Continuous vs. Finite Calculus

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1 Introduction

Most of us are familiar with calculus over $\mathbb{R}^n$, however the concepts of continuity, differentiation, and integration as we learned them in high school don't quite translate directly to finite spaces like $\mathbb{Z}^n$. For stochastic processes, understanding how to operate over countably infinite spaces is essential.

2 Finite vs. Continuous

What exactly is the difference between finite and continuous calculus? Really, it boils down to what we consider the smallest possible difference between two elements of the set.
For instance, take any distinct $m, n \in \mathbb{N}$. We can easily show that $|m - n| \geq 1$. That is, every natural number is a distance of at least 1 from either of its closest neighbors, so the "resolution" of the space is 1.

However, in the real numbers, this isn't the case. In fact, the real numbers don't have a smallest resolution. We can prove this by contradiction:

Theorem 2.1. *For any $x, y \in \mathbb{R}$ with $x < y$, $\exists z \in \mathbb{R}$ such that $x < z < y$.*

Proof. Suppose, by way of contradiction, that for any choice of z, either $z < x$ or $y < z$.
Fix $z = \frac{x+y}{2}$, which is clearly in $\mathbb{R}$.
Now, $x = \frac{2x}{2}$, $y = \frac{2y}{2}$, and since $y > x$, $2x < x + y < 2y$.
Then $x = \frac{2x}{2} < \frac{x+y}{2} < \frac{2y}{2} = y$, therefore $x < z < y$, which is a contradiction. $\square$

This means that any two real numbers can be as close as we'd like, and there would still be a number in between them (actually, an uncountably infinite collection of numbers).

3 Continuity

So why do we care? Say we have a function f, and want to show that said function is continuous at a point c. One way we can understand this is that, for any distance ϵ, we can pick a separate distance δ such that for any number x, if $f(x)$ is within ϵ of $f(c)$, it must be true that x is within δ of c. Or, more formally,

Theorem 3.1. *A function f is continuous at c if for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|x - c| \leq \delta$ implies $|f(x) - f(c)| \leq \epsilon$.*

For a continuous set, δ and ϵ can be as small as we'd like, however in a finite set, the smallest either can be is 1.
So, the function

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is not continuous on $\mathbb{R}$, but is continuous on $\mathbb{Z}$.

4 Differentiation

This also means that we have to adjust our understanding of the derivative. We have the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

On a continuous space like $\mathbb{R}$, h can get as close to zero as we'd like, but in a finite space like $\mathbb{Z}$, we are limited to values of h outside of ± 1.
So if we have $f(x) = x^2$, on $\mathbb{R}$ the derivative is

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

while its derivative on $\mathbb{Z}$ is

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{(x+1)^2 - x^2}{1} = 2x + 1$$

since h can get no closer to zero than ± 1.

We denote this finite difference operator by $\Delta[f](x) = f(x+1) - f(x)$, with the necessary restriction that x must be less than the upper bound of our set (otherwise, $f(x+1)$ would not be defined).

A curious implication of this fact is 2^x is its own derivative on $\mathbb{Z}$ since $2^{x+1} - 2^x = 2(2^x) - 2^x = 2^x$, much like e^x is in $\mathbb{R}$.

5 Integration

What about integration? Recall the definition of a Riemann integral:

$$\int_a^b f(x)dx = \lim_{|\Delta x| \to 0} \sum_{i=1}^{n} f(x_i) \Delta x$$

, where (x_i) is the sequence of partitions of the interval $[a, b]$ of length Δx.
Once again, Δx can get as close to zero as we'd like when working in $\mathbb{R}$, but when working in $\mathbb{Z}$, it can get no closer than ± 1. So our "integral" in $\mathbb{Z}$ looks like:

$$\sum_{x=a}^{b-1} f(x)$$

6 Fundamental Theorem of Calculus

Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

and

$$\int_a^b f'(x)dx = f(a) - f(b)$$

Applying the unique definitions of differentiation and integration for finite spaces, we find that similar properties hold:

$$\Delta \sum_{x=a}^{b-1} f(x) = f(x)$$

and

$$\sum_{x=a}^{b-1} \Delta[f](x) = f(b) - f(a)$$

I won't prove these here, but the proofs aren't actually that difficult.

7 Unique application of the FTC

Suppose we want to evaluate:

$$\sum_{x=1}^{100} 2x + 1$$

We could put this into a calculator, or write out every term (gross), or we could apply the Fundamental Theorem of Calculus.
Earlier, we showed that if $f(x) = x^2$, then $\Delta[f](x) = 2x + 1$.
So, if $f(x) = x^2$, then $\sum_{x=1}^{100} \Delta[f](x) = \sum_{x=1}^{100} 2x + 1$.
Therefore, by the FTC:

$$\sum_{x=1}^{100} 2x + 1 = \sum_{x=1}^{100} 2x + 1 = f(100) - f(1) = 100^2 - 1^2 = 9999$$

Pretty neat.