Quantum Information A
Problem Set 3, Solutions

Problem 1
Since I is a unitary operator, the polar decomposition of a positive operator P is simply

$$P = IP = PI. \tag{1}$$

Similarly, since I is a positive operator, the polar decomposition of a unitary operator U is simply

$$U = UI = IU. \tag{2}$$

Let $H = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of a Hermitian operator H. Let us define

$$J \equiv \sum_i |\lambda_i| |\varphi_i\rangle\langle\varphi_i| \qquad\qquad U \equiv \sum_i \mathrm{sgn}(\lambda_i) |\varphi_i\rangle\langle\varphi_i|, \tag{3}$$

where $\mathrm{sgn}(\cdot)$ is the sign function. Then J is clearly a positive operator and U a unitary operator and

$$H = UJ = JU \tag{4}$$

is the polar decomposition of H.

Problem 2

We can find the polar decompositions by following the steps of the proof of Theorem 2.3 in Nielsen & Chuang. Let $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. As we have seen in earlier exercises

$$M^\dagger M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad MM^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{5}$$

We can find their spectral decompositions from their eigenvectors and eigenvalues as

$$M^\dagger M = \frac{3+\sqrt{5}}{2}|v_1\rangle\langle v_1| + \frac{3-\sqrt{5}}{2}|v_2\rangle\langle v_2| \tag{6}$$

$$MM^\dagger = \frac{3+\sqrt{5}}{2}|v_2\rangle\langle v_2| + \frac{3-\sqrt{5}}{2}|v_1\rangle\langle v_1|, \tag{7}$$

where

$$|v_1\rangle = \sqrt{\frac{5-\sqrt{5}}{10}}\left(\frac{1+\sqrt{5}}{2}|0\rangle + |1\rangle\right) \qquad |v_2\rangle = \sqrt{\frac{5-\sqrt{5}}{10}}\left(|0\rangle - \frac{1+\sqrt{5}}{2}|1\rangle\right) \tag{8}$$

$$|w_1\rangle = \sqrt{\frac{5-\sqrt{5}}{10}}\left(|0\rangle + \frac{1+\sqrt{5}}{2}|1\rangle\right) \qquad |w_2\rangle = \sqrt{\frac{5-\sqrt{5}}{10}}\left(\frac{1+\sqrt{5}}{2}|0\rangle - |1\rangle\right). \tag{9}$$

Then, we can define $J \equiv \sqrt{M^\dagger M}$ and $K \equiv \sqrt{MM^\dagger}$, giving

$$J = \frac{1+\sqrt{5}}{2}|v_1\rangle\langle v_1| + \frac{\sqrt{5}-1}{2}|v_2\rangle\langle v_2| = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \tag{10}$$

and

$$K = \frac{1+\sqrt{5}}{2}|w_1\rangle\langle w_1| + \frac{\sqrt{5}-1}{2}|w_2\rangle\langle w_2| = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}. \tag{11}$$

Since

$$\frac{M|v_1\rangle}{\|M|v_1\rangle\|} = \sqrt{\frac{5-\sqrt{5}}{10}}\left(|0\rangle + \frac{1+\sqrt{5}}{2}|1\rangle\right) = |w_1\rangle \tag{12}$$

$$\frac{M|v_2\rangle}{\|M|v_2\rangle\|} = \sqrt{\frac{5-\sqrt{5}}{10}}\left(\frac{1+\sqrt{5}}{2}|0\rangle - |1\rangle\right) = |w_2\rangle, \tag{13}$$

we can define

$$U \equiv |w_1\rangle\langle v_1| + |w_2\rangle\langle v_2| = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \tag{14}$$

which gives our polar decomposition as

$$M = UJ = KU. \tag{15}$$

Problem 3
First of all, let us note that

$$(v \cdot \sigma)^\dagger = v_1\sigma_1^\dagger + v_2\sigma_2^\dagger + v_3\sigma_3^\dagger = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 = v \cdot \sigma \tag{16}$$

so $v \cdot \sigma$ is Hermitian and can be given a spectral decomposition

$$v \cdot \sigma = \lambda_1 |v_1\rangle\langle v_1| + \lambda_2 |v_2\rangle\langle v_2|. \tag{17}$$

