Partial differential equations II

Kamil Belan

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Contents

1 Winter semester addendum

1.0.1 Weak* convergence

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is not reflexive, we cannot extract a (weakly) convergent subsequence; however, we know the predual of $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is reflexive, i.e.

$$\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*,$$

which means that balls in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ are weakly* compact. Moreover, $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ is *separable*, from which it follows $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ with the weak* topology is metrizable and thus there exists s weakly * converging subsequence (from the balls).

Example (For people without Functional Analysis I)**.** Let X be a linear normed space, $\{x_n\} \subset X$ a sequence in X. We say x_n converges weakly to $x \in X$ whenever

$$f(x_n) \to f(x), \forall f \in X^*.$$

Let $X*$ be the topological dual to X, $\{x_n\} \subset X^*$ a sequence in X. We say f_n converges weakly* to $f \in X^*$ whenever

$$f_n(x) \to f(x), \forall x \in X^*,\ i.e.\, x(f_n) \to x(f),$$

where by $x(y), x \in X, y \in X.*$ we understand

$$\varepsilon_x : X^* \to \mathbb{K}, y \mapsto y(x).$$

Since $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big) \cong \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, every point $x \in \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$ can be interpreted as a linear functional on $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, so given $\{x_n\} \subset \mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, we can interpret is as a $\{x_n\} \subset \Big(\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)\Big)^*$, meaning given a weakly converging sequence in $\mathrm{L}_\infty\Big((0,T);\mathrm{L}_2(\Omega)\Big)$, it is actually a weakly* converging sequence in $\mathrm{L}_1\Big((0,T);\mathrm{L}_2(\Omega)\Big)$.

Theorem 1. *Let the assumptions of the previous theorem hold and $\Omega \in C^{1,1}, \delta \in (0,1)$. Then $u \in L_2\Big((\delta,T);\, W^{2,2}(\Omega)\Big)$.*

Proof. Take the weak formulation in $t \in (\delta, T)$. WLOG further assume $d = 0$. Then

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi - bu\varphi - \mathbf{c}\cdot\nabla u\varphi - \int_\Omega \partial_t u\varphi = \int_\Omega (f - bu - \mathbf{c}\cdot\nabla u - \partial_t u)\varphi,$$

and the integrand of the last integral is in $\mathrm{L}_2(\Omega)$ for a.e. $t \in (\delta,T)$. We can thus use the elliptic regularity results and write:u

$$\|u\|^2_{\mathrm{W}^{2,2}(\Omega)} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\partial_t u\|^2_{\mathrm{L}_2(\Omega)}),$$

integrating both sides $\int_\delta^T \mathrm{d}t$ yields

$$\|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))} \le C(\|f\|^2_{\mathrm{L}_2(\Omega)} + \|u\|^2_{\mathrm{L}_2\big((0,T);\mathrm{W}^{1,2}(\Omega)\big)} + \|u\|^2_{\mathrm{L}_2((\delta,T);\mathrm{L}_2(\Omega))})$$

□

Theorem 2. *If data are smooth and satisfy the compatibility condiitons, then the weak solutions to the parabolic equation are smooth.*

Proof. no. □

Remark (Compatibility condition)*.* : Take the heat equation : $\partial_t u - \triangle u = f$ at time zero: $\triangle u(0) + f(0) = \partial_t u(0) \in \mathrm{W}^{1,2}_0(\Omega)$, so we need that $f(0) + \triangle u(0)$ has zero trace $\Rightarrow$ compatibility conditions.

Theorem 3 (Uniqueness of the solution to a hyperbolic equation)**.** *Let the assumptions on the data of the hyperbolic equations be standard (i.e. minimal). Further assume that* $\mathbf{c} \in W^{1,\infty}(\Omega)$*. Then the weak solution to the hyperbolic equation is unique.*

Proof. It is enough that if $u_0 = 0, u_1 = 0 \Rightarrow u = 0 \in Q_T$. To do that, take the equation, multiply it by $\varphi \in V$ fixed and integrate over Ω for $t \in (0,T)$ fixed:

$$< \partial_{tt} u(t), \varphi > + \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \varphi \, \mathrm{d}x - \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \varphi \, \mathrm{d}x = 0.$$

Now, take a special test function

$$\psi(t) = \left(\int_t^s u(\tau) \, \mathrm{d}\tau \right) \chi_{(0,s)}(t),$$

for some $s \in (0,T)$. Then $\partial_t \psi(t) = -u(t)$ on $t \in (0,s)$. Next, integrate the equation in time over $(0,s)$.

$$\int_0^s < \partial_{tt} u(t), \psi > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, \mathrm{d}x \, \mathrm{d}t - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \, \mathrm{d}x \, \underset{\cdot}{\mathrm{t}} = 0,$$

Now use per partes on the first term (deploy Gelfand triple):

$$\int_0^s < \partial_{tt} u(t), \varphi > \mathrm{d}t =< \partial_t u(s), \psi(s) > - < \partial_t u(0), \psi(0) > - \int_0^s < \partial_t u(t), \partial_t \psi(t) > \mathrm{d}t \,,$$

and realize $\psi(s) = 0, \partial_t u(0) = 0$, so

$$- \int_0^s < \partial_t u(t), \partial_t \psi(t) > \mathrm{d}t + \int_0^s \int_\Omega \mathbb{A}(t) \nabla u(t) \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega (bu(t) + \mathbf{c} \cdot \nabla u(t)) \psi \, \mathrm{d}x \, \mathrm{d}t - \int_0^s \int_\Omega u(t) \mathbf{d}(t) \cdot \nabla \psi \, \mathrm{d}x \, \underset{\cdot}{\mathrm{t}} = 0,$$

but since $\partial_t \psi(t) = -u(t)$, we can actually write (**notice we have kept** $u(t)$ **for Gronwall**:

$$\int_0^s < \partial_t u(t), u(t) > \mathrm{d}t - \int_0^s \int_\Omega \mathbb{A}(t) \nabla \partial_t \psi(t) \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t - \int_0^s \int_\Omega (-bu(t) + \mathbf{c} \cdot \nabla \partial_t \psi(t)) \psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega \psi(t) \mathbf{d}(t) \cdot \nabla \psi \, \mathrm{d}x \, \underset{\cdot}{\mathrm{t}} = 0,$$

and so

$$\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \left(\| u(t) \|^2_{\mathrm{L}_2(\Omega)} - \int_\Omega \mathbb{A}(t) \nabla \psi(t) \cdot \nabla \psi(t) \, \mathrm{d}x \right) = - \int_0^s \int_\Omega \psi(t) \mathbf{d}(t) \cdot \nabla \psi(t) \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega (-bu(t) + \mathbf{c} \cdot \nabla \psi(t)) \psi \, \mathrm{d}x \, \mathrm{d}t \,.$$

Integrating the LHS and realizing $u(0) = 0, \nabla \psi(s) = 0$,yields

$$\frac{1}{2} \left(\| u(s) \|^2_{\mathrm{L}_2(\Omega)} + \int_\Omega \mathbb{A}(0) \nabla \psi(0) \cdot \nabla \psi(0) \, \mathrm{d}x \right) = - \int_0^s \int_\Omega \psi(t) \mathbf{d}(t) \cdot \nabla \psi(t) \, \mathrm{d}x \, \mathrm{d}t + \int_0^s \int_\Omega (-bu(t) - \mathbf{c} \cdot \nabla \psi(t)) \psi \, \mathrm{d}x \, \mathrm{d}t \,.$$

Using the ellipticity of $\mathbb{A}$ and the regularity of the data:

$$\int_\Omega \mathbb{A}(0) \nabla \psi(0) \cdot \psi(0) \, \mathrm{d}x \geq C_1 \| \nabla \psi(0) \|_{\mathrm{L}_2(\Omega)} \Rightarrow - \int_\Omega \mathbb{A}(0) \nabla \psi(0) \cdot \psi(0) \, \mathrm{d}x \leq -C_1 \| \nabla \psi(0) \|_{\mathrm{L}_2(\Omega)} \leq -C_1 \| \psi(0) \|_V \,,$$

$$\int_\Omega \psi \mathbf{d} \cdot \nabla \psi \, \mathrm{d}x \leq \| \mathbf{d} \|_{\mathrm{L}_\infty(\Omega)} \| \psi \|_{\mathrm{L}_2(\Omega)} \| \nabla \psi \|_{\mathrm{L}_2(\Omega)} \leq \| \mathbf{d} \|_{\mathrm{L}_\infty(\Omega)} \| \psi \|^2_{\mathrm{W}^{1,2}(\Omega)} \,,$$

$$\int_\Omega (-bu + \mathbf{c} \cdot \nabla \psi) \, \mathrm{d}x \leq \left(\| b \|_{\mathrm{L}_\infty(\Omega)} \| u \|_{\mathrm{L}_2(\Omega)} + \| \mathbf{c} \|_{\mathrm{L}_\infty(\Omega)} \| u \|_{\mathrm{W}^{1,2}(\Omega)} \right) \| \psi \|_{\mathrm{W}^{1,2}(\Omega)} \,,$$

and so

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\psi(0)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C\Big(\int_0^s \|\psi\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)} + \|\psi(0)\|_{\mathrm{L}_2(\Omega)}\Big).$$

If we now choose a test function such that $\chi(t) = \int_0^t u(\tau)\,\mathrm{d}\tau\,, t \in (0,T]$, and repeat the whole procedure (again $\partial_t\chi(t) = u(t), \chi(0) = 0$, we obtain the estimates

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C\Big(\int_0^s \|\chi(t) - \chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u(t)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{L}_2(\Omega)}\Big),$$

and since $\|\chi(t) - \chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le 2\Big(\|\chi(t)\|_{\mathrm{W}^{1,2}(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)}\Big)$, the above inequality in fact implies

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + (1 - 2sC_1)\|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C_1 \int_0^s \Big(\|\chi\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|u\|^2_{\mathrm{L}_2(\Omega)}\Big)\,\mathrm{d}t\,.$$

If we now choose $T_1 \le T$ so small that $1 - 2T_1C_1 \ge \frac{1}{2}$, and restrict ourselves on $s \in [0, T_1]$, combining the two estimates yields

$$\|u(s)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(s)\|^2_{\mathrm{W}^{1,2}(\Omega)} \le C \int_0^s \Big(\|u(t)\|^2_{\mathrm{L}_2(\Omega)} + \|\chi(t)\|^2_{\mathrm{W}^{1,2}(\Omega)}\Big)\,\mathrm{d}t\,,$$

which implies $u = 0$ on $[0, T_1]$ by the Gronwall lemma: we have

$$\xi(t) \le \int_0^t \xi(s)\,\mathrm{d}s\,, \text{ for } a.a.\, t \in (0,T) \Rightarrow \xi(t) = 0\; a.e..$$

for $\xi \in \mathrm{L}_1((0,T))$ nonnegative. If we now boostrap on $[T_1, 2T_1], [2T_1, 3T_1]$ etc., we obtain $u = 0$ on $(0,T]$.

□

2 Sobolev spaces revisited

Let $\Omega \subset \mathbb{R}^d \quad open, p \in [1,+\infty], k \in N$. We define

$$\mathrm{W}^{\mathrm{k,p}}(\Omega) = \Big\{f \in \mathrm{L}_\mathrm{p}(\Omega)\,; D^\alpha f \in \mathrm{L}_\mathrm{p}(\Omega)\,, \forall|\alpha| \le k\Big\},$$

with the norm

$$\|f\|^p_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} = \|f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)} + \sum_{0<|\alpha|\le k} \|D^\alpha f\|^p_{\mathrm{L}_\mathrm{p}(\Omega)}.$$

Recall that:

- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is Banach $\forall p$ and Hilbert for $p = 2$.
- $\mathrm{W}^{\mathrm{k,p}}(\Omega)$ is separable if $p < \infty$ and reflexive if $p > 1, p < \infty$.

Our goal will be to prove embedding and trace theorems. We will use the density of smooth functions.

2.1 Tools from functional analysis

Definition 1 (Regularization kernel)**.** The function η is called the regularization kernel supposed:

- $\eta \in \mathcal{D}(\mathbb{R}^d)$
- $\operatorname{supp} \eta \subset \mathrm{U}(0,1)$
- $\eta \geq 0$
- η is radially symmetric
- $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$

Definition 2 (Regularization of a function)**.** Let η be a regularization kernel. Set $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(x/\varepsilon), \varepsilon > 0$. We define the smoothing of f by

$$f_\varepsilon(x) = (f \star \eta_\varepsilon)(x).$$

Remark (Propertios of regularization). The regularization has the following properties:

- $f \in \mathrm{L_p}(\Omega) \Rightarrow f_\varepsilon \to f \,\mathrm{in}\, \mathrm{L_p}(\Omega)$ and also a.e
- $f \in \mathrm{L}_\infty(\Omega) \Rightarrow f_\varepsilon \to f$ a.e and *-weak
- $f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathrm{U}(x,\varepsilon)} f_y \eta_\varepsilon(x-y)\,\mathrm{d}y$
- $\operatorname{supp} f_\varepsilon \subset \overline{U(\Omega,\varepsilon)}, f = 0 \,\mathrm{on}\, \mathrm{U}(x,\varepsilon) \Rightarrow f_\varepsilon(x) = 0$

Definition 3 ($\Omega' \subset\subset \Omega$)**.** $O \subset\subset \Omega$ means $\overline{O}$ is compact and $\overline{O} \subset \Omega$.

Lemma 1 (Approximation of Sobolev functions using regularization)**.** *Assume* $p \in [1,\infty), \Omega \subset \mathbb{R}^d$ *open*$, k \in \mathbb{N}, u \in W^{k,p}(\Omega), \Omega' \subset\subset \Omega$*. Then it holds*

1. $\operatorname{dist}(\overline{\Omega}', \partial\Omega) = D > 0$
2. $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$ *in* $\Omega', \forall \varepsilon \in (0,D), \forall |\alpha| \leq k$
3. $f_\varepsilon \to f$ *in* $W^{k,p}(\Omega), \varepsilon \to 0^+$

Proof.

1. disjoint compact and closed set
2. WLOG $\frac{\partial f_\varepsilon}{\partial x^k} = \frac{\partial \int_{\mathbb{R}^d} f_y \eta_\varepsilon(x-y)\mathrm{d}y}{\partial x^k} = \int_\Omega f_y \frac{\partial \eta_\varepsilon}{\partial x^k}\,\mathrm{d}y = -\int_\Omega f(y)\frac{\partial \eta_\varepsilon}{\partial y^k}\,\mathrm{d}y = -\int_\Omega \frac{\partial f}{\partial y^k}\eta_\varepsilon(x-y)\,\mathrm{d}y = (D^\alpha f)_\varepsilon(x)$.
3. follows from 2) and the remark above applied to $f, D^\alpha f, |\alpha| \leq k$.

□

Lemma 2 (Partition of unity)**.** *Let* $E \subset \mathbb{R}^d, \mathcal{G}$ *opencovering. Then there exists a countable system* $\mathcal{F}$ *of nonnegative functions* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *such that* $0 \leq \varphi \leq 1$ *and*

1. $\mathcal{F}$ *is subordinate to* $\mathcal{G} : \forall \varphi \exists U \in \mathcal{Q} : \operatorname{supp} \varphi \subset U$
2. $\mathcal{F}$ *is locally finite:* $\forall K \subset E$ *compact*$, \operatorname{supp} \varphi \cap K \neq \emptyset$ *for at most finitely many* $\varphi \in \mathcal{F}$.

3. $\sum_{\varphi\in\mathcal{F}}\varphi(x) = 1, \forall x \in E$.

Proof. (Sketch) *Step 1 (E is compact)*:

E compact $\Rightarrow \exists N \in \mathbb{N} : U_j \in \mathcal{Q} \quad s.t. E \subset \bigcup_{j=1}^m U_j$. Moreover, $\exists K_j \subset U_j$ compact such that $E \subset \cup_{j=1}^m K_j$. That follows from the exhaustion argument: for $U \subset \mathbb{R}^d$ open, you can approximate it by a compact set: $K_m = \left\{x \in U, \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{m}, \|x\| \leq m\right\}$. Then clearly $K_1 \subset K_2 \ldots$, and they "converge monotonously to U. Next, find $\phi_j \in C_c(U_j), \phi_j > 0 \,\mathrm{on}\, K_j$, e.g. $\phi_j = \theta(\operatorname{dist}(x, \partial U_j))$. Then use convolution: $\psi_j = (\phi_j)_\varepsilon, \varepsilon > 0$ small and take finally $\varphi_j = \frac{\psi_j}{\sum_j \psi_j}$.

