Thermodynamics and mechanics of solids

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Contents

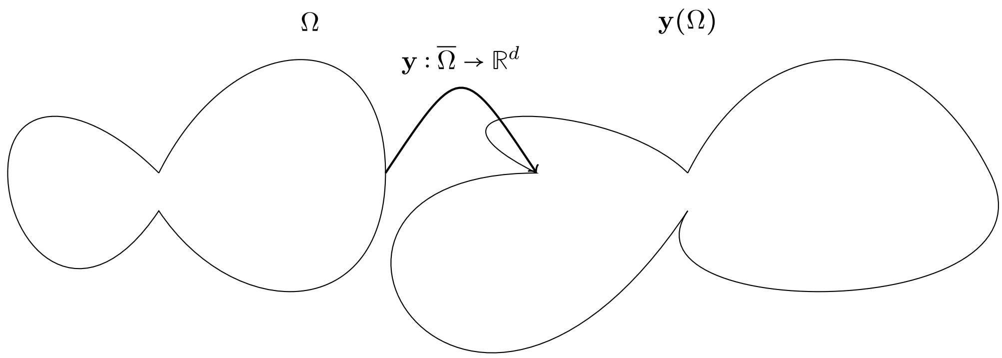
1 TODO

- include missing lecture about potential forces
- include missing lecture about rank one convexity
- include weak convergence symbol
- fix bold greek letters

2 Geometry

2.1 Deformation

Suppose we are given an abstract body $\Omega \subset \mathbb{R}^d, d = 2, 3$. Choosing a particular state, we denote it the **reference configuration**. After the acting of forces, the body deforms into the **current, deformed configuration**.



The mapping that produces the deformation is called the **deformation**, denoted $\mathbf{y}$, i.e.

$$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^d.$$

Of large interest will be the **deformation gradient**

$$\mathbb{F}(\mathbf{x}) = \nabla y(\mathbf{x}), (\nabla y)_{ij} = \frac{\partial y^i}{\partial x^j},$$

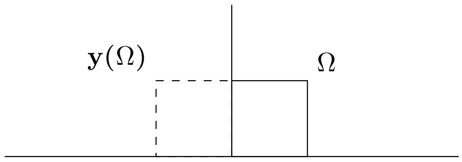
on which we put some physically sound restrictions, such asdet $\mathbb{F} > 0$. This means in particular that the determinant is nonzero, but also that preserves orientations of bases:

$$(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0 \Rightarrow (\mathbb{F}\mathbf{e}_1 \times \mathbb{F}\mathbf{e}_2) \cdot \mathbb{F}\mathbf{e}_3 > 0.$$

Example. Suppose the deformation is given as

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x},$$

i.e., $\mathbb{F} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det \mathbb{F} = -1$. This is an example of a deformation that is *forbidden*.



Imagine it is a sheet of paper in a plane - you cannot reflect it without lifting it from the plane.

2.2 Displacement

Another useful way of describing the deformation is by using the **displacement vector u**:

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x},$$

so taking the gradient gives

$$\nabla \mathbf{u}(\mathbf{x}) = \mathbb{F}(\mathbf{x}) - \mathbb{I}.$$

Remark. It is interesting that we do not place any restrictions on the determinant of the displacement gradient.

2.3 Changes of measures

We need to examine how the volume, area and lengths change under deformation. In what follows, for a set $\omega \subset \mathbb{R}^d$ in the reference configuration we denote $\omega^y \subset \mathbb{R}^d$ to be the deformed body in the current configuration, i.e.

$$\omega^y = \mathbf{y}(\omega).$$

2.3.1 Change of volume

Using the change of variable theorem we obtain

$$\lambda(\omega^y) = \int_{\omega^y} 1 \, \mathrm{d}\mathbf{x}^y = \int_{\omega} \det \mathbb{F}(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

so we write $\mathrm{d}\mathbf{x}^y = \det \mathbb{F} \, \mathrm{d}\mathbf{x}$. This motivates "our" definition of the determinant of the deformation gradient:

$$\det \mathbb{F}(\mathbf{x}) = \lim_{r \to 0} \frac{\lambda(\mathbf{y}(B(\mathbf{x}, r))}{\lambda(B(\mathbf{x}, r))}, \tag{1}$$

where $B(\mathbf{x}, r)$ is a (closed) ball centered at $\mathbf{x}$ of radius r.

2.3.2 Change of lengths

Suppose the line segment $\mathbf{x} + \Delta\mathbf{x}$ undergoes deformation. How does its length change? Taylor expansion yields:

$$\mathbf{y}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \mathbb{F}(\mathbf{x})\Delta\mathbf{x} + \text{h.o.t.},$$

where h.o.t. stands for higher order terms. Using the expansion, we can write

$$\|\mathbf{y}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{y}(\mathbf{x})\|^2 = (\Delta\mathbf{x})^\top \mathbb{F}^\top \mathbb{F} \Delta\mathbf{x} = (\Delta\mathbf{x})^\top \mathbb{C}(\mathbf{x})\Delta\mathbf{x},$$

where we have denoted

$$\mathbb{C}(\mathbf{x}) = \mathbb{F}^\top(\mathbf{x})\mathbb{F}(\mathbf{x}),$$

as the **Right Cauchy Green tensor**.

Example. Let the deformation $\mathbf{y}$ be given as $\mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v}, \mathbf{v} \in \mathbb{R}^d, \mathbb{R} \in \mathrm{SO(d)} = \left\{\mathbb{A} \in \mathbb{R}^{d\times d}, \mathbb{A}^\top\mathbb{A} = \mathbb{A}\mathbb{A}^\top = \mathbb{I}, \det\mathbb{A} = 1^1 \det\mathbb{A} > 0\right\}$. Then $\mathbb{F} = \mathbb{R}$, $\mathbb{C} = \mathbb{I}$.

2.3.3 Change of surfaces

For $\mathbb{A} \in \mathbb{R}^{d\times d}$ regular we define the **cofactor matrix** $\operatorname{cof}\mathbb{A}$ as

$$\operatorname{cof}\mathbb{A} = (\det\mathbb{A})\mathbb{A}^{-\top},$$

which is an interesting quantity whatsoever; we will use the following theorem

Theorem 1 (Piola's identity)**.** *Let* $\mathbf{y} \in C^2(\Omega; \mathbb{R}^d)$*, then* $\forall\mathbf{x} \in \Omega:$

$$\nabla \cdot (\operatorname{cof}\nabla\mathbf{y}(\mathbf{x})) = \mathbf{0}.$$

For a regular matrix $\mathbb{A}$, we also have the identity

$$\mathbb{A}^{-1} = \frac{1}{\det\mathbb{A}}(\operatorname{cof}\mathbb{A})^\top, \tag{2}$$

What about the determinant of the cofactor? Clearly

$$\det\operatorname{cof}\mathbb{A} = (\det\mathbb{A})^d \det\mathbb{A}^{-\top} = (\det\mathbb{A})^{d-1},$$

so we can also express equation 2 in a different way

$$\mathbb{A}^{-1} = \frac{(\operatorname{cof}\mathbb{A})^\top}{(\det\operatorname{cof}\mathbb{A})^{1/d-1}}. \tag{3}$$

From geometry, recall the change of variables for surface integration:

$$\int_{\partial\omega^y} \mathbf{n}^y \,\mathrm{d}S^y = \int_{\partial\omega} \operatorname{cof}\mathbb{F}\mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}^y$ is the outward unit normal to the deformed boundary ω^y. Informally, we write $\mathbf{n}^y\,\mathrm{d}S^y = \operatorname{cof}\mathbb{F}\mathbf{n}\,\mathrm{d}S$. We can also explicitely express the normal to the deformed boundary as

$$\mathbf{n}^y(\mathbf{x}^y) = \frac{\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})}{\|\operatorname{cof}\mathbb{F}(\mathbf{x})\mathbf{n}(\mathbf{x})\|}, \mathbf{x} \in \partial\omega, \mathbf{y}(\mathbf{x}) \in \partial\omega^y. \tag{4}$$

[1] From the fact $\mathbb{A}$ is orthogonal automatically follows $\det\mathbb{A} = \pm 1$.

2.4 Affine transformations

An example of deformation is the so called **affine transformation**.

Example. Suppose the folowing deformation:

$$\mathbf{y}(\mathbf{x}) = \mathbb{A}\mathbf{x} + \mathbf{v}, \mathbb{A} \in \mathbb{R}^{d\times d}, \mathbf{v} \in \mathbb{R}^d, \det \mathbb{F} > 0.$$

Clearly then $\mathbb{F}(\mathbf{x}) = \mathbb{A}$.

It is crucial to realize how $\mathbb{F}, \mathbb{F}^\top, \mathbb{F}^{-\top}$ work.

