MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 126, APRIL 1974, PAGES 581–583

Rational Chebyshev Approximations for the Modified Bessel Functions $I_0(x)$ and $I_1(x)$

By J. M. Blair

Abstract. This note presents nearly-best rational approximations for the functions $I_0(x)$ and $I_1(x)$, with relative errors ranging down to 10^{-23}.

The most useful set of approximations for $I_n(x)$ for $n = 0, 1$ are the Chebyshev series expansions given in [1] and [2]. The expansions in [1] apply to the functions $I_n(x)/x^n$ for the range $|x| \leqq 8$ and to the functions $(2\pi x)^{1/2}e^{-x}I_n(x)$ for the range $x \geqq 8$. The advantage of these expansions is that they can be truncated to give near-minimax approximations of arbitrary accuracy. However, they suffer from two minor defects, namely a loss of two digits of precision by cancellation for small values of x, and a lack of balance in the amount of computation required in the two ranges. For example, to compute 14S approximations, we need 15 terms of the series for $|x| \leqq 8$, and 17 terms together with the computation of $(2\pi x)^{-1/2}$ and e^x for $x \geqq 8$. Shortening the lower range reduces the amount of cancellation [3], but increases the imbalance. If we use rational function approximations and minimize the relative error, we find that we can reduce the cancellation and still increase the lower range, thereby obtaining a better balance.

Other rational function approximations for $I_0(x)$ and $I_1(x)$ are given in [4]. They are limited to nine or ten digits of accuracy and, since they minimize the absolute error, are less efficient than those presented here. A number of rational approximations are also given in [5], but they only apply to the range $|x| \leqq 1$.

This note gives nearly-best rational function approximations for the complete range of the argument, with relative errors ranging down to 10^{-23}. The approximation forms and intervals are

$$I_n(x) \simeq x^n R_{lm}(x^2), \qquad |x| \leqq 15.0,$$

$$I_n(x) \simeq x^{-1/2}e^x R_{lm}(1/x), \qquad x \geqq 15.0,$$

for $n = 0, 1$, where $R_{lm}(x)$ are rational functions of degree l in the numerator and m in the denominator. The details of the approximations are given in the tables that appear in the microfiche section of this issue. The format is similar to that used in [6].

Tables I to IV summarize the best approximations in the L_∞ Walsh arrays of the functions, and Tables V to XXV give the coefficients of selected approximations. The precision is defined as

Received January 26, 1973.

AMS (MOS) subject classifications (1970). Primary 65D20; Secondary 33A40, 41A50.

Key words and phrases. Rational Chebyshev approximations, Bessel functions.

AECL-4674

$$-\log_{10} \max_x \left| \frac{f(x) - R_{lm}(x)}{f(x)} \right|$$

where $f(x)$ is the function being approximated and the maximum is taken over the appropriate interval. The "cancellation" is a measure of the maximum number of decimal digits lost by cancellation over the range of the approximation. For a polynomial we divide the sum of the terms by the sum of the moduli of the terms. The negative logarithm of the modulus of this ratio gives the cancellation for one argument x. The maximum cancellation over the range is taken as the cancellation of the polynomial. The cancellation of a rational function is the maximum of the cancellations of the numerator and denominator. With this definition a value of 0.48 for cancellation corresponds to a loss of one binary digit, and a value of 0.85 corresponds to two binary digits.

In the range $|x| \leqq 15.0$, the rational functions are expressed in terms of power polynomials of the form

$$P_l(x) = \sum_{j=0}^{l} p_j x^j.$$

In the range $x \geqq 15.0$, the Chebyshev series form is better conditioned, and we use

$$P_l(1/x) = \tfrac{1}{2}p_0 + p_1T_1(\xi) + p_2T_2(\xi) + \cdots + p_lT_l(\xi) = \sum_{j=0}^{l}{}' p_jT_j(\xi)$$

where $\xi = 30/x - 1$.

In Tables I to IV, we list where possible the most accurate approximations of degree $l + m$ having cancellations not greater than 0.48. For the range $|x| \leqq 15.0$, the lowest degree approximations all have cancellations greater than 0.48, and we have selected those with the smallest cancellations. The approximations in Tables I and II have precisions only slightly smaller than the maxima of the same degree. The greatest difference in precision between the most accurate approximations and those given in Tables III and IV is about 2.0.

