Review of Probability Theory

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Contents

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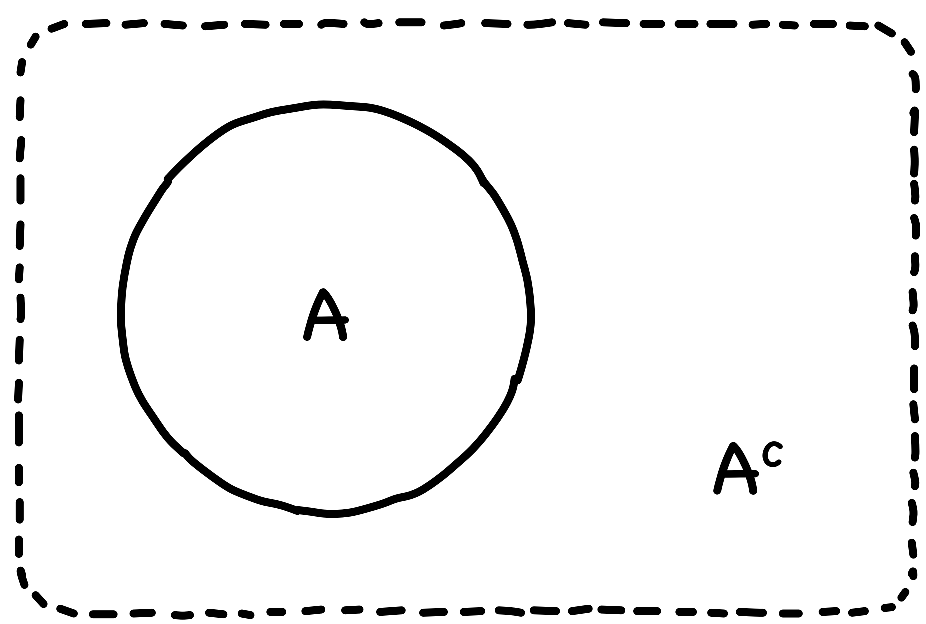
General Probability Rules

- Rules of complements

Notes: If A is any event, then we have

$$P(A) = 1 - P(A^c),$$

where A^c is the **complement** of A, i.e. the event that A does not occur.



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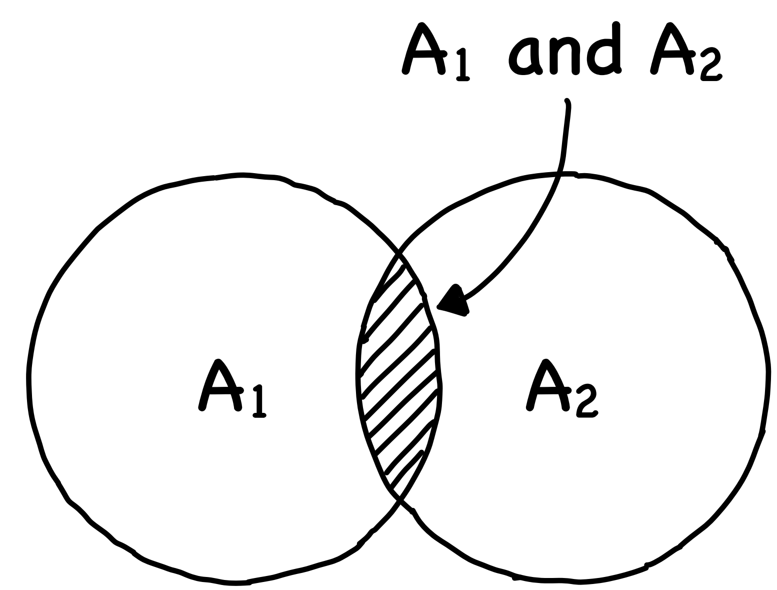
General Probability Rules

- General addition rule

Notes: For events A_1, and A_2,

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) - P(A_1 \text{ and } A_2)$$



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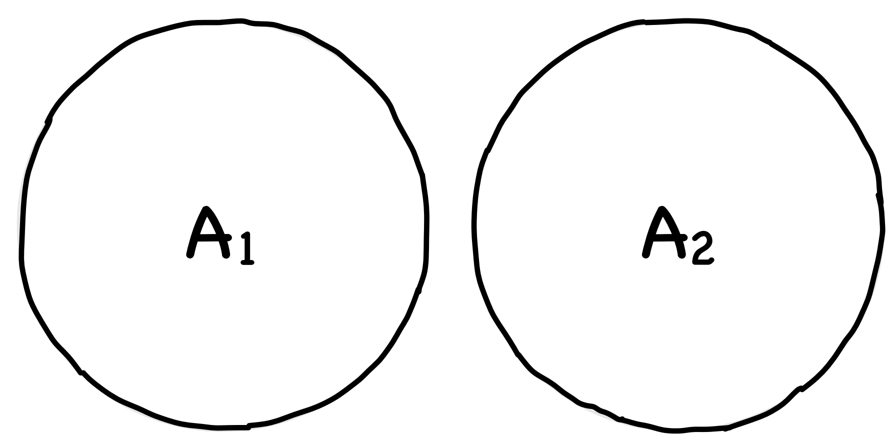
General Probability Rules

- General addition rule

> **Notes:** For events A_1, and A_2,
> $$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) - P(A_1 \text{ and } A_2)$$

- If A_1 and A_2 are mutually exclusive

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$



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General Probability Rules

- General addition rule

> **Notes:** For events A_1, and A_2,
> $$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) - P(A_1 \text{ and } A_2)$$

- If A_1, ..., A_n are n mutually exclusive events

$$P(\text{one of } A_1 \text{ through } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

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General Probability Rules

- Conditional probability and independence

Notes: The **conditional probability** of A given B is expressed as

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.$$

It also implies that

$$P(A \text{ and } B) = P(A \mid B)P(B).$$

Notes: If events A and B are **probabilistically independent**, then

$$P(A \text{ and } B) = P(A)P(B).$$

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Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable

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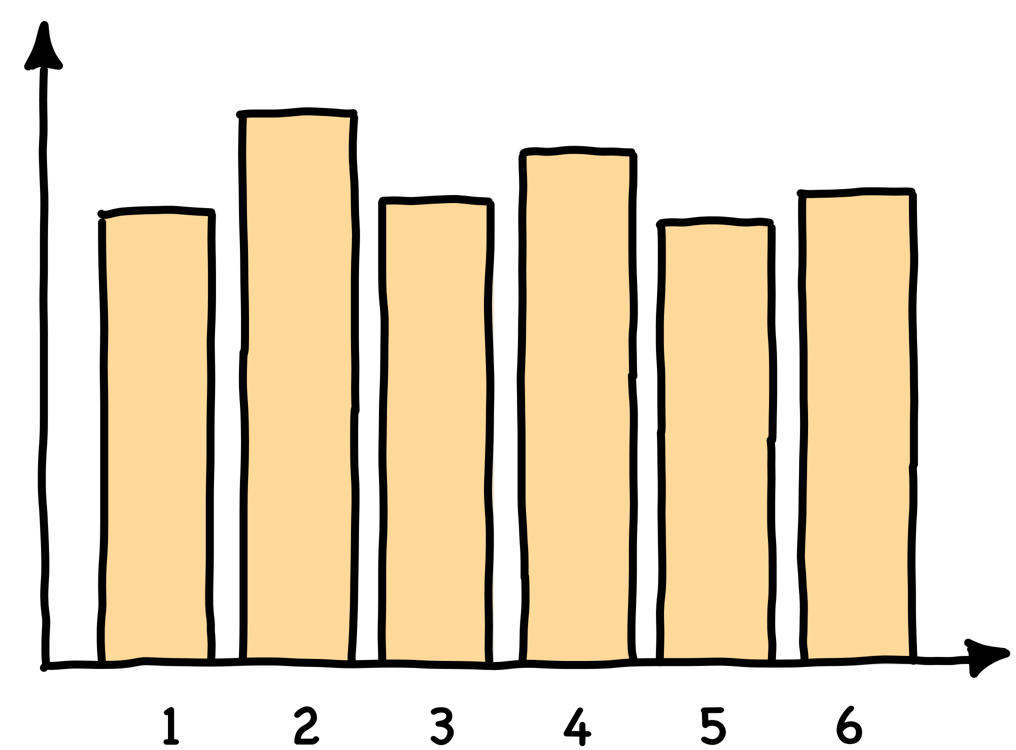
Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable
    - ✓ The result of rolling a die



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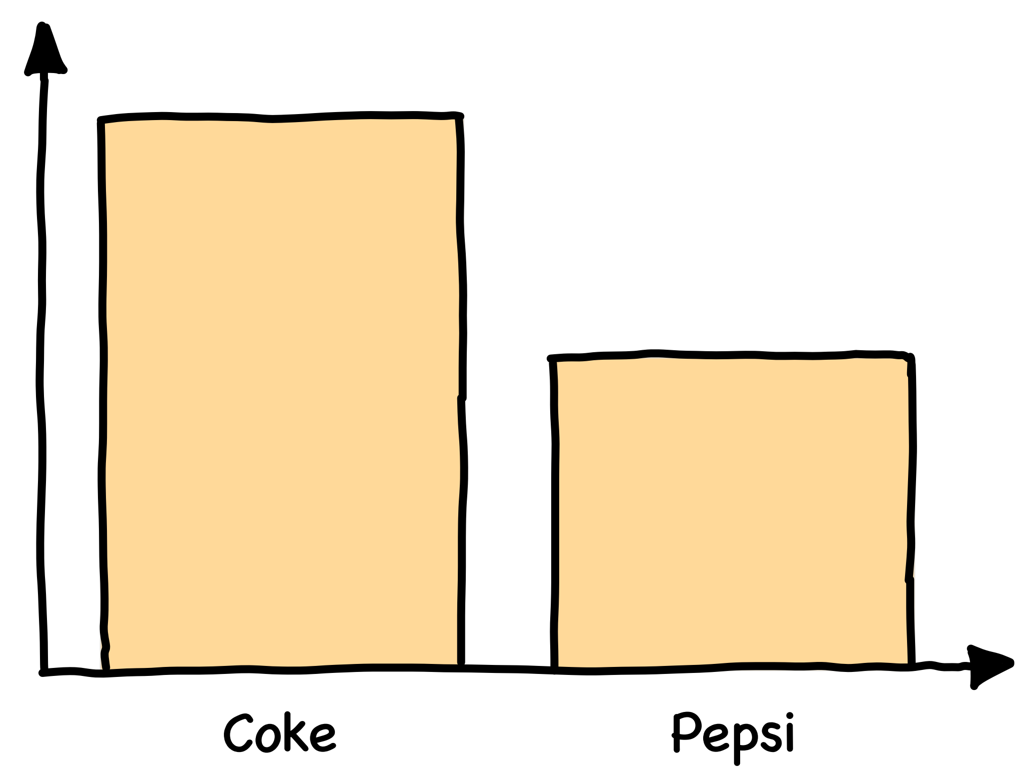
Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable
    - ✓ The preference of a randomly selected customer for Coke or Pepsi



