s.t.*

$$\|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \leq \|u_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j \cap \Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

We are on a good track. Since we know $\tau_\delta u_j$ is already close to $u_j$, we are done once we approximate $\tau_\delta u_j$ by a function from $\mathrm{C}^\infty\left(\overline{\Omega}\right)$. Notice that if we show $\overline{\Omega} \subset \Omega_j^\delta$, then clearly $\mathrm{C}^\infty\left(\Omega_j^\delta\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right)$.

*Show* $\Omega \subset\subset \Omega_j^\delta$*:* We already know $\Omega \subset \Omega_j^\delta$, so it suffices to show $\partial\Omega \subset \Omega_j^\delta$. Our parametrization of the boundary yields

$$\partial\Omega = \bigcup_{k=1}^m \mathbb{A}_k(\{(x', x_d) | x_d = a_k(x'), x' \in \mathrm{U}(0,\alpha)\}),$$

and the set $\Omega_j^\delta$ is given as $\Omega_j^\delta = \mathbb{R}^d / \overline{S_j}$, where

$$S_j = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta < x_d < a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right).$$

Realize it suffices to show $\partial\Omega \notin \overline{S_j}$, as then it wont be excluded from $\mathbb{R}^d$ and thus will end up in $\Omega_j^\delta$. *Thanks to continuity of* $a_j$, we may write

$$\overline{S_j} = \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

*i.e.*, the " $<$ " have changed to " $\leq$ ". Since we are doing everything locally, it is enough to show

$$\mathbb{A}_j(\{(x', x_d) | x_d = a_j(x'), x' \in \mathrm{U}(0,\alpha)\}`) \notin \mathbb{A}_j\left(\left\{(x', x_d) | a_j(x') - \frac{\beta}{2} - \delta \leq x_d \leq a_j(x') - \delta, x' \in \mathrm{U}(0,\alpha)\right\}\right),$$

which is equivalent to

$$\Big((a_j \le a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta)\Big) \vee \Big((a_j > a_j - \delta) \wedge (a_j \ge a_j - \frac{\beta}{2} - \delta)\Big).$$

Our choice has been $\delta \in \left(0, \frac{\beta}{2}\right)$, and $\beta > 0$ from the definition of $\Omega \in \mathrm{C}^{0,0}$, so the second statement is clearly true $\forall j \in 1, \dots, m$. Consequently $\partial\Omega \notin \overline{S}_j$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

*Approximation of* $\tau_\delta u_j$. Since $\Omega_j^\delta$ is open there $\exists w_j \in \mathrm{C}^\infty\left(\Omega_j^\delta\right)$ such that

$$\|\tau_\delta u_j - w_j\|_{\mathrm{W}^{k,p}(\Omega)} \le \|\tau_\delta u_j - w_j\|_{\mathrm{W}^{k,p}\left(\Omega_j^\delta\right)} < \frac{\varepsilon}{4(m+1)}.$$

What is more, since $\overline{\Omega} \subset \Omega_j^\delta$, we see $w_j \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ in fact. What is even more, we see $\operatorname{supp} w_j \subset \operatorname{supp} u_j - \delta$ so $\operatorname{supp} \tau_{-\delta} w_j \subset \operatorname{supp} u_j \subset\subset \Omega$, and since $w_j$ is defined on the whole $\mathbb{R}^d$, also $v_j := \tau_{-\delta} w_j$ is defined on the whole $\mathbb{R}^d$. Alltogether, $v_j = \tau_{-\delta} w_j \in \mathcal{D}\left(\mathbb{R}^d\right)$, and from the properties of the shift operator, there $\exists \delta > 0$ *s.t.*

$$\|v_j - w_j\|_{\mathrm{W}^{k,p}(\Omega)} < \frac{\varepsilon}{4(m+1)}.$$

*Approximation of* $u$.

Finally, let us set

$$v = \sum_{j=0}^{m} v_j.$$

Then $v \in \mathcal{D}\left(\mathbb{R}^d\right)$ and it holds

$$\begin{aligned}
\|u - v\|_{\mathrm{W}^{k,p}(\Omega)} &= \left\| \sum_{j=0}^{m} u_j - \sum_{j=0}^{m} v_j \right\|_{\mathrm{W}^{k,p}(\Omega)} = \left\| \sum_{j=0}^{m} u_j - v_j \right\|_{\mathrm{W}^{k,p}(\Omega)} \le \sum_{j=0}^{m} \|u_j - v_j\|_{\mathrm{W}^{k,p}(\Omega)} \le \\
&\le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - u_j\|_{\mathrm{W}^{k,p}(\Omega)} \le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - w_j\|_{\mathrm{W}^{k,p}(\Omega)} + \sum_{j=1}^{m} \|w_j - \tau_\delta u_j\|_{\mathrm{W}^{k,p}(\Omega)} + \sum_{j=1}^{m} \|\tau_\delta u_j - u_j\|_{\mathrm{W}^{k,p}(\Omega)} \\
&< \frac{\varepsilon}{m+1} + 2\sum_{j=1}^{m} \frac{\varepsilon}{4(m+1)} + \sum_{j=1}^{m} \frac{\varepsilon}{2(m+1)} = \varepsilon
\end{aligned}$$

Finally,

$$\mathcal{D}\left(\mathbb{R}^d\right) \subset \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) \subset \mathrm{C}^\infty\left(\overline{\Omega}\right).$$

*This proof may still have some flaws, but the author has decided to move on.* □

*Remark* (What is $\mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right)$). Recall

$$C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \left\{ u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d) \right\}.$$

## 2.3 Extension of Sobolev functions

*Problem of extension*: For $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, does there exist

$$\overline{u} \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right),\ s.t.\ \overline{u}|_\Omega = u,\ \|\overline{u}\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} \le C(\Omega)\|u\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$$

?

The answer is **yes**, if $\Omega$ is nice enough. Notice however this is not as simple as in the case of Lebesgue spaces, where we could just extend the function by zero trivailly. We are dealing with derivatives, that are somehow regular, and if we extend a nonzero function by zero, it might mess up the regularity of the derivatives.

We will be using $\mathrm{C}^1$ diffeomorphisms heavily, so we investigate some of their properties first.

**Lemma 4** (Properties of $C^1$ diffeomorphisms). *Let $U, V \subset \mathbb{R}^d$ be open, $\Phi : U \to V$ be a $C^1$ diffeomorphism and let $\tilde{U} \subset \mathbb{R}^d$ s.t. $\tilde{U} \subset\subset U$. Then*

1. $\Phi(\tilde{U}) \subset\subset V$,
2. *if moreover $\tilde{U}$ is compact, then* [5]

$$\exists C > 0 : \forall u \in C^1(V) : \|u \circ \Phi\|_{W^{1,p}(\tilde{U})} \le C\|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

*Proof. Ad 1.*: No proof has been given.

*Ad 2.*: Just a change of variables formula:

$$\|u \circ \Phi\|^p_{\mathrm{L_p}(\tilde{U})} = \int_{\tilde{U}} |u \circ \Phi|^p \,\mathrm{d}x = \int_{\Phi(\tilde{U})} |u|^p |\det \nabla\Phi|^{-1} \,\mathrm{d}x\,.$$

Since $\Phi$ is one-to-one, we know $|\det \nabla\Phi| > 0$ on $U$, and since $\Phi \in \mathrm{C}^1(U)$ and $\tilde{U} \subset U \Rightarrow \Phi \in \mathrm{C}^1\left(\tilde{U}\right) \Rightarrow \det \nabla\Phi \in \mathrm{C}^0\left(\tilde{U}\right)$, and since $\tilde{U}$ is compact, $|\det \nabla\Phi| \ge C_1 > 0 \Leftrightarrow |\det \nabla\Phi|^{-1} \le \frac{1}{C_1}$. In total

$$\|u \circ \Phi\|^p_{\mathrm{L_p}(\tilde{U})} \le \frac{1}{C_1} \int_{\Phi(\tilde{U})} |u|^p \,\mathrm{d}x = C\|u\|^p_{\mathrm{L_p}(\Phi(\tilde{U}))}.$$

As for the derivative, we have $\forall i \in \{1, \dots, d\}$ :

$$\begin{aligned}\int_{\tilde{U}} |\partial_i(u \circ \Phi)|^p \,\mathrm{d}x \le \int_{\tilde{U}} |\nabla(u \circ \Phi)|^p \,\mathrm{d}x = \int_{\tilde{U}} |\nabla\Phi\big((\nabla u) \circ \Phi\big)|^p \,\mathrm{d}x \le \\ \le \|\nabla\Phi\| \int_{\tilde{U}} |(\nabla u) \circ \Phi|^p \,\mathrm{d}x = \|\nabla\Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p |\det \nabla\Phi|^{-1} \,\mathrm{d}x \le C\|\nabla\Phi\| \int_{\Phi(\tilde{U})} |\nabla u|^p \,\mathrm{d}x \le \\ \le C\|\nabla u\|^p_{\mathrm{L}_p(\Phi(\tilde{U}))},\end{aligned}$$

where $\|\nabla\Phi\|$ is *e.g.* the operator norm of the matrix $\nabla\Phi$. □

**Lemma 5** (Flat extension). *Let $\alpha, \beta > 0, K \subset\subset U(0,\alpha) \times [0,\beta)$ be compact. Then there $\exists C > 0$, a linear operator*

$$E : C^1\left((B(0,\alpha) \times [0,\beta])\right) \to C^1\left((B(0,\alpha) \times [-\beta,\beta])\right),$$

*and the set $\tilde{K} \subset\subset U(0,\alpha) \times [-\beta,\beta)$ such that $\forall u \in C^1\left(B(0,\alpha) \times [0,\beta]\right)$ it holds*

[5] For $\tilde{U}$ compact: $\tilde{U} \subset\subset V \Leftrightarrow \tilde{U} \subset V$.

1. $Eu = u$ *on* $B(0,\alpha)\times[0,\beta]$,
2. $\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \le \|u\|_{W^{1,p}(U(0,\alpha)\times(0,\beta))}$. *Actually,*
$$\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)),W^{1,p}(U(0,\alpha)\times(-\beta,\beta)))} = 2^{\frac{1}{p}}$$
3. *if* $\operatorname{supp} u \subset K$ *then* $\operatorname{supp} Eu \subset \tilde{K}$

*Proof.* (The set $\mathrm{U}(0,\alpha)\times[0,\beta)$ is a cylinder of radius $\alpha$ and height $\beta$)

The proof is constructive: for the assumed $u$ we write ($x=(x',x_d)$, where $x'\in\mathrm{B}(0,\alpha)\subset\mathbb{R}^{d-1}$, $x_d\in[0,\beta]\subset\mathbb{R}$)
$$Eu(x',x_d) = \begin{cases} u(x',x_d), & x_d\ge 0\\ -3u(x',-x_d)+4u(x',-\frac{x_d}{2}), & x_d<0. \end{cases}$$
Does $Eu$ lie in $\mathrm{C}^1\left(\mathrm{B}(0,\alpha)\times[-\beta,\beta]\right)$? Since $u\in\mathrm{C}^1\left(\mathrm{B}(0,\alpha)\times[0,\beta]\right)$, it us continuous in the "lower cylinder", check only the transition through the origin plane: take some $a=(x',0)\in\mathrm{B}(0,\alpha)\times\{0\}$. Then
$$\lim_{x\to a} Eu(x) = \begin{cases} u(a), & x_d\ge 0\\ -3u(a)+4u(a)=u(a), & x_d<0, \end{cases}$$
so $Eu$ is continuous. The derivatives
$$\lim_{x\to a}\partial_d Eu(x',x_d) = \begin{cases} \partial_d u(x',0), & x_d\ge 0\\ \left(-3\partial_d u(x',-x_d)(-1)+4\partial_d u(x',-\frac{x_d}{2})(-\frac{1}{2})\right)\Big|_{x_d=0} = \partial_d u(x',0), & x_d<0, \end{cases}$$
and also for any $i\in\{1,\dots,d-1\}$
$$\lim_{x\to a}\partial_i Eu(x_1,\dots,x_d) = \begin{cases} =\partial_i u(x_1,\dots,0), & x_d\ge 0\\ =-3\partial_i u(x_1,\dots,0)+4\partial_i u(x_1,\dots,0)=\partial_i u(x_1,\dots,0) \end{cases}$$
so the the derivative is also continuous. Thus, we have
$$Eu\in C^1\subset \mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta)).$$
The first property is clear from the definition of $Eu$, the estimates of the norm: (all derivatives can in fact be assumed classical)
$$\begin{aligned}
\|Eu\|^p_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} &= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} + \sum_{|\alpha|=1}\|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} =\\
&= \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))}\\
&+ + \sum_{|\alpha|=1}\|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1}\|D^\alpha Eu\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} =\\
&= \|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \|4u-3u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))} +\\
&+ \sum_{|\alpha|=1}\|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + \sum_{|\alpha|=1}\|D^\alpha(4u-3u)\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(-\beta,0))}\\
&= 2\|u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} + 2\sum_{|\alpha|=1}\|D^\alpha u\|^p_{\mathrm{L_p}(\mathrm{U}(0,\alpha)\times(0,\beta))} = 2\|u\|^p_{\mathrm{W^{k,p}}(\mathrm{U}(0,\alpha)\times(0,\beta))}
\end{aligned}$$

and so

$$\|Eu\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = 2^{\frac{1}{p}}\|u\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))}.$$

Where we have used the obvious fact

$$\int_{\mathrm{U}(0,\alpha)\times(0,\beta)} |f|^p \,\mathrm{d}x = \int_{\mathrm{U}(0,\alpha)\times(-\beta,0)} |f|^p \,\mathrm{d}x\,,$$

We will assume (although this is *with* a loss of generality), that

$$Eu(x',x_d) = 0 \Leftrightarrow (u(x',x_d) = 0 \vee 3u(x',-x_d) = 0 \vee 4u(x',-\frac{x_d}{2}) = 0),$$

*i.e.*, that support lies in

$$\operatorname{supp} Eu \subset \{(x',x_d) \in \mathrm{U}(0,\alpha)\times[0,\beta] | u(x',x_d) = 0\} \cup \{(x',x_d) \in \mathrm{U}(0,\alpha)\times[-\beta,0] | u(x',-x_d) \neq 0\} \cup$$
$$\cup \left\{(x',x_d) \in \mathrm{U}(0,\alpha)\times[-\beta,0] | u(x',-\frac{x_d}{2}) \neq 0\right\}.$$

Denote $\Phi_1, \Phi_2 : \mathrm{U}(0,\alpha)\times(0,\beta) \to \mathrm{U}(0,\alpha)\times(-\beta,\beta)$ to be the mappings

$$\Phi_1(x',x_d) = (x',-x_d),$$
$$\Phi_2(x',x_d) = (x',-\frac{x_d}{2}),$$

then clearly $\Phi_1, \Phi_2$ are $\mathrm{C}^1$ diffeomorphisms and

$$u(x',-x_d) = (u\circ\Phi_1)(x',x_d), u\left(x',-\frac{x_d}{2}\right) = (u\circ\Phi_2)(x',x_d),$$

*i.e.*,

$$u(x',x_d) = \left(u\circ\Phi_1^{-1}\right)(x',-x_d) = \left(u\circ\Phi_2^{-1}\right)\left(x',-\frac{x_d}{2}\right),$$

and so

$$\operatorname{supp} Eu \subset \operatorname{supp} u \cup \Phi_1^{-1}(\operatorname{supp} u) \cup \Phi_2^{-1}(\operatorname{supp} u) \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

as $\operatorname{supp} u \subset\subset K$. Let us define

$$\tilde{K} := K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K),$$

Then we see

$$\operatorname{supp} Eu \subset K \cup \Phi_1^{-1}(K) \cup \Phi_2^{-1}(K) = \tilde{K},$$

And, finally, we have $K \subset\subset \mathrm{U}(0,\alpha)\times[0,\beta) \Rightarrow K \subset\subset \mathrm{U}(0,\alpha)\times(-\beta,\beta) \Rightarrow \Phi_1^{-1}(K), \Phi_2^{-1}(K) \subset\subset \mathrm{U}(0,\alpha)\times(0,\beta)$, which really implies

$$\tilde{K} \subset\subset \mathrm{U}(0,\alpha)\times[-\beta,\beta).$$

$\square$

Let us prove the main result.

**Theorem 6** (Extension of Sobolev functions)**.** *Let $\Omega \in C^{k-1,1}, k \in \mathbb{N}, p \in [1,\infty], V \subset \mathbb{R}^d$ open such that $\Omega \subset\subset V$. Then there is $E : W^{k,p}(\Omega) \to W^{k,p}\left(\mathbb{R}^d\right)$ bounded linear operator such that*

*1.* $\forall u \in W^{k,p}(\Omega) : Eu = u$ *a.e. in* $\Omega$*,*

*2.* $\forall u \in W^{k,p}(\Omega) : \operatorname{supp} Eu \subset V \supset\supset \Omega$*,*

*3.* $\|E\|_{\mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^d))} \le C, C = C(p, \Omega, V)$*.*

*Proof.* **Will only be presented for** $k = 1, \Omega \in C^1, p < \infty$**.** The strategy is:

1. covering of $\overline{\Omega}$ & partition of unity
2. obtain a diffeomorphism from the fact $\Omega \in \mathrm{C}^{1,0}$,
3. suitable composition & cut off,
4. flat extension,
5. show existence of $E : \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right) \to \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right)$ with the desired properties,
6. extend via density

*Covering of* $\Omega$*:* In the following, $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ are as in the definition of a $\mathrm{C}^{1,0}$ domain. Set $U_0 = \Omega$ and realize

$$\overline{\Omega} \subset \bigcup_{j=0}^{m} U_j,$$

*i.e.*, $\{U_j\}_{j=0}^m$ is an open covering of $\overline{\Omega}$. Denote $\{\varphi_j\}_{j=0}^m$ as the partition of unity subordinate to $\{U_j\}_{j=0}^m$.

*Diffeomorphism & flat extension* For $j \in \{1, \dots, m\}$ we define $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta, \beta) \to U_j$ by

$$\Phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

(A bit confusingly, we will however be "interpreting" $\Phi_j$ as $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to \mathbb{R}^d$, with it being extended by zero on $\mathbb{R}^d / \overline{U_j}$, as we need $\Phi_j^{-1}$ to be defined on the whole $\mathbb{R}^d$.)

As $\Omega \in \mathrm{C}^{1,0}$, we know $a_j \in \mathrm{C}^1(\mathrm{B}(0,\alpha)) \subset \mathrm{C}^1(\mathrm{U}(0,\alpha))$, and so $\phi_j$ is a $\mathrm{C}^1$ diffemorphism. Let us denote by $\tilde{E} : \mathrm{C}^1(\mathrm{B}(0,\alpha) \times [-\beta,\beta]) \to \mathrm{C}^1(\mathrm{B}(0,\alpha) \times [-2\beta,\beta])$ the extension operator from the Flat extension lemma. Then we for any $u \in C^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right) : u = \sum_{j=1}^m \varphi_j u$ define

$$Eu = \varphi_0 u + \sum_{j=1}^{m} \left(\eta \tilde{E}((\varphi_j u) \circ \Phi_j)\right) \circ \Phi_j^{-1},$$

where $\eta \in \mathrm{C}^{\infty}(\mathrm{U}(0,\alpha) \times \mathbb{R})$ is a cut-off function

$$\eta(y', y_d) \begin{cases} = 1 \text{ on} & y_d \ge 0, \\ = 0 \text{ on} & y_d \le -h, \\ \in (0,1) & \text{else}. \end{cases},$$

for some parameter $h > 0$ which will be defined later. With this definition, $\forall x \in \Omega$ it holds

$$(Eu)(x) = (\varphi_0 u)(x) + \sum_{j=1}^{m} \left(\eta \tilde{E}(u\varphi_j \circ \Phi_j)\right) \underbrace{\left(\Phi_j^{-1}(x)\right)}_{\in \mathrm{U}(0,\alpha) \times (-\beta,\beta)} =$$

$$= (\varphi_0 u)(x) + \sum_{j=1}^{m} (u\varphi_j \circ \Phi_j)\left(\Phi_j^{-1}(x)\right) = (\varphi_0 u)(x) + \sum_{j=1}^{m} (u\varphi_j)(x) = \sum_{j=0}^{m} (u\varphi_j)(x) = u(x) \sum_{j=0}^{m} \varphi_j(x) = u(x),$$

since $\eta(y) = 1, \tilde{E}(\varphi_j u \circ \Phi_j)(y) = (\varphi_j u \circ \Phi_j)(y)$ for $y \in \mathrm{U}(0,\alpha) \times (-\beta,\beta)$, according to our definition of $\eta$ and the properties of the extension $\tilde{E}$.

The motivation behind the cutoff is the following: we know $\operatorname{supp}\varphi_j \subset U_j$, so since $\Phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to U_j$, we have $\operatorname{supp} u\varphi_j \circ \Phi_j \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta)$ and from the properties of the flat extension operator we also have $\operatorname{supp}\tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-2\beta,\beta)$. Since moreover $\operatorname{supp}\eta = \mathrm{U}(0,\alpha) \times (-2h,\infty)$, in total

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) = \operatorname{supp}\eta \cap \operatorname{supp}\tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h,\infty) \cap (-\beta,\beta),$$

and also [6]

$$\operatorname{supp} Eu \subset \Phi_j\big(\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j)\big).$$

We need to prove $\operatorname{supp} Eu \subset V$, where $V$ is some neighbourhood of $\Omega$. If it holds

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta),$$

the desired property holds, as then

$$\operatorname{supp} Eu \subset \Phi_j\big(\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j)\big) \subset \Phi_j(\mathrm{U}(0,\alpha) \times (-\beta,\beta)) \subset U_j \subset \overline{\Omega} \subset V.$$

In total,

$$\operatorname{supp}\eta\tilde{E}(u\varphi_j \circ \Phi_j) = \mathrm{U}(0,\alpha) \times (-2h,\infty) \cap (-\beta,\beta) \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta).$$

So if $h$ is such that $-2h > -\beta \Leftrightarrow h < \frac{\beta}{2}$, we can guarantee $\operatorname{supp} Eu \subset V$ and that is what we want.

