Classical problems in continuum mechanics

Kamil Belan

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1 Curvilinear coordinates, tensor & vector calculus

How to write $\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u}$ etc. in polar, cylindrical and other coordinates? Notice that there are similiarities between change of coordinats $\mathbf{x} = \mathbf{x}(\boldsymbol{\gamma})$ and deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{x})$, the $\mathbb{B}$ tensor and the metric tensor $\mathbb{g}$.

1.1 Curvilinear coordinates

Let us have $x^1, x^2, \dots, x^n$ cartesian coordinates and a different set $\mathbf{x} = \mathbf{x}(\mathbf{y})$, for example $x = r\cos\varphi, y = r\sin\varphi, [x, y] = [x^1, x^2], [r, \varphi] = [y^1, y^2]$. That means every point on a plane can be described by using $[x^1, x^2]$ or $[r, \varphi]$. We are used to analysis in cartesian coordinates - how can i do it in a more general setting?

Remark. The name curvilinear coordinates come from the fact that the lines $y^k = \text{const}$ are not "straight lines"

Definition 1 (Coordinate lines)**.** Coordinate lines/curves are the curves

$$\boldsymbol{\gamma}^j(y^j) = \mathbf{x}(y^1, \dots, y^j, \dots, y^n).$$

1.1.1 Basis of a vector space

In cartesian coordinates: $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where the vectors are *tangent to the coordinate lines*, that is

$$\mathbf{e}_i = \frac{\partial \boldsymbol{\gamma}^i}{\partial x^i}. \tag{1}$$

In a curvilinear coordinate system, we can repeat the same construction. We can *define a vector tangent to the coordinate line*

$$\mathbf{g}_i(\mathbf{y}) = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}y^i}\left(y^i\right) = \frac{\partial \mathbf{x}}{\partial y^i}\left(y^1, \dots, y^i, \dots y^n\right) \tag{2}$$

The problem is that the vectors $\mathbf{g}_i$ are not constant in space! It is a vector field!.

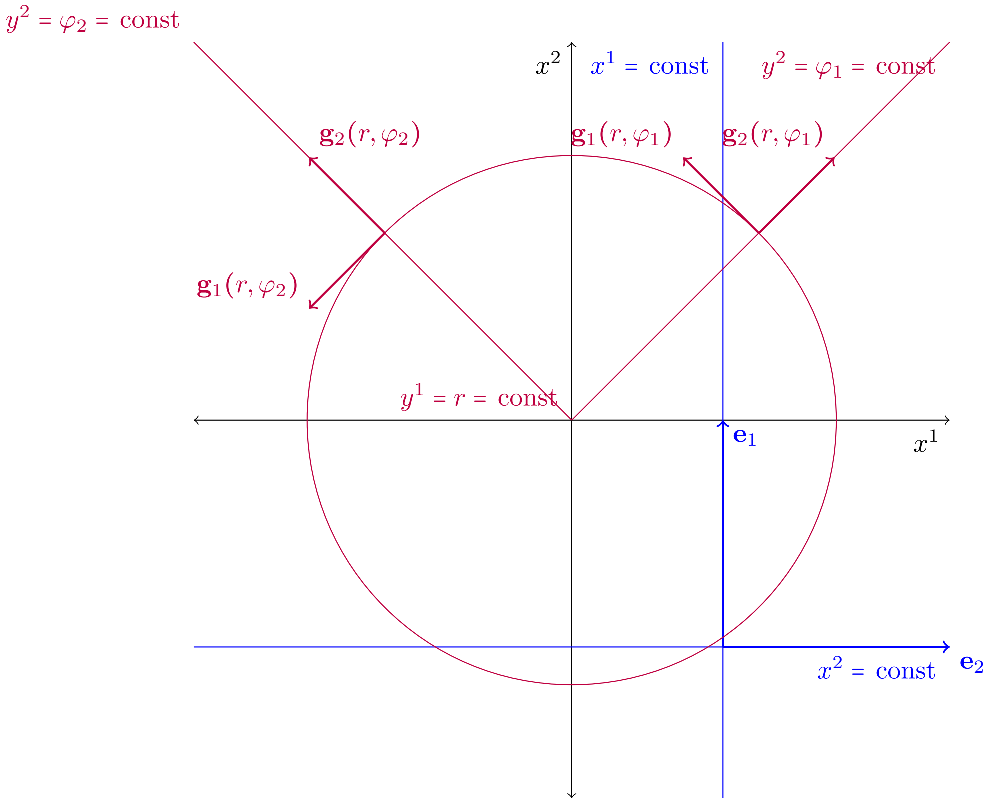

Figure 1: Coordinate lines and basis vectors in cartesian and polar coordinates (*the length of the vectors is the same...*)

1.1.2 Vector fields

A vector $\mathbf{v}$ is independent of a basis; i can write $\mathbf{v} = v^i \mathbf{e}_i = \nu^i \mathbf{g}_i$. (Note that in general $v^i \neq \nu^i$.) What about its derivatives?

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial \left(v^j \mathbf{e}_j \right)}{\partial x^i} = \frac{\partial v^j}{\partial x^i} \mathbf{e}_j,$$

works perfectly fine in cartesian coordinates, as $\mathbf{e}_j$ = const. In curvilinear setting

$$\frac{\partial \mathbf{v}}{\partial y^i} = \frac{\partial \left(v^j \mathbf{g}_j \right)}{\partial y^i} = \frac{\partial v^j}{\partial y^i} \mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial y^i}, \tag{3}$$

as generally $\frac{\partial \mathbf{g}_j}{\partial y^i} \neq \mathbf{0}$. We can identify the last term, as it must be a vector:

$$\frac{\partial \mathbf{g}_j}{\partial y^i} = \Gamma^k_{ji} \mathbf{g}_k,$$

where Γ^k_{ji} are the coefficients of the linear combinations. Thanks to the *commutation of the partial derivatives* [1], it holds

$$\Gamma^k_{ji} = \Gamma^k_{ij}, \tag{4}$$

i.e., Γ^k_{ij} is symmetric in ij. Well, that did not help *very* much, as we don't know Γ^k_{ji}, but at least we have the symmetry property. Going back to 3:

$$\frac{\partial \mathbf{v}}{\partial y^i} = \frac{\partial v^j \mathbf{g}_j}{\partial y^i} = \frac{\partial v^j}{\partial y^i} \mathbf{g}_j + v^j \frac{\partial \mathbf{g}_j}{\partial y^i} = \frac{\partial v^k}{\partial y^i} \mathbf{g}_k + v^j \Gamma^k_{ij} \mathbf{g}_k = \left(\frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j \right) \mathbf{g}_k. \tag{5}$$

In short $\frac{\partial \mathbf{v}}{\partial y^i} = \left(\frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j \right) \mathbf{g}_k$. Compare it to $\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial v^k}{\partial x^i} \mathbf{e}_k$. This leads us to the definition

Definition 2 (Covariant derivative of a vector field)**.** The quantity:

$$v^k|_i = \frac{\partial v^k}{\partial y^i} + \Gamma^k_{ij} v^j, \tag{6}$$

is called the **covariant derivative of the vector field v**

1.1.3 Dot product

The number $\mathbf{v} \cdot \mathbf{u}$ is obtained in a special manner:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{e}_i \cdot u^j \mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{e}_j) v^i u^j = \delta_{ij} v^i u^j.$$

I can of course write the vectors in a different basis:

$$\mathbf{v} \cdot \mathbf{u} = v^i \mathbf{g}_i \cdot u^j \mathbf{g}_j = (\mathbf{g}_i \cdot \mathbf{g}_j) v^i u^j = g_{ij} v^i u^j.$$

Definition 3 (Metric tensor)**.** The tensor $\mathbb{g}$ such that $\forall \mathbf{v} = v^i \mathbf{g}_i, \mathbf{u} = u^j \mathbf{g}_j$ it holds:

$$\mathbf{v} \cdot \mathbf{u} = g_{ij} v^i u^j, g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

is called the **metric tensor**.