- $\mathbb{F}$ takes a vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration* and maps it to the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ in the *current configuration*
- $\mathbb{F}^{-1}$ takes the vector $\mathbb{F}\mathbf{x} - \mathbb{F}\mathbf{0}$ from the *current configuration* and maps it to the vector $\mathbf{x} - \mathbf{0}$ from the *reference configuration*
- $\mathbb{F}^\top$ is defined through: $\mathbb{F}\mathbf{u}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbb{F}^\top\mathbf{w}$, and since $\mathbb{F}$ is defined on the reference configuration, $\mathbb{F}^\top$ must take something from the *current configuration* and return something from the *reference configuration.*
- $\mathbb{F}^{-\top}$ consequently takes something from the *referrence configuration* and maps it to something from the *current configuration.*

Example. What when $\mathbb{C} = \mathbb{I}$? Can we say something about $\mathbb{F}$? Write $\mathbb{C} = \mathbb{F}^\top\mathbb{F} = \mathbb{I}$, so $\mathbb{F}^\top = \mathbb{F}^{-1}, \det \mathbb{F} > 0$. From this we have $\mathbb{F}(\mathbf{x}) = \mathbb{R}(\mathbf{x}), \mathbf{x} \in \Omega$, where $\mathbb{R}$ is a rotation tensor. Investigate the cofactor of the deformation gradient:

$$\operatorname{cof} \mathbb{F} = \det \mathbb{F}\mathbb{F}^{-\top} = \operatorname{cof} \mathbb{R} = 1\mathbb{R}(\mathbf{x}) = \mathbb{F}(\mathbf{x}).$$

This implies $\operatorname{cof} \mathbb{F} = \mathbb{F}$. Recall Piola's identity:

$$\mathbf{0} = \nabla \cdot \operatorname{cof} \mathbb{F} = \nabla \cdot \mathbb{F}(\mathbf{x}) = \nabla^2 \mathbf{y}(\mathbf{x}).$$

We have the identity: and since the LHS is zero, we also have $\|\nabla\nabla\mathbf{y}\| = 0 \Rightarrow \mathbf{y}(\mathbf{x}) = \mathbb{R}\mathbf{x} + \mathbf{v}$. Let $\mathbb{R}$ be piecewise affine. Then $\mathbb{R}_1(\mathbb{I} - \mathbf{n}\otimes\mathbf{n}) = \mathbb{R}_2(\mathbb{I} - \mathbf{n}\otimes\mathbf{n})$, so $\mathbb{R}_1 - \mathbb{R}_2 = (\mathbb{R}_1 = \mathbb{R}_2) = (\mathbb{R}_1 - \mathbb{R}_2)\mathbf{n}\otimes\mathbf{n} = \mathbf{a}\otimes\mathbf{b}$, but that is not possible for two rotations; the rank of the RHS is one, whereas the LHS is not.

3 Forces

3.1 Forces in the deformed configuration

Recall $\mathbf{y} : \overline{\Omega} \to \overline{\Omega}^y$.We can define the **volume density of applied forces** $\mathbf{f}^y : \overline{\Omega}^y \to \mathbb{R}^3$ (in newtons per cubic meters, e.g. gravity). The same on the boundary $\mathbf{g}^y : \Gamma_N^y \to \mathbb{R}^3$ (**surface density of applied forces** (in newtons per square meters = Pascals, e.g. hydrostatic pressure.)

3.1.1 Cauchy stress tensor

Lemma 1 (Stress principle of Euler and Cauchy)**.** *There exists a (Cauchy) stress vector function* $\mathbf{t}^y : \overline{\Omega}^y \times \mathcal{S}^{d-1} \to \mathbb{R}^d$ *with the following properties.*

1. *If* $\mathbf{x}^y \in \Gamma_N^y$*, then* $\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbf{g}^y(\mathbf{x}^y)$*, where* $\mathbf{n}^y$ *is the unit outer normal vector to* $\partial\Omega^y$ *at* $\mathbf{x}^y$*.*
2. $\forall \omega^y \subset \Omega^y$ *it holds that* $\int_{\omega^y} \mathbf{f}(\mathbf{x^y}) \,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) \,\mathrm{d}S^y = 0$*. (Balance of forces in static equilibrium.*
3. $\forall \omega^y \subset \Omega^y$ *it holds that* $\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y) \,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) \,\mathrm{d}S^y = \mathbf{0}$*. (Balance of moment of forces in static equilibrium.)*

Euler says that the direct consequence of this is the existence of $\mathbb{T}^y(\mathbf{x}^y)$ *such that*

$$\mathbf{t}^y(\mathbf{x}^y, \mathbf{n}^y) = \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y, \tag{5}$$

where the tensorial quantity $\mathbb{T}$ *is called the* ***Cauchy stress tensor****.*

3.1.2 Balance equations in the deformed configuration

Classical physics gives us 2 fundamental relations: Newtons second law for momenta and for angular momenta. We examine these in the continuum mechanics setting.

From second property it follows:

$$\int_{\omega^y} \mathbf{f}^y(\mathbf{x}^y) \,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y \,\mathrm{d}S^y = \int_{\omega^y} \mathbf{f}^y(\mathbf{x})^y \,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \nabla \cdot \mathbb{T}^y(\mathbf{x}^y) \,\mathrm{d}\mathbf{x}^y = 0, \tag{6}$$

so using the localization theorem we get

$$\mathbf{f}^y(\mathbf{x}^y) + \nabla \cdot \mathbf{T}^y(\mathbf{x}^y) = \mathbf{0}, \forall \mathbf{x}^y \in \Omega^y.$$

From the third property it follows

$$\begin{aligned}
&\int_{\omega^y} \mathbf{x}^y \times \mathbf{f}^y(\mathbf{x}^y) \,\mathrm{d}\mathbf{x}^y + \int_{\partial\omega^y} \mathbf{x}^y \times \mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y \,\mathrm{d}S^y = \\
&= \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y \,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} x_j^y (T_{km}^y n_m^y) \,\mathrm{d}S^y = \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y \,\mathrm{d}\mathbf{x}^y = \int_{\omega^y} \varepsilon_{ijk} \frac{\partial (x_j^y T_{km}^y)}{\partial x_m^y} \,\mathrm{d}\mathbf{x}^y = \\
&= \int_{\omega^y} \varepsilon_{ijk} x_j^y f_k^y \,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} x_j^y \frac{\partial T_k m}{\partial x_m^y} \,\mathrm{d}\mathbf{x}^y + \int_{\omega^y} \varepsilon_{ijk} \delta_{jm} T_{km}^y \,\mathrm{d}\mathbf{x}^y = \mathbf{0}.
\end{aligned}$$

The last term implies

$$\int_{\omega^y} \varepsilon_{ijk} T_{kj}^y = 0,$$

and using the localization theorem, we obtain

$$T_{ij}^y(\mathbf{x}^y) = T_{ji}^y(\mathbf{x}^y), \quad i.e. \mathbb{T}^y(\mathbf{x}^y) = (\mathbb{T}^y(\mathbf{x}^y))^\top. \tag{7}$$

The **Cauchy stress tensor is symmetric.**

3.2 Forces in the undeformed configuration

We have obtained the equations in the deformed configuration. That is however unconvenient - we solve the equations to find the deformed configuration. This brings us to find a new way to write the equations - in the reference configuration. The construction is a bit synthetic, our intuition will be guided by the requirement to obtain similair equations as in the current configuration.

3.2.1 Piola-Kirchhoff stresses

Definition 1 (First Piola-Kirchhoff stress tensor)**.** Given the Cauchy stress tensor $\mathbb{T}^y(\mathbf{x^y})$, we define the **First Piola Kirchhoff stress tensor**

$$\mathbb{T} : \overline{\Omega} \to \mathbb{R}^{3\times3}, \mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{x}^y)\operatorname{cof}\mathbb{F}(\mathbf{x}) = \det\mathbb{F}(\mathbf{x})\mathbb{T}^y(\mathbf{x}^y)\mathbb{F}^{-\top}(\mathbf{x}).$$

Definition 2 (Second Piola-Kirchhoff stress tensor)**.** The quantity

$$\mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1}\mathbb{T}(\mathbf{x}) = \mathbb{S}(\mathbf{x})^\top,$$

is called the **second Piola-Kirchhoff stress tensor.**

Remark. The first PK tensor $\mathbb{T}$ is *not symmetric in general.*, but the second $\mathbb{S}(\mathbf{x}) = \mathbb{F}^{-1} = \det\mathbb{F}(\mathbf{x})\mathbb{F}^{-1}\mathbb{T}^y(\mathbf{x}^y)\mathbb{F}^{-\top}(\mathbf{x})$ *is.* Also, we see that not every matrix can serve as $\mathbb{T}$; it must hold $\mathbb{T}(\mathbf{x})\big(\operatorname{cof}\mathbb{F}^{-1}\big)$ is symmetric.

Remark. We have the following identity (using Piola's identity):

$$\nabla\cdot\mathbb{T}(\mathbf{x}) = \det\mathbb{F}(\mathbf{x})\nabla\cdot\mathbb{T}^y(\mathbf{x}^y)^y. \tag{8}$$

3.2.2 Balance equations in the deformed configuration

Recall the balance of momentum

$$-\nabla\cdot(\mathbb{T}^y(\mathbf{x}^y)) = \mathbf{f}^y(\mathbf{x}^y)\,\text{in}\,\Omega^y,$$

multiplying by $\det\mathbb{F} > 0$ yields

$$\det\mathbb{F}\nabla\cdot(\mathbb{T}^y(\mathbf{x}^y)) = \det\mathbb{F}(\mathbf{x})\mathbf{f}^y(\mathbf{x}^y), \tag{9}$$

which *begs* for the definition

$$\mathbf{f}(\mathbf{x}) = \det\mathbb{F}(\mathbf{x})\mathbf{f}^y(\mathbf{y}(\mathbf{x})),$$

as the force in the *referential configuration.*

In total, the total acting body force on the body can be written as

$$\int_{\mathbf{y}(\omega)}\mathbf{f}^y(\mathbf{x}^y)\,\mathrm{d}\mathbf{x}^y = \int_\omega\mathbf{f}^y(\mathbf{y}(\mathbf{x}))\det\mathbb{F}(\mathbf{x})\,\mathrm{d}x = \int_\omega\mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

We can do the same for the balance of surface forces; the total contact force acting on the body is

$$\begin{aligned}\int_{\Gamma_N^y}\mathbf{g}^y(\mathbf{x}^y)\,\mathrm{d}S^y &= \int_{\partial\omega^y}\mathbf{t}^y(\mathbf{x}^y,\mathbf{n}^y)\,\mathrm{d}S^y = \int_{\partial\mathbf{y}(\omega)}\mathbb{T}^y(\mathbf{x}^y)\mathbf{n}^y\,\mathrm{d}S^y = \\ &= \int_{\partial\omega}\mathbb{T}^y(\mathbf{y}(\mathbf{x}))\operatorname{cof}\mathbb{F}(t,\mathbf{x})\mathbf{n}\,\mathrm{d}S = \int_{\partial\omega}\mathbb{T}(\mathbf{x})\mathbf{n}\,\mathrm{d}S\,,\end{aligned}$$

so if we define

$$\mathbf{g}(\mathbf{x}) = \mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x}),$$

as the contact force in the *referential configuration*, we formally have a similiar expression.