Step 2 (E is open):

Use exhaustion argument, then finite → countable. ☐

2.2 Density of smooth functions

Theorem 4 (Density of smooth functions I)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}, p \in [1,\infty)$. Then $\Big\{f \in C^\infty(\Omega), \operatorname{supp} f \quad bounded\Big\} \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \varepsilon > 0$. I want to show $\exists v \in C^\infty(\Omega) \cap \mathrm{W}^{\mathrm{k,p}}(\Omega) \quad s.t \|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon$. Using the exhaustion argument, define

$$\Omega_j = \Big\{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\Big\}.$$

Clearly, $\Omega_j \subset \Omega_{j+1}, \cup_{j=1}^\infty \Omega_j = \Omega$. Next, set $U_j = \Omega_{j+1}\ \overline{\Omega_{j-1}}, j = 1, 2, \ldots$, where $\Omega_0 = \Omega_{-1} = \varnothing$. Using the partition of unity lemma, $\exists\{\varphi_j\}$ partition of unity subordinate to $\{U_j\}$. We can write $u = \sum_j u\varphi_j$, where $u\varphi_j \in \mathrm{W}^{\mathrm{k,p}}(\Omega), \operatorname{supp} u\varphi_j \subset U_j \subset \Omega_{j+1} \subset\subset \Omega$. This is ready for convolution with $\varepsilon_j > 0$ sufficiently small: set $v_j = (u\varphi_j)_{\varepsilon_j}$. By the properties of regularization, we now

$$\|u - u\varphi_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \frac{\varepsilon}{2^j},$$

by taking ε_j small enough. Set $v = \sum_j v_j$ and use the following trick:

Fix $N \in \mathbb{N}$ and estimate $\|v - u\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$. Observe $u - v = \sum_{j=1}^\infty (u\varphi_j - v_j)$, so taking $x \in \Omega_N$ i have $(u - v)(x) = \sum_{j=1}^{N+1}(u\varphi_j - v_j)$. The norm of this is

$$\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \leq \sum_{j=1}^{N+1} \|u\varphi_j - v_j\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)} < \varepsilon.$$

It only remains to let $N \to \infty$ and realize $\|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega_N)} \to \|u - v\|_{\mathrm{W}^{\mathrm{k,p}}(\Omega)}$ by Lévi's theorem: $\int_{\Omega_N} |D^\alpha f| \,\mathrm{d}x \to \int_\Omega |D^\alpha| \,\mathrm{d}x$. ☐

Remark. It is nice that we only require Ω to be open (no boundary regularity required), but on the other hand, we don't have any information about the function's behaviour near it.

Recall $\Omega \in C^0$ means $\exists U_j, j = 1, \ldots, m \quad open, \exists\alpha, \beta > 0, a_j : \overline{U(0,\alpha)} \to \mathbb{R}, \mathbb{A}_j : \mathbb{R}^d \to \mathbb{R}^d \quad aff.orthogonal, \quad suchthat \partial\Omega$ $\cup_{j=1}^m U_j, \partial\Omega \cap U_j = \Big\{(x', a(x'), x' \in U(0,\alpha)\Big\}$. Setting $G_j(x', b) = \mathbb{A}_j(x', a(x') + b)$ we moreover require $G_j(U(0,\alpha) \times (0,\beta)) \subset \Omega, G_j(U(0,\alpha) \times (-\beta, 0)) \subset \overline{\mathbb{R}^d/\Omega}$.

Definition 4 (Shift operator)**.** For $u \in \mathrm{L_p}(\Omega)\,, k \in \{1, \ldots, d\}, h > 0$, we introduce the shift operator

$$\tau_h u(x) = u(x + h\mathbf{e}_k)$$

Lemma 3 (Approximation property of the shift operator)**.** *For $u \in L_p(\Omega)$, it holds $\tau_h u \to u$ in $L_p(\Omega)\,, h \to 0^+$.*

Theorem 5 (Density of smooth functions II)**.** *Let $\Omega \in C^0$ bounded, $k \in \mathbb{N}, p \in [1, \infty)$. Then $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in \mathrm{W^{k,p}}(\Omega)\,, \varepsilon > 0 \quad given$, i am looking for $v \in C_c^\infty(\mathbb{R}^d) \quad such that \|u - v\|_{\mathrm{W^{k,p}}(\Omega)} < \varepsilon$.

The sketch is simple: covering of $\overline{\Omega}$, partition of unity. Clearly, $\Omega \subset \cup_{j=0}^m U_j$, where $U_0 = \Omega, U_j$ are from the definition of C^0 boundary. Take $\{\varphi_j\}$ to be the partition of unity on $\overline{\Omega}$, subordinate to this cover. Observe that $u\varphi_j \in \mathrm{W^{k,p}}(\Omega)\,, \operatorname{supp} u\varphi_j \subset U_j$. Find

$$v_j \in \mathcal{D}(\mathbb{R}^d) \quad s.t. \|v_j - u\varphi_j\|_{\mathrm{W^{k,p}}(\Omega)} < \frac{\varepsilon}{m+1}.$$

If i am able to do this, i am finished: just take

$$v = \sum_{j=0}^m v_j$$

.

Case $j = 0$. We have $\operatorname{supp} u\varphi_0 \subset\subset \Omega$, take $v_0 = (u\varphi_0)_\varepsilon$, so if we take $\varepsilon > 0$ small enough, i can use the previous lemma.

Case $j \in \{1, \ldots, m\}$. Set $w_j = u\varphi_j, \tau_\delta w_j(x', x_d) = w(x', x_d + \delta)$ (ignore $\mathbb{A}_{\mathbb{j}}$), observe $t_\delta u_j \in \mathrm{W^{k,p}}\big(U_j^\delta\big)\,, U_j \subset\subset U_j^\delta$. Finally, set $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$ small enough. From the properties of the shift $\tau_\delta w_j$ is close to w_j in $\mathrm{L_p}(U_j \cap \Omega)$ and $D^\alpha \tau_\delta w_j = \tau_\delta(D^\alpha w_j)$ close to $D^\alpha w_j$ in $\mathrm{L_p}(U_j \cap \Omega)$. Finally, set $v_j = (t_\delta w_j)_{\varepsilon_j}, \varepsilon_j > 0$ small enough $\Rightarrow v_j \in \mathcal{D}(\mathbb{R}^d), \operatorname{supp} v_j \subset U_j$ by the previous lemma $\|v_j - \tau_\delta w_j\|_{\mathrm{W^{k,p}}(\Omega)}$ small. ☐

Remark. Recall $C^\infty_{\overline{\Omega}}(\mathbb{R}^d) = \Big\{u|_{\overline{\Omega}}, u \in C^\infty(\mathbb{R}^d)\Big\}$.

2.3 Extension of Sobolev functions

Problem of extension: For $u \in \mathrm{W^{k,p}}(\Omega)$, does there exist $\overline{u} \in \mathrm{W^{k,p}}\big(\mathbb{R}^d\big)\,, \quad s.t. \overline{u}|_\Omega = u, \|\overline{u}\|_{\mathrm{W^{k,p}}(\mathbb{R}^d)} \leq C(\Omega)\|u\|_{\mathrm{W^{k,p}}(\Omega)}$?

The answer is **yes**, if Ω is nice enough.

Lemma 4. *Let $\alpha, \beta > 0, K \subset U(0, \alpha) \times [\alpha, \beta]$ be compact. Then*

$$\exists C > 0, \exists E : C^1(\overline{U(0,\alpha)} \times [0, \beta]) \to C^1(\overline{U(0,\alpha)} \times [-\beta, \beta]), \exists \tilde{K} \subset U(0, \alpha) \times [-\beta, b) \; compact$$

such that:

1. $\|Eu\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))} \leq \|u\|_{W^{1,p}(U(0,\alpha)\times(-\beta,\beta))}$
2. *if* $\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Eu \subset \tilde{K}$

Proof. Use the following trick:

$$\overline{u}(x)=\begin{cases}u(x), & x_d>0\\ -3u(x_1,\dots,x_{d-1},-x_d)+4u(x_1,\dots,x_{d-1},-\frac{x_d}{2}), & x_d<0.\end{cases}$$

Is this extension C^1? Take some $a=(x_1,\dots,x_{d-1},0)$. Then

$$u(x\to a)=\begin{cases}u(a), & x_d>0\\ -3u(a)+4u(a)=u(a), & x_d<0,\end{cases}$$

so $\overline{u}$ is continuous. Its derivative
$\partial_k\overline{u}, k=1,\dots,d-1$ is the same as for u, where as

$$\partial_d\overline{u}=\begin{cases}\partial_d u, & x_d>0\\ -3\partial_d u(x_1,\dots,x_{d-1},-x_d)(-1)+4\partial_d u(x_1,\dots,x_{d-1},-\frac{x_d}{2})(\frac{-1}{2})=3\partial_d u-2\partial_d u, & x_d<0,\end{cases}$$

so the the derivative is also continuous. Thus, we have $Eu=\overline{u}\in C^1\subset \mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta))$ and estimate of the norm $\|Eu\|_{\mathrm{W}^{1,p}(U(0,\alpha)\times(-\beta,\beta))}$ is clear, as the wanted term is just some linear combination.

Mr. Prazak is not sure how this should be correctly finished and i am not also. □

Lemma 5 (Change of variables under C^1 diffeomorphisms)**.** *Let $U,V\subset\mathbb{R}^d$ be open, $\phi:U\to V$ be C^1 diffeomorphism. Let $\tilde{U}\subset\subset U$. Then*

$$\phi(\tilde{U})\subset\subset V,\ \textit{and}\ \exists C>0:\forall u\in C^1(V):\|u\circ\phi\|_{W^{1,p}(\tilde{U})}\le C\|u\|_{W^{1,p}(\phi(\tilde{U}))}$$

Proof. $\|u\circ\phi\|^p_{\mathrm{L_p}(\tilde{U})}=\int_{\tilde{U}}(u\circ\phi)^p|\det\nabla\phi|\,\mathrm{d}x\le C_0^{-1}\int_{\tilde{U}}|u\circ\phi|^p|\det\nabla\phi|\,\mathrm{d}x$, where $\det\nabla\phi>0$ in U, so $\det\nabla\phi\ge C_0>0$ in $\tilde{U}$. Together $\|u\circ\phi\|^p_{\mathrm{L_p}(\tilde{U})}=C_0^{-1}\int_{\phi(\tilde{U})}|u|^p\,\mathrm{d}x=C_0^{-1}\|u\|_{\mathrm{L_p}(\phi(\tilde{U}))}$ □

Lemma 6. *Let $\alpha,\beta>0, K\subset U(0,\alpha)\times[0,\beta), K$ compact. Then there is $C>0, E:C^1\left(\overline{U(0,\alpha)\times[0,\beta)}\right)\to C^1\left(\overline{U(0,\alpha)}\times[-\beta,\beta]\right), \tilde{K}\subset U(0,\alpha)\times[-\beta,\beta)$ compact such that*

- $\|E\|_{\mathcal{L}(W^{1,p}(U(0,\alpha)\times(0,\beta)),W^{1,p}(U(0,\alpha)\times(-\beta,\beta))}\le C$
- $u\in C^1\left(\overline{U(0,\alpha)}\times[0,\beta]\right), \operatorname{supp}u\subset K\Rightarrow\operatorname{supp}Eu\subset\tilde{K}$

Proof. No proof. □

Lemma 7. *Let $U,V\subset\mathbb{R}^d$ open, $\Phi:U\to V, C^1$ diffeomoprhism, $\tilde{U}\subset\subset U$ compact. Then $\Phi(\tilde{U})\subset\subset V$ and*

$$\exists C>0:\forall u\in C^1(V):\|u\circ\Phi\|_{W^{1,p}(\tilde{U})}\le C\|u\|_{W^{1,p}(\Phi(\tilde{U}))}.$$

Proof. No proof. □

Theorem 6 (Extension of Sobolev functions)**.** *Let $\Omega\in C^{k-1,1}, k\in\mathbb{N}, p\in[1,\infty], V\subset\mathbb{R}^d$ open such that $\Omega\subset\subset V$. Then there is $E:W^{k,p}(\Omega)\to W^{k,p}(\mathbb{R}^d)$ bounded linear operator such that*

1. $\forall u \in W^{k,p}(\Omega) : Eu = u$ *a.e. in* Ω,

2. $\forall u \in W^{k,p}(\Omega) : \operatorname{supp} Eu \subset V$,

3. $\|E\| \leq C, C = C(p, \Omega, V)$.

Proof. Only for $k = 1, \Omega \in C^1, p < \infty$. We know $C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ is dense in $\mathrm{W}^{1,p}(\Omega)$, we show existence of E for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$ with properties 1),2),3) and then extend E to $\mathrm{W}^{1,p}(\Omega)$ by density.
Covering of Ω*:*

$$\overline{\Omega} \subset \Omega \cup \bigcup_{j=1}^{m} U_j$$

with $U_j, a_j, \mathbb{A}_j, \alpha, \beta$ as in the definition of a C^1 domain. In particular, $a_j \in C^1(\mathrm{U}(0,\alpha))$.
Construction of E*:* We denote $\{\varphi_j\}_{j=0}^m$ partition of unity subordinate to $\{U_j\}_{j=1}^m$. For $j \in \{1,\dots,n\}$ we define $\phi_j : \mathrm{U}(0,\alpha) \times (-\beta,\beta) \to U_j$ by

$$\phi_j(y', y_d) = \mathbb{A}_j(y', a_j(y') + y_d), y' \in \mathbb{R}^{d-1}, y_d \in \mathbb{R}.$$

Trivially ϕ_j is C^1 diffemorphism. Let us denote by $\tilde{E}$ the extension operator from the previous lemma. Then we have for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u = \sum_{j=1}^m \varphi_j u$. We define

$$Eu = \varphi_0 u + \sum_{j=1}^{m} \left(\eta \tilde{E}((\varphi_j u) \circ \phi_j)\right) \circ \phi_j^{-1},$$

where η is a cut-off function $\eta = 1 \,\text{on}\, y_d \geq 0, \in (0,1) \,\text{else}\,, = 0 \,\text{on}\, y_d \leq -h$, for some parameter $h > 0$ which will be defined later. We also take $\eta \in C^\infty$. Due to our construction,

$$\phi_j^{-1}(\mathrm{U}(0,\alpha) \times [-2h, \beta)) \subset \mathrm{U}(\Omega, \varepsilon) \subset \mathrm{U}(\Omega, 2\varepsilon) \subset V,$$

for some $\varepsilon > 0$.
Properties of E*:* It is clear that

- E is linear from its definition
- 1) holds, as ϕ_j and ϕ_j^{-1} cancel *somewhere*
- 2) holds for $h < \frac{\beta}{2}$
- 3) we use the previous lemma:

$$\begin{aligned} \left\| \underbrace{\left(\eta \tilde{E}(\varphi_j u \circ \phi_j)\right)}_{\mathrm{supp}() \subset \mathrm{U}(0,\alpha) \times (-\beta,\beta)} \circ \phi_j^{-1} \right\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &\leq C \left\| \eta \tilde{E}(\varphi_j u \circ \phi_j) \right\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha) \times (-\beta,\beta))} \\ &\underbrace{\leq}_{\text{previous lemma}} C \|\varphi_j u \circ \phi_j\|_{\mathrm{W}^{1,p}(\mathrm{U}(0,\alpha) \times (0,\beta))} \\ &\underbrace{\leq}_{\text{previous lemma}} C \|\varphi_j u\|_{\mathrm{W}^{1,p}(U_j \cap \Omega)} \leq \|u\|_{\mathrm{W}^{1,p}(\Omega)} \Rightarrow \|E\| \leq C. \end{aligned}$$

So all the properties hold for $u \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d)$. We need to show them also for $u \in \mathrm{W}^{1,p}(\Omega)$. Pick an arbitrary $u \in \mathrm{W}^{1,p}(\Omega)$, find $\{u_k\} \subset C^\infty_{\overline{\Omega}}(\mathbb{R}^d) : u_k \to u \,\text{in}\, \mathrm{W}^{1,p}(\Omega)$.
Ad 1): Since E is continuous, then $Eu_k \to Eu \,\text{in}\, \mathrm{W}^{1,p}(\mathbb{R}^d)$. Since $\Omega \subset \mathbb{R}^d \Rightarrow Eu = u \,\text{in}\, \mathrm{W}^{1,p}(\Omega)$.
Ad 2): $\operatorname{supp} Eu_k \subset \mathrm{U}(\Omega, \varepsilon) \Rightarrow \operatorname{supp} Eu \subset \overline{\mathrm{U}(\Omega,\varepsilon)} \subset V$.

