The Gamma Function

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1 Introduction

Many algorithms and combinatorial problems require the computation of factorials. While this can be acceptable for small numbers, computing the factorial of large numbers can be intensive. If you don't believe me, go ahead and figure out 100! by hand for me real quick, you'll see what I mean. Luckily, we have the gamma function for such cases, and Stirling's approximation for when "close enough" is acceptable.

2 The Gamma Function

The key functionality we're looking for from such a function is that $f(x+1) = (x+1)f(x)$, since this is how factorials work. Take the equation:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

So,

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x} dt = -e^{-t} t^x \mid_0^\infty + x \int_0^\infty e^{-t} t^{x-1} dt = 0 + x\Gamma(x)$$

Therefore, as long as x is a non-negative integer, $x! = \Gamma(x)$

3 Stirling's Approximation

Who wants to do integrals anyways?
For applications where a (very) good ballpark is acceptable, we have Stirling's Approximation, which is given by

$$\Gamma(x+1) \approx x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

NOTE: This approximation is better for larger numbers than smaller ones. For instance, Stirling's Approximation for $x = 5$ is approximately 24, but $5! = 120$.