All computations were done on a CDC 6600 in 29 decimal arithmetic, using a version of the second algorithm of Remes due to Ralston [7]. The master routines, based on the standard power series and asymptotic series expansions, were verified to be accurate to at least 27S by comparison with the values in [8] and by means of the Wronskian $I_0(x)K_1(x) + I_1(x)K_0(x) = 1/x$. The approximations in Tables V to XXV were verified by comparing them with the master routines for 5000 pseudorandom values of the argument.

Acknowledgment. I wish to thank the referees for bringing reference [8] to my attention.

Atomic Energy of Canada Limited
Chalk River Nuclear Laboratories
Mathematics and Computation Branch
Chalk River, Ontario, Canada

1. Y. L. LUKE, *The Special Functions and Their Approximations.* Vol. 2, Math. in Science and Engineering, vol. 53, Academic Press, New York, 1969. MR **40** #2909.

2. C. W. Clenshaw, *Chebyshev Series for Mathematical Functions*, Nat. Phys. Lab. Math. Tables, vol. 5, Her Majesty's Stationery Office, London, 1962. MR **26** #362.

3. J. Wimp, "Polynomial expansions of Bessel functions and some associated functions," *Math. Comp.*, v. 16, 1962, pp. 446–458. MR **26** #6452.

4. I. Gargantini, "On the application of the process of equalization of maxima to obtain rational approximations to certain modified Bessel functions," *Comm. ACM*, v. 9, 1966, pp. 859–863.

5. A. E. Russon & J. M. Blair, *Rational Function Minimax Approximations for the Bessel Functions* $K_0(x)$ *and* $K_1(x)$, Report AECL-3461, Atomic Energy of Canada Limited, Chalk River, Ontario, 1969.

6. J. F. Hart et al., *Computer Approximations*, Wiley, New York, 1968.

7. J. H. Johnson & J. M. Blair, *REMES 2—A FORTRAN Program to Calculate Rational Minimax Approximations to a Given Function*, Report AECL-4210, Atomic Energy of Canada Limited, Chalk River, Ontario, 1973.

8. B. S. Berger & H. McAllister, "A table of the modified Bessel functions $K_n(x)$ and $I_n(x)$ to at least 60S for $n = 0, 1$ and $x = 1, 2, \cdots, 40$," *Math. Comp.*, v. 24, 1970, p. 488, RMT 34.

TABLE I

$$I_0(x) \approx P_\ell(x^2)/Q_m(x^2)$$

RANGE	PRECISION	CANCELLATION	ℓ	m
[0,15]	1.10	1.09	4	1
	1.83	0.81	5	1
	2.64	.64	6	1
	3.53	.52	7	1
	4.50	.43	8	1
	5.53	.36	9	1
	6.63	.31	10	1
	7.78*	.27	11	1
	9.00*	.23	12	1
	10.76	.43	12	2
	12.11	.38	13	2
	13.51	.34	14	2
	15.32*	.47	14	3
	16.83	.43	15	3
	18.38*	.38	16	3
	20.25	.48	16	4
	21.90	.44	17	4
	22.09*	.70	15	6

*Coefficients given in tables V to IX.

TABLE II

$$I_1(x)/x \approx P_\ell(x^2)/Q_m(x^2)$$

RANGE	PRECISION	CANCELLATION	ℓ	m
[0,15]	1.04	1.03	5	0
	1.70	0.91	6	0
	3.02	.58	6	1
	3.95	.47	7	1
	4.95	.39	8	1
	6.01	.34	9	1
	7.14	.29	10	1
	8.32*	.25	11	1
	10.04	.46	11	2
	11.37	.41	12	2
	12.74	.36	13	2
	14.16*	.32	14	2
	16.00*	.45	14	3
	17.54	.41	15	3
	19.11*	.37	16	3
	21.00	.46	16	4
	22.67*	.42	17	4

*Coefficients given in tables X to XIV.