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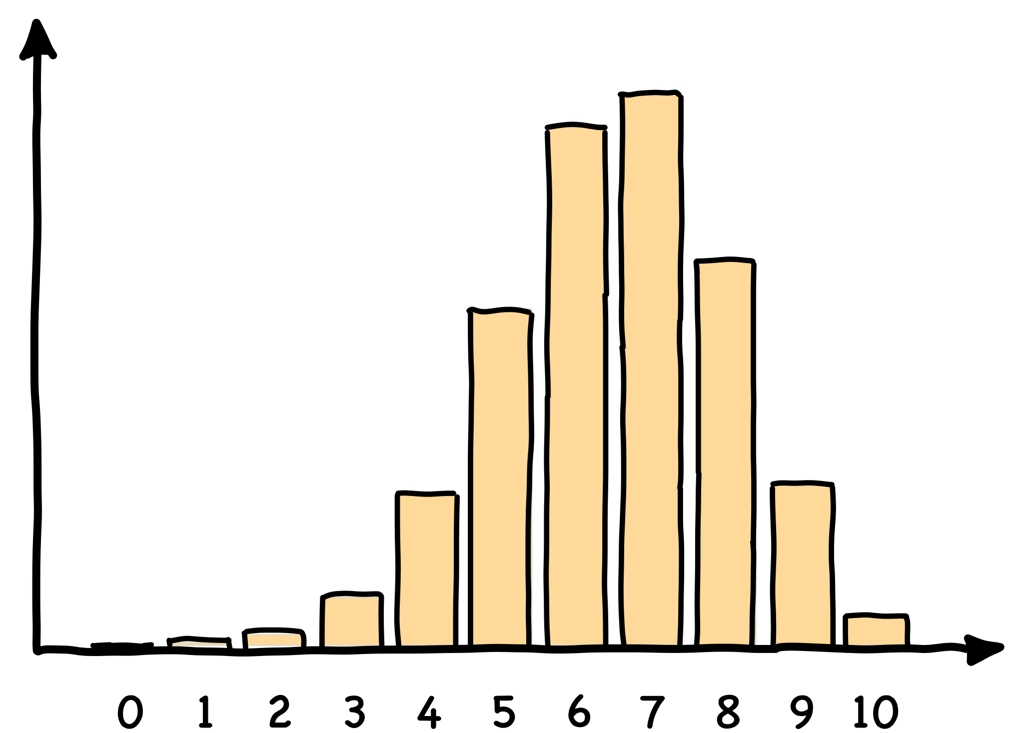
Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable
    - ✓ Among 10 randomly selected customers, the number of people who prefer Coke over Pepsi



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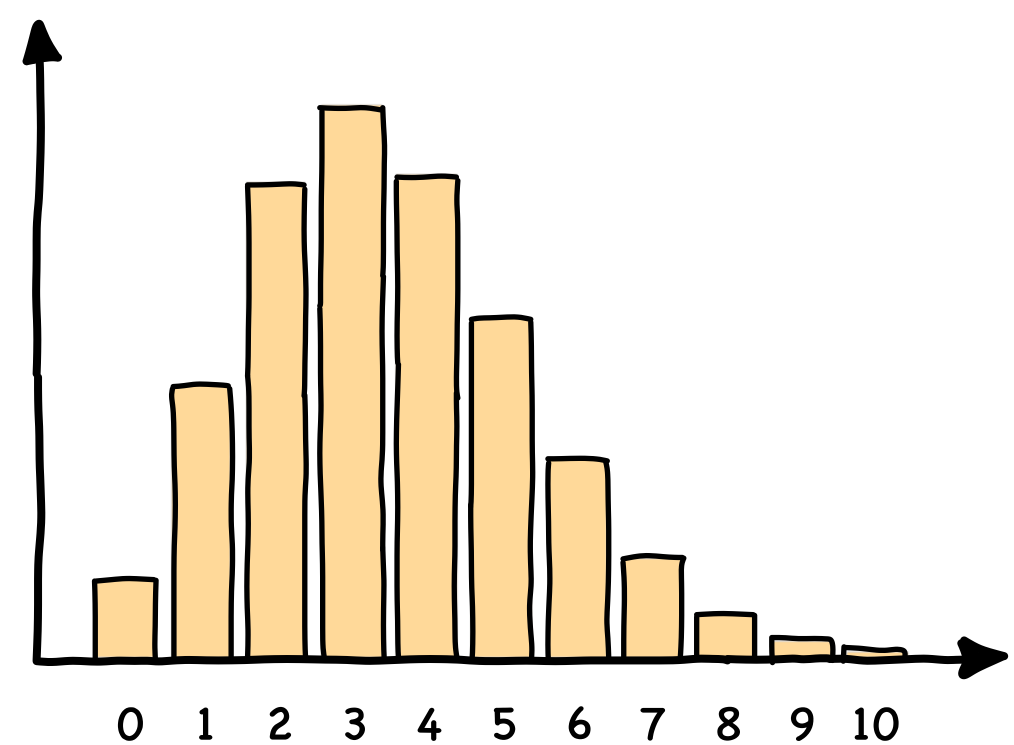
Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable
    - ✓ The number of patients arriving in an emergency room within a fixed time interval



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Random Variables

- Discrete random variables
  - Possible outcomes are finite or countable

Notes: For a discrete random variable X with k possible outcomes x_j, $j = 1,2,...k$, the **probability mass function (PMF)** is given as:

$$P(X = x_j) = p_j \text{ for each } j = 1,2,...,k,$$

where p_j is the probability of the outcome x_j, and all p_j must satisfy

$$\begin{cases} 0 \leq p_j \leq 1 \text{ for each } j = 1,2,...,k \\ \sum_{j=1}^{k} p_j = 1 \end{cases}$$

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Random Variables

- Discrete random variables

Example 1: Suppose that in Singapore, the percentage of customers who prefer Coke is $p = 65\,\%$, and the remaining $1 - p = 35\,\%$ customers prefer Pepsi. Now we randomly survey $n = 10$ customers, and the number of surveyed customers who prefer Coke is denoted by X. What is the probability that $X = 8$?

✓ Totally $n = 10$ independent experiments

✓ Each experiment has two different outcomes: Coke and Pepsi

✓ Each outcome has a fixed probability: $p = 65\,\%$ for Coke and $1 - p = 35\,\%$ for Pepsi

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Random Variables

- Discrete random variables

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```python
x = 8
n = 10
p = 0.65

pmf = binom.pmf(x, n, p)
print(f'The probability is {pmf}')
```

```python
from scipy.stats import binom
```

```
The probability is 0.17565295310595733
```


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Random Variables

- Discrete random variables

Example 1: Suppose that in Singapore, the percentage of customers who prefer Coke is $p = 65\,\%$, and the remaining $1 - p = 35\,\%$ customers prefer Pepsi. Now we randomly survey $n = 10$ customers, and the number of surveyed customers who prefer Coke is denoted by X. What is the probability that $X = 8$?

```python
x = 8
n = 10
p = 0.65

pmf = binom.pmf(x, n, p)
print(f'The probability is {pmf}')
```

PMF of the distribution

```
The probability is 0.17565295310595573
```


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Random Variables

- Discrete random variables

Example 1: Suppose that in Singapore, the percentage of customers who prefer Coke is $p = 65\,\%$, and the remaining $1 - p = 35\,\%$ customers prefer Pepsi. Now we randomly survey $n = 10$ customers, and the number of surveyed customers who prefer Coke is denoted by X. What is the probability that $X = 8$?

```
x = 8
n = 10
p = 0.65

pmf = binom.pmf(x, n, p)
print(f'The probability is {pmf}')
```

Value of the random variable

Shape parameters of the distribution

```
The probability is 0.17565295310595 73
```


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Random Variables

- Discrete random variables

Example 2: Let X be the random variable representing the number of surveyed customers who prefer Coke in **Example 1**, plot the distribution of X.

