Quantum Information A
Problem Set 4, Solutions

Problem 1
We will assume that V is finite dimensional. Let $|w_i\rangle$ be an orthonormal basis for W . Then we can use Gram-Schmidt procedure to extend this basis for V. We will denote this extended basis by a prime $|w_i'\rangle$ so that $|w_i'\rangle = |w_i\rangle$ for all $i \leq \dim(W)$.

Let $|\tilde{w}_i\rangle = U|w_i\rangle$. Then $\langle\tilde{w}_j|\tilde{w}_i\rangle = \langle w_j|U^\dagger U|w_i\rangle = \langle w_j|w_i\rangle = \delta_{ij}$, and we can again extend this basis using Gram-Schmidt denoting it by $|\tilde{w}_i'\rangle$.

Now, let $U' = \sum_i |\tilde{w}_i'\rangle\langle w_i'|$. Then, for any $|w\rangle \in W$

$$\begin{aligned} U'|w\rangle &= \sum_i |\tilde{w}_i'\rangle\langle w_i'|w\rangle = \sum_{i\leq\dim(W)} |\tilde{w}_i\rangle\langle w_i|w\rangle = \sum_{i\leq\dim(W)} U|w_i\rangle\langle w_i|w\rangle \\ &= U\sum_i |w_i'\rangle\langle w_i'|w\rangle = U|w\rangle, \end{aligned} \tag{1}$$

where we have used the fact that since $|w\rangle \in W$, then $\langle w_i'|w\rangle = 0$ for any $i > \dim(W)$. Also, for any $|v_1\rangle$, $|v_2\rangle \in V$, we have

$$\langle v_1|U'^\dagger U'|v_2\rangle = \langle v_1|\sum_{ij} |w_i'\rangle\langle\tilde{w}_i'|\tilde{w}_j'\rangle\langle w_j'|v_2\rangle = \langle v_1|\sum_i |w_i'\rangle\langle w_i'|v_2\rangle = \langle v_1|v_2\rangle, \tag{2}$$

so U' is clearly our desired extension of U.

Problem 2

1. A density matrix ρ is a positive operator with trace 1. This means that we can write an arbitrary single qubit density operator as

$$\rho = a|\psi_1\rangle\langle\psi_1| + (1-a)|\psi_2\rangle\langle\psi_2| \tag{3}$$

for some orthonormal $|\psi_1\rangle$, $|\psi_2\rangle$ and some $0 \leq a \leq 1$.

We showed in the last exercises that for a unit vector $\vec{v}$, we can write $\vec{v}\cdot\vec{\sigma} = |\psi_+\rangle\langle\psi_+| - |\psi_-\rangle\langle\psi_-|$ for some orthonormal $|\psi_+\rangle$ and $|\psi_-\rangle$. Let us now show that for an arbitrary orthonormal basis $|\psi_+\rangle$, $|\psi_-\rangle$, we can find a corresponding unit vector $\vec{v}$ satisfying the same equation. We don't need to worry about global phases (they would cancel out in the outer product) so we can write our basis as

$$|\psi_+\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle \qquad\qquad |\psi_-\rangle = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\varphi}|1\rangle \tag{4}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Now,

$$\begin{aligned}
|\psi_+\rangle\langle\psi_+| - |\psi_-\rangle\langle\psi_-| &= \begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} \\ 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} & \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \\
&= \cos\theta Z + \sin\theta\cos\varphi X + \sin\theta\sin\varphi Y \\
&= \vec{v}\cdot\vec{\sigma}
\end{aligned} \tag{5}$$

where $v_1 = \sin\theta\cos\varphi$, $v_2 = \sin\theta\sin\varphi$ and $v_3 = \cos\theta$. Now, we can write equation 3 as

$$\begin{aligned}
\rho &= a|\psi_1\rangle\langle\psi_1| + (1-a)|\psi_2\rangle\langle\psi_2| \\
&= \frac{(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) + (2a-1)(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|)}{2} \\
&= \frac{I + (2a-1)\vec{v}\cdot\vec{\sigma}}{2} \\
&= \frac{I + \vec{r}\cdot\vec{\sigma}}{2}
\end{aligned} \tag{6}$$

where we have chosen $\vec{r} = (2a-1)\vec{v}$. Note also that $\|\vec{r}\| = |2a-1| \leq 1$ like we wanted.

2. We can see straight from the previous result that the state $\rho = I/2$ corresponds to the zero vector $\vec{r} = 0$.

3. A state is pure if and only if it can be written as $|\psi\rangle\langle\psi|$ for some $|\psi\rangle$. We can see that this corresponds to the $a = 0$ or $a = 1$ cases above, which in turn means that $\|\vec{r}\| = |2a - 1| = 1$.

4. We can see that our unit vector $\vec{v}$ and therefore the pure states coincide with the Bloch sphere description of Section 1.2.

Problem 3
Let $\rho = \sum_i q_i|\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of ρ on the support of ρ, i.e., we sum only over elements for which $q_i > 0$. $|\varphi_i\rangle$ is then an orthonormal basis for the support of ρ and we can write $|\psi\rangle = \sum_i |\varphi_i\rangle\langle\varphi_i|\psi\rangle$ for a state $|\psi\rangle$ in the support of ρ. Then,

$$|\psi\rangle = \sum_i \frac{\langle\varphi_i|\psi\rangle}{\sqrt{q_i}}\sqrt{q_i}|\varphi_i\rangle. \tag{7}$$

We have $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| = \sum_j q_j|\varphi_j\rangle\langle\varphi_j|$ if and only if

$$\sqrt{p_i}|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}|\varphi_j\rangle \tag{8}$$

for some unitary matrix u_{ij} (see theorem 2.6 from Nielsen & Chuang). Since we have

$$\sqrt{p}|\psi\rangle = \sum_i \sqrt{p}\frac{\langle\varphi_i|\psi\rangle}{\sqrt{q_i}}\sqrt{q_i}|\varphi_i\rangle, \tag{9}$$

we can choose

$$p = \left(\sum_i \frac{|\langle\varphi_i|\psi\rangle|^2}{q_i}\right)^{-1}. \tag{10}$$

Then, setting

$$u_{1j} \equiv \sqrt{p}\frac{\langle\varphi_j|\psi\rangle}{\sqrt{q_j}}, \tag{11}$$

we have

$$\sum_j |u_{1j}|^2 = 1 \tag{12}$$

and with Gram-Schmidt procedure, we can extend u_{ij} to form a unitary matrix. Since $\{q_i, |\varphi_i\rangle\}$ must be a minimal ensemble, so is $\{p_i, |\psi_i\rangle\}$ defined by u_{ij} and equation 8. Also, since $|\psi_1\rangle = |\psi\rangle$, our minimal ensemble contains $|\psi\rangle$ like we wanted.

To show that p is of the given form, first note that

$$\rho^{-1} = \sum_i \frac{1}{q_i}|\varphi_i\rangle\langle\varphi_i| \tag{13}$$

on the support of ρ. Then, equation 10 becomes

$$p = \left(\sum_i \frac{\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle}{q_i}\right)^{-1} = \left(\langle\psi|\left[\sum_i \frac{|\varphi_i\rangle\langle\varphi_i|}{q_i}|\right]\psi\rangle\right)^{-1} = \frac{1}{\langle\psi|\rho^{-1}|\psi\rangle}. \tag{14}$$

Problem 4

The Bell states are

$$|\varphi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\varphi_1\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\varphi_2\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \quad |\varphi_3\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{15}$$

The reduced density operator for qubit 1 is

$$\langle 0_2|\rho|0_2\rangle + \langle 1_2|\rho|1_2\rangle \tag{16}$$

and similarly for qubit 2. As we don't want to do each of the eight cases separately, let us try to do them all at once. We can see that for each of the Bell states we have either

$$\langle 0_i|\varphi_j\rangle = \pm\frac{1}{\sqrt{2}}|0\rangle, \qquad \langle 1_i|\varphi_j\rangle = \pm\frac{1}{\sqrt{2}}|1\rangle \tag{17}$$

or

$$\langle 0_i|\varphi_j\rangle = \pm\frac{1}{\sqrt{2}}|1\rangle, \qquad \langle 1_i|\varphi_j\rangle = \pm\frac{1}{\sqrt{2}}|0\rangle. \tag{18}$$

Therefore

$$\langle 0_i|\rho|0_i\rangle + \langle 1_i|\rho|1_i\rangle = \langle 0_i|\varphi_j\rangle\langle\varphi_j|0_i\rangle + \langle 1_i|\varphi_j\rangle\langle\varphi_j|1_i\rangle = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{I}{2} \tag{19}$$

for all of the cases.

Let us demonstrate this by calculating the reduced density matrix for qubit 1 in the state $|\varphi_0\rangle$:

$$\begin{aligned}
\rho_1 &= \langle 0_2|\varphi_0\rangle\langle\varphi_0|0_2\rangle + \langle 1_2|\varphi_0\rangle\langle\varphi_0|1_2\rangle \\
&= \frac{\langle 0_2|0_2\rangle|0_1\rangle}{\sqrt{2}}\frac{\langle 0_2|0_2\rangle\langle 0_1|}{\sqrt{2}} + \frac{\langle 1_2|1_2\rangle|1_1\rangle}{\sqrt{2}}\frac{\langle 1_2|1_2\rangle\langle 1_1|}{\sqrt{2}} \\
&= \frac{1}{2}|0_1\rangle\langle 0_1| + \frac{1}{2}|1_1\rangle\langle 1_1| \\
&= \frac{I}{2}.
\end{aligned} \tag{20}$$

Note that the significance of this result is that if you obtain only one of the qubits in a Bell pair, you cannot obtain any information about the underlying Bell state.

Problem 5

The first state
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{21}$$
already satisfies the Schmidt decomposition in the computational basis. For the other two, we can follow the steps of Theorem 2.7 from N&C. Using the computational basis, we can write
$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \sum_{jk} a_{jk}|j\rangle|k\rangle \tag{22}$$
with
$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{23}$$
Since a happens to be a positive matrix, we can find its singular value decomposition relatively easily
$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]. \tag{24}$$
Defining
$$u = v = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad\qquad d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$
we can now write
$$\begin{aligned} |\psi\rangle &= \sum_{ijk} u_{ji} d_{ii} v_{ik}|j\rangle|k\rangle = \sum_i d_{ii}\left(\sum_j u_{ji}|j\rangle\right)\left(\sum_k v_{ik}|k\rangle\right) \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] = |+_A\rangle|+_B\rangle \end{aligned} \tag{26}$$
which is the Schmidt decomposition for the second state.

For the third state, we have
$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} = \sum_{jk} a_{jk}|j\rangle|k\rangle \tag{27}$$
with
$$a = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{28}$$

Again, a happens to be a Hermitian matrix, so finding the singular value decomposition is not too hard

$$\begin{aligned} a &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1+\sqrt{5}}{2\sqrt{3}} w_1 w_1^\dagger + \frac{1-\sqrt{5}}{2\sqrt{3}} w_2 w_2^\dagger \\ &= \begin{pmatrix} w_1 & -w_2 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{3}} & 0 \\ 0 & \frac{-1+\sqrt{5}}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} w_1^\dagger \\ w_2^\dagger \end{pmatrix} \end{aligned} \tag{29}$$

where

$$w_1 = \sqrt{\frac{5-\sqrt{5}}{10}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \qquad w_2 = \sqrt{\frac{5-\sqrt{5}}{10}} \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix} \tag{30}$$

are the eigenvectors of a. Now, we can define

$$u = \begin{pmatrix} w_1 & -w_2 \end{pmatrix} \qquad v = \begin{pmatrix} w_1^\dagger \\ w_2^\dagger \end{pmatrix} \qquad d = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{3}} & 0 \\ 0 & \frac{-1+\sqrt{5}}{2\sqrt{3}} \end{pmatrix} \tag{31}$$

so we get

$$\begin{aligned} |\psi\rangle &= \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle|k\rangle = \sum_i d_{ii} \left(\sum_j u_{ji} |j\rangle \right) \left(\sum_k v_{ik} |k\rangle \right) \\ &= \frac{1+\sqrt{5}}{2\sqrt{3}} |w_{1A}\rangle|w_{1B}\rangle + \frac{-1+\sqrt{5}}{2\sqrt{3}} (-|w_{2A}\rangle)|w_{2B}\rangle \end{aligned} \tag{32}$$

where $|w_i\rangle$ is the same vector as w_i but written in the ket form, and this gives the final Schmidt decomposition.

Problem 6

1. We must show that the given state reduces to $\sum_i p_i|\psi_i\rangle\langle\psi_i|$ when looking only at system A. Since $|i\rangle$ is an orthonormal basis,

$$\begin{aligned}\mathrm{tr}_R\left(\sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle \sum_j \sqrt{p_j}\langle\psi_j|\langle j|\right) &= \sum_{ij}\sqrt{p_i p_j}|\psi_i\rangle\langle\psi_j|\ \mathrm{tr}(|i\rangle\langle j|)\\ &= \sum_{ij}\sqrt{p_i p_j}|\psi_i\rangle\langle\psi_j|\delta_{ij}\\ &= \sum_i p_i|\psi_i\rangle\langle\psi_i|\\ &= \rho.\end{aligned} \tag{33}$$

2. The probability of outcome i is

$$\left(\sum_j \sqrt{p_j}\langle\psi_j|\langle j|\right)|i\rangle\langle i|\left(\sum_k \sqrt{p_k}|\psi_k\rangle|k\rangle\right) = p_i\langle\psi_i|\psi_i\rangle = p_i. \tag{34}$$

The state of the composite system after obtaining result i from system R is

$$\frac{|i\rangle\langle i|\sum_i \sqrt{p_j}|\psi_j\rangle|j\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i}|\psi_i\rangle|i\rangle}{\sqrt{p_i}} = |\psi_i\rangle|i\rangle \tag{35}$$

so the state of system A is $|\psi_i\rangle$.

3. $|AR\rangle$ is a purification of ρ so we can write it as

$$|AR\rangle = \sum_i \sqrt{q_i}|\psi_i'\rangle|i'\rangle \tag{36}$$

for some probability distribution q_i and orthonormal bases $|\psi_i'\rangle$ and $|i'\rangle$. Since $|AR\rangle$ is a purification of ρ it must reduce to ρ:

$$\mathrm{tr}_R(|AR\rangle\langle AR|) = \sum_i q_i|\psi_i'\rangle\langle\psi_i'| = \rho = \sum_j p_j|\psi_j\rangle\langle\psi_j|. \tag{37}$$

According to Theorem 2.6 of N&C, this implies

$$\sqrt{q_i}|\psi_i'\rangle = \sum_j u_{ij}\sqrt{p_j}|\psi_j\rangle \tag{38}$$

for some unitary matrix u. Inserting this into equation 36 gives

$$|AR\rangle = \sum_{ij} u_{ij}\sqrt{p_j}|\psi_j\rangle|i'\rangle = \sum_j \sqrt{p_j}|\psi_j\rangle\left(\sum_i u_{ij}|i'\rangle\right) = \sum_j \sqrt{p_j}|\psi_j\rangle|j\rangle \tag{39}$$

where we define the basis $|j\rangle = \sum_i u_{ij}|i'\rangle$ (note that this doesn't have anything to do with the similarly denoted basis of the previous parts). Since u_{ij} is unitary, we have

$$\langle i|j\rangle = \sum_{kl} u_{ki}^* u_{lj}\langle k'|l'\rangle = \sum_k (u^\dagger)_{ik} u_{kj} = (u^\dagger u)_{ij} = \delta_{ij} \tag{40}$$

so the basis $|i\rangle$ is orthonormal and by comparing equation 39 with part 2 of this exercise, we can now see that measuring R in the basis $|i\rangle$ leaves system A in state $|\psi_i\rangle$ with probability p_i.