Then, the anti-commutation relations of Pauli matrices tell us $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$, so we have

$$\begin{aligned}(v \cdot \sigma)^2 &= v_1^2\sigma_1^2 + v_2^2\sigma_2^2 + v_3^2\sigma_3^2 + v_1v_2\{\sigma_1,\sigma_2\} + v_1v_3\{\sigma_1,\sigma_3\} + v_2v_3\{\sigma_2,\sigma_3\} \\ &= v_1^2 + v_2^2 + v_3^2 = I.\end{aligned} \tag{18}$$

Since also $(v \cdot \sigma)^2 = \lambda_1^2 |v_1\rangle\langle v_1| + \lambda_2^2 |v_2\rangle\langle v_2|$, we see that $\lambda_i^2 = 1$. Since no linear combination of the three Pauli matrices can be I or $-I$, we know that the eigenvalues then have to be +1 and -1.

We can now write the spectral decomposition as

$$v \cdot \sigma = |v_+\rangle\langle v_+| - |v_-\rangle\langle v_-|. \tag{19}$$

Since $I = |v_+\rangle\langle v_+| + |v_-\rangle\langle v_-|$, we can write the projectors to the eigenspaces as

$$P_+ = |v_+\rangle\langle v_+| = (I + v \cdot \sigma)/2 \tag{20}$$

and

$$P_- = |v_-\rangle\langle v_-| = (I - v \cdot \sigma)/2. \tag{21}$$

Problem 4

The probability of a result m from a projective measurement measuring state $|\psi\rangle$ is given by

$$p(m) = \langle\psi|P_m|\psi\rangle. \tag{22}$$

Since

$$\langle 0|\sigma_i|0\rangle = \delta_{i,3} \tag{23}$$

we get for the probability of obtaining result $+1$ for a measurement of $v \cdot \sigma$ with initial state $|0\rangle$

$$p(m = +1) = \langle 0|(I + v \cdot \sigma)/2|0\rangle = (1 + v_3)/2. \tag{24}$$

This probability can of course also be written equivalently as $p(m = +1) = |\langle 0|v_+\rangle|^2$, where $|v_+\rangle$ is the eigenstate of $v \cdot \sigma$ corresponding to eigenvalue $+1$.

The state after outcome m is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}} \tag{25}$$

so in this case the final state, if we assume measurement result $+1$, is

$$\frac{(I + v \cdot \sigma)/2|0\rangle}{\sqrt{(1 + v_3)/2}} = \frac{(1 + v_3)/2|0\rangle + (v_1 + iv_2)/2|1\rangle}{\sqrt{(1 + v_3)/2}} \quad (= |v_+\rangle). \tag{26}$$

Problem 5
Note that we are dealing with real vector spaces here; in complex spaces we would need to replace the transposes with conjugate transposes.

First,

$$\begin{aligned} P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = A\left[(A^TA)^{-1}A^TA\right](A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T = P \end{aligned} \tag{27}$$

which proves that P is a projector (in the complex case we would also need P to be Hermitian). With our given u_1, u_2, we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{28}$$

$$\Rightarrow A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{29}$$

$$\Rightarrow (A^TA)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{30}$$

$$\Rightarrow P = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

With arbitrary v, we have then

$$P \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} \tag{32}$$

so it is clearly in the xy-plane.
Let us define $G = A^TA$. Since the columns of A are our basis vectors, the entries of G are then

$$G_{ij} = \sum_k (A^T)_{ik}A_{kj} = \sum_k A_{ki}A_{kj} = u_i \cdot u_j \tag{33}$$

so G is a Gram matrix.

More generally, in the complex case we could write $A = \sum |u_i\rangle\langle i|$ with an orthonormal basis $|i\rangle$ for some input space with the same dimension as the subspace spanned by $|u_i\rangle$. P is now

$$P = A(A^\dagger A)^{-1}A^\dagger = \sum_i |u_i\rangle\langle i| \left(\sum_{jk} \langle u_j|u_k\rangle |j\rangle\langle k|\right)^{-1} \sum_l |l\rangle\langle u_l|. \tag{34}$$