Finally, $E$ is clearly linear, its norm: (we are using the lemma about flat extension and the properties of $\mathrm{C}^1$ diffeomorphisms together with the facts $\eta \leq$ on $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, $\Phi_j(\mathrm{U}(0,\alpha) \times (0,\beta)) \subset U_j, \Phi_j^{-1}\big(\mathbb{R}^d\big) \subset$ [7], $\mathrm{U}(0,\alpha) \times (-\beta,\beta)$, [8])

$$\begin{aligned}\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &= \big\|\big(\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big) \circ \Phi_j^{-1}\big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq C\big\|\eta\tilde{E}(\varphi u_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} = \\ &= C\big\|\tilde{E}(u\varphi_j \circ \Phi_j)\big\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(-\beta,\beta))} \leq C\|u\varphi_j \circ \Phi_j\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha)\times(0,\beta))} \leq \\ &\leq C\|u\varphi_j\|_{\mathrm{W}^{1,p}(U_j\cap\Omega)} \leq C\|u\|_{\mathrm{W}^{1,p}(\Omega)},\end{aligned}$$

from which it clearly follows $\|E\|_{\mathcal{L}(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d))} \leq C$.

So all the properties hold for $u \in C^\infty_{\overline{\Omega}}\big(\mathbb{R}^d\big)$, let us show them also for $u \in \mathrm{W}^{1,p}(\Omega)$. Pick an arbitrary $u \in \mathrm{W}^{1,p}(\Omega)$, find $\{u_k\} \subset C^\infty_{\overline{\Omega}}\big(\mathbb{R}^d\big) : u_k \to u$ in $\mathrm{W}^{1,p}(\Omega)$. Since $E$ is continous, we know $\lim_{k\to\infty} Eu_k$ exists. Let us set

$$Eu \coloneqq \lim_{k\to\infty} Eu_k,$$

where $Eu_k$ is defined above for $u_k \in \mathrm{C}^\infty_{\overline{\Omega}}\big(\mathbb{R}^d\big)$.

Ad 3): Clearly $E(\alpha u) = \lim_{k\to\infty} E(\alpha u_k) = \alpha Eu, E(u+v) = \lim_{k\to\infty} E(u_k + v_k) = \lim_{k\to\infty} Eu_k + Ev_k = Eu + Ev$, since $E$ on the continous functions is linear. Also

[6] Generally, $\operatorname{supp} v \circ \Psi^{-1} = \overline{\{x \in \mathbb{R}^d | u(\Psi^{-1}(x)) \neq 0\}} = \overline{\{x \in \mathbb{R}^d | \Psi^{-1}(x) \in \operatorname{supp} u\}} = \overline{\{x \in \mathbb{R}^d | x \in \Psi(\operatorname{supp} u)\}} = \Psi(\operatorname{supp} u)$.

[7] We can somehow extend $\Phi_j^{-1}$ by zero from $U_j \subset \Omega \subset \mathbb{R}^d$ to be defined on the whole $\mathbb{R}^d$ (i guess)

[8] ...even though the assumptions to use those are not totally valid... but doc. Kaplicky is okay with that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} = \left\|\lim_{k\to\infty} Eu_k\right\|_{W^{1,p}(\mathbb{R}^d)} = \lim_{k\to\infty} \|Eu_k\|_{W^{1,p}(\mathbb{R}^d)} \le \|E\| \lim_{k\to\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^d)} = \|E\|\|u\|_{W^{1,p}(\mathbb{R}^d)},$$

(we are using $\{Eu_k\}$ has a limit); we see our above definition truly yields a continuous linear operator.
Ad 1): $\forall\, a.a.\, x \in \Omega : Eu(x) = \lim_{k\to\infty} Eu_k(x) = \lim_{k\to\infty} u_k(x) = u(x)$,
Ad 2): $\operatorname{supp} Eu_k \subset \mathrm{U}(\Omega,\varepsilon) \Rightarrow \operatorname{supp} Eu \subset \overline{\mathrm{U}(\Omega,\varepsilon)} \subset V$.

$$\operatorname{supp} Eu = \left\{x \in \mathbb{R}^d \middle| \lim_{k\to\infty} Eu_k \neq 0\right\} \subset \bigcap_{k=1}^{\infty} \underbrace{\operatorname{supp} Eu_k}_{\subset V} \subset V.$$

We are done.

$\square$

*Remark* ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $\Omega \in \mathrm{C}^{0,1}$ and $p \in (1,\infty), k \in \mathbb{N}$, but the construction of the extension must be different. "It seems the result is not known for $\Omega \in \mathrm{C}^{0,1}$ and $p = 1$, or $p = \infty$."

## 2.4 Embedding theorems

From the definition of $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ it immediately follows $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L_p}(\Omega)$. Can we obtain $\mathrm{W}^{\mathrm{k,p}}(\Omega) \subset \mathrm{L}_q(\Omega)$ for some $q > p$? The answer **yes**, if $\Omega$ is again nice enough (and there will also be some dependence on the dimension of $\mathbb{R}^d$.)

### 2.4.1 Theorems for $p \le d$

**Example.** Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1,x_2) = \int_{-\infty}^{x_1} \partial_1 u(s,x_2)\,\mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1,s)\,\mathrm{d}s\,,$$

so

$$\begin{aligned}
\|u\|^2_{\mathrm{L}_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \\
&\le \left(\int_{\mathbb{R}}\int_{-\infty}^{x_1} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{-\infty}^{x_2} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_1 u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_2 u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) \le \\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(s,x_2)|\,\mathrm{d}s\,\mathrm{d}x_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1,s)|\,\mathrm{d}s\,\mathrm{d}x_2\right) = \\
&= \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right) = \left(\int_{\mathbb{R}^2} |\nabla u|\,\mathrm{d}x\right)^2 = \\
&= \|\nabla u\|^2_{\mathrm{L}_1(\mathbb{R}^2)},
\end{aligned}$$

so

$$\|u\|_{\mathrm{L}_2(\mathbb{R}^2)} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^2)}.$$

This can be generalized in two ways:

- $d > 2$,
- less smoothness.

**Lemma 6** (Gagliardo)**.** *Let $d \geq 2$. Let $\hat{u}_i : \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative and measurable for $j \in \{1, \dots, d\}$. We define*

$$\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), \mathrm{d}\hat{\mathrm{x}}_j = \mathrm{d}x_1 \dots \mathrm{d}x_{j-1}\, \mathrm{d}x_{j+1} \dots \mathrm{d}x_d\,.$$

*Consider the functions $u_j : \mathbb{R}^d \to \mathbb{R}, u_j(x) = \hat{u}_j(\hat{x}_j)$. Then*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} u_j(x)\, \mathrm{d}x \leq \prod_{j=1}^{d} \left( \int_{\mathbb{R}^{d-1}} (\hat{u}_j(\hat{x}_j))^{d-1} \mathrm{d}\hat{\mathrm{x}}_j \right)^{\frac{1}{d-1}}. \tag{1}$$

*(Both integrals can be infinity.)*

*Proof.* Induction by dimension: *The case $d = 2$.*:

$$\int_{\mathbb{R}^2} u_1(x_1, x_2) u_2(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2)\hat{u}_2(x_1)\, \mathrm{d}x_1\, \mathrm{d}x_2 = \left( \int_{\mathbb{R}} \hat{u}_1(x_2)\, \mathrm{d}x_2 \right)\left( \int_{\mathbb{R}} \hat{u}_2(x_1)\, \mathrm{d}x_1 \right) =$$
$$= \left( \int_{\mathbb{R}} \hat{u}_1(\hat{x}_1) \mathrm{d}\hat{\mathrm{x}}_1 \right)\left( \int_{\mathbb{R}} \hat{u}_2(\hat{x}_2) \mathrm{d}\hat{\mathrm{x}}_2 \right).$$

(an equality in fact.) We have used Fubini once, which is permitted, as we have measurability + nonnegativity.

*The case $d \to d + 1$* Before we procceed, recall the "generalized Holder", all functions are non-negative

$$\int_{\Omega} \prod_{j=1}^{d} f_j\, \mathrm{d}x \leq \prod_{j=1}^{d} \left( \int_{\Omega} f_j^{p_j}\, \mathrm{d}x \right)^{\frac{1}{p_j}},$$

where $\sum_{j=1}^{d} \frac{1}{p_j} = 1$. See that if we take $p_j = d$, then $\sum_{j=1}^{d} \frac{1}{d} = \frac{1}{d} \sum_{j=1}^{d} 1 = 1$, so

$$\int_{\Omega} \prod_{j=1}^{d} f_j\, \mathrm{d}x \leq \prod_{j=1}^{d} \left( \int_{\Omega} f_j^{d}\, \mathrm{d}x \right)^{\frac{1}{d}}.$$

Let us compute:

$$\int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^{d} u_j \overbrace{u_{d+1}}^{=\hat{u}(\hat{x}_{d+1})} \overbrace{\mathrm{d}x_1 \dots \mathrm{d}x_d}^{=\mathrm{d}\hat{\mathrm{x}}_{d+1}} \,\mathrm{d}x_{d+1} = \int_{\mathbb{R}^d} \underbrace{\left( \int_{\mathbb{R}} \prod_{j=1}^{d} u_j \,\mathrm{d}x_{d+1} \right)}_{\text{Holder}} u_{d+1} \mathrm{d}\hat{\mathrm{x}}_{d+1} \leq$$

$$\leq \int_{\mathbb{R}^d} \underbrace{\left( \prod_{j=1}^{d} \int_{\mathbb{R}} u_j^d \,\mathrm{d}x_{d+1} \right)^{\frac{1}{d}} u_{d+1}}_{\text{Holder}} \,\mathrm{d}\hat{\mathrm{x}}_{d+1} \leq \left( \int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \left( \int_{\mathbb{R}^d} \left( \prod_{j=1}^{d} \int_{\mathbb{R}} u_j^d(x) \,\mathrm{d}x_{d+1} \right)^{\frac{1}{d}\frac{d}{d-1}} \mathrm{d}\hat{x}_{d+1} \right)^{\frac{d-1}{d}} \leq$$

$$\leq \left( \int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \left( \prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} u_j^d \,\mathrm{d}x_{d+1} \right)^{\frac{1}{d-1}\frac{d-1}{1}} \mathrm{d}\hat{x}_j \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{d-1}{d}\frac{1}{d-1}} =$$

$$= \left( \int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^{d} \left( \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d \,\mathrm{d}x_{d+1} \,\mathrm{d}\hat{x}_j \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} = \left( \int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \right)^{\frac{1}{d}} \prod_{j=1}^{d} \left( \int_{\mathbb{R}^d} u_j^d \,\mathrm{d}\hat{x}_j \right)^{\frac{1}{d}} =$$

$$= \prod_{j=1}^{d+1} \left( \int_{\mathbb{R}^d} u_j^d \,\mathrm{d}\hat{x}_j \right)^{\frac{1}{d}},$$

where the induction step is taken for the function

$$v_j = \left( \int_{\mathbb{R}} u_j^d \,\mathrm{d}x_{d+1} \right)^{\frac{1}{d-1}},$$

that is clearly nonnegative and measurable. □

*Remark.* Sometimes, the lemma is stated as: $\hat{u}_i \in \mathrm{L}_\infty\left(\mathbb{R}^{d-1}\right), \operatorname{supp} \hat{u}_i$ is compact $\forall i \in \{1, \dots, d\}$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} |u_j(x)| \,\mathrm{d}x \leq \prod_{j=1} \left( \int_{\mathbb{R}^{d-1}} |\hat{u}_j(\hat{x}_j)|^{d-1} \mathrm{d}\hat{\mathrm{x}}_{\mathrm{i}} \right)^{\frac{1}{d-1}} = \prod_{j=1}^{d} \|\hat{u}_j\|_{\mathrm{L}_{d-1}(\mathbb{R}^{d-1})}.$$

The difference is that in our version, we have nonnegativity in the assumptions and do not requiry compact supports and essential boundedness, as we work with integrals that are possibly infinite.

**Theorem 7** (Gagliardo-Nirenberg)**.** *Let* $p \in [1, d)$*. Then* $\forall u \in W^{1,p}(\mathbb{R}^d)$*:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L_p(\mathbb{R}^d)},$$

*where* $p^* = \frac{dp}{d-p}$.

*Proof.* Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$, then use density, as $\mathrm{W}^{1,p}(\mathbb{R}^d) = \mathrm{W}_0^{1,p}(\mathbb{R}^d)$.

$$\forall j \in \{1, \dots, d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d) \,\mathrm{d}s\,.$$

This estimate is independent of $j \in \{1, \dots, d\}$, so it holds

$$|u(x)| \leq \int_{\mathbb{R}} |\nabla u|(\dots, s, \dots) \,\mathrm{d}s\,.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \leq \left(\int_{\mathbb{R}^d} |\nabla u|(\dots, s, \dots)\,\mathrm{d}s\right)^{\frac{d}{d-1}} = \prod_{j=1}^{d}\left(\int_{\mathbb{R}} |\nabla u|(\dots, s, \dots)\,\mathrm{d}s\right)^{\frac{1}{d-1}}.$$

Denote

$$u_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) = \left(\int_{\mathbb{R}^d} |\nabla u| \overbrace{(\dots, x_j, \dots)}^{=x \text{ in fact}}\,\mathrm{d}x_j\right)^{\frac{1}{d-1}},$$

which is a function independent of $x_j, u_j \equiv u_j(\hat{x}_j)$. So the integral (the $\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)$ norm)

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}}\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \prod_{j=1}^{d} u_j\,\mathrm{d}x \underbrace{\leq}_{\text{Gagliardo lemma}} \left(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} u_j^{d-1}(\hat{x}_j)\,\mathrm{d}\hat{x}_j\right)^{\frac{1}{d-1}} =$$

$$= \left(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x)\,\mathrm{d}x_j\,\mathrm{d}\hat{x}_j\right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u|\,\mathrm{d}x\right)^{\frac{d}{d-1}},$$

and so

$$\|u\|^{\frac{d}{d-1}}_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla u\|^{\frac{d}{d-1}}_{\mathrm{L}_1(\mathbb{R}^d)},$$

meaning ($1^* = \frac{d}{d-1}$)

$$\|u\|_{\mathrm{L}_{1^*}(\mathbb{R}^d)} \leq 1\|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^d)}.$$

If now $p \in (1, d)$, we investigate for what $q$ can we obtain estimate of $\||u|^q\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)}$:

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\mathbb{R}^d)} = \||u|^q\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla(|u|^q)\|_{\mathrm{L}_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1}|\nabla u|\,\mathrm{d}x \underbrace{\leq}_{\text{Holder}} q\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}\left\||u|^{q-1}\right\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} =$$

$$= q\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}\|u\|^{q-1}_{\mathrm{L}_{(q-1)p'}(\mathbb{R}^d)}.$$

We want $(q-1)p' = \frac{qd}{d-1}$, so we can divide both sides:

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd - p - pd + d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$\frac{q}{d-1} = \frac{p}{d-p} \Leftrightarrow \frac{qd}{d-1} = \frac{dp}{d-p} = p^*, q = \frac{p(d-1)}{d-p}$$

and thus

$$\|u\|^q_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq q\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}\|u\|^{q-1}_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \Leftrightarrow \|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

$\Rightarrow$ statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\mathbb{R}^d)$ : let $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$, be arbitrary. Then $\exists\{u_k\} \subset \mathcal{D}(\mathbb{R}^d) : u_k \to u$ in $\mathrm{W}^{1,p}(\mathbb{R}^d)$. Moreover, we have showed that $\forall k \in \mathbb{N}$ :

$$\|u_k\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

so passing to the (strong) limit and using the continuity of the norm indeed yields

$$\|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

We are done.

□

*Remark.*

- It is evident that nonzero constants are not in $\mathrm{W}^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.
- the set $\mathbb{R}^d$ is of course unbounded, so we have no ordering of $\mathrm{L_p}(\Omega)$ spaces.
- of course, we require no smoothness of the domain

**Theorem 8.** *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W_0^{1,p}(\Omega)\,, \forall p \in [1,d)$ the statement of the previous theorem holds.*

*Proof.* An immediate corollary of the previous theorem: we have showed the inequality for $u \in \mathcal{D}(\mathbb{R}^d)$, but WLOG it holds also for $u \in \mathcal{D}(\Omega)$ (i can keep the integrals over $\mathbb{R}^d$, but in the end only values from $\Omega$ count) and since $\mathrm{W}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{\mathrm{W}^{1,p}(\Omega)}}$, i can again extend it on the whole $\mathrm{W}_0^{1,p}(\Omega)$. □

*Remark.* In the proof of theorem we showed that $\forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\Omega)} \le q\|\nabla u\|_{\mathrm{L_p}(\Omega)}\|u\|^{q-1}_{\mathrm{L}_{\frac{p(q-1)}{p-1}}(\Omega)},$$

for $q$ such that $\frac{qd}{d-1} \le p^*$.

**Definition 5** (Continuous & compact embeddings)**.** Let $X, Y$ be linear normed spaces. We say

- $X$ is continuously embedded into $Y$, $X \hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and
$$\forall x \in X : \|x\|_Y \le C\|x\|_X.$$
- $X$ is compactly embedded into $Y$, $X \hookrightarrow\hookrightarrow Y$, provided $X \subset Y$ (is a subspace) and
$$\forall A \subset X \,\text{bounded} : \overline{A}^Y \text{ is compact in}\, Y.$$

This is the same as saying[9] $X \subset Y$ (is a subspace) and the identity $\mathrm{id} : X \to Y$ is

- a bounded linear operator, $\mathrm{id} \in \mathcal{L}(X,Y)$
- is a compact linear operator, $\mathrm{id} \in \mathcal{K}(X,Y)$

**Theorem 9** (Embedding theorem for $p \le d$)**.** *Let $\Omega \in C^{0,1}, p^* = \frac{dp}{d-p}$. Then*

[9] Really, we have $\|x\|_Y = \|\,\mathrm{id}\,x\|_Y \le \|\mathrm{id}\|_{\mathcal{L}(X,Y)}\|x\|_X$, and if $A \subset X$ is bounded, than from the definition of $\mathrm{id} \in \mathcal{K}(X,Y) : \mathrm{id}(A) = A \subset Y$ is relatively compact in $Y$.

- *if* $p \in [1,d)$, *then*
$$W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)\; \forall q \in [1,p^*],$$

- *if* $p \in [1,d)$, *then*
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)\,, \forall q \in [1,p^*),$$

- *if* $p = d$, *then*
$$W^{1,d}(\Omega) \hookrightarrow L_q(\Omega)\; \forall q \in [1,\infty),$$

- *if* $p = d$, *then*
$$W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)\; \forall q \in [1,\infty),$$

  *(the same as above, i.e., every continuous embedding is also compact.)*

*Proof.* We would like to use the previous lemmas + extension.
*Ad continuity for* $p < d$:
Recall that the composition of continuous operators yields a continuous operator. In our case:

- the extension operator $E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$ is continuous
- identity $I_1 : \mathrm{W}^{1,p}\left(\mathbb{R}^d\right) \to \mathrm{L}_{p^*}\left(\mathbb{R}^d\right)$ is continous (Gagliardo-Nirenberg: $\|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$.)
- restriction $I_2 : \mathrm{L}_{p^*}\left(\mathbb{R}^d\right) \to \mathrm{L}_{p^*}(\Omega)$ is continuous (monotonicity of the L. integral: $\Omega \subset \mathbb{R}^d \Rightarrow \|u\|_{\mathrm{L}_{p^*}(\Omega)} \leq \|u\|_{\mathrm{L}_{p^*}(\mathbb{R}^d)}$.)
- identity $I_3 : \mathrm{L}_{p^*}(\Omega) \to \mathrm{L}_q(\Omega)$ is continous (embedding of Lebesgue spaces: $\Omega$ is bounded $\Rightarrow \mathrm{L}_{p^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\; \forall q \in [1,p^*]$)

Together, the mapping
$$\mathrm{id} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\Omega)\,, \mathrm{id} = I_3 \circ I_2 \circ I_1 \circ E$$
is continuous, and so $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1,p^*]$.

*Ad continuity for* $p = d$:
If $p = d$, we have (this holds generally) $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega)\; \forall r \in [1,d)$, (embedding of Lebesgue spaces, $\mathrm{L}_d(\Omega) \hookrightarrow \mathrm{L}_r(\Omega)\,, \forall r \in [1,d)$). Notice $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d-$, which means we can for all $q \in [1,\infty)$ find $r \in [1,d)$ s.t. $r^* > q$. Consequently,
$$\forall q \in [1,\infty) \exists r \in [1,d)\; s.t.\; \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,.$$
Notice also that $\forall r \in [1,d)$ we always have
$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega)$$
(that's just renaming $p$ with $r$ in embedding for $p < d$). Then it holds
$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{L}_{r^*}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,,$$
and so $\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)\,, \forall q \in [1,\infty)$.