□

Remark ($\Omega \in C^{0,1}$ suffices). The theorem is still valid if we assume only $C^{0,1}$ and $p \in (1,\infty), k > 1$.

2.4 Embedding theorems

Example. Let $u \in \mathcal{D}(\mathbb{R}^2)$. Then

$$u(x_1,x_2) = \int_{-\infty}^{x_1} \partial_1 u(s,x_2)\,\mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1,s)\,\mathrm{d}s\,,$$

so

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u(x_1,x_2)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \le \int_{\mathbb{R}} |\partial_1 u(s,x_2)|\,\mathrm{d}s \int_{\mathbb{R}} |\partial_2 u(x_1,s)\,|\,\mathrm{d}s\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \Big(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}\lambda^2 \Big)^2,$$

so

$$\|u\|_{\mathrm{L}_2(\mathbb{R}^2)} \le \|\nabla u\|_{\mathrm{L}_1(\mathbb{R}^2)}.$$

Lemma 8. *Let $d \ge 2$. Let $\hat{u}_i : \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative and measurable for $j \in \{1,\dots,d\}$. We define*

$$\hat{x_j} = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_d), \mathrm{d}\hat{\mathrm{x}}_\mathrm{j} = \mathrm{d}x_1 \dots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \dots \mathrm{d}x_d\,.$$

Consider the functions $u_j : \mathbb{R}^d \to \mathbb{R}, u_j(x) = \hat{u_j}(\hat{x_j})$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^d u_j(x)\,\mathrm{d}x \le \prod_{j=1}^d \Big(\int_{\mathbb{R}^{d-1}} (\hat{u_j}(\hat{x_j}))^{d-1} \mathrm{d}\hat{\mathrm{x}}_\mathrm{j} \Big)^{\frac{1}{d-1}}. \tag{1}$$

Proof. Induction by d.

1. $d = 2 : \int_{\mathbb{R}^d} u_1(x_1,x_2) u_2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\mathbb{R}^2} \hat{u}_1(x_2)\hat{u}_2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \underbrace{=}_{\text{Fubini}} \int_{\mathbb{R}} \hat{u}_1(x_2)\,\mathrm{d}x_2 \int_{\mathbb{R}} \hat{u}_2\,\mathrm{d}x_1\,.$

2.

$$\begin{aligned} d \to d+1 : \int_{\mathbb{R}^{d+1}} \prod_{j=1}^{d+1} u_j(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \prod_{j=1}^{d} u_j(x)\,\mathrm{d}x_{d+1}\, u_{d+1}\,\mathrm{d}x\,\mathrm{d}\hat{\mathrm{x}}_{\mathrm{d+1}} \\ &\underbrace{\le}_{\text{Holder}} \int_{\mathbb{R}^d} \Big(\prod_{j=1}^{d} \int_{\mathbb{R}} (u_j(x))^d \,\mathrm{d}x_{d+1} \Big)^{\frac{1}{d}} u_{d+1}(x)\mathrm{d}\hat{\mathrm{x}}_{\mathrm{d+1}} \\ &\underbrace{\le}_{\text{Holder}} \left(\int_{\mathbb{R}^d} \Big(\prod_{j=1}^{d} \int_{\mathbb{R}} u_j^d(x)\,\mathrm{d}x_{d-1} \Big)^{\frac{1}{d-1}} \mathrm{d}x_{\hat{d+1}} \right)^{\frac{d-1}{d}} \Big(\int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}x_{\hat{d+1}} \Big)^{\frac{1}{d}} \\ &\underbrace{\le}_{\text{induction step}^1} \Big(\int_{\mathbb{R}^d} u_{d+1}^d \,\mathrm{d}\hat{x}_{d+1} \Big)^{\frac{1}{d}} \Big(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u_j^d(x)\,\mathrm{d}x_{d+1}\,\mathrm{d}\hat{x}_j\,\mathrm{d}\hat{x}_{d+1} \Big)^{\frac{d-1}{d}\frac{1}{d-1}}. \end{aligned}$$

□

Theorem 7 (Gagliardo-Nirenberg)**.** *Let $p \in [1,d)$. Then $\forall u \in W^{1,p}(\mathbb{R}^d)$:*

$$\|u\|_{L_{p^*}(\mathbb{R}^d)} \le \frac{p(d-1)}{d-p}\|\nabla u\|_{L_p(\mathbb{R}^d)},$$

where $p^* = \frac{dp}{d-p}$.

Proof. Estimate for $u \in \mathcal{D}(\mathbb{R}^d)$:

$$\forall j \in \{1,\dots,d\}, x \in \mathbb{R}^d : u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_d)\,\mathrm{d}s$$

independet of x_j, so

$$|u(x)| \le \int_{\mathbb{R}} |\nabla u|(\dots,s,\dots)\,\mathrm{d}s\,.$$

Next, consider $p = 1, p^* = \frac{d}{d-1}$ and estimate:

$$|u|^{\frac{d}{d-1}} \le \prod_{j=1}^{d} \underbrace{\left(\int_{\mathbb{R}} |\nabla u|(\dots,s,\dots)\,\mathrm{d}s\right)^{\frac{1}{d-1}}}_{u_j \text{ independent of } x_j},$$

so the integral

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}}\,\mathrm{d}x \le \int_{\mathbb{R}^d} \prod_{j=1}^{d} u)j\,\mathrm{d}x \underbrace{\le}_{\text{previous lemma}} \left(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\nabla u|(x)\,\mathrm{d}x_j\,\mathrm{d}\hat{x}_j\right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u|\,\mathrm{d}x\right)^{\frac{d}{d-1}}.$$

If $p \in (1,d)$, compute

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\mathbb{R}^d)} = \||u|^q\|_{\mathrm{L}_{\frac{d}{d-1}}(\mathbb{R}^d)} \le \|\nabla(|u|^q)\|_{\mathrm{L}_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} q|u|^{q-1}|\nabla u|\,\mathrm{d}x \underbrace{\le}_{\text{Holder}} \|\nabla u\|_{\mathrm{L}_p(\mathbb{R}^d)}\|u\|^{q-1}_{\mathrm{L}_{(q-1)p'}(\mathbb{R}^d)}.$$

We want $\frac{(q-1)p'}{p-1} = \frac{qd}{d-1}$, so

$$q\left(\frac{p}{p-1} - \frac{d}{d-1}\right) = \frac{p}{p-1}, \Leftrightarrow q\frac{pd-p-pd+d}{(p-1)(d-1)} = \frac{d-p}{(p-1)(d-1)} = \frac{p}{p-1} \Leftrightarrow q = \frac{d-1}{d-p}p.$$

Also

$$q\frac{d}{d-1} = p^*.$$

$\Rightarrow$ statement holds for $u \in \mathcal{D}(\mathbb{R}^d)$. To finish, use density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\mathbb{R}^d)$. □

Remark.
- It is evident that nonzero constants are not in $\mathrm{W}^{1,p}(\mathbb{R}^d)$ and that also the inequality does not hold for them.
- the set $\mathbb{R}^d$ is of course unbounded, so we have no ordering of $\mathrm{L_p}(\Omega)$ spaces.

- of course, we require no smoothness of the domain

Theorem 8. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\forall u \in W_0^{1,p}(\Omega), \forall p \in [1,d)$ the statement of the previous theorem holds.*

Proof. An immediate corollary of the previous theorem. □

Remark. In the proof of theorem we showed that $\forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|^q_{\mathrm{L}_{\frac{qd}{d-1}}(\Omega)} \leq q\|\nabla u\|_{\mathrm{L_p}(\Omega)}\|u\|^{q-1}_{\mathrm{L}_{\frac{p(q-1)}{p-1}}(\Omega)},$$

for q such that $\frac{qd}{d-1} \leq p^*$.

Theorem 9 (Embedding theorem)**.** *Let $\Omega \subset C^{0,1}, p^* = \frac{dp}{1-p}$ If $p \in [1,d)$ then*

$$W^{1,p}(\Omega) \subset L_q(\Omega)\ \forall q \in [1,p^*].$$

Moreover, if $q < p^*$, then

$$W^{1,p}(\Omega) \subset\subset L_q(\Omega).$$

If $p = d$, then

$$W^{1,p}(\Omega) \subset L_q(\Omega)\ \forall q < \infty,\ W^{1,p}(\Omega) \subset\subset L_q(\Omega)\ \forall 1 \leq q < \infty.$$

Proof. We would like to use the previous theorem + extension.

Ad continuity for $p < d : E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ the extension is continuous. We also know

- identity $I_1 : \mathrm{W}^{1,p}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\mathbb{R}^d)$ is continous,
- restriction $I_2 : \mathrm{L}_{p^*}(\mathbb{R}^d) \to \mathrm{L}_{p^*}(\Omega)$ is continuous,
- identity $I_3 : \mathrm{L}_{p^*}(\Omega) \to \mathrm{L}_q(\Omega)$ is continous.

Together, the mapping $id : \mathrm{W}^{1,p}(\Omega) : \mathrm{L}_q(\Omega), id = I_3 \circ I_2 \circ I_1 \circ E$ identity is continuous. If p=d, then $\mathrm{W}^{1,d}(\Omega) \subset \mathrm{W}^{1,r}(\Omega)\ \forall r \in [1,d)$, and $r^* \to \infty$ as $r \to d-$. For $q \in [1,\infty)$ find $r \in [1,d)$ s.t. $r^* > q$. Then

$$\mathrm{W}^{1,d}(\Omega) \subset \mathrm{W}^{1,r}(\Omega) \subset \mathrm{L}_{r^*}(\Omega) \subset \mathrm{L}_q(\Omega),$$

using the previous results.

Ad compactness: We show $\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_q(\Omega)$ using Arzela-Ascoli and then it will get technical: show compactness in smooth functions, then show compactness in $\mathrm{L}_1(\Omega)$, then approximate the norm of $\mathrm{L}_q(\Omega)$ using the obtained quantities.

Consider $B = \mathrm{U}_{\mathrm{W}^{1,p}(\Omega)}(0,1)$ and extend it to EB. Fix $\delta > 0$ and let η be a regularization kernel. Then $\exists R > 0 : \operatorname{supp}(EB)_\delta \subset \overline{\mathrm{U}(0,R)} \subset \mathbb{R}^d$ (i.e. all the functions from EB have the support contained in the ball). Moreover, $(EB)_\delta \subset C^1\left(\overline{\mathrm{U}(0,R)}\right)$. Actually, it is bounded in $C^1\left(\overline{\mathrm{U}(0,R)}\right)$. $\underbrace{\subset\subset}_{\text{Arzela-Ascoli}}$ $C\left(\overline{\mathrm{U}(0,R)}\right)$ (uniform equicontinuity comes from uniform boundedness of the gradients, $\nabla(u * \eta_\delta) = u * \nabla\eta_\delta$.) Altogether $(EB)_\delta$ is relatively compact in

$$C\left(\overline{\mathrm{U}(0,R)}\right) \underbrace{\Rightarrow}_{\text{the space } C\left(\overline{\mathrm{U}(0,R)}\right) \text{ is complete}} \text{bounded in } C\left(\overline{\mathrm{U}(0,R)}\right) \underbrace{\Rightarrow}_{\text{bounded domain}} \text{bounded in } \mathrm{L}_1(\mathrm{U}(0,R)).$$

Next, take

$$u \in B : \|u - (Eu)_\delta\|_{\mathrm{L}_q(\Omega)} \le \|Eu - (Eu)_\delta\|_{\mathrm{L}_q(\mathrm{U}(0,R))} = \int_{\mathrm{U}(0,R)} |v - v_\delta| \,\mathrm{d}x = \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} v(x+y) - v(x)\eta_\delta(y)\,\mathrm{d}y\,|\,\mathrm{d}x \le$$
$$\le \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \frac{|v(x+y)-v(x)|}{|y|}|\eta_\delta(y)||y|\,\mathrm{d}y\,\mathrm{d}x \underbrace{\le}_{\text{Fubini}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y\,.$$

Estimate the inner integral: assume v is smooth and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|}|\int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(v(x+sy))}_{\nabla v(x+sy)\cdot y}\,\mathrm{d}s\,|\,\mathrm{d}x \underbrace{\le}_{\text{Cauchy Schwartz}} \int_{\mathbb{R}^d}\int_0^1 |\nabla v|(x+sy)\,\mathrm{d}s\,\mathrm{d}x \underbrace{\le}_{\text{Holder}} C(R)\Big(\int_{\mathbb{R}^d} |\nabla v|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

Now, take $v \in \mathrm{W}^{1,p}_0(\mathrm{U}(0,R))$, then $\exists \{v_k\} \subset \mathcal{D}(\mathrm{U}(0,R)) : v_k \to v \,\text{in}\, \mathrm{W}^{1,p}(\mathrm{U}(0,R))$. So

$$\forall y \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{|v_k(x+y)-v_k(x)|}{|y|}\,\mathrm{d}x \le C(R)\Big(\int_{\mathbb{R}^d}|\nabla v_k|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \to C(R)\Big(\int_{\mathbb{R}^d}|\nabla v|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

So finally

$$\|u-(Eu)_\delta\|_{\mathrm{L}_q(\Omega)} \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x+y)-v(x)|}{|y|}\,\mathrm{d}x\,|y|\eta_\delta(y)\,\mathrm{d}y \underbrace{\le}_{|y|\le\delta} C(R)\delta \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\nabla u|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\mathrm{d}x \le C_1\delta.$$

Fix $\varepsilon > 0$, find finite $\frac{\varepsilon}{2}$-net in $(EB)_\delta$ in $\mathrm{L}_1(\mathrm{U}(0,R))$ (that is possible since we have total boundedness in $\mathrm{L}_1(\mathrm{U}(0,R))$.) Set $\delta > 0\,\text{s.t.}\, C_1\delta\frac{\varepsilon}{4}$. [2]. Denote the $\frac{\varepsilon}{2}$-net as $\{Eu_k\}_{k=1}^m, m \in \mathbb{N}$. We show $\{u_k\}_{k=1}^m$ is a ε-net in B. Fix $u \in B$, find $j \in \{1,\dots,m\} : \|(Eu)_\delta - (Eu_j)_\delta\|_{\mathrm{L}_1(\mathrm{U}(0,R))}$. Compute

$$\|u-u_j\|_{\mathrm{L}_1(\Omega)} \le \|u-(Eu)_\delta\|_{\mathrm{L}_1(\Omega)} + \big\|(Eu)_\delta - (Eu_j)_\delta\big\|_{\mathrm{L}_1(\Omega)} + \big\|(Eu_j)_\delta - u_j\big\|_{\mathrm{L}_1(\Omega)} \le 2C_1\delta + \frac{\varepsilon}{2} \le \varepsilon.$$

Thus, we have shown

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}_1(\Omega)\,.$$

It remains to show the validity for a general q. Let $q \in [1,p^*) : \|v\|_{\mathrm{L}_q(\Omega)} \le \|v\|^{\alpha}_{\mathrm{L}_1(\Omega)}\|v\|^{1-\alpha}_{\mathrm{L}_{p^*}(\Omega)}$, for $\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}, \alpha \in (0,1]$. Is B totally bounded in $\mathrm{L}_q(\Omega)$? Let us compute

$$\|u-u_j\|_{\mathrm{L}_q(\Omega)} \le \|u-u_j\|^{\alpha}_{\mathrm{L}_1(\Omega)} \underbrace{\|u-u_j\|^{1-\alpha}_{\mathrm{L}_{p^*}(\Omega)}}_{\le C, \mathrm{W}^{1,p}(\Omega)\subset \mathrm{L}_{p^*}(\Omega)} \le C\varepsilon^{\alpha}.$$

□

[2] The order of the choices is not precise...

