Module 16: Decision Trees

Reading Material: 16.1 – Introduction

I am writing this book to support my usual MBA class, so I include this decision tree module even though it does not use the Solver and is quite different from previous modules.

My class has always been focused on the linear, deterministic optimization portion of Management Science – there is another area of the field that deals with stochastic phenomenon, that is, probabilistic events. Historically, I usually take the last sliver of time to talk about probabilistic decision making, focusing on sequential decision making using decision trees. Why? Decision trees are a very useful practical tool. Additionally, covering this useful topic provides the student a necessary broad perspective about the entire field and is a great way to end the class.

There are many sources for primers on the core probability concepts; I assume all are decently comfortable with them. I want to keep this module short, focus on sequential decisions, provide some basic calculations to support the decision-making framework, and then let you attack the practice problems.

Reading Material: 16.2 – Basic Probability Concepts: Preparing for Sequential Decisions

When a fair coin is flipped, we know two things can occur – a "heads" or a "tails." Each occurs with a 50% probability. Thus, the sum of the probabilities of all the possible independent events in a coin flip that can occur is 1.

If the weather forecast says there is a 40% chance of precipitation tomorrow, the implication is that there is a 60% chance that it will not rain or snow. Hopefully, we can agree that the sum of probabilities of independent events must be = 1.

Let us return to the coin. If we flip the coin 9 times, how many heads would we expect to occur? Well, the expected number of heads would be $.5*9 = 4.5$. This is the expected value of the process. We know that if we actually flip a coin 9 times, we will never actually have 4.5 heads occur – we will have 4 heads, or 5 heads, or whatever number. But on average, if we were to replicate this silly experiment an infinite number of times, the average number of heads that would occur each time (of 9 flips) is 4.5

The concept of expected value is the normal criterion used in assessing probabilistic decisions. However, sometimes decision makers are risk-averse or risk-takers, and they will choose or make decisions using different criteria.

Here is a crazy example that illustrates this point. A stranger comes up to you on a street corner with a suitcase full of money. He says it is a million dollars. He offers you the following situation:

He gives you a $5 bill. You can keep the $5 and walk away, or you can give him back the bill as an entry fee into his peculiar game. His game: He will write down an integer number between 1 and 100. If you guess it, you win the million dollars. If you don't guess it, you have to pay him $100 (you've already given up the $5 as well).

So, if you are risk-averse, you'd likely take the $5 payoff. It is a "sure thing" (with apologies to Jon Cusack). What is the expected payoff of playing the game with the wacko? You have a 1 in 100 chance of guessing the same number, so the expected value of that portion of the possible events would be the probability of 1/100 multiplied by the payoff of $1,000,000 – which is equal to $10,000. The other event that could happen – 99/100 times you would not guess the number, and would owe the person $100. The expected value of that would be 99/100 multiplied by (–100), which would be –99. So, the net expected value of playing the game would be $10,000 – $99 or $9,901. Rational decision making using decision analysis and expected value as the criterion would indicate that you should engage the wacko and take the risk of losing $100 to potentially gain $1,000,000.

Would you do this? Would you be willing to play the game risking $100 to win $1,000,000? Many wouldn't, and would just take the Cusack winnings (the $5). But expected value tells you the best choice is to play the game.

Watch (if you can stand it) the game show *Deal or No Deal*. We play this game in class sometimes and talk about risk-aversion, risk-taking, and expected value. Of particular interest is the manner in which the banker's offer to the contestant to quit the game toward the end starts approaching the expected value of the game to him or her. It still amazes me how they can stretch out one *Deal or No Deal* game into an entire hour. Pure TV torture, but valuable probability lessons.

We could model the situation with the wacko with the suitcase (I'll let you decide if that includes Howie Mandel) using a decision tree, the main topic of this module. Decision trees are useful in a couple of ways – one, to help us paint a visual picture of a set of sequential decisions so that decision makers can think about the relevant likelihoods and possible payoffs and consequences. Obviously, the calculations of a decision tree are also important because they suggest the optimal strategy in a particular circumstance, often with contingencies attached.

In the big scheme of things, the visuality of a decision tree and it requiring decision makers to consider possible consequences and dependencies of their actions and decisions is the most value-added aspect of this technique to the decision maker. A simple example follows that will serve as our starting point in decision trees.