Because $|u_i\rangle$ are linearly independent, the rank of $A^\dagger A$ and therefore also $(A^\dagger A)^{-1}$ must be the number of our basis vectors $|u_i\rangle$. Therefore

$$A^\dagger A|n\rangle = \sum_{jk} \langle u_j|u_k\rangle |j\rangle\langle k|n\rangle = \sum_j \langle u_j|u_n\rangle |j\rangle \tag{35}$$

implies

$$(A^\dagger A)^{-1} \left(\sum_l \langle u_l|u_n\rangle |l\rangle\right) = |n\rangle. \tag{36}$$

This means that if we act with P on one of the basis states $|u_n\rangle$, we get

$$P|u_n\rangle = \sum_i |u_i\rangle\langle i| \left(\sum_{jk} \langle u_j|u_k\rangle |j\rangle\langle k|\right)^{-1} \sum_l |l\rangle\langle u_l|u_n\rangle = \sum_i |u_i\rangle\langle i|n\rangle = |u_n\rangle. \tag{37}$$

On the other hand, if we act on a state orthogonal to all the $|u_i\rangle$, the result is clearly 0. Since P is a linear operator, it must indeed be a projector to the space spanned by $|u_i\rangle$.

Problem 6

Let V denote our vector space and W the subspace spanned by the set $\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$. Let us also denote by W_n, $1 \le n \le m$, the subspace spanned by the set $\{|\psi_1\rangle, \ldots, |\psi_{n-1}\rangle, |\psi_{n+1}\rangle, \ldots, |\psi_m\rangle\}$, i.e., all the $|\psi_i\rangle$ except $|\psi_n\rangle$. Since $|\psi_i\rangle$ are linearly independent, clearly $W \ne W_n$ for all n.

Let us define P as the projector onto W and P_n as the projector onto W_n. Now $P - P_n$ is clearly a projector to the subspace of W which is orthogonal to W_n. Therefore, for $j \ne n$

$$(P - P_n)|\psi_j\rangle = P|\psi_j\rangle - P_n|\psi_j\rangle = |\psi_j\rangle - |\psi_j\rangle = 0 \tag{38}$$

but also, since P and P_n must be different,

$$(P - P_n)|\psi_n\rangle \ne 0, \tag{39}$$

$$\langle\psi_n|(P - P_n)|\psi_n\rangle = [\langle\psi_n|(P - P_n)]\,[(P - P_n)|\psi_n\rangle] > 0. \tag{40}$$

We could equally well write this projector as

$$P - P_n = \frac{(I - P_n)|\psi_n\rangle\langle\psi_n|(I - P_n)}{1 - \langle\psi_n|P_n|\psi_n\rangle} \tag{41}$$

since $(I - P_n)|\psi_n\rangle = (P - P_n)|\psi_n\rangle$ is the part of $|\psi_n\rangle$ which is orthogonal to W_n and the denominator is just to normalize the vector.

Let us next define

$$D \equiv \max_{|\varphi\rangle \in V} \left(\sum_i \langle\varphi|(P - P_i)|\varphi\rangle \right). \tag{42}$$

Because $P - P_i$ is a positive operator, we must have $D > 0$. We will now define

$$E_i = \frac{1}{D}(P - P_i) \tag{43}$$

for all $1 \le i \le m$ and

$$E_{m+1} = I - \sum_i E_i. \tag{44}$$

Let us prove that these satisfy our requirements of POVM. First of all, if $j \ne n \le m$, then

$$\langle\psi_j|E_n|\psi_j\rangle = \frac{1}{D}\langle\psi_j|(P - P_n)|\psi_j\rangle = 0 \tag{45}$$

but

$$\langle\psi_n|E_n|\psi_n\rangle = \frac{1}{D}\langle\psi_n|(P - P_n)|\psi_n\rangle > 0 \tag{46}$$

like we wanted. E_i are clearly positive operators for all $1 \le i \le m$ so all that remains is to show that E_{m+1} is also a positive operator.

$$\langle\varphi|E_{m+1}|\varphi\rangle = \langle\varphi|(I - \sum_i E_i)|\varphi\rangle = 1 - \frac{1}{D}\sum_i \langle\varphi|(P - P_i)|\varphi\rangle \ge 1 - \frac{1}{D}D = 0 \tag{47}$$

where the inequality follows from our definition of D so this concludes the proof.