pages 115-125. For more explanation of construction of character tables, see Jones, section 4.4. You will work out some character tables in a problem set.

Again, the orthogonality of characters can be interpreted as an orthogonality relation for vectors, with useful consequences. Let $C_1, C_2, \ldots, C_k$ be the conjugacy classes of G, denote the number of elements of C_i by $|C_i|$. Then (10) implies

$$\sum_{\{C_i\}} |C_i| \chi^{(\alpha)*}(C_i) \chi^{(\beta)}(C_i) = |G| \; \delta_{\alpha\beta} \; . \tag{11}$$

Consider then the vectors $\vec{v}_\alpha = (\sqrt{|C_1|}\chi^{(\alpha)}(C_1), \ldots, \sqrt{|C_k|}\chi^{(\alpha)}(C_k))$. The number of such vectors is the same as the number of irreducible representations. On the other hand, (11) tells that the vectors are mutually orthogonal, so their number cannot be larger than the dimension of the vector space k, the number of conjugacy classes. Again, it can be shown that the numbers are actually the same:

Theorem 3.9 *The number of unitary irreducible representations of a finite group is the same as the number of its conjugacy classes.*

If the group is Abelian, the conjugacy class of each element contains only the element itself: $gg_0g^{-1} = g_0gg^{-1} = g_0$. So the number of conjugacy classes is the same as the order of the group $|G|$, this is then also the number of unitary irreducible representations. On the other hand, according to Burnside's theorem,

$$\sum_{\alpha=1}^{|G|} (\dim D^{(\alpha)})^2 = |G| \; .$$

Since there are $|G|$ terms on the left hand side, it must be $\dim D^{(\alpha)} = 1$ for all α. Hence:

Theorem 3.10 *All unitary irreducible representations of an Abelian group are one dimensional.*

This fact can be shown to be true even for continuous Abelian groups. (Hence no word "finite" in the above.)

Example: Let us see how characters work in practice. First we will construct the character table of S_3. The table will have rows for irreducible representations and columns for conjugacy classes. At their crossings we insert the values of the characters for a each conjugacy class in each irreducible representation. Conjugacy classes of the permutation groups consist of permutations of the same cycle type. For S_3, they are represented by the identity permutation e (3 1-cycles), transpositions (12) etc. (1 2-cycle, 1 1-cycle) and the cyclic permutations (123) (1 3-cycle). There is one

Table 1: The character table of S_3 so far.

	e	3(12)	2(123)
1	1	1	1
1′	1	a	b
2	2	c	d

element in the first conjugacy class, 3 in the second, 2 in the last one. We label the conjugacy classes as $1e, 3(12), 2(123)$ with the numbers in front denoting the number of elements. Accordingly, there are 3 irreducible representations. From our previous example in applying Burnside's theorem we then know that their dimensions must be 1,1 and 2. Let us denote the irreducible representations as $1, 1', 2$. One of the 1-dimensional representations is always the trivial representation with all group elements represented by 1. Also, the unit element is always represented with the unit matrix of the corresponding dimension. This information gives us the Table 1. We denote the unknown entries by a, b, c, d.

Let us introduce character vectors as follows:

$$\begin{aligned} \vec{\chi}_1 &= (1,1,1) \\ \vec{\chi}_{1'} &= (1,a,b) \\ \vec{\chi}_2 &= (2,c,d) \ . \end{aligned}$$

In computing their scalar products we must insert the number of elements in each conjugacy class, as in (11). The ortogonality theorem of characters (11) gives equations:

$$\begin{aligned} \vec{\chi}_1 \cdot \vec{\chi}_{1'} &= 1 + 3a + 2b = 0 \\ \vec{\chi}_1 \cdot \vec{\chi}_2 &= 2 + 3c + 2d = 0 \\ \vec{\chi}_{1'} \cdot \vec{\chi}_2 &= 2 + 3ac + 2bd = 0 \ . \end{aligned}$$