4 Elasticity

Definition 3 (Elasticity)**.** We say that a material is **elastic (or Cauchy elastic)** if there is a response function $\tilde{\mathbb{T}}^D : \Omega \times \mathbb{R}_+^{3\times 3} \to \mathbb{R}_{\text{sym}}^{3\times 3}$ such that

$$\mathbb{T}^y(\mathbf{x}^y) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}).$$

The response function is also called the **constitutive law**.

Remark. If we know the material is elastic, we also have the information about the First Piola-Kirchhoff stress, as $\mathbb{T}(\mathbf{x}) = \mathbb{T}^y(\mathbf{x}^y)\operatorname{cof}\mathbb{F}$, so

$$\mathbb{T}(\mathbf{x}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\operatorname{cof}\mathbb{F} = \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}). \tag{10}$$

4.1 Frame invariance principle

The frame invariance principle states:

$$\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{R}\mathbf{x}) = \mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{R}^\top, \forall \mathbb{R} \in \text{ SO}(3)\,, \forall \mathbf{x} \in \overline{\Omega},$$

from which it follows ($\tilde{\mathbb{T}}$ is defined in 10)

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \det(\mathbb{R}\mathbb{F})\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{R}\mathbb{F})(\mathbb{R}\mathbb{F})^{-\top} = \det(\mathbb{R}\mathbb{F})\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{R}^\top\mathbb{R}\mathbb{F}^{-\top} = \det\mathbb{F}\mathbb{R}\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F})\mathbb{F}^{-\top} = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}),$$

thus

$$\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \mathbb{R}\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}), \text{ i.e. } \mathbb{R}^\top\tilde{\mathbb{T}}(\mathbf{x}, \mathbb{R}\mathbb{F}) = \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}), \forall \mathbb{R} \in \text{ SO}(3)\,, \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}.$$

4.2 Isotropic material

Recall $\mathbb{T}^y(\mathbf{x}^y) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}), \mathbf{y} : \overline{\Omega} \to \Omega^y = \mathbf{y}(\Omega)$. Take $\mathbf{x}_0 \in \overline{\Omega}$ general but fixed, take $\mathbf{v}(\mathbf{z}) = \mathbf{x}_0 + R^\top(\mathbf{z} - \mathbf{x}_0)$ for some $\mathbb{R} \in \text{ SO}(3)$ and define a *new deformation*

$$\tilde{\mathbf{y}} = \mathbf{y} \circ \mathbf{v}^{-1} : \mathbf{v}\big(\overline{\Omega}\big) \to \mathbf{y}\big(\overline{\Omega}\big), \tilde{\mathbf{y}}(\tilde{\mathbf{x}}) = \mathbf{y}(\mathbf{x}_0 + \mathbb{R}(\tilde{\mathbf{x}} - \mathbf{x}_0)).$$

This implies

$$\mathbf{x}_0^y = \mathbf{x}_0^{\tilde{y}}, \mathbb{T}^y(\mathbf{x}_0^y) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \mathbb{F}) = \mathbb{T}^{\tilde{y}}(\mathbf{x}_0^{\tilde{y}}) = \tilde{\mathbb{T}}^D\big(\mathbf{x}_0, \tilde{\mathbb{F}}(\mathbf{x}_0)\big) = \tilde{\mathbb{T}}^D(\mathbf{x}_0, \mathbb{F}(\mathbf{x}_0)\mathbb{R}).$$

Definition 4 (Isotropic material)**.** We cal the material **isotropic** if it holds

$$\tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}) = \tilde{\mathbb{T}}^D(\mathbf{x}, \mathbb{F}\mathbb{R}), \forall \mathbb{R} \in \text{ SO}(3)\,, \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}.$$

Remark. For the first Piola-Kirchhoff we obtain: $\mathbb{T}^D(\mathbf{x}, \mathbb{F}\mathbb{R}) = \mathbb{T}^D(\mathbf{x}, \mathbb{F})\mathbb{R}$, which means

$$\mathbb{T}^D(\mathbf{x}, \mathbb{Q}\mathbb{F}\mathbb{R}) = \mathbb{Q}\tilde{\mathbb{T}}^D\mathbb{R}, \forall \mathbb{R}, \mathbb{Q} \in \text{ SO}(3)\,, \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}.$$

4.3 Hyperelastic materials

Definition 5. We say that a material is hyperelastic if there is a function $W : \overline{\Omega} \times \mathbb{R}_+^{3\times 3} \to \mathbb{R}$ such that

$$\mathbb{T}(\mathbf{x}) = \tilde{\mathbb{T}}(\mathbf{x}, \mathbb{F}) = \frac{\partial W(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}}, \mathbb{F} = \nabla \mathbf{y}(\mathbf{x}).$$

The function $W, [W] = \frac{J}{m^3} = \frac{Nm}{m^3} = \text{Pa}$ is called **stored energy density.**

Remark. Evidently, W has a potential.

4.4 Properties of W

It is physical to assume

1. $W \geq 0$ (energy is nonnegative)
2. $W(\mathbf{x}, \mathbb{F}) = W(\mathbf{x}, \mathbb{RF}), \forall \mathbb{R} \in \text{SO}(3), \forall \mathbf{x} \in \overline{\Omega}, \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}$. (energy does not change under rotations [2]
3. $W\big(\mathbf{x}, \tilde{\mathbb{R}}\mathbb{U}\big) = W(\mathbf{x}, \mathbb{U}), \mathbb{U} = \sqrt{\mathbb{C}}$. (matrices are from the polar decomposition)
4. $W(\mathbf{x}, \mathbb{F}) \to \infty$ if $\det \mathbb{F} \to 0_+$ (it takes infinite energy to deform the body to a point)
5. $W(\mathbf{x}, \mathbb{F}) \geq \alpha\big(\|\mathbb{F}\|^p + \|\text{cof}\,\mathbb{F}\|^q + (\det \mathbb{F})^r\big) - d, \forall \alpha > 0, \forall p, q, r \geq 1, \forall d \in \mathbb{R}, \forall \mathbf{x} \in \overline{\Omega}, \forall \mathbb{F} \in \mathbb{R}_+^{3\times 3}$.

Definition 6 (Natural state of the body)**.** The natural state of the body is the state in which

$$W(\mathbf{x}, \mathbb{F}) = 0 \wedge \frac{\partial W}{\partial \mathbb{F}}(\mathbf{x}, \mathbb{F}) = \mathbb{0}. \tag{11}$$

Remark (Unnatural vegetable)*.* Not all materials (bodies) have its natural states. Zum Beispiel, carrot does not have a natural state.

From the previous work, we can write $\mathbb{R}^\top \frac{\partial W(\mathbf{x}, \mathbb{RF})}{\partial \mathbb{F}} = \frac{\partial W(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}}$, and for brevity denote $W(\mathbf{x}, \mathbb{RF}) = W_R(\mathbf{x}, \mathbb{F})$. Next, we *suppose we can Taylor expand*:

$$\begin{aligned} W_R\big(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\big) &= W\big(\mathbf{x}, \mathbb{RF} + \mathbb{R}\tilde{\mathbb{F}}\big) = W(\mathbf{x}, \mathbb{RF}) + \frac{\partial W(\mathbf{x}, \mathbb{RF})}{\partial \mathbb{F}} : \big(\mathbb{R}\tilde{\mathbb{F}}\big) + \text{h.o.t.} \\ &= W_R(\mathbf{x}, \mathbb{F}) + \mathbb{R}^\top \frac{\partial W(\mathbf{x}, \mathbb{RF})}{\partial \mathbb{F}} : \tilde{\mathbb{F}} + \text{h.o.t.}. \end{aligned}$$

Moreover

$$W_R\big(\mathbf{x}, \mathbb{F} + \tilde{\mathbb{F}}\big) = W_R(\mathbf{x}, \mathbb{F}) + \frac{\partial W_R(\mathbf{x}, \mathbb{F})}{\partial \mathbb{F}} : \tilde{\mathbb{F}}.$$

Altogether

$$\frac{\partial}{\partial \mathbb{F}} (W_R(\mathbf{x}, \mathbb{F}) - W_R(\mathbf{x}, \mathbb{F})) = \mathbb{0},$$

[2] If this was not true, you could create infinite energy by just spinning a rubber.

from which it follows [3]

$$W(\mathbf{x}, \mathbb{RF}) = W(\mathbf{x}, \mathbb{F}) + k(\mathbb{R}).$$

Take $\mathbb{F} = \mathbb{I}$, then

$$W\left(\mathbf{x}, \mathbb{R}^2\right) = W(\mathbf{x}, \mathbb{RR}) = W(\mathbf{x}, \mathbb{R}) + k(\mathbb{R}) = W(\mathbf{x}, \mathbb{I}) + 2k(\mathbb{R}),$$

so

$$W(\mathbf{x}, \mathbb{R}^n) = W(\mathbf{x}, \mathbb{I}) + nk(\mathbb{R}).$$

Since the set of rotations is closed and bounded, it is compact, so there exists a convergent subsequence of $\{\mathbb{R}^n\}$. Moreover, we assume W to be continuous (we took the derivative...), so $\lim_{n\to\infty} W(\mathbf{x}, \mathbb{R}^n)$ exists and from the properties of W we get it is finite. But then $k(\mathbb{R}) = 0$, as otherwise $nk(\mathbb{R}) \to \infty$. All in all, we have shown

$$W(\mathbf{x}, \mathbb{RF}) = W(\mathbf{x}, \mathbb{F}). \tag{12}$$

Definition 7 (Energy functional)**.** Let us have $\partial\Omega = \Gamma_N \cup \Gamma_D, \Gamma_N \cap \Gamma_D = \varnothing$, where the parts of the boundary are those when Neumann/Dirichlet boudary conditions are prescribed. The energy functional of the material is the functional

$$I(\mathbf{y}) = \int_\Omega W(\mathbf{x}, \mathbb{F}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x})\,\mathrm{d}S\,,$$

where the first part corresponds to the stored energy and the remaining terms are the work done by the external loads.