TABLE III

$$x^{\frac{1}{2}} e^{-x} I_0(x) \approx P_\ell(z)/Q_m(z) \quad , \quad z = 1/x$$

RANGE	PRECISION	CANCELLATION	ℓ	m
$[0,1/15]_z$	2.36	0.00	0	0
	4.47	.00	0	1
	6.39	.02	1	1
	8.11*	.04	1	2
	9.70	.07	2	2
	11.15	.12	2	3
	12.49	.18	3	3
	13.65	.26	3	4
	14.71*	.38	4	4
	15.67*	.47	6	3
	16.59	.46	2	8
	17.49	.00	13	0
	18.05*	.00	15	0
	18.75	.00	16	0
	19.57	.00	17	0
	20.22	.00	19	0
	21.23	.00	20	0
	22.42	.00	23	0
	23.01*	2.82	10	11
	23.32*	0.00	25	0

*Coefficients given in tables XV to XX.

TABLE IV

$$x^{\frac{1}{2}} e^{-x} I_1(x) \approx P_\ell(z)/Q_m(z) \; , \; z = 1/x$$

RANGE	PRECISION	CANCELLATION	ℓ	m
$[0,1/15]_z$	1.89	0.00	0	0
	4.14	.01	1	0
	6.11	.02	1	1
	7.88	.04	2	1
	9.50*	.07	2	2
	10.98	.11	3	2
	12.33	.17	3	3
	13.53	.25	4	3
	14.59*	.36	4	4
	15.59*	.47	6	3
	16.47	.45	8	2
	16.94	.31	10	1
	17.48	.01	13	0
	18.02*	.01	15	0
	18.72	.01	16	0
	19.56	.01	17	0
	20.20	.01	19	0
	21.20	.01	20	0
	22.41	.01	23	0
	23.29*	.01	25	0

*Coefficients given in tables XXI to XXV.

TABLE V

$$I_0(x) \approx \sum_{j=0}^{11} p_j x^{2j} \Big/ \sum_{j=0}^{1} q_j x^{2j} \;, \quad |x| \le 15.0$$

P00	(3)	-.75281 10816 90069 24
P01	(3)	-.18720 28333 27321 123
P02	(2)	-.11512 63361 64299 623
P03	(0)	-.31112 48464 37021 41
P04	(-2)	-.46703 81175 57369 46
P05	(-4)	-.44322 16023 33460 62
P06	(-6)	-.28508 77048 31484 19
P07	(-8)	-.13704 36382 41021 20
P08	(-11)	-.42520 97159 55323 18
P09	(-13)	-.15716 37533 25118 95
P10	(-17)	-.93466 49519 95487
P11	(-19)	-.84368 25781 37484 9
Q00	(3)	-.75281 10941 09394 03
Q01	(1)	.10000 00000 00000

TABLE VI

$$I_0(x) \approx \sum_{j=0}^{12} p_j x^{2j} \Big/ \sum_{j=0}^{1} q_j x^{2j} \;, \quad |x| \le 15.0$$

P00	(3)	-.85520 70519 16347 4905
P01	(3)	-.21280 17582 21297 9710
P02	(2)	-.13112 61353 08378 9856
P03	(0)	-.35555 77639 60492 626
P04	(-2)	-.53658 16795 32952 437
P05	(-4)	-.51209 29908 79367 730
P06	(-6)	-.33519 03260 42599 559
P07	(-8)	-.15779 77232 84435 5146
P08	(-11)	-.57181 87604 17654 521
P09	(-13)	-.14409 54995 98008 316
P10	(-16)	-.39649 02134 85888 546
P11	(-19)	-.26775 69129 65770 68
P12	(-21)	-.14296 64241 68601 19
Q00	(3)	-.85520 70510 54649 748
Q01	(1)	.10000 00000 00000 0

TABLE VII

$$I_0(x) \approx \sum_{j=0}^{14} p_j x^{2j} \Big/ \sum_{j=0}^{3} q_j x^{2j}\ , \quad |x| \leq 15.0$$