*Ad compactness for $p < d$:*

It suffices to show $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$ is relatively compact in $\mathrm{L}_q(\Omega)$, which is, since $\mathrm{L}_q(\Omega)$ is complete, equivalent to $U$ being totally bounded in $\mathrm{L}_q(\Omega)$.[10] Extend the functions to $\mathrm{W}^{1,p}(\mathbb{R}^d)$ using the extension operator $E$, so $EU \subset \mathrm{W}^{1,p}(\mathbb{R}^d)$. Take some yet undetermined $\delta > 0$ and denote by $(EU)_\delta$ the set of regularized functions from $EU$ with some kernel $\eta$. Our next strategy is the following:

1. show $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0,R))$,
2. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_1(\Omega)$,
3. show $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega)$.

Since the supports of the functions from $EU$ are uniformly bounded[11], we know [12] $\exists R > 0$ *s.t.*

$$\forall v \in (EU)_\delta : \operatorname{supp} v \subsetneq \mathrm{B}(0,R) \subset \mathbb{R}^d.$$

(Also remember that $\Omega \subset \mathrm{B}(0,R)$.) Moreover, from the properties of mollification[13] it follows:

$$(EU)_\delta \subset \mathrm{C}^1\left((\mathrm{B}(0,R))\right).$$

Next up, calculate for $v \in (EU)_\delta$ the norm $\|v\|_{\mathrm{C}^1(\mathrm{B}(0,1))}$. Realize that in fact $v = (Eu)_\delta$ for some $u \in U$, and so

$$\left| \int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y \right| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}\left(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d)\right)}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,$$

as $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$. Also

$$\begin{aligned}\left|\nabla \int_{\mathbb{R}^d} Eu(y)\eta_\delta(x-y)\,\mathrm{d}y\right| = \left| \int_{\mathbb{R}^d} Eu(y)\nabla_x\eta_\delta(x-y)\,\mathrm{d}y\right| \le \|Eu\|_{\mathrm{L_p}(\mathbb{R}^d)}\|\nabla\eta_\delta\|_{\mathrm{L}_{p'}(\mathbb{R}^d)} \le \\ \le C\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \le C\|E\|_{\mathcal{L}\left(\mathrm{W}^{1,p}(\Omega),\mathrm{W}^{1,p}(\mathbb{R}^d)\right)}\|u\|_{\mathrm{W}^{1,p}(\Omega)} \le C,\end{aligned}$$

using the same arguments. In total

$$\forall v \in (EU)_\delta : \|v\|_{\mathrm{C}^1(\mathrm{B}(0,R))} \le C,$$

or in other words, all functions from $(EU)_\delta$ are uniformly bounded in $\mathrm{C}^1(\mathrm{B}(0,R)) \Rightarrow$ they are uniformly bounded and uniformly equicontinuous (that is implied by uniform boundedness of the derivatives). Thus we can use Arzela-Ascoli theorem and state

$$(EU)_\delta \subset\subset \mathrm{C}^0\left(\mathrm{B}(0,R)\right).$$

[10] A metric space $P$ is totally bounded if there exists a finite $\varepsilon$−net: a finite open covering of $P$ by balls centered in $P$ of radii smaller than $\varepsilon$.

[11] From the properties of extension, we know $\forall u \in \mathrm{W}^{1,p}(\Omega) : \operatorname{supp} Eu \subset V$ with $V$ open *s.t.* $\Omega \subset\subset V$.

[12] Properties of mollification include $\operatorname{supp}(Eu)_\delta \subset \overline{\delta + \operatorname{supp} Eu}$.

[13] We have $(EU)_\delta \subset \mathrm{C}^\infty(\mathbb{R}^d)$.

Since $\mathrm{C}^0(\mathrm{B}(0,R))$ is complete, this also means $(EU)_\delta$ is totally bounded in $\mathrm{C}^0(\mathrm{B}(0,R))$. Using the fact ($\mathrm{B}(0,R)$ is compact) [14]

$$\mathrm{C}^0(\mathrm{B}(0,R)) \hookrightarrow\hookrightarrow \mathrm{L}_1(\mathrm{U}(0,R)),$$

we also see that $(EU)_\delta$ is totally bounded in $\mathrm{L}_1(\mathrm{U}(0,R))$.

Next, take an arbitrary $u \in U$ and compute (we are using the fact $Eu = u$ *a.e.* in $\Omega, \Omega \subset \mathrm{U}(0,R)$.)

$$\begin{aligned}
\|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} &\le \Big\|\overbrace{Eu}^{:=v}-(Eu)_\delta\Big\|_{\mathrm{L}_1(\mathrm{U}(0,R))} = \int_{\mathrm{U}(0,R)}|v-v_\delta|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}\Big|v(x)-\int_{\mathbb{R}^d}v(y)\eta_\delta(x-y)\,\mathrm{d}y\Big|\,\mathrm{d}x = \\
&= \int_{\mathrm{U}(0,R)}\Big|v(x)-\int_{\mathbb{R}^d}v(x-y)\eta_\delta(y)\,\mathrm{d}y\Big|\,\mathrm{d}x = \Big[x\mapsto x+y\Big] = \int_{\mathrm{U}(0,R)}\Big|\int_{\mathbb{R}^d}v(x+y)-v(x)\underbrace{\eta_\delta(y)}_{\le 1}\,\mathrm{d}y\Big|\,\mathrm{d}x \le \\
&\le \int_{\mathrm{U}(0,R)}\Big|\int_{\mathbb{R}^d}(v(x+y)-v(y))\eta_\delta(y)\,\mathrm{d}y\Big|\,\mathrm{d}x \le \int_{\mathrm{U}(0,R)}\int_{\mathbb{R}^d}\frac{|v(x+y)-v(x)|}{|y|}|\eta_\delta(y)||y|\,\mathrm{d}y\,\mathrm{d}x \le \\
&\underbrace{\le}_{\text{Fubini}} \int_{\mathbb{R}^d}\int_{\mathrm{U}(0,R)}\frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y\,.
\end{aligned}$$

Estimate the inner integral: assume $v$ is smooth, $v \in \mathcal{D}(\mathrm{U}(0,R))$ and write

$$\begin{aligned}
\int_{\mathrm{U}(0,R)}\frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x &= \int_{\mathrm{U}(0,R)}\frac{1}{|y|}\Big|\int_0^1\underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(v(x+sy))}_{\nabla v(x+sy)\cdot y}\,\mathrm{d}s\Big|\,\mathrm{d}x \underbrace{\le}_{\text{C-S}} \int_{\mathrm{U}(0,R)}\frac{1}{|y|}\int_0^1|\nabla v(x+sy)||y|\,\mathrm{d}s\,\mathrm{d}x = \\
&= \int_{\mathrm{U}(0,R)}\int_0^1|\nabla v(x+sy)|\,\mathrm{d}s\,\mathrm{d}x = \int_0^1\int_{\mathrm{U}(0,R)}|v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

The last integral can be further manipulated by using the change of variables $z := x+sy \in \{x+sy | x \in \mathrm{U}(0,R)\} = \mathrm{U}(0,R)+sy = \mathrm{U}(sy,R)$. Since $\nabla v \in \mathcal{D}(\mathrm{U}(0,R))$, the integral is nonzero only for $z \in \mathrm{U}(sy,R)\cap\mathrm{U}(0,R) \subset \mathrm{U}(0,R)$ so we can write

$$\begin{aligned}
\int_0^1\int_{\mathrm{U}(0,R)}|\nabla v(x+sy)|\,\mathrm{d}x\,\mathrm{d}s &= \int_0^1\int_{\mathrm{U}(sy,R)\cap\mathrm{U}(0,R)}|\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \le \int_0^1\int_{\mathrm{U}(0,R)}|\nabla v(z)|\,\mathrm{d}z\,\mathrm{d}s \le \\
&\le \int_0^1\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}(\lambda(\mathrm{U}(0,R)))^{\frac{1}{p'}}\,\mathrm{d}s \le C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))},
\end{aligned}$$

and so we have shown

$$\forall v \in \mathcal{D}(\mathrm{U}(0,1)) : \int_{\mathrm{U}(0,R)}\frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x \le C(R)\|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

---

[14] Take some sequence $\{f_k\} \subset \mathrm{C}^0(\mathrm{B}(0,R))$ that is bounded, then

$$\lim_{k\to\infty}\int_{\mathrm{U}(0,R)}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}\lim_{k\to\infty}|f_k|\,\mathrm{d}x = \int_{\mathrm{U}(0,R)}|f|\,\mathrm{d}x = \int_{\mathrm{B}(0,R)}|f|\,\mathrm{d}x \le \infty,$$

the majorant being *e.g.* $\max_{k\in\mathbb{N}}\|f_k\|_\infty$. Hence every bounded sequence in $\mathrm{C}^0(\mathrm{B}(0,R))$ has a converging (sub)sequence in $\mathrm{L}_1(\mathrm{U}(0,R))$.

Now, take[15] $v \in \mathrm{W}_0^{1,p}(\mathrm{U}(0,R))$, then $\exists \{v_k\} \subset \mathcal{D}(\mathrm{U}(0,R)) : v_k \to v \,\text{in}\, \mathrm{W}_0^{1,p}(\mathrm{U}(0,R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y) - v_k(x)|}{|y|} \,\mathrm{d}x \le C(R) \|\nabla v_k\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \to C(R) \|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))}.$$

Putting it all together:

$$\begin{aligned} \|u - (Eu)_\delta\|_{\mathrm{L}_1(\Omega)} &\le \int_{\mathbb{R}^d} \int_{\mathrm{U}(0,R)} \frac{|v(x+y) - v(x)|}{|y|} \,\mathrm{d}x \, |y| \eta_\delta(y) \,\mathrm{d}y \le C(R) \|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \int_{\mathbb{R}^d} \underbrace{|y|}_{\le \delta} \eta_\delta(y) \,\mathrm{d}y \le \\ &\le C(R) \delta \|\nabla v\|_{\mathrm{L_p}(\mathrm{U}(0,R))} \int_{\mathbb{R}^d} \eta_\delta(y) \,\mathrm{d}y = C(R)\delta \|v\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} = C(R)\delta \|Eu\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,R))} \le C_1 \delta \|u\|_{\mathrm{W}^{1,p}(\Omega)} \le \\ &\le C_1 \delta. \end{aligned}$$

where we have used the properties of the reg. kernel $\eta_\delta$, the extension operator $E$ and the fact $u \in U$.

Now fix $\varepsilon > 0$, find $\{(Eu_k)_\delta\}_{k=1}^m$ a finite $\frac{\varepsilon}{2}$-net in $(EB)_\delta$ in $\mathrm{L}_1(\mathrm{U}(0,R))$ (which is possible, since we have total boundedness in $\mathrm{L}_1(\mathrm{U}(0,R))$.) We will show $\{u_k\}_{k=1}^m$is a (finite) $\varepsilon$–net in $\mathrm{L}_1(\Omega)$.

Up to now, $\delta > 0$ has been undetermined; now comes the time - set

$$\delta > 0 \,\text{s.t.}\, C_1 \delta < \frac{\varepsilon}{4}.$$

Fix an arbitrary $u \in U$, and find $j \in \{1, \dots, m\}$ *s.t.* $\|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L}_1(\mathrm{U}(0,R))} < \frac{\varepsilon}{2}$. Compute

$$\|u - u_j\|_{\mathrm{L}_1(\Omega)} \le \|u - (Eu)_\delta\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu)_\delta - (Eu_j)_\delta\right\|_{\mathrm{L}_1(\Omega)} + \left\|(Eu_j)_\delta - u_j\right\|_{\mathrm{L}_1(\Omega)} \le C_1\delta + \frac{\varepsilon}{2} C_1 \delta < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon,$$

where we have used the above estimate and the fact $\Omega \subset \mathrm{B}(0,R)$. Thus, we have shown $U$ is totally bounded in $\mathrm{L}_1(\Omega)$ and so

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_1(\Omega).$$

It remains to show the validity for a general $q \in [1, p^*)$. Using the interpolation theorem on Lebesgue spaces [16] we obtain

$$\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|_{\mathrm{L}_1(\Omega)}^\theta \|u\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta},$$

where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Let us now show $U$ is totally bounded in $\mathrm{L}_q(\Omega)$, *i.e.*, $\forall \epsilon > 0$ there exists a finite $\epsilon-$ net in $U$ in $\mathrm{L}_q(\Omega)$. Pick $\{u_j\}_{j=1}^m \subset U$ that is an $\beta > 0$ net in $\mathrm{L}_1(\Omega)$, where $\beta$ will be determined later. Then it holds

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \|u - u_j\|_{\mathrm{L}_1(\Omega)}^\theta \|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta} \le \beta^\theta \|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta}.$$

Since we have already shown $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_{p^*}(\Omega)$, we know (again $u, u_j$ are in $U$)

$$\|u - u_j\|_{\mathrm{L}_{p^*}(\Omega)}^{1-\theta} \le C_2 \|u - u_j\|_{\mathrm{W}^{1,p}(\Omega)}^{1-\theta} \le C_2 2^{1-\theta},$$

and so

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \le \beta^\theta C_2 2^{1-\theta}.$$

[15] Recall we have $v \in (EU)_\delta$ and so $\operatorname{supp} v \subsetneq \mathrm{B}(0,R)$, meaning it is "zero on $\mathrm{S}(0,R)$" - in the sense of traces.

[16] In the case $q \in [r,s)$ it holds $\|u\|_{\mathrm{L}_q(\Omega)} \le \|u\|_{\mathrm{L}_r(\Omega)}^\theta \|u\|_{\mathrm{L}_s(\Omega)}^{1-\theta}$, $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$.

We see that if we choose $\beta$ *s.t.*

$$\beta < \left(\frac{\epsilon}{C_2 2^{1-\theta}}\right)^{\frac{1}{\theta}},$$

we obtain

$$\|u - u_j\|_{\mathrm{L}_q(\Omega)} \leq \epsilon,$$

*i.e.*, $\{u_j\}_{j=1}^m$ is a $\epsilon$–set in $\mathrm{L}_q(\Omega)$. Thus $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*)$.

*Ad compactness for $p = d$*

Finally, let us show the last result. It holds

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega), \forall r \in [1, d).$$

Moreover, we have just shown

$$\mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_s(\Omega), \forall s \in [1, r^*).$$

Notice that $r^* = \frac{rd}{r-d} \to \infty$ as $r \to d^-$, so $\forall q \in [1, \infty)$ fixed $\exists r \in [1, d) : r^* > q$, *i.e.*, $q \in [1, r^*)$. But then

$$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty).$$

And realize that implies $\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega) \, \forall q \in [1, \infty)$ : if $\{u_n\}$ is bounded in $\mathrm{W}^{1,d}(\Omega)$, then it is bounded in $\mathrm{W}^{1,r}(\Omega)$, as the identity between those spaces is continuous, and the above compact embedding tells us $\{u_{n_k}\}$ is convergent in $\mathrm{L}_q(\Omega)$ for some $\{n_k\}$. In total, $\{u_n\} \subset \mathrm{W}^{1,d}(\Omega)$ has a subsequence $\{u_{n_k}\}$ convergent in $\mathrm{L}_q(\Omega)$. We are done. □

### 2.4.2 Theorems for $p > d$

We know we will encounter Holder spaces. Let us recall some of their properties in a remark.

*Remark* (Properties of Holder spaces). Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $k \in \mathbb{N}_0, \lambda \in [0, 1]$. The norm on the space $\mathrm{C}^{0,1}\left(\overline{\Omega}\right)$ is defined as

$$\|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} = \|u\|_{\mathrm{C}^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x \neq y, x, y, \in \overline{\Omega}} \frac{|D^\alpha u(x) - D^\alpha(y)|}{|x - y|^\lambda},$$

and the space

$$\mathrm{C}^{k,\lambda}\left(\overline{\Omega}\right) \coloneqq \left\{u \in \mathrm{C}^k\left(\overline{\Omega}\right) | \|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} \leq \infty\right\},$$

where we identify

$$\mathrm{C}^{k,0}\left(\overline{\Omega}\right) = \mathrm{C}^k\left(\overline{\Omega}\right).$$

Moreover, we have the following embeddings: $\forall \alpha \in [0, 1]$ it holds

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\beta}\left(\Omega\right), \forall \beta \in [1, \alpha],$$

and

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [1, \alpha).$$

A fresh start of a new chapter calls for a fresh new lemma.

**Lemma 7** (Morrey)**.** *Let $u \in \mathcal{D}\left(\mathbb{R}^d\right)$. Then $\forall x_1, x_2 \in \mathbb{R}^d, \forall \mu \in (0,1]$ it holds*

$$|u(x_1) - u(x_2)| \le \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^{\mu}[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)},$$

*with*

$$[\nabla u]_{L_{1,\mu}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}\sup_{\rho>0}\int_{[0,\rho]^d}\frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\,\mathrm{d}y\,.$$

*Proof.* Pick arbitrary but fixed $x_1, x_2 \in \mathbb{R}^d$. Denote by $C_\rho$ the closed cube with a side of length $\rho$ *s.t.* $x_1$ and $x_2$ lie on opposite faces. Then $\rho \le |x_1 - x_2| \le \rho\sqrt{d}$ (it is not closer then the height and not further then the diagonal.)

Let us begin by first computing the deviation of $u(x_i)$ from the mean value of $u$ on $C_\rho$:

$$\left|\frac{1}{\lambda(C_\rho)}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_i)\right| = \left|\int_{C_\rho}\frac{u(x)-u(x_i)}{\rho^d}\,\mathrm{d}x\right| \le \int_{C_\rho}\frac{|u(x)-u(x_i)|}{\rho^d}\,\mathrm{d}x\,.$$

What other can we use when estimating differences than Newton - Leibniz, right? Since $u \in \mathcal{D}\left(\mathbb{R}^d\right) \subset \mathrm{C}^1\left(\mathbb{R}^d\right)$ it holds for $i \in \{1,2\}$ and $\forall x \in C_\rho$ :

$$\begin{aligned}|u(x) - u(x_i)| &\le \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}u(x_i + s(x-x_i))\,\mathrm{d}s\right| = \left|\int_0^1 \nabla u(x_i + s(x-x_i))\cdot(x-x_i)\,\mathrm{d}s\right| \le \\ &\le \int_0^1 |\nabla u(x_i + s(x-x_i))||x-x_i|\,\mathrm{d}s \le \rho\sqrt{d}\int_0^1|\nabla u(x_i+s(x-x_i))|\,\mathrm{d}s\,.\end{aligned}$$

Notice that it is important $x \in C_\rho$ and $x_1, x_2$ are on the opposite sides. With this estimate for $|u(x)-u(x_i)|$ and Fubini we can write for the deviattion

$$\left|\frac{1}{\rho^d}\int_{C_\rho}u(x)\,\mathrm{d}x - u(x_i)\right| \le \sqrt{d}\int_{C_\rho}\frac{1}{\rho^{d-1}}\int_0^1|\nabla u(x_i+s(x-x_i)|\,\mathrm{d}s\,\mathrm{d}x = \sqrt{d}\int_0^1\int_{C_\rho}\frac{|\nabla u(x_i+s(x-x_i))|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s\,.$$

This calls for a sensible change of variables. Denote $z = x_i + s(x-x_i)$, then under this transformation the cube $C_\rho$ becomes

$$z \in x_i + s(C_\rho - x_i) = x_i(1-s) + C_{s\rho} = x_i(1-s) + [0, s\rho]^d := C^i_{s\rho}$$

which, since $x_i$ is taken from the faces and $s \le 1$, is a cube with its "origin" somewhere in $C_\rho$ and a side of length $s\rho \le \rho$, which implies $C^i_{s\rho} \subset [0,R]^d$ for some $R$. The integral then becomes (clearly $|\det\nabla_x z| = s^d$),

$$\sqrt{d}\int_0^1\int_{C_\rho}\frac{|\nabla u(x_i+s(x-x_i)|}{\rho^{d-1}}\,\mathrm{d}x\,\mathrm{d}s = \sqrt{d}\int_0^1\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{s^d\rho^{d-1}}\,\mathrm{d}z\,\mathrm{d}s = \sqrt{d}\rho^\mu\int_0^1 s^{\mu-1}\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z\,\mathrm{d}s\,,$$

where we are being a bit suggestively imaginative. The "deviation estimate" then becomes

$$\left|\frac{1}{\rho^d}\int_{C_\rho}u(x)\,\mathrm{d}x - u(x_i)\right| \le \sqrt{d}\rho^\mu\int_0^1 s^{\mu-1}\underbrace{\int_{C^i_{s\rho}}\frac{|\nabla u(z)|}{(s\rho)^{d-1+\mu}}\,\mathrm{d}z}_{\le[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}}\,\mathrm{d}s \le \sqrt{d}\rho^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}\int_0^1 s^{\mu-1}\,\mathrm{d}s = \frac{\sqrt{d}}{\mu}\rho^\mu[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)},$$

where we used that $C^i_{s\rho} \subset [0,R]^d$ and that $0 \in \mathbb{R}^d$, and so we could estimate the integral over $C^i_{s\rho}$ by $[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}$. Triangle inequality and the fact $\rho \leq |x_1 - x_2|$ concludes our proof:

$$|u(x_1)-u(x_2)| \leq |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_1)| + |\frac{1}{\rho^d}\int_{C_\rho} u(x)\,\mathrm{d}x - u(x_2)| \leq \frac{2\sqrt{d}}{\mu}\rho^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \leq \frac{2\sqrt{d}}{\mu}|x_1-x_2|^\mu [\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)}.$$

□

*Remark.* It is sufficient when $u \in \mathrm{C}^1_0(\mathbb{R}^d)$.

Gagliardo had in fact two lemmas, so let us even the game for Morrey.