```
n = 10
p = 0.65
x = np.arange(n+1)

pmfs = binom.pmf(x, n, p)

plt.bar(x, pmfs, color='b', alpha=0.5)
plt.xlabel('X', fontsize=12)
plt.ylabel('Probability', fontsize=12)
plt.show()
```

An array of possible outcomes of the random variable

An array of values of PMFs

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Random Variables

- Discrete random variables

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pmfs = binom.pmf(x, n, p)

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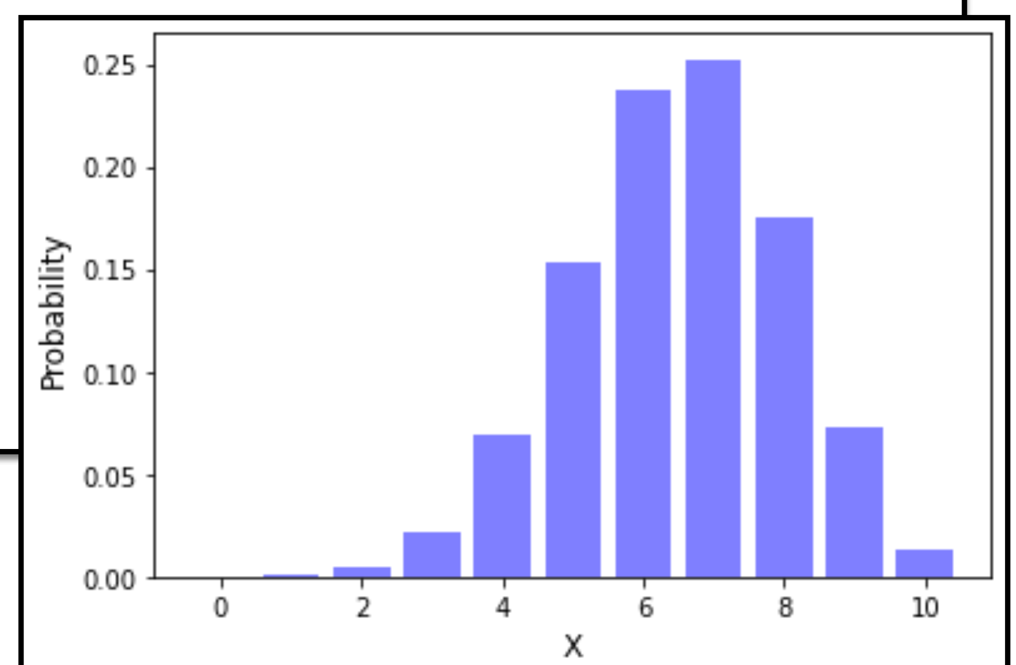

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Random Variables

- Discrete random variables

Notes: The **cumulative distribution function (CDF)** of a random variable X is defined as

$$F(x) = P(X \leq x) \,.$$

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Random Variables

- Discrete random variables

Example 3: Consider the survey in **Example 1**, what is the probability that the majority of the surveyed customers prefer Coke.

$$P(X \geq 6) = 1 - P(X \leq 5) = 1 - F(5)$$

```
x = 5
n = 10
p = 0.65

prob = 1 - binom.cdf(x, n, p)
print(f'The probability is {prob}')
```

```
The probability is 0.7514955091185547
```


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Random Variables

- Discrete random variables

Question 1: Plot the CDF of the random variable X described in **Example 1**.

```
step = 0.01
x = np.arange(0, n+step, step)
n = 10
p = 0.65

cdf = binom.cdf(x, n, p)

plt.plot(x, cdf, linewidth=2.5, color='b', alpha=0.6)
plt.xlabel('X', fontsize=12)
plt.ylabel('Cumulative Proability', fontsize=12)
plt.show()
```


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Random Variables

- Discrete random variables

Question 1: Plot the CDF of the random variable X described in **Example 1**.

```
step = 0.01
x = np.arange(0, n+step, step)
n = 10
p = 0.65

cdf = binom.cdf(x, n, p)

plt.plot(x, cdf, linewidth=2.5, color='
plt.xlabel('X', fontsize=12)
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plt.show()
```

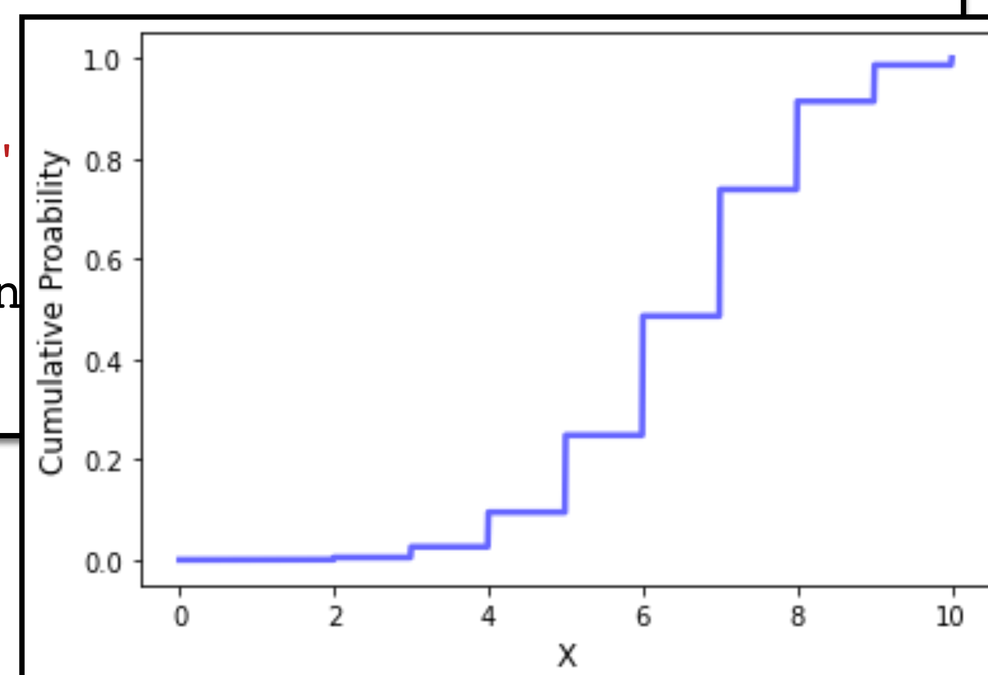

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Random Variables

- Continuous random variables
  - All values in an interval of numbers

Notes: Let $F(x) = P(X \leq x)$ be the CDF of a continuous random variable X, then

- The derivative $f(x) = \dfrac{dF(x)}{dx}$ of the CDF $F(x)$ is called the **probability density function (PDF)** of X. This definition also implies that $F(x) = \int_{-\infty}^{x} f(t)dt$.
- The inverse of CDF $F(x)$, denoted by $F^{-1}(q)$, is called the **Percent Point Function (PPF)**, where q is the given cumulative probability. This function is sometimes referred to as the **inverse cumulative distribution function** or the **quantile function**.

Notes: For a continuous random variable X and given values $x_1 \leq x_2$, then the probability

$$P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1) = F(x_2) - F(x_1).$$

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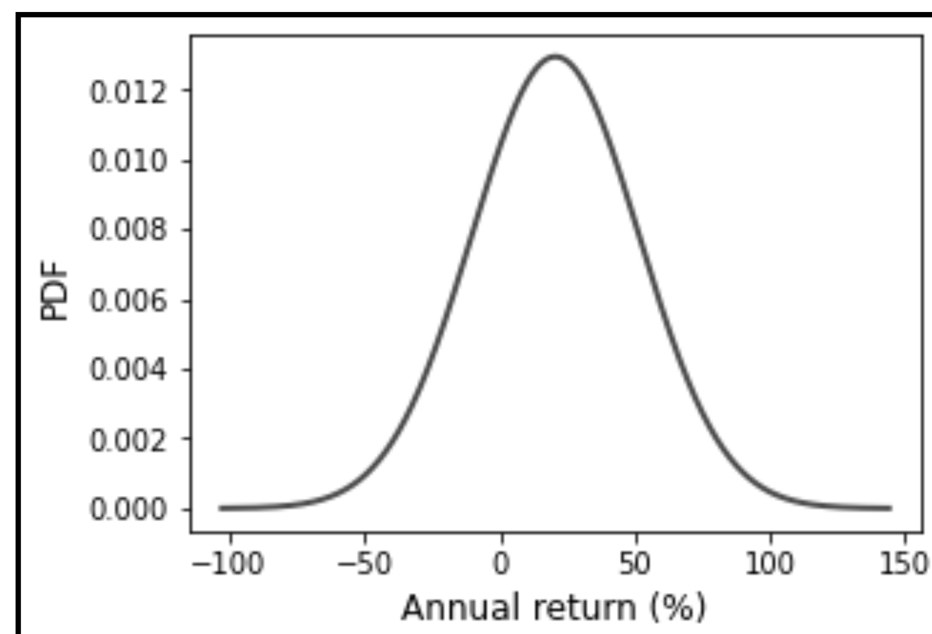
Random Variables

- Continuous random variables

Example 4: The annual returns of an emerging markets exchange-traded fund (ETF) have an expected return of $20.60\,\%$ and a standard deviation of $30.85\,\%$. Assume the returns are normally distributed.

- What is the probability that returns are worse than $-30\,\%$?

$$P(X \leq -30\%) = F(-30\%)$$



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Random Variables

- Continuous random variables

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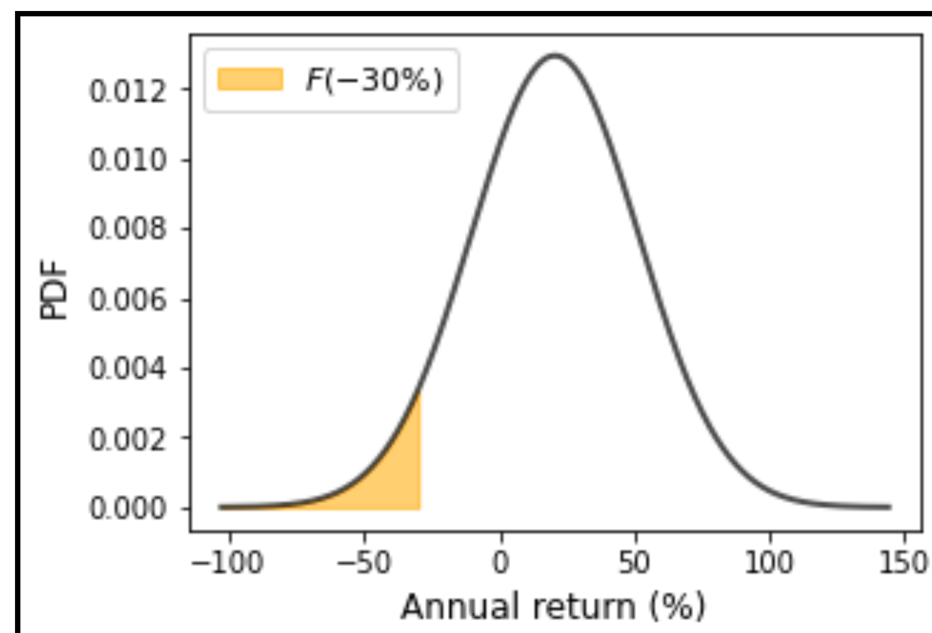

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Random Variables

- Continuous random variables

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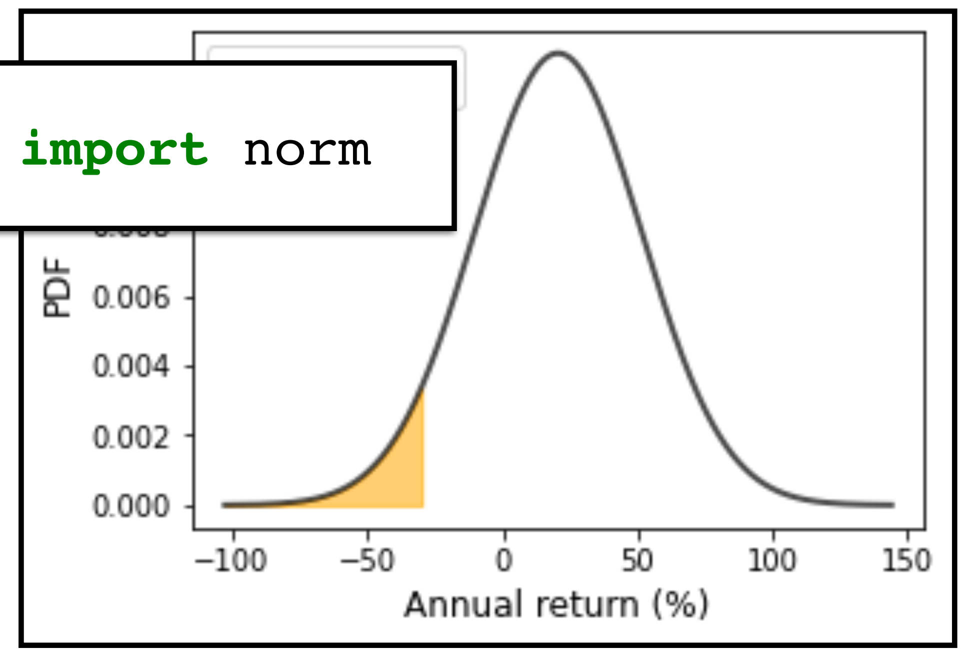
- What is the probability that returns are worse than $-30\,\%$?

