Thus the total number of *distinct* reorderings of the elements of X is

$$\left(\begin{array}{c} N \\ m_1, m_2, \dots, m_k \end{array} \right) \equiv \frac{N!}{m_1! m_2! \cdots m_k!} , \tag{2.40}$$

where k is the number of distinct elements of X, and $\sum_{i=1}^{k} m_i = N$ is a partition of N. Equation (2.40) defines a *multinomial coefficient*, which is a generalization of the binomial coefficient. The name comes from the generalization of the binomial theorem to k variables, the *multinomial theorem*

$$(x_1 + \cdots + x_k)^N = \sum_{\{m_i\}} \left(\begin{array}{c} N \\ m_1, m_2, \dots, m_k \end{array} \right) x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} \tag{2.41}$$

where the sum is over all partitions $\{m_i\}$ of N.

Now we can compute how many different anagrams of ABRACADABRA there are. The answer is

$$\frac{11!}{5!2!2!1!1!} = 83160 . \tag{2.42}$$

2.7 Free groups, presentations of groups, braid groups

This section introduces a new way to construct groups.

2.7.1 Free groups, presentations

We begin by defining *free groups*. Let G be a group and $X = \{g_1, g_2, \dots, g_n\}$ a subset of elements of G. If *every* element $g \in G \setminus \{e\}$ (excluding the unit element e) can be *uniquely* written as a product

$$g = g_{j_1}^{i_1} g_{j_2}^{i_2} \cdots g_{j_m}^{i_m} \tag{2.43}$$

of elements g_{j_k} taken only from the set X with exponents $i_k \in \mathbb{Z} \setminus \{0\}$ such that no two adjacent elements are equal (*i.e.*, $g_{j_i} \neq g_{j_{i+1}}$), we say that G is a *free group* and X is a *free set of generators* (of G).

The elements of X are called *letters*. An *arbitrary* product of letters

$$g = g_{j_1}^{i_1} g_{j_2}^{i_2} \cdots g_{j_m}^{i_m} \tag{2.44}$$

where the exponents $i_k \in \mathbb{Z}$ (note: $i_k = 0$ is allowed) is called a *word*. If it satisfies the additional conditions of the previous definition, $i_k \neq 0$ and $g_{j_i} \neq g_{j_{i+1}}$, the word is called a *reduced word*.

Note that this is otherwise like the familiar construction of words with letters (with $a^3 b^2 = aaabb$ *etc.*), except that group elements also have inverses. Zero exponent gives the unit element e. If a word is not a reduced word, we can perform a *reduction* to rewrite it in a reduced form (combining adjacent elements and removing unit elements) . Note also that the product is in general not commutative: $ab \neq ba$.

Example—Let $X = \{a, b, c\}$ be a collection of elements of a group G (excluding the unit element). For example, $g = a^3b^{-1}c^2b^4c$ is a reduced word, and $h = c^{-1}b^3b^{-2}a^0$ is a word, but not reduced. The reduction of h produces the reduced word $h' = c^{-1}b$.

Words can be joined together by forming *products*. For example,

$$gh = a^3b^{-1}c^2b^4cc^{-1}b^3b^{-2}a^0 \ . \tag{2.45}$$

The reduction of this gives the reduced word $(gh)' = a^3b^{-1}c^2b^5$. If we replace in the product the word h by its reduced form h' and then perform a reduction, we obtain the same reduced form: $(bh')' = (bh)'$. We can now define a free group G in an alternative way.
A *free group generated by* X is the set of of all reduced words formed from the letters of X and the empty word e, with products of words (joining of words) followed by reduction as the multiplication rule. To emphasize the generators X, we denote the free group generated by X by $F(X)$.