**Lemma 8.** *Let $p \in (d,\infty)$, and let $\mu = 1 - \frac{d}{p}$. Then $\forall u \in \mathcal{D}(\mathbb{R}^d)$ it holds*

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu}\right)\|u\|_{W^{1,p}(\mathbb{R}^d)},$$

*where*

$$\|u\|_{C^{0,\mu}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}|u(x)| + \sup_{x,y\in\mathbb{R}^d, x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\mu}.$$

*Proof.* We prove the assertion by estimating both terms in the above norm. To obtain those specific constants, we will pay some more attention to our procceeding. Let us also state the trivial: $\mu = 1 - \frac{d}{p} \in (0,1)$ for $p \in (d,\infty)$.

Begin with the differences: choose an arbitrary $\rho > 0$ and compute

$$\int_{[0,\rho]^d}\frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\,\mathrm{d}y \leq \left(\int_{[0,\rho]^d}\left(\frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}}\right)^p \mathrm{d}y\right)^{\frac{1}{p}}\left(\lambda\left([0,\rho]^d\right)\right)^{\frac{p-1}{p}} = \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}\frac{\rho^{\frac{d(p-1)}{p}}}{\rho^{d-1+\mu}} = \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

because $\frac{d(p-1)}{p} - d + 1 - \mu = \frac{dp-d}{p} - d + 1 - 1 + \frac{d}{p} = \frac{dp}{p} - d = 0$. Taking the suprema yields

$$[\nabla u]_{\mathrm{L}_{1,\mu}(\mathbb{R}^d)} \leq \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

Going with this into the first Morrey lemma, we see

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu}|x_1 - x_2|^\mu \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

meaning

$$\sup_{x_1,x_2\in\mathbb{R}^d, x_1\neq x_2}\frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\mu} \leq \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

To estimate the infinity norm, we can actually exploit the above result aswell: pick $x \neq y \in \mathbb{R}^d$ and write

$$|u(x)| - |u(y)| \leq |u(x) - u(y)| \leq \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

and so

$$|u(x)| \leq |u(y)| + \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

Now fix $\rho \geq 2|x-y| > 0$, integrate both sides w.r.t $y$ over $[0,\rho]^d$ and obtain

$$|u(x)|\rho^d \leq \int_{[0,\rho]^d} |u(y)|\,\mathrm{d}y + \frac{2\sqrt{d}}{\mu}\rho^d \|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)},$$

which upon using Holder in the integral becomes

$$|u(x)|\rho^d \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)}\rho^{\frac{d(p-1)}{p}} + \frac{2\sqrt{d}}{\mu}\rho^d \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}.$$

Since we have lost $y$, we can in fact choose $\rho = 1$, and upon taking the supremum write

$$\sup_{x\in\mathbb{R}^d} |u(x)| \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{2\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)},$$

and so in total

$$\|u\|_{\mathrm{C}^{0,\mu}(\mathbb{R}^d)} \leq \|u\|_{\mathrm{L_p}(\mathbb{R}^d)} + \frac{4\sqrt{d}}{\mu}\|\nabla u\|_{\mathrm{L_p}(\mathbb{R}^d)} = \left(1 + \frac{4\sqrt{d}}{\mu}\right)\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

$\square$

*Remark.* Since $\mathcal{D}\left(\mathbb{R}^d\right)$ is dense in $\mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$, the above lemma holds also for $u \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$. We have to be careful to pick a good representant though.

**Theorem 10** (Embedding theorems for $p > d$)**.** *Let $\Omega \in C^{0,1}, d \in \mathbb{N}, p > d$, i.e., $p \in (d,\infty]$. Denote $\mu^* = 1 - \frac{d}{p} \in (0,1)$. Then*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*],$$

*and*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*).$$

*Proof. Ad continuous:*

Since $\Omega \in \mathrm{C}^{0,1}$, we are able to use the extension theorem; recall $\forall u \in \mathrm{W}^{1,p}(\Omega) : \operatorname{supp} Eu \subset V$, where $\overline{\Omega} \subset V$. Let us deal with the case $p \in (d,\infty)$ first. We have shown

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} \leq \|u\|_{\mathrm{C}^{0,\mu^*}(\mathbb{R}^d)} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}, \forall u \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right).$$

Realize that in fact

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})},$$

as $\partial\Omega$ is a set of zero Lebesgue measure and so in fact $Eu = u$ on $\overline{\Omega}$, which together with the obvious fact $Eu \in \mathrm{W}^{1,p}\left(\mathbb{R}^d\right)$ gives

$$\|u\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} = \|Eu\|_{\mathrm{C}^{0,\mu^*}(\overline{\Omega})} \leq \left(1 + \frac{4\sqrt{d}}{\mu^*}\right)\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\Omega)}.$$

This exactly means

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right).$$

Realize also that

$$\mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*],$$

and so

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*].$$

If now $p = \infty$, realize that (by embedding of Lebesgue spaces)

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{W}^{1,q}(\Omega), \forall q \in [1,\infty).$$

From the previous result it follows

$$\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in \left[0, 1-\frac{d}{q}\right],$$

and notice that $1-\frac{d}{q} \to 1$ as $q \to \infty$. This means that $\forall q \in [1,\infty)$ it holds

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, 1-\frac{d}{q}].$$

Since $q \in [1,\infty)$ was arbitrary, we conclude it must be

$$\mathrm{W}^{1,\infty}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,1].$$

[17]

*Ad compactness* This will be a bit of a cheating: we know

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\beta}\left(\overline{\Omega}\right), \forall \beta \in [0,\mu^*],$$

and for Holder spaces it also holds

$$\mathrm{C}^{0,\beta}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\beta),$$

which means if we choose $\beta = \mu^*$, we in fact obtain

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\mu^*}\left(\overline{\Omega}\right) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*).$$

Using the same arguments as in the case of the compact embedding for $p < d$, we can conclude

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu^*).$$

We are done. $\square$

[17] The norm

$$\|u\|_{\mathrm{C}^{0,1-\frac{d}{q}}\left(\overline{\Omega}\right)} = \sup_{x\in\overline{\Omega}}|u(x)| + \sup_{x_1\neq x_2\in\overline{\Omega}} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^{1-\frac{d}{q}}},$$

actually allows one to pass to the limit $q \to \infty$, since the supremum is independent of the exponent in the denonimanotor, the function on the RHS is continuous. So if we know $\|u\|_{\mathrm{C}^{0,1-\frac{d}{q}}\left(\overline{\Omega}\right)} \le C\|u\|_{\mathrm{W}^{1,\infty}(\Omega)}, \forall q \in [1,\infty)$, we can pass to the limit on the LHS and obtain $\|u\|_{\mathrm{C}^{0,1}\left(\overline{\Omega}\right)} \le C\|u\|_{\mathrm{W}^{1,\infty}(\Omega)}$, which is the only missing possibility.

*Remark.* Note that in the case of $p > d$, from

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$

and[18]

$$\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right) \hookrightarrow \mathrm{L}_\infty(\Omega), \forall \alpha \in [0, 1]$$

it follows[19]

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega), \forall p > d.$$

But that of course means ($\Omega$ is bounded) that

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty],$$

whenever $p > d$.

*Remark* (Summary). Let us summarize the obtained results. If $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, then upon denoting

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p},$$

it holds

- if $p < d$, then

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*],$$

  and

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, p^*),$$

- if $p = d$, then

  $$\mathrm{W}^{1,d}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty)$$

  and

  $$\mathrm{W}^{1,d}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega), \forall q \in [1, \infty),$$

- if $p > d$, then

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*],$$

  and

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*),$$

  which also imply

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty(\Omega),$$

  and

  $$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\infty(\Omega).$$

[18] Clearly,

$$\|u\|_{\mathrm{L}_\infty(\Omega)} \le \|u\|_{\mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right)} = \|u\|_{\mathrm{L}_\infty(\Omega)} + \sup_{x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}.$$

[19] Again, composition of a compact and continuous (linear) operators yields a compact operator independently of the order.

*Remark* (Summary - embedding into Lebesgue spaces). It can be guiding to look only at the embeddings into some Lebesgue spaces. Let $d \in \mathbb{N}, \Omega \in \mathrm{C}^{0,1}$, denote

$$p^* = \frac{dp}{d-p}, \mu^* = 1 - \frac{d}{p}.$$

Then it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_q(\Omega),$$

where $q$ is given:

- if $p < d$, then $q \in [1, p^*]$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

Also, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_q(\Omega),$$

where $q$ is given as

- if $p < d$, then $q \in [1, p^*)$,
- if $p = d$, then $q \in [1, \infty)$,
- if $p > d$, then $q \in [1, \infty]$.

## 2.5 Trace theorems

There are many many troubles with the boundary. Another one we have yet not encountered arises with *e.g.* homogenous Dirichlet boundary coniditons: it should hold

$$u = 0 \,\mathrm{on}\, \partial\Omega,$$

but typically, $u$ is an element of some Sobolev space on $\Omega$, and $\lambda_d(\partial\Omega) = 0$. We cannot sensibly talk about pointwise values on a set of measure zero, they can be arbitrary. This can be dealt with provided - as expected - when $\Omega$, *i.e.*, $\partial\Omega$ is benevolent enough.

Realize also that for $\Omega$ at least $\mathrm{C}^{0,0}$ we also have the density of $\mathrm{C}^{\infty}_{\overline{\Omega}}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\Omega)$, meaning the values on $\partial\Omega$ are well defined at least for a dense subset of $\mathrm{W}^{1,p}(\Omega)$. It should not be too difficult to extend to the whole $\mathrm{W}^{1,p}(\Omega)$, right...?

Moreover, in the case $p > d, \Omega \in \mathrm{C}^{0,1}$ we already know $\forall u \in \mathrm{W}^{1,p}(\Omega)$ there exists $u^* \in \mathrm{C}^0(\overline{\Omega})$, such that $u = u^*$ *a.e.* in $\Omega$, and so in these cases, the values $u^* \restriction_{\partial\Omega}$ are well defined. What if $p \leq d$?

*Remark* (The space $\mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega)$). In the winter semester, we have defined

$$\mathrm{L_p}(\partial\Omega) \equiv \mathrm{L_p}(\mathcal{H}_{d-1}, \partial\Omega),$$

*i.e.*, the lebesgue spaces are taken w.r.t the $d-1$ dimensional (normalized complete) Hausdorff measure $\mathcal{H}_{d-1}$.

**Theorem 11** (Continuous trace theorem). *Let $\Omega \in C^{0,1}, p \in [1, \infty]$, denote $p^{\#} = \frac{dp-p}{d-p}$. Then there is a continuous linear operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L_q(\partial\Omega)$, *with $q$ being*

- *if* $p < d$, *then* $q \in [1, p^{\#}]$,
- *if* $p = d$, *then* $q \in [1, \infty)$,
- *if* $p > d$, *then* $q \in [1, \infty]$.

*Moreover,* $\forall u \in C^{\infty}\left(\overline{\Omega}\right)$ *it holds*

$$\operatorname{tr} u = u \restriction_{\partial\Omega},$$

*meaning* $\operatorname{tr} : W^{1,p}(\Omega) \to L_p(\partial\Omega)$ *is an extension of* $\tilde{\operatorname{tr}} : C^{\infty}\left(\overline{\Omega}\right) \to C^0(\partial\Omega)$.

*Proof.* The strategy is the following

1. define $\tilde{\operatorname{tr}}$ for smooth functions,
2. obtain estimates for $\tilde{\operatorname{tr}}$ using embedding theorems,
3. extend $\tilde{\operatorname{tr}}$ to the whole space, which defines tr.

*Case* $p < d$*:*
As we have mentioned, the case for functions smooth up to the boundary is evident. Let us so define $\tilde{\operatorname{tr}} : \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right) \to \mathrm{C}^0(\partial\Omega)$, by

$$\tilde{\operatorname{tr}} u = u \restriction_{\partial\Omega} .$$

Then clearly $\tilde{\operatorname{tr}}$ is linear continuous operator.

(We are using the notation from the definition of a $\mathrm{C}^{0,1}$ domain). Let us for clarity define (and also recall)

$$\begin{aligned} G_j &= \mathbb{A}_j(\{(x', a_j(x')|x' \in \mathrm{U}(0,\alpha)\}), \\ G_j^+ &= \mathbb{A}_j(\{x', a_j(x') + b|x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}), \\ G_j^- &= \mathbb{A}_j(\{x', a_j(x') - b|x' \in \mathrm{U}(0,\alpha), b \in (0,\beta)\}). \end{aligned}$$

Within this notation, $G_j \subset \partial\Omega, G_j^+ \subset \Omega, G_j^- \subset \mathbb{R}^d/\overline{\Omega}$ and $U_j = G_j \cup G_j^+ \cup G_j^-$. Moreover, $\{U_j\}_{j=1}^m$ are open sets *s.t.* $\partial\Omega \subset \bigcup_{j=1}^m U_j$. Denote $\{\varphi_j\}_{j=1}^m \subset \mathcal{D}\left(\mathbb{R}^d\right)$ to be the partition of unity subordinate to this (open) covering. Realize moreover that since $\mathcal{H}_{d-1}(\partial\Omega) < \infty$, it holds

$$\mathrm{L}_{p^{\#}}(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega), \forall q \in [1, p^{\#}].$$

So if we are able to show

$$\|u\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \leq C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

we have the rest of the estimates for free.

Take $u \in \mathrm{C}^{\infty}_{\overline{\Omega}}\left(\mathbb{R}^d\right)$, and denote $u_j = u\varphi_j, j \in \{1, \ldots, m\}$. Let for the moment $p > 1$, so $p^{\#} = \frac{dp-p}{d-p} > \frac{d-1}{d-1} = 1$. As $u_j \in \mathrm{C}^{\infty}\left(\mathbb{R}^d\right)$, it holds $u_j \in \mathrm{C}^1\left(\overline{G_j^+}\right)$. Moreover, since $\operatorname{supp} u_j \subset U_j$ and as $U_j \cap \mathbb{A}_j(\{(x', a_j(x') + b|x' \in \mathrm{U}(0,\alpha), b \in [0,\infty)\}) = \emptyset$, it holds

$$u_j(\mathbb{A}_j(x', a_j(x') + \beta)) = 0.$$

With those qualities, if we denote (for an arbitrary $j \in \{1, \dots, m\}$; this will become $\|u_j\|_{\mathrm{L}_{p^\#}(\partial\Omega)}$)

$$v(x') = |u_j(\mathbb{A}_j(x', a_j(x')))|^{p^\#} = |u_j(\mathbb{A}_j(x', a_j(x')))|^{\frac{dp-p}{d-p}},$$

we can write (recall $\mathbb{A}_j$ is orthogonal and $x' \in \mathrm{U}(0,\alpha)$)

$$\begin{aligned}|v(x')| &= \left| |u_j(\mathbb{A}_j(x', a_j(x')))|^{\frac{dp-p}{d-p}} - |u_j(\mathbb{A}_j(x', a_j(x') + \beta))|^{\frac{dp-p}{d-p}} \right| \le \left| \int_{a_j(x')+\beta}^{a_j(x')} \frac{\partial}{\partial s} |u_j(\mathbb{A}_j(x', s))| \,\mathrm{d}s \right| = \\ &= \left| \int_{a_j(x')+\beta}^{a_j(x')} p^\# |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} \operatorname{sign}(u_j(\mathbb{A}_j(x', s))) \nabla u_j(\mathbb{A}_j(x', s)) \cdot \mathbb{A}_j(x', 1) \,\mathrm{d}s \right| \le \\ &\le p^\# \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(\mathbb{A}_j(x', s))| \underbrace{|\mathbb{A}_j(x', 1)|}_{=|(x',1)|} \,\mathrm{d}s \le \\ &\le p^\# \sqrt{1+\alpha^2} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(x', s)| \,\mathrm{d}s\,.\end{aligned}$$

Integrate this inequality over $\mathrm{U}(0,\alpha)$ and write (recall the definition of $G_j^+$, we are using Fubini)

$$\begin{aligned}\int_{\mathrm{U}(0,\alpha)} |v(x')| \,\mathrm{d}x' &\le p^\# \sqrt{1+\alpha^2} \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))|^{\frac{dp-d}{d-p}} |\nabla u_j(\mathbb{A}_j(x', s))| \,\mathrm{d}s \,\mathrm{d}x' \le \\ &\le p^\# \sqrt{1+\alpha^2} \int_{\mathbb{A}_j(\{(x', a_j(x')+b) | x' \in \mathrm{U}(0,\alpha), b \in [0,\beta]\})} |u_j|^{\frac{dp-d}{d-p}} |\nabla u_j(x)| \,\mathrm{d}x = \\ &= p^\# \sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{\frac{dp-d}{d-p}} |\nabla u(x)| \,\mathrm{d}x \le p^\# \sqrt{1+\alpha^2} \|\nabla u\|_{\mathrm{L}_p\left(G_j^+\right)} \left\| |u_j|^{\frac{dp-d}{d-p}} \right\|_{\mathrm{L}_{p'}\left(G_j^+\right)},\end{aligned}$$

and since $\frac{dp-d}{d-p} p' = \frac{dp-d}{d-p} \frac{p}{p-1} = \frac{dp}{d-p} = p^*$, we have

$$\int_{\mathrm{U}(0,\alpha)} |v(x')| \,\mathrm{d}x' = p^\# \sqrt{1+\alpha^2} \|\nabla u\|_{\mathrm{L}_p\left(G_j^+\right)} \|u_j\|_{\mathrm{L}_{p^*}\left(G_j^+\right)}^{\frac{dp-d}{d-p}}.$$

Since $G_j^+ \in \mathrm{C}^{0,1}$, the last term can be estimated using the continuous embedding theorems:

$$\|u_j\|_{\mathrm{L}_{p^*}\left(G_j^+\right)}^{\frac{dp-d}{d-p}} \le C \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d}{d-p}},$$

whereas the integral on the LHS actually is

$$\int_{\mathrm{U}(0,\alpha)} |v(x')| \,\mathrm{d}x' = \int_{\mathrm{U}(0,\alpha)} |u_j(\mathbb{A}_j(x', a_j(x')))|^{p^\#} \,\mathrm{d}x = \int_{\mathbb{A}_j(\{(x', a_j(x')) | x' \in \mathrm{U}(0,\alpha)\})} |u_j(x)|^{p^\#} \,\mathrm{d}x = \|u_j\|_{\mathrm{L}_{p^\#}(G_j)}^{p^\#},$$

and so we write

$$\|u_j\|_{\mathrm{L}_{p^\#}(G_j)}^{p^\#} \le C \|\nabla u_j\|_{\mathrm{L}_p\left(G_j^+\right)} \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d}{d-p}} \ge C \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{\frac{dp-d+d-p}{d-p}} = C \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}^{p^\#},$$

and so

$$\|u_j\|_{\mathrm{L}_{p^\#}(G_j)} \le C \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)}.$$

This has been done for $p > 1$, but in fact taking the limit $p \to 1^+$ is allowed here (without a proof). Hence the above estimate holds $\forall p \in [1, d)$.

The estimates have so far been local - let us glue them together. Recall $\partial\Omega = \bigcup_{j=1}^m G_j, G_j^+ \subset \Omega$, and so

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} = \left\|\sum_{j=1}^m u_j\right\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_{p^\#}(\partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(\operatorname{supp} u_j \cap \partial\Omega)} = \sum_{j=1}^m \|u_j\|_{\mathrm{L}_p(G_j)} \le$$

$$\le C\sum_{j=1}^m \|u_j\|_{\mathrm{W}^{1,p}\left(G_j^+\right)} \le C\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where we have used the fact $0 \le \varphi_j \le 1$ in the last inequality. And so we have shown

$$\|u\|_{\mathrm{L}_{p^\#}(\partial\Omega)} \le \|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall u \in \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right).$$

Now let $u \in \mathrm{W}^{1,p}(\Omega)$ be arbitrary. Since $\Omega \in \mathrm{C}^{0,1}$, there $\exists\{u_k\} \subset \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right) s.t.\, u_k \to u \,\mathrm{in}\, \mathrm{W}^{1,p}(\Omega)$. Set

$$\operatorname{tr} u = \lim_{k\to\infty} \tilde{\operatorname{tr}}\, u_k.$$

The definition is sensible, as $\tilde{\operatorname{tr}}$ is continuous and $\{u_k\}$ converges. Also, from the arithmetic of the limits and linearity of $\tilde{\operatorname{tr}}$ we see tr is linear. Next, check

$$\|\operatorname{tr} u\|_{\mathrm{L}_q(\partial\Omega)} = \left\|\lim_{k\to\infty} \tilde{\operatorname{tr}} u_k\right\|_{\mathrm{L}_q(\partial\Omega)} = \lim_{k\to\infty} \|u_k \restriction_{\partial\Omega}\|_{\mathrm{L}_q(\partial\Omega)} \le C \lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\Omega)} = C\|u\|_{\mathrm{W}^{1,p}(\Omega)}, \forall q \in [1, p^\#].$$

and so inded $\operatorname{tr} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, p^\#]$ and it surely is bounded.