2.5 Trace theorems

2.6 Composition of sobolev functions

2.7 Difference quotients

3 Nonlinear elliptic equations as compact perturbations

Theorem 10 (Nemytskii). *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}, N \in N, \Omega \subset \mathbb{R}^d$ measurable, f Caratheodory. Then*

1. *if $u : \Omega \to \mathbb{R}^N$ is measurable then $f(\cdot, u)$ is also measurable*
2. *If there is $p_i \in [1, +\infty), i \in \{1, \dots, N\}, q \in [1, \infty), g \in L_q(\Omega), C > 0$ such that for almost all*

$$x \in \Omega, \forall y \in \mathbb{R}^N : |f(x,y)| \le g(x) + c \sum_{i=1}^{N} |y_i|^{p_i/q}$$

   , then $u \mapsto f(\cdot, u)$ is continuous from $L_{p_i}(\Omega) \times \dots \times L_{p_N}(\Omega)$ to $L_q(\Omega)$. Moreover, it maps bounded sets to bounded sets.

Proof. No proof □

Definition 5 (Compact operator — Drábek, Milota: Methods of Nonlinear Analysis, Def 5.2.2)**.** Let X, Y be normed linear spaces, $M \subset X$. The mapping $F : M \to Y$ is called a compact operator on M into Y if F is continuous and $F(M \cap K)$ is relatively compact in Y for any bounded $K \subset X$.

Remark. We have no linearity of F! So continuity cannot follow from compactness (we have compactness $\Rightarrow$ boundedness $\ne$ continuity for nonlinear operators)

Theorem 11 (Brouwer fixed point theorem)**.** *Let $K \subset \mathbb{R}^N, N \in \mathbb{N}$ be a nonempty convex closed bounded. Assume that $F : K \to K$ is continuous. Then F has a fixed point in K, i.e.,*

$$\exists x_0 \in K : F(x_0) = x_0.$$

Proof. No proof □

Theorem 12 (Schauder fixed point theorem)**.** *Let $K \subset X$ be a nonempty convex closed bonded subset of a linear normed space X. Assume that F is compact on K into K and $F(K) \subset K$. Then there is fixed point of F in K.*

Proof. No proof □

- for Brouwer, $K \subset \mathbb{R}^N$ so since it is closed and bouded, it is automatically compact, and since $F : K \to K$ is continuous, F is compact. For Schauder, we have to assume this extra.
- proof of Brouwer with N=1 is easy, based on Darboux property.

3.0.1 Problem protypes

In this chapter some nonlinear elliptic equations are discussed.

Example. Suppose the following problem:

$$\begin{cases} -\triangle u + g(u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

where

$$g:\mathbb{R}\to\mathbb{R}, f\in\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*, \text{continuous}\,, \exists\alpha\in[0,1):\forall s\in\mathbb{R}:|g(s)|\le C(1+|s|^\alpha).$$

Theorem 13 (Existence)**.** *Let $\Omega\in C^{1,1}, f\in\left(W_0^{1,2}(\Omega)\right)^*, g$ is as above. Then there is a weak solution to the above problem, i.e., it holds:*

$$\forall\varphi\in W_0^{1,2}(\Omega):\int_\Omega \nabla u\cdot\nabla\varphi\,\mathrm{d}x =< f,\varphi>_{(W_0^{1,2}(\Omega))^*}\,.$$

If $f\in L_2(\Omega)$, then the solution $u\in W^{2,2}(\Omega)$.

Proof. We define $S:\mathrm{L}_2(\Omega)\to\mathrm{L}_2(\Omega)$ such that

$$Sw = u \Leftrightarrow \forall\varphi\in\mathrm{W}_0^{1,2}(\Omega):\int_\Omega\nabla u\cdot\nabla\varphi\,\mathrm{d}x =< f,\varphi> - \int_\Omega g(w)\varphi\,\mathrm{d}x\,.$$

S is well defined:

$$|\,\mathrm{RHS}\,|\le\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|\varphi\|_{\mathrm{W}^{1,2}(\Omega)}+\|\varphi\|_{\mathrm{L}_2(\Omega)}\|g(w)\|_{\mathrm{L}_2(\Omega)},$$

and

$$\int_\Omega|g(w)|^2\,\mathrm{d}x\le\int_\Omega C(1+|w|^\alpha)^2\,\mathrm{d}x\le\int_\Omega C\left(1+|w|^{2\alpha}\right)\mathrm{d}x\le\int_\Omega C\left(1+|w|^2\right)\mathrm{d}x\le\infty,$$

where we used the Young inequality and $\alpha\le 1$. We have thus shown the mapping $w\mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by Nemytskii. Next, S is continuous:

- $w\mapsto g(w)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$
- $w\mapsto\left(\varphi\mathrm{W}_0^{1,2}(\Omega)\to< f,\varphi> - \int_\Omega g(w)\varphi\,\mathrm{d}x\right)$ is continuous from $\mathrm{L}_2(\Omega)$ to $\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*$
- $F\to u$, where u is the weak solution of

$$\begin{cases} -\triangle u = F & in\Omega \\ u = 0 & on\partial\Omega, \end{cases}$$

  , is linear and continuous from $(\mathrm{W}_0^{1,2}(\Omega))^*$ to $\mathrm{W}_0^{1,2}(\Omega)$.

In total, the composition is continuous and yields S. Next, we would like to show S is compact. We start with showing S maps bounded sets in $\mathrm{L}_2(\Omega)$ to bounded sets in $\mathrm{W}_0^{1,2}(\Omega)$; for that we need apriori estimates: test the weak formulation with u:

$$\|\nabla u\|^2_{\mathrm{L}_2(\Omega)}\le\varepsilon\|u\|^2_{\mathrm{W}^{1,2}(\Omega)}+C\Big(\|f\|^2_{(\mathrm{W}^{1,2}(\Omega))^*}+\|g(w)\|^2_{\mathrm{L}_2(\Omega)}\Big)\underbrace{\le}_{\text{Younge}} C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*)}+1+\|w\|^2_{\mathrm{L}_2(\Omega)}\Big),$$

from which follows S is compact from $\mathrm{L}_2(\Omega)$ to $\mathrm{L}_2(\Omega)$ by compact embedding. Now we need to show $S(\mathrm{U}(0,R)) \subset \mathrm{U}(0,R)$ for some $R > 0$. From the previous we know:

$$\frac{C}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\Big(\|f\|_{(\mathrm{W}^{1,2}_0(\Omega))^*} + \|g\|^2_{\mathrm{L}_2(\Omega)}\Big),$$

so since

$$\tilde{C}\int_\Omega |g(w)|^2\,\mathrm{d}x \le \int_\Omega C\Big(1+|w|^{2\alpha}\Big)\,\mathrm{d}x \underbrace{\le}_{\text{Younge}} \int_\Omega \Big(C + \frac{c}{4}|w|^2\Big)\,\mathrm{d}x$$

we know

$$\frac{c}{2}\|u\|^2_{\mathrm{L}_2(\Omega)} \le \frac{c}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)} \le \tilde{C}\|f\|^2_{(\mathrm{W}^{1,2}_0(\Omega))^*} + C\lambda(\Omega) + \frac{c}{4}\|w\|^2_{\mathrm{L}_2(\Omega)},$$

and thus

$$\|u\|^2_{\mathrm{L}_2(\Omega)} \le \underbrace{\frac{2\tilde{C}}{c}\|f\|^2_{(\mathrm{W}^{1,2}_0(\Omega))^*} + 2\frac{C}{c}}_{=\overline{C}} + \frac{1}{2}\|w\|^2_{\mathrm{L}_2(\Omega)}.$$

so if $\overline{C} + \frac{1}{2}R^2 < R^2$, we are done [3]. But such an R of course exists (says doc. Kaplicky) $\Rightarrow$ the image of a ball is in a ball for some $R \Rightarrow S$ is compact and using Schauder we get the solution exists.

For the regularity part of the assertion, realize that u_0 solves $\begin{cases} -\triangle u_0 = f - g(u_0) \in \mathrm{L}_2(\Omega) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$.
So from the regularity theory for elliptic equations we get

$$u \in \mathrm{W}^{2,2}(\Omega)\,.$$

□

Theorem 14 (Uniqueness)**.** *Let $u_1, u_2 \in W^{1,2}_0(\Omega)$ be weak solutions to the above problem. Let $f \in \big(W^{1,2}_0(\Omega)\big)^*$, g be continuous. Let either*

1. *g is nondecreasing*
2. *$g \in C^1(\mathbb{R})$, $\|g'\|_\infty$ small.*

Then $u_1 = u_2$.

Proof. We subtract the equations for u_1, u_2 and test with $u_1 - u_2$.:

$$\int_\Omega |\nabla(u_1-u_2)|^2 + (g(u_1)-g(u_2))(u_1-u_2)\,\mathrm{d}x = 0.$$

In the first case, the second term is nonnegative and so

$$0 = \|\nabla(u_1-u_2)\|_{\mathrm{L}_2(\Omega)} \ge C\|u_1-u_2\|^2_{\mathrm{W}^{1,2}(\Omega)} \Rightarrow u_1 - u_2 = 0.$$

$$|\int_\Omega (g(u_1)-g(u_2)(u_1-u_2))\,\mathrm{d}x| \le \int_\Omega \|g'\|_\infty |u_1-u_2|^2\,\mathrm{d}x \le \|g'\|_\infty C_P\|\nabla(u_1-u_2)\|^2_{\mathrm{L}_2(\Omega)} = 0 \Rightarrow u_1 = u_2,$$

whenever $C\|g'\|_\infty < 1$. □

[3] The constants are most probably messed up.

Example. Suppose the following problem

$$\begin{cases} -\triangle u + b(\nabla u) = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}$$

where $f \in (\mathrm{W}_0^{1,2}(\Omega))^*, b$ is continuous and bounded. The weak formulation is

$$u \in \mathrm{W}_0^{1,2}(\Omega) \wedge \forall \varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla \varphi + b(\nabla u)\varphi \,\mathrm{d}x =< f, \varphi >,$$

and the first apriori estimates (test with u)

$$\|\nabla u\|_{\mathrm{L}_2(\Omega)} \le \|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*}\|u\|_{\mathrm{W}_0^{1,2}(\Omega)} + \int_\Omega |u|\,\mathrm{d}x\,\|b\|_{\mathrm{L}_\infty(\Omega)}.$$

Theorem 15. *Let $f \in \left(W_0^{1,2}(\Omega)\right)^*, \Omega \in C^{0,1}, b: \mathbb{R}^d \to \mathbb{R}$ continuous and bounded. Then there is a weak solution to the above problem.*

Proof. $S: \mathrm{W}_0^{1,2}(\Omega) \to \mathrm{W}_0^{1,2}(\Omega)\,, Sw = u$ iff u solves

$$\begin{cases} -\triangle u = f - b(\nabla w) & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}, \text{ i.e.}$$

it holds

$$\forall \varphi \in \mathrm{W}_0^{1,2}(\Omega): \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x =< f, \varphi > - \int_\Omega b(\nabla w)\varphi \,\mathrm{d}x\,.$$

Clearly, S is well defined and

$$\|Sw\|_{\mathrm{W}_0^{1,2}(\Omega)} \le \underbrace{C\Big(\|f\|_{(\mathrm{W}_0^{1,2}(\Omega))^*} + \|b\|_{\mathrm{L}_\infty(\Omega)}\Big)}_{:=R},$$

meaning $S\Big(\overline{\mathrm{U}(0,R)}\Big) \subset \overline{\mathrm{U}(0,R)}$. Moreover, S]s continuous, as S is the composiiton of a Nemytskii operator and the solution operator of the Laplace equation. It remains to show S is compact: we already have continuity, consider $\{w_k\}_{k\in\mathbb{N}} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded. Then $\exists\{u_k\} \subset \mathrm{W}_0^{1,2}(\Omega)$ bounded: $u_k \to u$ in $\mathrm{L}_1(\Omega)$ by embedding up to a subsequence. Next, uss the following trick: substitue equation for u_k from equation for u_l and test with $u_l - u_k$

$$C\|u_l - u_k\|^2_{\mathrm{W}_0^{1,2}(\Omega)} \le \|\nabla(u_l - u_k)\|^2_{\mathrm{L}_2(\Omega)} \le \int_\Omega |b(\nabla u_l) - b(\nabla u_k)\,||u_l - u_k|\,\mathrm{d}x \le 2\|b\|_{\mathrm{L}_\infty(\Omega)}\|u_l - u_k\|_{\mathrm{L}_1(\Omega)}.$$

All in all, S has a fixed point by Schauder, which is of course the weak solution. □

But this says $\{u_k\}$ is Cauchy in $\mathrm{W}_0^{1,2}(\Omega)$.

4 Nonlinear elliptic equations - monotone operator theory

Lemma 9. *Let $g: B(0,R) \subset \mathbb{R}^n \to \mathbb{R}^N$ be continuous, $N \in \mathbb{N}, R > 0,$ and $\forall c \in S(0,R): g(c)\cdot c \ge 0$. Then, there is $c_0 \in B(0,R): g(c_0) = 0$.*

Proof. By contradiction. Let $g \neq 0$ in $\mathrm{U}(0,R)$. Let us define

$$h(x) = -R\frac{g(x)}{|g(x)|}.$$

Then $h \in C(\mathrm{B}(0,R)), h(\mathrm{B}(0,R)) \subset \mathrm{S}(0,R)$, so by Brouwer there $\exists x_0 \in \mathrm{B}(0,R) : h(x_0 = x_0 \Rightarrow -R\frac{g(x_0)}{|g(x_0)|} = x_0$. Take the dot product with x_0 and write

$$\underbrace{-R\frac{g(x_0)\cdot x_0}{|g(x_0)|}}_{\leq 0} = \underbrace{|x_0|^2}_{=R^2} \wedge x_0 \in \mathrm{S}(0,R),$$

so that is a contradiction. □

Consider the following problem:

$$\begin{cases} -\sum_{i=1}^d \partial_i(a_i(x,u(x),\nabla u(x)) + a_0(x,u(x),\nabla u(x))) = f(x) & \text{in}\,\Omega \\ u = u_0 & \text{on}\,\partial\Omega \end{cases}$$

The date are

- $\Omega \in C^{0,1}$,
- $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, i \in \{1,\dots,d\}$ are Caratheodory in x and $(u,\nabla u)$.
- $f \in \left(\mathrm{W}_0^{1,r}(\Omega)\right)^*$,

and the unknown is $u : \Omega \to \mathbb{R}$.

Remark. The function $(u,p) \mapsto a_i(\cdot,u,p)$ is continuous from $(\mathrm{L}_r(\Omega))^{d+1}$ to $\mathrm{L}_{r'}(\Omega)$. by Nemystkii theorem.

Definition 6 (Coercivity)**.** We say that $\{a_i\}_{i=0}^d$ are coercive if $\exists C_1 > 0, C_2 \in \mathrm{L}_1(\Omega)$: a.e. $x \in \Omega, \forall(z,p) \in \mathbb{R}^{d+1}$:

$$\sum_{i=1}^d a_i(x,z,p)p_i + a_0(x,z,p) \geq C_1|p|^r - C_2(x), \text{ i.e. } a(x,z,p)\cdot p \geq C_1|p|^r - C_2(x)$$

Definition 7 (Monotonicity)**.** We say that $\{a_i\}_{i=0}^d = a$ is monotone if for almost all

$$x \in \Omega, \forall(z_1,p_1),(z_2,p_2) \in \mathbb{R}^{d+1} : (a(x,z_1,p_1) - a(x,z_2,p_2))\cdot(p_1-p_2)+(a_0(x,z_1,p_1)-a_0(x,z_2,p_2))\cdot(z_1-z_2) \geq 0.$$

Very similiarly we define strict monotonicity.

Definition 8 (Weak solution)**.** We say that $u \in \mathrm{W}^{1,r}(\Omega)$ is a weak solution to the above problem if

- $u = u_0$ in the sense of traces on $\partial\Omega$,
- $$\int_\Omega a(\cdot,u,\nabla u)\cdot\nabla\varphi + a_0(\cdot,u,\nabla u)\varphi\,\mathrm{d}x = < f,\varphi >, \forall\varphi \in \mathrm{W}_0^{1,r}(\Omega).$$

Theorem 16 (Existence and uniqueness). *Let $\Omega \in C^{0,1}$, $u_0 \in W^{1,r}(\Omega)$, $r \in (1,\infty)$, $\{a_i\}_{i=1}^d$ be Caratheodory, coercive and m and let them also satisfy the growth conditions. Finally, let $f \in \left(W^{1,r}(\Omega)\right)^*$. Then, there is a weak solution to the problem. If, moreover, $\{a_i\}_{i=1}^d$ is strictly monotone, then the weak solution is unique.*

Proof. The strategy is the following:

1. Galerkin Approximation
2. uniform estimates
3. limit passage
4. identification of limits

One of the issues we will face is that nonlinearities may destroy weak convergence, see the below example.

