Module 4: A Picture of Insight

Reading Material: 4.1 – Introduction

When creating models of any kind, sometimes the most valuable outcome is the insight gained during their formulation, as well as that gained by critiquing initial solutions to see if they pass the 'smell test', perhaps finding poor logic, missing constraints, etc. One tends to iteratively improve these models, using the insight gained to properly refine their behavior. The same can certainly be true when using linear programming.

In LP models, there are two primary forms of immediate insight gained after a model is solved that provide additional information beyond the optimal solution. This supplemental information comes directly from the algebra of the LP model. One such item of insight includes identifying the inhibiting (binding) constraints and the corresponding marginal value of additional units of that scarce resource.

Another insightful question of interest regards determining how "touchy" the optimal solution may be to changes in our objective function. As decision makers, knowledge about this "touchiness" provides us with either a great deal or very little confidence in the robustness of our optimal solution to changes in environmental factors, model estimates, or simply errors made that impact model parameters of cost, sales price, profits, and so forth. This is true as one recalls that the optimal extreme point of our LP graph is determined strictly by the objective function values (such as profits, sales, costs, etc.) of our decision variables.

When a person is "touchy," they might also be called sensitive. So, because we are examining how "touchy" our LP model may be, this might be why this area of study has historically been called Sensitivity Analysis (okay, that's a good try, right?). When we use the EXCEL Solver to implement LP models, we can view an automatically generated report called "Sensitivity Analysis." The Solver takes advantage of the algebraic solution method employed in linear programming and automatically generates this insightful information.

The major learning objective in this module is to provide a pictorial foundation and explanation for our future Sensitivity Analysis (SA) insight report. This module is meant to be quick, to build more foundational understanding of this basic insight information while in "picture" mode (simple graphs of two decision variables). This will allow us to understand SA reports in general without requiring us to learn the solution algorithm employed (SIMPLEX) by the EXCEL Solver. The Simplex method is a classic algorithm, but it is beyond the scope of this book. Therefore, it is advantageous to introduce these concepts in pictures, and it gives us a haven to return to if we need additional clarification with the Solver SA reports.

Reading Material: 4.2 – More Than a Solution – Constraints at Our Optimal Point

We will continue using our LEGO Production Scenario – producing LEGO tables and chairs, constrained by the amount of small and large LEGO blocks and a measure of capacity/space.

In the previous module, we found the following extreme points (which defined our feasible region) and identified Point C as our optimal production point – 120 tables and 60 chairs.

Extreme Point	Tables – Profit = $16	Chairs – Profit = $9	Total
A	0	200	1800
B	80	120	2360
C	120	60	2460
D	150	0	2400

So, let us examine the three constraints at Point C. How many small LEGO blocks, large LEGO blocks, and units of capacity are used in producing 120 tables and 60 chairs?

Small LEGO blocks: 2 TAB + 2 CHR = 2(120) + 2(60) = 360 (400 available)

Large LEGO blocks: 2 TAB + CHR = 2(120) + (60) = 300 (300 available)

Capacity: 6 TAB + 4 CHR = 6(120) + 4(60) = 960 (960 available)

Note that we use all available large LEGO blocks and units of capacity, whereas we have 40 unused small LEGO blocks.

By definition, large LEGO blocks and capacity are said to be binding constraints. Small LEGO blocks are termed, not surprisingly, *non-binding constraints*.

At our optimal production point, we use all available large LEGO blocks and units of capacity. They inhibit making any more furniture; thus they are termed binding constraints. It is reasonable to think that if we could somehow obtain more units of our binding constraints, we could produce more furniture and gain more profit or sales. The reverse (fewer units of binding resources leading to less profit or sales) also seems reasonable – and both are true.

We have 40 unused or “slack” units of small LEGO blocks. Additional small LEGO blocks would not seem to alter our production plan (there would just be more left over) whereas reducing small LEGO blocks would also have no effect except when that reduction exceeds 40 (the unused amount).

It is not a coincidence that large LEGO blocks and capacity were binding constraints, and point C, the optimal extreme point (120 tables and 60 chairs), fell at the intersection of those same two constraints. Later in this module, we are going to use this information to gain some additional insight about the marginal value of additional units of our binding constraints –very useful information to a decision maker. But we will get back to this in a bit.

To summarize so far, we previously found the optimal solution to an LP model. At the optimal solution point, we have now also identified binding and non-binding constraints, useful additional information.

