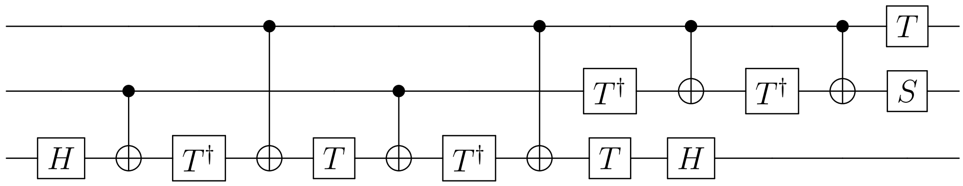
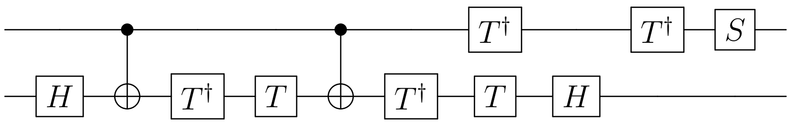
Quantum Information A
Problem Set 6, Solutions

Problem 1

We have the circuit



Let us first check, what it does when the first qubit is set to 0. Since $T|0\rangle = |0\rangle$, we see that nothing happens to the first qubit and for the other two, we have

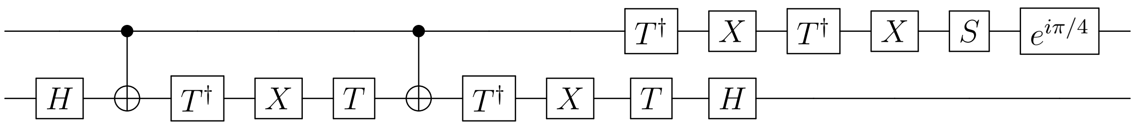


Since $TT^\dagger = ST^\dagger T^\dagger = I$ (remember that the circuits are read from left to right but matrix operations to the state are from right to left), and two consecutive CNOTs cancel as well, we can easily see that the circuit above does nothing. We now have

Table 1: Action of the circuit.

State before	State after	State before	State after
000	000	100	?
001	001	101	?
010	010	110	?
011	011	111	?

If the first qubit is set to 1, we get a phase factor from the first qubit and can write the circuit for the other two qubits (adding also the phase factor) as



If now the first of these is set to 0, we have for the second qubit

$$HTX\overbrace{T^\dagger T}^{I}XT^\dagger H = HT\overbrace{XX}^{I}T^\dagger H = H\overbrace{TT^\dagger}^{I}H = HH = I \tag{1}$$

and for the first qubit

$$e^{i\pi/4}SXT^\dagger XT^\dagger|0\rangle = e^{i\pi/4}SXT^\dagger X|0\rangle = e^{i\pi/4}SXT^\dagger|1\rangle = e^{i\pi/4}SXe^{-i\pi/4}|1\rangle = S|0\rangle = |0\rangle. \tag{2}$$

Table 2: Action of the circuit.

State before	State after	State before	State after
000	000	100	100
001	001	101	101
010	010	110	?
011	011	111	?

It remains to check, what the above circuit does, when the first of the two remaining qubits is set to 1. Then we have for the first qubit

$$e^{i\pi/4}SXT^\dagger XT^\dagger|1\rangle = SXT^\dagger X|1\rangle = SXT^\dagger|0\rangle = SX|0\rangle = S|1\rangle = i|1\rangle \tag{3}$$

and for the second qubit

$$HTXT^\dagger XTXT^\dagger XH. \tag{4}$$

One can easily verify that $XT^\dagger X = e^{-i\pi/4}T$, so this series of operations becomes

$$HT\overbrace{XT^\dagger X}^{e^{-i\pi/4}T}T\overbrace{XT^\dagger X}^{e^{-i\pi/4}T}H = e^{-i\pi/2}HTTTTH = e^{-i\pi/2}HZH = -iX. \tag{5}$$

We see that the phase factor in front cancels with the one we got for the other qubit so finally we have

Table 3: Action of the circuit.

State before	State after	State before	State after
000	000	100	100
001	001	101	101
010	010	110	111
011	011	111	110

and due to linearity, we can see that the circuit implements the Toffoli gate.