Galerkin: Since $\mathrm{W}_0^{1,r}(\Omega)$ is separable $\Rightarrow \exists\{w_i\}_{i=1}^\infty$ that is a dense[4] linearly independent subset of $\mathrm{W}_0^{1,r}(\Omega)$. We search for $n \in \mathbb{N}$ such that

$$u^n(x) := u_0(x) + \sum_{j=1}^n \alpha_j^n w_j(x),$$

where $\alpha_j \in \mathbb{R}$ and u^n satisfy

$$\forall j \in \{1,\dots,n\} : \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla w_j + a_0(\cdot,u^n,\nabla u^n)w_j \,\mathrm{d}x = < f, w_j > .$$

We claim such $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}^n$ exist $\forall n \in \mathbb{N}$ by the previous lemma. We define a vector function

$$F(\alpha^n) := \{\int_\Omega a\cdot\nabla w_j + a_0 w_j \,\mathrm{d}x - < f, w_j >\}_{j=1}^n,$$

from Nemystkii $F : \mathbb{R}^n \to \mathbb{R}^n$, F is continuous on $\mathbb{R}^n$. Moreover, it holds

$$\begin{aligned} F(\alpha^n)\cdot\alpha^n &\ge \int_\Omega a(\cdot,u^n,\nabla u^n)\nabla(u^n-u_0) + a_0(u^n-u_0)\,\mathrm{d}x - < f, u^n-u_0 > \\ &\underbrace{\ge}_{\text{coercivity}} \int_\Omega C_1|\nabla u^n|^r - (C_2(\cdot) + |a||\nabla u_0| + |a_0||u_0|)\,\mathrm{d}x - \|u^n\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*)} - \|u_0\|_{\mathrm{W}^{1,r}(\Omega)}\|f\|_{(\mathrm{W}_0^{1,r}(\Omega)^*)}, \end{aligned}$$

together with the fact

$$\|\nabla u^n\|_{\mathrm{L}_r(\Omega)}^r \ge \left(|\|\nabla(u-u_0)\|_{\mathrm{L}_r(\Omega)} - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}\right)^r \ge \|\nabla(u^n-u_0)\|_{\mathrm{L}_q(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r \ge C\|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)}^r - \|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r,$$

Next, realize that $\alpha^n \in \mathbb{R}^n \mapsto \|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)}$ is a norm equivalent to $|\alpha^n|$ (Euclidian norm). So that means $\exists K_1(n) > 0 : \forall\alpha \in \mathbb{R}^n : K_1(n)|\alpha^n| \le \|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)}$. For $|\alpha^n| = R, R > 0$ determined later estimate $F(\alpha^n)\cdot\alpha^n \ge c\|u^n-u_0\|_{\mathrm{W}^{1,r}(\Omega)} - \tilde{c}\left(\|\nabla u_0\|_{\mathrm{L}_r(\Omega)}^r + 1 + \|u_0\|_{\mathrm{L}_r(\Omega)}^r + \|f\|_{(\mathrm{W}_0^{1,r}(\Omega))^*}^{r'}\right)$ (which is not a trivial computation). And so $\exists R > 0, \forall\alpha^n \in \mathrm{S}(0,R) \subset \mathbb{R}^n : F(\alpha^n)\cdot\alpha^n > 0$, so from the

[4] It can be chosen such that it is itself dense, not only its span

previous lemma $\exists \alpha^n \in \mathrm{S}(0,R) : F(\alpha^n) = 0$, and we fix these α^n. **Uniform estimates** They follow from the previous manipulation:

$$\|u^n - u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

and

$$\|u^n\|_{\mathrm{W}^{1,r}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

$$\forall j \in \{0,\dots,d\} : \|a_j(\cdot,u^n,\nabla u^n)\|^{r'}_{\mathrm{L}_{r'}(\Omega)} \leq C\Big(1 + \|u_0\|^r_{\mathrm{W}^{1,r}(\Omega)} + \|f\|_{(\mathrm{W}^{1,r}(\Omega))^*}\Big),$$

Limit passage From the separability of the spaces, we can extract sequences (not renamed):

$$u^n \rightharpoonup u \,\mathrm{in}\, \mathrm{W}^{1,r}(\Omega)\,, a_j \rightharpoonup \alpha_j \,\mathrm{in}\, \mathrm{L}_{r'}(\Omega)\,.$$

We pass to the limit in the estimates and are able to write:

$$\forall j \in \mathbb{N} : \int_\Omega \alpha \cdot \nabla w_j + \alpha_0 w_j \,\mathrm{d}x =< f, w_j >,$$

and from density of $\{w_j\}_{j\in\mathbb{N}}$ in $\mathrm{W}^{1,r}(\Omega)$ we have

$$\forall \varphi \in \mathrm{W}^{1,r}_0(\Omega) : \int_\Omega \alpha \cdot \nabla\varphi + \alpha_0\varphi \,\mathrm{d}x =< f.\varphi >\,.$$

Identification of α's We want to show $\alpha_j = a_j(\cdot,u,\nabla u), j \in \{0,\dots,d\}$. For that, we use the *Minty trick*:

$$\begin{aligned}
0 &\leq \int_\Omega (a(\cdot,u^n,\nabla u^n) - a(\cdot,v,V)) \cdot (\nabla u^n - V) + (a_0(\cdot,u^n,\nabla u^n) - a_0(\cdot,v,V)) \cdot (u^n - v)\\
&\leq \int_\Omega a(\cdot,u^n,\nabla u^n) \cdot \nabla u^n + a_0(\cdot,u^n,\nabla u^n) \cdot u^n \,\mathrm{d}x +\\
&- \int_\Omega (a(\cdot,u^n,\nabla u^n)V + a_0(\cdot,u^n,\nabla u^n)v - a(\cdot,v,V) + a_0(\cdot,v,V) \cdot (u^n - v))\,\mathrm{d}x\,.
\end{aligned}$$

Denote

$$I^n = \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot\nabla(u^n - u_0) + a_0(\cdot,u^n,\nabla u^n)\cdot(u^n - u_0))\,\mathrm{d}x + \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot u_0 + a_0(\cdot,u^n,\nabla u^n)u_0\,\mathrm{d}x\,,$$

by using the equation we obtain

$$I^n =< f, u^n - u_0 > + \int_\Omega a(\cdot,u^n,\nabla u^n)\cdot u_0 + a_0(\cdot,u^n,\nabla u^n)u_0\,\mathrm{d}x \to < f, u - u_0 > + \int_\Omega \alpha\nabla u_0 + \alpha_0 u_0\,\mathrm{d}x = \int_\Omega \alpha\nabla u + \alpha_0 u\,\mathrm{d}x\,,$$

as the rest has subtracted. In total, we have

$$0 \leq \int_\Omega (\alpha - a(\cdot,v,V)) \cdot (\nabla u - V) + (\alpha_0 - a_0(\cdot,v,V))(u - v)\,\mathrm{d}x\,.$$

So far, v, V have been arbitrary. If we take

$$V = \nabla u - \lambda\psi, \psi \in \mathrm{L}_r(\Omega)\,, v = u,$$

then $0 \leq \int_\Omega (\alpha - a(\cdot, \nabla u + \lambda\psi))\lambda\psi \,\mathrm{d}x$, so if we take $\lambda > 0$ and pass to the limit $\lambda \to 0_+$ (using Nemytskii theorem) we can write

$$0 \leq \int_\Omega (\alpha - a(\cdot, u, \nabla u))\psi \,\mathrm{d}x\,.$$

Since ψ was arbitrary, we could have taken $\psi \to -\psi$, which in total means

$$0 = \int_\Omega (\alpha - a(\cdot, u, \nabla u))\psi \,\mathrm{d}x$$

Finally, from the previous results, we obtain

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega a(\cdot, u, \nabla u)\nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \,\mathrm{d}x = < f, \varphi >,$$

and we are almost done. It only remains to show the traces are correct, bt since $u^n \rightharpoonup u$ in $\mathrm{W}^{1,r}(\Omega)$ and from the continuity of the traces, we obtain

$$\operatorname{tr} u = \operatorname{tr} u_0.$$

Uniqueness: Let u_1, u_2 be two solutions. Use strict monotonicity, subtract the weak formulation and test with $u_2 - u_1$:

$$\int_\Omega \underbrace{(a(\cdot, u_2, \nabla u_2) - a(\cdot, u_1, \nabla u_1)) \cdot \nabla(u_2 - u_1) + (a_0(\cdot, u_2, \nabla u_2) - a_0(\cdot, u_1, \nabla u_1))(u_2 - u_1)}_{:=T} \,\mathrm{d}x = 0,$$

where $T \geq 0$, so from strict monotonicity we obtain $T = 0$ a.e. in Ω but that means $u_1(x) = u_2(x) \wedge \nabla u_1(x) = \nabla u_2(x)$, a.e. in $\Omega \Rightarrow u_1 = u_2$ in $\mathrm{W}^{1,r}(\Omega)$. $\square$

Example (Nonlinearities vs weak convergence)**.** Let $g_n(x) = \sin(nx)$, then $g \rightharpoonup 0$ in $\mathrm{L}_2((0,4))$ (Riemann-Lebesgue lemma). However,

$$\int_0^4 \sin^2(nx)\varphi \,\mathrm{d}x \geq \int_2^4 \sin^2(nx) \,\mathrm{d}x \to \frac{1}{2} \neq 0, \forall \varphi \in \mathrm{L}_2((0,4))\,,$$

so $\{u_n^2\} = \{\sin^2(nx)\}$ **does not converge weakly to** $0 = 0^2$.

Remark. The method of the presented proof is ***very important.***

Theorem 17. *Let $\Omega \in C^{0,1}$. Let $X = W_0^{1,r}(\Omega)\,, r \in (1,\infty)$ with equivalent norm $|||u||| = \|\nabla u\|_{W_0^{1,r}(\Omega)}$. Then*

$$\forall \in X^* \exists \mathbf{F} \in L_{r'}(\Omega) \text{ s.t. } : \forall \varphi \in W_0^{1,r}(\Omega) : \Phi(\varphi) = \int_\Omega \mathbf{F} \cdot \nabla\varphi \,\mathrm{d}x\,, \|\Phi\|_{X^*} = \|\mathbf{F}\|_{L_{r'}(\Omega)}.$$

Proof. We solve the problem

$$\begin{cases} -\nabla \cdot \left(|\nabla u|^{r-2}\nabla u\right) = \Phi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}. \tag{2}$$

Such $u \in \mathrm{W}_0^{1,r}(\Omega)$ exists and is unique by the above theorem. In this case: $a(x,z,p) = |p|^{r-2}p, a_0() = 0$. Coercivity is clear, monotonicity will be shown in the tutorials[5]. Write the weak formulation of the above problem:

$$\forall \varphi \in \mathrm{W}_0^{1,r}(\Omega) : \int_\Omega |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \Phi(\varphi).$$

Set $\mathbf{F} = |\nabla u|^{r-2}\nabla u$, and test the weak formulation with u itself:

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^r = \Phi(u) \leq \|\Phi\|_{X^*} \|\nabla u\|_{\mathrm{L}_r(\Omega)}.$$

If now $\|\nabla u\|_{\mathrm{L}_r(\Omega)} = 0$, then $\Phi = 0$ and we are finished, if it is nonzero, then

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} \leq \|\Phi\|_{X^*}.$$

Realize now

$$\|\nabla u\|_{\mathrm{L}_r(\Omega)}^{r-1} = \left\| |\nabla u|^{r-1} \right\|_{\mathrm{L}_{\frac{r}{r-1}}(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \Rightarrow \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \leq \|\Phi\|_{X^*}.$$

On the other hand:

$$\|\Phi\|_{X^*} = \sup_{\mathrm{B}_X(0,1)} |\Phi(\varphi)| = \sup_{\mathrm{B}_X(0,1)} |\int_\Omega \mathbf{F} \cdot \nabla\varphi| \, \mathrm{d}x \leq \sup_{\mathrm{B}_X(0,1)} \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)} \|\nabla\varphi\|_{\mathrm{L}_r(\Omega)} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)},$$

so $\|\Phi\|_{X^*} = \|\mathbf{F}\|_{\mathrm{L}_{r'}(\Omega)}$. □

5 Calculus of variations

Our motivation is the following: search for a point of minimum for a mapping

$$I : X \subset \mathrm{W}^{1,r}(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega F(\cdot, u, \nabla u) \, \mathrm{d}x \,,$$

with the basic assumptions $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + \mathrm{W}_0^{1,r}(\Omega), u_0 \in \mathrm{W}^{1,r}(\Omega), F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Caratheodory. Moreover,

$$\exists C_1 > 0, c_2 \in \mathrm{L}_1(\Omega), \text{ a.e. } x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : F(x,z,p) \geq C_1 |p|^r - c_2(x).$$

Remark. From the assumptions it follows $\int_\Omega F(\cdot, u, \nabla u) \, \mathrm{d}x$ is defined $\forall u \in \mathrm{W}^{1,r}(\Omega)$.

Hold on, we are interested in PDE's. Why should we care about calculus of variations...?

Lemma 10. *Let $\Omega \in C^{0,1}, r \in (1,\infty), X = u_0 + W_0^{1,r}(\Omega), u_0 \in W^{1,r}(\Omega), F$ Caratheodory. Moreover, let the following condition hold*

$$\exists C > 0, h \in L_1(\Omega) : \forall \text{ a.a} x \in \Omega, \forall z \in \mathbb{R}, \forall p \in \mathbb{R}^d : |\nabla_p F(x,z,p)| + |\partial_z F(x,z,p)| \leq C(|z|^r + |p|^r) + |h(x)|, F(x,\cdot,\cdot) \in C^1(\mathbb{R}^{d+1}).$$

Let now $u \in u_0 + W_0^{1,r}(\Omega)$ be a local minimizer of I over X, i.e.,

$$\exists \rho > 0 : \forall v \in \mathcal{D}(\Omega), \|v\|_{W^{1,r}(\Omega)} < \rho : \int_\Omega F(\cdot, u, \nabla u) \, \mathrm{d}x \leq \int_\Omega F(\cdot, u+v, \nabla(u+v)) \, \mathrm{d}x \,, F(\cdot, u, \nabla u) \in L_1(\Omega) \,.$$

[5] This was a lie

Then u is the weak solution to the ***Euler-Lagrange equations****:*

$$\begin{cases} -\nabla \cdot (\nabla_p F(\cdot, u, \nabla u) + \partial_z F(\cdot, u, \nabla u)) = 0, & in\, \Omega \\ u = u_0, & on\, \partial\Omega \end{cases},$$

i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi + \partial_z F(\cdot, u, \nabla u)\varphi \,\mathrm{d}x = 0, \operatorname{tr} u = \operatorname{tr} u_0 \; on\, \partial\Omega.$$

Proof. First $\operatorname{tr} u = \operatorname{tr} u_0$ holds, so we are fine. Now fix $\varphi \in \mathcal{D}(\Omega)$ and define

$$\iota : \mathbb{R} \to \mathbb{R}^*, \iota(\tau) = \int_\Omega \underbrace{F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))}_{:= l(\tau, \cdot)} \,\mathrm{d}x\,.$$

Now ι has a local minimum in 0. We show that $\iota'(0)$ exists and is equal to the of Euler-Lagrange equations.