Next, we will explore two more detailed aspects of insight – how sensitive is this optimal solution point to changes in our objective function coefficients (our sales or profits) and how valuable are the resources that bind our solution (marginal values of our resources). This parallels the EXCEL Solver SA printout mentioned before that will be explored in the future.

Reading Material: 4.3 – Impact of Objective Function Coefficient Changes

First, a word about what we are calling Sensitivity Analysis. It is basic analysis, looking at the impacts of varying one model parameter at a time on our LP model solution. So, when we are investigating a parameter of interest, the assumption is that, besides the item under analysis, all other model parameters remain constant. In class, I have used an unpronounceable acronym to remind us of this: AOMPC – All Other Model Parameters Constant.

The great thing about this simple insight is that it just “falls out” of the model’s algebra. We will explore a little of this, but not too much, here. Our one parameter, AOMPC analysis, provides us some initial information about our modeling environment in addition to the optimal solution. This initial information is very valuable because it is easily obtained and provides a starting point for more detailed scenario analysis that the decision maker may find pertinent (i.e., running various different models with altered parameters, etc.). So, the simple SA information we find is to help drive deeper exploration of our model and not intended to keep us from exploring multiple simultaneous changes through different scenarios.

Back to our LEGO production example. The extreme point table is repeated again below:

Extreme Point	**Tables – Profit = $16**	**Chairs – Profit = $9**	**Total**
A	0	200	1800
B	80	120	2360
C	120	60	2460
D	150	0	2400

In Module 3, we suggested that if the sales price of tables increased (through a new policy, through higher demand, whatever the reason), we might find that the optimal extreme point would change away from Point C to another point (say Point D, because it represents producing more tables). Let’s explore the impact of increasing profit of tables beyond the original $16.

To start, consider an alteration to the sales price of tables of $16 to $17 (yes, AOMPC!!!). For each table, we therefore gain an additional $1. We can easily calculate the new objective function values for the four extreme points: A= 1800, B = 2440, C= 2580, D = 2550. Note that Point D is “catching up” in total sales to Point C, but Point C remains the optimal production mix. It would seem reasonable that at some point of increasing the sales price for tables, Point D (making more tables than Point C) would be equal, then more preferred, to Point C.

We can use basic algebra to find this crossover point. (This is the same algebra that the Solver uses). Define X to be the sales price amount of tables when the objective function value of Point C and Point D are equal. Recall that the sales price of chairs is $9. The objective function value at Point C is X * 120 + 9 * 60, whereas the objective function value at Point D is X * 150. We can then solve for X, finding the sales price of tables at which the two extreme points have equal sales dollars, as follows:

$120X + 540 = 150X$, so

$30X = 540$

$X = 18$.

Thus, if the sales price of tables is 18, both extreme points C and D have the equivalent objective function value (18 * 150 = 18 * 120 + 540 = 2700).

So, a quick pause in the discussion to summarize. The original LP model was solved with the sales price (objective function coefficient) of tables at $16, and we found the optimal solution to be 120 tables and 60 chairs. The sales price of tables can increase up to 18, and 120 tables and 60 chairs remains the optimal solution. At a sales price of $18, TWO solutions are equal: 120 tables and 60 chairs, and Point D, 150 tables. (This is called *multiple optimal solutions* – more below). With the sales price beyond $18, the optimal solution will change to Point D, to produce only 150 tables (you can verify this if you'd like).

The term *multiple optimal solutions* was mentioned earlier. This phenomenon simply means there is more than one best answer. It happens, and it is nothing bad. Sometimes, in practice, if multiple optimal solutions exist in a model, we may try to add additional objective(s) to our modeling scenario in attempts to further refine our search for the best solution. We will revisit this when we talk *about multiple objective modeling* later in the book. Also, in this case, it is worth noting that all the solutions along the line segment between Point C and Point D are optimal – all of them, when the sales price of tables is $18, providing the same objective function value. So, there are actually an infinite number of optimal solutions. OK, so it was just barely worth mentioning.