P00	(10)	-.14404	82982	27235	62383	88363	6
P01	(9)	-.35664	44822	44025	17892	93529	
P02	(8)	-.21641	55723	61227	43218	18449	
P03	(6)	-.57166	11305	63785	39597	87284	
P04	(4)	-.83079	25418	09429	47015	50684	
P05	(2)	-.75433	73289	48189	35480	4898	
P06	(0)	-.46307	62847	21000	98098	86078	
P07	(-2)	-.20259	10841	43397	79773	56747	
P08	(-5)	-.65485	83700	96785	17773	8182	
P09	(-7)	-.16022	46793	95361	44460	18457	
P10	(-10)	-.30093	11271	12960	75338	6880	
P11	(-13)	-.43512	59712	62668	45955	419	
P12	(-16)	-.47944	02575	48299	85775	741	
P13	(-19)	-.38071	52423	45326	46053	47	
P14	(-22)	-.21058	07228	90567	28120	1	
Q00	(10)	-.14484	82982	27235	55486	29722	
Q01	(7)	.34762	63324	05882	68510	243	
Q02	(4)	-.30764	69126	82801	54629	3	
Q03	(1)	.10000	00000	00000	0000		

TABLE VIII

$$I_0(x) \approx \sum_{j=0}^{16} p_j x^{2j} \Big/ \sum_{j=0}^{3} q_j x^{2j} \;, \quad |x| \leq 15.0$$

P00	(10)	-.27288	44657	27379	51578	78952	3409
P01	(9)	-.67685	49084	67382	48943	40380	223
P02	(8)	-.41302	96432	63047	68292	74339	869
P03	(7)	-.11016	59514	61646	11763	17178	7004
P04	(5)	-.16241	00026	42783	70075	03320	319
P05	(3)	-.15038	41142	33544	44058	93518	061
P06	(0)	-.94744	91499	75326	60441	69670	31
P07	(-2)	-.42873	50374	76200	71055	16581	810
P08	(-4)	-.14478	96113	29836	90095	81404	138
P09	(-7)	-.37511	40237	44978	94525	96428	50
P10	(-10)	-.76014	75596	24348	25650	10948	32
P11	(-12)	-.12199	28315	43841	16256	56770	55
P12	(-15)	-.15587	38720	78529	91014	83867	9
P13	(-18)	-.15795	54421	14788	23152	99226	9
P14	(-21)	-.12478	19710	17500	40580	44059	
P15	(-25)	-.72585	40693	58759	57424	755	
P16	(-28)	-.28840	54480	36473	13855	232	
Q00	(10)	-.27288	44657	27379	51567	46641	315
Q01	(7)	.53562	55851	06629	04759	87259	
Q02	(4)	-.38305	19168	28025	36272	760	
Q03	(1)	.10000	00000	00000	00000	00	

TABLE IX

$$I_0(x) \approx \sum_{j=0}^{15} p_j x^{2j} / \sum_{j=0}^{6} q_j x^{2j} \quad , \quad |x| \leq 15.0$$

P00	(19)	.56035	01719	34324	03368	11660	9544
P01	(19)	.13806	85421	12231	06913	76699	6891
P02	(17)	.82540	25773	81152	14978	52860	4642
P03	(16)	.21248	36383	73507	28373	91307	0301
P04	(14)	.29746	62948	32557	11859	95774	2759
P05	(12)	.25694	43760	73959	38520	83963	4293
P06	(10)	.14804	60962	33746	37341	20116	0871
P07	(7)	.59908	87391	05831	36330	46096	7729
P08	(5)	.17629	79404	44569	56337	54326	7387
P09	(2)	.38597	09109	84984	05313	09950	7569
P10	(-1)	.63663	45454	66140	73749	66853	0955
P11	(-4)	.79290	10113	77581	64597	25735	7979
P12	(-7)	.73740	74581	91379	92953	28487	7443
P13	(-10)	.49544	15190	52039	77182	06307	329
P14	(-13)	.22186	57889	38938	11487	13869	966
P15	(-17)	.52410	73806	85406	29571	21247	6
Q00	(19)	.56035	01719	34324	03368	12109	7615
Q01	(17)	-.20190	00871	34993	92830	44127	3068
Q02	(14)	.33045	55175	19329	29120	12101	989
Q03	(11)	-.31517	07498	60031	86627	77695	63
Q04	(8)	.18511	41700	37901	98376	08237	
Q05	(4)	-.63598	88065	49827	59009	332	
Q06	(1)	.10000	00000	00000	00000	000	