*Case $p = d$*

In this case, we have $\mathrm{id} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega)\,, \mathrm{W}^{1,r}(\Omega)), \forall r \in [1, d)$ and the previous result tells us $\operatorname{tr} \in \mathcal{L}\left(\mathrm{W}^{1,r}(\Omega)\,, \mathrm{L}_q(\partial\Omega)\right), \forall q \in [1, r^\#]$. Observe that $r^\# = \frac{dr-r}{d-r} \to \infty$ as $r \to d^-$, meaning $\forall q \in [1, \infty)$ there exists $r \in [1, d)\; s.t.\, r^\# > q$, $i.e.\, q \in [1, r^\#)$, which in fact means

$$\operatorname{tr} \in \mathcal{L}\left(\mathrm{W}^{1,r}(\Omega)\,, \mathrm{L}_q(\partial\Omega)\right), \forall r \in [1, d), \forall q \in [1, \infty).$$

But then $\operatorname{tr} \circ \mathrm{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is a continous linear operator $\forall q \in [1, \infty)$, as it as a composition of continuous linear operators.

*Case $p > d$*

This is the easiest case: we know that $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0, \mu^*]$, so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{C}^0\left(\overline{\Omega}\right) \subset \mathrm{C}^0\left(\partial\Omega\right) \subset \mathrm{L}_\infty\left(\partial\Omega\right),$$

and so in total

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}_\infty\left(\partial\Omega\right),$$

which together with the fact $(\mathcal{H}_{d-1}(\partial\Omega) < \infty)$

$$\mathrm{L}_\infty\left(\partial\Omega\right) \hookrightarrow \mathrm{L}_q(\partial\Omega)\,, \forall q \in [1, \infty),$$

concludes the proof. □

*Remark.* These results are very similiar to the results obtained in the case of embedding theorems - and it is not surprising, since it might seem we have in fact shown the embeddings of some Sobolev spaces into some Lebesgue spaces. One needs to be careful however, as it *is not an embedding* - we are taking $\Omega$ open, so $\partial\Omega \notin \Omega$. It only makes sense in the case of $p > d$.

The chapter will be concluded by stating the compact analogue to the embedding theorems.

**Theorem 12** (Compact trace theorem)**.** *Let $d \in \mathbb{N}, \Omega \in C^{0,1}$, denote $p^{\#} = \frac{dp-p}{d-p}$, and let* tr *be the trace operator from the previous theorem. Then*

$$\operatorname{tr} \in \mathcal{K}\Big( W^{1,p}(\Omega)\,, L_q(\partial\Omega)\Big),$$

*where $q$ is*

- *if $p < d$, then $q \in [1, p^{\#})$,*
- *if $p = d$, then $q \in [1, \infty)$,*
- *if $p > d$, then $q \in [1, \infty]$,*

*Proof. Case $p < d$:* Let us adopt the custom that when talking about the properties of $u \in \mathrm{W}^{1,p}(\Omega)$, in the space $\mathrm{L}_q(\partial\Omega)$, we are always talking about the properties of $\operatorname{tr} u$ in $\mathrm{L}_q(\partial\Omega)$...

It will be pretty similiar to the continuous case, so let us skip only to the key estimate. We are not apriori sure in which space we will be able to comfortably show the compactness, so let first $q \in [1, \infty)$ and compute. Then we might use some interpolation estimates...

$$\begin{aligned}\int_{\mathrm{U}(0,\alpha)} |u_j(\mathbb{A}_j(x', a_j(x')))|^q \,\mathrm{d}x' &\le \left| \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} \frac{\partial}{\partial s} |u_j(\mathbb{A}_j(x', s))|^q \,\mathrm{d}s\,\mathrm{d}x' \right| \le \\ &\le q\sqrt{1+\alpha^2} \int_{\mathrm{U}(0,\alpha)} \int_{a_j(x')}^{a_j(x')+\beta} |u_j(\mathbb{A}_j(x', s))| |\nabla u_j(\mathbb{A}_j(x', s))| \,\mathrm{d}s\,\mathrm{d}x' \le \\ &\le q\sqrt{1+\alpha^2} \int_{G_j^+} |u_j(x)|^{q-1} |\nabla u_j(x)| \,\mathrm{d}x \le q\sqrt{1+\alpha^2} \|\nabla u_j\|_{\mathrm{L}_q\left(G_j^+\right)} \|u_j\|^{q-1}_{\mathrm{L}_{q'(q-1)}\left(G_j^+\right)},\end{aligned}$$

where $q'(q-1) = \frac{q}{q-1}(q-1) = q$, so all in all

$$\|u_j\|_{\mathrm{L}_q(G_j)} \le C(q,\Omega) \|\nabla u_j\|^{\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \le C(q,\Omega) \|u_j\|^{\frac{1}{q}}_{\mathrm{W}^{1,q}\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)}$$

which leads to

$$\begin{aligned}\|u\|_{\mathrm{L}_q(\partial\Omega)} = \left\| \sum_{j=1}^m u_j \right\|_{\mathrm{L}_q(\partial\Omega)} &\le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(\operatorname{supp} u_j \cap \partial\Omega)} \le \sum_{j=1}^m \|u_j\|_{\mathrm{L}_q(G_j)} \le \\ &\le C \sum_{j=1} \|u_j\|^{\frac{1}{q}}_{\mathrm{W}^{1,q}\left(G_j^+\right)} \|u_j\|^{1-\frac{1}{q}}_{\mathrm{L}_q\left(G_j^+\right)} \le C_1 \|u\|^{\frac{1}{q}}_{\mathrm{W}^{1,p}(\Omega)} \|u\|^{1-\frac{1}{q}}_{\mathrm{L}_q(\Omega)}.\end{aligned}$$

Denote $U = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$; we will show $\operatorname{tr} U$ is totally bounded in $\mathrm{L}_q(\partial\Omega)$ for *some* $q \in [1, \infty)$. Let $\varepsilon > 0$ be given. Realize that $\forall p \in [1, d)$ it is $p < p^*$, and so it always holds $\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_\mathrm{p}(\Omega)$. For the moment, we pick $q = p$. Denote $\{u_k\}_{k=1}^m$ to be the $\delta$–net from $U$ in $\mathrm{L}_\mathrm{p}(\Omega)$, where $\delta$ will be chosen suitably later. Let now $u \in U$ be arbitrary and find $u_i \in U$ *s.t.* $\|u - u_i\|_{\mathrm{L_p}(\Omega)} \le \delta$. Using the estimate from above we have

$$\|u - u_i\|_{\mathrm{L}_p(\partial\Omega)} \le C_1 \|u - u_i\|^{\frac{1}{q}}_{\mathrm{W}^{1,p}(\Omega)} \|u - u_i\|^{1-\frac{1}{q}}_{\mathrm{L}_p(\Omega)} \le C_1 2^{\frac{1}{q}} \delta^{1-\frac{1}{q}},$$

where we have used the fact $u, u_i \in U$. We see that upon choosing

$$\delta < \left( \frac{\varepsilon}{C_1 2^{\frac{1}{q}}} \right)^{\frac{1}{1-q}},$$

we in fact have

$$\|u - u_i\|_{\mathrm{L_p}(\partial\Omega)} < \varepsilon,$$

and so $\{\operatorname{tr} u_i\}_{i=1}^m \subset U$ is a $\varepsilon$–net in $\mathrm{L_p}(\partial\Omega)$, meaning

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L_p}(\partial\Omega)\big).$$

Since now $\mathrm{L}_p(\partial\Omega) \hookrightarrow \mathrm{L}_q(\partial\Omega), \forall q \in [1,p]$, we have also shown

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1,p].$$

The remaining case is when $q \in (p, p^{\#})$. As in the case of compact embedding of Sobolev spaces, a suitable interpolation theorem will do the job for us. It holds

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \le \|u\|^{\theta}_{\mathrm{L}_p(\partial\Omega)} \|u\|^{1-\theta}_{\mathrm{L}_{p^{\#}}(\partial\Omega)},$$

where $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^{\#}}$. Let now $\{u_i\}_{i=1}^m \subset U$ be such that $\{\operatorname{tr} u_i\}_{i=1}^m$ is the $\beta$–net in $\mathrm{L_p}(\partial\Omega)$ whose existence we have just proven $\forall \beta > 0$. Recall also that since $U$ is bounded and $\operatorname{tr} \in \mathcal{L}\big(\mathrm{W}^{1,p}(\Omega), \mathrm{L}_q(\partial\Omega)\big) \forall q \in [1, p^{\#}]$, there exists $0 < C_2 < \infty$ *s.t.*

$$\|u\|_{\mathrm{L}_q(\partial\Omega)} \le C_2, \forall u \in U.$$

and so $\|u\|_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \le C_2, \forall u \in U$ in particular.

Finally, $\forall \in U$ it holds

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \le \|u - u_i\|^{\theta}_{\mathrm{L_p}(\partial\Omega)} \|u - u_i\|^{1-\theta}_{\mathrm{L}_{p^{\#}}(\partial\Omega)} \le \beta^{\theta} (2C_2)^{1-\theta},$$

so if we choose

$$\beta < \left( \frac{\varepsilon}{(2C_2)^{1-\theta}} \right)^{\frac{1}{\theta}},$$

then

$$\|u - u_i\|_{\mathrm{L}_q(\partial\Omega)} \le \varepsilon, \forall u \in U.$$

which concludes the proof $\{\operatorname{tr} u_i\}_{i=1}^m$ is an $\varepsilon$–net in $U$ in $\mathrm{L}_q(\partial\Omega)$ also for $q \in (p, p^{\#})$. In total, we have showed the compactness of the trace operator for all $q \in [1, p^{\#})$.

*Case* $p = d$

In this case $\operatorname{tr} \in \mathcal{L}(\mathrm{W}^{1,d}(\Omega), \mathrm{L}_q(\partial\Omega)), \forall q \in [1,\infty)$. We also know $\mathrm{id} \in \mathcal{L}\big(\mathrm{W}^{1,d}(\Omega), \mathrm{W}^{1,r}(\Omega)\big), \forall r \in [1,d)$ and that

$$\operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big), \forall q \in [1, r^{\#}).$$

Repating the same arguments, we see $r^{\#} \to \infty$ as $r \to d^{-}$, meaning $\forall q \in [1,\infty)$ there exists $r \in [1,d)$ *s.t.* $r^{\#} > q$, and consequently this implies

$$\forall q \in [1,\infty) \exists r \in [1,d) \; s.t. \; \operatorname{tr} \in \mathcal{K}\big(\mathrm{W}^{1,r}(\Omega), \mathrm{L}_q(\partial\Omega)\big).$$

But then the operator $\operatorname{tr} \circ \mathrm{id} : \mathrm{W}^{1,d}(\Omega) \to \mathrm{L}_q(\partial\Omega)$ is compact $\forall q \in [1,\infty)$, as it is a composition of a (linear) continuous and compact operator.

*Case* $p > d$

This case is again trivial: from the embedding theorems, it holds

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0,\alpha}\left(\overline{\Omega}\right), \forall \alpha \in [0,\mu),$$

so in particular

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{C}^{0}\left(\overline{\Omega}\right) \subset \mathrm{C}^{0}\left(\partial\Omega\right) \subset \mathrm{L}_{\infty}\left(\partial\Omega\right),$$

meaning

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_{\infty}\left(\partial\Omega\right),$$

which together with the embedding

$$\mathrm{L}_q(\partial\Omega) \hookrightarrow \mathrm{L}_{\infty}\left(\partial\Omega\right), \forall q \in [1,\infty).$$

completes the proof for any $q \in [1,\infty]$. We are done once again.

□

## 2.6 Composition of Sobolev functions

In the chapter about extension of functions from Sobolev spaces we have seen: $U, V \subset \mathbb{R}^d$ open, $u \in \mathrm{W}^{1,p}(U), \Phi : U \to V$ a $\mathrm{C}^1$-diffemorphism $\Rightarrow u \circ \Phi \in \mathrm{W}^{1,p}(V)$. Is there any other class of mappings that guarantee the composition remains in some Sobolev spaces?

**Theorem 13** (Derivative of superposition)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $p \in [1,\infty]$, $u : \Omega \to \mathbb{R}^d$ be in $W^{1,p}(\Omega)$. Denote for an arbitrary $a \in \mathbb{R}$ the set*

$$\Omega_a = \{x \in \Omega | u(x) = a\}.$$

*Then*

1. $\nabla u = 0$ *on* $\Omega_a$,
2. *if $f \in C^{0,1}\left(\mathbb{R}^d\right)$ s.t. $\|f\|_{L_\infty(\Omega)} < \infty$ (it is a Lipschitz continuous function), then $f \circ u - f(0) \in W^{1,p}(\Omega)$ and it holds*

$$\nabla(f \circ u) = \begin{cases} (f' \circ u)\nabla u, & a.e. \; in \{u \notin S\}, \\ 0, & a.e. \; in \{u \in S\} \end{cases},$$

*where*

$$S = \{s \in \mathbb{R} | f'(s) \; does \; not \; exist\}$$

*(in the strong sense).*

*Proof.* The proof has been presented for the case $f \in \mathrm{C}^1(\mathbb{R})$, $\|f\|_{\mathrm{L}_\infty(\mathbb{R})} < \infty$. In the general setting, one can play with *a.e.* convergence and Rademacher theorem.

Denote $L = \|f\|_{\mathrm{L}_\infty(\mathbb{R})}$. Since $f$ is continuous, $f \circ u$ is measurable. Moreover, from the Lipschitzity on the whole $\mathbb{R}$ it follows

$$|(f \circ u)(x) - f(0)| \leq L|u(x) - 0| = L|u(x)|,$$

and since $u \in \mathrm{L}_\mathrm{p}(\Omega)$, also $f \circ u - f(0) \in \mathrm{L}_\mathrm{p}(\Omega)$. To obtain the wanted result, we need to show $\nabla(f \circ u)$ exists, so let us move on to the second claim. The strategy is to work with some regularization and then pass to the limit with the support size. In concrete terms, pick $\varphi \in \mathcal{D}(\Omega)$, and regularization $u_\varepsilon$ *s.t.* $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > 2\varepsilon$. Then we can write

$$-\int_\Omega (f \circ u_\varepsilon)(x)\partial_i\varphi(x)\,\mathrm{d}x = \int_\Omega (f' \circ u_\varepsilon)(x)\partial_i u_\varepsilon(x)\varphi\,\mathrm{d}x\,,$$

using per partes and chain rules (those are strong derivatives). Now we would like to pass to the limit $\varepsilon \to 0^+$. Take the LHS:

$$\lim_{\varepsilon\to 0^+} \Big| \int_\Omega (f \circ u_\varepsilon - f \circ u)\partial_i\varphi\,\mathrm{d}x \Big| \leq \lim_{\varepsilon\to 0^+} \int_\Omega |f \circ u_\varepsilon - f \circ u||\partial_i\varphi|\,\mathrm{d}x \leq L\|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\operatorname{supp}\varphi} |u_\varepsilon - u|\,\mathrm{d}x = 0,$$

as $u_\varepsilon \to u$ in $\mathrm{L}_1(\Omega \cap \operatorname{supp}\varphi)$ (notice from the theorem on mollification we actually obtain this convergence in $\mathrm{L}_1(\Omega_\varepsilon)$, but we have made a good choice of $\varphi$) The RHS can be manipulated

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \left| \int_\Omega (f' \circ u_\varepsilon)\partial_i u_\varepsilon\varphi - (f' \circ u)\partial_i u\varphi\,\mathrm{d}x \right| \leq \lim_{\varepsilon\to 0^+} \int_\Omega |(f' \circ u_\varepsilon - f' \circ u)||\partial_i u||\varphi|\,\mathrm{d}x + \lim_{\varepsilon\to 0^+} \int_\Omega |f' \circ u_\varepsilon||\partial_i u_\varepsilon - \partial_i u||\varphi|\,\mathrm{d}x \leq \\
&\leq \|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\operatorname{supp}\varphi} |f' \circ u_\varepsilon - f' \circ u||\partial_i u|\,\mathrm{d}x + L\|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_{\Omega\cap\operatorname{supp}\varphi} |\partial_i u_\varepsilon - \partial_i u|\,\mathrm{d}x = \\
&= \|\varphi\|_{\mathrm{L}_\infty(\Omega)} \lim_{\varepsilon\to 0^+} \int_\Omega |f' \circ u_\varepsilon - f' \circ u||\partial_i u|\,\mathrm{d}x\,,
\end{aligned}$$

where we have used $\nabla u_\varepsilon \to \nabla u$ in $\mathrm{L}_1(\Omega \cap \operatorname{supp}\varphi)$. As for the second integral, recall that $u_\varepsilon \to u$ *a.e.* in $\Omega \cap \operatorname{supp}\varphi$ and that $f'$ is globally continous. Thus with the majorant

$$|f' \circ u_\varepsilon - f' \circ u||\partial_i u| \leq L\|u\|_{\mathrm{W}^{1,p}(\Omega)}|u_\varepsilon - u|,$$

that is integrable on the bounded set $\operatorname{supp}\varphi \cap \Omega$ we see the second integral is zero aswell. All in all, we have shown

$$-\int_\Omega (f \circ u)\partial_i\varphi\,\mathrm{d}x = \int_\Omega (f' \circ u)\partial_i u\varphi\,\mathrm{d}x\,, \forall\varphi \in \mathcal{D}(\Omega)\,,$$

which is exactly what we need to state that the $\partial_i$-weak derivative of $f \circ u$ exists and is equal to $(f' \circ u)\partial_i u$. Finally, the following estimate holds:

$$\Bigg\| \underbrace{\nabla(f \circ u)}_{=\nabla(f\circ u - f(0))} \Bigg\|_{\mathrm{L}_\mathrm{p}(\Omega)} = \|(f' \circ u)\nabla u\|_{\mathrm{L}_\mathrm{p}(\Omega)} \leq L\|\nabla u\|_{\mathrm{L}_\mathrm{p}(\Omega)} \leq L\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

and so

$$f \circ u - f(0) \in \mathrm{W}^{1,p}(\Omega)\,.$$

Let us now deal with the first assertion. We will first show $\nabla u = 0$ *a.e.* on $\Omega_0$. For that, choose a special function

$$f(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and so $f \circ u = u^+$. If we now set

$$f_\varepsilon(x) = \begin{cases} \left(x^2 + \varepsilon^2\right)^{1/2} - \varepsilon, & x > 0, \\ 0, & x \le 0, \end{cases}$$

with the derivative being

$$f_\varepsilon'(x) = \begin{cases} \frac{x}{(x^2+\varepsilon^2)^{1/2}}, & x > 0, \\ 0, x \le 0 \end{cases}$$

and so $\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = f(x)$, and $\lim_{\varepsilon \to 0^+} f_\varepsilon'(x) = \chi_{\mathbb{R}^+}(x)$ *a.e.* in $\mathbb{R}$, meaning

$$\lim_{\varepsilon \to 0^+} f_\varepsilon \circ u = u^+, \lim_{\varepsilon \to 0^+} f_\varepsilon' \circ u = \chi_{x \in \Omega | u(x) > 0},$$

*a.e.* in $\Omega$. Using the preivously obtained general expression and passing to the limit yields

$$- \int_\Omega (f \circ u) \nabla \varphi \, \mathrm{d}x = \int_\Omega \nabla u \varphi \chi_{x \in \Omega | u(x) > 0} \, \mathrm{d}x \,,$$

and realizing $u^- = (-u)^+$, we also have

$$- \int_\Omega (f \circ u) \nabla \varphi \, \mathrm{d}x = \int_\Omega \nabla u \varphi \chi_{x \in \Omega | u(x) < 0} \, \mathrm{d}x \,,$$

□

### 2.7 Difference quotients

## 3 Nonlinear elliptic equations as compact perturbations

**Theorem 14** (Nemytskii)**.** *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}, N \in N, \Omega \subset \mathbb{R}^d$ measurable, $f$ Caratheodory. Then*

1. *if $u : \Omega \to \mathbb{R}^N$ is measurable then $f(\cdot, u)$ is also measurable*
2. *If there is $p_i \in [1, +\infty), i \in \{1, \dots, N\}, q \in [1, \infty), g \in L_q(\Omega), C > 0$ such that for almost all*

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x,y)| \le g(x) + c \sum_{i=1}^N |y_i|^{p_i/q}$$

*, then $u \mapsto f(\cdot, u)$ is continuous from $L_{p_i}(\Omega) \times \dots \times L_{p_N}(\Omega)$ to $L_q(\Omega)$. Moreover, it maps bounded sets to bounded sets.*

*Proof.* No proof □

**Definition 6** (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2)**.** Let $X, Y$ be normed linear spaces, $M \subset X$. The mapping $F : M \to Y$ is called a compact operator on $M$ into $Y$ if $F$ is continuous and $F(M \cap K)$ is relatively compact in $Y$ for any bounded $K \subset X$.

*Remark.* We have no linearity of $F$! So continuity cannot follow from compactness (we have compactness $\Rightarrow$ boundedness $\neq$ continuity for nonlinear operators)

**Theorem 15** (Brouwer fixed point theorem)**.** *Let $K \subset \mathbb{R}^N, N \in \mathbb{N}$ be a nonempty convex closed bounded. Assume that $F : K \to K$ is continuous. Then $F$ has a fixed point in $K$, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

*Proof.* No proof ☐

**Theorem 16** (Schauder fixed point theorem)**.** *Let $K \subset X$ be a nonempty convex closed bonded subset of a linear normed space $X$. Assume that $F$ is compact on $K$ into $K$ and $F(K) \subset K$. Then there is fixed point of $F$ in $K$.*

*Proof.* No proof ☐

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bouded, it is automatically compact, and since $F : K \to K$ is continuous, $F$ is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with N=1 is easy, based on Darboux property.

#### 3.0.1 Problem protypes

In this chapter some nonlinear elliptic equations are discussed.