Remark. If $\mathbf{y}$ is the minimizer of I, then $I(t\boldsymbol{\varphi} + \mathbf{y}) \geq I(\mathbf{y}), \forall t, \boldsymbol{\varphi}$. If we denote

$$a(t) := I(t\boldsymbol{\varphi} + \mathbf{y}),$$

then it most hold

$$0 = a'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega W(\mathbb{F} + t\nabla\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g}(\mathbf{x}) \cdot (\mathbf{y}(\mathbf{x}) + t\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}x\right)\Bigg|_{t=0},$$

calculating the derivatives yields

$$\begin{aligned}
0 &= \int_\Omega \frac{\partial W(\mathbb{F})}{\partial \mathbb{F}} : \nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\varphi_i\right)\mathrm{d}\mathbf{x} - \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right)\varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S = \\
&= \int_{\Gamma_N} \frac{\partial W(\mathbb{F})}{\partial F_{ij}}\varphi_i n_j\,\mathrm{d}S - \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right)\varphi_i\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\varphi}\,\mathrm{d}S\,,
\end{aligned}$$

so it must hold

$$-\frac{\partial}{\partial x_j}\left(\frac{\partial W(\mathbb{F})}{\partial F_{ij}}\right) = f_i \text{ in}\,\Omega, \frac{\partial W(\mathbb{F})}{\partial F_{ij}}n_j = g_i \,\text{on}\,\Gamma_N.$$

This is exactly

$$-\nabla \cdot \mathbb{T} = \mathbf{f}\,\text{in}\,\Omega, \mathbb{T}\mathbf{n} = \mathbf{g},\,\text{on}\,\Gamma_N.$$

This implies that $\mathbf{y}$ minimizes energy $\Leftrightarrow$ $\mathbf{y}$ is governed by the equations of classical mechanis.

[3] The set of matrices with positive determinant is connected.

Are there some other qualities of W? It is natural to assume

$$W(\mathbb{I}) = 0 \Rightarrow W(\mathbb{R}) = 0, \forall \mathbb{R} \in \mathrm{SO}(3)$$

and $W(\mathbb{F}) > 0$ whenever $\mathbb{F} \notin \mathrm{SO}(3)$ This however implies W is not convex! Assume

$$\mathbb{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbb{R}_2 = \mathbb{I},$$

then

$$W\left(\frac{1}{4}\mathbb{R}_1 + \frac{3}{4}\mathbb{R}_2\right) > \frac{1}{4}W(\mathbb{R}_1) + \frac{3}{4}W(\mathbb{R}_2) = 0.$$

Example (Minimizer does not exist)**.** Assume $J(u) = \int_0^1 \left(1 - (u'(x))^2\right)^2 + u(x)^2 \,\mathrm{d}x\,, u \in \mathrm{W}^{1,4}(0,1), u(0) = u(1) = 0$, and find the minimum of J. First of all, $J > 0$, so the minimum also. I can take u_k such that $u_k'(x) = 1$ on $(0, 1/2)$ and $u_k'(x) = -1$ on $(1/2, 1)$. Then $J(u_k) \to 0 \Rightarrow \inf J = 0$ but there is no minimizer.

Not everything is lost...

Definition 8 (Polyconvexity, 1977 J.M. Ball)**.** $W : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ is polyconvex provided there exists convex and lower-semicontinuous function $h : \mathbb{R}^{19} \to \mathbb{R} \cup \{\infty\}$:

$$W(\mathbb{A}) = h(\mathbb{A}, \operatorname{cof}\mathbb{A}, \det\mathbb{A}).$$

Example.

- If W is convex and lower-semicontinuous then W is polyconvex.
- $W(\mathbb{A}) = \det\mathbb{A}$ is polyconvex but not convex.

Remark (Weak convergence in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$). Let $1 < p < \infty$ and $\{\mathbf{u_k}\} \subset \mathrm{L_p}(\Omega;\mathbb{R}^3)$. We say $\{\mathbf{u_k}\}$ converges weakly to $\mathbf{u}$ in $\mathrm{L_p}(\Omega;\mathbb{R}^3)$ provided

$$\int_\Omega \mathbf{u_k}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,, \forall\boldsymbol{\varphi} \in \mathrm{L}_{p'}(\Omega;\mathbb{R}^3).$$

Theorem 2 (Magic)**.** *Assume that* $\mathbf{y}^k$ *converges weakly to* $\mathbf{y}$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3 \in C^{0,1}, p > 3$. *Then* $\det\nabla\mathbf{y}^k$ *converges weakly to* $\det\nabla\mathbf{y}$ *in* $L_{\frac{p}{3}}(\Omega)$. *Moreover* $\operatorname{cof}\nabla\mathbf{y}^l$ *converges weakly to* $\operatorname{cof}\nabla\mathbf{y}$ *in* $L_{\frac{p}{2}}(\Omega;\mathbb{R}^{3\times 3})$.

Proof. Only in 2 dimensions. The determinant can be written as:

$$\det\nabla\mathbf{y} = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(y_1\frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1\frac{\partial y_2}{\partial x_1}\right) = \nabla\cdot\left(y_1, -\frac{\partial y_2}{\partial x_1}\right),$$

so then

$$\int_\Omega \det\nabla\mathbf{y}^k\varphi\,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1^k\frac{\partial y_2^k}{\partial x_2}\right)\varphi\,\mathrm{d}x - \int_\Omega \frac{\partial}{\partial x_2}\left(y_1^k\frac{\partial y_2^k}{\partial x_1}\right)\varphi\,\mathrm{d}x = -\int_\Omega y_1^k\frac{\partial y_2^k}{\partial x_2}\frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x + \int_\Omega \frac{\partial\varphi}{\partial x_2}y_1^k\frac{\partial y_2^k}{\partial x_1}\,\mathrm{d}x\,,$$

and the result follows from the embedding theorems and strong convergence (strong times weak gives weak convergence). $\square$

4.5 Rank-one convexity

Assume the following domain: $\Omega = (1,2) \times (0,4\pi) \times (1,2)$ and the deformation $\mathbf{y} : \overline{\Omega} \to \mathbb{R}^3, \mathbf{y}(x_1,x_2,x_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3), \mathbb{F} = \begin{bmatrix} \cos x_2 & -x_1 \sin x_2 & 0 \\ \sin x_2 & x_1 \cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
We can calculate $\det \mathbb{F} = x_1 \cos^2 x_2 + x_1 \sin^2 x_2 = x_1$. But even though the deformation has positive determinant, we still face self-penetration issues, i.e., $\mathbf{y}$ *is not injective.*

Theorem 3 (Ciarlet-Nečas condition)**.** *Let $p > 3$ and let $\det \mathbb{F} > 0$ a.e. in $\Omega \subset \mathbb{R}^3, \mathbf{y} \in W^{1,p}(\Omega; \mathbb{R}^3)$. If*

$$\int_\Omega \det \mathbb{F}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq \lambda(\mathbf{y}(\Omega))$$

then $\mathbf{y}$ is injective almost everywhere in Ω, i.e., $\exists \omega \subset \Omega : \lambda(\omega) = 0, \mathbf{y}|_{\Omega/\omega}$ is injective.

Is the determinant condition of any use? Let us compute, assuming $\mathbf{y} = \mathbf{0}$ on $\partial\Omega$.

$$\int_\Omega \det \mathbb{F} \,\mathrm{d}\mathbf{x} = \int_\Omega \frac{\partial}{\partial x_1}\left(y_1 \frac{\partial y_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(y_1 \frac{\partial y_2}{\partial x_1}\right) \mathrm{d}\mathbf{x} = \int_{\partial\Omega} y_1 \frac{\partial y_2}{\partial x_2} n_1 - y_1 \frac{\partial y_2}{\partial x_1} n_2 \,\mathrm{d}S \underbrace{\Rightarrow}_{y=0 \text{ on } \partial\Omega} \int_\Omega \det \mathbb{F}(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0.$$

This is powerful! Assume that $\mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}$ on $\partial\Omega$, then

$$\int_\Omega \det \mathbb{F}(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \lambda(\Omega) \det \mathbb{F}.$$

Now let

$$I(\mathbf{y}) = \int_\Omega \det \mathbb{F}(\mathbf{x}) \,\mathrm{d}\mathbf{x}, \mathbf{y} \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^3), \mathbf{y}(\mathbf{x}) = \mathbb{F}\mathbf{x}.$$

Then I is constant[4] and it holds

$$I(\mathbf{y}) = \lambda(\Omega) \det \mathbb{F}.$$

5 Linearized elasticity

Recall the Right Cauchy-Green tensor: $\mathbb{C} = \mathbb{F}^\top \mathbb{F}$. Using it, we can define

Theorem 4 (Green-Lagrange strain tensor/Green-St.-Venaint strain tensor)**.** *Let $\mathbb{C}$ be the Right Cauchy-Green tensor. We define the Green-Lagrange strain tensor/Green-St.-Venaint strain tensor as*

$$\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I}).$$

Remark. The Green-St.-Venaint strain tensor can be rewritten as:

$$\mathbb{E} = \frac{1}{2}\left((\mathbb{I} + \nabla\mathbf{u})^\top (\mathbb{I} + \nabla\mathbf{u}) - \mathbb{I}\right) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) + \frac{1}{2}(\nabla\mathbf{u})^\top \nabla\mathbf{u} = \mathbb{e}(\mathbf{u}) + \frac{1}{2}\mathbb{C}(\nabla\mathbf{u}).$$

[4] All constant functionals are convex.