[1] We are still in flat $\mathbb{R}^d$, i.e. euclidian space. No curvature, torsion, that would obstruct the comutation properties.

1.1.4 Dual space

The (vector) dual space is the space of all linear forms on the underlying vector space. In particular it is a vector basis itselfs, so $\forall \mathbf{l} \in V^* : \mathbf{l} = l_i \mathbf{e}^i$, where $\mathbf{e}^i$ is the i-th basis vector. The action of the forms can be described as

$$\mathbf{l}(\mathbf{v}) = l_i \mathbf{e}^i (v^j \mathbf{e}_j) = l_i v^j \mathbf{e}^i (\mathbf{e}_j), \forall \mathbf{v} \in V.$$

If it holds $\mathbf{e^i}(\mathbf{e}_j) = \delta^i_j$, we call the basis $\mathbf{e}^i$ dual to $\mathbf{e}_j$. What about curvilinear setting? We can adopt the same definition

Definition 4. We call the basis $\mathbf{g}^j$ of V^* the dual basis to $\mathbf{g}_i$ iff

$$\mathbf{g}^j(\mathbf{g}_i) = \delta^j_i.$$

For the original basis we had $\mathbf{g}_i = \frac{\partial \gamma}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\mathbf{e}_j$, in the dual case (using the chain rule):

$$\delta^i_j = \frac{\partial y^i}{\partial y^j} = \frac{\partial y^i}{\partial x^k}\frac{\partial x^k}{\partial y^j},$$

so i can conclude

$$\mathbf{g}^j = \frac{\partial y^j}{\partial x^k}\mathbf{e}^k.$$

Recall that we have the *Riesz representation theorem*:

$$\forall \mathbf{l} \in V^*, \exists \quad unique \mathbf{u} \in V : \mathbf{l}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{u}, \forall \mathbf{v} \in V.$$

This implies

$$l_i \mathbf{g}^i (v^j \mathbf{g}_j) = u^m u^n g_{mn}, \quad i.e. l_i v^j \delta^i_j = u^m v^n g_{mn}, \quad i.e. l^i v_i = u^m v^i g_{mi}, \quad i.e. l_i = g_{im} u^m.$$

So $l_i = g_{im}u^m$, where $\mathbf{u}$ represents $\mathbf{l}$. It is common to write

$$l_i = g_{im} l^m.$$

1.1.5 Covector fields

How to compute $\frac{\partial \mathbf{l}}{\partial y^i}$? Just change the location of the index :)

$$\frac{\partial \mathbf{l}}{\partial y^i} = \frac{\partial (l_j \mathbf{g}^j)}{\partial y^i} = \frac{\partial l^j}{\partial y^i}\mathbf{g}^j + \frac{\partial \mathbf{g}^j}{\partial y^i} l_j.$$

Again, the last term must be expressable in the dual basis, so

$$\frac{\partial \mathbf{l}}{\partial y^i} = \frac{\partial l_j}{\partial y^i}\mathbf{g}^j + \tilde{\Gamma}^j_{im} l_j \mathbf{g}^m, \tag{7}$$

where again $\Gamma^{\tilde{j}}_{im} = \Gamma^{\tilde{j}}_{mi}$ are the coefficients of the linear combinations, "that are symmetric".

What is the relation between Γ^k_{im} and $\Gamma^{\tilde{k}}_{im}$? Recall $\delta^i_j = \mathbf{g}^i(\mathbf{g}_j)$, so differentiating can lead us to

$$\Gamma^j_{im} = -\Gamma^j_{im}. \tag{8}$$

Definition 5. Let $\mathbf{l}$ be a covector field. The quantity

$$\mathbf{l}_m|_j = \frac{\partial \mathbf{l}}{\partial y^j} - \Gamma^l{}_{jm} v_l, \tag{9}$$

is called **the covariant derivative of the covector field l**.

TODO $\mathbb{A} = A_{mn} \mathbf{g}^m \otimes \mathbf{g}^n$.

1.1.6 Direct expression of the Christoffel symbols

With the above relation, we can express $g_{mn}|_j$. Moreover, we can directly differentiate.

$$\frac{\partial g_{mn}}{\partial y^j} = \frac{\partial (\mathbf{g}_m \cdot \mathbf{g}_n)}{\partial y^j} = \frac{\partial \mathbf{g}_m}{\partial y^j} \cdot \mathbf{g}_n + \mathbf{g}_m \frac{\partial \mathbf{g}_n}{\partial y^j} = \Gamma^k{}_{mj} \mathbf{g}_k \cdot \mathbf{g}_n + \mathbf{g}_m \cdot \Gamma^k{}_{nj} \mathbf{g}_k = \Gamma^k{}_{mj} g_{kn} + \Gamma^k{}_{nj} g_{mk}$$

,

$$g_{mn}|_j = \frac{\partial g_{mn}}{\partial y^j} - g_{kn} \Gamma^k{}_{jm} - g_{mk} \Gamma^k{}_{jn}.$$

From this, it follows

$$g_{mn}|_j = 0. \tag{10}$$

This property is particularly useful, as it allows us to express the Christoffel symbols. Using cyclic permuation, we can write

$$\begin{aligned} A &= \frac{\partial g_{mn}}{\partial y^j} = \Gamma^k{}_{mj} g_{kn} + \Gamma^k{}_{nj} g_{mk}, \\ B &= \frac{\partial g_{jm}}{\partial y^j} = \Gamma^k{}_{jn} g_{kn} + \Gamma^k{}_{mn} g_{jk}, \\ C &= \frac{\partial g_{nj}}{\partial y^n} = \Gamma^k{}_{nm} g_{kj} + \Gamma^k{}_{jm} g_{nk}. \end{aligned}$$

Taking $A - B - C$ yields

$$\frac{\partial g_{mn}}{\partial y^j} - \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} = -2\Gamma^k{}_{nm} g_{jk},$$

multiplying by g^{jl} gives

$$-2\Gamma^k{}_{nm} \delta^l_k = g^{jl} \left(\frac{\partial g_{mn}}{\partial y^j} - \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} \right),$$

from which it follows

$$\Gamma^l{}_{nj} = \frac{1}{2} g^{lm} \left(\frac{\partial g_{mn}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^n} - \frac{\partial g_{nj}}{\partial y^m} \right). \tag{11}$$

1.1.7 Interchangability of the derivatives

In euclidian space:

$$\frac{\partial^2 \mathbf{v}}{\partial y^j \partial y^i} = \frac{\partial^2 (v^k \mathbf{e}_k)}{\partial y^j \partial y^i} = \left(\frac{\partial^2 v^k}{\partial x^j \partial x^i} \right) \mathbf{e}_k,$$

when $\mathbf{e}_k$ are basis vectors of *cartesian coordinate system.* Will it hold even in curvilinear coordinate systems?

$$0 = \frac{\partial^2 \mathbf{v}}{\partial x^j \partial x^k} - \frac{\partial^2 \mathbf{v}}{\partial x^j \partial x^k} = \text{apply the covariant derivative two times} =$$
$$= \left(v^k|_{ij} - v^k|_{ji}\right)\mathbf{g}_k = \left(\frac{\partial \Gamma^i{}_{jm}}{\partial y^k} - \frac{\partial \Gamma^i{}_{km}}{\partial y^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}\right)v^m \mathbf{g}_i.$$

Definition 6 (Riemann curvature tensor)**.** The *tensor*

$$R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial y^k} - \frac{\partial \Gamma^i{}_{km}}{\partial y^i} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km}, \tag{12}$$

is called the **Riemann curvature tensor**

We see that if the Riemann curvature tensor is zero, then effectively, we are in the case of a flat euclidian space, as the derivatives commute (?). In other words, in flat euclidian space, the Riemann curvature tensor is always zero. If we flip this, we see that if we have a space with zero Riemann curvature tensor, *we have a chance* that the derivatives commute, i.e. that the structure is euclidian.