TABLE X

$$I_1(x) \approx x \sum_{j=0}^{11} p_j x^{2j} / \sum_{j=0}^{1} q_j x^{2j} \quad , \quad |x| \leq 15.0$$

P00	(3)	-.40032	98826	67768	67
P01	(2)	-.49541	24244	74871	71
P02	(1)	-.20225	47838	75190	51
P03	(-1)	-.40835	08773	78415	10
P04	(-3)	-.48866	81859	50422	0
P05	(-5)	-.38496	32541	06941	7
P06	(-7)	-.21196	23439	06934	17
P07	(-10)	-.88174	75938	02476	1
P08	(-12)	-.24879	45760	64717	2
P09	(-15)	-.77820	17611	91706	7
P10	(-18)	-.53191	53090	54784	
P11	(-20)	-.34191	18851	68608	
Q00	(3)	-.80065	97691	33453	7
Q01	(1)	.10000	00000	00000	

TABLE XI

$$I_1(x) \approx x \sum_{j=0}^{14} p_j x^{2j} / \sum_{j=0}^{2} q_j x^{2j} \quad , \quad |x| \leq 15.0$$

P00	(6)	.68146	79652	62501	95948	277
P01	(5)	.84070	57728	77836	02831	90
P02	(4)	.34106	97522	84422	32612	997
P03	(2)	.68210	05679	80207	87138	91
P04	(0)	.80614	48788	21295	14762	78
P05	(-2)	.62472	61951	27003	40172	48
P06	(-4)	.33947	28903	08516	60126	24
P07	(-6)	.13545	52288	41095	97662	470
P08	(-9)	.41006	89068	47159	16798	81
P09	(-12)	.96362	88915	18449	63125	5
P10	(-14)	.17846	93614	10091	27254	10
P11	(-17)	.26137	27721	58124	56371	2
P12	(-20)	.30627	92836	56135	18796	
P13	(-23)	.25709	19055	84414	35529	
P14	(-26)	.20717	57672	32792	4824	
Q00	(7)	.13629	35930	52499	46125	803
Q01	(4)	-.22258	36740	00860	39089	0
Q02	(1)	.10000	00000	00000	0000	

TABLE XII

$$I_1(x) \approx x \sum_{j=0}^{14} p_j x^{2j} \Big/ \sum_{j=0}^{3} q_j x^{2j} \;, \quad |x| \le 15.0$$

P00	(9)	-.84559 44405 55196 08585 79195
P01	(9)	-.10376 21475 64256 29176 22829
P02	(7)	-.41636 18032 23204 12542 2064
P03	(5)	-.81866 13312 14633 21882 3416
P04	(3)	-.94511 69098 47806 24626 2937
P05	(1)	-.71038 73105 79512 23513 3392
P06	(-1)	-.37144 90925 43642 44201 2956
P07	(-3)	-.14133 48131 35574 18029 1363
P08	(-6)	-.40370 79233 30095 34750 8726
P09	(-9)	-.88380 41416 11642 41792 493
P10	(-11)	-.15000 77999 54130 34418 3297
P11	(-14)	-.19758 65825 00615 33379 341
P12	(-17)	-.19948 52769 79604 57955 01
P13	(-20)	-.14602 22418 73480 17027 31
P14	(-24)	-.73798 34201 81379 53673
Q00	(10)	-.16911 88881 11039 20051 30879
Q01	(7)	.38743 15010 28548 49252 075
Q02	(4)	-.32500 18299 01020 88467 0
Q03	(1)	.10000 00000 00000 0000

TABLE XIII

$$I_1(x) \approx x \sum_{j=0}^{16} p_j x^{2j} / \sum_{j=0}^{3} q_j x^{2j} \quad , \quad |x| \leq 15.0$$