**Example.** Suppose the following problem:

$$\begin{cases} -\triangle u + g(u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

where

$$g : \mathbb{R} \to \mathbb{R}, f \in \left(\mathrm{W}_0^{1,2}(\Omega)\right)^*, \text{continuous}\,, \exists \alpha \in [0,1) : \forall s \in \mathbb{R} : |g(s)| \le C(1 + |s|^\alpha).$$

**Theorem 17** (Existence)**.** *Let $\Omega \in C^{1,1}, f \in \left(W_0^{1,2}(\Omega)\right)^*, g$ is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = < f, \varphi >_{(W_0^{1,2}(\Omega))^*} .$$

*If $f \in L_2(\Omega)$, then the solution $u \in W^{2,2}(\Omega)$.*

*Proof.* We define $S : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ such that

$$Sw = u \Leftrightarrow \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = < f, \varphi > - \int_\Omega g(w) \varphi \,\mathrm{d}x\,.$$

$S$ is well defined:

$$|\,\mathrm{RHS}\,| \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} \|\varphi\|_{\mathrm{W}^{1,2}(\Omega)} + \|\varphi\|_{\mathrm{L}_2(\Omega)} \|g(w)\|_{\mathrm{L}_2(\Omega)},$$

and
$$\int_\Omega |g(w)|^2\,\mathrm{d}x \le \int_\Omega C(1+|w|^\alpha)^2\,\mathrm{d}x \le \int_\Omega C\big(1+|w|^{2\alpha}\big)\,\mathrm{d}x \le \int_\Omega C\big(1+|w|^2\big)\,\mathrm{d}x \le \infty,$$
where we used the Young inequality and $\alpha \le 1$. We have thus shown the mapping $w \mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by Nemytskii. Next, $S$ is continuous:

- $w \mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$
- $w \mapsto \big(\varphi \mathrm{W}_0^{1,2}(\Omega) \to < f,\varphi > - \int_\Omega g(w)\varphi\,\mathrm{d}x\big)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\big(\mathrm{W}_0^{1,2}(\Omega)\big)^*$
- $F \to u$, where $u$ is the weak solution of
$$\begin{cases} -\triangle u = F & in\Omega \\ u = 0 & on\partial\Omega, \end{cases}$$
, is linear and continuous from $(\mathrm{W}_0^{1,2}(\Omega))^*$ to $\mathrm{W}_0^{1,2}(\Omega)$.

In total, the composition is continuous and yields $S$. Next, we would like to show $S$ is compact. We start with showing $S$ maps bounded sets in $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega)$; for that we need apriori estimates: test the weak formulation with $u$:
$$\|\nabla u\|^2_{\mathrm{L}_2(\Omega)} \le \varepsilon \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + C\Big(\|f\|^2_{(\mathrm{W}^{1,2}(\Omega))^*} + \|g(w)\|^2_{\mathrm{L}_2(\Omega)}\Big) \underbrace{\le}_{\text{Younge}} C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*)} + 1 + \|w\|^2_{\mathrm{L}_2(\Omega)}\Big),$$
from which follows $S$ is compact from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by compact embedding. Now we need to show $S(\mathrm{U}(0,R)) \subset \mathrm{U}(0,R)$ for some $R > 0$. From the previous we know:
$$\frac{C}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|g\|^2_{\mathrm{L}_2(\Omega)}\Big),$$
so since
$$\tilde{C}\int_\Omega |g(w)|^2\,\mathrm{d}x \le \int_\Omega C\big(1+|w|^{2\alpha}\big)\,\mathrm{d}x \underbrace{\le}_{\text{Younge}} \int_\Omega \Big(C + \frac{c}{4}|w|^2\Big)\,\mathrm{d}x$$
we know
$$\frac{c}{2}\|u\|^2_{\mathrm{L}_2(\Omega)} \le \frac{c}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + C\lambda(\Omega) + \frac{c}{4}\|w\|^2_{\mathrm{L}_2(\Omega)},$$
and thus
$$\|u\|^2_{\mathrm{L}_2(\Omega)} \le \underbrace{\frac{2\tilde{C}}{c}\|f\|^2_{(\mathrm{W}_0^{1,2}(\Omega))^*} + 2\frac{C}{c}}_{=\overline{C}} + \frac{1}{2}\|w\|^2_{\mathrm{L}_2(\Omega)}.$$
so if $\overline{C} + \frac{1}{2}R^2 < R^2$, we are done [20]. But such an $R$ of course exists (says doc. Kaplicky) $\Rightarrow$ the image of a ball is in a ball for some $R \Rightarrow S$ is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that $u_0$ solves $\begin{cases} -\triangle u_0 = f - g(u_0) \in \mathrm{L}_2(\Omega) & \text{in}\,\Omega \\ u_0 = 0 & \text{on}\,\partial\Omega. \end{cases}$.
So from the regularity theory for elliptic equations we get
$$u \in \mathrm{W}^{2,2}(\Omega)\,.$$
□

[20] The constants are most probably messed up.

**Theorem 18** (Uniqueness)**.** *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be weak solutions to the above problem. Let $f \in \left(W_0^{1,2}(\Omega)\right)^*$, $g$ be continuous. Let either*

1. *$g$ is nondecreasing*
2. *$g \in C^1(\mathbb{R}), \|g'\|_\infty$ small.*

*Then $u_1 = u_2$.*

*Proof.* We subtract the equations for $u_1, u_2$ and test with $u_1 - u_2$.:

$$\int_\Omega |\nabla(u_1 - u_2)|^2 + (g(u_1) - g(u_2))(u_1 - u_2)\,\mathrm{d}x = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1 - u_2)\|_{\mathrm{L}_2(\Omega)} \geq C\|u_1 - u_2\|^2_{\mathrm{W}^{1,2}(\Omega)} \Rightarrow u_1 - u_2 = 0.$$

$$|\int_\Omega (g(u_1) - g(u_2)(u_1 - u_2))\,\mathrm{d}x| \leq \int_\Omega \|g'\|_\infty |u_1 - u_2|^2\,\mathrm{d}x \leq \|g'\|_\infty C_P \|\nabla(u_1 - u_2)\|^2_{\mathrm{L}_2(\Omega)} = 0 \Rightarrow u_1 = u_2,$$

whenever $C\|g'\|_\infty < 1$. □

**Example.** Suppose the following problem

$$\begin{cases} -\triangle u + b(\nabla u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}$$

where $f \in (\mathrm{W}_0^{1,2}(\Omega))^*$, $b$ is continuous and bounded. The weak formulation is

$$u \in \mathrm{W}_0^{1,2}(\Omega) \wedge \forall\varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla\varphi + b(\nabla u)\varphi\,\mathrm{d}x =< f, \varphi >,$$

and the first apriori estimates (test with $u$)

$$\|\nabla u\|_{\mathrm{L}_2(\Omega)} \leq \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \int_\Omega |u|\,\mathrm{d}x\,\|b\|_{\mathrm{L}_\infty(\Omega)}.$$

**Theorem 19.** *Let $f \in \left(W_0^{1,2}(\Omega)\right)^*, \Omega \in C^{0,1}, b: \mathbb{R}^d \to \mathbb{R}$ continuous and bounded. Then there is a weak solution to the above problem.*

*Proof.* $S: \mathrm{W}_0^{1,2}(\Omega) \to \mathrm{W}_0^{1,2}(\Omega)\,, Sw = u$ iff $u$ solves

$$\begin{cases} -\triangle u = f - b(\nabla w) & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall\varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla\varphi\,\mathrm{d}x =< f, \varphi > - \int_\Omega b(\nabla w)\varphi\,\mathrm{d}x\,.$$

Clearly, $S$ is well defined and

$$\|Sw\|_{\mathrm{W}_0^{1,2}(\Omega)} \le \underbrace{C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|b\|_{\mathrm{L}_\infty(\Omega)}\Big)}_{:=R},$$

meaning $S\Big(\overline{\mathrm{U}(0,R)}\Big) \subset \overline{\mathrm{U}(0,R)}$. Moreover, $S$ ]s continuous, as $S$ is the composiiton of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show $S$ is compact: we already have continuity, consider $\{w_k\}_{k\in\mathbb{N}} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded. Then $\exists\{u_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded: $u_k \to u$ in $\mathrm{L}_1(\Omega)$ by embedding up to a subsequence. Next, uss the following trick: substitue equation for $u_k$ from equation for $u_l$ and test with $u_l - u_k$

$$C\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \le \|\nabla(u_l - u_k)\|^2_{\mathrm{L}_2(\Omega)} \le \int_\Omega |b(\nabla u_l) - b(\nabla u_k)|\,|u_l - u_k|\,\mathrm{d}x \le 2\|b\|_{\mathrm{L}_\infty(\Omega)}\|u_l - u_k\|_{\mathrm{L}_1(\Omega)}.$$

All in all, $S$ has a fixed point by Schauder, which is of course the weak solution. ☐

But this says $\{u_k\}$ is Cauchy in $\mathrm{W}_0^{1,2}(\Omega)$.

# 4 Nonlinear elliptic equations - monotone operator theory

**Lemma 9.** *Let $g : B(0,R) \subset \mathbb{R}^n \to \mathbb{R}^N$ be continuous, $N \in \mathbb{N}, R > 0$, and $\forall c \in S(0,R) : g(c)\cdot c \ge 0$. Then, there is $c_0 \in B(0,R) : g(c_0) = 0$.*

*Proof.* By contradiction. Let $g \ne 0$ in $\mathrm{U}(0,R)$. Let us define

$$h(x) = -R\frac{g(x)}{|g(x)|}.$$

Then $h \in C(\mathrm{B}(0,R)), h(\mathrm{B}(0,R)) \subset \mathrm{S}(0,R)$, so by Brouwer there $\exists x_0 \in \mathrm{B}(0,R) : h(x_0 = x_0 \Rightarrow -R\frac{g(x_0)}{|g(x_0)|} = x_0$. Take the dot product with $x_0$ and write

$$\underbrace{-R\frac{g(x_0)\cdot x_0}{|g(x_0)|}}_{\le 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in \mathrm{S}(0,R),$$

so that is a contradiction. ☐

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i(a_i(x,u(x),\nabla u(x)) + a_0(x,u(x),\nabla u(x))) = f(x) & \text{in}\,\Omega \\ u = u_0 & \text{on}\,\partial\Omega \end{cases}$$

The date are

- $\Omega \in C^{0,1}$,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, i \in \{1,\dots,d\}$ are Caratheodory in $x$ and $(u,\nabla u)$.

- $f \in \left(\mathrm{W}_0^{1,r}(\Omega)\right)^*$,

and the unknown is $u : \Omega \to \mathbb{R}$.

*Remark.* The function $(u, p) \mapsto a_i(\cdot, u, p)$ is continuous from $(\mathrm{L}_r(\Omega))^{d+1}$ to $\mathrm{L}_{r'}(\Omega)$. by Nemystkii theorem.

**Definition 7** (Coercivity)**.** We say that $\{a_i\}_{i=0}^d$ are coercive if $\exists C_1 > 0, C_2 \in \mathrm{L}_1(\Omega)$ : a.e. $x \in \Omega, \forall (z, p) \in \mathbb{R}^{d+1}$ :

$$\sum_{i=1}^{d} a_i(x, z, p)p_i + a_0(x, z, p) \geq C_1|p|^r - C_2(x), \text{ i.e. } a(x, z, p) \cdot p \geq C_1|p|^r - C_2(x)$$

**Definition 8** (Monotonicity)**.** We say that $\{a_i\}_{i=0}^d = a$ is monotone if for almost all

$x \in \Omega, \forall (z_1, p_1), (z_2, p_2) \in \mathbb{R}^{d+1} : (a(x, z_1, p_1) - a(x, z_2, p_2)) \cdot (p_1 - p_2) + (a_0(x, z_1, p_1) - a_0(x, z_2, p_2)) \cdot (z_1 - z_2) \geq 0.$

Very similiarly we define strict monotonicity.

**Definition 9** (Weak solution)**.** We say that $u \in \mathrm{W}^{1,r}(\Omega)$ is a weak solution to the above problem if

- $u = u_0$ in the sense of traces on $\partial\Omega$,
- $$\int_\Omega a(\cdot, u, \nabla u) \cdot \nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \, \mathrm{d}x = < f, \varphi >, \forall \varphi \in \mathrm{W}_0^{1,r}(\Omega).$$

**Theorem 20** (Existence and uniqueness)**.** *Let $\Omega \in C^{0,1}$, $u_0 \in W^{1,r}(\Omega)$, $r \in (1, \infty)$, $\{a_i\}_{i=1}^d$ be Caratheodory, coercive and m and let them also satisfy the growth conditions. Finally, let $f \in \left(W^{1,r}(\Omega)\right)^*$. Then, there is a weak solution to the problem. If, moreover, $\{a_i\}_{i=1}^d$ is strictly monotone, then the weak solution is unique.*

*Proof.* The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

**Galerkin**: Since $\mathrm{W}_0^{1,r}(\Omega)$ is separable $\Rightarrow \exists \{w_i\}_{i=1}^\infty$ that is a dense[21] linearly independent subset of $\mathrm{W}_0^{1,r}(\Omega)$. We search for $n \in \mathbb{N}$ such that

$$u^n(x) := u_0(x) + \sum_{j=1}^{n} \alpha_j^n w_j(x),$$

[21]It can be chosen such that it is itself dense, not only its span

where $\alpha_j \in \mathbb{R}$ and $u^n$ satisfy

$$\forall j \in \{1,\dots,n\} : \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla w_j + a_0(\cdot,u^n,\nabla u^n)w_j \,\mathrm{d}x =< f, w_j > .$$

We claim such $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$ exist $\forall n \in \mathbb{N}$ by the previous lemma. We define a vector function

$$F(\alpha^n) \coloneqq \{\int_\Omega a\cdot\nabla w_j + a_0 w_j \,\mathrm{d}x - < f, w_j >\}_{j=1}^n,$$

from Nemystkii $F : \mathbb{R}^n \to \mathbb{R}^n, F$ is continuous on $\mathbb{R}^n$. Moreover, it holds

$$\begin{aligned}F(\alpha^n)\cdot\alpha^n &\geq \int_\Omega a(\cdot,u^n,\nabla u^n)\nabla(u^n-u_0) + a_0(u^n-u_0)\,\mathrm{d}x - < f, u^n-u_0 > \\ &\underbrace{\geq}_{\text{coercivity}} \int_\Omega C_1|\nabla u^n|^r - (C_2(\cdot) + |a||\nabla u_0| + |a_0||u_0|)\,\mathrm{d}x - \|u^n\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*}) - \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*}),\end{aligned}$$

together with the fact

$$\|\nabla u^n\|_{\mathrm{L}_r(\Omega)}^r \geq \Big(|\|\nabla(u-u_0)\|_{\mathrm{L}_r(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\Big)^r \geq \|\nabla(u^n-u_0)\|_{\mathrm{L}_q(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r \geq C\|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r,$$

Next, realize that $\alpha^n \in \mathbb{R}^n \mapsto \|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}$ is a norm equivalent to $|\alpha^n|$ (Euclidian norm). So that means $\exists K_1(n) > 0 : \forall\alpha \in \mathbb{R}^n : K_1(n)|\alpha^n| \leq \|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}$. For $|\alpha^n| = R, R > 0$ determined later estimate $F(\alpha^n)\cdot\alpha^n \geq c\|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)} - \tilde{c}\Big(\|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r + 1 + \|u_0\|_{\mathrm{L}_r(\Omega)}^r + \|f\|_{(\mathrm{W}_0^{1,r}(\Omega))^*}^{r'}\Big)$ (which is not a trivial computation). And so $\exists R > 0, \forall\alpha^n \in \mathrm{S}(0,R) \subset \mathbb{R}^n : F(\alpha^n)\cdot\alpha^n > 0$, so from the previous lemma $\exists\alpha^n \in \mathrm{S}(0,R) : F(\alpha^n) = 0$, and we fix these $\alpha^n$. **Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r \leq C\Big(1 + \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

and

$$\|u^n\|_{\mathrm{W}^{1,r}(\Omega)} \leq C\Big(1 + \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

$$\forall j \in \{0,\dots,d\} : \|a_j(\cdot,u^n,\nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}^{r'} \leq C\Big(1 + \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

**Limit passage** From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \,\mathrm{in}\, \mathrm{W}^{1,r}(\Omega)\,, a_j \rightharpoonup \alpha_j \,\mathrm{in}\, \mathrm{L}_{r'}(\Omega)\,.$$

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_\Omega \alpha\cdot\nabla w_j + \alpha_0 w_j \,\mathrm{d}x =< f, w_j >,$$

and from density of $\{w_j\}_{j\in\mathbb{N}}$ in $\mathrm{W}^{1,r}(\Omega)$ we have

$$\forall\varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega \alpha\cdot\nabla\varphi + \alpha_0\varphi \,\mathrm{d}x =< f.\varphi > .$$

**Identification of $\alpha$'s** We want to show $\alpha_j = a_j(\cdot, u, \nabla u), j \in \{0, \dots, d\}$. For that, we use the *Minty trick*:

$$\begin{aligned}
0 &\le \int_\Omega (a(\cdot, u^n, \nabla u^n) - a(\cdot, v, V)) \cdot (\nabla u^n - V) + (a_0(\cdot, u^n, \nabla u^n) - a_0(\cdot, v, V)) \cdot (u^n - v) \\
&\le \int_\Omega a(\cdot, u^n, \nabla u^n) \cdot \nabla u^n + a_0(\cdot, u^n, \nabla u^n) \cdot u^n \,\mathrm{d}x + \\
&- \int_\Omega (a(\cdot, u^n, \nabla u^n) V + a_0(\cdot, u^n, \nabla u^n) v - a(\cdot, v, V) + a_0(\cdot, v, V) \cdot (u^n - v)) \,\mathrm{d}x \,.
\end{aligned}$$

Denote

$$I^n = \int_\Omega a(\cdot, u^n, \nabla u^n)\cdot\nabla(u^n - u_0) + a_0(\cdot, u^n, \nabla u^n)\cdot(u^n - u_0)) \,\mathrm{d}x + \int_\Omega a(\cdot, u^n, \nabla u^n)\cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \,\mathrm{d}x \,,$$

by using the equation we obtain

$$I^n = < f, u^n - u_0 > + \int_\Omega a(\cdot, u^n, \nabla u^n)\cdot u_0 + a_0(\cdot, u^n, \nabla u^n) u_0 \,\mathrm{d}x \to < f, u - u_0 > + \int_\Omega \alpha \nabla u_0 + \alpha_0 u_0 \,\mathrm{d}x = \int_\Omega \alpha \nabla u + \alpha_0 u \,\mathrm{d}x \,,$$

as the rest has subtracted. In total, we have

$$0 \le \int_\Omega (\alpha - a(\cdot, v, V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot, v, V))(u - v) \,\mathrm{d}x \,.$$

So far, $v, V$ have been arbitrary. If we take

$$V = \nabla u - \lambda\psi, \psi \in \mathrm{L}_r(\Omega)\,, v = u,$$

then $0 \le \int_\Omega (\alpha - a(\cdot, \nabla u + \lambda\psi))\lambda\psi \,\mathrm{d}x$, so if we take $\lambda > 0$ and pass to the limit $\lambda \to 0_+$ (using Nemytskii theorem) we can write

$$0 \le \int_\Omega (\alpha - a(\cdot, u, \nabla u))\psi \,\mathrm{d}x \,.$$

Since $\psi$ was arbitrary, we could have taken $\psi \to -\psi$, which in total means

$$0 = \int_\Omega (\alpha - a(\cdot, u, \nabla u))\psi \,\mathrm{d}x$$

Finally, from the previous results, we obtain

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega a(\cdot, u, \nabla u)\nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \,\mathrm{d}x = < f, \varphi >,$$

and we are almost done. It only remains to show the traces are correct, bt since $u^n \rightharpoonup u$ in $\mathrm{W}^{1,r}(\Omega)$ and from the continuity of the traces, we obtain

$$\operatorname{tr} u = \operatorname{tr} u_0.$$

**Uniqueness:** Let $u_1, u_2$ be two solutions. Use strict monotonicity, subtract the weak formulation and test with $u_2 - u_1$:

$$\int_\Omega \underbrace{(a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla(u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)}_{:=T} \,\mathrm{d}x = 0,$$

where $T \ge 0$, so from strict monotonicity we obtain $T = 0$ a.e. in $\Omega$ but that means $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$, a.e. in $\Omega \Rightarrow u_1 = u_2$ in $\mathrm{W}^{1,r}(\Omega)$. $\square$

**Example** (Nonlinearities vs weak convergence)**.** Let $g_n(x) = \sin(nx)$, then $g \rightharpoonup 0$ in $\mathrm{L}_2((0,4))$ (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx)\varphi \, \mathrm{d}x \geq \int_2^4 \sin^2(nx) \, \mathrm{d}x \to \frac{1}{2} \neq 0, \forall \varphi \in \mathrm{L}_2((0,4)),$$

so $\{u_n^2\} = \{\sin^2(nx)\}$ **does not converge weakly to** $0 = 0^2$.