Reading Material: 16.3 – Only So Much Time in the Day . . .

16.3.1

Example: A diligent MBA student faces a dilemma late in their last semester in the program. Two of their classes, a finance class and a management science class have homework assignments due on the same day. There is not enough time to complete them both.

The finance faculty member has shown some previous tendencies for succumbing to pressure and has numerous times during the semester allowed extensions to homework due dates at the last minute. The management science faculty member is a crusty old ?!?##! who is as inflexible as an ice cube.

The MBA student wishes to decide which homework to focus on, based on the student's key objective. Assume the choice is based on trying to maximize their expected GPA for the semester.

Some numerical estimations to help the MBA student decide: If the student chooses to complete the finance homework (and thus, not the management science homework), there is an equal likelihood (probability) that the student will earn a 3.5 semester GPA or a 3.75 semester GPA.

If the student chooses to complete the management science homework, there is a 70% chance that the finance instructor will allow an extension. If there is an extension given for the finance homework, the student will have a 66.66% chance of earning a 3.7 GPA and a 33.33% chance of earning a 4.0 GPA. Without an extension, the assumption is that the student will earn a 3.4 GPA when all is said and done.

Determine the optimal choice for the student based on the expected GPA criterion. Would that decision be different if the person was a pessimistic/risk-averse/risk-taking decision maker?

16.3.2 Setting Up a Decision Tree to Model the Decision

Let's model this using squares/quadrilaterals to represent decisions to be made and ovals to represent events that occur with a certain probability. The only decision faced by the student is whether to do the finance homework or the management science homework. Figure 16.1 shows the picture of the initial decision.

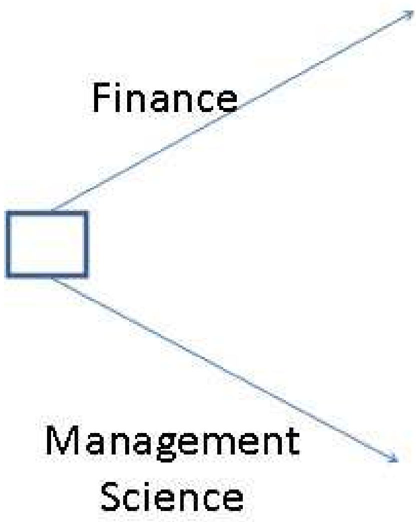


Figure 16.1

If the student chooses to do the finance homework, we reach an “event” – there is a 50% chance that the student will earn a 3.5 GPA, and a 50% chance that the student will earn a 3.75 GPA. The GPA values are the payoffs – the criteria for our decision. Figure 16.2 shows the addition of two branches from the event node illustrating what happens if finance is chosen as the initial decision. The probability of each event branch is also shown as well and, along with the payoffs associated with the ending of the tree (the leaves?).

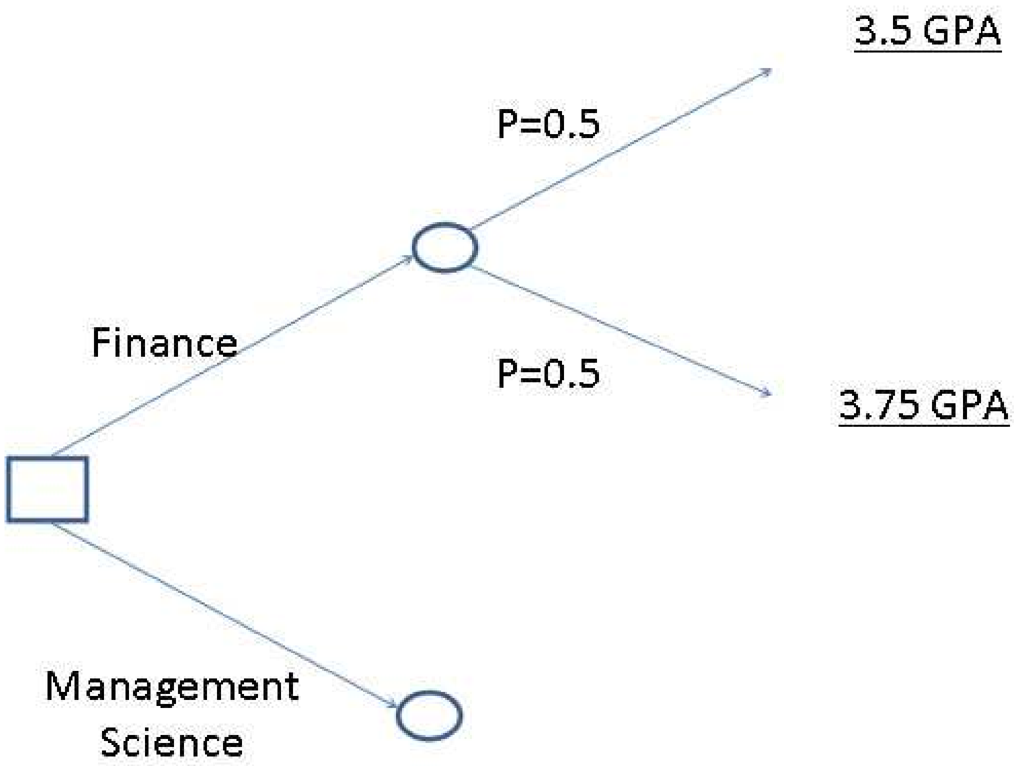


Figure 16.2

On the “choose Management Science” branch, similar branches are constructed depicting the events that could occur after the initial choice. First, an extension will happen with a 70% likelihood, but no extension occurs with a 30% chance. With the no extension branch, there is a payoff of +3.4 GPA. On the extension branch, there is a 66.67% (2/3) likelihood of achieving a payoff (GPA) of 3.7, and a 1/3 (33.33%) likelihood of achieving a payoff (GPA) of 4.0

Figure 16.3 shows the complete decision tree – one initial decision, and then a non-symmetric sequence of possible events. The event nodes are numbered for future reference.

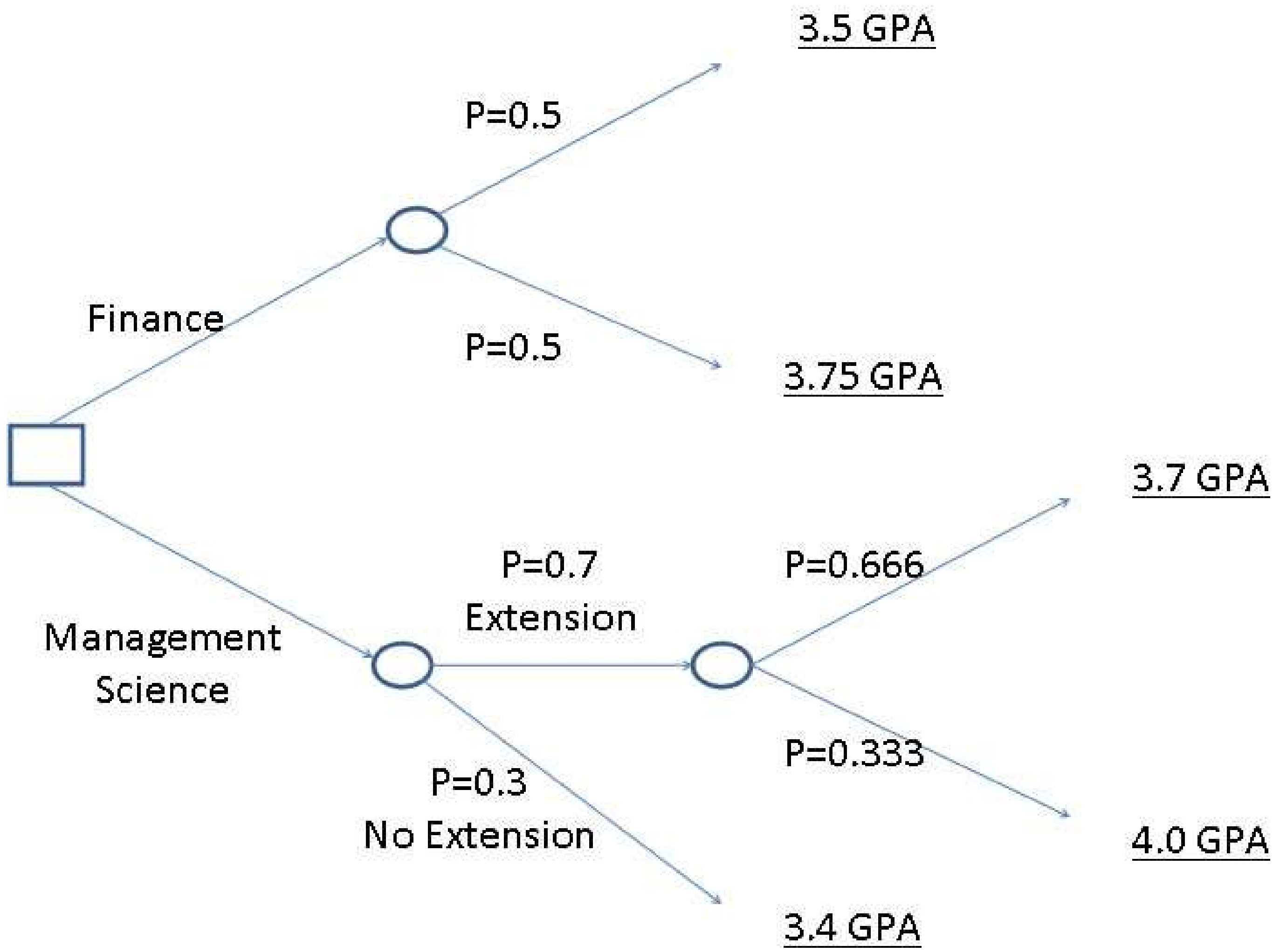


Figure 16.3

Next, the expected value will be used to determine the optimal strategy.

16.3.3 The Wal-Mart Solution Approach: Rolling Back the Expected Value

As the section title implies, we will work from the end of the tree back toward the start to determine our course of action (no pun intended). Let's look at Event Node 3.

There is a 2/3 likelihood that a payoff (GPA) of 3.7 will occur and a 1/3 likelihood that a payoff (GPA) of 4.0 will occur. Thus, the expected value of Event Node 3 is $2/3*(3.7) + 1/3*(4.0) = 3.8$. This expected value is now associated as the payoff of the branch that reaches node 3. (The expected value is 'rolled-back' to that event node).

Now the expected value for Event Node 2 can be calculated. There is a 70% likelihood that there would be an extension for the Finance homework, leading to Event Node 3. We have calculated the expected payoff of Event Node 3 as 3.8. There was a 30% likelihood that there was no extension, which lead to a payoff of 3.4. Therefore, the total expected value of Event Node 2 is found by: .7 * (3.8) + .3 * (3.4) = 3.68.

Because this is the expected value of Event Node 2, it represents the value of choosing to do the management science homework.

On the upper branch, we need to do similar analysis with Event Node 1, the event that occurs if we choose to do the finance homework. There is a 50% chance of earning a GPA of 3.5 and a 50% chance of earning a GPA of 3.75. That results in an expected GPA of 3.625.

So, these calculations are then "rolled-back" to the initial decision node. The finance branch has an expected GPA of 3.625, and the management science branch has an expected value of 3.68. The highest number wins (in the expected value criteria), so the optimal decision would be to focus on the management science homework. But you kind of anticipated that didn't you?

Note that by choosing management science, we will NOT end up with a GPA of 3.68. That value is our metric for decision making. There is actually a 30% chance of earning a 3.4, a 14/30 (46.66%) chance of earning a 3.7, and a 7/30 (23.333%) chance of earning a 4.0. If one was a pessimistic decision maker, one might opt to do the finance homework because the lowest possible GPA in that scenario was 3.5, an improvement over the lowest possible GPA by selecting to work on the management science homework. Thus, the criteria can definitely influence the decision.

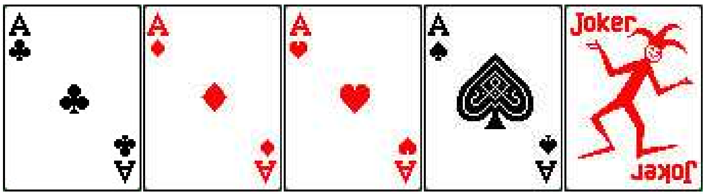
Finally, the next example is a simple game that has sequential decisions and cost involved as we broaden our exposure to decision trees.