Problem 2
We want to implement the following permutation

$$U = (1234567) \tag{6}$$

where we are using the cyclic notation: (abc) means $a \to b$, $b \to c$, $c \to a$ and we have written the computational basis states in decimal numbers. Remembering the properties of cyclic permutations, we can write this as a product of 2-cycles

$$U = (1234567) = (12)(23)(34)(45)(56)(67). \tag{7}$$

One more useful fact which we will use is that

$$(ab) = (ab)(bc)(bc) = (abc)(bc) = (bca)(bc) = (bc)(ca)(bc) = (bc)(ac)(bc). \tag{8}$$

Now, let us write the action of all the possible CNOT and Toffoli gates as permutations. For example, a CNOT gate with the first qubit as control qubit and second qubit as target qubit will switch the states $|100\rangle \leftrightarrow |110\rangle$ and $|101\rangle \leftrightarrow |111\rangle$, so we would write it as the permutation (46)(57). In the table below, we will give the permutations for all of the operations.

Table 4: Action of CNOT and Toffoli gates

Gate	Control qubit	Target qubit	Permutation
C_{12}	1	2	(46)(57)
C_{21}	2	1	(26)(37)
C_{13}	1	3	(45)(67)
C_{31}	3	1	(15)(37)
C_{23}	2	3	(23)(67)
C_{32}	3	2	(13)(57)
T_1	2,3	1	(37)
T_2	1,3	2	(57)
T_3	1,2	3	(67)

By combining the operations, we can easily create some 2-cycles, e.g.,

$$C_{12}T_2 = (46)(57)(57) = (46). \tag{9}$$

Now, we can write the 2-cycles we need as

$$(12) = (23)(13)(23) = [C_{23}T_3][C_{32}T_2][C_{23}T_3] \tag{10}$$

$$(23) = C_{23}T_3 \tag{11}$$

$$(34) = (45)(35)(45) = (45)(37)(57)(37)(45) = [C_{13}T_3]T_1T_2T_1[C_{13}T_3] \tag{12}$$

$$(45) = C_{13}T_3 \tag{13}$$

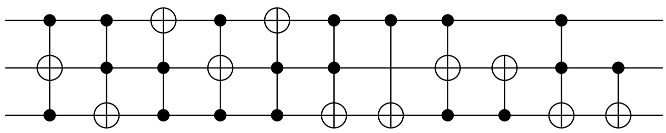
$$(56) = (67)(57)(67) = T_3T_2T_3 \tag{14}$$

$$(67) = T_3 \tag{15}$$

Combining these and getting rid of operations that cancel each other, we have for the final operation

$$U = (1234567) = C_{23}T_3C_{32}T_2C_{13}T_3T_1T_2T_1T_3T_2 \tag{16}$$

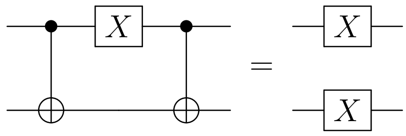
which is the circuit



(We could optimize this a bit further if we wanted since at least the first two T_3 gates actually cancel out.)

Problem 3

We will try to use slightly different methods in proving the three identities. First, we want to prove



Let us see how the circuit on the left acts when the first qubit is in a computational basis state. If the original state is $|0\psi\rangle$ then the circuit transforms it as

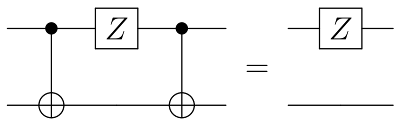
$$|0\psi\rangle \quad \rightarrow \quad |0\psi\rangle \quad \rightarrow \quad |1\psi\rangle \quad \rightarrow \quad |1\rangle \otimes X|\psi\rangle = (X \otimes X)|0\psi\rangle. \tag{17}$$

If the original state is $|1\varphi\rangle$, then the circuit transforms it as

$$|1\varphi\rangle \quad \rightarrow \quad |1\rangle \otimes X|\varphi\rangle \quad \rightarrow \quad |0\rangle \otimes X|\varphi\rangle \quad \rightarrow \quad |0\rangle \otimes X|\varphi\rangle = (X \otimes X)|1\varphi\rangle. \tag{18}$$

Since we are dealing with linear operators and any state can be written as a linear combination of $|0\psi\rangle$ and $|1\varphi\rangle$ for some $|\psi\rangle$ and $|\varphi\rangle$, it follows that the action of the two circuits must be identical.

Let us next prove



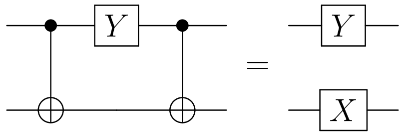
We can write the matrix representations of the gates as

$$C = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \qquad\qquad Z \otimes I = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{19}$$

Then

$$CZ_1C = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -X^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = Z_1. \tag{20}$$

For the last one, we have



Since $Y = iXZ$ we can write the operation on the left as

$$CY_1C = iCX_1Z_1C = iCX_1CCZ_1C = iX_1X_2Z_1 = iX_1Z_1X_2 = Y_1X_2, \tag{21}$$

where we have used the two previous results with the facts that $C^2 = I$ and the Pauli matrices acting on different qubits must commute.