Example (Interpretation in continuum mechanics)**.**

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) \text{ vs } \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}),$$
$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial y^i} \text{ vs } \mathbf{g}_i = \frac{\partial \boldsymbol{\chi}}{\partial X^i}, \text{ i.e. } (\mathbf{g}_m)^i = F^i{}_m = \frac{\partial \chi^i}{\partial X^m},$$
$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = (\mathbb{F}^\top \mathbb{F})_{ij} = (\mathbb{C})_{ij},$$

So

$$\mathbb{g} = \mathbb{C}. \tag{13}$$

We can calculate:

$$\frac{\partial \mathbf{g}_m}{\partial X^j} = \Gamma^k{}_{mj}\mathbf{g}_k,$$

which is a system of equations for the basis vectors. There are some solvability conditions

$$\partial_{JI}\mathbf{g}_m = \partial_{IJ}\mathbf{g}_m = \frac{\partial\left(\Gamma^k{}_{im}\mathbf{g}_k\right)}{\partial X^j} = \frac{\partial\left(\Gamma^k{}_{mj}\mathbf{g}_k\right)}{\partial X^i},$$

this is equivalent to

$$\partial_{JI}\mathbf{g}_m - \partial_{IJ}\mathbf{g}_n = 0 \Leftrightarrow \dots R^i{}_{jkm} = 0.$$

So this implies that all physically admissable deformations produce a deformed space with zero Riemann curvature.

1.2 Calculus

1.2.1 Gradient

Remember $(\nabla\varphi)_i = \frac{\partial\varphi}{\partial x^i}$, so that means

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{e}^i.$$

The gradient is a covector.

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\mathbf{e}^i = \frac{\partial\varphi}{\partial\xi^j}\underbrace{\frac{\partial\xi^j}{\partial x^i}\mathbf{e}^i}_{=\mathbf{g}^j} = \frac{\partial\varphi}{\partial\xi^j}\mathbf{g}^j,$$

What about the gradient of a vector field? In cartesian coordinate system:

$$\nabla\mathbf{v} = \nabla\left(v^i\mathbf{e}_i\right) = \frac{\partial v^i}{\partial x^j}\mathbf{e}_i\otimes\mathbf{e}^j = \mathbf{v}\otimes\nabla.$$

In curvilinear coordinates:

$$\begin{aligned}
\nabla\left(v^i\mathbf{g}_i\right) &= \nabla\left(v^i\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right) = \frac{\partial}{\partial x^j}\left(v^i\frac{\partial x^m}{\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \left(\frac{\partial v^i}{\partial x^j}\frac{\partial x^m}{\partial\xi^i} + v^i\frac{\partial^2 x^m}{\partial x^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \left(\frac{\partial v^i}{\partial\xi^n}\frac{\partial\xi^n}{\partial x^j}\frac{\partial x^m}{\partial\xi^i} + v^i\frac{\partial^2 x^m}{\partial x^j\partial\xi^i}\right)\mathbf{e}_m\otimes\mathbf{e}^j = \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial x^m}{\partial\xi^u}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial x^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial x^m}{\partial\xi^i}\right)\frac{\partial\xi^l}{\partial x^j}\mathbf{e}_m\otimes\mathbf{e}^j = \\
&= \frac{\partial v^i}{\partial\xi^n}\left(\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^n}{\partial x^j}\mathbf{e}^j\right) + v^i\frac{\partial}{\partial\xi^l}\left(\frac{\partial x^m}{\partial\xi^i}\mathbf{e}_m\right)\otimes\left(\frac{\partial\xi^l}{\partial x^j}\mathbf{e}^j\right) = \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\frac{\partial\mathbf{g}^i}{\partial\xi^l}\otimes\mathbf{g}^l = \\
&= \frac{\partial v^i}{\partial\xi^n}\mathbf{g}_i\otimes\mathbf{g}^n + v^i\Gamma^s{}_{il}\mathbf{g}_s\otimes\mathbf{g}^l = \left(\frac{\partial v^s}{\partial\xi^l} + \Gamma^s{}_{il}v^i\right)\mathbf{g}_s\otimes\mathbf{g}^l = \\
&= v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l.
\end{aligned}$$

Until now, we have not discussed the fact $|\mathbf{g}_i| \neq 1$, which is a kind of a problem. Let us define

$$\mathbf{v} = v^i\mathbf{g}_i = v^i|\mathbf{g}_i|\frac{\mathbf{g}_i}{|\mathbf{g}_i|} = v^{\hat{i}}\mathbf{g}_{\hat{i}},$$

where we have defined

$$v^{\hat{i}} = |\mathbf{g}^i|v^i, \mathbf{g}_{\hat{i}} = \frac{\mathbf{g}_i}{|\mathbf{g}_i|}.$$

But! the differential formulas work for $v^i, \mathbf{g}_i$, **not for** $v^{\hat{i}}, \mathbf{g}_{\hat{i}}$**!**

$$\nabla\varphi = \frac{\partial\varphi}{\partial\xi^j}\mathbf{g}^j = |\mathbf{g}^i|\frac{\partial\varphi}{\partial\xi^i}\mathbf{g}^{\hat{i}}, \tag{14}$$

$$\nabla\mathbf{v} = v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l = |\mathbf{g}_s||\mathbf{g}^l|v^s|_l\mathbf{g}_{\hat{s}}\otimes\mathbf{g}^{\hat{l}}, \tag{15}$$

For the divergence of a vector field, we know: $\operatorname{tr}(\mathbf{u}\otimes\mathbf{v})$, so

$$\nabla\cdot\mathbf{v} = \operatorname{tr}(\nabla\mathbf{v}) = \operatorname{tr}\left(v^s|_l\mathbf{g}_s\otimes\mathbf{g}^l\right) = v^s|_s.$$

The divergence of a tensor field a can be tricky, but be guided by the summation convention; for the tensor $\mathbb{A} = A^{is}\mathbf{g}_i \otimes \mathbf{g}_s$ we can define

$$\nabla \cdot \mathbb{A} = A^{is}|_s \mathbf{g}_i.$$

For the tensors of a different type, we need to change the position of the indices to obtain a bivector.

1.2.2 Laplace-Beltrami operator

$$\triangle \varphi = \frac{1}{\sqrt{\det \mathbb{g}}} \frac{\partial}{\partial \xi^i}\left(\sqrt{\det \mathbb{g}} g^{ij} \frac{\partial \varphi}{\partial \xi^j}\right),$$

on one hand:

$$\triangle \varphi = \nabla \cdot \nabla \varphi = (\nabla \varphi)^i|_i = \left(g^{ij} \frac{\partial \varphi}{\partial \xi^j}\right)\bigg|_i =$$

where we have rised the index $(\nabla\varphi)^i = g^{ij}(\nabla\varphi)_j = g^{ij}\frac{\partial\varphi}{\partial\xi^j}$, so using the covariant derivative definition

$$\nabla \cdot \nabla \varphi = \frac{\partial}{\partial \xi^i}\left(g^{ij} \frac{\partial \varphi}{\partial \xi^j}\right) + \Gamma^i{}_{il} g^{lj} \frac{\partial \varphi}{\partial \xi^j},$$

on the other

$$\begin{aligned}
\triangle \varphi &= \frac{1}{\sqrt{\det \mathbb{g}}}\left(\frac{\partial}{\partial \xi^i}\left(\sqrt{\det \mathbb{g}}\right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} \frac{\partial g^{ij}}{\partial \xi^i} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j}\right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}}\left(\frac{1}{2\sqrt{\det \mathbb{g}}} \frac{\partial}{\partial \xi^i}(\det \mathbb{g}) g^{ij} \frac{\partial \varphi}{\partial \xi^j} + \sqrt{\det \mathbb{g}} \frac{\partial g^{ij}}{\partial \xi^i} + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j}\right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}}\left(\frac{1}{2} \operatorname{tr}\left(\mathbb{g}^{-1} \frac{\partial \mathbb{g}}{\partial \xi^i}\right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} - \sqrt{\det \mathbb{g}}\left(\Gamma^j{}_{kn} g^{in} - \Gamma^i{}_{km} g^{mj}\right) + \sqrt{\det \mathbb{g}} g^{ij} \frac{\partial^2 \varphi}{\partial \xi^i \partial \xi^j}\right) = \\
&= \frac{1}{\sqrt{\det \mathbb{g}}}\left(\frac{1}{2}\left(g^{mn} \frac{\partial g_{mn}}{\partial \xi^i}\right) g^{ij} \frac{\partial \varphi}{\partial \xi^j} - \right)
\end{aligned}$$

1.2.3 Bipolar coordinates

Define $\boldsymbol{\xi} = [\alpha, \beta]$, where

$$\alpha + i\beta = \log \frac{y + i(x+a)}{y + i(x-a)}.$$

This can be inversed and write

$$x = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta},$$
$$y = \frac{a \sin \beta}{\cosh \alpha - \cos \beta},$$

moreover,

$$
\begin{aligned}
(x - a\coth\alpha)^2 + y^2 &= \frac{a^2}{\sinh^2\alpha},\\
x^2 + (y - a\cot\beta)^2 &= \frac{a^2}{\sin^2\beta}.
\end{aligned}
$$

Calculate *everything* for this coordinate system.