P00	(10)	-.15798	51426	87218	41569	35585	113
P01	(9)	-.19452	41820	68837	17310	52731	023
P02	(7)	-.78607	51032	13881	55732	64375	91
P03	(6)	-.15627	37728	42972	22834	02210	301
P04	(4)	-.18323	55806	36384	11680	27011	370
P05	(2)	-.14061	65890	38505	00787	96376	962
P06	(-1)	-.75542	39530	31591	51868	77938	18
P07	(-3)	-.29763	57895	78760	88217	43211	43
P08	(-6)	-.88923	77011	80942	76730	83500	31
P09	(-8)	-.20638	67005	91213	55034	54497	93
P10	(-11)	-.37848	46313	53910	03474	86888	8
P11	(-14)	-.55425	83398	23134	36305	87016	0
P12	(-17)	-.65064	06956	06576	25880	17162	
P13	(-20)	-.60916	04945	28158	70649	3877	
P14	(-23)	-.44649	94661	65005	39258	0707	
P15	(-26)	-.24193	54960	03820	50802	986	
P16	(-30)	-.88988	57637	22376	75307	8	
Q00	(10)	-.31597	02853	74436	83136	23744	729
Q01	(7)	.59144	92580	37169	13638	17312	
Q02	(4)	-.40279	40238	83927	02625	8113	
Q03	(1)	.10000	00000	00000	00000	00	

TABLE XIV

$$I_1(x) \approx x \sum_{j=0}^{17} p_j x^{2j} / \sum_{j=0}^{4} q_j x^{2j} \ , \quad |x| \leq 15.0$$

P00	(13)	.36803	48925	48553	66447	73065	2953
P01	(12)	.45216	38584	07976	38816	55905	1765
P02	(11)	.18190	38573	46810	61606	33866	0237
P03	(9)	.35915	90413	24533	38283	63649	9974
P04	(7)	.41722	11085	73308	86016	56558	7496
P05	(5)	.31641	22921	12507	70035	07635	2242
P06	(3)	.16755	34294	29529	44644	93553	1931
P07	(0)	.64904	62231	91330	30664	35368	8159
P08	(-2)	.19017	04506	54869	91599	85344	7490
P09	(-5)	.43184	09760	24169	82899	07904	5896
P10	(-8)	.77332	04358	70381	92084	46771	8106
P11	(-10)	.11045	78886	95968	08966	69860	2545
P12	(-13)	.12653	68822	35424	69943	50411	0601
P13	(-16)	.11608	76083	05791	92013	44010	4579
P14	(-20)	.84326	01669	28007	47262	74908	63
P15	(-23)	.47128	23565	32195	81795	48203	3
P16	(-26)	.18926	27772	09016	28179	89651	9
P17	(-30)	.45418	18255	21763	13951	0914	
Q00	(13)	.73606	97850	97107	32895	46286	4045
Q01	(11)	-.15759	51455	54313	84863	29581	1341
Q02	(8)	.13742	81498	33729	38950	55238	284
Q03	(4)	-.57970	35309	58498	00721	13094	1
Q04	(1)	.10000	00000	00000	00000	0000	

TABLE XV

$$I_0(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{1}{}' p_j T_j(30/x-1) / \sum_{j=0}^{2}{}' q_j T_j(30/x-1) \quad , \; x \geq 15.0$$

P00	(5)	.15560 92418 71	
P01	(3)	-.30894 68457	
Q00	(5)	.38842 47200 002	
Q01	(3)	-.85490 53305 7	
Q02	(1)	.10000 0000	

TABLE XVI

$$I_0(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{4}{}' p_j T_j(30/x-1) / \sum_{j=0}^{4}{}' q_j T_j(30/x-1) \quad , \; x \geq 15.0$$

P00	(6)	.69303 96667 14758 1424
P01	(6)	-.14346 46313 13583 4367
P02	(4)	.78403 42490 05088 33
P03	(3)	-.11559 19781 04434 7
P04	(0)	.24392 60769 778
Q00	(7)	.17313 30549 66411 06067
Q01	(6)	-.36184 77792 19653 1384
Q02	(5)	.20312 84361 00794 270
Q03	(3)	-.32519 73333 69824 1
Q04	(1)	.10000 00000 0000

TABLE XVII

$$I_0(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{6}{}' p_j T_j(30/x-1) / \sum_{j=0}^{3}{}' q_j T_j(30/x-1) \ , \ x \geq 15.0$$