Example—

1. Let $X = \{a\}$. It generates the free group $F(X) = \{a^n | n \in \mathbb{Z}\}$ which is isomorphic with $(\mathbb{Z}, +)$. The isomorphism is $a^n \leftrightarrow n$, $a^n a^m \leftrightarrow n + m$.
2. Let $X = \{a, b\}$. Now $F(X) = \{e, a^n, b^m, a^{\cdot}b^m, b^n a^m, a^n b^m a^k, b^n a^m b^k, \ldots\}$

We can define a constraint by setting a reduced word to be equal to the unit element by an equation

$$r \equiv g_{j_2}^{i_1} g_{j_2}^{i_2} \cdots g_{j_m}^{i_m} = e \ . \tag{2.46}$$

Such a constraint is called a *relation*. There can be several independent relations $r_1, r_2, \ldots, r_n$.

Example— Let $X = \{a\}$, set $r \equiv a^n = e$. With this relation, the set X generates the cyclic group $\{e, a, a^2, \ldots, a^n\} \cong \mathbb{Z}_n$.

A definition of a group now consists of the set of generators $X = \{g_1, g_2, \ldots, g_n\}$ and the complete list of independent relations $r_1, r_2, \ldots, r_m$. We use the notation $\langle g_1, g_2, \ldots, g_n | r_1, r_2, \ldots, r_m\rangle$ to denote this group, called the *presentation* of the group. For previous example, the presentation of the group is

$$\langle a | a^n \rangle \cong \mathbb{Z}_n \ . \tag{2.47}$$

Additional examples:

1. Let $X = \{a, b\}$, $r = aba^{-1}b^{-1} = e$, to define the group $\langle a, b, |aba^{-1}b^{-1}\rangle$. Note that the relation is equivalent to the equation $ab = ba$, meaning that the generators commute. Thus $\langle a, b, |aba^{-1}b^{-1}\rangle = \{a^n b^m | ab = ba;\ n, m \in \mathbb{Z}\} \cong (\mathbb{Z} \times \mathbb{Z}, +)$. (Verify the isomorphism.)
2. The dihedral group D_4 is the group of symmetries of a square. Consider the operations $r=$ rotate the square by $\pi/2$ and $f=$ reflect the square about the symmetry axis passing through the midpoints of opposite sides. The following relations are easy to see: $r^4 = e$ (rotation by 2π) and $f^2 = e$ (reflecting twice). A bit less obvious is $rfrf = e$, which is illustrated in Figure 1. One can check that there are only these three independent relations. The dihedral group has then presentation $D_4 = \langle r, f | r^4, f^2, rfrf\rangle$. In general dihedral groups can be defined via the presentation $D_n = \langle r, f | r^n, f^2, rfrf\rangle$.

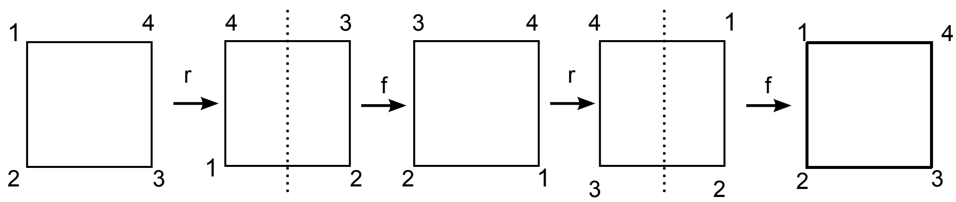


Fig. 2.1 Illustration of the relation $rfrf = e$.

2.7.2 Braids

Next, we turn to consider something familiar from knitting. A *braid* consists of *strands* which run forward and can pass under or over each other. In physics context an important related situation is met when we consider worldines of point particles moving in two space dimensions. The particle worldlines then form strands that become entangled just like knitting strands. This is depicted in Figure 2.3. Here we adopt a convention where the braid is drawn upright (another alternative would be to draw it sideways), with braiding of strands beginning from the bottom and proceeding upwards, as indicated by the arrow in Figure 2.3.