In general $\mathbf{g}_i = \frac{\partial x^j}{\partial \xi^i}\mathbf{e}_j$, so in our case

$$
\begin{aligned}
\mathbf{g}_\alpha &= \frac{\partial x}{\partial \alpha}\mathbf{e}_x + \frac{\partial y}{\partial \alpha}\mathbf{e}_y\\
&= \left(\frac{(a\cosh\alpha)(\cosh\alpha - \cos\beta) - a\sinh\alpha\sinh\alpha}{(\cosh\alpha - \cos\beta)^2}\right)\mathbf{e}_x + \left(\frac{a\cos\beta(\cosh\alpha - \cos\beta) - a\sin\beta\sinh\alpha}{(\cosh\alpha - \cos\beta)^2}\right)\mathbf{e}_y =\\
&= \frac{a}{(\cosh\alpha - \cos\beta)^2}((1 - \cosh\alpha\cos\beta)\mathbf{e}_x - (\sin\beta\sinh\alpha)\mathbf{e}_y),\\
\mathbf{g}_\beta &= \frac{\partial x}{\partial \beta}\mathbf{e}_x + \frac{\partial y}{\partial \beta}\mathbf{e}_y\\
&= \cdots =\\
&= \frac{a}{(\cosh\alpha - \cos\beta)^2}(-(\sin\beta\sinh\alpha)\mathbf{e}_x + (-1 + \cosh\alpha\cos\beta)\mathbf{e}_y).
\end{aligned}
$$

We can see that $\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = 0$ and so

$$
\mathbb{g} = \left(\frac{a}{\cosh\alpha - \cos\beta}\right)^2 \mathbb{I}, \mathbb{g}^{-1} = \left(\frac{\cosh\alpha - \cos\beta}{a}\right)^2 \mathbb{I}.
$$

Coming back to Laplace-Beltrami operator, we can calculate

$$
\left(\sqrt{\det \mathbb{g}}\,\mathbb{g}^{-1}\right) = \left(\frac{a}{\cosh\alpha - \cos\beta}\right)^2 \left(\frac{\cosh\alpha - \cos\beta}{\alpha}\right)^2 \mathbb{I} = \cdots = \mathbb{I},
$$

and calculating a bit more yields

$$
\triangle\varphi \to \left(\frac{\cosh\alpha - \cos\beta}{a}\right)^2 \triangle_{\alpha\beta}\varphi.
$$

Remark (Relation to complex analysis). This can be seen as a conformal transformation

$$
\gamma = f(z),
$$

where

$$
\begin{aligned}
\gamma &= \alpha + i\beta,\\
z &= x + iy,
\end{aligned}
$$

Let us write

$$
f(z) = f^x(x, y) + i f^y(x, y) \Leftrightarrow \mathbf{f}(\mathbf{x}) = [f^x(\mathbf{x}), f^y(\mathbf{x})], z = x + iy,
$$

and compute

$$
\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ \frac{\partial f^y}{\partial x} & \frac{\partial f^y}{\partial y} \end{bmatrix}.
$$

Recall Cauchy-Riemmann conditions:

$$\frac{\partial f^x}{\partial x} = \frac{\partial f^y}{\partial y},$$
$$\frac{\partial f^x}{\partial y} = -\frac{\partial f^y}{\partial x},$$

using which the gradient becomes:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f^x}{\partial x} & \frac{\partial f^x}{\partial y} \\ -\frac{\partial f^x}{\partial y} & \frac{\partial f^y}{\partial y} \end{bmatrix},$$

which is an **orthogonal matrix:**

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\top} = \left(\left(\frac{\partial f^x}{\partial x}\right)^2 + \left(\frac{\partial f^y}{\partial y}\right)^2\right)\mathbb{I}.$$

Realize that all this structure comes just from the fact that the transformation is given through a holomorfic function.

1.2.4 Compatibility conditions in linearised elasticity

$$R^i{}_{jkm} = \frac{\partial \Gamma^i{}_{jm}}{\partial \xi^k} - \frac{\partial \Gamma^i{}_{km}}{\partial \xi^j} + \Gamma^i{}_{lk}\Gamma^l{}_{jm} - \Gamma^i{}_{lj}\Gamma^l{}_{km},$$

and we know

$$R^i{}_{jkm} = 0 \Leftrightarrow \mathbb{C} = \mathbb{F}^{\top}\mathbb{F}, \mathbb{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}.$$

All this works in fully *nonlinear setting!*. In the classical lecture, we have been able to obtain compatibility condition in *linearised elasticity*: $\nabla \times \varepsilon = \mathbb{O}, \mathbb{e} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\top}\right)$.

Consider the following setting:

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}), \\ \mathbf{u} &= \chi(\mathbf{X}) - \mathbf{X}, \\ \nabla \mathbf{u} &= \mathbb{F} - \mathbb{I}, \\ \mathbb{F} &= \mathbb{I} + \nabla \mathbf{u}, \end{aligned}$$

then

$$\mathbb{C} = \mathbb{F}^{\top}\mathbb{F} = \left(\mathbb{I} + (\nabla \mathbf{u})^{\top}\right)(\mathbb{I} + \nabla \mathbf{u}) = \mathbb{I} + 2\varepsilon + \text{h.o.t.},$$

and so

$$\mathbb{g}^{-1} = \mathbb{I} - 2\varepsilon.$$

The Christoffel symbols are

$$\begin{aligned} \Gamma^l{}_{nj} &= \frac{1}{2} g^{lm}\left(\frac{\partial g_{mn}}{\partial X^j} + \frac{\partial g_{jm}}{\partial X^n} - \frac{\partial g_{nj}}{\partial X^m}\right) \\ &\approx \frac{1}{2}(\mathbb{I} - 2\varepsilon)^{lm}\left(\frac{\partial}{\partial X^j}(\mathbb{I} + 2\varepsilon)_{mn} + \frac{\partial}{\partial X^n}(\mathbb{I} + 2\varepsilon)_{jm} - \frac{\partial}{\partial X^m}(\mathbb{I} + 2\varepsilon)_{nj}\right), \\ &\approx \delta^{lm}\left(\frac{\partial \varepsilon_{mn}}{\partial X^j} + \frac{\partial \varepsilon_{jm}}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}\right) = \frac{\partial \varepsilon^l_n}{\partial X^j} + \frac{\partial \varepsilon^l_j}{\partial X^n} - \frac{\partial \varepsilon_{nj}}{\partial X^m}, \end{aligned}$$

the Riemann curvature tensor is (linear approximation)

$$
\begin{aligned}
0 = R^i{}_{jkm} &\approx \frac{\partial \Gamma^i{}_{jm}}{\partial X^k} - \frac{\partial \Gamma^i{}_{km}}{\partial X^j} \\
&= \frac{\partial}{\partial X^k}\left(\frac{\partial \varepsilon^i_j}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^j} - \frac{\partial \varepsilon_{mj}}{\partial X^i}\right) - \frac{\partial}{\partial X^j}\left(\frac{\partial \varepsilon^i_k}{\partial X^m} + \frac{\partial \varepsilon^i_m}{\partial X^k} - \frac{\partial \varepsilon_{km}}{\partial X^i}\right) = \\
&= \frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i},
\end{aligned}
$$

so the compatibility conditions are

$$
\frac{\partial^2 \varepsilon_{ij}}{\partial X^k \partial X^m} - \frac{\partial^2 \varepsilon_{mj}}{\partial X^k \partial X^i} - \frac{\partial^2 \varepsilon_{ik}}{\partial X^j \partial X^m} + \frac{\partial^2 \varepsilon_{km}}{\partial X^j \partial X^i} = 0.
$$

1.3 Surface geometry

In this part, we will work with surfaces embedded in $\mathbb{R}^3$.