P00	(4)	-.14643 44752 21320 50875 1
P01	(3)	.36708 72668 87521 81490
P02	(2)	-.23646 90648 85843 1210
P03	(0)	.35541 13803 76086 64
P04	(-3)	.47168 78247 6977
P05	(-5)	.23102 48135 09
P06	(-7)	.13880 16318
Q00	(4)	-.36588 69668 89217 11808 4
Q01	(3)	.92422 61776 52793 61236
Q02	(2)	-.60938 00222 80924 7298
Q03	(1)	.10000 00000 00000 000

TABLE XVIII

$$I_0(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{15}{}' p_j T_j(30/x-1) \ , \ x \geq 15.0$$

P00	(0)	.80130 85461 96987 16121 06
P01	(-2)	.17290 97766 12134 60475
P02	(-4)	.17434 43037 31276 666
P03	(-6)	.33910 02544 19661 2
P04	(-7)	.10138 66762 44447
P05	(-9)	.42129 87673 402
P06	(-10)	.23058 71786 27
P07	(-11)	.16147 70821 5
P08	(-12)	.14383 54373
P09	(-13)	.16049 3342
P10	(-14)	.22704 188
P11	(-15)	.38393 17
P12	(-16)	.67545 9
P13	(-17)	.91190
P14	(-18)	-.3544
P15	(-18)	-.9272

TABLE XIX

$$I_0(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{10}{}' p_j T_j(30/x-1) \Big/ \sum_{j=0}^{11}{}' q_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(13)	-.46184	46623	62571	55757	21300	6952
P01	(13)	.37417	39968	85384	07787	32923	9892
P02	(13)	-.20962	53184	18692	53117	64107	6660
P03	(12)	.82412	10739	00083	80538	26705	4247
P04	(12)	-.22680	98748	62310	96955	24823	4641
P05	(11)	.42368	33252	33927	93593	49026	900
P06	(10)	-.50595	30572	35637	62841	63443	57
P07	(9)	.34935	00129	73644	99308	00583	9
P08	(8)	-.12122	21929	43243	15345	30242	
P09	(6)	.17220	49528	57118	31639	564	
P10	(3)	-.65006	08746	26265	84765		
Q00	(14)	-.11567	49473	26007	63737	58851	7431
Q01	(13)	.93751	31151	90561	02731	22136	8485
Q02	(13)	-.52565	10102	15634	17239	71303	7783
Q03	(13)	.20693	10832	87715	51798	40719	4447
Q04	(12)	-.57068	39665	10022	84577	38993	6296
Q05	(12)	.10696	41035	92848	01240	04156	5350
Q06	(11)	-.12849	31589	14381	83185	29366	205
Q07	(9)	.89763	51140	23068	15591	34479	6
Q08	(8)	-.31953	48778	97911	54011	67544	
Q09	(6)	.48356	83731	77316	91079	478	
Q10	(4)	-.22312	31202	98989	95868	3	
Q11	(1)	.10000	00000	00000	000		

TABLE XX

$$I_0(x) \approx x^{-\frac{1}{2}}\, e^{x} \sum_{j=0}^{25}{}' \, p_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(0)	.80130 85461 96987 16121 06457 692
P01	(-2)	.17290 97766 12134 60475 12697 6
P02	(-4)	.17434 43037 31276 66567 8151
P03	(-6)	.33910 02544 19661 23538 07
P04	(-7)	.10138 66762 44446 81651 4
P05	(-9)	.42129 87673 40184 4259
P06	(-10)	.23058 71786 26748 963
P07	(-11)	.16147 70821 52564 85
P08	(-12)	.14303 54372 50355 8
P09	(-13)	.16049 33421 52869
P10	(-14)	.22704 18862 8639
P11	(-15)	.38393 18408 111
P12	(-16)	.67545 69299 62
P13	(-17)	.91151 38185 5
P14	(-18)	-.37296 87231
P15	(-18)	-.86194 37387
P16	(-18)	-.34664 00954
P17	(-19)	-.70624 0530
P18	(-20)	.30556 730
P19	(-20)	.76603 496
P20	(-20)	.23745 709
P21	(-22)	.11983 8
P22	(-21)	-.24309 46
P23	(-22)	-.72086 8
P24	(-23)	.69870
P25	(-23)	.96880