We found that the optimal solution stays the same combination of tables and chairs for sales price values for tables from $16 to $18. This information is the part of sensitivity analysis that addresses, for lack of a better term (and trust me, I wish I had one!) "solution robustness." This addresses the question of how sensitive our solution is to model parameters question. Why is this important? The decision maker benefits by having a fundamental appreciation for the level of confidence he or she can have in the universality of the determined optimal solution (call it decision maker warm fuzzies). If the optimal solution is fairly invariant to model parameter changes (specifically, the objective function values we are looking at here), the decision maker has a high degree of confidence in implementing the solution (strong warm fuzzies). Likewise, if small changes in model parameters would lead to a solution change, much more caution (and perhaps analysis) would be prudent for the decision maker (weak warm fuzzies). Decision makers want to look smart. Using sophisticated models is a start. But understanding their limitations (or the impacts of errors or parameters changes on proposed solutions) is also just as important. SA provides the decision maker just such information.

In this case, we determined that the upper bound for the price of tables that does not have an impact on the optimal solution to be \$18, a difference of +2 from the original price. Is that a big difference or a small difference? Hard to say, as such an assessment is contextual – meaning that the decision maker will need to assess the magnitude of robustness in this domain. Not a cop-out response, but a \$2 change in some environments might be small, in others, big!

For completeness, let us calculate the lower bound for the sales price robustness of tables as well. Define J as the sales price of Tables where Point B and Point C would have the same objective function value (Point B is the next contiguous extreme point and thus, as sales price of tables would decrease, it becomes the next viable candidate for the optimal extreme point).

Similarly to the previous analysis, total dollars at Point B would be 80J + 9 * 120, where total dollars at Point C would be 120J + 9 * 60. Below, we solve for J such that Point B and Point C are both optimal.

$80J + 1080 = 120J + 540$

$540 = 40J$

$J = 13.5$

So, if the sales price of tables was lowered to \$13.5, we would be indifferent between Points B and C in determining the optimal production mix. Between \$13.5 and \$16, Point C would remain optimal. Lower than \$13.5, Point B would be optimal (to a certain point, but at some sales price level, Point A would become preferred to Point B – this is beyond the scope of our present analysis).

So, let's summarize this analysis of the sales price (objective function coefficient) of tables. An LP model was created using \$16 as the sales price of tables, and the optimal production mix was determined to be 120 tables and 60 chairs. Considering solution robustness based on the sales price of tables, it was determined that the originally determined optimal solution (120 tables and 60 chairs) remains optimal for a sales price variance between \$13.5 and \$18. The optimal solution (mix of furniture) would change if the sales price fell outside of this range. This analysis was conducted under the assumption of AOMPC. It is up to the decision maker to assess whether this implies a robust optimal solution based on the context of the problem.

We could easily calculate the same information for the sales price of chairs. That is left as the end of module exercises.

For the future, when the EXCEL Solver Sensitivity Analysis printout is first introduced and if you need some additional explanation of where the data comes from, this is the place to come. We have mimicked the math used by the Solver here while still dealing with very small (and visual) models.

Next up – assessing the marginal value of our resources – looking at the impact of our binding and non-binding constraints.

Reading Material: 4.4 – Marginal Value of Resources – The Shadow Price

Again, going back to our original model, recall that our optimal solution used all available large LEGO blocks and all available capacity in producing 120 tables and 60 chairs, while there were 40 unused small LEGO blocks. We also speculated that additional large LEGO blocks would increase our sales return if we could obtain more, as would additional units of capacity. Below, we analyze what additional units of these resources are worth – identifying their marginal value (which, for historical reasons and not really any other reason, we will call its shadow price).

A decision maker would find this information useful as they analyze the determined optimal production mix. Knowing the impact of additional resources would help provide insight to the decision maker in the environment and could influence future procurement decisions as they seek to be more profitable or effective in their operation. And, as in the previous section, the algebra of an LP model can be easily used to determine a resource's marginal value. As before, this is just a quick and dirty analysis, but it directly relates to the second table of the EXCEL Solver Sensitivity Analysis report.

A modified graph of the original LEGO problem (Figure 4.1) is shown below. The dotted line represents the impact of increasing the available numbers of large LEGO blocks by one unit. Note how the originally determined optimal solution Point C moves to C' (in essence, it moves to the southeast if you think in N/S/E/W terms).