For the stored energy density, we can write

$$W(\mathbb{F}) = W(\mathbb{R}\mathbb{F}) = \overline{W}(\mathbb{C}(\mathbb{F})) = \hat{W}(\mathbb{E}(\mathbb{F})).$$

and also

$$W(\mathbb{F}) = \hat{W}(\mathbb{e}(\mathbf{u}) + \mathbb{C}(\nabla\mathbf{u})).$$

It is our assumption that

$$\hat{W}(\mathbb{O}) = 0, \hat{W}(\mathbb{E}) > 0 \,\mathrm{if}\, \mathbb{E} \neq \mathbb{O},$$

and also that

$$\mathbb{C}(\nabla\mathbf{u}) = \mathbf{0}.$$

Using Taylor expansion, we can write

$$\hat{W}(\mathbb{e}(\mathbf{u})) = \hat{W}(\mathbb{O}) + \frac{\partial\hat{W}}{\partial\mathbb{e}}(\mathbb{O})\mathbb{e}(\mathbf{u}) + \frac{1}{2}\frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{O})\mathbb{e}(\mathbf{u})\mathbf{e}(u) + \text{h.o.t.}.$$

Since $\hat{W}(\mathbb{O}) = \frac{\partial\hat{W}}{\partial\mathbb{O}}(\mathbb{O}) = 0$ the above (formal) manipulation leads us to the definition

Definition 9 (Tensor of elastic constants)**.**

$$\mathcal{C} = \frac{\partial^2\hat{W}}{\partial\mathbb{e}^2}(\mathbb{O}), C_{ijkl} = \frac{\partial^2\hat{W}}{\partial e_{ij}\partial e_{kl}}.$$

Remark. Since we assume $\hat{W}$ is smooth, we have some symmetries, and from the general 81 components of C_{ijkl} only 21 are unique.

With the notion of a tensor of elastic constants, we can then write the stored energy density as

$$w(\mathbb{e}) = \frac{1}{2}(\mathcal{C}\mathbb{e}) : \mathbb{e}.$$

Following our definition $\mathbb{T} = \frac{\partial\hat{W}}{\partial\mathbb{F}}$ we see

$$\sigma = \frac{\partial w(\mathbb{e})}{\partial\mathbb{e}} = \mathbb{C}\mathbb{e}, \sigma_{ij} = C_{ijkl}e_{kl}.$$

Is a useful notion of stress. It is denoted as the *Cauchy stress.* or in components

$$\sigma_{ij} = C_{ijkl}e_{kl}.$$

5.1 Equations

Rewritting the equations in the linearized elasticity setting we obtain the system

$$\begin{aligned} -\nabla\cdot\sigma = -\nabla\cdot(\mathcal{C}\mathbb{e}) &= \mathbf{f} \,\mathrm{in}\, \Omega \\ \sigma\mathbf{n} &= \mathbf{g} \,\mathrm{on}\, \Gamma_N, \\ \mathbf{u} &= \mathbf{0} \,\mathrm{on}\, \Gamma_D. \end{aligned}$$

The weak formulation can be obtained as

$$\int_\Omega \frac{\partial}{\partial x_j}(C_{ijkl}e_{kl})v_i \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,, \forall\mathbf{v} \in \mathrm{W}^{1,2}\big(\Omega;\mathbb{R}^3\big)\,, u = 0 \,\mathrm{on}\, \Gamma_D,$$

so

$$\int_\Omega C_{ijkl}e_{kl}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} C_{ijkl}e_{kl}v_i n_j\,\mathrm{d}S = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,,$$

which can be rewritten as

$$\underbrace{\int_\Omega \mathbb{C}\mathbb{e}(\mathbf{u})\cdot\mathbb{e}(\mathbf{v})\,\mathrm{d}\mathbf{x}}_{:=B(u,v)} = \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}S,}_{:=L(v)}$$

where we have denoted

$$\mathbb{e}(\mathbf{v}) = \operatorname{sym}(\nabla\mathbf{v}).$$

We are looking for

$$u\in V = \{u\in \mathrm{W}^{1,2}\left(\Omega;\mathbb{R}^3\right), \operatorname{tr} u = 0 \,\mathrm{on}\, \Gamma_D\} : B(u,v) = L(v)\forall v\in V,$$

and to prove the existence, we will use the Lax-Milgram lemma. Show that

- $L \in V^*$
- $B : V\times V \to \mathbb{R}$ is V-bounded and V-coercive

Realize that in order to show the properties, we would have to be able to control $\nabla\mathbf{u}$ by $\operatorname{sym}(\nabla\mathbf{u})$. Is that even possible?

Example. Let $u = 0 \,\mathrm{on}\, \partial\Omega$. In particular, let us take $\mathbf{u}\in\mathcal{D}(\Omega;\mathbb{R}^n)$. Then

$$\exists C > 0 : \int_\Omega |\mathbb{e}(\mathbf{u})|^2\,\mathrm{d}\mathbf{x} \geq c\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,.$$

Can this hold? Make a quick test: Take $\mathbf{u}$ such that $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, so $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0$, so of course:

$$\nabla\mathbf{u} = -(\nabla\mathbf{u})^\top,$$

and $\nabla\mathbf{u}$ must have the form

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix},$$

where $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = 0$, but since $\mathbf{u} = \mathbf{0}$ at the boundary, it also holds that $\mathbf{u} = \mathbf{0}$ in Ω. Okay, so that not disprove the above inequality.

Let us try something else (although unsure what this means):

$$\begin{aligned}\int_\Omega |\mathbb{e}(\mathbf{u})|^2\,\mathrm{d}x &= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \left(\frac{\partial u_j}{\partial x_i}\right)^2\mathrm{d}x = \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\,\mathrm{d}x\,,\end{aligned}$$

where we used the symmetry property. Integrating by parts two times to obtain "$\partial_i u_i \partial_j u_j = (\partial_j u_j)^2$ [5]. All in all

$$\frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_i}{\partial x_i}\right)^2\mathrm{d}x \geq 0.$$

[5] Sign does not change as we integrate 2 times. Also, we have homogenous Dirichlet

Theorem 5 (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain ($\Omega \in C^{0,1}$). Then there exists $C > 0$ such that $\forall \mathbf{u} \in W^{1,2}((\Omega;\mathbb{R}^n))$ it holds*

$$\left(\|\mathbb{e}(\mathbf{u})\|^2_{L_2((\Omega;\mathbb{R}^{n\times n}))} + \|\mathbf{u}\|^2_{L_2((\Omega;\mathbb{R}^n))}\right) \geq c\|\mathbf{u}\|_{W^{1,2}((\Omega;\mathbb{R}^n))}.$$

Definition 10 (Axial vectors)**.** Let $\mathbb{A} = -\mathbb{A}^\top, \mathbb{A} \in \mathbb{R}^{n\times n}$. Then there is $b \in \mathbb{R}^n$ such that $\mathbb{A}\mathbf{v} = \mathbf{b}\times\mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{b}$ is called the axial vector of $\mathbb{A}$.

Remark ($\mathbb{R}^n$)*.* This truly holds in $\mathbb{R}^n$, not only in $\mathbb{R}^3$. We only have to replace $\times$ by $\wedge$, the outter product.

Assume that $\mathbf{u} \in C^2(\Omega;\mathbb{R}^3)$. Then

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial e_{ik}}{\partial x_j}(\mathbf{u}) + \frac{\partial e_{ij}}{\partial x_k}(\mathbf{u}) - \frac{\partial e_{jk}}{\partial x_i}(\mathbf{u}).$$

If now $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, then $\mathbf{u}$ is an affine function, because $\frac{\partial^2 u_i}{\partial x_j \partial x_k}, \forall i,j,k \in \{1,2,3\}$. [6] It must thus hold

$$u_i(x) = a_i + b_{ij}x_j,$$

and $\frac{\partial u_i}{\partial x_j} = b_{ij} = -b_{ji}$, because $\mathbb{e}(\mathbf{u}) = \mathbb{0}$, so it must be skew symmetric. The skew-symmetry also means it can be written

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d}\times\mathbf{x}.$$

If additionaly we assume that $\mathbf{u} = \mathbf{0}$ on some $\Gamma_D \subset \partial\Omega, \mathcal{H}(\Gamma_D) > 0$ and $\mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{d}\times\mathbf{x}$, then $\mathbf{u} = \mathbf{0}$ identically in Ω This moreover means that

$$\mathbf{u} \mapsto \|\mathbb{e}(\mathbf{u})\|_{\mathrm{L}_2((\Omega;\mathbb{R}^{n\times n}))}$$

is a norm on

$$V = \left\{\mathbf{w} \in \mathrm{W}^{1,2}\left((\Omega;\mathbb{R}^3)\right), \mathbf{w} = \mathbf{0} \,\mathrm{on}\, \Gamma_D\right\}$$

which is equivalent to the norm of $\mathrm{W}^{1,2}\left((\Omega;\mathbb{R}^3)\right)$.