```python
mean = 20.60
std = 30.85
x = -30

prob = norm.cdf(x, mean, std)
print(f'The probability is {prob}')
```

```python
from scipy.stats import norm
```

```
The probability is 0.0504823673606 6858
```



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Random Variables

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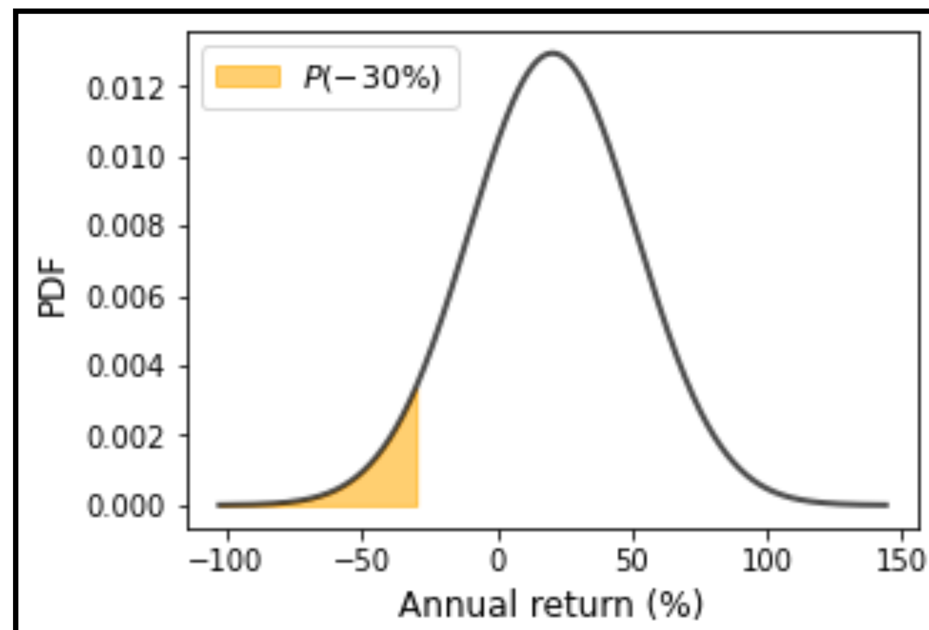
mean = 20.60
std = 30.85
x = -30

prob = norm.cdf(x, mean, std)
print(f'The probability is {prob}')
```

```
The probability is 0.05048236736066858
```



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Random Variables

- Continuous random variables

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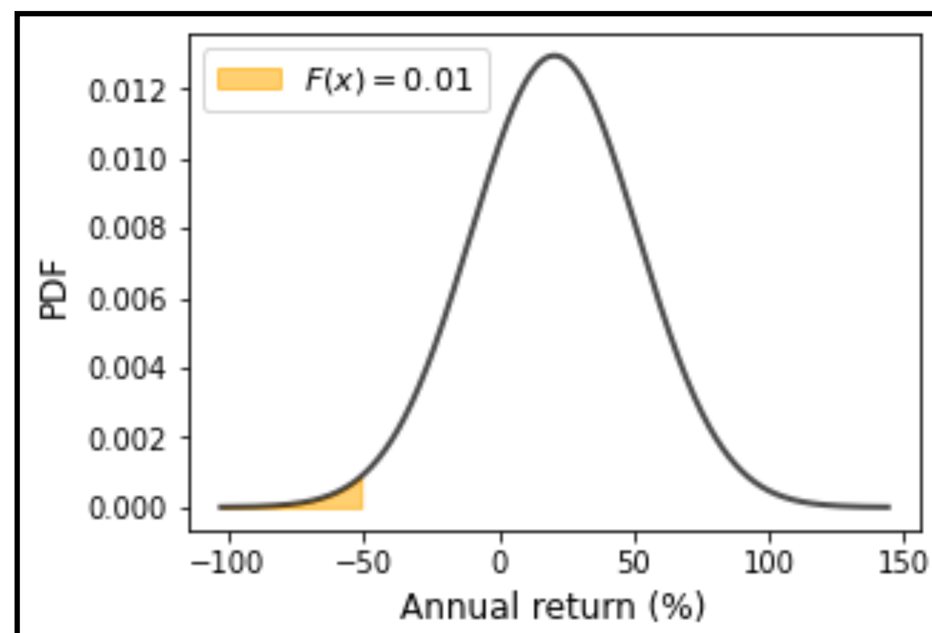
- An investor is interested in finding the value of x, such that the probability of having a return worse than x is no larger than a given probability $\alpha = 0.01$. What is the value of x?

$$P(X \leq x) = F(x) = \alpha$$

$$\Downarrow$$

$$x = F^{-1}(\alpha)$$



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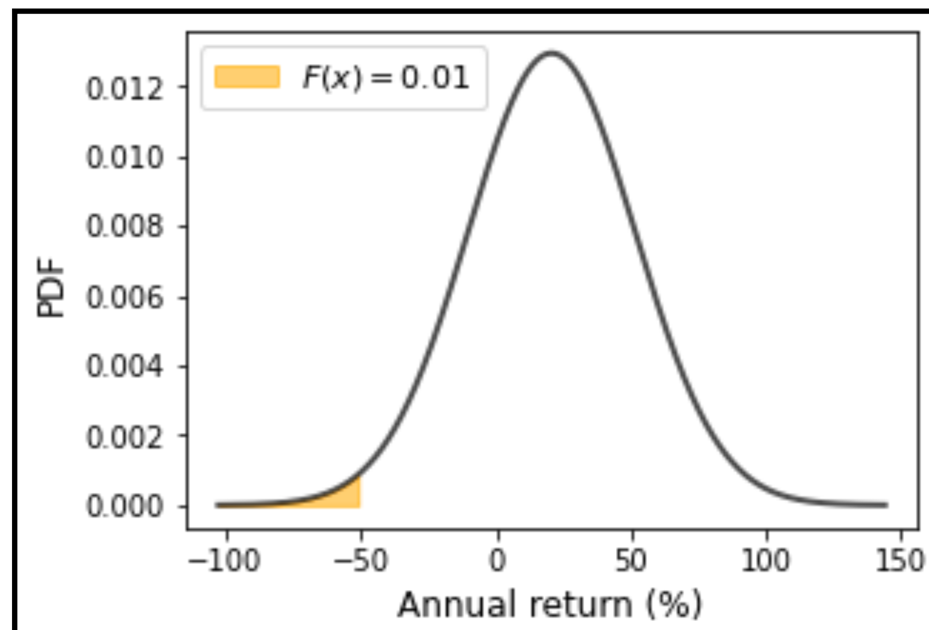
- An investor is interested in finding the value of x, such that the probability of having a return worse than x is no larger than a given probability $\alpha = 0.01$. What is the value of x?

```
mean = 20.60
std = 30.85
alpha = 0.01

x = norm.ppf(alpha, mean, std)
print(f'The value of x is {x}')
```

```
The value of x is -51.16783191415994
```



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The `scipy.stats` Module

- Distribution objects
  - Imported from the `scipy.stats` module

```
from scipy.stats import <object>
```