Problem 4

Let us assume that a measurement leaves the density matrix of a system as ρ_i with probability p_i. If we do the measurement without learning the result, then the density matrix is effectively

$$\rho' = \sum_i p_i \rho_i. \tag{22}$$

Here, we have initially a density matrix ρ for a two qubit system and we are performing a projective measurement with projectors $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ acting on the second qubit. The probabilities are now (see section 2.4.1)

$$p_i = \mathrm{tr}(P_i \rho P_i) \tag{23}$$

and the corresponding density operators

$$\rho_i = \frac{P_i \rho P_i}{\mathrm{tr}(P_i \rho P_i)} \tag{24}$$

so the final density operator if we don't learn the measurement result is

$$\rho' = \mathrm{tr}(P_0 \rho P_0)\frac{P_0 \rho P_0}{\mathrm{tr}(P_0 \rho P_0)} + \mathrm{tr}(P_1 \rho P_1)\frac{P_1 \rho P_1}{\mathrm{tr}(P_1 \rho P_1)} = P_0 \rho P_0 + P_1 \rho P_1. \tag{25}$$

You might also notice that this result follows directly from equation 2.152.

Now,

$$\mathrm{tr}_2(\rho') = \mathrm{tr}_2(P_0 \rho P_0) + \mathrm{tr}_2(P_1 \rho P_1) = \langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle = \mathrm{tr}_2(\rho) \tag{26}$$

so the reduced density matrix for the first qubit is not affected.

Problem 5

We will follow here the same steps as taken for the 3×3 matrix in the book. Let

$$U = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}. \tag{27}$$

Now, setting

$$V_1 = \begin{bmatrix} \frac{U_{11}^*}{\sqrt{|U_{11}|^2+|U_{21}|^2}} & \frac{U_{21}^*}{\sqrt{|U_{11}|^2+|U_{21}|^2}} & 0 & 0 \\ \frac{U_{21}}{\sqrt{|U_{11}|^2+|U_{21}|^2}} & \frac{-U_{11}}{\sqrt{|U_{11}|^2+|U_{21}|^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & I \end{bmatrix} \tag{28}$$

we have

$$V_1 U = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \frac{1+i}{\sqrt{2}} & 0 & \frac{1-i}{\sqrt{2}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}. \tag{29}$$

Continuing similarly, always getting rid of one entry of the matrix, we get

$$V_2 = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad V_2 V_1 U = \frac{1}{2} \begin{bmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 0 & \frac{3+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{3-i}{\sqrt{6}} \\ 1 & -i & -1 & i \end{bmatrix} \tag{30}$$

$$V_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \qquad V_3 V_2 V_1 U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-i}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1+i}{2\sqrt{2}} \\ 0 & \frac{3+i}{2\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{3-i}{2\sqrt{6}} \\ 0 & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{bmatrix} \tag{31}$$

$$V_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}(1+i)}{4} & \frac{3-i}{4} & 0 \\ 0 & \frac{3+i}{4} & \frac{\sqrt{3}(-1+i)}{4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad V_4V_3V_2V_1U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \end{bmatrix} \tag{32}$$

$$V_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-i}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{i}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix} \qquad V_5V_4V_3V_2V_1U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \tag{33}$$

$$V_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \qquad V_6V_5V_4V_3V_2V_1U = I_4 \tag{34}$$

Therefore, we have

$$U = V_1^\dagger V_2^\dagger V_3^\dagger V_4^\dagger V_5^\dagger V_6^\dagger \tag{35}$$

with the following two-level unitaries

$$V_1^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad V_2^\dagger = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

$$V_3^\dagger = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \qquad V_4^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}(1-i)}{4} & \frac{3-i}{4} & 0 \\ 0 & \frac{3+i}{4} & -\frac{\sqrt{3}(1+i)}{4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{37}$$

$$V_5^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-i}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{i}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix} \qquad V_6^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \tag{38}$$

Problem 6
We are trying to implement the transformation

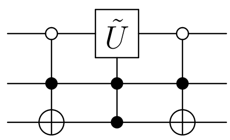
$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & d \end{bmatrix} \tag{39}$$

using only single qubit and CNOT operations, where

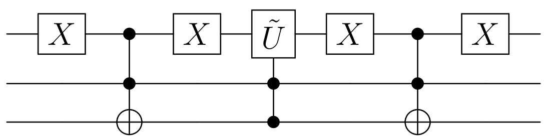
$$\tilde{U} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{40}$$

is an arbitrary unitary matrix.