Reading Material: 16.4 – "Some People Call Me the Space Cowboy . . ."

16.4.1 Example: "The Joker"

A popular game at a local bar that features 1970's music is called "Joker." (Legend has it that the game was invented by one of the old veteran bartenders who is rumored to be a retired management science professor from the local university).

The basic premise: It costs $1 to enter the contest. The bartender has five cards (shown below) – a joker and the aces of each suit (from a normal playing deck).



He shuffles the cards and asks you to select one.

If you select the joker, you have the following option: you can either accept $2 in winnings (and the game ends) or choose the “Triple or Nothing” path (which does NOT cost anything additional, but you do NOT win $2). In Triple or Nothing, you randomly select one of the four remaining cards. If you select the ace of spades, you win $6. If you don’t, you win nothing. In either case, the game ends.

If in your original selection you did NOT select the joker, you are now faced with another decision. You can stop playing immediately (and thus have only contributed $1 to the bartender’s retirement fund), or you can pay another $1 and enter the “second chance” portion of the game.

In the “second chance” option, the bartender reshuffles the entire five-card deck and asks you to draw again. If you draw the joker this time, you win $10 (and the game stops). But if you don’t draw the joker, the game also stops and you have contributed $2 to the unreported income of the bartender.

Starting with the initial decision to play the game or not play the game, outline your optimal strategy and any contingencies that may or may not occur.

Also answer the following two questions:

1. On average, how much would you expect to win or lose each time you play the joker?

2. What percentage of time in playing the Joker would you expect to win money?

These are of course two different questions.

16.4.2 The Tree for the Joker

The starting point for the analysis has really been given to us – do we play the game or not? That is the first decision in the decision tree. Figure 16.4 shows the entire tree that we will discuss below. Note the cost of (–1) associated with the “Play” branch. We will subtract the cost of taking that branch (decision) after the expected value of all subsequent events and decisions have been calculated. Obviously, the expected value of “Not Play” is 0 – nothing ventured, nothing gained, nothing lost.

Figure 16.4

Following the “Play” branch – one has a 20% (.2) chance of selecting the joker on the draw (1 chance in 5). Thus, the two event branches after the initial decision are shown as “Draw Joker,” with probability of 0.2, and “Did Not Draw Joker,” with probability of 0.8.

For the “Draw Joker” branch, the next decision faced by the player is to either accept $2 in winnings (or stop), or play Triple or Nothing. The decision “Accept” leads to a payoff of +$2. The decision “Triple or Nothing” leads to an event node, where the player will either draw the ace of spades (probability of .25) or not (probability of .75). Note that there is a 1 in 4 chance of drawing the ace of spades. If the ace of spades is drawn, a payoff of +$6 occurs and is shown as a leaf of the tree. If it is not drawn, the payoff of 0 is shown.

Similarly, for the event branch “Did Not Draw Joker,” another decision awaits the joker player. Play again (for another cost of $1) or stop and lick your wounds. If the player does not play again, then that decision leads to a payoff of 0. If one opts to play again, we see the cost reflected in the branch (–1) and then another event node. At this event node, with all five cards again to be used, the player has a .2 chance of drawing the joker, which would lead to the $10 payoff. The alternative of course is an .8 chance of having a payoff of 0.

16.4.3 Evaluating the Joker’s Decision Tree

Figure 16.5 illustrates the roll-back of tree expected values and contingent decisions. We will briefly summarize the calculations.

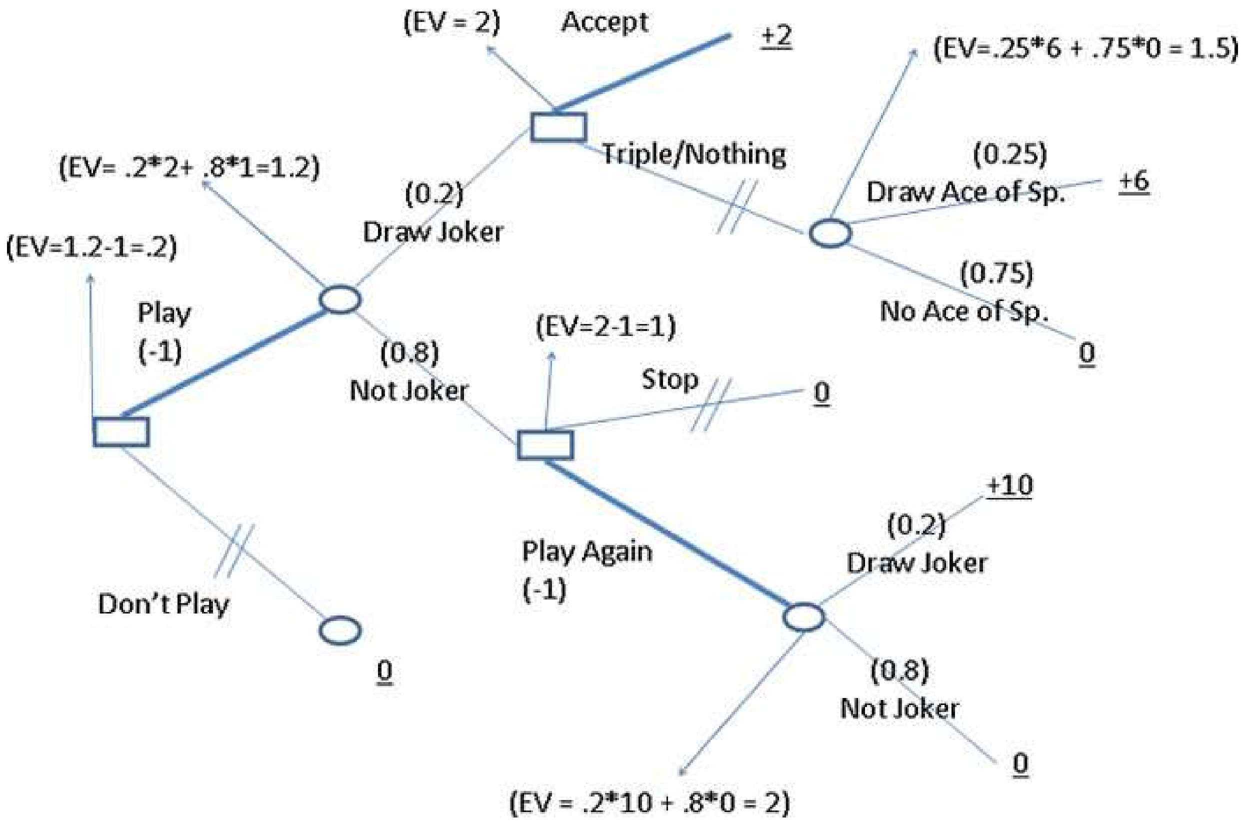


Figure 16.5

We arbitrarily start at the upper branch (the "Play" branch, followed by the event "Draw Joker"). Consider first the Triple or Nothing decision, which leads to the event note involving the ace of spades. The expected value of the event node after the Triple or Nothing decision is the probability of drawing the ace of spades multiplied by the payoff of \$6 + probability of NOT drawing the ace of spades multiplied by the payoff of \$0. This is .25*6 + .75*0 = \$1.5.

Thus, the expected value of opting to play Triple or Nothing after an initial draw of the joker is \$1.5. We contrast this with the expected value of simply accepting \$2 in winnings, and the correct decision (using expected value) is obvious: the optimal decision is if we draw a joker, we should stop and accept \$2 in winning. This now becomes the expected value of this decision node, and will be rolled-back with other information from the other part of the tree toward the initial decision.

Note the slash marks on Figure 16.5 indicating our analysis shows the optimal second-stage decision in this scenario is to accept winnings instead of playing Triple or Nothing.

Focus now on the branches indicating that the joker was NOT drawn. We are faced with the decision to stop (expected value of 0, obviously) or play again, with card replacement (cost of – 1). The expected value of the "Play Again" decision is the probability of drawing the joker the on the second draw multiplied by the new payoff ($10) plus the probability of not drawing the joker multiplied by the payoff of $0. The calculation: .2* 10 + .8*0 = $2.