- $l(\tau, \cdot)$ is measurable for τ from some neighbourhood of 0.
- $l(\tau, \cdot)$ is differentiable

Moreover

$$\begin{aligned} \partial_\tau l(\tau, \cdot) &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi = \\ &= \partial_z F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi))\varphi + \nabla_p F(\cdot, u + \tau\varphi, \nabla(u + \tau\varphi)) \cdot \nabla\varphi. \end{aligned}$$

Also

$$i(0) = \int_\Omega F(\cdot, u, \nabla u) \,\mathrm{d}x < \infty$$

and

$$|\partial_\tau l(\tau, \cdot)| \le (C(|u|^r + |\varphi|^r + |\nabla u|^r + |\nabla\varphi|^r) + |h(x)|)(|\varphi| + |\nabla\varphi|) \in \mathrm{L}_1(\Omega)\,.$$

Altogether, $\iota(\tau)$ is finite on $(-1, 1), \iota'(\tau)$ exists and

$$\iota'(0) = \int_\Omega \partial_z F(\cdot, u, \nabla u)\varphi + \nabla_p F(\cdot, u, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x\,.$$

□

Example. Let

$$F(x, z, p) = \frac{1}{r}(1) + |p|^2)^{\frac{r}{2}} - gz - Gp,$$

then

$$-\nabla_p F(x, z, p) = \left(\frac{r}{2}\frac{1}{r}2\left(1 + |p|^2\right)^{\frac{r-2}{2}}\right)p - G = \left(1 + |p|^2\right)^{\frac{r-2}{2}} p - G, \partial_z F(x, z, p) = -g.$$

We have

$$|\left(1 + |p|^{2\,\frac{r-2}{2}}\right)p| \le \left(1 + |p|^2\right)^{\frac{r-2}{2}}\left(1 + |p|^2\right)^{\frac{1}{2}} = \left(1 + |p|^2\right)^{\frac{r-1}{2}} \le C(1 + |p|^r).$$

So the estimates are met (somehow with some fantasy). The Euler-Lagrange equations are

$$\begin{cases} -\nabla \cdot \left(\left(1 + |\nabla u|^2\right)^{\frac{r-2}{2}} \nabla u\right) = -\nabla \cdot G + g, & \text{in}\, \Omega \\ u = u_0, & \text{on}\, \partial\Omega. \end{cases},$$

whereas their weak form:

$$\forall\varphi \in \mathcal{D}(\Omega) : \int_\Omega \left(1+|\nabla u|^2\right)^{\frac{r-2}{2}} \nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega (G\cdot\nabla\varphi + g\varphi)\,\mathrm{d}x\,.$$

Remark. We have $\{u_n\} \subset X$ s.t. $\lim_{n\to\infty} I(u_n) = \inf_X I$. Then use

- compacntess: $u_n \to u$ in some sonse (weak convergence)
- weak lower semicontinuity $I(u) \le \liminf_{n\to\infty} I(u_n)$

Lemma 11. *Let $F : \mathbb{R}^N \to \mathbb{R}, F \in C^1\left(\mathbb{R}^N\right), N \in \mathbb{N}$. Then*

1. *F is (strictly) convex $\Leftrightarrow$ ∇F is (strictly) monotone*
2. *If F is (strictly) convex, then*

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \ne \xi_2 : F(\xi_1) - F(\xi_2) \ge \nabla F(\xi_2)\cdot(\xi_1-\xi_2).$$

Proof. Fix $\xi_1,\xi_2,\xi_1\ne\xi_2$, define $\varphi(t) = F(\xi_2 + t(\xi_1-\xi_2))$. Then $\varphi\in C^1(\mathbb{R})$ and

$$\varphi'(t) = \nabla F(\xi_2 + t(\xi_1-\xi_2))\cdot(\xi_1-\xi_2).$$

So

$$" \Rightarrow " : (\nabla F(\xi_1) - \nabla F(\xi_2))\cdot(\xi_1-\xi_2) = \varphi'(1) - \varphi'(0) \underbrace{\ge}_{\varphi\text{convex or strictly convex}} 0.$$

And $" \Leftarrow "$: Fix $t_1 > t_2$ and compute

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\xi_2 + t_1(\xi_1-\xi_2)) - \nabla F(\xi_2 + t_2(\xi_1-\xi_2)))\cdot(\xi_1-\xi_2)(t_1-t_2),$$

define

$$\eta_1 - \eta_2 = (\xi_1-\xi_2)(t_1-t_2)$$

and we obtain

$$\varphi'(t_1) - \varphi'(t_2) = (\nabla F(\eta_1) - \nabla F(\eta_2))\cdot(\eta_1-\eta_2)$$

and we are in the same situation as before. For 2) we already know F (strictly) convex $\Rightarrow$ φ (strictly) convex

$$\Rightarrow \forall t\in(0,\frac{1}{2}) : \frac{\varphi(1)-\varphi(0)}{1} \ge \frac{\varphi(t)-\varphi(0)}{t} \to \varphi'(0),$$

as $t\to 0_+$. And so

$$F(\xi_1) - F(\xi_2) \ge \nabla F(\xi_2)\cdot(\xi_1-\xi_2).$$

$\square$

Theorem 18. *Let $M,N\in\mathbb{N},\Omega$ open, $F:\Omega\times\mathbb{R}^{N+M}\to\mathbb{R}$ Caratheodory, F convex in $p\in\mathbb{R}^n$, i.e. $\forall$ a.e. $x\in\Omega, \forall z\in\mathbb{R}^M : F(x,z,\cdot)$ is convex and $\exists c_2\in L_1(\Omega), \forall$ a.e. $x\in\Omega, \forall z\in\mathbb{R}^M, \forall p\in\mathbb{R}^N : F(x,z,p)\ge c_2(x)$. Let $u_n\to u$ in $L_1(\Omega), U_n \rightharpoonup U$ in $L_1(\Omega)$. Then*

$$\int_\Omega F(\cdot,u,U)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega F(\cdot,u_n,U_n)\,\mathrm{d}x\,.$$

Proof. The proof will be given only if moreover $\forall\, a.e.\, x \in \Omega, \forall z \in \mathbb{R}^M : F(x,z,\cdot) \in C^1\big(\mathbb{R}^N\big)$. Idea: by the previous lemma:

$$\int_\Omega F(\cdot, u_n, U_n)\,\mathrm{d}x \geq \int_\Omega \left(F(\cdot, u_n, U) + \nabla_p F(\cdot, u_n, U)\cdot (U_n - U)\right)\mathrm{d}x\,,$$

and we have uniform convergence in the first term and second term and weak convergence in $\mathrm{L}_1(\Omega)$ in the last term. If Ω is bounded, we can find $K_k \subset K_{k+1} \subset \Omega\, s.t.\, \lambda(\Omega\ \bigcup_{k\in\mathbb{N}} K_k) = 0$, and moreover $\forall k \in \mathbb{N} : K_k \subset \overline{K_k} \subset \Omega, \overline{K_k}$ are compact $, u_n \rightrightarrows u \,\mathrm{on}\, K_k, \|u\|_{\mathrm{L}_\infty(K_k)} + \|U\|_{\mathrm{L}_\infty(K_k)} \leq k$ up to a subsequence. We can now extract a subsequence $u_n \to u\, a.e.$ and apply the Egorov theorem

$$\forall k \in \mathbb{N}, \exists \tilde{E_k}\; s.t.\; u_n \rightrightarrows u \,\mathrm{on}\, \tilde{E_k} \wedge \lambda\big(\Omega\ \tilde{E_k}\big) < \frac{1}{k}.$$

Now define

$$\hat{E_k} = \bigcup_{j=1}^{k} \tilde{E_j}, E_k = \hat{E_k} \cap \{x \in \Omega, \mathrm{dist}\,(x, \partial\Omega) > \frac{1}{k}\},$$

and E_k satisfy [6]

$$\lambda\left(\Omega\ \bigcup_k E_k\right) = 0.$$

Finally, set

$$F_k = \{x \in \Omega, |u(x)| \leq k \wedge |U(x)| \leq k\}$$

and we also have $\lambda(\Omega\ \bigcup_k F_k) = 0$. FINALLY, set

$$K_k = E_k \cap F_k \Rightarrow \lambda\left(\Omega\ \bigcup_k K_k\right) = 0.$$

□

Remark.
- if $U_n \to U$ strongly $\Rightarrow u_n \to u, U_n \to U\;\; a.e.$ (up to a subsequence) and the claim follows from the Fatou lemma. [7]
- norm is weakly lower semicontinuous:

$$\nabla u_n \rightharpoonup \nabla u \,\mathrm{in}\, \mathrm{L_p}(\Omega) \Rightarrow \int_\Omega |\nabla u|^p\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p\,\mathrm{d}x\,.$$

Lemma 12 (Arzela-Ascoli)**.** *Let X, Y be Banach spaces, $X \subset\subset Y$. Then*

$$C^1([0,T]; X) \subset\subset C([0,T]; Y).$$

Lemma 13 (Ehrling)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3$. Then*

$$\forall \varepsilon > 0 \exists C > 0 : \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + C\|u\|_{V_3}.$$

[6]"This is homework", says doc. Kaplicky

[7]For Fatou, we need nonnegativity of the integrand, but that can be met from the assumptions $F - c_2 \geq 0, F - c_2 \in \mathrm{L}_1(\Omega)$

Proof. By contradicition, assume

$$\exists \varepsilon > 0\, s.t.\, \forall n \in N \exists u_n \in V_1 : \|u_n\|_{V_2} > \varepsilon \|u_n\|_{V_1} + n\|u_n\|_{V_3}.$$

WLOG we can assume $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$: truly, the inequaility is 1-homogenous and holds if $u_n = 0$. In particular we see $\|u_n\|_{V_3} < \frac{1}{n}$, so $u_n \to 0$ in V_3. Moreover, $\{u_n\}$ is bounded in V_1 and since $V_1 \subset\subset V_2$ there exists $\{u_{n_k}\} \subset \{u_n\}$ *s.t.*: $u_{n_k} \to u$ in V_2 strongly. Since $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$, also $\|u\|_{V_2} = 1$. Finally, taking the limit passage yields $0 \geq \|u\|_{V_3}$ and so $u = 0$ in V_3 and also in V_2. But that is a contradiction with the fact $\{u_n\} \subset \mathrm{S}_{V_2}(0,1)$. □

Theorem 19 (Aubin-Lions)**.** *Let V_1, V_2, V_3 be Banach spaces s.t. $V_1 \subset\subset V_2 \subset V_3, p \in [1,\infty)$. Then the space*

$$\mathcal{U} = \{u \in L_p\Big((0,T);V_1\Big), \partial_t u \in L_1\Big((0,T);V_3\Big)\},$$

with the norm

$$|\|u\|| = \|u\|_{L_p((0,T);V_1)} + \|\partial_t u\|_{L_1((0,T);V_3)},$$

satisfies

$$\mathcal{U} \subset\subset L_p\Big((0,T);V_2\Big).$$

Proof. Strategy: I want to fix $M \subset \mathcal{U}$ bounded and show that it is precompact in $\mathrm{L_p}\Big((0,T);V_2\Big)$. That will be done in the following way:

1. Mollify M by convolution
2. use Arzela-Ascoli
3. show compactness in $\mathrm{L_p}\Big((0,T);V_3\Big)$
4. apply Ehrling lemma and show compactness in $\mathrm{L_p}\Big((0,T);V_2\Big)$.

Fix $M \subset \mathcal{U}$ bounded. Then $\exists C^* > 0 : \forall u \in M : |\|u\|| \geq C^*$.
Next, take

$$\varphi : \mathbb{R} \to [0,\infty), \varphi \in C^\infty(\mathbb{R}), \operatorname{supp}\varphi \subset (-1,0), \int_{\mathbb{R}} \varphi \,\mathrm{d}x = 1,$$

a regularization kernel, then $\forall \delta > 0$ define $\varphi_\delta(t) :== \frac{1}{\delta}\varphi\big(\frac{t}{\delta}\big)$.
Now, extend functions from M to $(0,2T)$ in the following way:

$$\forall u \in M : \tilde{u}(t) := \begin{cases} u(t), & t \in (0,T) \\ u(2T-t), & t \in (T,2T) \end{cases}.$$

Now mollify: for $\delta > 0, \delta < T$ fixed define

$$M_\delta = \{\, (\tilde{u} \star \varphi_\delta)\Big|_{(0,T)} |u \in M\}.$$

From the properties of regularization it follows $M_\delta \subset C^1([0,T];V_1) \underbrace{\subset\subset}_{\text{A.A.}} C([0,T];V_2) \subset \mathrm{L_p}\Big((0,T);V_2\Big)$.

Now estimate the distance of M and M_δ in $\mathrm{L_p}\Big((0,T);V_3\Big)$: for

$$u \in M, t \in (0,T) : \tilde{u}(t) - \tilde{u}_\delta(t) = \tilde{u}(t) - \int_{-\delta}^{0} \tilde{u}(t-s)\varphi_\delta)(s)\,\mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t-s))\varphi_\delta(s)\,\mathrm{d}s = \int_{-\delta}^{0} (\tilde{u}(t) - \tilde{u}(t-s))\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\delta}^{s-\delta}$$

and this is equal to

$$(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^{s} \varphi_\delta(\sigma)\,\mathrm{d}\sigma \Big|_{s=-\delta}^{0} - \int_{-\delta}^{0} \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{u}(t) - \tilde{u}(t-s)) \int_{-\delta}^{s} \varphi_\delta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\,,$$

since the first bracket is 0 and by denoting the first term in the second integrand by $\tilde{u'}(t-s)$ this becomes (using Fubini)

$$= -\int_{-\delta}^{0}\int_{\sigma}^{0} \tilde{u'}(t-s)\,\mathrm{d}s\,\varphi_\sigma(\sigma)\,\mathrm{d}\sigma\,,$$

and we see

$$\|\tilde{u}(t) - \tilde{u_\delta}(t)\|_{V_3} \le \int_{-\delta}^{0}\int_{\sigma}^{0} \left\|\tilde{u'}(t-s)\right\|_{V_3} \mathrm{d}s\,\varphi_\sigma(\sigma)\,\mathrm{d}\sigma\,.$$

$\mathrm{L_1}\Big((0,T);V_3\Big)$ estimate:

$$\int_0^T \|u(t) - u_\delta(t)\|_{V_3}\,\mathrm{d}t \le \int_0^T \int_{-\delta}^{0}\int_{\sigma}^{0} \left\|u'(\tilde{t-s})\right\|_{V_3} \mathrm{d}s\,\varphi_\delta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t \le 2\delta\|u'\|_{\mathrm{L_1}((0,T);V_3)} \le 2\delta C^*$$

$\mathrm{L_\infty}\Big((0,T);V_3\Big)$ estimate:

$$\|u - u_\delta\|_{\mathrm{L_\infty}((O,T);V_3)} \le 2\|u'\|_{\mathrm{L_1}((0,T);V_3)} \le 2C^*$$

It remains to show $M_\delta \subset\subset \mathrm{L_p}\Big((0,T);V_2\Big)$:

$$\|u - u_\delta\|_{\mathrm{L_p}((0,T);V_3)} \le \|u - u_\delta\|^{1/p}_{\mathrm{L_1}((0,T);V_3)} \|u - u_\delta\|^{1-1/p}_{\mathrm{L_\infty}((O,T);V_3)} \le 2C^*\delta^{1/p}.$$

Finally, from Ehrling we have

$$\forall \mu > 0 \exists C_\mu > 0 : \forall u \in \mathcal{U} : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le \mu\|u - u_\delta\|_{\mathrm{L_p}((0,T);V_1)} + C_\mu\|u - u_\delta\|_{\mathrm{L_p}((0,T);V_3)}.$$

This means

$$\forall u \in M : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} \le C^* + C\mu 2C^*\delta^{1/p}.$$

Now fix $\varepsilon > 0$ and find

$$\mu > 0 : \mu C^* < \frac{\varepsilon}{2}, \delta > 0, C_\mu 2C^*\delta^{1/p} < \frac{\varepsilon}{2} \Rightarrow \forall u \in M : \|u - u_\delta\|_{\mathrm{L_p}((0,T);V_2)} < \varepsilon.$$

This means $\exists \{w_k\}_{k=1}^N \subset M : \{(w_k)_\delta\}_{k=1}^n$ is ε–net in M in $\mathrm{L_p}\Big((0,T);V_2\Big)$. If we now fix $u \in M$, then

$$\exists K \in \{1,\ldots,N\} : \left\|u_{\delta-(w_K)_\delta}\right\|_{\mathrm{L_p}((0,T);V_2)} < \varepsilon.$$

□

Remark. The pair $(\mathcal{U}, |||\cdot|||)$ is a Banach space.