Coming back to our equation $B(u,v) = L(v), \forall v \in V$, we have showed everything to use Lax-Milgram $\Rightarrow \exists! u \in V$. This also means the functional

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega (\mathbb{C}\mathbb{e}(\mathbf{v}) : \mathbb{e}(\mathbf{v}) - L(\mathbf{v}))\,\mathrm{d}\mathbf{x}, \forall v \in V.$$

has an unique minimizer.

5.2 Convex analysis

We will deal with the analysis of the functions $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}, f$ is convex.

Definition 11 (Epigraph of a set)**.** The epigraph of a function f is the set

$$\operatorname{epi} f = \{(x,y) : y \geq f(x)\}$$

Remark. With the notion of $\operatorname{epi} f$ we can work with sets instead of functions. Moreover, it holds

[6] Recall that Ω is simply connected.

- $\operatorname{epi} f$ is closed $\Leftrightarrow$ f is lower-semicontinuous,
- f is convex $\Leftrightarrow$ $\operatorname{epi} f$ is convex

From one of the consequences of Hahn-Banach theorem (oddělovací věty), we obtain the existence of such $\xi \in \mathbb{R}^n$ (dependent of x) that for fixed x it yields

$$f(z) \geq f(x) + \xi \cdot (z - x), \forall z \in \mathbb{R}^n.$$

If f is differentiable at x, then

$$\xi = \nabla f(x).$$

But in general it does not have to be differentiable. This motivates the following definition

Definition 12 (Subgradient, subdifferential)**.** The function $\xi(x)$ such that

$$f(z) \geq f(x) + \xi(x) \cdot (z - x), \forall z \in \mathbb{R}^n,$$

is called the **subgradient** of f at x. The set of all subgradients of f at x is called the **subdifferential** of f at x and it is denoted $\partial f(x)$.

Let $f : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$, convex and lower semicontinuous [7], $f \neq \infty$. The function $\xi(\mathbb{X})$ such that

$$f(\mathbb{Y}) \geq f(\mathbb{X}) + \xi(\mathbb{X}) \cdot (\mathbb{Y} - \mathbb{X}), \forall \mathbb{Y} \in \mathbb{R}^{n\times m},$$

is called the subgradient of f at $\mathbb{X}$. The set of all subgradients of f at $\mathbb{X}$ is called the subdifferential and denoted $\partial f(\mathbb{X})$.

Remark.
- If $\partial f(\mathbb{X})$ is a singleton, then $\nabla f(\mathbb{X})$ exist.
- $\partial f(\mathbb{X})$ is convex
- $0 \in \partial f(x) \forall x \in \mathbb{R}^n$ is a condition for the minimizer.

Definition 13 (Indicator function)**.** Let $K \subset \mathbb{R}^{n\times m}$ be a closed convex nonempty set. The function $I_K(\mathbb{X})$ given as

$$I_K(\mathbb{X}) = \begin{cases} 0, & \text{if } \mathbb{X} \in K \\ +\infty, & \text{otherwise} \end{cases},$$

is called the indicator function of K

The indicator function is hepful for constraint minimization. If f is reasonably (at least finitely valued on K) then it holds:

$$\min_K f = \min_{\mathbb{R}^{n\times m}} (f + I_K).$$

Example (Unit interval)**.** Let $K = [0, 1]$. What is $\partial I_K(x)$?

If $x \in (0, 1)$, then $I_K(x) = 0$ so the only ξ such that $I_K(y) \geq 0 + \xi(y - x)$ holds is $\xi = 0$.

If $x = 0, x = 1$ then $\partial I_K(0) = (-\infty, 0], \partial I_K(1) = [0, \infty)$. This resembles a normal "vector", but in fact it is not a single vector and more a "cone" of vectors.

[7] $f(x) \leq \liminf_{k\to\infty} f(x_k), x_k \to x$

Definition 14 (Normal cone to a set)**.** Let K be closed convex nonempty set. The subdifferential of the indicator function I_K is called the normal cone to the set K and it is denoted by N_K.

Example. Minimize x^2 on $[1,2]$. We are looking for

$$\min_{[1,2]} x^2 = \min_{\mathbb{R}} \big(x^2 + I_{[1,2]}(x)\big).$$

It must hold at the minimum

$$0 \in \partial\big(x^2 + I_{[1,2]}(x)\big) \Leftrightarrow -\partial I_{[1,2]}(x) \subset \partial x^2 \Leftrightarrow \big(x^2\big)' \in -N_{[1,2]}(x)$$

Example. Take a square $K = [0,1] \times [0,1] \subset \mathbb{R}^2$. We know $K \in C^{0,1}$ so the outter normal exist at a.a. points on the boundary. The outter normal does not exist in the corners, but the normal cone does.

Definition 15 (Fenchel (convex) conjugate — Legendre transformation)**.** Let x^* be a slope i have chosen (it is given). I require

$$f(x) \geq x^* \cdot x - k, \forall x \in \mathbb{R}^{n\times m},$$

which means $k \geq x^* \cdot x - f(x), \forall x \mathbb{R}^{n\times m}$, and so we can define

$$f^*(x^*) \coloneqq \sup_{x\in\mathbb{R}^{n\times m}} (x^* \cdot x - f(x)).$$

Remark. f^* is always convex even if f is not. But when f is convex and lower-semicontinuous, then

$$f^{**} = f, \text{ (biconjugate)}.$$

Theorem 6 (Fenchel identity)**.** *Let $x^* \in \partial f(x)$. Then*

$$x^* \cdot x = f(x) + f^*(x^*).$$

Proof. Let us assume that $x^* \in \partial f(x)$. Then it must hold

$$f(y) \geq f(x) + x^* \cdot (y - x), \forall y,$$

so

$$x^* \cdot x - f(x) \geq x^* \cdot y - f(y),$$

and taking the supremum over y yields[8]

$$x^* \cdot x - f(x) = \sup_y (x^* \cdot y - f(y)) = f^*(x^*).$$

We have thus obtained

$$x^* \cdot x = f(x) + f^*(x^*).$$

□

Remark (Minimization of $f \Leftrightarrow$ minimization of f^*). We see that it holds:

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*).$$

[8]The inequality becomes equality, as it can be reached by taking $y = x$.

5.3 Problem of a man...

Assume a person is pulling a box of weight m of weight m of weight m of weight m by a spring. If he is pulling just a little, the box does not move, only the spring is deformed - but in a reversible, elastic way. To move the box, the man needs to pull at least with the force $\sigma_0 = mgc$, where c is some friction coefficient. When he us pulling with force greater than σ_0, the box is moving and does not require any extra force to be moved (the system to be deformed). The deformation can be decomposed as

$$\mathbb{e} = \tilde{\mathbb{e}} + \mathbb{p},$$

where $\mathbb{e}$ is the total strain, $\tilde{\mathbb{e}}$ is the elastic strain and $\mathbb{p}$ is the plastic strain.

5.4 von Mieses elatoplasticity

The elasticity part is described as

$$\begin{cases} -\nabla\cdot\sigma = \mathbf{f}, & \text{in the bulk} \\ \sigma\mathbf{n} = \mathbf{g}, & \text{on the boundary} \end{cases},$$

with some constituive relation $\sigma = \mathcal{C}\tilde{\mathbb{e}} = \mathcal{C}(\mathbb{e} - \mathbb{p})$. What about the plastic part?

$$\begin{cases} \dot{\mathbb{p}}(t) \in N_K(\sigma), \\ \mathbb{p}(0) = \mathbb{p}_0, \end{cases}$$

where K is a convex closed subset such that $0 \in K$. This means that the plastic deformation is zero inside K, i.e. for some stresses.

Remark. Very often, the deformation is considered "incompressible", i.e.,

$$\det\mathbb{F} = 1,$$

which in linear case translates into

$$\operatorname{tr}\varepsilon = 0.$$

In most cases, the set K is given as

$$K = \{\sigma : \varphi(\sigma) \le 0\},$$

where φ is the **yield function.** The set

$$\{\sigma|\varphi(\sigma) = 0\}$$

is called the **yield surface.** Very often we have

$$\varphi(\sigma) = |\sigma^D| - c_0,$$

where $|\cdot|$ denotes the Frobenius norm and

$$\sigma^D = \sigma - \frac{1}{3}(\operatorname{tr}\sigma)\mathbb{I},$$

is the *deviatoric part of the stress tensor.*

5.4.1 Plastic evolution

From the previous we have

$$\dot{\mathbb{p}} = \begin{cases} \mathbb{0}, & \text{if } \varphi(\sigma) < 0, \\ \frac{\lambda}{|\sigma^D|}\sigma^D, & \text{if } \varphi(\sigma) = 0, \lambda \geq 0 \end{cases}.$$