  - Syntax of conducting calculations

```
<object>.<method>(value, shape_param1, shape_param2, ...)
```

✓ `<method>` specifies the distribution function

✓ `value` is the value(s) of the random variable

✓ `shape_param1`, `shape_param2`, … are the distribution parameters

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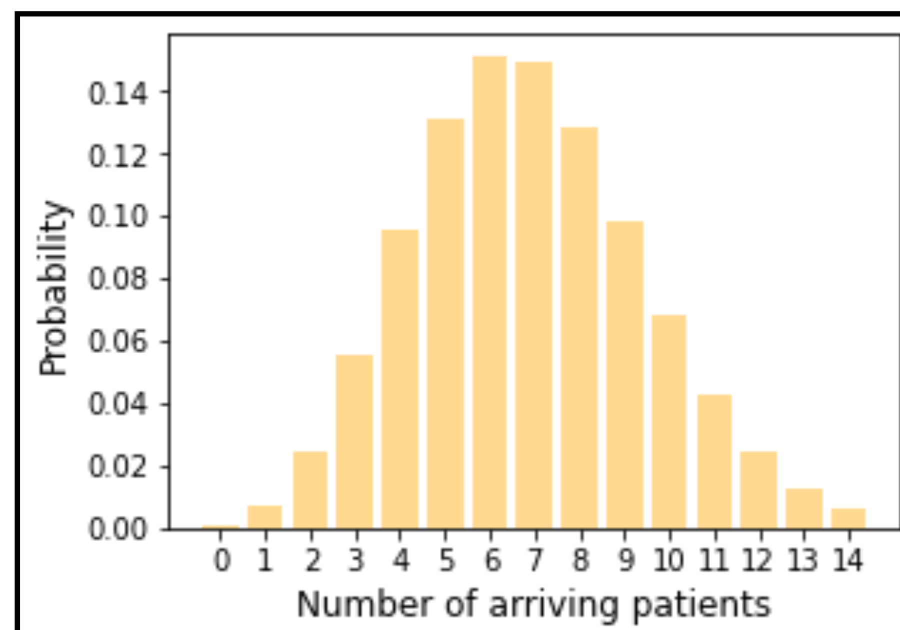
The `scipy.stats` Module

- Distribution objects

Example 5: The records of a hospital show that the number of patients arriving between 6:00 P.M. and 7:00 P.M. has a Poisson distribution with parameter $\mu = 6.9$. Determine the probability that between 6:00 P.M. and 7:00 P.M.,
- The number of patients arriving at the hospital is exactly four.



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The `scipy.stats` Module

- Distribution objects

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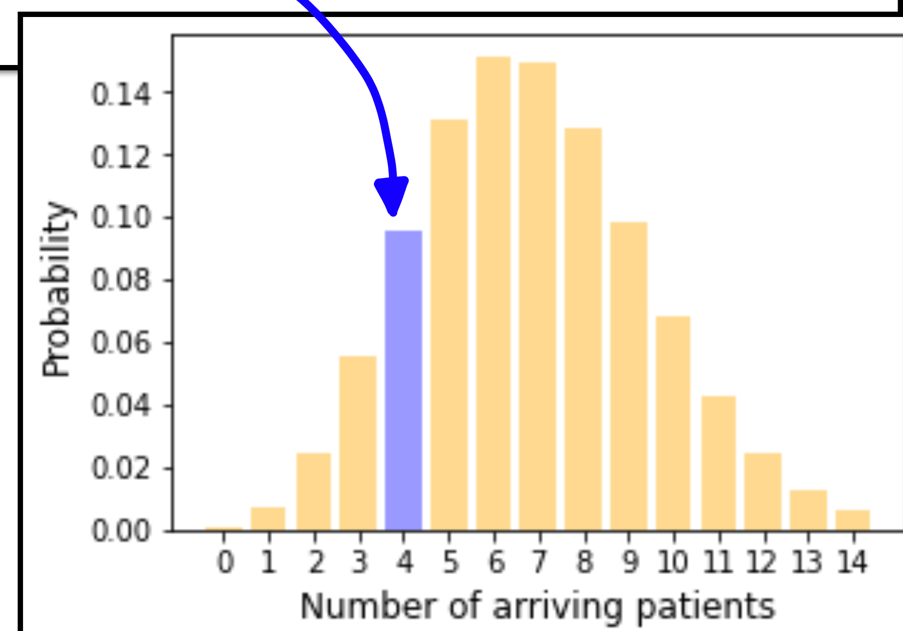
- The number of patients arriving at the hospital is exactly four.

```
mu = 6.9
prob = poisson.pmf(4, mu)
print(f'P(X=4) = {prob}')
```

```
P(X=4) = 0.09518164275949798
```



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The `scipy.stats` Module

- Distribution objects

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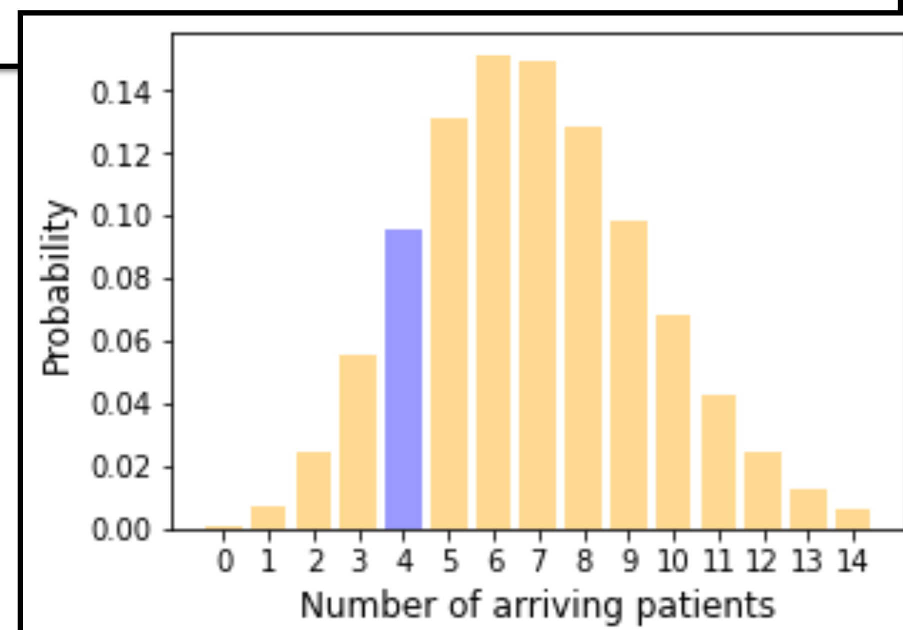
```

```
P(X=4) = 0.09518164275949798
```

Shape parameter of the Poisson distribution



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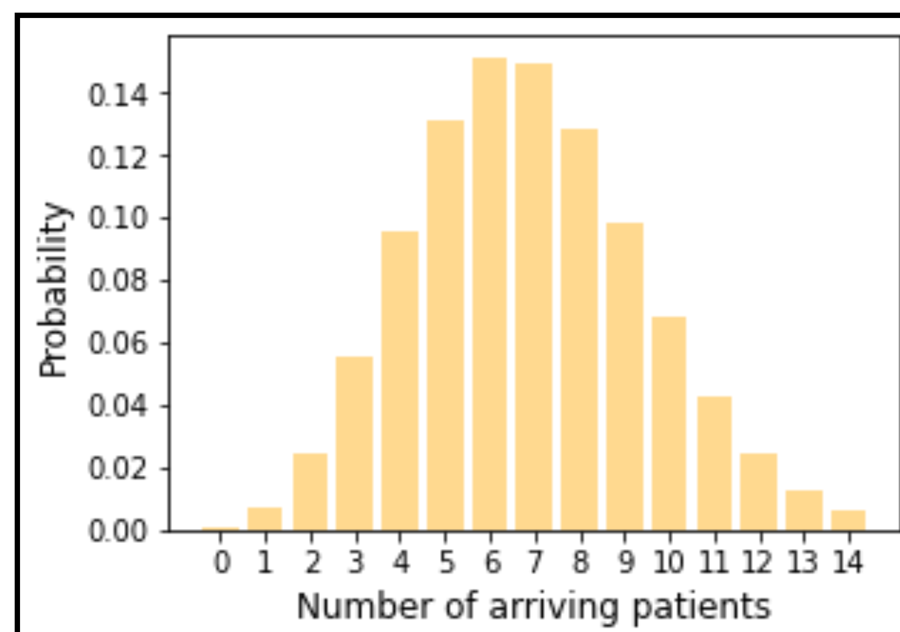
The `scipy.stats` Module

- Distribution objects

Example 5: The records of a hospital show that the number of patients arriving between 6:00 P.M. and 7:00 P.M. has a Poisson distribution with parameter $\mu = 6.9$. Determine the probability that between 6:00 P.M. and 7:00 P.M.,
- The number of patients arriving at the hospital is between four and ten, inclusive.



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The `scipy.stats` Module

- Distribution objects

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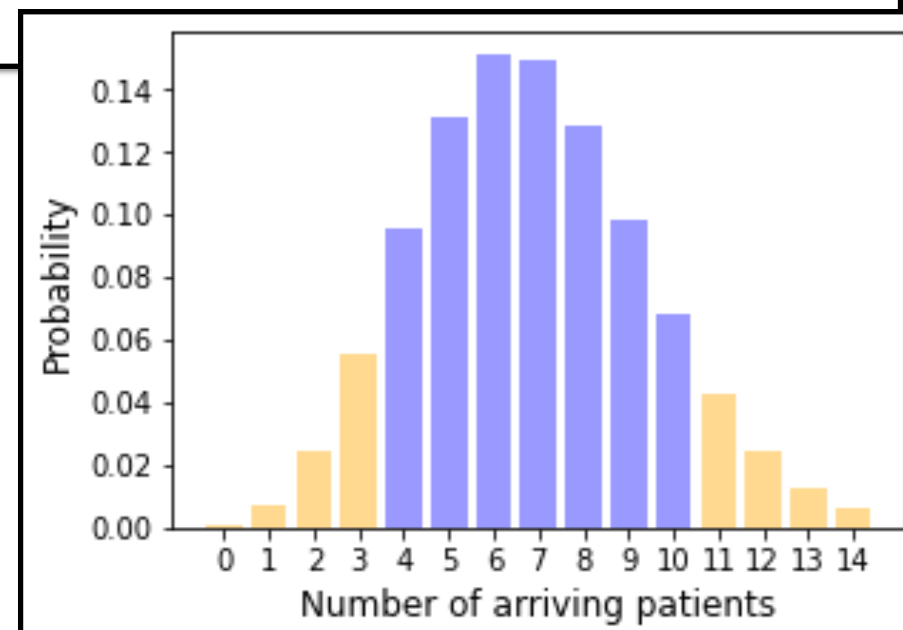
- The number of patients arriving at the hospital is between four and ten, inclusive.