To find the expected value of the “Play Again” branch, the cost to play must also be considered. The net expected value then is the Event Node expected value (+2) less the costs to choose that decision (–1). This leads to the net expected value for “Play Again” to be +1. Because +1 > 0, the optimal decision at this decision point is to pay $1 and play the game again.

Now, we continue to roll-back the expected values. We are now analyzing the expected value of the entire “Play” decision. This can be found by taking the probability of initially drawing the joker (0.2) multiplied by the expected value of that entire future branch of activities and future decisions, plus the probability of initially NOT drawing the joker (.8) multiplied by the expected value of its entire future branch of activities. The two branches had determined expected values of +$2 and +1, respectively. Therefore, the calculation of expected value aggregately would be:

2 * +2 + .8 * +1 = $1.2.

The cost to play must be subtracted from this value to arrive at the overall expected payoff to play this game. Thus, $1 is subtracted from $1.2, leaving us with a new expected value of $0.20, or 20 cents. This is the expected value of playing the game.

Decision-wise, then, the expected payoff to play the game ($0.20) exceeds the payoff to not play the game ($0), so the decision tree analysis indicates the following decision:

Play the game. If you draw the joker, stop and accept the $2 in winnings. If you initially do not draw the joker, pay $1 and play again.

This kind of decision, expressing the contingencies of what is done when faced with different scenarios is the complete answer. Decision trees typically involve sequential decisions, so the answer tends to be a form of IF-THE-ELSE statements.

What does the expected value of $0.20 mean? This means that if we played the game an infinite number of times, on average, we would win 20 cents per game. Not bad for a $1 investment (20% return?). Having said that, that doesn’t mean (of course) that each time we play we win 20 cents. That is the average over the long run. Each game we play, based on the optimal strategy, will lead to different results.

For instance, if we draw a joker on the first card, the decision says – stop and collect your $2. This occurs 20% of the time, and it represents a net payoff of $1 ($2 less the entry fee).

If we do not draw a joker on the first card (80% likelihood), the optimal decision says – play again (another $1 cost). Then, 20% of the time we would win $10 on the second draw (a net of $8) and 80% of the time we’d lose again.

Put another way, if we play this game and follow the optimal strategy, 20% of the time we'd win $1 (.80*.20 = 16%) , 16% of the time we'd win net $8 (two entry fees and the $10 winnings), and the rest of the time (.80*.80 = 64%) we'd lose money, actually a loss of $2. So, just 36% of the time we would win money in playing this game. Perhaps a different decision-making criterion might lead to a different initial decision?

And, if you're wondering, the simplification in the preceding paragraph does still work out as a net expected value for the game as $0.20. Consider the payoffs/losses and probabilities:

(.2* +1) + (.16*+8) + (.64*-2) = .2 + 1.28 – 1.28 = .2.

Thus ends the analysis of the simple game of "The Joker."

Reading Material: 16.5 – Conclusion and Final Comments

I can see an army of management science students descending on Las Vegas armed with their new-found confidence in decision trees! In some ways, that wouldn't be a bad thing.

One example we have not seen here (but you will in the practice problems) is creating a decision tree in which you are modeling a competition (picking baseball pitchers, deciding what to do in product introduction, and considering competitor reaction, etc.). When one has this kind of competitive scenario, remember that each participant is going to make decisions based on what THEY feel is best. With decision tree software one might assume that all decisions are made from the same perspective. Beware.

Speaking of decision tree software, there are products out there (such as TreePlan, www.treeplan.com) that can be implemented in Excel. I used TreePlan with another text, and it works pretty well for our purposes. Because of initial copyright issues, I am not including it with this text, but perhaps in the future. It is a time and cost issue, nothing against the developer or company.

Thinking about explicit events, actions, decisions, and likelihood of occurrence in picture form is the major practical benefit of decision tree analysis. Sure, we can do the calculations and find the right answer. Perhaps a bigger important impact is that using such an analysis forces decision makers to think about future actions and possible consequences – which lead to better insight about the problem under study. After all, that is what this class is all about – insight.

This was a really quick introduction to decision analysis and decision trees, but it gives you a flavor of the plethora of tools and techniques out there just waiting for you to discover them so you can help make better decisions for your organization and save the world. Peace and good will!