which is equivalent to

$$\Big((a_j \le a_j - \delta) \wedge (a_j < a_j - \frac{\beta}{2} - \delta)\Big) \vee \Big((a_j > a_j - \delta) \wedge (a_j \ge a_j - \frac{\beta}{2} - \delta)\Big).$$

Our choice has been $\delta \in \left(0, \frac{\beta}{2}\right)$, and $\beta > 0$ from the definition of $\Omega \in \mathrm{C}^{0,0}$, so the second statement is clearly true $\forall j \in 1, \ldots, m$. Consequently $\partial\Omega \notin \overline{S}_j$ which leads to $\partial\Omega \subset \Omega_j^\delta$, and since also $\Omega \subset \Omega_j^\delta$, we have $\overline{\Omega} \subset \Omega_j^\delta$.

Approximation of $\tau_\delta u_j$. Since Ω_j^δ is open there $\exists v_j \in \mathrm{C}^\infty\left(\Omega_j^\delta\right)$ such that

$$\|\tau_\delta u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \|\tau_\delta w_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}\left(\Omega_j^\delta\right)} < \frac{\varepsilon}{2(m+1)}.$$

What is more, since $\overline{\Omega} \subset \Omega_j^\delta$, we see $v_j \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ in fact.

Approximation of u.

Finally, let us set

$$v = \sum_{j=0}^{m} v_j.$$

Then $v \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ and it holds

$$\begin{aligned}
\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} &= \left\|\sum_{j=0}^{m} u_j - \sum_{j=0}^{m} v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \left\|\sum_{j=0}^{m} u_j - v_j\right\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \sum_{j=0}^{m} \|u_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \\
&\le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \le \frac{\varepsilon}{m+1} + \sum_{j=1}^{m} \|v_j - \tau_\delta u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} + \sum_{j=1}^{m} \|\tau_\delta u_j - u_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} \\
&< \frac{\varepsilon}{m+1} + 2\sum_{j=1}^{m} \frac{\varepsilon}{2(m+1)} = \varepsilon
\end{aligned}$$

□

Remark (What is $\mathrm{C}^\infty_{\overline{\Omega}}(\mathbb{R}^d)$). Recall

$$C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \left\{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\right\}.$$

In other literature, it is stated that also $\mathrm{C}^\infty\left(\overline{\Omega}\right)$ is dense in $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ if $\Omega \in \mathrm{C}^{0,0}$. This probably means

$$\mathrm{C}^\infty\left(\overline{\Omega}\right) \subset \mathrm{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^d\right).$$

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega)$, does there exist $\overline{u} \in \mathrm{W}^{\mathrm{k,p}}\left(\mathbb{R}^d\right), \quad s.t. \overline{u}|_\Omega = u, \|\overline{u}\|_{\mathrm{W}^{\mathrm{k,p}}(\mathbb{R}^d)} \le C(\Omega)\|u\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$?

The answer is **yes**, if Ω is nice enough.

Lemma 4. *Let* $\alpha, \beta > 0, K \subset U(0,\alpha) \times [\alpha, \beta]$ *be compact. Then*

$$\exists C > 0, \exists E : C^1(\overline{U(0,\alpha)} \times [0,\beta]) \to C^1(\overline{U(0,\alpha)} \times [-\beta,\beta]), \exists \tilde{K} \subset U(0,\alpha) \times [-\beta, b) \text{ compact}$$

such that: