EVENTS AND PROBABILITY

Lecture 1B

See Book Chapter 3

Probabilistic/Statistical View of Data

- As we saw earlier, we can view the data in Table 1 as a collection of column *random* variables using the following mapping:
  - {Age. Gender, Education, Credit Score, Income, Net Worth, Sales} → {X_1, X_2, X_3, X_4, X_5, X_6, X_7}.
- A random variable is a numerical representation of an **event** and its **probability**.
  - For example, Income is a random variable and the values it takes follow a probability distribution
- BUT, first, we have to understand *Events* and *Probability*

Table 1

ID	Name	Age	Gender	Education	Credit Score	Income	Net Worth	Sales
001	Adams, John	36	M	HS	350	38,900	65,924	1,535
002	Ramesh, Jyoti	23	F	Bachelors	600	172,000	178,154	2,196
003	Mendez, Nick	67	M	Bachelors	700	218,000	265,209	1,287
004	Mendez, Joan	38	F	PhD	550	182,000	85,277	2,143
005	Ritter, Jake	24	M	Masters	625	434,000	193,760	707
006	Rao, Eric	61	M	PhD	770	82,000	314,953	2,170
007	Blake, Ann	26	F	HS	490	112,000	192,946	1,229
008	Bishop, Marge	44	F	Masters	540	242,000	339,705	520
009	Ahmed, Mo	31	M	Masters	680	111,000	185,767	2,326
010	Shultz, Dante	44	M	Bachelors	280	66,000	97,778	588

Outcomes and Experiment

- Consider the **casting of 2 dice,** a red one and an yellow one. We generally call this an *experiment*.
- Every experiment has an outcome of interest. In our case, the outcome is the two numbers on the face of the dice.
- Enumerating all possible outcomes in an experiment is called *exhaustive enumeration* and results in the *sample space* of the experiment.
- The *sample space* of this experiment has 36 possible *outcomes* summarized in the first table.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Outcomes, Events and Complement of Events

- Experiment: Toss one die and look at the number on the face.

- There are six (mutually exclusive) outcomes which make up the set {1,2,3,4,5,6}
- An **event** is a set (collection) of one or more outcomes from the sample space
  - Event **A** = "An even numbered outcome from the toss of a single die" {2,4,6}
  - Event **B** = "An odd numbered outcome from the toss of a single die" {1,3,5}
- **Complement** of an event is the set of all outcomes in the sample space that do not belong to that event i.e., all the outcomes in the sample space that are ***not in*** **A.** It is often denoted as $\mathbf{A^c}$**.**
  - Complement of Event **A** = $\mathbf{A^c}$ = "An odd numbered outcome" {1,3,5}; That is: $\mathbf{A^c}$ = **B**
  - Complement of Event **B** = $\mathbf{B^c}$ = "An even numbered outcome" {2,4,6}; That is: $\mathbf{B^c}$ = **A**
- Two events **A** and **B** are **mutually exclusive** events if they do not share any outcomes
  - Events **A** and **B** defined above are mutually exclusive (they don't have any outcomes in common)
- By definition, an **event and its complement** are mutually exclusive. i.e., **A** = $\mathbf{A^c}$ are mutually exclusive
- Events need not always be mutually exclusive
  - <u>Example</u>: In tossing **two** dice,:
  - **Event A** = {(*2,1*), (4,4), (*5,5*), (3,6)}
  - **Event B** = {(*2,1*), (3,3), (*5,5*), (2,6), (1,6), (6,2)}.
  - Notice that the two events have the italicized outcomes in common. **A** and **B** are NOT mutually exclusive.

Combining Events

- Events or outcomes may be combined to create other events
- **"OR Event"** - The event "**A OR B**" is denoted as "**A U B**" and consists of outcomes that are in one, or the other, or both. Duplicates are removed
  - **A**: "Set of even outcomes" {2, 4, 6};
  - **B**: "Set of outcomes greater than 3" {4, 5, 6}
  - Event **C** = **A U B** = "Set of even outcomes **OR** outcomes greater than 3"; {2, 4, 5, 6}
- "**AND** Event" - The event "**A AND B**" is denoted as "**A ∩ B**" and consists of outcomes common to both.
  - **A**: "Set of even outcomes" {2, 4, 6};
  - **B**: "Set of outcomes greater than 3" {4, 5, 6}
  - Event **D** = **A ∩ B** = "Set of even outcomes **AND** outcomes greater than 3"; {4, 6}

Outcomes

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Two Notions of Probability – Equally Likely Outcomes

- Consider the **casting of 2 dice,** a red one and an yellow one. We generally call this an *experiment*.
- The outcomes are shown in the first table.
- Outcomes have probabilities (of occurring).
- If we assume *Equally Likely* Outcomes, since there 36 outcomes in this *sample space*, the probability of any outcome occurring (such as **2**, **1**) = 1/36.
- This is shown in the second table.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Two Notions of Probability – Relative Frequency

- Rather than deriving probability based on exhaustive enumeration of equally likely outcomes (sample space), probability can also be understood as the result of *repeatable experiments*.
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	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

- We can view the probability numbers in the above table as the result of long-term observations *over thousands of repeated casting of the two dice* and observing the *relative frequency* of outcomes. This is a more convenient and preferred view for this course.
- The **law of large numbers** states that as the number of repetitions of an experiment is increased, the relative frequency obtained in the experiment tends to become closer and closer to the theoretical probability.

Probability Axioms

- An axiom is “a statement or proposition that is regarded as being established, accepted, or self-evidently true”. Axioms are used to derive more advanced results through, for example, deductive logic

- Let ***F*** denote the set of all possible outcomes (such as the table for the roll of two dice). That is, it denotes the sample space or event space. Let **E** denote any particular outcome. Then, all probability results are built on the following three axioms:
   1. The probability of an outcome is a non-negative real number: i.e., P (**E**) ≥ 0.
   2. The probability that at least one of the outcomes in the entire sample space will occur is 1 **(or)** the sum of probabilities of all possible outcomes in the sample space always equals 1.
   3. The probability of the union of outcomes $\mathbf{E_1, E_2, \ldots, E_n}$ denoted as $P(\bigcup_{I=1}^{n} \mathbf{E_i}) = P(\mathbf{E_1}) + P(\mathbf{E_2}) +, \ldots, + P(\mathbf{E_n})$.
      1. The outcomes are *mutually exclusive*; if one outcome occurs, the others cannot occur.

Outcomes

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Probabilities

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Calculating Probability of Events

- The probability of a null event i.e., an event with *no outcomes* P(φ) = 0.
- The probability of the event with *all outcomes* i.e., the whole sample space = 1
- The probability of an event is the *sum of the probabilities of all outcomes* belonging to the event
  - **A**: "Set of even outcomes" {2, 4, 6}; **B**: "Set of outcomes greater than 3" {4, 5, 6}
  - P(**A**) = P{2} + P{4} + P{6} = 1/6 + 1/6 + 1/6 = 1/2
  - P(**B**) = P{4} + P{5} + P{6} = 1/6 + 1/6 + 1/6 = 1/2
- "**Addition** Rule" (from the definition of the **OR** Event): P(**A U B**) = P(**A**) + P(**B**) - P(**A ∩ B**)
  - Event **C** = **A U B** = "Set of even outcomes **OR** outcomes greater than 3"; {2, 4, 5, 6}
  - Event **D** = **A ∩ B** = "Set of even outcomes **AND** outcomes greater than 3"; {4, 6}
  - P(**D**) = P(**A ∩ B**) = 1/3
  - P(**C**) = P(**A U B**) = 2/3 = P(**A**) + P(**B**) - P(**A ∩ B**) = 1/2 + 1/2 – 1/3 = 2/3
- When **A** and **B** are **mutually exclusive**, P(**A ∩ B**) i.e., P(φ)= 0 so that P(**A U B**) = P(**A**) + P(**B**)

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Conditional Probability of Events

- **Conditional Probability** – The conditional probability **P(A|B)** is the probability of observing event **A** after observing event **B**.
- Going back to the two dice problem, given:
  - Event **A** is "all outcomes where the red die is even and the yellow die is > 4)
    A = {(2,5), (2,6), (4,5), (4,6), (6,5), (6,6)}
  - Event **B** is "all outcomes where the sum of the faces of the two dice is 8"
    B = {(2,6), (6,2), (3, 5), (5,3), (4,4)}
  - The conditioning on **B** reduces the *sample space* for **A** to that of **B** i.e., {(**2,6**), (6,2), (3, 5), (5,3), (4,4)}.

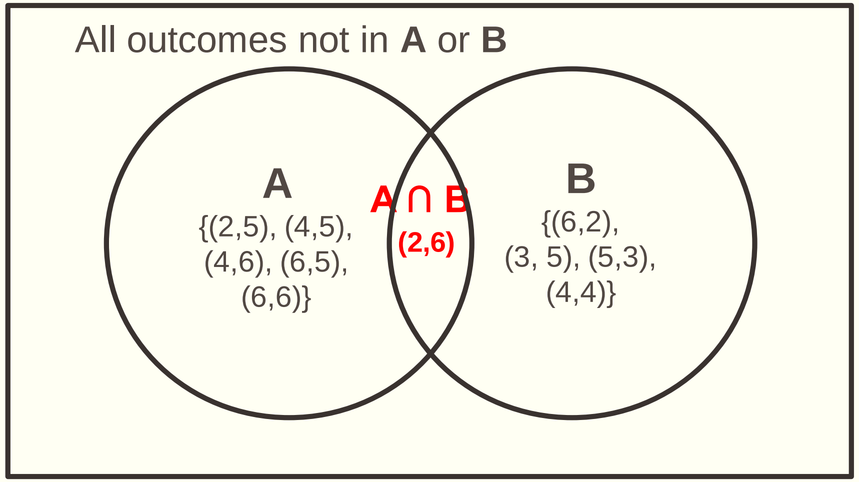
    Out of these, only one outcome (2,6) belongs to **A** so that **P(A|B)** = 1/5.
- The general formula, therefore, is
  - P(# outcomes belonging to both **A** and **B** / # outcomes in **B**) i.e.,
  - P(**A|B**) = P(**A ∩ B**)/P(**B**).
- **Multiplication** Rule" – Because P**(A|B)** = P(**A ∩ B**)/P(**B**), we have
  - P(**A ∩ B**) = P**(A|B)**)*P(**B**)

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Calculating Probability of Events – Venn Diagram

- Venn diagrams are useful when you can enumerate outcomes of two events.
- Event **A** is “all outcomes where the red die is even and the yellow die is > 4)
  - i.e., **A** = {(2,5), (2,6), (4,5), (4,6), (6,5), (6,6)} and
- Event **B** is “all outcomes where the sum of the faces of the two dice is 8”
  - i.e., **B** = {(2,6), (6,2), (3, 5), (5,3), (4,4)}
- **Probabilities**:
  - P(**A ∩ B**) = 1 / 36
  - P(**A U B**) = P(**A**) + P(**B**) - P(**A ∩ B**) = 6/36 + 5/36 – 1/36 = 10/36
  - P**(A|B**) = P(**A ∩ B**) / P(**B**) = 1/36 / 5/36 = 1/5 = 0.2



Sample Space = 36 outcomes

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Independent and Mutually Exclusive events

- Two events **A** and **B** are **independent** if the knowledge that one occurred does not affect the *probability* that the other occurs.
  - For example, the outcomes of two rolls of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. If two events are NOT independent, then we say that they are **dependent**.
- This means that P(**A**) knowing that event **B** has occurred is the same as it originally was.
- That is, P**(A|B)** = P(**A**).
- This also means that = P(**A**) = P**(A|B)** = P(**A ∩ B**)/P(**B**),
  - Thus, for ***independent events*** P(A ∩ B) = P(A)*P(B)
- Recall that ***mutually exclusive events*** have no outcomes in common. That is, **A ∩ B** = φ, the null set. A consequence of the second axiom of probability is that P(φ) = 0.
  - Thus, for ***mutually exclusive events***, P(**A ∩ B**) = 0 which means that P(**A|B**) = 0.

Complementary Events

- Remember: The *complement* of an event means that the event *did not occur*. The complement of event **A** is denoted as **A**^{c}.
- Going back to the two dice problem, given:
  - Event **A** is “all outcomes where the red die is even and the yellow die is > 4)
    - i.e., **A** = {(2,5), (2,6), (4,5), (4,6), (6,5), (6,6)}
  - So **A**^{c} = All the other 30 outcomes that do not include the outcomes in A
  - Event **B** is “all outcomes where the sum of the faces of the two dice is 8”
    - i.e., **B** = {(2,6), (6,2), (3, 5), (5,3), (4,4)}
  - So, **B**^{c} = All the other 31 outcomes that do not include the outcomes in B

A^c

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	~~(2,5)~~	~~(2,6)~~
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	~~(4,5)~~	~~(4,6)~~
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	~~(6,5)~~	~~(6,6)~~

B^c

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	~~(2,6)~~
(3,1)	(3,2)	(3,3)	(3,4)	~~(3,5)~~	(3,6)
(4,1)	(4,2)	(4,3)	~~(4,4)~~	(4,5)	(4,6)
(5,1)	(5,2)	~~(5,3)~~	(5,4)	(5,5)	(5,6)
(6,1)	~~(6,2)~~	(6,3)	(6,4)	(6,5)	(6,6)

Probability of Complementary Events

- **A** and A^c are mutually exclusive i.e., $P(\mathbf{A} \cap \mathbf{A^c}) = 0$ i.e., if one occurs, the other does not occur.

- $P(\mathbf{A} \cup \mathbf{A^c}) = P(\mathbf{A}) + P(\mathbf{A^c}) - P(\mathbf{A} \cap \mathbf{A^c}) = P(\mathbf{A}) + P(\mathbf{A^c}) = 1$ i.e., either **A** or its complement always occurs. So, $P(\mathbf{A^c}) = 1 - P(\mathbf{A})$
  - So, Probability ("all outcomes where the red die is not even and the yellow die is not > 4) = 1 – 6/36 = 30/36.

- $P(\mathbf{A} \cap \mathbf{B}) = 1 / 36$ (only the outcome (2,6)) so $P(\mathbf{A} \cap \mathbf{B})^c = 35/36$. This is also, $P(\mathbf{A^c} \cup \mathbf{B^c})$ i.e., probability of outcomes that are not in **A** or not in **B**.
  - i.e., $P(\mathbf{A} \cap \mathbf{B})^c = P(\mathbf{A^c} \cup \mathbf{B^c})$

- Similarly, $P(\mathbf{A} \cup \mathbf{B}) = 10/36$, so $P(\mathbf{A} \cup \mathbf{B})^c = 26/36 = P(\mathbf{A^c} \cap \mathbf{B^c})$ i.e., probability of outcomes that are not in **A** and not in **B.**
  - i.e., $P(\mathbf{A} \cup \mathbf{B})^c = P(\mathbf{A^c} \cap \mathbf{B^c})$

- Also, $P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A} \cap \mathbf{B^c}) = 1/36$ {only outcome (2,6)} + 5/36 {other outcomes in A namely, (2,5), (4,5), (4,6), (6,5), (6,6)} = 6/36 = 1/6 = $P(\mathbf{A})$. Note that $(\mathbf{A} \cap \mathbf{B})$ and $(\mathbf{A} \cap \mathbf{B^c})$ are mutually exclusive.

A^c

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	~~(2,5)~~	~~(2,6)~~
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	~~(4,5)~~	~~(4,6)~~
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	~~(6,5)~~	~~(6,6)~~

B^c

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	~~(2,6)~~
(3,1)	(3,2)	(3,3)	(3,4)	~~(3,5)~~	(3,6)
(4,1)	(4,2)	(4,3)	~~(4,4)~~	(4,5)	(4,6)
(5,1)	(5,2)	~~(5,3)~~	(5,4)	(5,5)	(5,6)
(6,1)	~~(6,2)~~	(6,3)	(6,4)	(6,5)	(6,6)

Summary of Complementary Event Probabilities

- $P(\mathbf{A} \cap \mathbf{A}^c) = 0$
- $P(\mathbf{A} \cup \mathbf{A}^c) = 1$
- $P(\mathbf{A}^c) = 1 - P(\mathbf{A})$
- $P(\mathbf{A} \cap \mathbf{B})^c = 1 - P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}^c \cup \mathbf{B}^c)$
- $P(\mathbf{A} \cup \mathbf{B})^c = 1 - P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}^c \cap \mathbf{B}^c)$
- $P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A})$ similarly $P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A}^c \cap \mathbf{B}) = P(\mathbf{B})$

Recommended Calculator – TI 30X IIS

- Youtube – Calculating factorials using TI 30X IIS (https://www.youtube.com/watch?v=0zJ-fT_ckHY)
- Youtube – Calculating Combinations and Permutations using TI 30X IIS (https://www.youtube.com/watch?v=wV9IE8cGuWM)
- Youtube – Calculating Binomial Probabilities using TI 30X IIS (https://www.youtube.com/watch?v=UYgsvTM9tBA)
- Youtube – Calculating Mean and Standard Deviation using TI 30X IIS(https://www.youtube.com/watch?v=31iAcHjq_G0)

- You will need the calculator for doing problems on the exam. Familiarize yourself fully with its operations as you do assignments.
- On the overheads, when you see Practice with calculator  it means that you should use your calculator to replicate the results and that you will have such problems in the exam for which you need your calculator

Practice with calculator

Calculating Probability of Events

- Carlos plays college soccer. He makes a goal 65% of the time he shoots. Carlos is going to attempt two goals in a row in the next game. *A* = the event Carlos is successful on his first attempt. *P*(*A*) = 0.65. *B* = the event Carlos is successful on his second attempt. *P*(*B*) = 0.65. Carlos tends to shoot in streaks. The probability that he makes the second goal **GIVEN** that he made the first goal is 0.90.

- So: Outcomes in Sample Space are:
  - Goal on First Attempt and Goal on Second Attempt (A **AND** B) or (A ∩ B)
  - Goal on First Attempt and No Goal on second attempt (A **AND** B^C) or (A ∩ B^C)
  - No Goal on First attempt and Goal on Second Attempt (A^C **AND** B) or (A^C ∩ B)
  - No Goal on First attempt and No Goal on second attempt (A^C **AND** B^C) or (A^C ∩ B^C)

- Given:
  - P(A) = 0.65; P(B) = 0.65; P(B|A) = 0.90

- a. What is the probability that he makes both goals?
  - The problem is asking you to find P(A **AND** B) = P(A ∩ B).
  - Since P(B|A) = 0.90: P(B ∩ A) = P(B|A)*P(A) = (0.90)(0.65) = 0.585.
  - Carlos makes a goal on the first attempt **AND** makes a goal on the second attempt with probability 0.585.

- b. What is the probability that Carlos makes *either* the first goal or the second goal?
  - The problem is asking you to find P(A **OR** B) = P(A **U** B)
  - P(A **U** B) = P(A) + P(B) - P(A ∩ B) = 0.65 + 0.65 - 0.585 = 0.715.
  - Carlos makes a goal either on the first or the second attempt with probability 0.715.

Practice with calculator

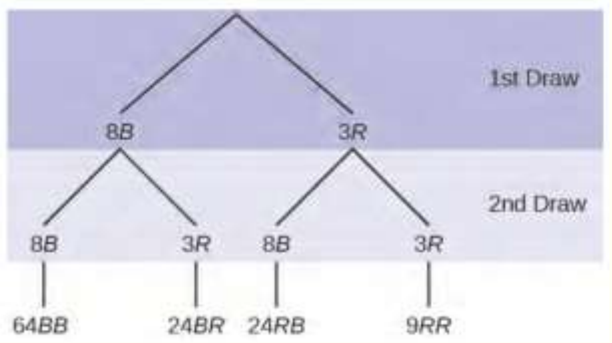
Calculating Probability of Events

- c. What is the probability that he **does not** score either on the first or second attempt?
  - This is $P(A^c \cap B^c) = P(\mathbf{A} \cup \mathbf{B})^c = 1 - 0.715 = 0.285$
  - So, the probability that Carlos does not make a goal on either kick = 1 - 0.715 = 0.285.
- d. Are A and B independent?
  - No, they are not, because $P(B \cap A) = 0.585$.
  - $P(B)P(A) = (0.65)(0.65) = 0.423$ and $0.423 \neq 0.585 = P(B \cap A)$
  - So, $P(B \cap A)$ is **not** equal to $P(B)P(A)$.
- e. Are A and B mutually exclusive?
  - No, they are not because P(A and B) = 0.585. To be mutually exclusive, $P(A \cap B)$ must equal zero.
- So, for the 4 outcomes:
  - $P(A \cap B) = 0.585$
  - $P(A \cap B^C) = ?$; Remember, $P(A) = P(A \cap B) + P(A \cap B^C)$; So, $P(A \cap B^C) = P(A) - P(A \cap B) = 0.65 - 0.585 = 0.065$
  - $P(A^C \cap B^C) = P(\mathbf{A} \cup \mathbf{B})^c = 1 - P(\mathbf{A} \cup \mathbf{B}) = 1 - 0.715 = 0.285$
  - $P(A^C \cap B) = P(B) - P(A \cap B) = 0.65 - 0.585 = 0.065$

- $P(\mathbf{A} \cap \mathbf{A^c}) = 0$
- $P(\mathbf{A} \cup \mathbf{A^c}) = 1$
- $P(\mathbf{A^c}) = 1 - P(\mathbf{A})$
- $P(\mathbf{A} \cap \mathbf{B})^c = 1 - P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A^c} \cup \mathbf{B^c})$
- $P(\mathbf{A} \cup \mathbf{B})^c = 1 - P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A^c} \cap \mathbf{B^c})$
- $P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A} \cap \mathbf{B^c}) = P(\mathbf{A})$ similarly $P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A^c} \cap \mathbf{B}) = P(\mathbf{B})$

Calculating Probability of Events – Tree Diagrams

- In an urn, there are 11 balls. Three balls are red (*R*) and eight balls are blue (*B*). Draw two balls, one at a time, **with replacement**. "With replacement" means that you put the first ball back in the urn before you select the second ball. The tree diagram using frequencies that show all the possible outcomes follows.
- The first set of branches represents the first draw.
- The second set of branches represents the second draw.
- Each of the outcomes is distinct.
- In fact, we can list each red ball as *R*1, *R*2, and *R*3 and each blue ball as *B*1, *B*2, *B*3, *B*4,*B*5, *B*6, *B*7, and *B*8. Then the nine *RR* outcomes can be written as:
  - *R*1*R*1; *R*1*R*2; *R*1*R*3; *R*2*R*1; *R*2*R*2; *R*2*R*3; *R*3*R*1; *R*3*R*2; *R*3*R*3
- The other outcomes are similar.
- There are a total of 11 balls in the urn. Draw two balls, one at a time, with replacement. There are 11(11) = 121 outcomes, the size of the **sample space**.



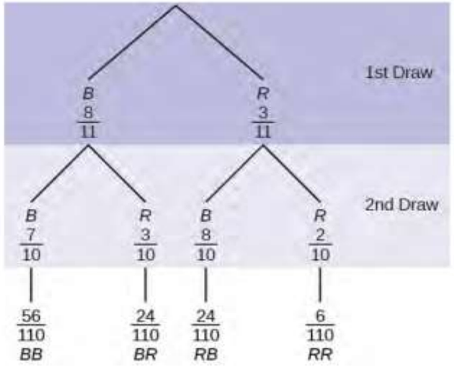
Probabilities:

- P(RR) = 9/121
- P(RB OR BR) = (3/11)(8/11) + (8/11)(3/11) = 48/121
- P(R on 1st draw AND B on 2nd draw) = (3/11)(8/11) = 24/121
- P(R on 2nd draw GIVEN B on 1st draw) = P(R on 2nd|B on 1st) = 24/88 = 3/11

Practice with calculator

Calculating Probability of Events – Tree Diagrams

- An urn has three red marbles and eight blue marbles in it. Draw two marbles, one at a time, this time without replacement, from the urn. **"Without replacement"** means that you do not put the first ball back before you select the second marble.
- The tree diagram for this situation is shown.
  - The branches are labeled with probabilities instead of frequencies.
  - The numbers at the ends of the branches are calculated by multiplying the numbers on the two corresponding branches, for example, (3/11)(2/10) = 6/110
- **Note:** If you draw a red on the first draw from the three red possibilities, there are two red marbles left to draw on the second draw. You do not put back or replace the first marble after you have drawn it. You draw **without replacement**, so that on the second draw there are ten marbles left in the urn.



Probabilities:

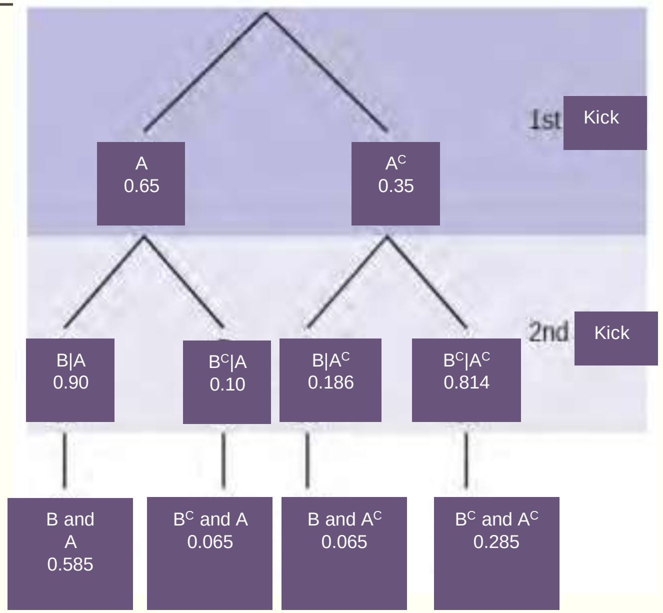
- P(RR) = (3/11)(2/10) = 6/110
- P(RB OR BR) = (3/11)(8/10) + (8/11)(3/10) = 48/110
- P(R on 1st draw AND B on 2nd draw) = (3/11)(8/10) = 24/110
- P(R on 2nd draw GIVEN B on 1st draw) = P(R on 2nd|B on 1st) = 3/10

Practice with calculator

Calculating Probability of Events – Tree Diagrams

- The Carlos Goal Problem: A = the event Carlos is successful on his first attempt. $P(A) = 0.65$. B = the event Carlos is successful on his second attempt. $P(B) = 0.65$. The probability that he makes the second goal **GIVEN** that he made the first goal is 0.90.