Let $G = \{\mathbf{u}\} \subset \mathbb{R}^2$ be the parametrization space and $\mathbf{\Phi} : G \subset \mathbb{R}^2 \to \mathbb{R}^3$ is the parametrization, so the points of the surface are

$$
\mathbf{x} = \mathbf{\Phi}(\mathbf{u}), \mathbf{x} \in \mathbb{R}^3.
$$

Definition 7. The indices $i, j, k, \cdots \in \{1,2,3\}$ will denote objects from $\mathbb{R}^3$ and indices $\alpha, \beta, \gamma, \cdots \in \{1,2\}$ will denote indices of objects from $\mathbb{R}^2$.

1.3.1 Tangent and normal vectors

As in the previous story, we can define (basis) tangent vectors:

$$
\mathbf{t}_1 = \frac{\partial \mathbf{\Phi}}{\partial u^1}, \mathbf{t}_2 = \frac{\partial \mathbf{\Phi}}{\partial u^2},
$$

and on surfaces, of importance is also the normal vector

$$
\mathbf{n} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|}.
$$

1.3.2 Distances and angles

The metric tensor on the surface is given by

$$
\mathbb{g}_s = \begin{bmatrix} \mathbf{t}_1 \cdot \mathbf{t}_1 & \mathbf{t}_1 \cdot \mathbf{t}_2 \\ \mathbf{t}_1 \cdot \mathbf{t}_2 & \mathbf{t}_2 \cdot \mathbf{t}_2 \end{bmatrix},
$$

or in context of diff. geo. it is called **the first fundamental form**.

1.3.3 Derivatives

In $\mathbb{R}^3$, we know how to differentiate tangent vectors (using Christoffel symbols). The metric tensor in $\mathbb{R}^3$ is given by

$$
\mathbb{g} = \begin{bmatrix} \mathbb{g}_s & 0 \\ 0 & 1 \end{bmatrix}
$$

Realize that, veiwed in $\mathbb{R}^3$ the relation can be written as

$$\frac{\partial \mathbf{t}_\alpha}{\partial u^\beta} = \Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + b_{\alpha\beta}\mathbf{n},$$

because viewed from $\mathbb{R}^3$, the basis vectors are $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ and we have just denoted $b_{\alpha\beta} = \Gamma^3{}_{\alpha\beta}$.

What about the derivative of the normal vector? From the length of $\mathbf{n}$ we know

$$\mathbf{n}\cdot\mathbf{n} = 1 \Rightarrow \frac{\partial \mathbf{n}}{\partial u^\alpha}\cdot\mathbf{n} = 0,$$

so that means the derivative is perpendicular to the normal direction, so

$$\frac{\partial \mathbf{n}}{\partial u^\alpha} = A^\gamma{}_\alpha \mathbf{t}_\gamma.$$

Next trick is to realize

$$0 = \frac{\partial}{\partial u^\alpha}(\mathbf{n}\cdot\mathbf{t}_\beta) = \frac{\partial \mathbf{n}}{\partial u^\alpha}\cdot\mathbf{t}_\beta + \mathbf{n}\cdot\frac{\partial \mathbf{t}_\beta}{\partial u^\alpha} = A^\gamma{}_\alpha\mathbf{t}_\gamma\cdot\mathbf{t}_\beta + \mathbf{n}\cdot\left(\Gamma^\delta{}_{\alpha\beta}\mathbf{t}_\delta + b_{\alpha\beta}\mathbf{n}\right) = A^\gamma{}_\alpha g_{s,\gamma\beta} + b_{\alpha\beta},$$

from which it follows

$$A^\gamma{}_\alpha = -g^{\gamma\beta}b_{\beta\alpha}.$$

1.3.4 Commutation of derivatives

What are the *implications* of

$$\frac{\partial^2\mathbf{t}_\alpha}{\partial u^\beta\partial u^\gamma} = \frac{\partial^2\mathbf{t}_\alpha}{\partial u^\beta\partial u^\alpha}?$$

Write

$$0 = \frac{\partial^2\mathbf{t}_\alpha}{\partial u^\beta\partial u^\gamma} - \frac{\partial^2\mathbf{t}_\alpha}{\partial u^\beta\partial u^\alpha} = (\text{ something })\mathbf{t}_\delta + (\text{ something different })\mathbf{n},$$

so we see the whole thing splits into two parts. It can be shown

Theorem 1 (Gauss relation)**.**

$$R_{\psi\beta\delta\alpha} = b_{\alpha\beta}b_{\psi\delta} - b_{\alpha\delta}b_{\psi\beta}.$$

Theorem 2 (Codazzi-Mainardi relation)**.**

$$b_{\alpha\beta}|_\delta - b_{\alpha\delta}|_\beta = 0$$

1.3.5 Surfaces evolving in time

Now the points of the surface are given by

$$\mathbf{x} = \mathbf{\Phi}(t,\mathbf{u}), \text{ where } \mathbf{\Phi} : \mathbb{R}\times G \to \mathbb{R}^3.$$

We can define the **velocity of the surface**:

$$\mathbf{v}_s = \frac{\partial \mathbf{\Phi}}{\partial t}(t,\mathbf{u}).$$

The basis of everything has always been Gauss theorem; we will be interested in the quantity of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \psi(t,\mathbf{x})\,\mathrm{d}S\,,$$

where $S(t)$ is a time-dependent surface. Let us try the approach from Reynolds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \psi(t,\mathbf{x})\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\boldsymbol{\Phi}(t,\mathbf{x}(t))^{-1}} \psi(t,\boldsymbol{\Phi}(t,\mathbf{u}))\sqrt{\det \mathbb{g}_s}\,\mathrm{d}u^1\,\mathrm{d}u^2 =,$$

and now we need to calculate the derivatives. Start slow:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{t}_\alpha}{\mathrm{d}t} &= \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\Phi}}{\partial u^\alpha}(t,\mathbf{u})\right) = \frac{\partial}{\partial u^\alpha}\underbrace{\left(\frac{\partial \boldsymbol{\Phi}}{\partial t}(t,\mathbf{u})\right)}_{=\mathbf{v}_s(t,\mathbf{u})} = \frac{\partial}{\partial u^\alpha}\left(\mathbf{v}_\parallel + v_\perp \mathbf{n}\right) = \\
&= \frac{\partial \mathbf{v}_\parallel}{\partial u^\alpha} + \frac{\partial (v_\perp \mathbf{n})}{\partial u^\alpha} = \frac{\partial\left(v_\parallel^\beta \mathbf{t}_\beta\right)}{\partial u^\alpha} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} + v_\perp \frac{\partial \mathbf{n}}{\partial u^\alpha} = \\
&= \frac{\partial v_\perp^\beta}{\partial u^\alpha}\mathbf{t}_\beta + \frac{\partial \mathbf{t}_\beta}{\partial u^\alpha} v_\parallel^\beta + \frac{\partial v_{per}}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta} b_{\beta\alpha}\mathbf{t}_\gamma = \\
&= \frac{\partial v_\parallel^\beta}{\partial u^\alpha}\mathbf{t}_\beta + v_\parallel^\beta \Gamma^\gamma{}_{\alpha\beta}\mathbf{t}_\gamma + v_\parallel^\beta b_{\alpha\beta}\mathbf{n} + \frac{\partial v_\perp}{\partial u^\alpha}\mathbf{n} - v_\perp g^{\gamma\beta} b_{\alpha\beta}\mathbf{t}_\gamma = \\
&= v_\parallel^\beta|_\alpha \mathbf{t}_\beta - v_\perp g^{\gamma\beta} b_{\alpha\beta}\mathbf{t}_\gamma + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n} = \\
&= \left(v_\parallel^\beta|_\alpha - v_\perp g^{\beta\gamma} b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n}.
\end{aligned}$$