We will be dealing with the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot a(\cdot, u, \nabla u) + a_0(\cdot, u, \nabla u) = f & \text{in } (0,T) \times \Omega, \\ u = u_0, & \text{on } \{0\} \times \Omega, \\ u = 0, & \text{on } (0,T) \times \partial\Omega \end{cases} .$$

The unknown is the function $u : (0,T) \times \Omega \to \mathbb{R}$, and we are given $\Omega \in C^{0,1}, T > 0, Q_T = (0,T) \times \Omega, f : Q \to \mathbb{R}$ or $f : (0,T) \to X$ a Banach space, $u_0 : \Omega \to \mathbb{R}, a : \Omega \times \mathbb{R}\mathbb{R}^d \to \mathbb{R}^d, a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are Caratheodory (the last 2). Moreover, the functions satisfy the following growth conditions: $\exists r > 1, \exists C > 0 : \ a.e.\, x \in \Omega, \forall (z,p) \in \mathbb{R}^{d+1} : |a_0(x,z,p)| + |a(x,z,p)| \le C(1 + |z|^{r-1} + |p|^{r-1})$ and $\exists C_1, C_2 > 0, q \in (1, \max(2,r))\ a.e.\, x \in \Omega, \forall (z,p) \in \mathbb{R}^{d+1} : a(x,z,p)p + a_0(\dots)z \ge C_1|p|^r - C_2(1 + |z|q)$.

Theorem 20. *Let $\Omega \in C^{0,1}, a, a_0$ satisfy growth conditions and coercivity, let $\{a_i\}_{i=0}^d$ be monotone. Denote $V = W_0^{1,r}(\Omega) \cap L_2(\Omega)$. Then $\forall f \in L_{r'}\Big((0,T); V^*\Big), u_0 \in L_2(\Omega)\ \exists u \in L_r\Big((0,T); V\Big)$ s.t. $\partial_t u \in L_{r'}\Big((0,T); V^*\Big), u \in C([0,T]; L_2(\Omega)), u(0) = u_0$ and moreover*

$$a.e.\, t \in (0,T), \forall \varphi \in V :< \partial_t u, \varphi > + \int_\Omega a(\cdot, u, \nabla u)\nabla\varphi + a_0(\cdot, u, \nabla u)\varphi \,\mathrm{d}x =< f, \varphi > .$$

Finally, the solution is unique.

Proof. The strategy is the following

1. approximate: either using Galerkin or using the Rothe method
2. a-priori estimates
3. convergences
4. limit passage
5. identification of the limits

Rothe method: Fix $h \in \{\frac{T}{n}, n \in \mathbb{N}\}$ and approximate the derivative with

$$\partial_t u(t,x) \approx \frac{1}{h}(u(t,x) - u(t-h,x)).$$

Define $u_0 = u_0, u_{k+1} \in V$ as a solution of

$$\frac{1}{h}(u_{k+1} - u_k) - \nabla \cdot a(\cdot, u_{k+1,}, \nabla u_{k+1}) + a_0(\cdot, u_{k+1}, \nabla u_{k+1}) = f_{k+1} \,\text{in}\, \Omega, u_{k+1} = 0 \,\text{on}\, \partial\Omega.$$

Define

$$f_{k+1} \coloneqq \int_{kh}^{(k+1)h} f \,\mathrm{d}t \,,$$

then the weak formulation becomes

$$\int_\Omega \frac{u_{k+1} - u_k}{h}\varphi + a(\cdot, u_{k+1}, \nabla u_{k+1}) \cdot \nabla\varphi a_0(\cdot, u_{k+1}, \nabla u_{k+1})\varphi \,\mathrm{d}x =< f_{k+1}, \varphi > .$$

We *claim without a proof* that the solutions $\{u_k\}_{k=0}^n \subset V$ exist.

To obtain a-priori estimates, tes the equation with u_{k+1}. This yields:

$$\int_\Omega |u_{k+1}|^2 - u_k u_{k+1}\,\mathrm{d}x = \int_\Omega \frac{1}{2}|u_{k+1}|^2 + \frac{1}{2}(u_{k+1}-u_k) - \frac{|u_k|^2}{2}\,\mathrm{d}x \Rightarrow \sum_{k=0}^{j-1}\int_\Omega |u_{k+1}|^2 - u_{k+1}u_k\,\mathrm{d}x = \frac{1}{2}\|u_j\|^2_{\mathrm{L}_2(\Omega)} - \frac{1}{2}\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1}(u_{k+}$$

so

$$\int_\Omega a(\dots)\nabla\cdot u_{k+1} + a_0(\dots)u_{k+1}\,\mathrm{d}x \geq C_1\int_\Omega |\nabla u_{k+1}|^r\,\mathrm{d}x - C_2\int_\Omega (1+|u_{k+1}|^q)\,\mathrm{d}x\,,$$

$$< f_{k+1}, u_{k+1} > \leq \|f_{k+1}\|_{V^*}\Big(\|u_{k+1}\|_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|_{\mathrm{L}_2(\Omega)}\Big) \leq \varepsilon\Big(\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)} + \|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)}\Big) + C\Big(\|f_{k+1}\|^{r'}_{V^*} + \|f_{k+1}\|^2_{V^*}\Big).$$

So together $\|u_j\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1}\Big[(u_{k+1}-u_k)^2 + h\|u_{k+1}\|^r_{\mathrm{W}_0^{1,r}(\Omega)}\Big] \leq C\Big(\|u_0\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\|u_{k+1}\|^2_{\mathrm{L}_2(\Omega)} + \sum_{k=0}^{j-1} h\Big(\|f\|^{r'}_{V^*}\|f\|^2_{V^*}\Big)\Big)$ □

Let us now define $u^n(t) = u_k$ for $t \in (h(k-1), hk)$, then

$$\|u^n\|^2_{\mathrm{L}_\infty((O,T);\mathrm{L}_2(\Omega))} + \|u^n\|^2_{\mathrm{L}_2((O,T);\mathrm{W}_0^{1,r}(Omega))} < C(\,\text{data}\,).$$

Now set $\tilde{u}^n(t) = u_{k-1} + \frac{t-t_{k-1}}{h}(u_k - u_{k-1})$ for $t \in (t_{k-1}, t_k)$ and

$$k \in \{1,\dots,n\}.$$

It holds

$$\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{h}, t \in (t_{k-1}, t_k).$$

Using these quantities, we rewrite the quation to the form

$$\int_\Omega \partial_t \tilde{u}^n\varphi + a(\cdot, u^n, \nabla u^n)\cdot\nabla\varphi + a_0(\cdot, u^n, \nabla u^n)\varphi\,\mathrm{d}x =< f^n, \varphi >,$$

where $f^n(t) := f_k$ in in

$$(t_{k-1}, t_k), k \in \{1,\dots,\}.$$

We are now ready to use growth and apriori estimates:

$$\|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(Q_T)} \leq C(\,\text{data}\,).$$

For the norm of the time derivative:

$$\sup_{\varphi\in \mathrm{S_V}(0,1)} < \partial_t\tilde{u}^n(t), \varphi >= \sup_{\varphi\in \mathrm{S_V}(0,1)} < f^n, \varphi > - \int_\Omega (a(\cdot, u^n, \nabla u^n)\cdot\nabla f + a_0(\cdot, u^n, \nabla u^n)\varphi)\,\mathrm{d}x\,,$$

at any $t \in (0,T)$. So using Holder:

$$\|\partial_t\tilde{u}^n(t)\|_{V^*} \leq \|f^n\|_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)},$$

and integrating

$$\int_0^T \|\partial_t\tilde{u}^n(t)\|^{r'}_{V^*}\,\mathrm{d}t \leq C\bigg(\int_0^T \|f^n\|^{r'}_{V^*} + \|a(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}(t) + \|a_0(\cdot, u^n, \nabla u^n)\|_{\mathrm{L}_{r'}(\Omega)}, \mathrm{d}t\bigg) \leq TC(\,\text{data}\,)$$

6 Semigroup theory

We consider the equation

$$
\begin{aligned}
u' &= Au, A \text{ is linear} \\
u(0) &= u_0,
\end{aligned}
$$

where $u : [0, \infty) \to \mathbb{R}$. We know that for example if $Au = au, a \in \mathbb{R}$ then

$$u(t) = u_0 e^{at}.$$

If $\mathbf{u} : [0, \infty) \to \mathbb{R}^d, A\mathbf{u} = \mathbb{A}\mathbf{u}, \mathbb{A} \in \mathbb{R}^{d \times d}$, then

$$\mathbf{u}(t) = \exp(t\mathbb{A})\mathbf{u}_0, \exp(t\mathbb{A}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{A}^k t^k.$$

This can be extended to $u : [0, \infty) \to X, X$ a banach space, $A \in \mathcal{L}(X)$, then

$$u(t) = \exp(tA) u_0,$$

where the operator exponential is the same. This works well for unbounded operators, but suppose now

$$X = \mathrm{L}_2(\Omega), Au = \triangle u.$$

We *guess* the solution should be

$$u(t) = \exp(\triangle t) u_0,$$

but what is

$$\exp(\triangle t)?$$

Definition 9 (Linear operator and its domain)**.** Let X be a Banach space over $\mathbb{K}$. Linear operator on X is a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace of X and $A : \mathcal{D}(A) \to X$ is linear.

Definition 10. A family $\{S(T)\}_{t \geq 0} \subset \mathcal{L}(X)$ is called a semigroup if

1. $S(0) = \mathrm{id}$
2. $\forall s, t \geq 0 : S(t)S(s) = S(t+s)$.

If moreover $\forall x \in X : S(t)x \to x$, as $t \to 0_+$, we call $\{S(t)\}$ a c_0- semigroup (strongly continuous).

Remark. $\{s(t)\}_{t \in \mathbb{R}}$ with the two conditions is an Abelian group $(\{S(t)\}_{t \in \mathbb{R}}, \circ)$ with

$$(S(t))^{-1} = S(-t). \tag{3}$$

Remark (X = Banach). **In the following, X is always a Banach space.**

Lemma 14. *Let $\{S(t)\}_{t \geq 0}$ be a c_0-semigroup in X. Then*

1. $\exists M \geq 1, \omega \in \mathbb{R}, \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$,
2. $\forall x \in X, t \mapsto S(t)x \in C([0, \infty); X)$.

Proof. $1 \Rightarrow 2$. Fix $t > 0, x \in X$ compute

$$\lim_{h\to 0_+} \|S(t+h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t)(S(h)x - x)\|_X \le \lim_{h\to 0_+} \|S(t)\|_{\mathcal{L}(X)} \|S(h)x - x\|_X \to 0.$$

now compute $\lim_{h\to 0_+} \|S(t-h)x - S(t)x\|_X = \lim_{h\to 0_+} \|S(t-h)(x - S(h)x)\|_X \le \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X \to 0$. □

Definition 11 (Infinitesimal generator)**.** A linear operator $(A, \mathcal{D}(A))$ is called a infinitesimal generator of the semigroup $\{S(t)\}_{t\ge 0}$, if

$$\forall x \in \mathcal{D}(A) : Ax = \lim_{h\to 0_+} \frac{S(h)x - x}{h},$$

where

$$\mathcal{D}(A) = \left\{ x \in X | \lim_{h\to 0_+} \frac{S(h)x - x}{x} \text{ exists in } X \right\},$$

Theorem 21. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\ge 0}$ in X. Then*

1. $x \in \mathcal{D}(A) \Rightarrow \forall t \ge 0 : S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)x)$,
2. $x \in X \wedge t \ge 0 \Rightarrow x_t = \int_0^t S(s)x \,\mathrm{d}s \in \mathcal{D}(A) \wedge A(x_t) = S(t)x - x$.

Proof. Fix $x \in \mathcal{D}(A), t \ge 0$. Calculate

$$\lim_{h\to 0_+} \frac{S(h)S(t)x - S(t)x}{h} =^{8} \lim_{h\to 0_+} S(t)\frac{S(h)x - x}{h} = S(t)Ax,$$

(convergence is in the norm of the Banach space X). This means $S(t)x \in \mathcal{D}(A) \wedge AS(t)x = S(t)Ax$, moreover, if $t > 0$:

$$\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right),$$

estimate,

$$\left\|\lim_{h\to 0_+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t-h)\left(\frac{x - S(h)x}{-h} - S(h)Ax\right)\right\| = \lim_{h\to 0_+} \left\|\frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0_+} S(t\right.$$

as $S(t)$ is continous and $S(0) = \mathrm{id}$. Clearly, $t \mapsto S(t)x$ is $C^1([0,\infty))$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)x) = S(t)S'(0)x = S(t)Ax.$$

To show the second part, compute

$$\lim_{h\to 0_+} \frac{1}{h}(S(h)x_t - x_t) = \lim_{h\to 0_+} \frac{1}{h}\left(\int_h^{t+h} S(s)x \,\mathrm{d}s - \int_0^t S(s)x \,\mathrm{d}s\right),$$

[8] $S(h)S(t) = S(h+t) = S(t+h) = S(t)S(h)$

realize that

$$S(h)x_t = \int_0^t S(s+h)x\,\mathrm{d}s = \int_h^{t+h} S(s)x\,\mathrm{d}s\,,$$

so the previous computation continues as follows

$$= \lim_{h\to 0_+} \frac{1}{h}\left(\int_t^{t+h} S(s)x\,\mathrm{d}s - \int_0^h S(s)x\,\mathrm{d}s\right) = S(t)x - x \wedge x_t \in \mathcal{D}(A).$$

□

Definition 12 (Closed operator)**.** We say that a linear operator $(A, \mathcal{D}(A))$ is closed if $\forall\{u_n\} \subset \mathcal{D}(A) : u_n \to u \wedge Au_n \to v$, for some $u, v \in X$, then it most hold

$$u \in \mathcal{D}(A) \wedge Au = v.$$

This also means that $\{(x, Ax)|x \in \mathcal{D}(A)\} \subset X \times X$ is closed in $(X \times X, \|\cdot\|_1)$.

Example. Let $\Omega \in C^{1,1}, X = \mathrm{L}_2(\Omega)\,, \mathcal{D}(A) = \mathrm{W}^{2,2}(\Omega) \cap \mathrm{W}^{1,2}_0(\Omega)\,, Au = \triangle\, u$. Then $(A, \mathcal{D}(A)$ is closed. Really, take $\{u_n\} \subset \mathrm{L}_2(\Omega) : u_n \to u$ in $\mathrm{L}_2(\Omega)$ for some $u \in \mathrm{L}_2(\Omega)$. Suppose $Au_n \to v$ in $\mathrm{L}_2(\Omega)\,, v \in \mathrm{L}_2(\Omega)$. Suppose the following equation: find

$$u_n\; s.t. -\triangle\, u_n = Au_n, u_n \,\mathrm{on}\, \partial\Omega.$$

From the regularity theory for elliptic problems, we know that $\|u_n\|_{\mathrm{W}^{2,2}(\Omega)} \le C\|Au_n\|_{\mathrm{L}_2(\Omega)} \le C$, so we can extract $u_{n_k} \rightharpoonup u$ in $\mathrm{W}^{2,2}(\Omega)$. Realize moreover

$$\int_\Omega \triangle\, u_n\varphi\,\mathrm{d}x = \int_\Omega u_n \triangle\,\varphi\,\mathrm{d}x\,, \forall\varphi \in \mathcal{D}(\Omega),$$

and the limit of this is

$$\int_\Omega v\varphi\,\mathrm{d}x = \int_\Omega u \triangle\,\varphi\,\mathrm{d}x = \int_\Omega \triangle\, u\varphi\,\mathrm{d}x\,,$$

which means $\triangle\, u = v$ *a.e.* in Ω and that $u \in \mathcal{D}(A), Au = v$.