Instead of proceeding to add a fourth equation to solve the 4 unknowns, we do a short-cut. The other 1-dimensional representation is given by choosing a sign according to the parity of the permutations. The conjugacy class 3(12) has the odd permutations, the other ones have the even permutations. Then we represent the even permutations with 1 and the odd permutations with -1. Thus $a = -1, b = 1$, which solves the first equation above. The remaining two equations become:

$$\begin{aligned} 2 - 3c + 2d &= 0 \\ 2 + 3c + 2d &= 0 \end{aligned}$$

with solution $c = 0, d = -1$. So we arrive at the end result for the character table (Table 2.) It is easy to check that the vectors have the correct lengths according to

Table 2: The character table of S_3.

	e	3(12)	2(123)
1	1	1	1
$1'$	1	-1	1
2	2	0	-1

the nonvanishing scalar products of (11):

$$\vec{\chi}_1^2 = 1 + 3 \cdot 1 + 2 \cdot 1 = 6 = |S_3|$$
$$\vec{\chi}_{1'}^2 = 1 + 3 \cdot (-1) + 2 \cdot 1 = 6$$
$$\vec{\chi}_2^2 = 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2 = 6 \ .$$

As the last demonstration, we show how to use characters to solve the reduction problem for a representation. Consider the following 3-dimensional representation D of S_3:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ; \ D(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$D(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ ; \ D(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$D(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ ; \ D(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ .$$

This representation is completely reducible. One way to find the irreducible representations that are its building blocks would be to move a to a basis which (block) diagonalizes the matrices. But we can now easily solve the reduction problem with characters. Computing the traces gives the character vector (components are for the conjugacy classes in the same order as previously)

$$\vec{\chi}_D = (3, 1, 0) \ .$$

We could use the "algorithm" that we discussed to compute the multiplicities n_α, but in this case the answer can be found just by comparing the character vector with those of the irreducible representations: it is easy to see that

$$\vec{\chi} = (3, 1, 0) = (1, 1, 1) + (2, 0, -1) = \vec{\chi}_1 + \vec{\chi}_2$$

so that the representation decomposes to a direct sum of the trivial and the twodimensional irreducible representation:

$$D = 1 \oplus 2 \; .$$

We conclude this section by continuing on the theme of vector spaces to introduce dual vectors and tensors, which are a generalization of the concept of a vector.

3.6 Dual Vectors and Tensors

Let V be a complex vector space and f a linear function $V \to \mathbb{C}$. Now $V^* = \{f | f \text{ is a linear function } V \to \mathbb{C}\}$ is also a complex vector space, the **dual vector space** to V:

- $(f_1 + f_2)(\vec{v}) = f_1(\vec{v}) + f_2(\vec{v})$
- $(af)(\vec{v}) = a(f(\vec{v}))$
- $\vec{0}_{V^*}(\vec{v}) = 0 \quad \forall \vec{v} \in V$

The elements of V^* are called the **dual vectors**.
Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis of V. Then any vector $\vec{v} \in V$ can be written as $\vec{v} = v^i \vec{e}_i$. We define a **dual basis** in V^* such that $e^{*i}(\vec{e}_j) = \delta^i{}_j$. From this it follows that $\dim V = \dim V^* = n$ (dual basis $= \{e^{*1}, \ldots, e^{*n}\}$). We can then expand any $f \in V^*$ as $f = f_i e^{*i}$ for some coefficients $f_i \in \mathbb{C}$. Now we have

$$f(\vec{v}) = f_i e^{*i}(v^j \vec{e}_j) = f_i v^j e^{*i}(\vec{e}_j) = f_i v^i.$$

This can be interpreted as an **inner product**:

$$\begin{aligned} \langle \; , \; \rangle \; &: \; V^* \times V \to \mathbb{C} \\ \langle f, \vec{v} \rangle \; &= \; f_i v^i. \end{aligned}$$

(Note that this is not the same inner product $\langle | \rangle$ which we discussed before: $\langle \; , \rangle \; : \; V^* \times V \to \mathbb{C}$ but $\langle \; | \; \rangle \; : \; V \times V \to \mathbb{C}$.)

Pullback: Let $f : V \to W$ and $g : W \to \mathbb{C}$ be linear maps ($g \in W^*$). It follows that $g \circ f : V \to \mathbb{C}$ is a linear map, i.e. $g \circ f \in V^*$.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ & \searrow & \downarrow g \\ g \circ f & & \mathbb{C} \end{array}$$

Now f induces a map $f^* : W^* \to V^*, g \mapsto g \circ f$ i.e. $f^*(g) = g \circ f \in V^*$. $f^*(g)$ is called the **pullback** (takaisinveto) of g.

Dual of a Dual: Let $\omega : V^* \to \mathbb{C}$ be a linear function ($\omega \in (V^*)^*$). Every $\vec{v} \in V$ induces via inner product $\omega_{\vec{v}} \in (V^*)^*$ defined by $\omega_{\vec{v}}(f) = \langle f, \vec{v} >$. On the other hand, it can be shown this gives all $\omega \in (V^*)^*$. So we can identify $(V^*)^*$ with V.

Tensors: A tensor of type (p, q) is a function of p dual vectors and q vectors, and is linear in its every argument[1]

$$T : \overbrace{V^* \times \ldots \times V^*}^{p} \times \overbrace{V \times \ldots \times V}^{q} \to \mathbb{C}.$$

Examples: (0,1) tensor = dual vector : $V \to \mathbb{C}$
(1,0) tensor = (dual of a dual) vector
(1,2) tensor: $T : V^* \times V \times V \to \mathbb{C}$. Choose basis $\{\vec{e}_i\}$ in V and $\{e^{*i}\}$ in V^*:

$$T(f, \vec{v}, \vec{w}) = T(f_i e^{*i}, v^j \vec{e}_j, w^k \vec{e}_k) = f_i v^j w^k \overbrace{T(e^{*i}, \vec{e}_j, \vec{e}_k)}^{\equiv T^i{}_{jk}} = T^i{}_{jk} f_i v^j w^k,$$

where $T^i{}_{jk}$ are the components of the tensor and they uniquely determine the tensor. Note the positioning of the indices.
In general, (p, q) tensor components have p upper and q lower indices.

Tensor product: Let R be a (p, q) tensor and S be a (p', q') tensor. Then $T = R \otimes S$ is defined as the $(p + p', q + q')$ tensor:

$$\begin{aligned} &T(f_1, \ldots, f_p; f_{p+1}, \ldots, f_{p+p'}; \vec{v}_1, \ldots, \vec{v}_q; \vec{v}_{q+1}, \ldots, \vec{v}_{q+q'}) \\ &\qquad = R(f_1, \ldots, f_p; \vec{v}_1, \ldots, \vec{v}_q) S(f_{p+1}, \ldots, f_{p+p'}; \vec{v}_{q+1}, \ldots, \vec{v}_{q+q'}). \end{aligned}$$

In terms of components:

$$T^{i_1 \ldots i_p i_{p+1} \ldots i_{p+p'}}_{j_1 \ldots j_q j_{q+1} \ldots j_{q+q'}} = R^{i_1 \ldots i_p}_{j_1 \ldots j_q} S^{i_{p+1} \ldots i_{p+p'}}_{j_{q+1} \ldots j_{q+q'}}$$

Contraction: This is an operation that produces a $(p-1, q-1)$ tensor from a (p, q) tensor:

$$\underbrace{T}_{(p,q)} \mapsto \underbrace{T_{c(ij)}}_{(p-1,q-1)} ,$$

where the $(p-1, q-1)$ tensor $T_{c(ij)}$ is

$$T_{c(ij)}(f_1, \ldots, f_{p-1}; \vec{v}_1, \ldots, \vec{v}_{q-1}) = T(f_1, \ldots, \overbrace{e^{*k}}^{i^{th}}, \ldots, f_{p-1}; \vec{v}_1, \ldots, \overbrace{\vec{e}_k}^{j^{th}}, \ldots, \vec{v}_{q-1}).$$

Note the sum over k in the formula above. In component form this is

$$T_{c(ij)}{}^{l_1 \ldots l_{p-1}}_{m_1 \ldots m_{q-1}} = T^{l_1 \ldots l_{i-1} k l_i \ldots l_{p-1}}_{m_1 \ldots m_{j-1} k m_j \ldots m_{q-1}}$$

[1]So T is a multilinear object.

4 Differentiable Manifolds

The theme of this chapter is differential topology. We begin by introducing basic topological concepts, such as topology (the notion of open sets, which are then used to define continuous mappings), homeomorphism, and topological invariants. We then introduce differential structure to extend calculus to more general setting, that of differentiable manifolds.

4.1 Topological Spaces and Manifolds

The **topology** of a space X is defined via its open sets. "Topology" is basically a declaration of which subsets within a set are by definition open sets. There are some consistency conditions that we need to impose, extending the properties of open sets familiar from calculus in Euclidean spaces.

Definition: Let X be a set, $\tau = \{X_\alpha\}_{\alpha \in I}$ a (finite or infinite) collection of subsets of X. (X, τ) is a **topological space**, if

T1 $\emptyset \in \tau, \ X \in \tau$

T2 <u>all</u> possible unions of X_α's belong to τ $\left(\bigcup_{\alpha \in I'} X_\alpha \in \tau, \ I' \subseteq I\right)$

T3 all intersections of a <u>finite</u> number of X_α's belong to τ. $(\bigcap_{i=1}^{n} X_{\alpha_i} \in \tau)$

The X_α are called the **open sets** of X in topology τ, and τ is said to give a **topology** to X.
So: topology $\hat{=}$ specify which subsets of X are open.
Within the same set X there are several possible definitions of topologies, below are some common examples.

Examples:

(i) $\tau = \{\emptyset, X\}$ This is the smallest possible choice, called the *trivial topology*

(ii) $\tau = \{\text{all subsets of } X\}$ This is the maximal choice, called the *discrete topology*

(iii) Let $X = \mathbb{R}$, $\tau = \{\text{open intervals }]a, b[\text{ and their unions}\}$ This is known as the *usual topology*

(iv) $X = \mathbb{R}^n$, $\tau = \{ \ ]a_1, b_1[\times \ldots \times]a_n, b_n[$ and unions of these.$\}$ is the usual topology in higher dimensions.

(v) If (X, τ) and (Y, σ) are topological spaces, the product space $X \times Y = \{(x, y) | x \in X, y \in Y\}$ is a topological space with the natural topology being the *product topology* $\tau \times \sigma = \{(X_\alpha, Y_\beta) | X_\alpha \in \tau, Y_\beta \in \sigma\}$. A natural example of these are the n-dimensional Euclidean spaces $\mathbb{R}^n$.

So far there is no notion of a distance between two points in a set. We define it next by defining a *metric* in a set. We will then proceed to use the metric to define a natural topology.

Definition: A **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that

M1 $d(x, y) = d(y, x)$ (symmetry)

M2 $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$. (positive definitness)

M3 $d(x, y) + d(y, z) \geq d(x, z)$, this is called the *triangle inequality*.

A space with a metric (X, d) is called a **metric space**.

Example: $X = \mathbb{R}^n$,

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} \ , \text{ where } p \geq 1.$$

For $p = 2$, this defines the **Euclidean** metric. In general, $d_p(x, y) = ||x - y||_p$, where $|| \cdot ||_p$ is the p-**norm** (or L^p-**norm**) in $\mathbb{R}^n$. If X has a metric, then we can use it to define the **metric topology** on X, by choosing all the open disks

$$U_\epsilon(x) = \{ \ y \in X | \ d(x, y) < \epsilon \ \}$$

and all their unions to define the collection τ of open sets.
One can show that the metric topology of $\mathbb{R}^n$ with metric d_p is equivalent with the usual topology (for all $p \geq 1$!)

Let (X, τ) be a topological space, $A \subset X$ a subset. The topology τ induces the **relative topology** τ' in A,

$$\tau' = \{ \ U_i \cap A \ | \ U_i \in \tau \ \}$$

This is how we obtain a topology for all subsets of $\mathbb{R}^n$ (such as the sphere S^n).

Now that we have defined open sets (the topology), we can use them to define what functions are *continuous*.

Definition: Let (X, τ) and (Y, σ) be topological spaces. A map $f : X \to Y$ is **continuous** if and only if the inverse image of every open set $V \in \sigma$, $f^{-1}(V) = \{ \ x \in X \ | \ f(x) \in V \}$, is an open set in X: $f^{-1}(V) \in \tau$.

Specifically, continuous bijections give us a way to decide which topological spaces are "the same", or *homeomorphic*.

Definition: A function $f : X \to Y$ is a **homeomorphism** if f is continuous, and has an inverse $f^{-1} : Y \to X$ which is also continuous.

Note: There are continuous functions with a non-continuous inverse. As an example, take a set X and equip it with two topologies τ_1, τ_2 so that τ_2 is a proper subset of τ_1. The identity map $id : (X, \tau_1) \to (X, \tau_2)$ is obviously invertible and continuous, but the inverse map is not continuous.

Definition: If there exists a homeomorphism $f : X \to Y$, then we say that X is **homeomorphic** to Y and vice versa. We denote this by $X \approx Y$.
We leave it as an exercise to show that $\approx$ is an equivalence relation.

Intuitively speaking, X and Y are homeomorphic if we can continuously deform X to Y (without cutting or pasting).
The canonical way to describe this to a layman is to explain that a coffee cup (of the usual design) can be continuously deformed into a donut shape, thus the two are homeomorphic.

The collection of all possible topological spaces is then partitioned into equivalence classes of homeomorphic spaces. Next we would like to classify, or at least describe different non-homeomorphic spaces. We need some adjectives, words that can be used as labels. To this end, we will next define various **topological invariants**, i.e. quantities which are invariant under homeomorphisms. Conversely, if we find that a topological invariant for $X_1 \neq$ the same topological invariant for X_2, then $X_1 \not\approx X_2$.

Definition: The **neighborhood** N of a point $x \in X$ is a subset $N \subset X$ such that there exists an open set $U \in \tau,\ x \in U$ and $U \subset N$.
(N does not have to be an open set).

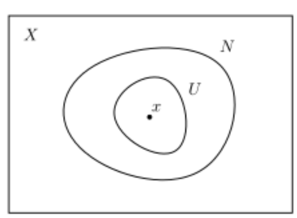

Figure 1: Neighbourhood.

Definition: Let S be a subset of a topological space X. A point $x \in S$ is a **limit point** of S if every neighborhood of x contains at least one point $y \in S$ with $y \neq x$. Limit points are sometimes also called **cluster points** or **accumulation points**.

Definition: The **closure** $\bar{S}$ of S is the union of S and all of its limit points.

It can be shown that if S is connected, then $\bar{S}$ is also connected. Furthermore, all subsets S' between S and $\bar{S}$, $S \subset S' \subset \bar{S}$, are also connected.

Example: The limit points of the open interval $S = (0, 1) \subset \mathbb{R}$ are 0 and 1, the closure $\bar{S}$ is the closed interval $[0, 1]$. The half-open interval $(0, 1]$ is an example of a subset S'.

Definition: (X, τ) is a **Hausdorff** space if for an arbitrary pair $x, x' \in X,\ x \neq x'$, there always exist neighborhoods $N \ni x,\ N' \ni x'$ such that $N \cap N' = \emptyset$.

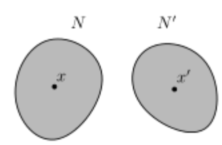

Figure 2: Hausdorff property.

We'll assume from now on that all topological spaces (that we'll consider) are Hausdorff.

Examples: $\mathbb{R}^n$ with the usual topology is Hausdorff.
All spaces X with metric topology are Hausdorff.
As a simple counterexample, a topological space X with the trivial topology is not Hausdorff.

Definition: A subset $A \subset X$ is **closed** if its complement $X - A = \{x \in X \mid x \notin A\}$ is open.
N.B. X and $\emptyset$ are both open and closed.

Definition: A collection $\{A_i\}$ of subsets $A_i \subset X$ is called a **cover** of X if $\bigcup_i A_i = X$.

If all A_i are open sets in the topology τ of X, $\{A_i\}$ is an **open cover**.

Definition: Let a collections $U = \{A_i\}_{i\in I}$ and $V = \{B_j\}_{j\in J}$ be two covers of X. The cover V is a **refinement** of the cover U, if and only if for every B_j, $j \in J$, there exists an A_i in U such that $B_j \subset A_i$.
Example: Let consider the open interval $X = (-10, 10) \subset \mathbb{R}$. The collection $U = \{(n, n+3)\}_{n=-10,-9,\ldots,8}$ is an open cover of X, and the collection $V = \{(n, n+2)\}_{n=-10,-9,\ldots,8}$ is a refinement of U.

Definition: A cover $U = \{A_i\}_{i\in I}$ of X is **locally finite** is for every $x \in X$ there exists a neighbourhood V_x of x such that only finitely many A_i's have a nonempty intersection with V_x.

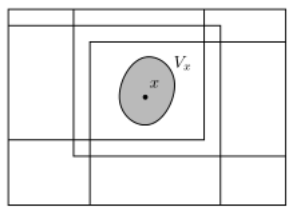

Figure 3: Locally finite cover.

Definition: A topological space (X, τ) is **paracompact** if every open cover of X has a refinement which is a locally finite open cover.

Loosely speaking, a nice thing about paracompact spaces is that they allow one to construct partitions of unity: collections of functions that are non-vanishing only locally, with their sum equal to one (more about them later). Another reason to introduce the definition is that we can now define a (topological) manifold.

Definition: A topological space (X, τ) is a **manifold** if it is a paracompact and Hausdorff, and locally homeomorphic to an open subset of $\mathbb{R}^n$. That is, every point $p \in X$ is contained in an open set U_i such that there exists a homeomorphism $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^n$. The natural number n is the **dimension** of the manifold X, denoted $\dim X = n$.

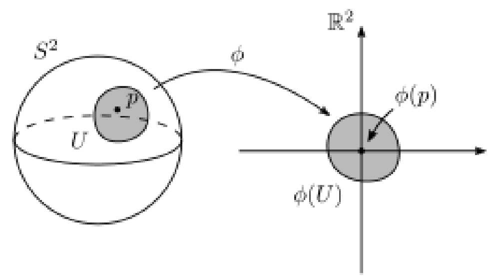

Figure 4: Manifold.

Examples:

(i) A trivial example is $\mathbb{R}^n$, with dimension n.

(ii) S^1 is a one-dimensional manifold, the map $\theta \mapsto e^{i\theta}$ restriced to open segments of S^1 gives the homomorphisms ϕ_i.

(iii) By similar arguments, n-spheres S^n are n-dimensional manifolds.

(iv) Intersecting lines do not define a manifold: neighbourhoods of the intersection point are not homeomorphic to open subsets of $\mathbb{R}$.

(v) If M and N are manifolds of dimension m and n, the product $M \times N$ with the product topology is a manifold (sometimes called a **product manifold**) of dimension $m + n$. Simple examples of these are the n-torus $T^n = S^1 \times \cdots \times S^1$ (with n factors) and the infinite two-dimensional cylinder $\mathbb{R} \times S^1$.

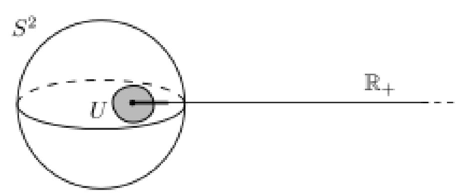

Figure 5: A sphere with a half line sticking out is not a manifold, because the open subset containing the intersection point is not homeomorphic to an open subset of $\mathbb{R}^1$ or $\mathbb{R}^2$.

Let $\mathbb{H}^n$ denote the Euclidean (upper) half-space $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$, with the relative topology induced by the usual topology of $\mathbb{R}^n$. The boundary is the $(n-1)$-dimensional plane $\partial\mathbb{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n = 0\}$, and the interior is $\text{int}(\mathbb{H}^n) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$. Note that if U is an open set in $\mathbb{R}^n$ such that a part of it is "chopped off" from the intersection $U \cap \mathbb{H}^n$, the intersection is by definition an open set in $\mathbb{H}^n$ but may not be open in $\mathbb{R}^n$.

Definition: A topological space (X, τ) is a **manifold with boundary** if it is paracompact and Hausdorff, and locally homeomorphic to an open subset of $\mathbb{H}^n$. That is, every point $p \in X$ is contained in an open set U_i such that there exists a homeomorphism $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{H}^n$. Points of X that map to the boundary of $\mathbb{H}^n$ are called boundary points. The **boundary** of X, denoted ∂X, is the set of all boundary points.

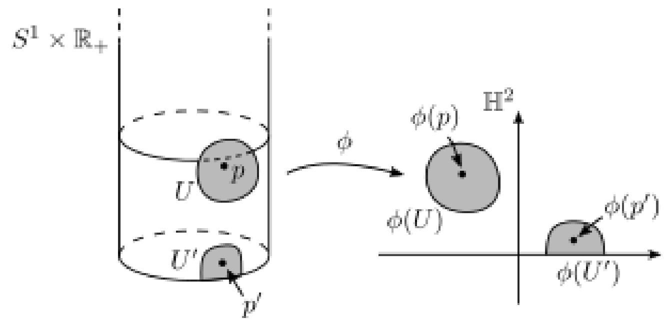

Figure 6: Manifold with a boundary.

Examples:

(i) The closed cube $\bar{C^n} = [0,1] \times \cdots [0,1] \subset \mathbb{R}^n$ (n factors in the product) is a manifold with boundary. Its boundary $\partial\bar{C^n}$ is a $(n-1)$-dimensional manifold homeomorphic to S^{n-1}.

(ii) More generally, if M is a n-dimensional manifold with boundary, ∂M is a $(n-1)$-dimensional manifold. (Show this!)

(iii) If M is a $(n-1)$-dimensional manifold, then $M \times [0,1]$ is a n-dimensional manifold with boundary.

(iv) If M is a manifold with boundary ∂M, then $M \times \mathbb{R}$ is a manifold with boundary with $\partial(\mathrm{M}\times\mathbb{R}) = \partial M \times \mathbb{R}$.

Definition: A topological space (X, τ) is **compact** if, for every open covering $\{\ U_i \mid i \in I\}$ there exists a finite subset $J \subset I$ such that $\{\ U_i \mid i \in J\}$ is also a covering of X, i.e. every open covering has a finite subcovering.

The above definition is not very intuitive. For metric spaces (in particular Euclidean spaces $\mathbb{R}^n$) the situation is a bit better.

Definition: Let (X, d) be a metric space. We say that it is **bounded**, if there is a radius $R > 0$ and a (center) point $x_0 \in X$ such that $d(x_0, x) < R \; \forall x \in X$. Similarly, a subspace $S \subset X$ is bounded, if $\exists R > 0, x_0 \in S$ such that $d(x_0, x) < R \; \forall x \in S$. (The space X then needs not be bounded.)
Next, we find some help in identifying compact spaces from the following theorem:

Theorem 4.1 (Heine-Borel Theorem) *A subspace S of an Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

In more general metric spaces, the situation is more complicated. Compactness implies closedness and boundedness, but the converse is not necessarily true. A counterexample is for example the metric space of rational numbers $\mathbb{Q}$ (with the Euclidean metric). If the converse statement (closedness and boundedness implies compactness) is true, the metric space is said to have the Heine-Borel property.

Let us now return to topological spaces.

Definition: X is **connected** if it cannot be written as $X = X_1 \bigcup X_2$, with X_1, X_2 both open, nonempty and disjoint, i.e. $X_1 \bigcap X_2 = \emptyset$.

Up to now, we have defined the following examples of topological invariants, *i.e.* quantities or properties invariant under homeomorphisms:

1. Connectedness
2. Compactness
3. Hausdorff

Euler characteristic (additional material, skip for now):
Let $X \subset \mathbb{R}^3$, $X \approx$ polyhedron K. (monitahokas)
Euler characteristic:

$$\chi(X) = \chi(K) = \text{(\# vertices in } K) - (\text{\# edges in } K) + (\text{\# faces in } K)$$
$$(= K\text{:n kärkien lkm.} - K\text{:n sivujen lkm.} + K\text{:n tahkojen lkm.})$$

Example: $\chi(T^2) = 16 - 32 + 16 = 0$.
$\chi(S^2) = \chi(\text{cube}) = 8 - 12 + 6 = 2$.