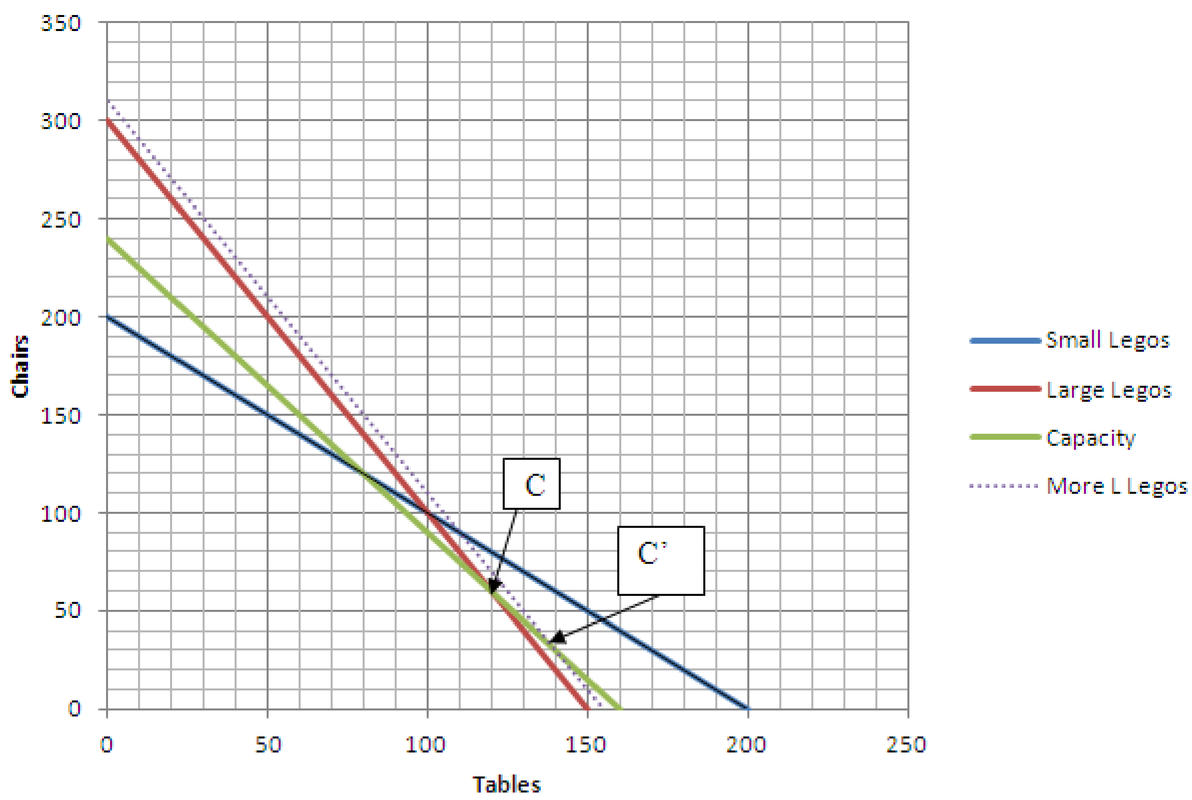

Figure 4.1

Another "trust me" statement – as long as the basic shape of the feasible region does not change when a change is made to a binding constraint (AOMPC), the originally determined optimal solution point (extreme point) will still be the optimal extreme point – it will have just moved slightly, representing the new solution that occurs as a result of the change. In our case, that means that as one unit of large LEGO blocks (AOMPC) is added; the feasible region still has four extreme points, and C' remains the optimal extreme point. Please note that C' has moved from C, so the optimal numbers of tables and chairs are affected (changed) by one additional large LEGO block.

At first glance, it might seem reasonable that our model would produce more tables AND more chairs with additional units of a binding resource/constraint. This is not actually what happens. Previously, we mentioned that C' moved southeast from C. That means that C' represents more tables than C but FEWER chairs. Counterintuitive? The picture shows it. How about the algebra?

Again, I hope you believe me when we say C' is the optimal solution in the model that has available 301 large LEGO blocks instead of 300. (Well, I'm going on like you do, so too late!).

Point C' is found at the intersection of the following two lines:

2TAB + CHR = 301 (the one additional LEGO we are analyzing)

6TAB + 4CHR = 960 (Capacity unchanged).

Rewrite the first equation as 6TAB + 3CHR = 903

Substitute in 6TAB = 903 – 3CHR and it simplifies to: CHR = 57, which results in T = 122.

