$$\ell = \sum_{i=1}^{n} \left(-\frac{1}{2} \log \det \Sigma - \frac{1}{2} \left(f(x_i) - \Phi G_{i,\cdot} \right)^T \Sigma^{-1} \left(f(x_i) - \Phi G_{i,\cdot} \right) \right)$$

$$+ \sum_{k=1}^{K} -\frac{1}{2} \log \det C - \frac{1}{2} G_{\cdot k}^T C^{-1} G_{\cdot k} .$$

$$\text{ELBO} = \mathbb{E}_q \log p(F \mid G) + \mathbb{E}_q \log p(G) - \mathbb{E}_q \log q(G) \qquad G_{ik} \sim \mathcal{N}(M_{ik}, V_{ik})$$

$$\mathbb{E}_q \log q(G) = -\frac{1}{2} 1_n^T \log(V) 1_K + \text{const.}$$

$$\mathbb{E}_q \log p(F \mid G) = -\frac{n}{2} \log \det \Sigma - \frac{1}{2} \sum_{i=1}^{n} \left(\mathbb{E}_q \left[(F_i - \Phi G_i)^T \Sigma^{-1} (F_i - \Phi G_i) \right] \right)$$

$$= -\frac{n}{2} \log \det \Sigma - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}_q \left[F_i^T \Sigma^{-1} F_i - (\Phi G_i)^T \Sigma^{-1} F_i - F_i^T \Sigma^{-1} \Phi G_i + G_i^T \Phi^T \Sigma^{-1} \Phi G_i \right]$$

$$= -\frac{n}{2} \log \det \Sigma - \frac{1}{2} \sum_{i=1}^{n} \left[F_i^T \Sigma^{-1} F_i - F_i^T \Sigma^{-1} \Phi M_i - M_i^T \Phi^T \Sigma^{-1} F_i + \operatorname{tr}\left(\Phi^T \Sigma^{-1} \Phi (\operatorname{diag} V_i) \right) + M_i^T \Phi^T \Sigma^{-1} \Phi M_i \right]$$

$$= -\frac{n}{2} \log \det \Sigma - \frac{1}{2} \sum_{i=1}^{n} \left[(F_i - \Phi M_i)^T \Sigma^{-1} (F_i - \Phi M_i) + \operatorname{tr}\left(\Phi^T \Sigma^{-1} \Phi \operatorname{diag} V_i \right) \right]$$

$$\mathbb{E}_q \log p(G) = \sum_{k=1}^{K} -\frac{1}{2} \log |C_k| - \frac{1}{2} \mathbb{E}_q G_{\cdot k}^T C_k^{-1} G_{\cdot k}$$

$$= \sum_{k=1}^{K} -\frac{1}{2} \log |C_k| - \frac{1}{2} M_{\cdot k}^T C_k^{-1} M_{\cdot k} - \frac{1}{2} \operatorname{tr}\left(C_k^{-1} \operatorname{diag}(V_k) \right)$$