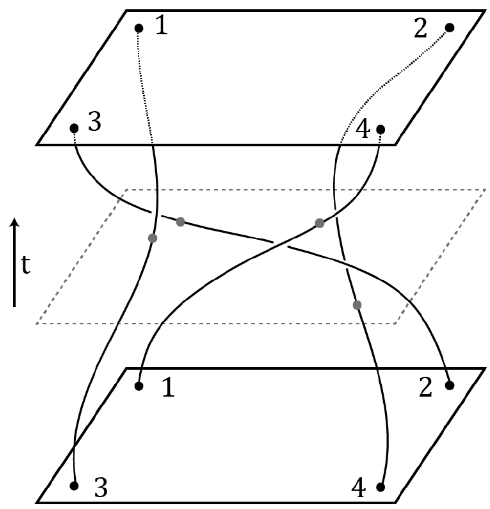


Fig. 2.2 Worldlines of 4 particles moving on a two-dimensional plane.

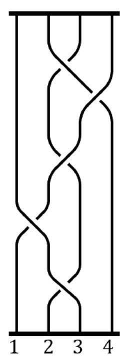


Fig. 2.3 A braid with 4 strands.

One can imagine that the strands are like pieces of string or cord, and thus they can be moved and deformed continuously as depicted in Figure 2.4.

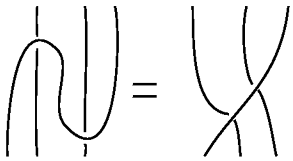


Fig. 2.4 The left is the same as the right.

or as in the example in Figure 2.5.

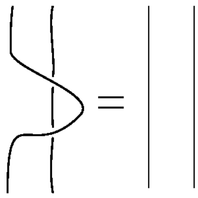


Fig. 2.5 The left is the same as the right.

In a braid of n strands, the strands are indexed with i running from 1 to n left to right. In forming a braid, a basic operation is $\sigma_i =$"move the ith strand over the $(i+1)$th strand, depicted in Figure 2.6.

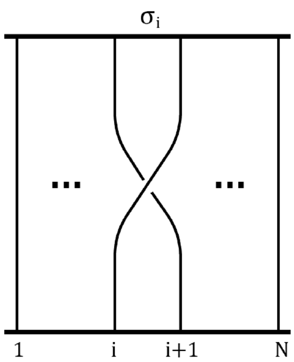


Fig. 2.6 The basic operation σ_i.

The inverse operation σ_i^{-1} moves the $(i+1)$th strand over the ith strand, as in Figure 2.7.

If we follow up these operations in a sequence, after deforming the strands the end result is the same as the identity operation, called e: $\sigma_i\sigma_i^{-1} = e$, as depicted in Figure ??. Note the convention: in drawing the braid we start from the bottom, read the sequence of operations $\sigma_i\sigma_i^{-1}$ from left to right, and perform the operations

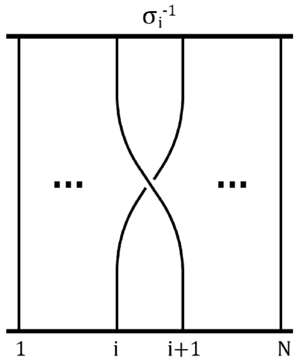
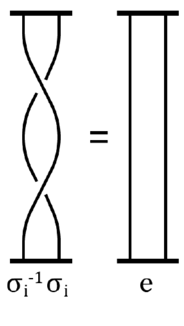


Fig. 2.7 The inverse operation σ_i^{-1}.

one by one while proceeding to draw the braid step by step. Another crucial rule to note is that after every operation, we index the strands again left to right from 1 to n and perform the next operation with the new indexing. Thus, in Figure **??**, after the first operation σ_i, the ith strand becomes the $(i+1)$th strand, and vice versa. With the new indexing, the next operation σ_i^{-1} then acts as in Figure 2.7 so that the end result is that of Figure **??**. In the final drawing, we can move the strands when possible. Similarly, one can verify $\sigma_i^{-1}\sigma_i = e$.



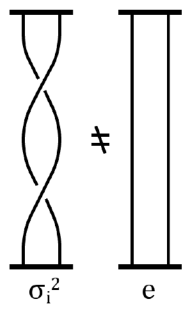


Fig. 2.8 The braid $\sigma_i\sigma_i^{-1}$ becomes trivial.

Fig. 2.9 The braid σ_i^2 does not.

