Quantum Information A
Problem Set 5, Solutions

Problem 1
Usually the logarithms are base 2 in N&C but since it isn't specified here, let the logarithms be in the base of some arbitrary $b > 1$. We will denote the natural logarithm with ln. Let us start by noting that

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} > \frac{1}{6}\left(1 + \frac{x}{4}\right) x^3 \geq x^3 \quad \forall x \geq 20. \tag{1}$$

Then also

$$x > (\ln x)^3 \quad \forall x \geq e^{20} \tag{2}$$

Let us define $n_0 = \max\left(\exp(20), \exp(\frac{1}{\ln b \ln c})\right)$. Now, for any $n \geq n_0$, we have

$$n > (\ln n)^3 \geq \ln n_0 (\ln n)^2 \geq \frac{1}{\ln b \ln c}(\ln n)^2 \tag{3}$$

$$\Rightarrow \frac{\ln c}{\ln b} n > \left(\frac{\ln n}{\ln b}\right)^2 \tag{4}$$

$$\Rightarrow n \log c > (\log n)^2 \tag{5}$$

$$\Rightarrow c^n > n^{\log n} \tag{6}$$

so we can see that c^n is $\Omega(n^{\log n})$.

Let us assume that there exists some $k > 0$ and n_1 s.t. for every $n \geq n_1$ we have $n^{\log n} \geq kc^n$. Clearly we must have $k < 1$ based on our previous result. Let us pick some m for which

$$m > -\frac{\ln k}{\ln c} \quad (> 0). \tag{7}$$

Then, we can define

$$a \equiv k^{1/m} c \quad (> k^{-\ln c/\ln k} c = 1) \tag{8}$$

and clearly $a < c$. Then, for any $n \geq m$,

$$kc^n = a^m c^{n-m} \geq a^n \tag{9}$$

and based on our previous result, there exists some n_2 s.t. for any $n \geq \max(n_1, n_2, m)$

$$n^{\log n} \geq kc^n \geq a^n > n^{\log n} \tag{10}$$

which gives a contradiction. Therefore, $n^{\log n}$ is never $\Omega(c^n)$.

Problem 2

Looking at the Bloch sphere picture, it should be clear that just $R_x(\alpha)R_z(\beta)$ or vice versa will not be enough to do the H gate: the action of H takes eigenstates of X to eigenstates of Z and vice versa. The R_z rotation would keep the Z eigenstates constant and the R_x operation would keep them at a constant distance from the X axis so we could never reach the X eigenstates. If we take the rotations the other way around, we could not reach Z eigenstates from X eigenstates. Therefore we need at least three rotations.

Let us then try to find the operation in the form $e^{i\varphi}R_z(\alpha)R_x(\beta)R_z(\gamma)$. Similarly to the above observations, we could find from the Bloch sphere picture that the needed rotations are $\varphi = \alpha = \beta = \gamma = \pi/2$ (or alternatively they could all be $-\pi/2$ giving the inverse operation, which is of course also H). Let us however use the matrix representations to prove this:

$$\begin{aligned} e^{i\varphi}R_z(\alpha)R_x(\beta)R_z(\gamma) &= e^{i\varphi}\begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix}\begin{bmatrix} \cos(\beta/2) & -i\sin(\beta/2) \\ -i\sin(\beta/2) & \cos(\beta/2) \end{bmatrix}\begin{bmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{bmatrix} \\ &= e^{i\varphi}\begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -ie^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ -ie^{-i(\gamma-\alpha)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{bmatrix}. \end{aligned} \tag{11}$$

We need the off-diagonal elements to have the same phase, so this gives us $\alpha = \gamma$. We also need the magnitude of all the entries to be the same, so this gives us $\beta = \pi/2$. Now we have

$$e^{i\varphi}R_z(\alpha)R_x(\pi/2)R_z(\alpha) = \frac{e^{i\varphi}}{\sqrt{2}}\begin{bmatrix} e^{-i\alpha} & -i \\ -i & e^{i\alpha} \end{bmatrix}. \tag{12}$$