TABLE XXI

$$I_1(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{2}{}' p_j T_j(30/x-1) \Big/ \sum_{j=0}^{2}{}' q_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(4)	.41537 46640 7387
P01	(3)	-.16494 52295 082
P02	(1)	.11369 88812 2
Q00	(5)	.10541 35504 26804
Q01	(3)	-.35057 82524 536
Q02	(1)	.10000 00000 0

TABLE XXII

$$I_1(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{4}{}' p_j T_j(30/x-1) \Big/ \sum_{j=0}^{4}{}' q_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(7)	.20555 33847 43048 7588
P01	(6)	-.40892 80849 44275 165
P02	(5)	.22054 97222 60335 71
P03	(3)	-.36202 64202 42263
P04	(1)	.14940 52814 740
Q00	(7)	.52057 75357 82099 0989
Q01	(7)	-.10042 54281 33695 1894
Q02	(5)	.49681 19495 33398 24
Q03	(3)	-.63188 32005 51590
Q04	(1)	.10000 00000 000

TABLE XXIII

$$I_1(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{6}{}' p_j T_j(30/x-1) / \sum_{j=0}^{3}{}' q_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(4)	-.16368 76343 41363 12564 4
P01	(3)	.41157 00072 02944 29692
P02	(2)	-.27944 49234 97539 2067
P03	(0)	.54470 16641 60367 61
P04	(-2)	-.18389 80940 13777
P05	(-5)	-.47142 31599 20
P06	(-7)	-.23063 43584
Q00	(4)	-.41426 77934 86515 60841 5
Q01	(4)	.10177 44903 06698 90059 7
Q02	(2)	-.64505 82590 23547 4191
Q03	(1)	.10000 00000 00000 000

TABLE XXIV

$$I_1(x) \approx x^{-\frac{1}{2}} e^x \sum_{j=0}^{15}{}' p_j T_j(30/x-1) \quad , \quad x \geq 15.0$$

P00	(0)	.78774 68851 57837 07001 06
P01	(-2)	-.50969 19601 75714 57492
P02	(-4)	-.28535 39754 52550 786
P03	(-6)	-.46592 29288 53057 6
P04	(-7)	-.12784 20111 59393
P05	(-9)	-.50457 17922 263
P06	(-10)	-.26675 86209 19
P07	(-11)	-.18213 91325 6
P08	(-12)	-.15817 68617
P09	(-13)	-.17462 5363
P10	(-14)	-.24386 758
P11	(-15)	-.40897 92
P12	(-16)	-.71816 8
P13	(-17)	-.98087
P14	(-18)	.2929
P15	(-18)	.9575

TABLE XXV

$$I_1(x) \approx x^{-\frac{1}{2}}\, e^{x} \sum_{j=0}^{25}{}' \, p_j T_j(30/x-1) \;, \quad x \geq 15.0$$

P00	(0)	.78774	68851	57837	07001	06153	735
P01	(-2)	-.50969	19601	75714	57491	81031	3
P02	(-4)	-.28535	39754	52550	78612	1011	
P03	(-6)	-.46592	29288	53057	59709	62	
P04	(-7)	-.12784	20111	59392	70416	1	
P05	(-9)	-.50457	17922	26263	7985		
P06	(-10)	-.26675	86209	19291	042		
P07	(-11)	-.18213	91325	55912	90		
P08	(-12)	-.15817	68617	27728	7		
P09	(-13)	-.17462	53626	31575			
P10	(-14)	-.24386	75810	0565			
P11	(-15)	-.40897	93211	351			
P12	(-16)	-.71816	70134	94			
P13	(-17)	-.98050	61055	9			
P14	(-18)	.31337	76171				
P15	(-18)	.88229	88936				
P16	(-18)	.35998	27547				
P17	(-19)	.74578	6124				
P18	(-20)	-.26000	276				
P19	(-20)	-.78331	290				
P20	(-20)	-.24765	041				
P21	(-22)	-.29435	5				
P22	(-21)	.24765	00				
P23	(-22)	.75113	4				
P24	(-23)	-.66051					
P25	(-23)	-.99414					