So all in all
$$\frac{\mathrm{d}\mathbf{t}_\alpha}{\mathrm{d}t} = \left(v_\parallel^\beta|_\alpha - v_\perp g^{\beta\gamma} b_{\alpha\gamma}\right)\mathbf{t}_\beta + \left(v_\parallel^\beta b_{\alpha\beta} + \frac{\partial v_\perp}{\partial u^\alpha}\right)\mathbf{n}.$$

Next ingredient is the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{g}_s$, so in components:

$$\frac{\mathrm{d}g_{\alpha\beta}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{t}_\alpha \cdot \mathbf{g}_\beta) = \cdots = v_\parallel^\delta|_\alpha g_{\delta\beta} + v_\parallel^\delta|_\beta g_{\delta\alpha} - 2v_\perp b_{\alpha\beta}.$$

After some further manipulation, the final formula becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \psi(t,\mathbf{x})\,\mathrm{d}S = \int_{S(t)} \frac{\mathrm{d}\psi}{\mathrm{d}t}(t,\mathbf{x}) + \psi(t,\mathbf{x})\left(\nabla\cdot\mathbf{v}_{\parallel S} - 2v_\perp(t,\mathbf{x})K(t,\mathbf{x})\right)\mathrm{d}S\,, \tag{16}$$

where

$$\nabla\cdot\mathbf{v}_{\parallel S} - 2v_\perp K := v^\beta(t,\mathbf{u})_\parallel|_\beta - 2v_\perp(t,\mathbf{u})K(t,\mathbf{u})\Big|_{\mathbf{u}=\boldsymbol{\Phi}(t,\mathbf{x})^{-1}}\,,$$

$$K = \frac{1}{2}g^{\beta\alpha}b_{\alpha\beta}$$

is the mean curvature.

2 Linearised elasticity

The static setting of the linearised elasticity theory is

$$\nabla \cdot \tau + \mathbf{f} = \mathbf{0}, \tag{17}$$

and for now we will want to solve for the stress, that is

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ & \tau_{yy} & \tau_{yz} \\ & & \tau_{zz} \end{bmatrix},$$

since τ is symmetric. Recall the compatibility conditions

$$\tau = \lambda(\operatorname{tr}\varepsilon)\mathbb{I} + 2\mu\varepsilon, \tag{18}$$

$$\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top}\right), \tag{19}$$

$$\nabla \times \left((\nabla \times \varepsilon)^{\top}\right) = \mathbb{O}, \tag{20}$$

$$\triangle\,\tau + \frac{1}{1+\nu}\nabla\nabla\operatorname{tr}\tau = -\left(\nabla\mathbf{f}(\nabla\mathbf{f})^{\top}\right) - \frac{\nu}{1-\nu}(\nabla\cdot\mathbf{f})\mathbb{I} \tag{21}$$

2.1 Plane stress/strain problems

In each of the cases, the stress/strain tensors have a *special structure*:

$$\tau(x,y) = \begin{bmatrix} \tau_{xx}(x,y) & \tau_{xy}(x,y) & 0 \\ \tau_{xy}(x,y) & \tau_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

and the same for the strain tensor. Inverting the stress-strain relation yields

$$\varepsilon = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau)\mathbb{I}\right),$$

but since $I_{zz} = 1$, in general we have

$$\varepsilon(x,y) = \begin{bmatrix} \varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0 \\ \varepsilon_{xy}(x,y) & \varepsilon_{yy}(x,y) & 0 \\ 0 & 0 & \varepsilon_{zz}(x,y) \end{bmatrix},$$

for $\varepsilon_{zz}(x,y) \neq 0$.

Remark (Notation)*.* Note that in the following, operators acting on tensors will always respect the dimensionality of the tensor (so i will write $\operatorname{tr}\tau_{2D}$ instead of $\operatorname{tr}_{2D}\tau_{2D}$. And the same for the laplacian, divergence and so on

2.2 Plane stress problem

The stress is given as

$$\tau = 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)\mathbb{I},$$

where τ has the structure 22. It must hold

$$0 = \tau_{zz} = 2\mu\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}),$$

so

$$0 = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + (2\mu + \lambda)\varepsilon_{zz},$$

and that yields a condition on ε_{zz}:

$$\varepsilon_{zz} = -\frac{\lambda}{2\mu + \lambda} \operatorname{tr}_{2D} \varepsilon_{2D}.$$

The constitutive relation can than be rewritten as

$$\tau_{2D} = 2\mu\varepsilon_{2D} + \lambda(\operatorname{tr}\varepsilon_{2D} + \varepsilon_{zz})\mathbb{I}_{2D} = 2\mu\left(\varepsilon_{2D} + \frac{\lambda}{2\mu + \lambda}(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right).$$

The "2D Beltrami-Michel equations" can be derived from:

$$\triangle\,\tau + \frac{1}{1 + \nu}\nabla\nabla \operatorname{tr}\tau = -\left(\nabla\mathbf{f} + (\nabla\mathbf{f})^{\top}\right) - \frac{\nu}{1 - \nu}(\nabla \cdot \mathbf{f})\mathbb{I},$$

but there is a problem: the zz equation yields:

$$0 + \frac{1}{1 + \nu}\frac{\partial^2}{\partial z^2}\left(\operatorname{tr}\tau_{2D} = 0 - \frac{\nu}{1 - \nu}(\nabla \cdot \mathbf{f}_{2D})\right),$$

but in our case τ_{2D} is not a function of z, so of course we would have

$$\nabla \cdot \mathbf{f}_{2D} = 0,$$

which is not generally true! The forces are given to us. Try something different: take the trace of the Beltrami-Michell equation and obtain (after some calculation)

$$\triangle \operatorname{tr}\tau = -\frac{1 + \nu}{1 - \nu}\nabla \cdot \mathbf{f},$$

so rewritting in "2D" view:

$$\left(\triangle_{2D} + \frac{\partial^2}{\partial z^2}\right)\operatorname{tr}\tau_{2D} = -\frac{1 + \nu}{1 - \nu}\nabla \cdot \mathbf{f}_{2D}.$$

That maybe did not help much, because the z derivative is still zero, but here comes the time for some handwaving: what about we use the above equation to replace the troublemaking term? We would obtain

$$\triangle_{2D} \operatorname{tr}\tau_{2D} - \nu\frac{1 + \nu}{1 - \nu}\nabla \cdot \mathbf{f}_{2D} = -\frac{1 + \nu}{1 - \nu}(\nabla \cdot \mathbf{f}_{2D}),$$

so after some manipulation

$$\triangle \operatorname{tr}\tau_{2D} = -(1 + \nu)\nabla \cdot \mathbf{f}_{2D}.$$

In total, the problem is described as

$$\mathbf{0}_{2D} = \nabla \cdot \tau_{2D} + \mathbf{f}_{2D}, \tag{23}$$

$$\tau_{2D} = 2\mu\left(\varepsilon_{2D} + \frac{\lambda}{2\mu + \lambda}(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right) \tag{24}$$

$$\triangle \operatorname{tr}\tau_{2D} = -(1 + \nu)\nabla \cdot \mathbf{f}_{2D}. \tag{25}$$