So, adding one more large LEGO block results in producing two more tables (120 to 122) but three fewer chairs (60 to 57). The net impact on objective function value? Two more tables adds 2 * $16 = +32 and three fewer chairs subtracts 3 * $9 or –27, a net impact of +$5. Thus, the marginal value of large LEGO blocks has been determined to be $5.

Note that if we looked at the impact of decreasing available large LEGO blocks by one unit, we would find it decreases the objective function value by $5 – thus, the marginal value "shadows" both sides of the original value; we call it a *shadow price* (sounds like a reasonable explanation for such an obtuse term, right?). So, large LEGO blocks have a shadow price (marginal value) of $5, implying the per-unit impact of small changes in that constraint. (Keep that 'small change' idea in the back of your mind).

Why did this happen (two more tables and three fewer chairs) with just a change of one unit in available LEGO blocks? The algebra of the model "stores" the relationships of the decision variables and thus determined that the best use of reshuffling resources when we added one unit of large LEGO block(and kept all other constraints CONSTANT!!) was adding two tables while subtracting three chairs. You'll note that the two for three tradeoff was a 'wash' with respect to capacity usage – two tables used 12 units of capacity, and giving up three chairs freed up those 12 units of capacity.

Thankfully, the Solver does these types of calculations for us (shadow price determinations).. Again, in the future, if you're struggling with where the autogenerated values came from in the Solver, come back to this module for a pictorial view.

We could do a similar analysis for calculating the shadow price of Capacity – that is left for the questions at the end of the module.

Earlier, I mentioned that the shadow price was valid for a small change. We did not define what is meant by a small change, and we are not going to calculate anything here either. But we do need to clarify.

Visualize adding additional units of large LEGO block; eventually, the large LEGO constraint will move outside, or beyond the capacity constraint, effectively changing the shape of the feasible region. In essence, point C' goes away. This is what I meant previously when I mentioned "the shape of the feasible region stays the same." The determined shadow price has a range of applicability (wish I could say that with fewer syllables, but I can't). Inside the range, we know the marginal impact. Outside the range, we have no idea. When we look at a change in a binding constraint's availability (its right-hand side value) *within this range*, we can comment on how it affects the objective function value (shadow price) and know that the optimal solution changes.

Outside the range, which would occur if a change alters the shape of the feasible region, all of the relationships in the model are different and the algebra cannot tell us anything. We are left with having to re-execute the LP model with the change to really know the impact. This is not as a bad thing, it just is a limit of what our basic sensitivity analysis can tell us.