$\tilde{U}$ affects the states $|010\rangle$ and $|111\rangle$. We can first swap the states $|010\rangle$ and $|011\rangle$ so that our unitary operation is performed on the states $|011\rangle$ and $|111\rangle$ instead which can be done with just controlled operation. I.e., our circuit becomes



With some X operations on the first qubit, the first and last controlled operations in the above circuit become just Toffoli gates:



We already saw in Problem 1 how to implement the Toffoli gates with just single qubit and CNOT gates so all that remains is the $C^2(\tilde{U})$ operation.

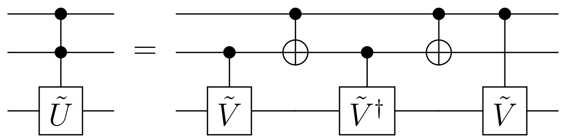
Since $\tilde{U}$ is unitary, we can write it as

$$\tilde{U} = e^{i\theta_1}|\varphi_1\rangle\langle\varphi_1| + e^{i\theta_2}|\varphi_2\rangle\langle\varphi_2| \tag{41}$$

in terms of its eigenvalues $e^{i\theta_i}$ and (orthogonal) eigenvectors $|\varphi_i\rangle$. Let us then define

$$\tilde{V} = e^{i\theta_1/2}|\varphi_1\rangle\langle\varphi_1| + e^{i\theta_2/2}|\varphi_2\rangle\langle\varphi_2| \tag{42}$$

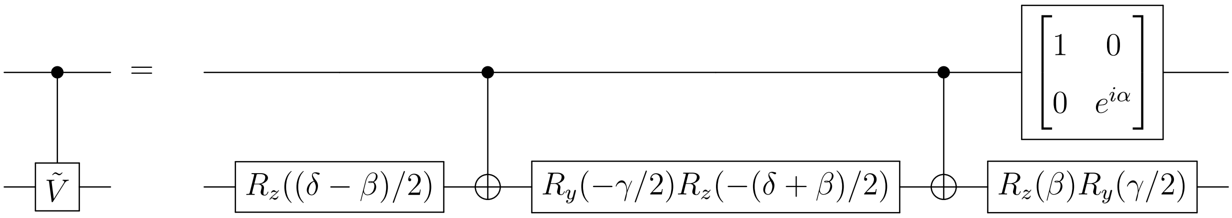
so that we clearly have $\tilde{V}^2 = \tilde{U}$ and $\tilde{V}$ is also a unitary operation. Now, as shown in figure 4.8 in N&C,



Since $\tilde{V}$ is unitary, we can decompose it as Z and Y rotations (Theorem 4.1)

$$\tilde{V} = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta) \qquad \tilde{V}^\dagger = e^{-i\alpha}R_z(-\delta)R_y(-\gamma)R_z(-\beta) \tag{43}$$

and then write the controlled $\tilde{V}$ operations as (Corollary 4.2)



and similarly for $\tilde{V}^\dagger$. Combining these gives us the desired operation with just single qubit operations and CNOTs.

If we really wanted, we could write α, β, γ and δ in terms of a, b, c, d as something like

$$\alpha = -\frac{i}{4}\ln\frac{ad}{|ad|} \qquad \gamma = 2\cos^{-1}\left(\frac{\left|\sqrt{\frac{d}{a}}|a|+1\right|}{\sqrt{\sqrt{\frac{d}{a}}|a|+\sqrt{\frac{a}{d}}|d|+2}}\right) \tag{44}$$

$$\tag{45}$$

$$\beta = -i\ln\left(\frac{-\sqrt{1-|ad|}\left(\sqrt{\frac{d}{a}}|a|+1\right)}{\sqrt{-\frac{c}{b}}|b|\left|\sqrt{\frac{d}{a}}|a|+1\right|}\right) \tag{46}$$

$$\delta = -i\ln\left(\frac{-\sqrt{-\frac{c}{b}}|b|\left(\sqrt{\frac{d}{a}}|a|+1\right)}{\sqrt{1-|ad|}\left|\sqrt{\frac{d}{a}}|a|+1\right|}\right) \tag{47}$$

where it doesn't really matter which values of ln and square root you pick, as long as you are consistent. The forms of α, β, γ and δ are much easier to find, when you know the actual values of a, b, c and d so finding the above formulas (or something similar) isn't really necessary in this exercise.