2.3 Plain strain problem

This time, the structure of the stress and strain are:

$$\varepsilon(x,y) = \begin{bmatrix} \varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0 \\ \varepsilon_{xy}(x,y) & \varepsilon_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tau(x,y) = \begin{bmatrix} \tau_{xx}(x,y) & \tau_{xy}(x,y) & 0 \\ \tau_{xy}(x,y) & \tau_{yy}(x,y) & 0 \\ 0 & 0 & \tau_{zz}(x,y) \end{bmatrix}.$$

Using a similiar approach, we can calculate, using $\varepsilon = \frac{1}{2\mu}\Big(\tau - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau)\mathbb{I}\Big)$,

$$\tau_{zz} = \lambda \operatorname{tr}\varepsilon_{2D},$$

so the constituive relation is

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau_{2D} + \tau_{zz})\mathbb{I}_{2D}\right) = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D} - \frac{\lambda}{3\lambda+2\mu}\lambda(\operatorname{tr}\varepsilon_{2D})\mathbb{I}_{2D}\right),$$

so taking the trace we can obtain: $\tau_{zz} = \frac{\lambda}{\lambda(\lambda+\mu)}\operatorname{tr}\tau_{2D}$ and plugging it into the original equation yields

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{2(\lambda+\mu)}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D}\right).$$

As for the Beltami-Michell equations, taking the trace gives us again

$$\triangle \operatorname{tr}\tau = -\frac{1+\nu}{1-\nu}\nabla\cdot\mathbf{f},$$

and in *plain strain*, we are able to simply do

$$\triangle \operatorname{tr}\tau_{2D} = -\frac{1}{1-\nu}\nabla\cdot\mathbf{f}_{2D},$$

without any magic. In total, the equations we are solving are

$$\mathbf{0}_{2D} = \nabla\cdot\tau_{2D} + \mathbf{f}, \tag{26}$$

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\tau_{2D} - \frac{\lambda}{2(\lambda+\mu)}(\operatorname{tr}\tau_{2D})\mathbb{I}_{2D}\right), \tag{27}$$

$$\triangle \operatorname{tr}\tau_{2D} = -\frac{1}{1-\nu}\nabla\cdot\mathbf{f}_{2D}. \tag{28}$$

2.4 Airy stress function

Let us assume that the force is given as

$$\mathbf{f}_{2D} = -\nabla\varphi,$$

i.e. the force is conservative. Moreover, let us use the following ansatz:

$$\tau_{2D} = \begin{bmatrix} \frac{\partial^2\Phi}{\partial y^2} + \varphi & \frac{\partial^2\Phi}{\partial x\partial y} \\ \frac{\partial^2\Phi}{\partial x\partial y} & \frac{\partial^2\Phi}{\partial x^2} + \varphi \end{bmatrix},$$

for some function $\Phi(x, y)$ called the *Airy stress function*. Why that would be useful? Calculate the divergence of the stress:

$$\nabla \cdot \tau_{2D} = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial y^2} + \varphi\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x \partial y}\right) \\ -\frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x^2} + \varphi\right) \end{bmatrix} = \nabla \varphi,$$

so identically we have

$$\mathbf{0}_{2D} = \nabla \cdot \tau_{2D} - \nabla \varphi,$$

and one of our equations is solved. What about the remaining ones? Beltrami-Michell:

$$\triangle \operatorname{tr} \tau_{2D} = \triangle \left(\triangle \Phi + 2\varphi\right) = \triangle \triangle \Phi + \triangle \varphi.$$

Using this in plain strain case:

$$\triangle \triangle \Phi + \frac{1 - 2\nu}{1 - \nu} \triangle \varphi = 0,$$

and in plain stress case:

$$\triangle \triangle \Phi + (1 - \nu) \triangle \varphi = 0.$$

Let us take a glimpse at the biharmonic equation.

2.5 Bending of a narrow rectangular beam by uniform load

Assume we have a narrow rectangular beam of length L, height h and width b, subjected to the load $q\mathbf{e}_y$, which is constant in the x-direction. $[q] = \frac{\mathrm{N}}{\mathrm{m}}$.

Boundary conditions are *essential*: they specify the problem. In our case, the **front/back face** is traction free:

$$\pm \tau \mathbf{e}_z = \mathbf{0}, \text{ on } \{z = \pm \frac{b}{2}\},$$

the **bottom face** is also stress free:

$$\tau \mathbf{e}_y = \mathbf{0}, \text{ on } \{y = \frac{h}{2}\},$$

the **top face** is subjected to the load

$$\tau \mathbf{e}_y = \frac{-q}{b} \mathbf{e}_y, \text{ on } \{y = -\frac{h}{2}\}.$$

On the lateral faces, we would like *something like*

$$\pm \tau \mathbf{e}_x = \mathbf{f}(y, z), \text{ on } \{x = \pm \frac{L}{2}\},$$

however, in our analysis, we are only interested in the fact whether the force can support the beam - but we dont care about the exact distribution of it. Thus, we require the *balance of forces:*

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{f}(y, z) \, \mathrm{d}y \, \mathrm{d}z = \frac{qL}{2} \mathbf{e}_y, \tag{29}$$

and moreover we require the *balance of torques:*

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathbf{r} \times \mathbf{f}(y,z)\,\mathrm{d}y\,\mathrm{d}z = \mathbf{0}, \tag{30}$$

So the roadplan is to find the stress τ and check whether 29 and 30 are satisfied.

From the symmetry of the load, we assume that

$$\tau^{zz} = 0,$$

so our problem is essentialy a *plane stress problem.* Let us sum up our analysis (this takes some work)

$$\begin{aligned}
t^{xy}\left(x, y = \frac{h}{2}\right) &= 0\\
t^{yy}\left(x, y = \frac{h}{2}\right) &= 0\\
t^{xy}\left(x, y = -\frac{h}{2}\right) &= 0\\
t^{yy}\left(x, y = -\frac{h}{2}\right) &= -\frac{q}{b}\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{xy}\left(x = \pm\frac{L}{2}, y\right)\mathrm{d}y &= \mp\frac{qL}{2},\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{xx}\left(x = \pm\frac{L}{2}, y\right)\mathrm{d}y &= 0,\\
b\int_{-\frac{h}{2}}^{\frac{h}{2}} y\tau^{xx}\left(x = \pm\frac{L}{2}, y\right)\mathrm{d}y &= 0^2
\end{aligned}$$

Remember, that on the lateral sides, x is fixed, so *the coordinates are y and z*; some manipulation with the cross product and stuff is needed, for example:

$$\mathbf{r} \times \tau\mathbf{e}_x = \pm\left(z\tau^{xx}\mathbf{e}_z \times \mathbf{e}_x + z\tau^{xy}\mathbf{e}_z \times \mathbf{e}_y + y\tau^{xx}\mathbf{e}_y \times \mathbf{e}_x + y\tau^{xy}\mathbf{e}_y \times \mathbf{e}_y\right)$$

Evidently, the system is complicated enough. We thus make the following assumptions:

- the material of interest is a homogenous isotropc elastic solid
- the beam is massless $\Leftrightarrow$ the predominant force is the external load (not the body force)

From our work on the plain-stress problem, we know the Airy-stress function will be helpful for us. It will be convenient to find Φ in the form

$$\Phi = \Phi(x,y) = Ay^3 + by^5 + Cyx^2 + Dx^2y^3 + Ex^2,$$

where A, B, C, D, E are some constants fitted so that Φ solves the homogenous biharmonic equation:

$$\triangle \triangle \Phi = 0,$$

(recall that since we have no body forces, $\varphi = 0$.) Once we solve for the stress field, we can obtain the strain field using the constitutive relation and then solve for the displacement (solve a linear PDE) using the definition of the linearised strain tensor; see 2.2.