Also $\dot{\mathbb{p}} \in N_K(\sigma) = \partial I_K(\sigma)$ so

$$\sigma \in \partial I_K^*(\dot{\mathbb{p}}),$$

where

$$I_K^*(\dot{\mathbb{p}}) = \sup_{\mathbb{q}\in\mathbb{R}^{3\times 3}} (\dot{\mathbb{p}} : \mathbb{q} - I_K(\mathbb{q})) = \sup_{\mathbb{q}\in K} \dot{\mathbb{p}} : \mathbb{q},$$

is the Fenckel transformation of I_K, also called the **supporting function** of $\dot{\mathbb{p}}$. We are able to rewrite the supremum to take the form[9]

$$I_K^*(\dot{\mathbb{p}}) = \dot{\mathbb{p}} : \frac{c_0}{|\dot{\mathbb{p}}|}\dot{\mathbb{p}},$$

if however the second term lies in K. Realize now that if $\operatorname{tr}\dot{\mathbb{p}} = 0$ then

$$I_K^*(\dot{\mathbb{p}}) = c_0|\dot{\mathbb{p}}|,$$

and if $\operatorname{tr}\dot{\mathbb{p}} \neq 0$, then $I_K^*(\dot{\mathbb{p}}) = +\infty$. If we now define the **dissipation potential** D as

$$D(\dot{\mathbb{p}}) = \begin{cases} c_0|\dot{\mathbb{p}}|, & \text{if } \operatorname{tr}\dot{\mathbb{p}} = 0 \\ +\infty, & \text{otherwise} \end{cases},$$

we get the following condition

$$\sigma \in \partial D(\dot{\mathbb{p}}).$$

Let us summarise a bit. For the stress tensor we have $\sigma = \mathcal{C}(\mathbb{e} - \mathbb{p}) \in D(\dot{\mathbb{p}})$. The general relation also yields $\sigma = \frac{\partial w(\tilde{\mathbb{e}})}{\partial \tilde{\mathbb{e}}} = \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \tilde{\mathbb{e}}}$, where $w(\tilde{\mathbb{e}}) = \frac{1}{2}C\tilde{\mathbb{e}} : \tilde{\mathbb{e}}$ is the free energy density. Using the chain rule we obtain the condition

$$\frac{\partial w(\mathbb{e} - \mathbb{p})}{\partial \mathbb{p}} \in \partial D(\dot{\mathbb{p}}).$$

In total, we are solving the following system

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D(\dot{\mathbb{p}}), & \text{in } \Omega (\textit{flow rule}) \\ \mathbb{p}(0) = \mathbb{p}_0, & \text{in } \Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e} - \mathbb{p})) = \mathbf{f}, & \text{in } \Omega \\ \text{boundary condiitons}, & \text{on } \partial\Omega \end{cases}.$$

How to solve the system?

[9] To utilize Cauchy-Schwarz later.

5.4.2 Discrete time setting

Let us take $t \in [0,T]$ and fix $\tau = \frac{T}{N}, N \in \mathbb{N}$ for some $N >> 1$. Assume that using some discrete scheme, we are able to calculate $\mathbb{p}$ at a certain time. Then we must solve

$$\begin{cases} 0 \in \frac{\partial w(\mathbb{e}-\mathbb{p})}{\partial \mathbb{p}} + \partial D\Big(\frac{\mathbb{p}-\dot{\mathbb{p}}_{\mathbb{k}-\mathbb{1}}}{\tau}\Big), & \text{in}\,\Omega \\ -\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k, & \text{in}\,\Omega \end{cases}.$$

Which are the E-L equations of the functional [10]

$$I(\mathbf{u},\mathbb{p}) = \int_\Omega w(\mathbb{e}(\mathbf{u}) - \mathbb{p})\,\mathrm{d}x + \tau\int_\Omega D\Big(\frac{1}{\tau}(\mathbb{p}-\mathbb{p}_{k-1})\Big)\,\mathrm{d}x - \int_\Omega \mathbf{f}_k\cdot\mathbf{u}\,\mathrm{d}x - \int_{\Gamma_N} \mathbf{g}_k\cdot u\,\mathrm{d}S\,.$$

Really, taking the variaton with respect to $\mathbf{u}$ gives us

$$-\nabla \cdot (\mathcal{C}(\mathbb{e}_k - \mathbb{p}_k)) = \mathbf{f}_k,$$

and the variation with respect to $\mathbb{p}$ gives us

$$0 \in -\sigma + \partial D\Big(\frac{1}{\tau}(\mathbb{p}-\mathbb{p}_{k-1})\Big).$$

If we want to minimize this functional, *i.e.*, solve the equations, it must hold [11] $D(\mathbb{q}) \neq +\infty$ (for $\mathbb{q}$ being the argument). From our assumptions on the dissipation potential this however implies.

$$D(\mathbb{q}) = c_0|\mathbb{q}|, \operatorname{tr}\mathbb{q} = 0,$$

and we say the evolution is **rate-independent**. We see that D is 1-homogenous:

$$D(\alpha\mathbb{q}) = \alpha D(\mathbb{q}).$$

Rewritting the functional now yields:

$$I(\mathbf{u},\mathbb{p}) = \frac{1}{2}\int_\Omega \mathcal{C}(\mathbb{e}(\mathbf{u}) - \mathbb{p}) : (\mathbb{e}(\mathbf{u}) - \mathbb{p})\,\mathrm{d}x + \int_\Omega c_0|\mathbb{p}-\mathbb{p}_{k-1}|\,\mathrm{d}x - L_k(\mathbf{u}), \mathbb{p}(0) = \mathbb{p}_0,$$

where $L_k(\mathbf{u})$ is the loading (at the k-th time step.) The sought solution is the pair $(\mathbf{u}_k,\mathbb{p}_k)$ which satisfies

$$I(\mathbf{u}_k,\mathbb{p}_k) = \min_{\mathbf{u},\mathbb{p}} I(\mathbf{u},\mathbb{p}).$$

5.5 Rheological models

5.5.1 Dashpots

Or *tlumič* in Czech. The stress is assumed to take the form

$$\sigma = \mathcal{D}\dot{\mathbb{e}}(\nabla\mathbf{u}), \sigma_{ij} = D_{ijkl}\dot{e}_{kl}(\nabla\mathbf{u}),$$

where $\mathcal{D}$ is the **tensor of viscosity constants.** [12]

[10] We have guessed it.
[11] If not, we have no chance of minimizing it.
[12] People say viscosity stresses or viscous stress. This is used, but nonetheless it is wrong.

5.5.2 Kelvin-Voigt mateial

The response of some materials can be modelled as a "parallel composition of a spring and a dashpot." Then, the total stress is

$$\sigma = \sigma_p + \sigma_e,$$

that is the sum of the plastic and the elastic stresses. The strain is of course the same:

$$\mathbb{e} = \mathbb{e}_p = \mathbb{e}_e.$$

The governing equations thus are

$$\begin{aligned} -\nabla\cdot(\mathcal{C}\mathbb{e}(\mathbf{u}) + \mathcal{D}\dot{\mathbb{e}}(\mathbf{u})) &= \mathbf{f},\ \text{in}\,\Omega \\ (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}})\mathbf{n} &= \mathbf{0},\ \text{on}\,\Gamma_N \\ \mathbf{u} &= \mathbf{0},\ \text{on}\,\Gamma_D \\ \mathbb{e}(t=0) &= \mathbb{e}_0,\ \text{in}\,\Omega. \end{aligned}$$

Let us obtain the energy *formally* balance. As usual, multiply the first equation by $\dot{\mathbf{u}}$ and integrate $\int_\Omega \mathrm{d}\mathbf{x}$.

$$\int_\Omega -\nabla\cdot(\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}})\cdot\dot{\mathbf{u}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

using Gauss

$$\int_\Omega (\mathcal{C}\mathbb{e} + \mathcal{D}\dot{\mathbb{e}}) : \nabla\dot{\mathbf{u}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x =^{13} \int_\Omega \mathcal{C}\mathbb{e} : \dot{\mathbb{e}}\,\mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}} : \dot{\mathbb{e}}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,,$$

and now we rewrite

$$= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\Big)\mathrm{d}x + \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,,$$

and integrate in time:

$$\int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u})\Big)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

Remember that

$$w(\mathbb{e}(\mathbf{u})) = \frac{1}{2}\mathcal{C}\mathbb{e}(\mathbf{u}) : \mathbb{e}(\mathbf{u}),$$

so we have obtained

$$\int_\Omega w(\mathbb{e}(\mathbf{u}(T)))\,\mathrm{d}x - \int_\Omega w(\mathbb{e}(\mathbf{u}(0)))\,\mathrm{d}x + \int_0^T \int_\Omega \mathcal{D}\dot{\mathbb{e}}(\mathbf{u}) : \dot{\mathbb{e}}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\,.$$

5.5.3 Maxwell material

This is the case when we "put the spring and the dashpot in serial composition". The total stress is

$$\sigma = \sigma_p = \sigma_e,$$

and the total strain is

$$\varepsilon = \mathbb{e}_p + \mathbb{e}_e.$$

[13] It holds $\dot{\mathbb{e}}(\mathbf{u}) = \mathbb{e}(\dot{\mathbf{u}})$.

5.6 Internal parameters

A lot of materials can be described using some internal parameters z (scalars, vectos, tensors); for example, plastic strain, fatique, damage, length of a crack, delamination.

The model

$$\sigma = \partial_{\dot{\mathbb{e}}}\zeta(\dot{\mathbb{e}}, \dot{z}) + \partial_{\mathbb{e}} w(\mathbb{e}, z),$$

with the flow rule $0 \in \partial_{\dot{z}}\zeta(\dot{\mathbb{e}}, \dot{z}) + \partial_z w(\mathbb{e}, z)$. is called the **generalized Kelvin-Voigt.**

6 (Some) tutorials

6.1 Change of observer

The requirement of material frame indifference yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}), \forall \mathbb{Q} \in \operatorname{orth}.$$

6.2 Change of reference configuration

The requiremenent of material symmetry yields:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{P}), \forall \mathbb{P} \in \mathcal{G},$$

where $\mathcal{G}$ is the symmetry group of the material.