*Remark.* The method of the presented proof is ***very important.***

**Theorem 21.** *Let $\Omega \in C^{0,1}$. Let $X = W_0^{1,r}(\Omega), r \in (1,\infty)$ with equivalent norm $|\|u\|| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$. Then*

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t.} : \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_\Omega \mathbf{F} \cdot \nabla \varphi \, \mathrm{d}x \, , \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

*Proof.* We solve the problem

$$\begin{cases} -\nabla \cdot \left( |\nabla u|^{r-2} \nabla u \right) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}. \tag{2}$$

Such $u \in \mathrm{W}_0^{1,r}(\Omega)$ exists and is unique by the above theorem. In this case: $a(x,z,p) = |p|^{r-2}p, a_0() = 0$. Coercivity is clear, monotonicity will be shown in the tutorials[22]. Write the weak formulation of the above problem:

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \Phi(\varphi).$$

Set $\mathbf{F} = |\nabla u|^{r-2} \nabla u$, and test the weak formulation with $u$ itself:

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{\mathrm{L}_r(\Omega)}.$$

If now $\|\nabla u\|_{\mathrm{L}_r(\Omega)} = 0$, then $\Phi = 0$ and we are finished, if it is nonzero, then

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} = \left\| |\nabla u|^{r-1} \right\|_{\mathrm{L}_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{\mathrm{B}_X(0,1)} |\Phi(\varphi)| = \sup_{\mathrm{B}_X(0,1)} \left| \int_\Omega \mathbf{F} \cdot \nabla \varphi \right| \mathrm{d}x \leq \sup_{\mathrm{B}_X(0,1)} \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \|\nabla \varphi\|_{\mathrm{L}_r(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)},$$

so $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}$. $\square$

[22]This was a lie

# 5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset \mathrm{W}^{1,r}(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega F(\cdot, u, \nabla u)\,\mathrm{d}x\,,$$

with the basic assumptions $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + \mathrm{W}_0^{1,r}(\Omega)\,, u_0 \in \mathrm{W}^{1,r}(\Omega)\,, F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in \mathrm{L}_1(\Omega)\,,\ \text{a.e.}\, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x,z,p) \geq C_1 |p|^r - c_2(x).$$

*Remark.* From the assumptions it follows $\int_\Omega F(\cdot, u, \nabla u)\,\mathrm{d}x$ is defined $\forall u \in \mathrm{W}^{1,r}(\Omega)$.

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

**Lemma 10.** *Let $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + W_0^{1,r}(\Omega)\,, u_0 \in W^{1,r}(\Omega)\,, F$ Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall\ a.a\, x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x,z,p)| + |\partial_z F(x,z,p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x,\cdot,\cdot) \in C^1\big(\mathbb{R}^{d+1}\big).$$

*Let now $u \in u_0 + W_0^{1,r}(\Omega)$ be a local minimizer of $I$ over $X$, i.e.,*

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_\Omega F(\cdot, u, \nabla u)\,\mathrm{d}x \leq \int_\Omega F(\cdot, u + v, \nabla(u + v))\,\mathrm{d}x\,, F(\cdot, u, \nabla u) \in L_1(\Omega)\,.$$

*Then $u$ is the weak solution to the **Euler-Lagrange equations**:*

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u) + \partial_z F(\cdot, u, \nabla u)) = 0, & in\,\Omega \\ u = u_0, & on\,\partial\Omega \end{cases},$$

*i.e.,*

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi + \partial_z F(\cdot, u, \nabla u)\varphi\,\mathrm{d}x = 0, \operatorname{tr} u = \operatorname{tr} u_0\ on\,\partial\Omega.$$

*Proof.* First $\operatorname{tr} u = \operatorname{tr} u_0$ holds, so we are fine. Now fix $\varphi \in \mathcal{D}(\Omega)$ and define

$$\iota : \mathbb{R} \to \mathbb{R}^*, \iota(\tau) = \int_\Omega \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:= l(\tau,\cdot)}\,\mathrm{d}x\,.$$

Now $\iota$ has a local minimum in 0. We show that $\iota'(0)$ exists and is equal to the of Euler-Lagrange equations.

- $l(\tau,\cdot)$ is measurable for $\tau$ from some neighbourhood of 0.
- $l(\tau,\cdot)$ is differentiable

Moreover

$$\begin{aligned} \partial_\tau l(\tau,\cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi = \\ &= = \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi. \end{aligned}$$

Also

$$i(0) = \int_\Omega F(\cdot, u, \nabla u)\,\mathrm{d}x < \infty$$

and

$$|\partial_\tau l(\tau, \cdot)| \le (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla\varphi|^r) + |h(x)|)(|\varphi| + |\nabla\varphi|) \in \mathrm{L}_1(\Omega)\,.$$

Altogether, $\iota(\tau)$ is finite on $(-1,1), \iota'(\tau)$ exists and

$$\iota'(0) = \int_\Omega \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi\,\mathrm{d}x\,.$$

□

**Example.** Let

$$F(x,z,p) = \frac{1}{r}(1) + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x,z,p) = \left(\frac{r}{2}\frac{1}{r}2\left(1 + |p|^2\right)^{\frac{r-2}{2}}\right)p - G = \left(1 + |p|^2\right)^{\frac{r-2}{2}} p - G, \partial_z F(x,z,p) = -g.$$

We have

$$|\left(1 + |p|^{2\,\frac{r-2}{2}}\right)p| \le \left(1 + |p|^2\right)^{\frac{r-2}{2}}\left(1 + |p|^2\right)^{\frac{1}{2}} = \left(1 + |p|^2\right)^{\frac{r-1}{2}} \le C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left(\left(1 + |\nabla u|^2\right)^{\frac{r-2}{2}} \nabla u\right) = -\nabla \cdot G + g, & \text{in}\,\Omega \\ u = u_0, & \text{on}\,\partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \left(1 + |\nabla u|^2\right)^{\frac{r-2}{2}} \nabla u \cdot \nabla\varphi\,\mathrm{d}x = \int_\Omega (G \cdot \nabla\varphi + g\varphi)\,\mathrm{d}x\,.$$

*Remark.* We have $\{u_n\} \subset X$ s.t. $\lim_{n\to\infty} I(u_n) = \inf_X I$. Then use

- compacntess: $u_n \to u$ in some sonse (weak convergence)
- weak lower semicontinuity $I(u) \le \liminf_{n\to\infty} I(u_n)$

**Lemma 11.** *Let* $F : \mathbb{R}^N \to \mathbb{R}, F \in C^1\left(\mathbb{R}^N\right), N \in \mathbb{N}$. *Then*

1. *$F$ is (strictly) convex $\Leftrightarrow$ $\nabla F$ is (strictly) monotone*
2. *If $F$ is (strictly) convex, then*

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \ne \xi_2 : F(\xi_1) - F(\xi_2) \ge \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

*Proof.* Fix $\xi_1, \xi_2, \xi_1 \ne \xi_2$, define $\varphi(t) = F(\xi_2 + t(\xi_1 - \xi_2))$. Then $\varphi \in C^1(\mathbb{R})$ and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1 - \xi_2)) \cdot (\xi_1 - \xi_2).$$

So

$$" \Rightarrow ": (\nabla F(\xi_1) - \nabla F(\xi_2)) \cdot (\xi_1 - \xi_2) = \varphi'(1) - \varphi'(0) \underbrace{\geq}_{\varphi \text{convex or strictly convex}} 0.$$

And $" \Leftarrow "$ : Fix $t_1 > t_2$ and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1 - \xi_2)) - \nabla F(\xi_2 + t_2(\xi_1 - \xi_2))) \cdot (\xi_1 - \xi_2)(t_1 - t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1 - \xi_2)(t_1 - t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2)) \cdot (\eta_1 - \eta_2)$$

and we are in the same situation as before. For 2) we already know $F$ (strictly) convex $\Rightarrow \varphi$ (strictly) convex

$$\Rightarrow \forall t \in (0, \frac{1}{2}) : \frac{\varphi(1) - \varphi(0)}{1} \geq \frac{\varphi(t) - \varphi(0)}{t} \to \varphi'(0),$$

as $t \to 0_+$. And so

$$F(\xi_1) - F(\xi_2) \geq \nabla F(\xi_2) \cdot (\xi_1 - \xi_2).$$

□

**Theorem 22.** *Let $M, N \in \mathbb{N}, \Omega$ open, $F : \Omega \times \mathbb{R}^{N+M} \to \mathbb{R}$ Caratheodory, $F$ convex in $p \in \mathbb{R}^n$, i.e. $\forall$ a.e. $x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot)$ is convex and $\exists c_2 \in L_1(\Omega)$, $\forall$ a.e. $x \in \Omega, \forall z \in \mathbb{R}^M, \forall p \in \mathbb{R}^N : F(x, z, p) \geq c_2(x)$. Let $u_n \to u$ in $L_1(\Omega)$, $U_n \rightharpoonup U$ in $L_1(\Omega)$. Then*

$$\int_\Omega F(\cdot, u, U)\, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega F(\cdot, u_n, U_n)\, \mathrm{d}x\,.$$

*Proof.* The proof will be given only if moreover $\forall$ *a.e.* $x \in \Omega, \forall z \in \mathbb{R}^M : F(x, z, \cdot) \in C^1\big(\mathbb{R}^N\big)$. Idea: by the previous lemma:

$$\int_\Omega F(\cdot, u_n, U_n)\, \mathrm{d}x \geq \int_\Omega (F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U) \cdot (U_n - U))\, \mathrm{d}x\,,$$

and we have uniform convergence in the first term and second term and weak convergence in $\mathrm{L}_1(\Omega)$ in the last term. If $\Omega$ is bounded, we can find $K_k \subset K_{k+1} \subset \Omega$ *s.t.* $\lambda(\Omega\ \bigcup_{k\in\mathbb{N}} K_k) = 0$, and moreover $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$ are compact, $u_n \rightrightarrows u$ on $K_k$, $\|u\|_{\mathrm{L}_\infty(K_k)} + \|U\|_{\mathrm{L}_\infty(K_k)} \leq k$ up to a subsequence. We can now extract a subsequence $u_n \to u$ *a.e.* and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E}_k \; s.t.\; u_n \rightrightarrows u \,\text{on}\, \tilde{E}_k \wedge \lambda\big(\Omega\ \tilde{E}_k\big) < \frac{1}{k}.$$

Now define

$$\hat{E}_k = \bigcup_{j=1}^{k} \tilde{E}_j, E_k = \hat{E}_k \cap \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}\},$$

and $E_k$ satisfy [23]

$$\lambda\left(\Omega\ \bigcup_k E_k\right) = 0.$$

[23]"This is homework", says doc. Kaplicky

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \le k \wedge |U(x)| \le k\}$$

and we also have $\lambda(\Omega \setminus \bigcup_k F_k) = 0$. FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\Big(\Omega \setminus \bigcup_k K_k\Big) = 0.$$

□

*Remark.*

- if $U_n \to U$ strongly $\Rightarrow u_n \to u, U_n \to U$ *a.e.* (up to a subsequence) and the claim follows from the Fatou lemma. [24]
- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \,\text{in}\, \mathrm{L_p}(\Omega) \Rightarrow \int_\Omega |\nabla u|^p \,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x\,.$$

**Lemma 12** (Arzela-Ascoli)**.** *Let $X, Y$ be Banach spaces, $X \subset\subset Y$. Then*

$$C^1([0,T]; X) \subset\subset C([0,T]; Y).$$

**Lemma 13** (Ehrling)**.** *Let $V_1, V_2, V_3$ be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3$. Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \le \varepsilon \|u\|_{V_1} + C\|u\|_{V_3}.$$

*Proof.* By contradicition, assume

$$\exists \varepsilon > 0 \, s.t. \, \forall n \in N \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n\|u_n\|_{V_3}.$$

WLOG we can assume $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$: truly, the inequaility is 1-homogenous and holds if $u_n = 0$. In particular we see $\|u_n\|_{V_3} < \frac{1}{n}$, so $u_n \to 0$ in $V_3$. Moreover, $\{u_n\}$ is bounded in $V_1$ and since $V_1 \subset\subset V_2$ there exists $\{u_{n_k}\} \subset \{u_n\}$ *s.t.*: $u_{n_k} \to u$ in $V_2$ strongly. Since $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$, also $\|u\|_{V_2} = 1$. Finally, taking the limit passage yields $0 \ge \|u\|_{V_3}$ and so $u = 0$ in $V_3$ and also in $V_2$. But that is a contradiction with the fact $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$. □

**Theorem 23** (Aubin-Lions)**.** *Let $V_1, V_2, V_3$ be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3, p \in [1,\infty)$. Then the space*

$$\mathcal{U} = \{u \in L_p\Big((0,T); V_1\Big), \partial_t u \in L_1\Big((0,T); V_3\Big)\},$$

*with the norm*

$$|\|u\|| = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

*satisfies*

$$\mathcal{U} \subset\subset L_p\Big((0,T); V_2\Big).$$

*Proof.* Strategy: I want to fix $M \subset \mathcal{U}$ bounded and show that it is precompact in $\mathrm{L_p}\Big((0,T); V_2\Big)$. That will be done in the following way:

[24] For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions $F - c_2 \ge 0, F - c_2 \in \mathrm{L}_1(\Omega)$

1. Mollify $M$ by convolution
2. use Arzela-Ascoli
3. show compactness in $\mathrm{L_p}\Big((0,T);V_3\Big)$
4. apply Ehrling lemma and show compactness in $\mathrm{L_p}\Big((0,T);V_2\Big)$.

Fix $M \subset \mathcal{U}$ bounded. Then $\exists C^* > 0 : \forall u \in M : |\|u\|| \geq C^*$.
Next, take

$$\varphi : \mathbb{R} \to [0,\infty), \varphi \in C^\infty(\mathbb{R}), \operatorname{supp}\varphi \subset (-1,0), \int_{\mathbb{R}} \varphi \,\mathrm{d}x = 1,$$

a regularization kernel, then $\forall \delta > 0$ define $\varphi_\delta(t) :== \frac{1}{\delta}\varphi\big(\frac{t}{\delta}\big)$.

Now, extend functions from $M$ to $(0,2T)$ in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0,T) \\ u(2T-t), & t \in (T,2T) \end{cases}.$$

Now mollify: for $\delta > 0, \delta < T$ fixed define

$$M_\delta = \{ \, (\tilde{u} \star \varphi_\delta)\Big|_{(0,T)} |u \in M\}.$$

From the properties of regularization it follows $M_\delta \subset C^1([0,T];V_1) \underbrace{\subset\subset}_{\text{A.A.}} C([0,T];V_2) \subset \mathrm{L_p}\Big((0,T);V_2\Big)$.

Now estimate the distance of $M$ and $M_\delta$ in $\mathrm{L_p}\Big((0,T);V_3\Big)$ : for

$$u \in M, t \in (0,T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^{0} \tilde{u}(t-s)\varphi_\delta)(s)\,\mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t-s))\varphi_\delta(s)\,\mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t-s)) \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^{s} \varphi_\delta(\sigma)\,\mathrm{d}\sigma \Big|_{s=-\delta}^{0} - \int_{-\delta}^{0} \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^{s} \varphi_\delta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\,,$$

since the first bracket is 0 and by denoting the first term in the second integrand by $\tilde{u'}(t-s)$ this becomes (using Fubini)

$$= -\int_{-\delta}^{0}\int_{\sigma}^{0} \tilde{u'}(t-s)\,\mathrm{d}s\,\varphi_\sigma(\sigma)\,\mathrm{d}\sigma\,,$$

and we see

$$\|\tilde{u}(t) - \tilde{u_\delta}(t)\|_{V_3} \leq \int_{-\delta}^{0}\int_{\sigma}^{0} \left\|\tilde{u'}(t-s)\right\|_{V_3} \mathrm{d}s\,\varphi_\sigma(\sigma)\,\mathrm{d}\sigma\,.$$

$\mathrm{L_1}\Big((0,T);V_3\Big)$ estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3}\,\mathrm{d}t \leq \int_0^T \int_{-\delta}^{0}\int_{\sigma}^{0} \left\|u'(\tilde{t-s})\right\|_{V_3} \mathrm{d}s\,\varphi_\delta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t \leq 2\delta\|u'\|_{\mathrm{L_1}((0,T);V_3)} \leq 2\delta C^*$$

$\mathrm{L}_\infty\Big((0,T);V_3\Big)$ estimate:

$$\|u-u_\delta\|_{\mathrm{L}_\infty((O,T);V_3)} \le 2\|u'\|_{\mathrm{L}_1((0,T);V_3)} \le 2C^*$$

It remains to show $M_\delta \subset\subset \mathrm{L}_\mathrm{p}\Big((0,T);V_2\Big)$:

$$\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_3)} \le \|u-u_\delta\|^{1/p}_{\mathrm{L}_1((0,T);V_3)}\|u-u_\delta\|^{1-1/p}_{\mathrm{L}_\infty((O,T);V_3)} \le 2C^*\delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall\mu>0\exists C_\mu>0:\forall u\in\mathcal{U}:\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_2)} \le \mu\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_1)} + C_\mu\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_3)}.$$

This means

$$\forall u\in M:\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_2)} \le C^* + C\mu 2C^*\delta^{1/p}.$$

Now fix $\varepsilon>0$ and find

$$\mu>0:\mu C^*<\frac{\varepsilon}{2},\delta>0,C_\mu 2C^*\delta^{1/p}<\frac{\varepsilon}{2}\Rightarrow\forall u\in M:\|u-u_\delta\|_{\mathrm{L}_\mathrm{p}((0,T);V_2)}<\varepsilon.$$

This means $\exists\{w_k\}_{k=1}^N\subset M:\{(w_k)_\delta\}_{k=1}^n$ is $\varepsilon-$net in $M$ in $\mathrm{L}_\mathrm{p}\Big((0,T);V_2\Big)$. If we now fix $u\in M$, then

$$\exists K\in\{1,\dots,N\}:\Big\|u_{\delta-(w_K)_\delta}\Big\|_{\mathrm{L}_\mathrm{p}((0,T);V_2)}<\varepsilon.$$

$\square$

*Remark.* The pair $(\mathcal{U},|||\cdot|||)$ is a Banach space.

We will be dealing with the following problem:

$$\begin{cases}\partial_t u-\nabla\cdot a(\cdot,u,\nabla u)+a_0(\cdot,u,\nabla u)=f & \text{in } (0,T)\times\Omega,\\ u=u_0, & \text{on } \{0\}\times\Omega,\\ u=0, & \text{on } (0,T)\times\partial\Omega\end{cases}.$$

The unknown is the function $u:(0,T)\times\Omega\to\mathbb{R}$, and we are given $\Omega\in C^{0,1},T>0,Q_T=(0,T)\times\Omega,f:Q\to\mathbb{R}$ or $f:(0,T)\to X$ a Banach space, $u_0:\Omega\to\mathbb{R},a:\Omega\times\mathbb{R}\mathbb{R}^d\to\mathbb{R}^d,a_0:\Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions: $\exists r>1,\exists C>0:\ a.e.\,x\in\Omega,\forall(z,p)\in\mathbb{R}^{d+1}:|a_0(x,z,p)|+|a(x,z,p)|\le C(1+|z|^{r-1}+|p|^{r-1})$ and $\exists C_1,C_2>0,q\in(1,\max(2,r))\ a.e.\,x\in\Omega,\forall(z,p)\in\mathbb{R}^{d+1}:a(x,z,p)p+a_0(\dots)z\ge C_1|p|^r-C_2(1+|z|q)$.

**Theorem 24.** *Let $\Omega\in C^{0,1},a,a_0$ satisfy growth conditions and coercivity, let $\{a_i\}_{i=0}^d$ be monotone. Denote $V=W_0^{1,r}(\Omega)\cap L_2(\Omega)$. Then $\forall f\in L_{r'}\Big((0,T);V^*\Big),u_0\in L_2(\Omega)\,\exists u\in L_r\Big((0,T);V\Big)$ s.t. $\partial_t u\in L_{r'}\Big((0,T);V^*\Big),u\in C([0,T];L_2(\Omega)),u(0)=u_0$ and moreover*

$$a.e.\,t\in(0,T),\forall\varphi\in V:<\partial_t u,\varphi>+\int_\Omega a(\cdot,u,\nabla u)\nabla\varphi+a_0(\cdot,u,\nabla u)\varphi\,\mathrm{d}x=<f,\varphi>.$$

*Finally, the solution is unique.*

*Proof.* The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

*Rothe method:* Fix $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$ and approximate the derivative with

$$\partial_t u(t,x) \approx \frac{1}{h}(u(t,x) - u(t-h,x)).$$

Define $u_0 = u_0, u_{k+1} \in V$ as a solution of

$$\frac{1}{h}(u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1,}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \,\mathrm{in}\, \Omega, u_{k+1} = 0 \,\mathrm{on}\, \partial\Omega.$$

Define

$$f_{k+1} := \int_{kh}^{(k+1)h} f \,\mathrm{d}t\,,$$

then the weak formulation becomes

$$\int_\Omega \frac{u_{k+1} - u_k}{h}\varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla\varphi a_0(\cdot, u_{k+1}, \nabla u_{k+1})\varphi \,\mathrm{d}x =< f_{k+1}, \varphi > .$$

We *claim without a proof* that the solutions $\{u_k\}_{k=0}^n \subset V$ exist.