Theorem 22. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\ge 0} \subset X$. Then*

1. *$\mathcal{D}(A)$ is dense in X,*
2. *$(A, \mathcal{D}(A)$ is closed.*

Proof. Ad 1.:

$$\frac{1}{t}x_t = \frac{1}{t}\int_0^t S(s)x\,\mathrm{d}s \underbrace{\in \mathcal{D}(A)}_{\text{prev. thm}}, \frac{x_t}{t} \to x \,\mathrm{in}\, X,$$

Ad 2.: Take $\{x_n\} \subset \mathcal{D}(A) : x_n \to x$ in $X, Ax \to v \,\mathrm{in}\, X$. Compute[9]

$\dfrac{(S(h) - \mathrm{id}\,)x_n}{h} = \dfrac{1}{h}\displaystyle\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}(S(s)x_n)\,\mathrm{d}s = \frac{1}{h}\int_0^h AS(s)x_n\,\mathrm{d}s = \frac{1}{h}\int_0^h S(s)\underbrace{Ax_n}_{\to v}$, so taking the limit yields $\dfrac{(S(h) - \mathrm{id})x}{h} =$

Altogether, $x \in \mathcal{D}(A), Ax = v$. □

[9]This "Newton-Leibniz formula" does not hold trivially, but doc. Kaplicky says it does; you have to realize that X is a Banach space and work with some functionals and Bochner integrals or whatever

Lemma 15. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of c_0-semigroups $\{S(t)\}_{t\geq 0}, \{\tilde{S}(t)\}_{t\geq 0}$. Then*

$$\{S(t)\}_{t\geq 0} = \{\tilde{S}(t)\}_{t\geq 0}.$$

Proof. We want to show

$$\forall x \in X, \forall t \geq 0 : S(t)x = \tilde{S}(t)x.$$

Fix $x \in \mathcal{D}(A), t > 0$. Then $g(s) := S(s)\tilde{S}(t-x)x$ satisfies $g \in C^1([0,t]; X), g'(s) = S'(s)\tilde{S}(t-s)x - S(s)\tilde{S}'(t-s)x = AS(s)\tilde{S}(t-s)x - S(s)A\tilde{S}(t-s)x = 0$, as A, S commute. This means $g(0) = g(1)$ and from this it follows $S(t)x = \tilde{S}(t)x, \forall x \in \mathcal{D}(A)$. Since $\overline{\mathcal{D}(A)} = X, S$ continous $\Rightarrow S(t)x = \tilde{S}(t)x \forall x \in X$, and since $t \geq 0$ was arbitrary, we are done. □

Definition 13 (Resolvent of a linear operator)**.** Let $(A, \mathcal{D}(A))$ be a linear (possibly unbounded) operator on X. We define

1. resolvent set

$$\rho(A) = \left\{\lambda \in \mathbb{K} | \lambda \,\mathrm{id} - A \,\text{is invertible and}\, (\lambda \mathrm{id} - A)^{-1} \in \mathcal{L}(X)\right\},$$

2. resolvent operator $R(\lambda, A) : X \to \mathcal{D}(A) : R(\lambda, A) = (\lambda \mathrm{id} - A)^{-1}$, for

$$\lambda \in \rho(A).$$

Remark. If $(A, \mathcal{D}(A)$ is a closed linear operator: $\lambda \in \rho(A) \Leftrightarrow \lambda \mathrm{id} - A$ is a bijection of $\mathcal{D}(A)$ onto X.

Lemma 16. *Let $(A, \mathcal{D}(A))$ be a linear operator on X. It holds*

1. $\forall x \in X, \forall \lambda \in \rho(A) : AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$
2. $\forall x \in \mathcal{D}(A), \forall \lambda \in \rho(A) : R(\lambda, A)Ax = \lambda R(\lambda, A)x - x,$
3. $\forall \lambda, \eta \in \rho(A) : R(\lambda, A) - R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$, *and* $R(\lambda, A)R(\eta, A) = R(\eta, A)R(\lambda, A)$,
4. *If moreover $(A, \mathcal{D}(A)$ is the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ s.t. $\forall t \geq 0$: $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, then*

$$\forall \lambda > \omega : \lambda \in \rho(A) \wedge R(\lambda, A) = \int_0^\infty e^{-\lambda t} S(t)\,\mathrm{d}t \wedge \|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{M}{\lambda - \omega}.$$

Remark. The point 4 says that under some conditions, the resolvent operator is the Laplace transformation of the semigroup operator.

Proof. Ad 1.:

$$AR(\lambda, A)x = (A - \lambda \mathrm{id}) \underbrace{R(\lambda, A)}_{=(\lambda \mathrm{id} - A)^{-1}} x + \lambda R(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Ad 2.: The same as 1.

Ad 3.:

$$R(\lambda, A) - R(\eta, A) = R(\lambda, A)(\mathrm{id} - (\lambda \mathrm{id} - A))R(\eta, A) = R(\lambda, A)(\eta \mathrm{id} - A - \lambda \mathrm{id} + A)R(\eta, A) = (\eta - \lambda)R(\lambda, A)R(\eta, A)$$

For $\lambda \neq \eta$ we also have

$$R(\lambda, A)R(\eta A) = \frac{R(\lambda, A) - R(\eta, A)}{\eta - \lambda} = \frac{R(\eta, A) - R(\lambda, A)}{\lambda - \eta} = R(\eta, A)R(\lambda, A).$$

Ad 4.: WLOG asume $\omega = 0$, meaning $\|S(t)\|_{\mathcal{L}(X)} \leq M \forall t \geq 0$. Denote $\tilde{S}(t) = e^{-\omega t}S(t)$. Define

$$\tilde{R}x = \int_0^\infty e^{-\lambda t} S(t)x \,\mathrm{d}t\,.$$

First of all, this is well defined as

$$\left\|\tilde{R}x\right\|_X \leq \int_0^\infty e^{-\lambda t} M\|x\|_X \,\mathrm{d}T = \frac{M}{\lambda}\|x\|_X,$$

and so $\left\|\tilde{R}\right\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda}, \tilde{R} \in \mathcal{L}(X)$. Next, we want to show

$$\forall x \in X : \tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Leftrightarrow \mathrm{id} = (\lambda\mathrm{id} - A)\tilde{R}.$$

For $x \in X, h > 0$ fixed compute

$$\begin{aligned}
\frac{1}{h}\big(S(h)\tilde{R}x - \tilde{R}x\big) &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t}S(t+h)x - e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \\
&= \frac{1}{h}\left(\int_h^\infty e^{-\lambda(t-h)}S(t)x\,\mathrm{d}t - \int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \\
&= \int_h^\infty \frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h} S(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^h e^{-\lambda t}S(t)x\,\mathrm{d}t = \\
&= e^{-\lambda t}\frac{e^{\lambda h} - 1}{h} \to \lambda e^{-\lambda t}, \text{ as } h \to 0_+
\end{aligned}$$

This implies

$$\chi_{(h,\infty)}(t)e^{-\lambda t}\frac{e^{h\lambda} - 1}{h}S(t)x \to \lambda e^{-\lambda t}S(t)x \,\text{on}\,(0,\infty)\,\text{as}\,h \to 0_+.$$

The norm of this can be estimated as $\left\|\lambda e^{-\lambda t}S(t)x\right\| \leq Ce^{-\lambda t}M\|x\|_X \in \mathrm{L}_1((0,\infty))$. Altogether, we obtain $\tilde{R}x \in \mathcal{D}(A) \wedge A\tilde{R}x = \lambda\tilde{R}x - x \Rightarrow (\lambda\mathrm{id} - A)\tilde{R}x = x$.

To procced further, we need the following theorem:

$$x \in \mathcal{D}(A), A\,\text{closed} : A\tilde{R}x = A\left(\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t\right) = \int_0^\infty e^{-\lambda t}\underbrace{AS(t)}_{=S(t)A}x\,\mathrm{d}t = \tilde{R}Ax,$$

which has been stated but not proved [10]. Finally, we can write: $\forall x \in \mathcal{D}(A) : \tilde{R}(\lambda\mathrm{id} - A)x = x \Rightarrow \lambda \in \rho(A) \wedge \tilde{R} = R(\lambda, A)$. Moreover, we have also shown the mapping is a bijection. ☐

Definition 14 (Contraction semigroup)**.** We say that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup if

$$\forall t \geq: \|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

[10]It could be shown by first constructing a approximating sequence of the Bochner integral, like a Riemann sum, do the calculation on this level and then pass to the limit.

Theorem 23 (Hille-Yosida)**.** *Let $M \geq 1, \omega \in \mathbb{R}$. A linear $(A, \mathcal{D}(A))$ on a Banach space X generates a c_0-semigroup (meaning it is its infinitesimal generator) satysfing $\forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$* ***if and only if***

1. *$(A, \mathcal{D}(A))$ is closed,*
2. *$\mathcal{D}(A)$ is dense in X,*
3. *$\forall \lambda > \omega, n \in \mathbb{N} : \lambda \in \rho(A) \wedge \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda-\omega)^n}$.*

Proof. If $M = 1, \omega = 0$, then $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \Rightarrow \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}$. " $\Rightarrow$ " has been proven, now show the other direction. The plan is to

1. approximate A by $\{A_n\} \subset \mathcal{L}(X)$,
2. construct S_n for A_n as previously,
3. estimate and limit passage.

Approximation: See the analogy: $a \in \mathbb{R} : \frac{n}{n-a} \to 1$, we would like $nR(n, A) \to \operatorname{id}$. Calculate the norm of $nAR(n, A) = n(nR(n, A) - \operatorname{id}) \in \mathcal{L}(X) \forall n \in \mathbb{N}$, (This approx. is called the Yosida approximation.) For $x \in \mathcal{D}(A)$ fixed:

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leq \|R(n, A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \frac{1}{n} \|Ax\|_X \to 0 \,\text{as}\, n \to \infty.$$

If

$$y \in X : \|nR(n, A)y - y\|_X \leq \|nR(n, A)(y - x)\|_X + \|nR(n, A)x - x\|_X + \|x - y\|_X \leq 2\|y - x\| + \underbrace{\|nR(n, a)x - x\|_X}_{\to 0},$$

but $\|y - x\|_X$ can be made arbitrarily small from density of $\mathcal{D}(A)$ in X, so in fact

$$nR(n, A)y \to y \,\text{in}\, X, \forall y \in X.$$

And so $nR(n, A)$ really approximates id.

Using this gives us

$$\forall x \mathcal{D}(A) : A_n x = nAR(n, A)x = n \overbrace{R(n, A)A}^{=AR(n,A)} x \to Ax \,\text{in}\, X$$

pointwisely. Define now

$$S_n(t) = \sum_{k=0}^{\infty} \frac{(A_n t)^k}{k!} \in \mathcal{L}(X) \forall t > 0,$$

which has a norm

$$\|S_n(t)\|_{\mathcal{L}(X)} \leq \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (tA_n)^k \right\|_{\mathcal{L}(X)} = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} \left(-nt\operatorname{id} + n^2 t R(n, A)\right)^k \right\|_{\mathcal{L}(X)}$$

and we claim this is equal to

$$= \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (-nt\mathrm{id})^k \sum_{k=0}^{\infty} \frac{\left(n^2 t R(n,A)\right)^k}{k!} \right\|_{\mathcal{L}(X)},$$

which follows from the Cauchy theorem on products of series. Estimating this gives $\leq e^{-nt}\,\mathrm{id}\,\sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n,A)\|_X^k \leq e^{-nt} e^n t = 1$, as $\|nR(n,A)\|^k \leq 1$. This means $\{S_n(t)\}_{\mathcal{L}(X)} \leq 1$.

Now show that this converges: fix $x \in \mathcal{D}(A)$, compute

$$\|S_n(t)x - S_m(t)x\|_X = \left\| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} (S_n(s)S(m)(t-s)x)\,\mathrm{d}s \right\|_X = \left\| \int_0^t S_n(s)(A_n - A_m)S_m(t-s)x\,\mathrm{d}s \right\|_X \underbrace{\leq}_{\|S_l\|_{\mathcal{L}(X)} \leq 1} t\|(A_n - A_m)$$

and since X is Banach, it is convergent also. Finally, for $y \in X$, we have

$$\|S_n(t)y - S_m(t)y\|_X \leq \|S_n(t)(y-x)\|_X + \|S_n(t)x - S_m(t)x\|_X + \|S_m(x-y)\|_X \leq 2\|x-y\|_X + t\|(A_n - A_m)x\|_X.$$

We claim that $\{S_n(t)y\}$ is Cauchy uniformly on $[0,T], T > 0 \Rightarrow \exists S(t) : S_n(t)y \to S(t)y \forall y \in X, t > 0$. And using Banach-Steinhaus (princip stejnoměrné omezenosti) we obtain $\{S(t)\}_{t \geq 0}$ is a c_0-semigroup.

It remains to answer this question. Is $(A, \mathcal{D}(A))$ the infinitesimal generator of $\{S(t)\}_{t\geq 0}$? Let $(\tilde{A}, \mathcal{D}(\tilde{A})$ be the infinitesimal generator of $\{S(t)\}_{t \geq 0}$. Compute

$$S_n(t)x - x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} S_n(s)x\,\mathrm{d}s = \int_0^t S_n(x)A_n x\,\mathrm{d}s\,,$$

realize that

$$\|S_n(t)A_n x - S(s)Ax\|_X \leq \|S_n(s)(A_n - A)x\|_X + \|S_n(s) - S(s)Ax\|_X \to 0,$$

from the previously shown convergences, and so (we have taken the limit of the LHS also)

$$S(t)x - x = \int_0^t S(s)Ax\,\mathrm{d}s\,.$$

This allows us to compute

$$\forall x \in \mathcal{D}(A) : \lim_{t \to 0_+} \frac{S(t)x - x}{t} = Ax \Rightarrow \mathcal{D}(A) \subset \mathcal{D}(\tilde{A} \wedge A = \tilde{A}\,\mathrm{on}\,\mathcal{D}(A).$$

The opposite inclusion is simple: fix $\lambda > 0 : \lambda \in \rho(A) \cap \rho(\tilde{A})$, and so $\lambda\mathrm{id} - A : \mathcal{D}(A) \to X$ is onto, but also $\lambda\mathrm{id} - A = \lambda\mathrm{id} - \tilde{A}$ on $\mathcal{D}(A)$, and so $\lambda\mathrm{id} - \tilde{A} : \mathcal{D}(A) \to X$ is onto. From the previous theorem, we know $\lambda\mathrm{id} - \tilde{A}$ is one-to-one, so $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Altogether, $A = \tilde{A}, \mathcal{D}(A) = \mathcal{D}(\tilde{A})$.

□

7 (Some) exercises

7.1 4.3.2025

Example (Coefficients for smooth extension)**.** Define

$$Eu(x', x_d) = u(x', x_d), x \geq 0, = \sum_{j=1}^{k+1} u\left(x', -\frac{x_d}{j}\right) c_j, x_d < 0.$$

for $u \in \mathcal{D}\left(\mathbb{R}^d\right)$. Find $\{c_j\}_{j=1}^{k=1}$ in such a way that $Eu \in C^k\left(\mathbb{R}^d\right)$. Moreover, take $d = 1$.

Proof. For $k = 0, j = 1$ we take $c_1 = 1, c_j = 0, j \neq 1$. For $k = 1$, compute the derivative:

$$\partial_{d^n} Eu(x', x_d) = \partial_{d^n} u(x', x_d), x_d \geq 0, = \sum_{j=1}^{k=1} (-1)^n \frac{\partial_{d^n} u\left(x', \frac{x_d}{j}\right)}{j^n} c_j, x_d < 0.$$

If we take $x_d = 0$ in particular:

$$\partial_{d^n} u(x', 0) = \sum_{j=1}^{k+1} \partial_{d^n} u(x', 0) \left(-\frac{1}{j}\right)^n c_j \Leftrightarrow 1 = \sum_{j=1}^{k+1} c_j \left(-\frac{1}{j}\right)^n, \forall n \in \{0, \dots, k\}.$$

That is a linear system of $k + 1$ equations. Is it solvable? □

7.2 8.4.2025

Example (Laplace)**.** Let $a_0 = 0, a(\cdot, z, p) = p$. Then $|a(\dots)| \leq |p|$,growth can be accomplished for $r = 2$, $a(\dots) \cdot p \geq |p|^2$. We can thus apply the theorem to our laplace equation

Example. Let $a_0 = 0, a(\cdot, z, p) = p \operatorname{atan}\left(1 + |p|^2\right)$. Then it is clearly Caratheodory, it is bounded $|a(\dots)| \leq |p| \frac{\pi}{2}$, so the growth conditions yield, it is coercive as $\operatorname{atan}\left(1 + |p|^2\right) \geq \frac{\pi}{4} |p|^2$, and it is monotone

$$\left(\operatorname{atan}\left(1 + |p_1|^2\right) p_1 - \operatorname{atan}\left(1 + |p_2|^2\right) p_2\right)(p_1 - p_2) = \int_0^1 \sum_{j=1}^{d} \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{atan}\left(1 + |p_2 + s(p_1 - p_2)|^2\right)(p_2 + s(p_1 - p_2))\, \mathrm{d}s\, (p_1 - p_2)_j$$