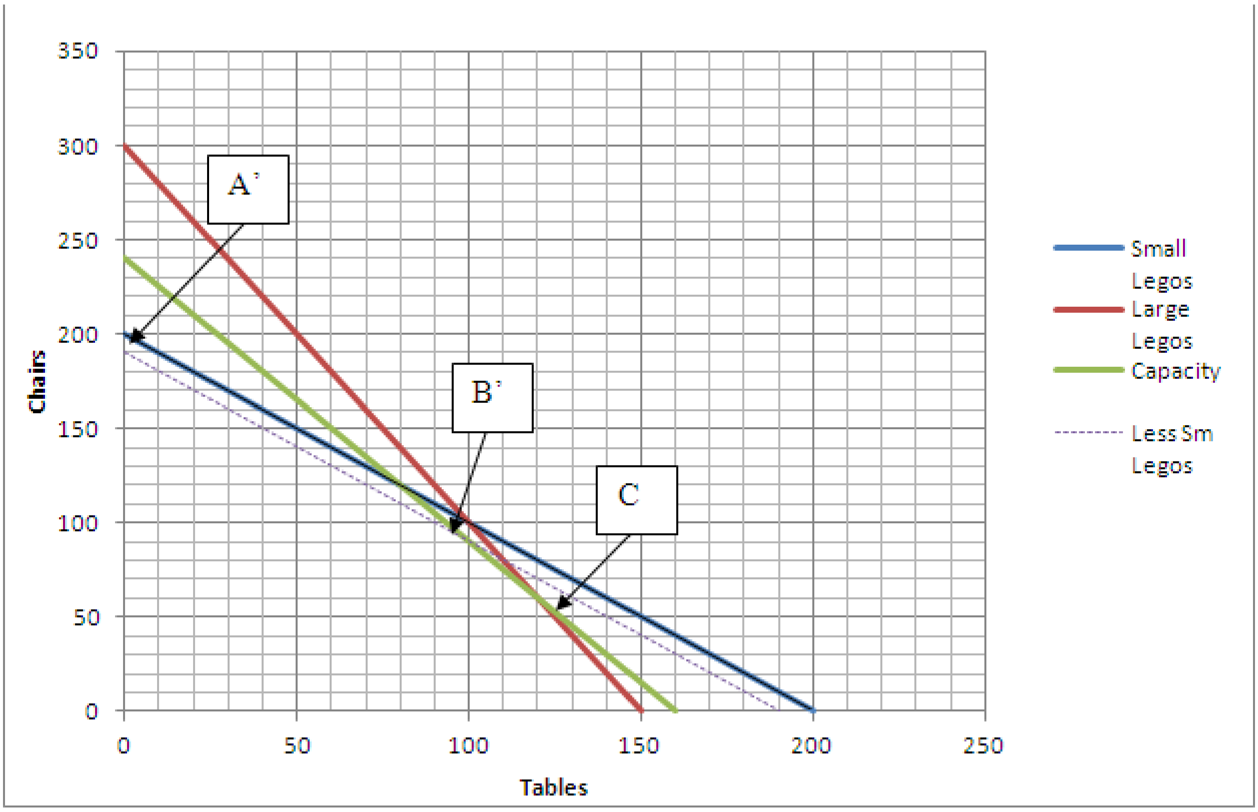

Figure 4.2

Perhaps another way to see this concept of range of applicability is by examining the one non-binding constraint in the model – the small LEGO blocks. We determined we had 40 extra small LEGO blocks in our optimal solution. Basically, if we added more small LEGO blocks to our production situation, what would be the impact on our total sales? Nothing. Just more left over.

There would also be no impact on the optimal solution if we decreased the number of small LEGO blocks up to a reduction of 40. It doesn't affect our solution because it would just be reducing the number of extra (slack) LEGO blocks. Thus, we have a non-binding constraint has a shadow price (marginal value) of 0!!

But perhaps we can more easily see this range of shadow price validity. We can increase the number of small LEGO blocks to infinity (and beyond!) and the impact remains the same – nothing. So the shadow price of 0 has an infinite upper bound of applicability.

Reducing the number of small LEGO blocks by less than 40 – again, no impact. The feasible region still would have a point A' and B' (thus, the shape is not changed, see figure 4.2), and it would not impact point C (the optimal point). Once the small LEGO constraint is reduced by 40 or more, the shape of the feasible region changes and all bets are off. So, the shadow price of 0 for small LEGO blocks has a lower bound of 360 (40 less than the original 400 units).

Let's summarize the insight analysis as it relates to LP model constraints. Binding constraints have non-0 shadow prices (marginal values). Non-binding constraints have 0 shadow prices. We can calculate these shadow prices for problems in two dimensions with some basic algebra.

Shadow prices have upper and lower bounds of applicability. It is easy to see this in non-binding constraints, but, as we will see, all constraints have these limits. We will not examine how to determine these limits – we will trust Excel Solver down the road.

The information discussed here appears in the second table of the EXCEL Solver Sensitivity Analysis (both shadow price and range information) printout and is a helpful first step for decision makers as they gain insight about the environment they are modeling and what strategies they may employ to increase profitability based on the marginal values of their scarce resources.

Reading Material: 4.5 – Summary

What have we learned in this module? The algebra behind the scenes in the EXCEL Solver is very powerful, all because LP modeling is just linear algebra with a twist. Thank goodness we have EXCEL to do the dirty work for us. But we learned a little bit about where the information comes from in case we need some warm fuzzies down the road when discussing Sensitivity Analysis in a more general way.

Objective function coefficients in LP models can change and NOT have an impact on the optimal solution (they also can change the solution as well). It is useful to know how robust our optimal solution is for implementation reasons. How confident can we as the decision maker be in our prescribed results?

In the spirit of sensitivity analysis, we can find the marginal value (shadow price) of our constraints/scarce resources, and this can be very helpful as well in gaining insight about our entire problem environment and where to focus future efforts.

All quantitative analysis of this type assumes we are making only one change, and all other model parameters remain constant (AOMPC). Should we want to look at simultaneous changes, we can just rerun the EXCEL model. Nonetheless, the simple yet powerful information discussed here in graphical form comes naturally from the algebraic model and can lead to greater decision maker insight into the scenario under study.