```
mu = 6.9
prob = poisson.cdf(10, mu) - poisson.cdf(3, mu)
print(f'P(4 <= X <= 10) = {prob}')
```

```
P(4 <= X <= 10) = 0.8212956204866553
```



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The `scipy.stats` Module

- Distribution objects

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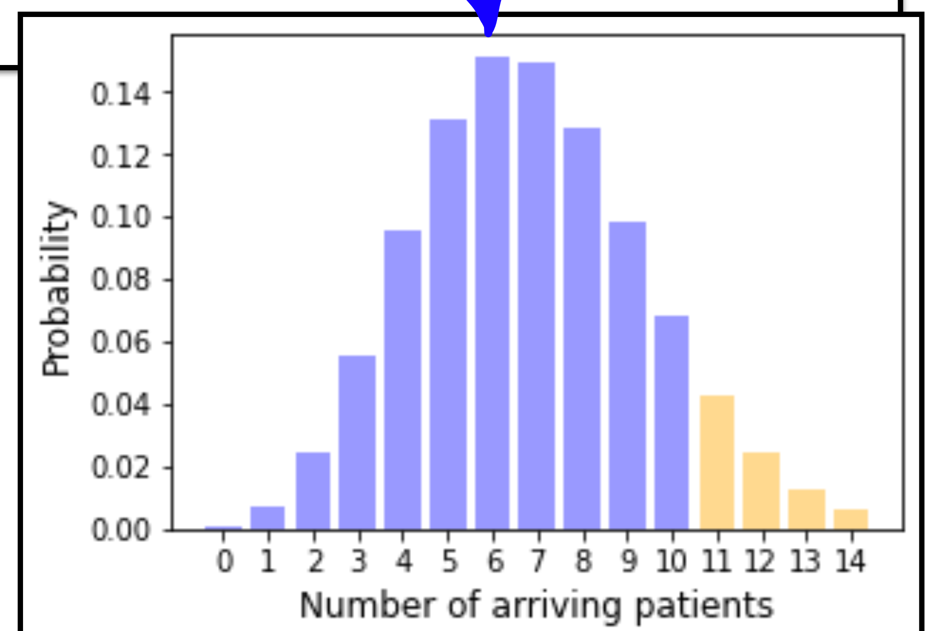

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- Distribution objects

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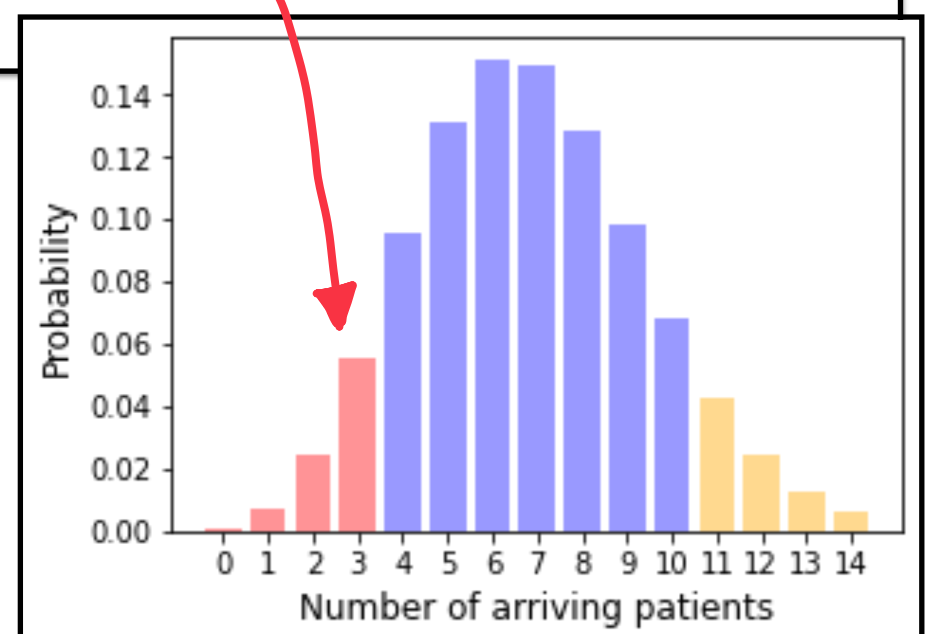

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The `scipy.stats` Module

- Distribution objects

Example 5: The records of a hospital show that the number of patients arriving between 6:00 P.M. and 7:00 P.M. has a Poisson distribution with parameter $\mu = 6.9$. Determine the probability that between 6:00 P.M. and 7:00 P.M.,

- Let X denote the number of patients arriving at the hospital. Plot the distribution of X, as X is between zero and 20, inclusive.

```
mu = 6.9
x = np.arange(21)

pmfs = poisson.pmf(x, mu)
plt.bar(x, pmfs, color='b', alpha=0.5)
plt.xlabel('Number of arriving patients', fontsize=12)
plt.ylabel('Probability', fontsize=12)
plt.show()
```


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The `scipy.stats` Module

- Distribution objects

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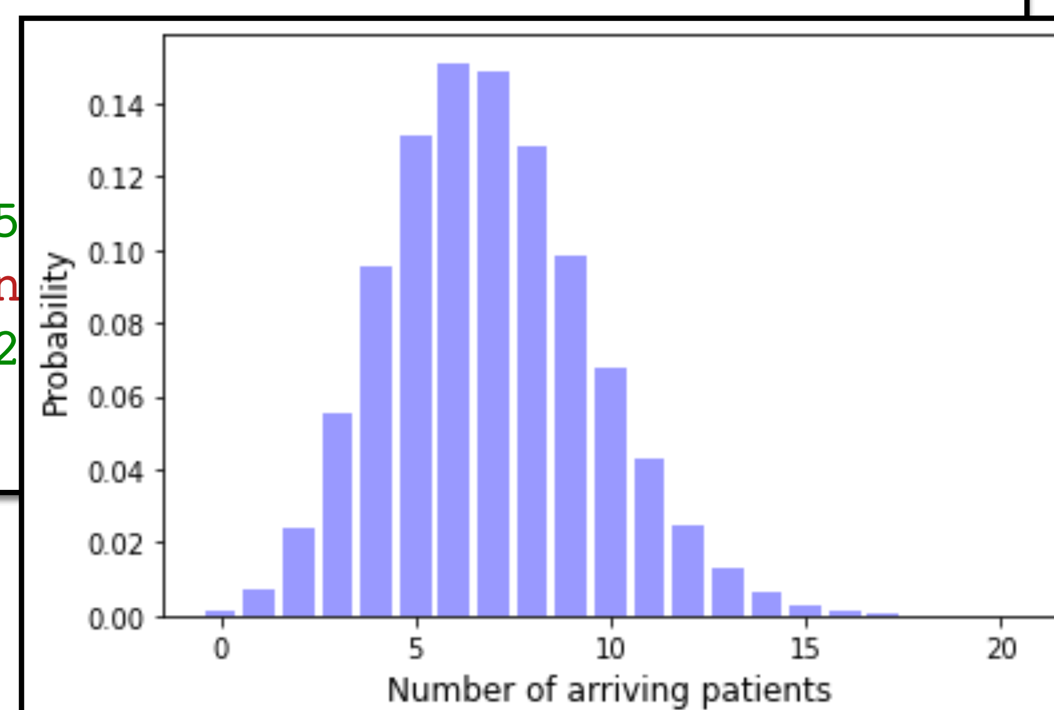

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Expected Values and Variances

- Definition
  - Discrete random variables

$$\begin{cases} \mathbb{E}(X) = \sum_{i=1}^{k} x_i p_i \\ \text{Var}(X) = \sum_{i=1}^{k} (x_i - \mathbb{E}(X))^2 p_i \end{cases}$$

  - Continuous random variables

$$\begin{cases} \mathbb{E}(X) = \int_{x\in\mathcal{X}} x f(x) dx \\ \text{Var}(X) = \int_{x\in\mathcal{X}} (x - \mathbb{E}(X))^2 f(x) dx \end{cases}$$

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Expected Values and Variances

- Expected values and variances of commonly used distributions

Distribution	Parameters	Expected value	Variance	SciPy object	Remarks
Binomial	n as a positive integer $0 < p < 1$ as a probability	np	$np(1-p)$	`binom`	-
Poisson	$\mu > 0$	μ	μ	`poisson`	-
Uniform	a as the lower bound b as the upper bound	$\frac{a+b}{2}$	$\frac{1}{12}(b-a)^2$	`uniform`	$a = 0$ and $b = 1$, by default
Normal	μ as the mean value $\sigma > 0$ as the standard deviation	μ	σ^2	`norm`	$\mu = 0$ and $\sigma = 1$, by default

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Expected Values and Variances

- Expected values and variances of commonly used distributions

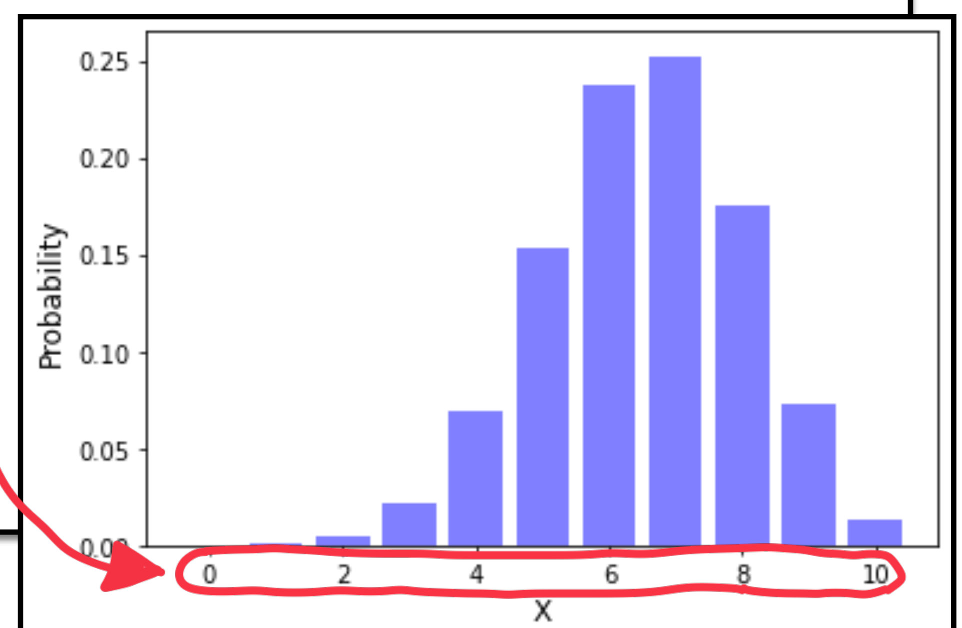
Example 6: Use the definition of expected values and variances to verify that for the binomial distribution described in **Example 1**, the expected value is $np = 6.5$ and the variance is $np(1-p) = 2.275$.