For a braid of n strands the basic operations are $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, forming the letters of a set X. The words

$$\sigma = \sigma_{j_1}^{i_1}\sigma_{j_2}^{i_2}\cdots\sigma_{j_m}^{i_m} \tag{2.48}$$

with exponents $i_i \in \mathbb{Z}$ represent the various braids; when drawing the braid we start from the bottom, read the instructions from the word left to right and proceed the drawing upwards (in our convention, a different convention would be *e.g.* to draw the braid sideways from left to right). Note that there are relations among the generators because one can move and deform the strands continuously. One relation is due to the fact that braiding of strands well apart from one another is independent, so the corresponding generators commute:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad , \quad \text{when } |i-j| \geq 2 \ . \tag{2.49}$$

A less obvious relation is

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{2.50}$$

which is depicted in Figure 2.10:

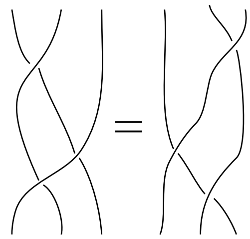


Fig. 2.10 A figure illustrating the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ arising from moving the strands.

Zooming into the relevant crossing region, we can show this again more clearly in Figure 2.11. You are encouraged to verify this experimentally.

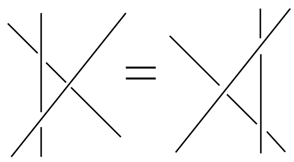


Fig. 2.11 A figure illustrating the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ arising from moving the strands.

One can show that (2.49) and (2.50) are the only relations between the generators σ_i. So we are ready to define the *braid group of n strands* B_n by its presentation:

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \ \sigma_i \sigma_j = \sigma_j \sigma_i, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle \ , \tag{2.51}$$

where $|i-j| \geq 2$ in the first relation, and $1 \leq i \leq n-2$ in the second relation.

We conclude by listing some properties of the braid groups.

1. B_1 is trivial, $B_2 \cong (\mathbb{Z}, +)$ (so it is abelian).
2. B_3 is non-abelian. Later in the course, after you have learned what are homotopy groups, you may be interested to know that B_3 is isomorphic to the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ where K is the **trefoil knot** depicted in Figure 2.12.
3. The braid groups form a nested sequence of subgroups $B_n \subset B_{n+1}$.

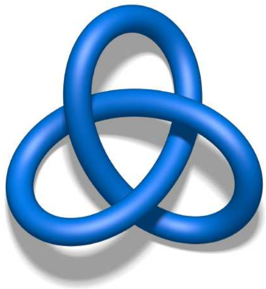


Fig. 2.12 A trefoil knot K in $\mathbb{R}^3$.

2.8 Continuous groups and Lie groups

Continuous groups are groups having an infinite and uncountable number of elements. Since the order of a continuous group is not a meaningful concept, the "size" of a continuous group can be characterized in terms of its *dimension.*

The *dimension* of a continuous group G, denoted as $\dim G$, is the minimum number of real parameters which are needed to uniquely identify its elements.

The real parameters identifying an element of a continuous group are sometimes called its *coordinates.* Note that each coordinate may take values within the whole $\mathbb{R}$, or within a real interval.

The definition of group dimension introduced above is distinct from (and more general than) the definition of dimension of a linear vector space. Indeed, continuous groups may, but do not necessarily have to, be realized in terms of linear vector spaces.

As it will be discussed more in detail in chapter **??**, given the product of two elements of a continuous group, $g'' = g'g$, the coordinates of g'' must be continuous functions of the coordinates of g and g'. This means that the set of coordinates parametrizing a group form a *manifold*, called the *group manifold.*

The most interesting continuous groups are the *Lie groups*, named after the Norwegian mathematician Marius Sophus Lie: a *Lie group* is a continuous group whose group manifold is *differentiable*, i.e. it is possible to take smooth partial derivatives of the group elements with respect to the coordinates.

Examples

- The set of real numbers $\mathbb{R}$, with the addition $+$ as the group product, is a a realization of a continuous group of dimension $\dim \mathbb{R} = 1$.
- A straightforward generalization of the group of real numbers can be constructed,