- For the 4 outcomes:
  - $P(A, B) = 0.585$
  - $P(A, B^C) = 0.65*0.1 = 0.065$
  - $P(A^C, B^C) = 1 – P(A \text{ OR } B) = 0.285$; So,
    - $P(B^C|A^C) = 0.285/0.35 = 0.814$;
    - $P(B|A^C) = 0.186$
  - $P(A^C, B) = P(B^C).P(A|\ B^C)$
  - $= 1 – (0.585 + 0.065 + 0.285) = 0.065$ or
  - $= P(B|A^C)*P(A^C) = 0.186*0.35 = 0.065$



Contingency Table

- The probability that the outcomes or events (or measurements) **A and B** occur together i.e., P(**A ∩ B**), is also called their *joint probability*.
  - For example, in the two dice example, the probability that the “Red die shows greater than 5” and the “Yellow die shows an even number” is their joint probability.

- In many situations we can show the outcomes from an experiment involving *two different measurements* in the form of a **contingency table** of joint probabilities that enables us to calculate **conditional probabilities**.

- Suppose a study of speeding violations and drivers who use cell phones produced the following fictional data for the last year:

	Speeding violation	**No speeding violation**	**Total**
Cell phone user	25	280	305
Not a cell phone user	45	405	450
Total	70	685	755

Practice with calculator

Contingency Table

- We convert the data to a contingency table of joint probabilities for the measurement “C = Cell phone use” with outcomes “User” and “Not User” and S = “Speeding Violation” with outcomes “Speeding violation” and “No Speeding violation” by diving all cells by the Total (755)

	Speeding violation (S)	**No speeding violation (S^c)**	**Total**
Cell phone user (C)	25/755 = **0.033**	280/755 = **0.371**	305/755 = 0.404
Not cell phone user (C^c)	45/755 = **0.060**	405/755 = **0.536**	450/755 = 0.596
Total	70/755 = 0.093	685/755 = 0.907	755/755 = 1

- From the table, the *joint probabilities* are:
  - P(C = “User” **and** S = “Speeding”) = 0.033; P(C = “User” **and** S^c = “Not Speeding”) = 0.371
  - P(C^c = “Not User” **and** S = “Speeding”) = 0.060; P(C^c = “Not User” **and** S^c = “Not Speeding”) = 0.536

- From the table, the *marginal probabilities (because they are in the margins of the table)* are:
  - P(S = “**Speeding**”) = P(C = “User” and S = “Speeding”) + P(C^c = “Not User” and S = “Speeding”) = **0.093**
  - P(S^c = “**Not Speeding**”) = P(C = “User” and S^c = “Not Speeding”) + P(C^c = “Not User” and S^c = “Not Speeding”) = **0.907**
  - P(C = “**User**”) = P(C = “User” and S = “Speeding”) + P(C = “User” and S^c = “Not Speeding”) = **0.404**
  - P(C^c = “**Not User**”) = P(C^c = “Not User” and S = “Speeding”) + P(C^c = “Not User” and S^c = “Not Speeding”) = **0.596**

Practice with calculator

Contingency Table

- We can use the contingency table to calculate conditional probabilities and determine whether the two measurements are independent or not
- **Given** that a person does not have a speeding violation, what is the probability that the person is a cell phone user?
  - We need P(C = “User”) | P(S = “Not Speeding”)
  - = P(C = “User” **and** S = “Not Speeding”) / P(S = “Not Speeding”) = **0.371 / 0.907 = 0.409**
- **Given** that a person was not a cell phone user, what is the probability that they have a speeding violation?
  - = 0.06 / 0.596 = 0.1
- Is speeding violation independent of cell phone use?
  - If speeding violation was independent of cell use, *then <u>every</u> joint probability in the table will be a product of the respective marginals*
  - Example: P(C = “User” **and** S = “Speeding”) must be = P(C = “User”).P(S = “Speeding”)

Practice with calculator

Contingency Table

- Actual Table in terms of Conditional and Marginal Probabilities

	Speeding violation	No speeding violation	Total
Cell phone user	25/755 = **0.033**	280/755 = **0.371**	305/755 = 0.404
Not a cell phone user	45/755 = **0.060**	405/755 = **0.536**	450/755 = 0.596
Total	70/755 = 0.093	685/755 = 0.907	755/755 = 1

- Expected (if Independent) Table in terms of Conditional and Marginal Probabilities

	Speeding violation	No speeding violation	Total
Cell phone user	0.093*0.404 = **0.036**	0.907*0.404 = **0.366**	0.404
Not a cell phone user	0.093*0.596 = **0.055**	0.907*0.596 = **0.542**	0.596
Total	0.093	0.907	1

- This shows that they are not independent (if we consider this as a population)