```
n = 10
p = 0.65

x = np.arange(n+1)
pmfs = binom.pmf(x, n, p)

exp = (x * pmfs).sum()
var = ((x - exp)**2 * pmfs).sum()

print(f'The expected value: {exp:.5f}')
print(f'The variance: {var:.5f}')
```




```
The expected value: 6.50000
The variance: 2.27500
```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Expected Values and Variances

- Expected values and variances of commonly used distributions

Example 6: Use the definition of expected values and variances to verify that for the binomial distribution described in **Example 1**, the expected value is $np = 6.5$ and the variance is $np(1 - p) = 2.275$.

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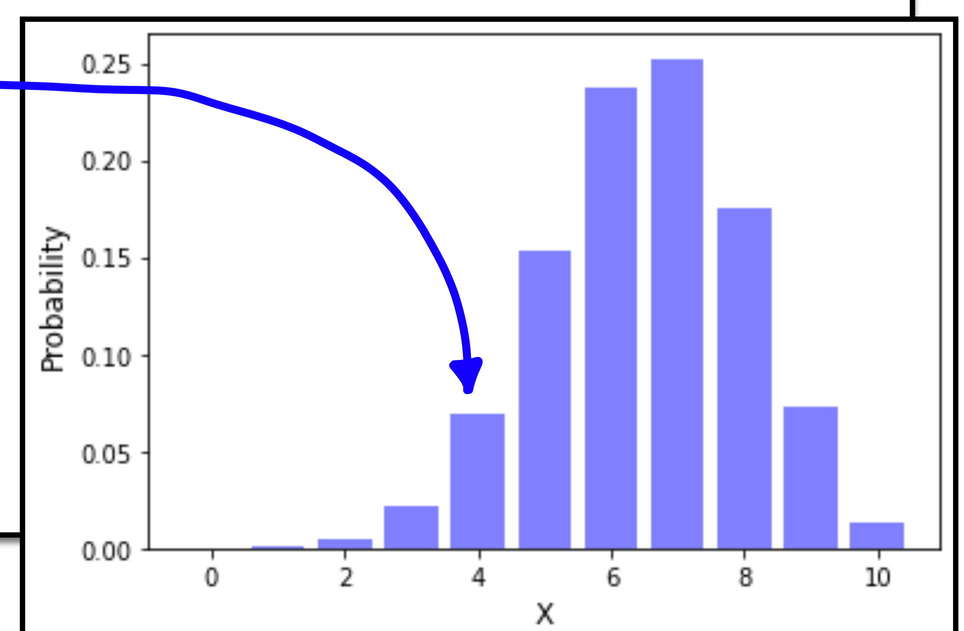
var = ((x - exp)**2 * pmfs).sum()

print(f'The expected value: {exp:.5f}')
print(f'The variance: {var:.5f}')
```

```
The expected value: 6.50000
The variance: 2.27500
```



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Expected Values and Variances

- Expected values and variances of commonly used distributions

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The expected value: 6.50000
The variance: 2.27500
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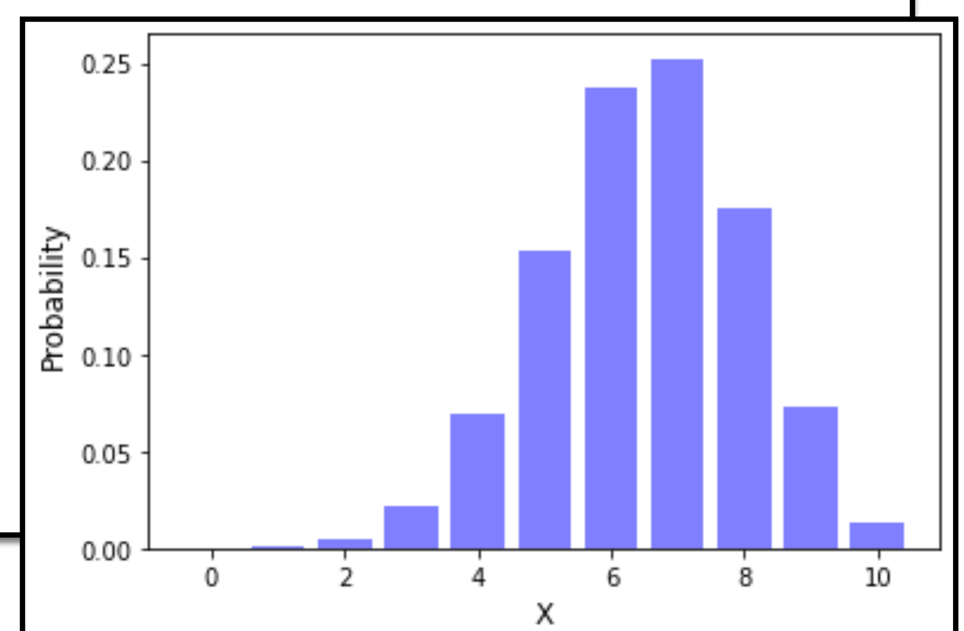

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Expected Values and Variances

- Properties of expected values and variances
  - Expected values
    - ✓ $\mathbb{E}(c) = c$
    - ✓ $\mathbb{E}(aX + c) = a\mathbb{E}(X) + c$
    - ✓ $\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \mathbb{E}(X_i)$

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Expected Values and Variances

- Properties of expected values and variances
  - Variances

    ✓ $\text{Var}(c) = 0$

    ✓ $\text{Var}(aX + c) = a^2\text{Var}(X)$

    ✓ $\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$

All random variables are pairwise uncorrelated

$$\text{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i)$$

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Expected Values and Variances

- Properties of expected values and variances

Question 2: Two retail chains COURTS and Challenger are planning to sell Apple iPads. Demands at these two retail chains are represented by random variables X and Y, respectively. Suppose that both variables are normally distributed and their means are $\mu_X = 800$ and $\mu_Y = 160$, and their standard deviations are $\sigma_X = 500$ and $\sigma_Y = 100$. The correlation between X and Y is $\text{Corr}(X, Y) = 0.23$.

- Both retail chains would like to stock enough iPads so that the probability of supplying the demands is at least $98\,\%$. What is the minimum number of iPads that COURTS and Challenger need to stock per day?

```
mu_x = 800
sigma_x = 500

stock_x = norm.ppf(0.98, mu_x, sigma_x)
print(f'The minimum stock of COURTS is {np.ceil(stock_x)}')
```

```
The minimum stock of COURTS is 1827.0
```


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Expected Values and Variances

- Properties of expected values and variances

Question 2: Two retail chains COURTS and Challenger are planning to sell Apple iPads. Demands at these two retail chains are represented by random variables X and Y, respectively. Suppose that both variables are normally distributed and their means are $\mu_X = 800$ and $\mu_Y = 160$, and their standard deviations are $\sigma_X = 500$ and $\sigma_Y = 100$. The correlation between X and Y is $\text{Corr}(X, Y) = 0.23$.

- Both retail chains would like to stock enough iPads so that the probability of supplying the demands is at least $98\,\%$. What is the minimum number of iPads that COURTS and Challenger need to stock per day?

```
mu_y = 160
sigma_y = 100

stock_y = norm.ppf(0.98, mu_y, sigma_y)
print(f'The minimum stock of Challenger is {np.ceil(stock_y)}')
```

```
The minimum stock of Challenger is 366.0
```


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Expected Values and Variances

- Properties of expected values and variances

Question 2: Two retail chains COURTS and Challenger are planning to sell Apple iPads. Demands at these two retail chains are represented by random variables X and Y, respectively. Suppose that both variables are normally distributed and their means are $\mu_X = 800$ and $\mu_Y = 160$, and their standard deviations are $\sigma_X = 500$ and $\sigma_Y = 100$. The correlation between X and Y is $\text{Corr}(X, Y) = 0.23$.

