Next we construct an $N \times N$ matrix $M(\sigma)$ by taking its rows to be the permuted unit vectors:

$$M(\sigma) = \begin{pmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ \vdots \\ e_{\sigma(N)} \end{pmatrix} \Rightarrow [M(\sigma)]_{ij} = \delta_{\sigma(i),j}$$

Thus in every row and in every column there is precisely one element 1, all other elements are 0.

Example: consider $\sigma = (1234) \in S_4$

$$M(\sigma) = \begin{pmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ e_{\sigma(3)} \\ e_{\sigma(4)} \end{pmatrix} = \begin{pmatrix} e_2 \\ e_3 \\ e_4 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}.$$

The matrices $M(\sigma)$ are called <u>permutation matrices</u>. When they act on vectors of $\mathbb{R}^N$, they permute its components by σ:

$$M(\sigma)\vec{v} = \begin{pmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ \vdots \\ e_{\sigma(N)} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} v_{\sigma(1)} \\ \vdots \\ v_{\sigma(N)} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \\ v_4 \\ v_1 \end{pmatrix}.$$

Finally, the composition of permutations maps to the product of matrices, but we have to be careful about the order...

next →

Let us consider an example:

Let $\pi = (12) \in S_3$, $\sigma = (23) \in S_3$

The corresponding matrices are:

$$M(\pi) = \begin{pmatrix} e_{\pi(1)} \\ e_{\pi(2)} \\ e_{\pi(3)} \end{pmatrix} = \begin{pmatrix} e_2 \\ e_1 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$$

$$M(\sigma) = \begin{pmatrix} e_1 \\ e_3 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$$

Matrix product:

$$M(\pi)\,M(\sigma) = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_1 \\ e_2 \end{pmatrix} = \underline{\underline{M((132))}}$$

$$M(\pi)\,M(\sigma)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_1 \\ v_2 \end{pmatrix} = M(132)\begin{pmatrix} v_3 \\ v_2 \\ v_1 \end{pmatrix}$$

or: $$= \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_3 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_1 \\ v_2 \end{pmatrix}$$

but:

$$\pi \circ \sigma = (12)(23) = \begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}\begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix} = \begin{pmatrix} 1&3&2 \\ 2&3&1 \end{pmatrix}\begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix} = \begin{pmatrix} 1&2&3 \\ 2&3&1 \end{pmatrix} = \underline{\underline{(123)}}$$

So we get a different result! However,

$$\sigma \circ \pi = (23)(12) = \begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix}\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix} = \begin{pmatrix} 2&1&3 \\ 3&1&2 \end{pmatrix}\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix} = \begin{pmatrix} 1&2&3 \\ 3&1&2 \end{pmatrix} = (132)$$

Thus, $M(\pi)\,M(\sigma) = M(\sigma \circ \pi)$. Order of products is reversed. However, there is an alternative definition.

Define permutation matrices P_σ :

$$\sigma \mapsto P_\sigma \equiv M(\sigma)^T \qquad \text{(transpose of } M(\sigma)\text{)}$$

Thus $(P_\sigma)_{ij} = M(\sigma)_{ji} = \delta_{\sigma(j),i} = \delta_{i,\sigma(j)}$

Then $P_{\sigma\circ\pi} = [M(\sigma\circ\pi)]^T = [M(\pi)M(\sigma)]^T$

$$= M(\sigma)^T M(\pi)^T = P_\sigma P_\pi .$$

In other words, this definition preserves the order of products.

Example:

$$\pi = (12) \mapsto P_\pi = M(\pi)^T = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}^T = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}$$

$$\sigma = (23) \mapsto P_\sigma = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}^T = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$$

$$P_\sigma P_\pi = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix} = \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}^T = \begin{pmatrix} e_3\\ e_1\\ e_2 \end{pmatrix}^T$$

$$= M(132)^T = P_{(132)} = P_{\sigma\circ\pi} \qquad \underline{\text{OK.}}$$

[Note: $P_\sigma = (e^T_{\sigma(1)}\ e^T_{\sigma(2)}\ e^T_{\sigma(3)})$ with
also

$$e_1^T = \begin{pmatrix} 1\\0\\0 \end{pmatrix} \quad e_2^T = \begin{pmatrix} 0\\1\\0 \end{pmatrix} \quad e_3^T = \begin{pmatrix} 0\\0\\1 \end{pmatrix} \quad]$$