To obtain a-priori estimates, tes the equation with $u_{k+1}$. This yields:

$$\int_\Omega |u_{k+1}|^2 - u_k u_{k+1} \,\mathrm{d}x = \int_\Omega \frac{1}{2}|u_{k+1}|^2 + \frac{1}{2}(u_{k+1} - u_k) - \frac{|u_k|^2}{2} \,\mathrm{d}x \Rightarrow \sum_{k=0}^{j-1} \int_\Omega |u_{k+1}|^2 - u_{k+1}u_k \,\mathrm{d}x = \frac{1}{2}\|u_j\|^2_{\mathrm{L}_2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} (u_{k+$$

so

$$\int_\Omega a(\dots)\nabla \cdot u_{k+1} + a_0(\dots)u_{k+1} \,\mathrm{d}x \geq C_1 \int_\Omega |\nabla u_{k+1}|^r \,\mathrm{d}x - C_2 \int_\Omega (1 + |u_{k+1}|^q) \,\mathrm{d}x\,,$$

$< f_{k+1}, u_{k+1} >\leq \|f_{k+1}\|_{V^*}\Big(\|u_{k+1}\|_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|_{\mathrm{L}_2(\Omega)}\Big) \leq \varepsilon\Big(\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)}\Big) + C\Big(\|f_{k+1}\|^{r'}_{V^*} + \|f_{k+1}\|^2_{V^*}\Big).$

So together $\|u_j\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1}\Big[(u_{k+1} - u_k)^2 + h\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)}\Big] \leq C\Big(\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\Big(\|f\|^{r'}_{V^*}\|f\|^2_{V^*}\Big)\Big)$ □

Let us now define $u^n(t) = u_k$ for $t \in (h(k-1), hk)$, then

$$\|u^n\|^2_{\mathrm{L}_\infty((O,T);\mathrm{L}_2(\Omega))} + \|u^n\|^2_{\mathrm{L}_2\left((O,T);\mathrm{W}_0^{1,r}(Omega)\right)} < C(\,\mathrm{data}\,).$$

Now set $\tilde{u}^n(t) = u_{k-1} + \frac{t - t_{k-1}}{h}(u_k - u_{k-1})$ for $t \in (t_{k-1}, t_k)$ and

$$k \in \{1, \dots, n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the quation to the form

$$\int_\Omega \partial_t \tilde{u}^n \varphi + a(\cdot, u^n, \nabla u^n) \cdot \nabla \varphi + a_0(\cdot, u^n, \nabla u^n) \varphi \,\mathrm{d}x =< f^n, \varphi >,$$

where $f^n(t) := f_k$ in in

$$(t_{k-1}, t_k), k \in \{1, \dots, \}.$$

We are now ready to use growth and apriori estimates:

$$\|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} \le C(\text{ data }).$$

For the norm of the time derivative:

$$\sup_{\varphi \in \mathrm{S_V}(0,1)} < \partial_t \tilde{u}^n(t), \varphi >= \sup_{\varphi \in \mathrm{S_V}(0,1)} < f^n, \varphi > - \int_\Omega (a(\cdot, u^n, \nabla u^n) \cdot \nabla f + a_0(\cdot, u^n, \nabla u^n) \varphi) \,\mathrm{d}x\,,$$

at any $t \in (0, T)$. So using Holder:

$$\|\partial_t \tilde{u}^n(t)\|_{V^*} \le \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)},$$

and integrating

$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{r'} \,\mathrm{d}t \le C\left( \int_0^T \|f^n\|_{V^*}^{r'} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}, \mathrm{d}t \right) \le TC(\text{ data })$$

# 6 Semigroup theory

We consider the equation

$$\begin{aligned} u' &= Au, A \,\text{is linear} \\ u(0) &= u_0, \end{aligned}$$

where $u : [0, \infty) \to \mathbb{R}$. We know that for example if $Au = au, a \in \mathbb{R}$ then

$$u(t) = u_0 e^{at}.$$

If $\mathbf{u} : [0, \infty) \to \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d \times d}$, then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^\infty \frac{1}{k!} \mathbb{A}^k t^k.$$

This can be extended to $u : [0, \infty) \to X, X$ a banach space, $A \in \mathcal{L}(X)$, then

$$u(t) = \exp(tA) u_0,$$

where the operator exponential is the same. This works well for unbounded operators, but suppose now

$$X = \mathrm{L}_2(\Omega)\,, Au = \triangle\, u.$$

We *guess* the solution should be

$$u(t) = \exp{(\triangle\, t)} u_0,$$

but what is

$$\exp(\triangle\, t)?$$

**Definition 10** (Linear operator and its domain)**.** Let $X$ be a Banach space over $\mathbb{K}$. Linear operator on $X$ is a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace of $X$ and $A : \mathcal{D}(A) \to X$ is linear.

**Definition 11.** A family $\{S(T)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if

1. $S(0) = \mathrm{id}$
2. $\forall s, t \geq 0 : S(t)S(s) = S(t+s)$.

If moreover $\forall x \in X : S(t)x \to x$, as $t \to 0_+$, we call $\{S(t)\}$ a $c_0-$ semigroup (strongly continuous).

*Remark.* $\{s(t)\}_{t\in\mathbb{R}}$ with the two conditions is an Abelian group $(\{S(t)\}_{t\in\mathbb{R}}, \circ)$ with

$$(S(t))^{-1} = S(-t). \tag{3}$$

*Remark* ($X$ = Banach)*.* **In the following, $X$ is always a Banach space.**

**Lemma 14.** *Let $\{S(t)\}_{t\geq 0}$ be a $c_0$-semigroup in $X$. Then*

1. $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$,
2. $\forall x \in X, t \mapsto S(t)x \in C([0,\infty); X)$.

*Proof.* $1 \Rightarrow 2$. Fix $t > 0, x \in X$ compute

$$\lim_{h\to 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t)(S(h)x - x)\|_X \leq \lim_{h\to 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \to 0.$$

now compute $\lim_{h\to 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t-h)(x - S(h)x)\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \to 0$. ◻

**Definition 12** (Infinitesimal generator)**.** A linear operator $(A, \mathcal{D}(A))$ is called a infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$, if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h\to 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X | \lim_{h\to 0_+} \frac{S(h)x - x}{x} \,\text{exists in}\, X \right\},$$

**Theorem 25.** *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a $c_0$-semigroup $\{S(t)\}_{t\geq 0}$ in $X$. Then*

1. $x \in \mathcal{D}(A) \Rightarrow \forall t \geq 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)x)$,

2. $x \in X \wedge t \geq 0 \Rightarrow x_t = \int_0^t S(s)x \,\mathrm{d}s \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x$.

*Proof.* Fix $x \in \mathcal{D}(A), t \geq 0$. Calculate

$$\lim_{h \to 0_+} \frac{S(h)S(t)x - S(t)x}{h} =^{25} \lim_{h \to 0_+} S(t)\frac{S(h)x - x}{h} = S(t)Ax,$$

(convergence is in the norm of the Banach space $X$). This means $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$, moreover, if $t > 0$:

$$\lim_{h \to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right),$$

estimate,

$$\left\|\lim_{h \to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right)\right\| = \lim_{h \to 0_+} \left\|\frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \to 0_+} S(t$$

as $S(t)$ is continous and $S(0) = \mathrm{id}$. Clearly, $t \mapsto S(t)x$ is $C^1([0,\infty))$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h \to 0_+} \frac{1}{h}(S(h)x_t - x_t) = \lim_{h \to 0_+} \frac{1}{h}\left(\int_h^{t+h} S(s)x \,\mathrm{d}s - \int_0^t S(s)x \,\mathrm{d}s\right),$$

realize that

$$S(h)x_t = \int_0^t S(s+h)x \,\mathrm{d}s = \int_h^{t+h} S(s)x \,\mathrm{d}s\,,$$

so the previous computation continues as follows

$$= \lim_{h \to 0_+} \frac{1}{h}\left(\int_t^{t+h} S(s)x \,\mathrm{d}s - \int_0^h S(s)x \,\mathrm{d}s\right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

**Definition 13** (Closed operator)**.** We say that a linear operator $(A, \mathcal{D}(A))$ is closed if $\forall \{u_n\} \subset \mathcal{D}(A) : u_n \to u \wedge Au_n \to v$, for some $u, v \in X$, then it most hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that $\{(x, Ax) | x \in \mathcal{D}(A)\} \subset X \times X$ is closed in $(X \times X, \|\cdot\|_1)$.

[25] $S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$

**Example.** Let $\Omega \in C^{1,1}, X = \mathrm{L}_2(\Omega), \mathcal{D}(A) = \mathrm{W}^{2,2}(\Omega) \cap \mathrm{W}_0^{1,2}(\Omega), Au = \triangle u$. Then $(A, \mathcal{D}(A)$ is closed. Really, take $\{u_n\} \subset \mathrm{L}_2(\Omega) : u_n \to u$ in $\mathrm{L}_2(\Omega)$ for some $u \in \mathrm{L}_2(\Omega)$. Suppose $Au_n \to v$ in $\mathrm{L}_2(\Omega), v \in \mathrm{L}_2(\Omega)$. Suppose the following equation: find

$$u_n \; s.t. - \triangle u_n = Au_n, u_n \,\mathrm{on}\, \partial\Omega.$$

From the regularity theory for elliptic problems, we know that $\|u_n\|_{\mathrm{W}^{2,2}(\Omega)} \le C\|Au_n\|_{\mathrm{L}_2(\Omega)} \le C$, so we can extract $u_{n_k} \rightharpoonup u$ in $\mathrm{W}^{2,2}(\Omega)$. Realize moreover

$$\int_\Omega \triangle u_n \varphi \,\mathrm{d}x = \int_\Omega u_n \triangle \varphi \,\mathrm{d}x \,, \forall \varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_\Omega v\varphi \,\mathrm{d}x = \int_\Omega u \triangle \varphi \,\mathrm{d}x = \int_\Omega \triangle u\varphi \,\mathrm{d}x \,,$$

which means $\triangle u = v \; a.e. \,\mathrm{in}\, \Omega$ and that $u \in \mathcal{D}(A), Au = v$.

**Theorem 26.** *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a $c_0$-semigroup $\{S(t)\}_{t\ge0} \subset X$. Then*

1. *$\mathcal{D}(A)$ is dense in $X$,*
2. *$(A, \mathcal{D}(A)$ is closed.*

*Proof.* Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t}\int_0^t S(s)x \,\mathrm{d}s \underbrace{\in \mathcal{D}(A)}_{\text{prev. thm}}, \frac{x_t}{t} \to x \,\mathrm{in}\, X,$$

Ad 2.: Take $\{x_n\} \subset \mathcal{D}(A) : x_n \to x$ in $X, Ax \to v \,\mathrm{in}\, X$. Compute[26]

$\dfrac{(S(h) - \mathrm{id})x_n}{h} = \dfrac{1}{h}\displaystyle\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}(S(s)x_n) \,\mathrm{d}s = \dfrac{1}{h}\displaystyle\int_0^h AS(s)x_n \,\mathrm{d}s = \dfrac{1}{h}\displaystyle\int_0^h S(s) \underbrace{Ax_n}_{\to v}$, so taking the limit yields $\dfrac{(S(h) - \mathrm{id})x}{h} =$

Altogether, $x \in \mathcal{D}(A), Ax = v$. ☐

**Lemma 15.** *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of $c_0$-semigroups $\{S(t)\}_{t\ge0}, \{\tilde{S}(t)\}_{t\ge0}$. Then*

$$\{S(t)\}_{t\ge0} = \{\tilde{S}(t)\}_{t\ge0}.$$

*Proof.* We want to show

$$\forall x \in X, \forall t \ge 0 : S(t)x = \tilde{S}(t)x.$$

Fix $x \in \mathcal{D}(A), t > 0$. Then $g(s) := S(s)\tilde{S}(t - x)x$ satisfies $g \in C^1([0,t]; X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$, as $A, S$ commute. This means $g(0) = g(1)$ and from this it follows $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X, S$ continous $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$, and since $t \ge 0$ was arbitrary, we are done. ☐

**Definition 14** (Resolvent of a linear operator)**.** Let $(A, \mathcal{D}(A))$ be a linear (possibly unbounded) operator on $X$. We define

[26] This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that $X$ is a Banach space and work with some functionals and Bochner integrals or whatever

1. resolvent set

$$\rho(A) = \Big\{\lambda \in \mathbb{K} | \lambda\,\mathrm{id} - A \,\text{is invertible and}\, (\lambda\mathrm{id} - A)^{-1} \in \mathcal{L}(X)\Big\},$$

2. resolvent operator $R(\lambda, A) : X \to \mathcal{D}(A) : R(\lambda, A) = (\lambda\mathrm{id} - A)^{-1}$, for

$$\lambda \in \rho(A).$$

*Remark.* If $(A, \mathcal{D}(A)$ is a closed linear operator: $\lambda \in \rho(A) \Leftrightarrow \lambda\mathrm{id} - A$ is a bijection of $\mathcal{D}(A)$ onto $X$.

**Lemma 16.** *Let $(A, \mathcal{D}(A))$ be a linear operator on $X$. It holds*

1. $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$
2. $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x,$
3. $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$, *and* $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A),$
4. *If moreover $(A, \mathcal{D}(A)$ is the infinitesimal generator of a $c_0$-semigroup $\{S(t)\}_{t\geq 0}$ s.t. $\forall t \geq 0$ : $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, then*

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t)\,\mathrm{d}t \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

*Remark.* The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

*Proof.* Ad 1.:

$$AR(\lambda, A)x = (A - \lambda\mathrm{id}) \underbrace{R(\lambda, A)}_{=(\lambda\mathrm{id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\mathrm{id} - (\lambda\mathrm{id} - A))R(\eta, A) = R(\lambda, A)(\eta\mathrm{id} - A - \lambda\mathrm{id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For $\lambda \neq \eta$ we also have

$$R(\lambda, A)R(\eta A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG asume $\omega = 0$, meaning $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$. Denote $\tilde{S}(t) = e^{-\omega t}S(t)$. Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x\,\mathrm{d}t\,.$$

First of all, this is well defined as

$$\big\|\tilde{R}x\big\|_X \leq \int_0^\infty e^{-\lambda t} M \|x\|_X \,\mathrm{d}T = \frac{M}{\lambda}\|x\|_X,$$

and so $\|\tilde{R}\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}, \tilde{R} \in \mathcal{L}(X)$. Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Leftrightarrow \mathrm{id} = (\lambda\mathrm{id} - A)\tilde{R}.$$

For $x \in X, h > 0$ fixed compute

$$\begin{aligned}\frac{1}{h}\big(S(h)\tilde{R}x - \tilde{R}x\big) &= \frac{1}{h}\bigg(\int_0^\infty e^{-\lambda t}S(t+h)x - e^{-\lambda t}S(t)x\,\mathrm{d}t\bigg) = \\ &= \frac{1}{h}\bigg(\int_h^\infty e^{-\lambda(t-h)}S(t)x\,\mathrm{d}t - \int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\bigg) = \\ &= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h}S(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t = \\ &= e^{-\lambda t}\frac{e^{\lambda h} - 1}{h} \to \lambda e^{-\lambda t}, \text{ as } h \to 0_+\end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t)e^{-\lambda t}\frac{e^{h\lambda} - 1}{h}S(t)x \to \lambda e^{-\lambda t}S(t)x \text{ on } (0,\infty) \text{ as } h \to 0_+.$$

The norm of this can be estimated as $\|\lambda e^{-\lambda t}S(t)x\| \leq Ce^{-\lambda t}M\|x\|_X \in \mathrm{L}_1((0,\infty))$. Altogether, we obtain $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda\mathrm{id} - A)\tilde{R}x = x$.

To procced further, we need the following theorem:

$$x \in \mathcal{D}(A), A \text{ closed} : A\tilde{R}x = A\bigg(\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\bigg) = \int_0^\infty e^{-\lambda t}\underbrace{AS(t)}_{=S(t)A}x\,\mathrm{d}t = \tilde{R}Ax,$$

which has been stated but not proved [27]. Finally, we can write: $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda\mathrm{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$. Moreover, we have also shown the mapping is a bijection. □

**Definition 15** (Contraction semigroup)**.** We say that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup if

$$\forall t \geq: \|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

**Theorem 27** (Hille-Yosida)**.** *Let $M \geq 1, \omega \in \mathbb{R}$. A linear $(A, \mathcal{D}(A))$ on a Banach space $X$ generates a $c_0$-semigroup (meaning it is its infinitesimal generator) satysfing $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$* ***if and only if***

1. *$(A, \mathcal{D}(A))$ is closed,*
2. *$\mathcal{D}(A)$ is dense in $X$,*
3. *$\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda-\omega)^n}$.*

*Proof.* If $M = 1, \omega = 0$, then $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}$. " $\Rightarrow$ " has been proven, now show the other direction. The plan is to

1. approximate $A$ by $\{A_n\} \subset \mathcal{L}(X)$,

[27] It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

2. construct $S_n$ for $A_n$ as previously,

3. estimate and limit passage.

*Approximation:* See the analogy: $a \in \mathbb{R} : \frac{n}{n-a} \to 1$, we would like $nR(n,A) \to \mathrm{id}$. Calculate the norm of $nAR(n,A) = n(nR(n,A) - \mathrm{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$, (This approx. is called the Yosida approximation.) For $x \in \mathcal{D}(A)$ fixed:

$$\|nR(n,A)x - x\|_X = \|R(n,A)Ax\|_X \leq \|R(n,A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \frac{1}{n}\|Ax\|_X \to 0 \,\text{as}\, n \to \infty.$$

If

$$y \in X : \|nR(n,A)y - y\|_X \leq \|nR(n,A)(y-x)\|_X + \|nR(n,A)x - x\|_X + \|x-y\|_X \leq 2\|y-x\| + \underbrace{\|nR(n,a)x - x\|_X}_{\to 0},$$

but $\|y-x\|_X$ can be made arbitrarily small from density of $\mathcal{D}(A)$ in $X$, so in fact

$$nR(n,A)y \to y \,\text{in}\, X, \forall y \in X.$$

And so $nR(n,A)$ really approximates $\mathrm{id}$.

Using this gives us

$$\forall x \mathcal{D}(A) : A_n x = nAR(n,A)x = n\overbrace{R(n,A)A}^{=AR(n,A)}x \to Ax \,\text{in}\, X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \leq \left\| \sum_{k=0}^{\infty} \frac{1}{k!}(tA_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!}\left(-nt\mathrm{id} + n^2 tR(n,A)\right)^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!}(-nt\mathrm{id})^k \sum_{k=0}^{\infty} \frac{\left(n^2 tR(n,A)\right)^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives $\leq e^{-nt}\,\mathrm{id} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n,A)\|_X^k \leq e^{-nt} e^n t = 1$, as $\|nR(n,A)\|^k \leq 1$. This means $\{S_n(t)\}_{\mathcal{L}(X)} \leq 1$.

Now show that this converges: fix $x \in \mathcal{D}(A)$, compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_n(s)S(m)(t-s)x)\,\mathrm{d}s \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x\,\mathrm{d}s \right\|_X \underbrace{\leq}_{\|S_l\|_{\mathcal{L}(X)} \leq 1} t\|(A_n - A_m)$$

and since $X$ is Banach, it is convergent also. Finally, for $y \in X$, we have

$$\|S_n(t)y - S_m(t)y\|_X \leq \|S_n(t)(y-x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x-y)\|_X \leq 2\|x-y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that $\{S_n(t)y\}$ is Cauchy uniformly on $[0,T], T > 0 \Rightarrow \exists S(t) : S_n(t)y \to S(t)y \forall y \in X, t > 0$. And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain $\{S(t)\}_{t\geq 0}$ is a $c_0$-semigroup.

It remains to answer this question. Is $(A, \mathcal{D}(A))$ the infinitesimal generator of $\{S(t)\}_{t\geq 0}$? Let $(\tilde{A}, \mathcal{D}(\tilde{A})$ be the infinitesimal generator of $\{S(t)\}_{t\geq 0}$. Compute

$$S_n(t)x - x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} S_n(s)x \,\mathrm{d}s = \int_0^t S_n(x)A_n x \,\mathrm{d}s\,,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \leq \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \to 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax \,\mathrm{d}s\,.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t\to 0_+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A} \,\mathrm{on}\, \mathcal{D}(A).$$

The opposite inclusion is simple: fix $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$, and so $\lambda\mathrm{id} - A : \mathcal{D}(A) \to X$ is onto, but also $\lambda\mathrm{id} - A = \lambda\mathrm{id} - \tilde{A}$ on $\mathcal{D}(A)$, and so $\lambda\mathrm{id} - \tilde{A} : \mathcal{D}(A) \to X$ is onto. From the previous theorem, we know $\lambda\mathrm{id} - \tilde{A}$ is one-to-one, so $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Altogether, $A = \tilde{A}, \mathcal{D}(A) = \mathcal{D}(\tilde{A})$.

□

# 7 (Some) exercises

## 7.1 4.3.2025

**Example** (Coefficients for smooth extension)**.** Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for $u \in \mathcal{D}(\mathbb{R}^d)$. Find $\{c_j\}_{j=1}^{k=1}$ in such a way that $Eu \in C^k(\mathbb{R}^d)$. Moreover, take $d = 1$.

*Proof.* For $k = 0, j = 1$ we take $c_1 = 1, c_j = 0, j \neq 1$. For $k = 1$, compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k=1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take $x_d = 0$ in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \ldots, k\}.$$

That is a linear system of $k+1$ equations. Is it solvable? □

### 7.2 8.4.2025

**Example** (Laplace)**.** Let $a_0 = 0, a(\cdot, z, p) = p$. Then $|a(\dots)| \le |p|$,growth can be accomplished for $r = 2$, $a(\dots) \cdot p \ge |p|^2$. We can thus apply the theorem to our laplace equation

**Example.** Let $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}\left(1 + |p|^2\right)$. Then it is clearly Caratheodory, it is bounded $|a(\dots)| \le |p| \frac{\pi}{2}$, so the growth conditions yield, it is coercive as $\operatorname{atan}\left(1 + |p|^2\right) \ge \frac{\pi}{4}|p|^2$, and it is monotone

$$\left(\operatorname{atan}\left(1 + |p_1|^2\right)p_1 - \operatorname{atan}\left(1 + |p_2|^2\right)p_2\right)(p_1 - p_2) = \int_0^1 \sum_{j=1}^d \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{atan}\left(1 + |p_2 + s(p_1 - p_2)|^2\right)(p_2 + s(p_1 - p_2))\,\mathrm{d}s\,(p_1 - p_2)_j$$