Practice with calculator

Contingency Table

- Actual Table in terms of **Counts**

	Speeding violation	No speeding violation	Total
Cell phone user	25	280	305
Not a cell phone user	45	405	450
Total	70	685	755

- Expected (if Independent) Table in terms of **Counts**

	Speeding violation	No speeding violation	Total
Cell phone user	305*70/755 = **28.3**	305*685/755 = **276.7**	305
Not a cell phone user	450*70/755 = **41.7**	450*685/755 = **408.3**	450
Total	70	685	755

- This (again) shows that they are not independent (if we consider this as a population)

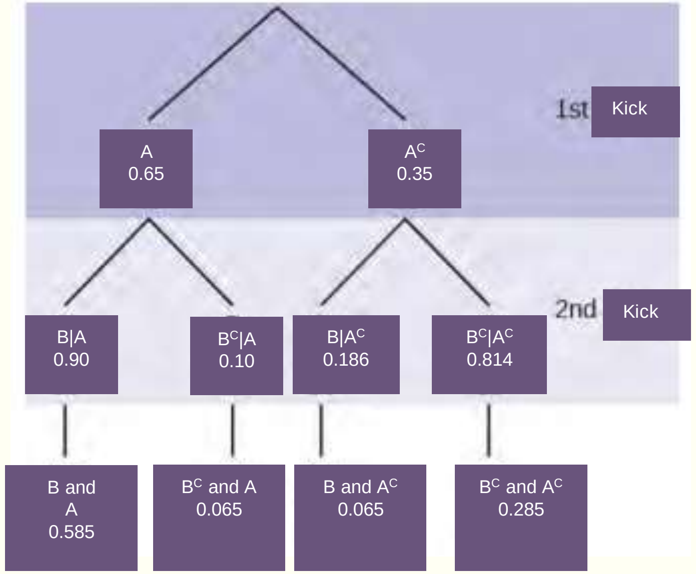
Practice with calculator

Contingency Table

- The Carlos Goal Problem: *A* = the event Carlos is successful on his first attempt. *P*(*A*) = 0.65. *B* = the event Carlos is successful on his second attempt. *P*(*B*) = 0.65.

	A	A^c	
B	0.585	0.065	0.65
$\mathbf{B^c}$	0.065	0.285	0.35
	0.65	0.35	

  - $P(A, B) = 0.585$
  - $P(A, B^C) = 0.065$
  - $P(A^C, B^C) = 0.285$;
  - $P(A^C, B) = 0.065$



- Calculate all the conditional probabilities.

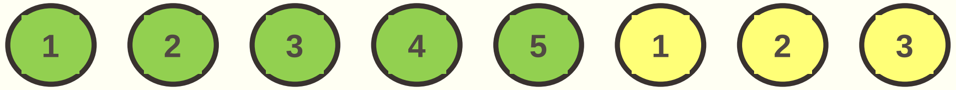
EVENTS AND PROBABILITY – PROBLEMS AND SOLUTIONS

Lecture 1B – Problems and Solutions

Practice with calculator

Book Page 220 – Problem 85

- Suppose that you have eight balls. Five are green and three are yellow. The five green balls are numbered 1, 2, 3, 4, and 5. The three yellow balls are numbered 1, 2, and 3. You randomly draw **one ball**.
  - *G* = ball drawn is green
  - *E* = ball drawn is even-numbered



a. List the sample space. – There are 8 outcomes in the sample space – G1,..,G5 and Y1,..Y3

b. *P*(*G*) = 5/8 and P(*E*) = 3/8 (Outcomes G2, G4 and Y2)

c. *P*(*G*|*E*) = Here E consists of {G2, G4, Y2}; So *P*(*G*|*E*) = 2/3 consisting of outcomes G2 and G4.

d. *P*(*G* AND *E*) = This event consists of outcomes (out of 8 possible) that have both G and even numbers in them = {G2, G4}; So, *P*(*G* AND *E*) = 2/8 = 1/4

e. *P*(*G* OR *E*) = This event consists of outcomes (out of 8 possible) that contain G OR an even number = {G1,G2,G3,G4,G5,Y2); *P*(*G* OR *E*) = 6/8 or ¾.

We can verify that *P*(*G* OR *E*) = P(G) + P(E) – P(G AND E) = 5/8 +3/8 – 2/8 = 6/8 or ¾.

f. Are *G* and *E* mutually exclusive? Justify your answer numerically.
To be mutually exclusive, P(G OR E) must be = P(G) + P(E).
But P(G) + P(E) = 8/8 = 1 but P(G OR E) = 6/8. So, they are **NOT** mutually exclusive.
Also, if they are mutually exclusive P(G AND E) = 0 which it is not.

g. Are *G* and *E* independent? Justify your answer numerically.
To be independent, P(G|E) = P(G). But P(G|E) = 2/3 and P(G) = 5/8. So they are **NOT** independent.

Practice with calculator

Book Page 222 – Problem 97

- United Blood Services is a blood bank that serves more than 500 hospitals in 18 states. According to their website, a person with type O blood **and** a negative Rh factor (Rh-) can donate blood to any person with any bloodtype. Their data show that 43% of people have type O blood and 15% of people have Rh- factor; 52% of people have type O OR Rh- factor.
  a. Find the probability that a person has both type O blood and the Rh- factor.
  b. Find the probability that a person does NOT have **both** type O blood **and** the Rh- factor.
  c. Find the probability that a person does NOT have **either** type O blood **or** the Rh- factor

- Here the data gives us
  - P(Type O) = 0.43, P(Rh-) = 0.15 and P(Type O **OR** Rh-) = 0.52

  a. We need P(Type O **AND** Rh-).
  We know P(Type O **OR** Rh-) = 0.52 = P(Type O) + P(Rh-) - P(Type O **AND** Rh-) = 0.43 +0.15 - P(Type O **AND** Rh-); 0.52 = 0.58 - P(Type O **AND** Rh-).
  So, P(Type O **AND** Rh-) = 0.06.

  b. We need P(Type O **AND** Rh-)^c = 1 - P(Type O **AND** Rh-) = 1 -0.06 = 0.94

  c. We need P(Type O **OR** Rh-)^c = 1 - P(Type O **OR** Rh-) = 1 – 0.52 = 0.48

Practice with calculator

Book Page 222 – Problem 97

- Previous Problem as a Contingency table. In the table, the cell entries are intersections (i.e., **AND**)

	Rh-	**Rh-^{c}**	
Type O	0.06	0.37	0.43
Type O^c	0.09	0.48	0.57
	0.15	0.85	1

a. We need P(Type O **AND** Rh-) = 0.06

b. We need P(Type O **AND** Rh-)^c = 1 - P(Type O **AND** Rh-) = 1 -0.06 = 0.94

c. We need P(Type O **OR** Rh-)^c = 1 - P(Type O **OR** Rh-) = 1 – P(Type O **OR** Rh-) = 1 – (0.43 +0.15 - P(Type O **AND** Rh-)) = 1 – (0.43 +0.15 – 0.06) = 0.48

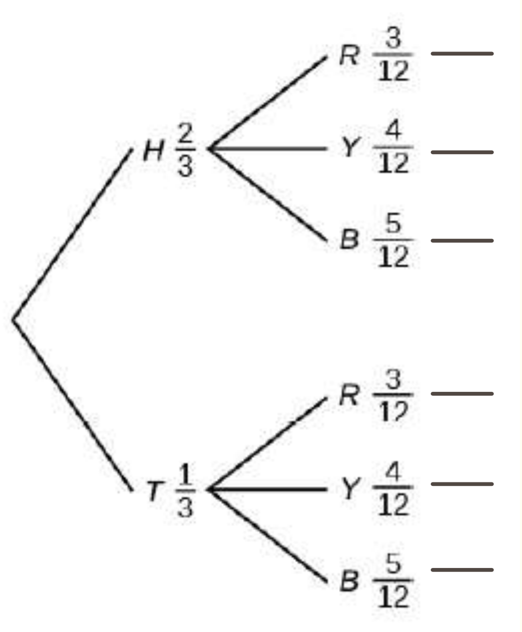
Practice with calculator

Book Page 225 - – Problems 114 and 115

- The tree diagram shows the tossing of an unfair coin, followed by drawing one bead from a cup containing three red (*R*), four yellow (*Y*) and five blue (*B*) beads. For the coin,
  - *P*(*H*) = 2/3; and *P*(*T*) = 1/3, where *H* is heads and *T* is tails.
- From the tree, we have:
  - P(R|H) = 3/12, P(Y|H) = 4/12 and P(B|H) = 5/12
  - P(R|T) = 3/12, P(Y|T) = 4/12 and P(B|T) = 5/12

a. Find *P*(tossing a Head on the coin AND a Red bead)
= P(H AND R) = P(R|H) * P(H) = (3/12)*(2/3) = 6/36 = 1/6

b. Find *P*(Blue bead).
= P(B) = P(H AND B) + P(T AND B) (see overhead 14 – Here T = H^c)
= P(B|H)*P(H) + P(B|T)*P(T)
= (5/12)*(2/3) + (5/12)*(1/3) = 5/12 (or 15/36)



In a tree diagram, the first level branches are *conditional probabilities*.

Practice with calculator

Book Page 228 - Problem 125

- The percent of licensed U.S. drivers (from a recent year) that are female is 48.60. Of the females, 5.03% are age 19 and under; 81.36% are age 20–64; 13.61% are age 65 or over. Of the licensed U.S. male drivers, 5.04% are age 19 and under; 81.43% are age 20–64; 13.53% are age 65 or over.
  a. Develop a contingency table
  b. Find P(driver is female). – Answer: 0.486
  c. Find P(driver is age 65 or over|driver is female) – Answer: 0.0661/0.486 = 0.136
  d. Find P(driver is age 65 or over AND female). – Answer: 0.0661
  e. In words, explain the difference between the probabilities in part c and part d. – Answer: c is the percentage of female drivers who are 65 or older and d) is the percentage of drivers who are female and over 65
  f. Find P(driver is age 65 or over). – Answer: 0.1356
  g. Are being age 65 or over and being female mutually exclusive events? How do you know? – Answer: To be mutually exclusive P(female **AND** 65 and older) must be 0, which it is not. Hence, not mutually exclusive. That is, they can occur at the same time.

a. Solution:

   a. Contingency Table

	<20	20–64	>64	Totals
Female	0.0244	0.3954	0.0661	0.486
Male	0.0259	0.4186	0.0695	0.514
Totals	0.0503	0.8140	0.1356	1