6.3 Consequences of isotropic hyperelastic solid

Remark (Groups unim, orth). The "biggest sensible" symmetry group is the unimodular group:

$$\operatorname{unim} = \{\mathbb{P}, \det \mathbb{P} = \pm 1\}.$$

There exists another common group:

$$\operatorname{orth}\{\mathbb{Q}, \mathbb{Q}\mathbb{Q}^\top = \mathbb{Q}^\top\mathbb{Q} = \mathbb{I}\} \subset \operatorname{unim}.$$

We thus have $W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}), \forall \mathbb{Q} \in \operatorname{orth}, \forall \mathbb{F}$.

Use *polar decomposition*: $\mathbb{F} = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R}, \mathbb{R} \in \operatorname{orth}, \mathbb{U}, \mathbb{V}\,\text{positively definite}, \mathbb{U} = \sqrt{\mathbb{C}}, \mathbb{V} = \sqrt{\mathbb{B}}$.

To make use of it, we first use m.f.i. and then isotropy and then first use isotropy and then m.f.i. So from material frame indifference

$$W = \hat{F}(\mathbb{F}) = \hat{W}(\mathbb{Q}\mathbb{F}) = \hat{W}\big(\mathbb{R}^\top\mathbb{R}\mathbb{U}\big) = \hat{W}(\mathbb{U}) = \overline{W}(\mathbb{C}),$$

where we have taken $\mathbb{Q} = \mathbb{R}^\top$. Note that this works universaly (without the need of isotropy), as it comes from the objectivity consideration.

From isotropy

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \overline{W}(\mathbb{C}) = \overline{W}\big((\mathbb{F}\mathbb{Q})^\top(\mathbb{F}\mathbb{Q})\big) = \overline{W}\big(\mathbb{Q}^\top\mathbb{F}^\top\mathbb{F}\mathbb{Q}\big) = \overline{W}\big(\mathbb{Q}^\top\mathbb{C}\mathbb{Q}\big), \forall Q \in \operatorname{orth}, \forall \mathbb{C}\,\text{admissable}.$$

Now backwards using the second polar decomposition:

$$W = \hat{W}(\mathbb{F}) = \hat{W}(\mathbb{F}\mathbb{Q}) = \hat{W}(\mathbb{V}\mathbb{R}\mathbb{Q}) = \hat{W}\big(\mathbb{V}\mathbb{R}\mathbb{R}^\top\big) = \hat{W}\Big(\sqrt{\mathbb{B}}\Big) = \tilde{W}(\mathbb{B}),$$

$$W = \tilde{W}(\mathbb{B}) = \tilde{W}\big(\mathbb{QF}(\mathbb{QF})^\top\big) = \tilde{W}\big(\mathbb{QBQ}^\top\big).$$

So far, we have shown

$$W(t,\mathbf{X}) = \overline{W}(\mathbb{C}(t,\mathbf{X}),\mathbf{X}) = \overline{W}\big(\mathbb{QCQ}^\top\big),$$
$$W(t,\mathbf{X}) = \tilde{W}(\mathbb{B}(t,\mathbf{X}),\mathbf{X}) = \tilde{W}\big(\mathbb{QBQ}^\top\big),$$

In HW, we will know

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}}, T_{iJ} = 2\frac{\partial \hat{W}}{\partial F_{iJ}}$$

and we can show

$$\mathbb{T} = 2\frac{\partial \hat{W}(\mathbb{F})}{\partial \mathbb{F}} = 2\mathbb{F}\frac{\partial \tilde{W}(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}, \mathbb{T}^y = \dots, \mathbb{S} = 2\frac{\partial \overline{W}(\mathbb{C})}{\partial \mathbb{C}}.$$

Definition 16 (Isotropic functions)**.** We say the functions $\hat{a}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha), \hat{\mathbf{a}}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha), \hat{\mathbb{A}}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha), \alpha = 1,\dots,N$ are isotropic functions (of their respective arguments) if it holds

$$\hat{a}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha) = \hat{a}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),$$
$$\mathbb{Q}\hat{\mathbf{a}}(y_\alpha,\mathbf{y}_\alpha,\mathbb{Y}_\alpha) = \hat{\mathbf{a}}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),$$
$$\mathbb{Q}\hat{\mathbb{A}}\mathbb{Q}^\top = \hat{\mathbb{A}}\big(y_\alpha,\mathbb{Q}\mathbf{y}_\alpha,\mathbb{Q}\mathbb{Y}_\alpha\mathbb{Q}^\top\big),$$

.

So we see that $\overline{W}(\mathbb{C}), \tilde{W}(\mathbb{B})$ are **scalar isotropic functions of 1 tensorial (symmetric) argument.**

Theorem 7 (Representation theorem for scalar isotropic functions)**.** *Let $\psi = \hat{\psi}(\mathbb{A}) = \hat{\psi}(\mathbb{QAQ}^\top)$ be a scalar isotropic function of a single symmetric tensorial variable. Then it must hold*

$$\hat{\psi}(\mathbb{A}) \equiv \hat{\psi}(\mathrm{I}_1(\mathbb{A}), \mathrm{I}_2(\mathbb{A}), \mathrm{I}_3(\mathbb{A})),$$

where

$$\mathrm{I}_1(\mathbb{A}) = \operatorname{tr}\mathbb{A},$$
$$\mathrm{I}_2(\mathbb{A}) = \frac{1}{2}\Big((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}\mathbb{A}^2\Big),$$
$$\mathrm{I}_3(\mathbb{A}) = \det\mathbb{A},$$

are the invariants of $\mathbb{A}$.

Proof. $\det(\mathbb{A} - \lambda\mathbb{I}) = -\lambda^3 + \lambda^2\,\mathrm{I}_1 - \lambda\,\mathrm{I}_2 + \mathrm{I}_3 = p_\lambda(\mathbb{A})$ We will prove a different assertion:

$\mathbb{A}, \mathbb{B}$ are symmetric with the same invariants $\Leftrightarrow \exists \mathbb{Q} : \mathbb{A} = \mathbb{QBQ}^\top$ " $\Leftarrow$ " is trivial, as then the matrices are similliar, so they have the same char. polynomial, so they have the same invariants. $\Rightarrow$ have same eigenvalues, so if i write the spectral decomposiiton, i can write

$$\mathbb{A} = \mathbb{Q}\Lambda\mathbb{Q}^\top, \mathbb{B} = \mathbb{Q}\Lambda\mathbb{R}^\top = \mathbb{R}\mathbb{Q}^\top\mathbb{A}\mathbb{Q}\mathbb{R}^\top.$$

Now suppose that the function is not a function of the invariants: $\hat{\psi} \neq \tilde{\psi}(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$. That means $\exists \mathbb{A}_, , \mathbb{A}_2$ such that $\mathrm{I}_1(\mathbb{A}_1) = \mathrm{I}_1(\mathbb{A}_2)$ and the same for the remaining invariants. Using the previous assertion, we have

$$\exists \mathbb{Q} : \mathbb{A}_1 = \mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top \Rightarrow \hat{\psi}(\mathbb{A}_1) = \hat{\psi}\left(\mathbb{Q}\mathbb{A}_2\mathbb{Q}^\top\right) = \hat{\psi}(\mathbb{A}_2), \text{ but } \hat{\psi}(\mathbb{A}_1) \neq \hat{\psi}(\mathbb{A}_2).$$

□

Since using polar decomposition it can be shown the invariants of $\mathbb{B}, \mathbb{C}$ are the same we recieve

$$W = \tilde{W}(\mathrm{I}_1(\mathbb{B}), \mathrm{I}_2(\mathbb{B}), \mathrm{I}_3(\mathbb{B})) = \overline{W}(\mathrm{I}_1(\mathbb{C}), \mathrm{I}_2(\mathbb{C}), \mathrm{I}_3(\mathbb{C})).$$

6.4 Representation in terms of principial stresses

... in terms of the eigenvalues $\mathbb{U}, \mathbb{V}$. The invariants can be expressed as

$$\begin{aligned}
\mathrm{I}_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\
\mathrm{I}_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \\
\mathrm{I}_3 &= \lambda_1\lambda_2\lambda_3.
\end{aligned}$$

Often in materials science the quantites can be expressed in these variables:

Example (Ogden materials)**.**

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{n} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\lambda_k} + \lambda_3^{\alpha_k} - 3\right)$$

How to calculate e.g. $\mathbb{T}$ in this representation?

$$\mathbb{T} = 2\frac{\partial W(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)}{\partial \mathbb{B}}\mathbb{F} = 2\frac{\partial \hat{W}}{\partial \mathbb{B}}(\lambda_1(\mathbb{B}), \lambda_2(\mathbb{B}), \lambda_3(\mathbb{B}))2\frac{\partial \hat{W}}{\partial \lambda_i}\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}\mathbb{F}.$$

What (in the hell) is $\frac{\partial \lambda_i(\mathbb{B})}{\partial \mathbb{B}}$? [14]

$$\mathbb{B}(s) = \sum_{\alpha=1}^{3} \omega_\alpha(s)\mathbf{g}_\alpha(s) \otimes \mathbf{g}_\alpha(s), \forall s \in I$$

where I is some open interval and $\{\mathbf{g}_\alpha\}$ is an ON eigenbasis of $\mathbb{B}$. Next, realize

$$\omega_1(s) = \mathbf{g}_1(s) \cdot \mathbb{B}(s)\mathbf{g}_1(s),$$

and differentiate this:

$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{g}_1}{\mathrm{d}s} \cdot \mathbb{B}\mathbf{g}_1 + \mathbf{g}_1\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}s}\mathbf{g}_1 + \mathbf{g}_1 \cdot \mathbb{B}\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}s} = \frac{1}{2} + +0.$$

[14] Recall the Daleckii-Krein theorem: