**Quantum Information A**
**Problem Set 2, Solutions**

**Problem 1**
With

$$w_1 = (1,2,2) \qquad w_2 = (-1,0,2) \qquad w_3 = (0,0,1), \tag{1}$$

the entries $G_{ij}$ of the Gram matrix are

$$G_{11} = w_1 \cdot w_1 = 9 \qquad G_{12} = G_{21} = w_1 \cdot w_2 = 3 \qquad G_{13} = G_{31} = w_1 \cdot w_3 = 2 \tag{2}$$

$$G_{22} = w_2 \cdot w_2 = 5 \qquad G_{23} = G_{32} = w_2 \cdot w_3 = 2 \qquad G_{33} = w_3 \cdot w_3 = 1, \tag{3}$$

i.e.,

$$G = \begin{pmatrix} 9 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \tag{4}$$

The vectors $w_i$ are linearly independent if and only if the determinant of the Gram matrix is non-zero.

$$\det G = 9 \cdot (5 \cdot 1 - 2 \cdot 2) + 3 \cdot (2 \cdot 2 - 3 \cdot 1) + 2 \cdot (3 \cdot 2 - 5 \cdot 2) = 4, \tag{5}$$

so the vectors are linearly independent and form a basis of $\mathbb{R}^3$. This basis is clearly non-orthogonal since $w_1 \cdot w_2 \neq 0$.

In the Gram-Schmidt process, an orthonormal basis is found by defining $|v_1\rangle = |w_1\rangle / \| |w_1\rangle \|$ and

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \|}. \tag{6}$$

Since the order doesn't really matter, we could pick $w_3$ as our first basis vector since it is already normalized, and going in the order $w_3$, $w_2$, $w_1$ would result in basis vectors parallel to the regular axes. For practice, we will instead use the given order of the vectors. Then,

$$v_1 = \frac{|w_1\rangle}{\| |w_1\rangle \|} = \frac{(1,2,2)}{\sqrt{w_1 \cdot w_1}} = \frac{1}{3}(1,2,2) = \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right). \tag{7}$$

Now,

$$\langle v_1 | w_2 \rangle = \frac{1}{3} w_1 \cdot w_2 = 1, \tag{8}$$

and

$$\begin{aligned} v_2 &= \frac{|w_2\rangle - \langle v_1 | w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1 | w_2\rangle |v_1\rangle \|} = \frac{|w_2\rangle - |v_1\rangle}{\| |w_2\rangle - |v_1\rangle \|} = \frac{\frac{1}{3}(-4,-2,4)}{\| \frac{1}{3}(-4,-2,4) \|} = \frac{1}{6}(-4,-2,4) \\ &= \left( -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right). \end{aligned} \tag{9}$$

Finally, we have

$$\langle v_1|w_3\rangle = \frac{2}{3} \qquad\qquad \langle v_2|w_3\rangle = \frac{2}{3}, \tag{10}$$

so

$$\begin{aligned} v_3 &= \frac{|w_3\rangle - \langle v_1|w_3\rangle|v_1\rangle - \langle v_2|w_3\rangle|v_2\rangle}{\||w_3\rangle - \langle v_1|w_3\rangle|v_1\rangle - \langle v_2|w_3\rangle|v_2\rangle\|} = \frac{|w_3\rangle - \frac{2}{3}|v_1\rangle - \frac{2}{3}|v_2\rangle}{\||w_3\rangle - \frac{2}{3}|v_1\rangle - \frac{2}{3}|v_2\rangle\|} \\ &= \frac{\frac{1}{9}(2,-2,1)}{\|\frac{1}{9}(2,-2,1)\|} = \frac{1}{3}(2,-2,1) = \left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right). \end{aligned} \tag{11}$$

Therefore, we get the orthonormal basis

$$v_1 = \left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right) \qquad v_2 = \left(-\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right) \qquad v_3 = \left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right). \tag{12}$$

**Problem 2**
Let

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{13}$$

Then,

$$M^\dagger M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{14}$$

but

$$MM^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{15}$$

so clearly $M$ is not normal.

Let $A$ be Hermitian, i.e., $A^\dagger = A$. Then

$$A^\dagger A = AA = AA^\dagger \tag{16}$$

so $A$ is indeed normal.

**Problem 3**

First, since $\langle x|y\rangle = \langle y|x\rangle^*$, we have

$$\langle x|y\rangle + \langle y|x\rangle = 2\Re(\langle x|y\rangle) \leq 2|\langle x|y\rangle| \leq 2\sqrt{\langle x|x\rangle\langle y|y\rangle} = 2\|x\|\|y\|, \tag{17}$$

where $\Re(\langle x|y\rangle)$ denotes the real part of $\langle x|y\rangle$ and we have used the Cauchy-Schwarz inequality in the last inequality. Now,

$$\begin{aligned} \|x+y\|^2 &= (\langle x| + \langle y|)(|x\rangle + |y\rangle) = \langle x|x\rangle + \langle y|y\rangle + \langle x|y\rangle + \langle y|x\rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2. \end{aligned} \tag{18}$$

Taking the square root of both sides (remembering that the norm is always non-negative) gives us the final result

$$\|x+y\| \leq \|x\| + \|y\|. \tag{19}$$

**Problem 4**
Let us start with the Pauli $X$ matrix

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|. \tag{20}$$

The action of $X$ on an arbitrary state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ is

$$X|\varphi\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle. \tag{21}$$

If we assume that $|\varphi\rangle$ is an eigenstate of $X$, then, for some $c \in \mathbb{C}$,

$$\begin{cases} \alpha = c\beta \\ \beta = c\alpha \end{cases} \quad \Rightarrow \quad \alpha = c^2\alpha \quad \Rightarrow \quad c^2 = 1 \tag{22}$$

from which we can see that the un-normalized eigenstates are $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ with eigenvalues $\pm 1$. With normalization, these become the already familiar states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{23}$$

Since $|+\rangle$ and $|-\rangle$ form an orthonormal basis, we can find the diagonal representation from

$$X = X(|+\rangle\langle +| + |-\rangle\langle -|) = |+\rangle\langle +| - |-\rangle\langle -|. \tag{24}$$

Next, for the Pauli $Y$ matrix, we have

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|. \tag{25}$$

As before, let us act on an arbitrary state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ giving

$$Y|\varphi\rangle = i\alpha|1\rangle - i\beta|0\rangle. \tag{26}$$

Then, for an eigenstate we have

$$\begin{cases} \alpha = -ic\beta \\ \beta = ic\alpha \end{cases} \quad \Rightarrow \quad \alpha = c^2\alpha \quad \Rightarrow \quad c^2 = 1 \tag{27}$$

giving the un-normalized eigenstates $|0\rangle + i|1\rangle$ and $|0\rangle - i|1\rangle$ with eigenvalues $\pm 1$. Normalizing these gives the states

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\varphi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{28}$$

which are easily verified to be orthonormal. The diagonal representation is thus

$$Y = |\varphi_1\rangle\langle\varphi_1| - |\varphi_2\rangle\langle\varphi_2|. \tag{29}$$

For the Pauli $Z$ matrix, the eigenstates can be trivially seen to be $|0\rangle$ and $|1\rangle$ with eigenvalues $\pm 1$. The diagonal representation is just as trivially

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{30}$$

**Problem 5**

Let $|\varphi\rangle$ be a normalized eigenvector of a unitary $U$ with eigenvalue $\lambda$. Then

$$1 = \langle\varphi|\varphi\rangle = \langle\varphi|U^\dagger U|\varphi\rangle = \langle\varphi|\lambda^*\lambda|\varphi\rangle = |\lambda|^2\langle\varphi|\varphi\rangle = |\lambda|^2 \tag{31}$$

from which we can see that $|\lambda| = 1$, i.e., $\lambda = e^{i\theta}$ for some real $\theta$.

**Problem 6**

Let $H$ be Hermitian, and let $|v_1\rangle$ and $|v_2\rangle$ be two eigenstates of $H$ with eigenvalues $\lambda_1$ and $\lambda_2 \neq \lambda_1$. Then,

$$\langle v_1|H|v_2\rangle = \langle v_1|(H|v_2\rangle) = \langle v_1|(\lambda_2|v_2\rangle) = \lambda_2\langle v_1|v_2\rangle \tag{32}$$

but also

$$\langle v_1|H|v_2\rangle = \langle v_1|H^\dagger|v_2\rangle = (\langle v_1|H^\dagger)|v_2\rangle = (\langle v_1|\lambda_1^*)|v_2\rangle = \lambda_1^*\langle v_1|v_2\rangle. \tag{33}$$

Combining the two, we get

$$\lambda_2\langle v_1|v_2\rangle = \lambda_1^*\langle v_1|v_2\rangle. \tag{34}$$

However, looking at $\langle v_1|H|v_1\rangle$, with similar reasoning we can also prove that $\lambda_1 = \lambda_1^*$, i.e., $\lambda_1$ is real. Then our last equation becomes

$$\lambda_2\langle v_1|v_2\rangle = \lambda_1\langle v_1|v_2\rangle \tag{35}$$

but since $\lambda_1 \neq \lambda_2$, we must have $\langle v_1|v_2\rangle = 0$, i.e., $|v_1\rangle$ and $|v_2\rangle$ are orthogonal.