To fix the off-diagonal elements, we need to set $\varphi = \pi/2$ and now also we must have $\alpha = \pi/2$. Finally, this gives us

$$e^{i\pi/2}R_z(\pi/2)R_x(\pi/2)R_z(\pi/2) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H. \tag{13}$$

Problem 3

The columns of any unitary operator are orthonormal, so let us first try to form an orthonormal basis for a single qubit. Any two arbitrary normalized vectors can be written as

$$|v_1\rangle = \begin{bmatrix} e^{i\varphi_1}\cos(\mu) \\ e^{i\sigma_1}\sin(\mu) \end{bmatrix} \qquad |v_2\rangle = \begin{bmatrix} -e^{i\varphi_2}\sin(\nu) \\ e^{i\sigma_2}\cos(\nu) \end{bmatrix}. \tag{14}$$

where we take φ_1, φ_2, σ_1, $\sigma_2 \in [0,2\pi]$ and we can restrict μ, $\nu \in [0,\pi/2]$ since the phase factors can take care of any minus signs. For these to be orthogonal, we need to have

$$-e^{i(\varphi_1-\varphi_2)}\cos(\mu)\sin(\nu) + e^{i(\sigma_1-\sigma_2)}\sin(\mu)\cos(\nu) = 0. \tag{15}$$

This gives us two constraints: the phases of the terms must be opposite so $\varphi_1 - \varphi_2 = \sigma_1 - \sigma_2$ and also

$$\cos(\mu)\sin(\nu) = \sin(\mu)\cos(\nu) \qquad \Leftrightarrow \qquad \tan(\mu) = \tan(\nu) \qquad \Leftrightarrow \qquad \mu = \nu. \tag{16}$$

Therefore, we can write an arbitrary orthonormal basis as

$$|v_1\rangle = \begin{bmatrix} e^{i\varphi_1}\cos(\mu) \\ e^{i\sigma_1}\sin(\mu) \end{bmatrix} \qquad |v_2\rangle = \begin{bmatrix} -e^{i\varphi_2}\sin(\mu) \\ e^{i(\varphi_2+\sigma_1-\varphi_1)}\cos(\mu) \end{bmatrix} \tag{17}$$

and an arbitrary unitary operator as

$$U = \begin{bmatrix} e^{i\varphi_1}\cos(\mu) & -e^{i\varphi_2}\sin(\mu) \\ e^{i\sigma_1}\sin(\mu) & e^{i(\varphi_2+\sigma_1-\varphi_1)}\cos(\mu) \end{bmatrix}. \tag{18}$$

Now, we can introduce new variables

$$\gamma = 2\mu \qquad \alpha = \frac{\varphi_2+\sigma_1}{2} \qquad \beta = \sigma_1 - \varphi_1 \qquad \delta = \varphi_2 - \varphi_1 \tag{19}$$

$$\Leftrightarrow$$

$$\mu = \frac{\gamma}{2} \qquad \varphi_1 = \alpha - \beta/2 - \delta/2 \qquad \varphi_2 = \alpha - \beta/2 + \delta/2 \qquad \sigma_1 = \alpha + \beta/2 - \delta/2. \tag{20}$$

With these, our unitary operator becomes

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}. \tag{21}$$

Problem 4

We will use equation 4.8 from N&C, which states that a rotation through an angle θ about the $\hat{n}$ axis can be written as

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) \hat{n} \cdot \vec{\sigma}. \tag{22}$$

Now, a rotation through an angle β_1 about the axis $\hat{n}_1$ followed by a rotation through an angle β_2 about the axis $\hat{n}_2$ is

$$\begin{aligned} &R_{\hat{n}_2}(\beta_2) R_{\hat{n}_1}(\beta_1) \\ =&\left[\cos\left(\frac{\beta_2}{2}\right) I - i \sin\left(\frac{\beta_2}{2}\right) \hat{n}_2 \cdot \vec{\sigma}\right]\left[\cos\left(\frac{\beta_1}{2}\right) I - i \sin\left(\frac{\beta_1}{2}\right) \hat{n}_1 \cdot \vec{\sigma}\right] \\ =&c_1 c_2 I - s_1 s_2 (\hat{n}_2 \cdot \vec{\sigma})(\hat{n}_1 \cdot \vec{\sigma}) - i s_1 c_2 \hat{n}_1 \cdot \vec{\sigma} - i c_1 s_2 \hat{n}_2 \cdot \vec{\sigma} \end{aligned} \tag{23}$$

where we have used the notation $c_i = \cos(\beta_i/2)$ and $s_i = \sin(\beta_i/2)$. Now, remembering that $\sigma_i^2 = I$, $XY = iZ$, $YX = -iZ$ and similarly for other multiplications, we can write $(\hat{n}_2 \cdot \vec{\sigma})(\hat{n}_1 \cdot \vec{\sigma})$ as

$$\begin{aligned} &(\hat{n}_2 \cdot \vec{\sigma})(\hat{n}_1 \cdot \vec{\sigma}) \\ =&\hat{n}_1 \cdot \hat{n}_2 I + i[(n_{2y} n_{1z} - n_{2z} n_{1y}) X + (n_{2z} n_{1x} - n_{2x} n_{1z}) Y + (n_{2x} n_{1y} - n_{2y} n_{1x}) Z] \\ =&\hat{n}_1 \cdot \hat{n}_2 I + i(\hat{n}_2 \times \hat{n}_1) \cdot \vec{\sigma} \end{aligned} \tag{24}$$

With this, the composition of the rotations becomes

$$\begin{aligned} R_{\hat{n}_2}(\beta_2) R_{\hat{n}_1}(\beta_1) &= c_1 c_2 I - s_1 s_2 (\hat{n}_1 \cdot \hat{n}_2 I + i(\hat{n}_2 \times \hat{n}_1) \cdot \vec{\sigma}) - i s_1 c_2 \hat{n}_1 \cdot \vec{\sigma} - i c_1 s_2 \hat{n}_2 \cdot \vec{\sigma} \\ &= (c_1 c_2 - s_1 s_2 \hat{n}_1 \cdot \hat{n}_2) I - i(s_1 c_2 \hat{n}_1 + c_1 s_2 \hat{n}_2 + s_1 s_2 \hat{n}_2 \times \hat{n}_1) \cdot \vec{\sigma}. \end{aligned} \tag{25}$$

Comparing this with equation 22, we can see that the axis $\hat{n}_{12}$ and angle β_{12} of the overall rotation satisfy

$$\cos(\beta_{12}/2) = c_1 c_2 - s_1 s_2 \hat{n}_1 \cdot \hat{n}_2 \tag{26}$$

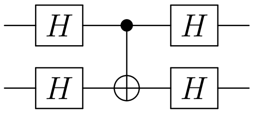
$$\sin(\beta_{12}/2) \hat{n}_{12} = s_1 c_2 \hat{n}_1 + c_1 s_2 \hat{n}_2 + s_1 s_2 \hat{n}_2 \times \hat{n}_1. \tag{27}$$

If we now set $\beta_1 = \beta_2$ and $\hat{n}_1 = \hat{z}$, then we will have $c_1 = c_2 = c$, $s_1 = s_2 = s$ and the above equations become

$$\cos(\beta_{12}/2) = c^2 - s^2\ \hat{z} \cdot \hat{n}_2 \tag{28}$$

$$\sin(\beta_{12}/2) \hat{n}_{12} = sc(\hat{z} + \hat{n}_2) + s^2\ \hat{n}_2 \times \hat{z}. \tag{29}$$

Problem 5
Let us denote the CNOT with C_{ij}, if qubit i is the control qubit and qubit j is the target qubit. Now, we can write the circuit



as

$$U = (H \otimes H) C_{12} (H \otimes H). \tag{30}$$

If we write the $H \otimes H$ as $(H \otimes I)(I \otimes H)$, the circuit becomes in the matrix representation

$$U = \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{31}$$

so the circuit swaps the states $|01\rangle$ and $|11\rangle$, so it clearly equals the C_{21} gate.

We can therefore also see that the effect of C_{12} equals $(H \otimes H) C_{21} (H \otimes H)$. If we write the operations in the outer product representation

$$H \otimes H = |++\rangle\langle 00| \ + \ |+-\rangle\langle 01| \ + \ |-+\rangle\langle 10| \ + \ |--\rangle\langle 11| \tag{32}$$
$$C_{21} = |00\rangle\langle 00| + |10\rangle\langle 10| + |11\rangle\langle 01| + |01\rangle\langle 11| \tag{33}$$

we can easily see that

$$\begin{aligned} C_{12} &= (H \otimes H) C_{21} (H \otimes H) = (H \otimes H) C_{21} (H \otimes H)^\dagger \\ &= |++\rangle\langle ++| \ + \ |+-\rangle\langle --| \ + \ |-+\rangle\langle -+| \ + \ |--\rangle\langle +-| \end{aligned} \tag{34}$$

so the effect is

$$|++\rangle \to |++\rangle \tag{35}$$
$$|-+\rangle \to |-+\rangle \tag{36}$$
$$|+-\rangle \to |--\rangle \tag{37}$$
$$|--\rangle \to |+-\rangle, \tag{38}$$

i.e., it flips the control qubit in the $|\pm\rangle$ basis if the target qubit is in state $|-\rangle$.

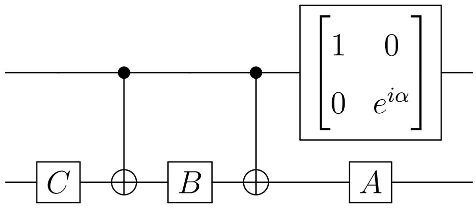
Problem 6
We can follow here the steps taken in N&C before the exercise. I.e., to construct a $C^1(U)$ operation, we can first decompose U into (see Theorem 4.1)

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta). \tag{39}$$

Then, we will define (see Corollary 4.2)

$$A \equiv R_z(\beta) R_y(\gamma/2) \qquad B \equiv R_y(-\gamma/2) R_z(-(\delta+\beta)/2) \qquad C \equiv R_z((\delta-\beta)/2) \tag{40}$$

and with these, the circuit

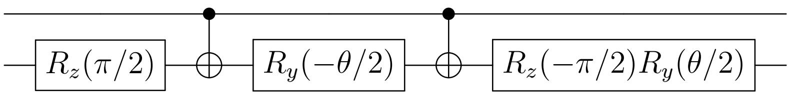


should implement $C^1(U)$ (see Figure 4.6 from N&C).

For $R_x(\theta)$, we will find the decomposition as (this can be done, e.g., by comparing equations 4.12 and 4.4 from the book or by noticing that the rotation $R_z(\pi/2)$ rotates between the x-axis and y-axis)

$$R_x(\theta) = R_z(-\pi/2) R_y(\theta) R_z(\pi/2). \tag{41}$$

Now, the circuit

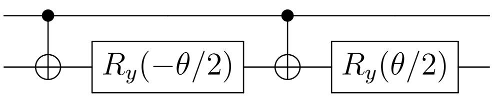


should implement $C^1(R_x(\theta))$.

For $R_y(\theta)$, we don't need to decompose it any further, now our R_z rotations are just through angle 0. Our single qubit operators are then

$$A = R_y(\theta/2) \qquad\qquad B = R_y(-\theta/2) \qquad\qquad C = I \tag{42}$$

so we can do it with just two single qubit operations, i.e., the circuit



gives us the $C^1(R_y(\theta))$ operation.