It can be shown the deflection of the middle point is

$$\delta = \frac{5}{384}\frac{qL^4}{EI_{zz}}\left(1 + \frac{12}{5}\frac{h^2}{L^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right),$$

where I_{zz} is a component of the inertia tensor:

$$I_{zz} = \frac{bh^3}{12}$$

2.6 Biharmonic equation in $\mathbb{R}^2$

Let $\Phi(x,y)$ be the Airy stress function. In the previous, we have come up to the problem of solving

$$\begin{cases} \triangle \triangle \Phi = 0, & \text{in}\, \Omega \subset \mathbb{R}^2, \\ \text{some boundary conditions}\,, & \text{on}\, \partial\Omega. \end{cases}$$

We are in $\mathbb{R}^2$, so we immedietaly use complex analysis: $z = x+iy, x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$, and for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we make the following identification

$$f(x,y) \Leftrightarrow f(z,\overline{z}),$$

and the derivatives are

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial z}(z,\overline{z})\frac{\partial z}{\partial x} + \frac{\partial f}{\partial z}(z,\overline{z})\frac{\partial \overline{z}}{\partial x} = \frac{\partial f}{\partial z}(z,\overline{z}) + \frac{\partial f}{\partial z}(z,\overline{z}),$$
$$\frac{\partial f}{\partial y}(x,y) = i\left(\frac{\partial f}{\partial z}(z,\overline{z}) - \frac{\partial f}{\partial \overline{z}}(z,\overline{z})\right),$$

from which it follows

$$\frac{\partial f}{\partial z}(z,\overline{z}) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x,y) - i\frac{\partial f}{\partial y}(x,y)\right),$$
$$\frac{\partial f}{\partial \overline{z}}(z,\overline{z}) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x,y) + i\frac{\partial f}{\partial y}(x,y)\right).$$

If we now take a look at the laplacian of a function $f(x,y)$, we can formally manipulate:

$$\triangle f(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f(x,y) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f(x,y) = 4\frac{\partial^2 f(z,\overline{z})}{\partial \overline{z}\partial z},$$

so in total

$$\triangle f(x,y) = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}(z,\overline{z}).$$

Using this, we can rewrite the *Laplace equation* to the form

$$\frac{\partial^2 g(z,\overline{z})}{\partial z \partial \overline{z}} = 0.$$

Let us solve it. It must be:

$$\frac{\partial g(z,\overline{z})}{\partial \overline{z}} = C_1(\overline{z}), g(z,\overline{z}) = \underbrace{\int C_1(\overline{z})\,\mathrm{d}\overline{z}}_{:=d_1(\overline{z})} + d_2(z)$$

so

$$g(z,\overline{z}) = d_1(\overline{z}) + d_2(z).$$

Now for the biharmonic equation, we need to solve

$$\frac{\partial^2 \Phi}{\partial z \partial \overline{z}} = d_1(\overline{z}) + d_2(z),$$

so that gives $\frac{\partial \Phi}{\partial \overline{z}} = z d_1(\overline{z}) + D_2(z) + e_1(\overline{z})$, and

$$\Phi(z,\overline{z}) = zD_1(\overline{z}) + \overline{z}D_2(z) + E_1(\overline{z}) + E_2(z).$$

In total, we have been able to derive:

$$\Phi(x,y) = \Re((\overline{z}\gamma(z) + \chi(z)))\Big|_{z=x+iy} = \Re\Big(\overline{(x+iy)}\gamma(x+iy) + \chi(x+iy)\Big).$$

2.7 Elliptic hole in uniformly stressed infinite plane

Suppose an infinite plane with a elliptic hole Ω with the standard paramateres a, b. The boundary conditions are

$$\tau \mathbf{n} = \mathbf{0}, \text{ on } \partial\Omega,$$
$$\lim_{x^2+y^2\to\infty} \tau(x,y) = S\mathbb{I},$$

where $S \in \mathbb{R}$ is given. The problem can be reformulated as

$$\triangle \triangle \Phi(x,y) = 0, \tau = \begin{bmatrix} \frac{\partial^2\Phi}{\partial y^2} & -\frac{\partial^2\Phi}{\partial x \partial y} \\ -\frac{\partial^2\Phi}{\partial x\partial y} & \frac{\partial^2\Phi}{\partial x^2} \end{bmatrix}, \tag{31}$$

plus the boundary conditions. The general representation of the solution is $\Phi = \Re(\overline{z}\psi(z) + \chi(z))$, moreover, we adopt a sensible coordinate system: **elliptical coordinates**

$$z = c\cosh\zeta,$$
$$z = x + iy.$$

Equivalently

$$x = c\cosh\xi\cos\eta,$$
$$y = c\sinh\xi\sin\eta,$$
$$\zeta = \xi + i\eta.$$

It follows immedietaly:

$$\left(\frac{x}{c\cosh\xi}\right)^2 + \left(\frac{y}{c\sinh\xi}\right)^2 = 1,$$
$$\left(\frac{x}{c\cos\eta}\right)^2 - \left(\frac{y}{c\sin\eta}\right)^2 = 1,$$

so the lines ξ = const are *ellipses* and the lines η = const are hyperbolas. This will be useful, as we can represent the boundary of the ellipse $\partial\Omega$ as some coordinate line ξ = const.

Through some simple calculation, we can show

$$\mathbf{g}_\eta = \sqrt{J}(-\sin\alpha\mathbf{e}_1 + +\cos\alpha\mathbf{e}_2),$$
$$\mathbf{g}_\zeta = \sqrt{J}(\cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2),$$

where α is the angle between the x axis and $\mathbf{g}_\xi(\xi,\eta)$ and

$$J = c^2\left(\sinh^2\zeta\cos^2\eta + \cosh^2\zeta\sin^2\eta\right).$$

We are interested in the quantity [3]

$$\exp(i2\alpha) = \frac{\sinh\zeta}{\sinh\overline{\zeta}},$$

Formally, for the normalised vectors, we can write something like

$$\begin{bmatrix}\mathbf{g}_{\hat{\xi}}\\ \mathbf{g}_{\hat{\eta}}\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} =$$

To solve for the stress, we need to express the stress in the elliptical coordinates:

$$\tau = \begin{bmatrix}\tau^{\xi\xi} & \tau^{\xi\eta}\\ \tau^{\xi\eta} & \tau^{\eta\eta}\end{bmatrix} = \tau^{xx}\mathbf{e}_x\otimes\mathbf{e}_x + \cdots = t^{\hat{\xi}\hat{\xi}}\mathbf{g}_{\hat{\xi}}\otimes\mathbf{g}_{\hat{\xi}} + \dots.$$

Thats just some similiarity transformation, we are representing the matrix in a different basis. The traces must be preserved:

$$\tau^{xx} + \tau^{yy} = \tau^{\hat{\xi}\hat{\xi}} + \tau^{\hat{\eta}\hat{\eta}}$$

and similiarly, it can be shown

$$\tau^{\hat{\eta}\hat{\eta}} - \tau^{\hat{\xi}\hat{\xi}} + 2i\tau^{\hat{\xi}\hat{\eta}} = \exp(i2\alpha)(\tau^{yy} - t^{xx} + 2i\tau^{xy}).$$

Combining **all of this** we obtain for the Airy stress function the following relations

$$\tau^{xx} + \tau^{yy} = 4\Re\frac{\mathrm{d}\psi}{\mathrm{d}z},$$
$$\tau^{yy} - \tau^{xx} + 2i\tau^{xy} = 2\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right).$$

[3] That describes the rotation of the coordinate lines, which could mean "the ripping of the ellipse" when pulling

Solving this system (heh) gives

$$
\begin{aligned}
\psi &= \frac{1}{2} S \sinh \zeta, \\
\chi &= \frac{1}{2} S c^2 \zeta \cosh(2\xi_0) \\
t^{\hat{\eta}\hat{\eta}} &= \frac{2S \sinh(2\xi_0)}{\cosh(2\xi_0) - \cos(2\eta)}, \\
\max_{\eta \in (0, 2\pi)} \tau^{\hat{\eta}\hat{\eta}} &= 2S \frac{a}{b}, \\
\min_{\eta \in (0, 2\pi)} \tau^{\hat{\eta}\hat{\eta}} &= 2S \frac{b}{a}.
\end{aligned}
$$

We see that if b is small (the ellipse is very flat), the maximum explodes; the quantity $2\frac{a}{b}$ is called the stress coefficient factor. Just note that even if the stress at infinity is controlled, the stress at the tips can be enormous.