1 Partitions, Young diagrams, and multisets

1.1 Partitions

We noticed that the elements of S_N fall into subsets where the permutations have a similar cycle structure (or they are of similar *cycle type*)[1]. *E.g.* in S_4 we had the following types

$(1234),\ etc.$	one $4-$ cycle	4
$(123)(4),\ etc.$	one $3-$ cycle, one $1-$ cycle	$3+1$
$(12)(34),\ etc.$	two $2-$ cycles	$2+2$
$(12)(3)(4),\ etc.$	one $2-$ cycle, two $1-$ cycles	$2+1+1$
$(1)(2)(3)(4),\ etc.$	four $1-$ cycles	$1+1+1+1$

The last column above lists the lengths of all cycles, including 1-cycles. We notice that adding up the lengths always gives 4. Of course, in permuting N elements, the sum of lengths of all cycles must be N for the permutation to map all of the N elements. In the above, the different sums are different *partitions* of 4.

Definition: A **partition** of N is a sum

$$N = \sum_i n_i \tag{1}$$

where all $n_i \in \mathbb{Z}_+$. The number of different partitions of N (different ways of breaking N into a sum of type (1)), denoted $p(N)$, is called the **partition function**.

For example, $p(4) = 5$. One way to compute $p(N)$ is to use a generating function. One can show that the following identity holds:

$$\sum_{N=0}^{\infty} p(N)x^N = \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k}\right) \ . \tag{2}$$

Now, expanding all terms on the right hand side as Taylor series:

$$\begin{aligned}
&(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \\
&(1-x^2)^{-1} = 1 + x^2 + x^4 + x^6 + \cdots \\
&(1-x^3)^{-1} = 1 + x^3 + x^6 + x^9 + \cdots \\
&etc.
\end{aligned} \tag{3}$$

then

$$\begin{aligned}
\prod_{k=1}^{\infty} \left(\frac{1}{1-x^k}\right) &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+\cdots)\cdots \\
&= 1 + x + 2x^2 + 3x^3 + \cdots
\end{aligned} \tag{4}$$

[1]We will also learn later that the subsets will be so-called *conjugacy classes* of S_N.

Matching the coefficients of x^N on both sides of (2) gives the values of $p(N)$: $p(0) = 1,\ p(1) = 1,\ p(2) = 2,\ p(3) = 3, \ldots$. For large values of N, Hardy and Ramanujan derived the famous asymptotic formula

$$p(N) \sim \frac{1}{4N\sqrt{3}} \exp\left(\pi\sqrt{\frac{2N}{3}}\right) \quad , \quad N \to \infty \,. \tag{5}$$

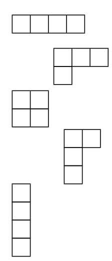
1.2 Young diagrams

The different partitions can be represented graphically with **Young diagrams**. Recall the partition sum (1). Assume that the summands have been indexed in descending order: $n_1 \geq n_2 \geq n_3 \geq \cdots$. Then draw a figure with n_1 adjacent boxes on the first row, n_2 boxes in the second row, and so on:

1	2	$\cdots$	$n_1 - 2$	$n_1 - 1$	n_1
1	2	$\cdots$	$n_2 - 1$	n_2	
1	2	$\cdots$	n_3		
$\vdots$	$\vdots$	$\ddots$			

The resulting figure is the Young diagram corresponding to the partition (1). For example, for $N = 4$ we have the partitions and Young diagrams

4
$3 + 1$
$2 + 2$
$2 + 1 + 1$
$1 + 1 + 1 + 1$



The Young diagrams then also represent graphically the different types of permutations (different conjugacy classes) of S_N.

1.3 Multisets

Permutations reorder the elements of a set X with *distinct* elements. If we represent the elements by alphabetic letters and orderings as *words*, permutations generate *anagrams*. E.g. for a set of 4 elements E,S,K,O we get 4!=24 different anagrams:

ESKO
SEKO
OKSE
EKOS
...

What about words where the same letter appears more than once, such as

ABRACADABRA

How many different anagrams would we generate now? Let us first define a **multiset**: a set where an element x_i can appear multiple times, specified by its **multiplicity** m_i. For example, in

$$X = \{x_1, x_2, x_2, x_2, x_3, x_3\}$$

the element x_1 has multiplicity $m_1 = 1$, x_2 has $m_2 = 3$, x_3 has $m_3 = 2$. The total number of elements of X is $N = \sum_i m_i$ ($N = 6$ in the above example), when we do not require that all the elements are distinct. The letters of the word ABRACADABRA form the multiset

$$X = \{A, A, A, A, A, B, B, R, R, C, D\}$$

with $m_A = 5, m_B = 2, m_R = 2, m_C = m_D = 1$, and $N = 11$. Different words formed by the letters are the the different permutations (reorderings) of the multiset. A priori, N elements can be reordered $N!$ times. But there are $m_1!$ ways to reorder the elements x_1 with no effect, $m_2!$ ways to reorder the elements x_2, and so on. Thus the total number of *distinct* reorderings of the elements of X is

$$\frac{N!}{m_1!m_2!\cdots m_k!} \equiv \begin{pmatrix} N \\ m_1, m_2, \ldots, m_k \end{pmatrix} , \tag{6}$$

where k is the number of distinct elements of X, and $\sum_{i=1}^{k} m_i = N$ is a partition of N. Equation (6) defines a **multinomial coefficient**, which is a generalization of the binomial coefficient. The name comes from the generalization of the binomial theorem to k variables, the multinomial theorem

$$(x_1 + \cdots + x_k)^N = \sum_{\{m_i\}} \begin{pmatrix} N \\ m_1, m_2, \ldots, m_k \end{pmatrix} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

where the sum is over all partitions $\{m_i\}$ of N.

Back to the question how many different anagrams of ABRACADABRA there are. The answer is

$$\frac{11!}{5!2!2!1!1!} = 83160 \; .$$