- If COURTS and Challenger decide to combine their warehouse to supply the total demand for iPads, what is the minimum number of iPads the combined warehouse needs to stock so that the total demand can be satisfied with a probability of $98\,\%$?

```
mu = mu_x + mu_y

corr = 0.23
cov = corr*sigma_x*sigma_y

sigma = (sigma_x**2 + sigma_y**2 + 2*cov) ** 0.5
```


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Expected Values and Variances

- Properties of expected values and variances

Question 2: Two retail chains COURTS and Challenger are planning to sell Apple iPads. Demands at these two retail chains are represented by random variables X and Y, respectively. Suppose that both variables are normally distributed and their means are $\mu_X = 800$ and $\mu_Y = 160$, and their standard deviations are $\sigma_X = 500$ and $\sigma_Y = 100$. The correlation between X and Y is Corr$(X, Y) = 0.23$.

- If COURTS and Challenger decide to combine their warehouse to supply the total demand for iPads, what is the minimum number of iPads the combined warehouse needs to stock so that the total demand can be satisfied with a probability of $98\,\%$?

```
stock = norm.ppf(0.98, mu, sigma)
print(f'The minimum stock is {np.ceil(stock)}')
```

```
The minimum stock is 2053.0
```


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Expected Values and Variances

- Properties of expected values and variances

Example 7: The **log return**, denoted by r_t, is defined as

$$r_t = \log\left(\frac{Q_t}{Q_{t-1}}\right),$$

where Q_t and Q_{t-1} are prices of an asset at time t and $t-1$, respectively, and the $\log(\cdot)$ is the natural logarithm function. Suppose that the daily log returns on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015.

- If you buy $\$1000$ worth of this stock at time $t=1$, what is the probability that after one trading day, i.e. at time $t=2$, your investment is worth less than $\$990$?

$$P(Q_2 \leq 990) \Rightarrow P\left(\frac{Q_2}{Q_1} \leq 0.99\right) \Rightarrow P(r_2 \leq \log(0.99))$$

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Expected Values and Variances

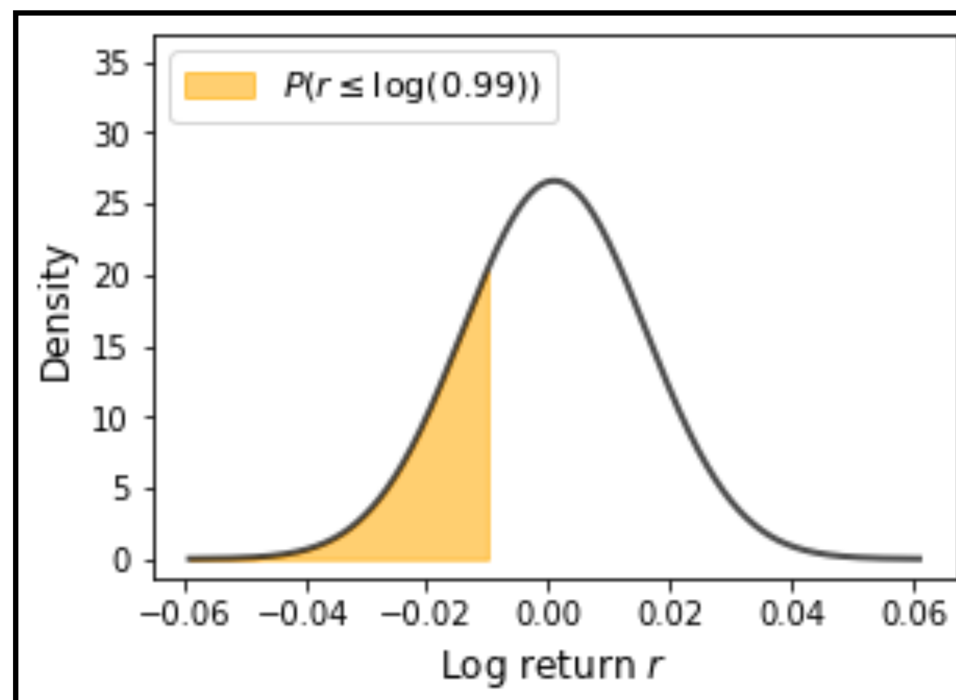
- Properties of expected values and variances

```
mean = 0.001
std = 0.015

prob = norm.cdf(np.log(0.99), mean, std)
print(f'The probability is {prob}')
```

```
The probability is 0.23065573155475771
```

$$P(r_2 \leq \log(0.99))$$



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Expected Values and Variances

- Properties of expected values and variances

Example 5: The **log return**, denoted by r_t, is defined as

$$r_t = \log\left(\frac{Q_t}{Q_{t-1}}\right),$$

where Q_t and Q_{t-1} are prices of an asset at time t and $t-1$, respectively, and the $\log(\cdot)$ is the natural logarithm function. Suppose that the daily log returns on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015.

- If you buy $\$1000$ worth of this stock at time $t=1$, what is the probability that after five trading day, i.e. at time $t=6$, your investment is worth less than $\$990$?

$$z = r_2 + r_3 + \ldots + r_6 = \log\left(\frac{Q_2}{Q_1}\right) + \log\left(\frac{Q_3}{Q_2}\right) + \ldots + \log\left(\frac{Q_6}{Q_5}\right)$$

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Expected Values and Variances

- Properties of expected values and variances

Example 5: The **log return**, denoted by r_t, is defined as

$$r_t = \log\left(\frac{Q_t}{Q_{t-1}}\right),$$

where Q_t and Q_{t-1} are prices of an asset at time t and $t-1$, respectively, and the $\log(\cdot)$ is the natural logarithm function. Suppose that the daily log returns on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015.

- If you buy $\$1000$ worth of this stock at time $t = 1$, what is the probability that after five trading day, i.e. at time $t = 6$, your investment is worth less than $\$990$?

$$z = r_2 + r_3 + \ldots + r_6 = \log\left(\frac{Q_2}{Q_1}\frac{Q_3}{Q_2}\cdots\frac{Q_6}{Q_5}\right) = \log\left(\frac{Q_6}{Q_1}\right)$$

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Expected Values and Variances

- Properties of expected values and variances

Example 5: The **log return**, denoted by r_t, is defined as

$$r_t = \log\left(\frac{Q_t}{Q_{t-1}}\right),$$

where Q_t and Q_{t-1} are prices of an asset at time t and $t-1$, respectively, and the $\log(\cdot)$ is the natural logarithm function. Suppose that the daily log returns on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015.

- If you buy $\$1000$ worth of this stock at time $t = 1$, what is the probability that after five trading day, i.e. at time $t = 6$, your investment is worth less than $\$990$?

$$P(Q_6 \leq 990) \Rightarrow P\left(\frac{Q_6}{Q_1} \leq 0.99\right) \Rightarrow P(z \leq \log(0.99))$$

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Expected Values and Variances

- Properties of expected values and variances

```
mean_z = mean * 5
std_z = std * (5**0.5)

prob = norm.cdf(np.log(0.99), mean_z, std_z)
print(f'The probability is {prob}')
```

```
The probability is 0.3268188763247845
```

$$z = r_2 + r_3 + \ldots + r_6$$

$$\Rightarrow$$

$$\begin{cases} \mathbb{E}(z) = \mathbb{E}(r_2) + \mathbb{E}(r_3) + \ldots + \mathbb{E}(r_6) \\ \text{Var}(z) = \text{Var}(r_2) + \text{Var}(r_3) + \ldots + \text{Var}(r_6) \end{cases}$$

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Expected Values and Variances

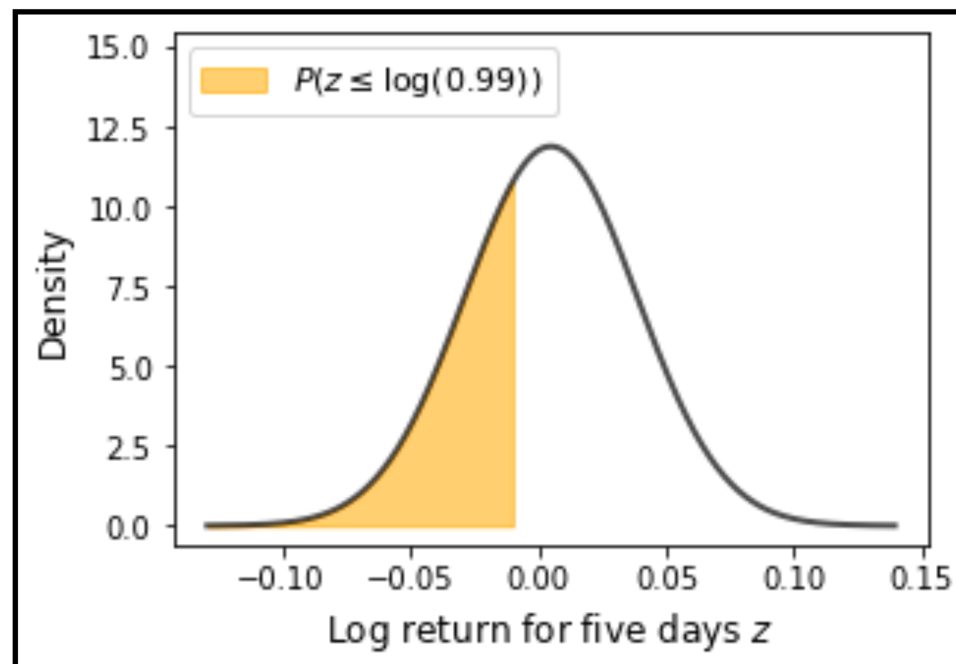
- Properties of expected values and variances

```
mean_z = mean * 5
std_z = std * (5**0.5)

prob = norm.cdf(np.log(0.99), mean_z, std_z)
print(f'The probability is {prob}')
```

```
The probability is 0.3268188763247845
```

$$P(z \leq \log(0.99))$$



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