Lecture 32 — November 1

Instructor: Stas Minsker *Scribe: Mose Wintner*

32.1 Structural Signal Recovery from Linear Measurements

$\lambda \in K \subseteq \mathbb{R}^p$ an unknown element of a known set; $\mathbb{X} \in \mathbb{R}^{n\times p}$ a matrix with iid $\mathcal{N}(0,1)$ entries. We observe $Y = \mathbb{X}\lambda \in \mathbb{R}^n$. We wish to recover λ in the situation when $n < p$ ($n \geq p$ solved by pseudo-inverse).

Let L be the subspace of $z \in \mathbb{R}^p$ satisfying $\mathbb{X}z = Y$ and let $\hat{\lambda} \in K \cap L$ be arbitrary. Then $\left|\left|\hat{\lambda} - \lambda\right|\right|_2 \leq \text{diam}(K \cap L)$.

Theorem 1. Let T be a bounded subset of $\mathbb{R}^p$. Let $\mathbb{X} \in \mathbb{R}^{n\times p}$ be a matrix with iid $\mathcal{N}(0,1)$ entries. Fix $\varepsilon \geq 0$ and define

$$T_\varepsilon = \{z \in T : \frac{1}{n} \left|\left|\mathbb{X}z\right|\right|_1 \leq \varepsilon\}.$$

Then

$$\mathbb{E} \sup_{z\in T_\varepsilon} \left|\left|z\right|\right|_2 \leq \sqrt{\frac{8\pi}{m}} \mathbb{E} \sup_{z\in T} |\langle g, z\rangle| + \sqrt{\frac{\pi}{2}}\varepsilon.$$

Corollary 1. When $\varepsilon = 0$, $T_\varepsilon = T \cap \ker \mathbb{X}$, so

$$\mathbb{E} \sup_{z\in T\cap \ker \mathbb{X}} \left|\left|z\right|\right|_2 \leq \sqrt{\frac{8\pi}{m}} \mathbb{E} \sup_{z\in T} |\langle g, z\rangle|.$$

Since $\hat{\lambda} - \lambda \in \ker \mathbb{X}$, theorem implies that

$$\mathbb{E} \left|\left|\hat{\lambda} - \lambda\right|\right|_2 \leq \sqrt{\frac{8\pi}{m}} \mathbb{E} \sup_{z\in K-K} |\langle g, z\rangle|$$

which is proportional to the Gaussian mean width of K.

Then note $z \mapsto \langle g, z\rangle$ is a Gaussian process when g is Gaussian.

Example 1. a) K a finite set.

$$\mathbb{E} \sup_{z\in K} |\langle g, z\rangle| \leq \max_{z\in K} \left|\left|z\right|\right|_2 \sqrt{2}\sqrt{\log 2|K|}.$$

b) $K \subset L$ a subspace of $\mathbb{R}^p$ of dimension d.

$$\mathbb{E} \sup_{z\in K} |\langle g, z\rangle| \leq \text{diam}(K)\sqrt{d/n}.$$

Proof of theorem: Note that $\frac{1}{n}\mathbb{X}z = \frac{1}{n}(\langle x_i, z\rangle)_i$. Suppose we can show

$$\mathbb{E}\sup_{z\in T}\left|\frac{1}{n}\sum_{i=1}^{n}|\langle x_i, z\rangle| - \sqrt{\frac{2}{\pi}}\,||z||_2\right| \leq \frac{4}{\sqrt{m}}\mathbb{E}\sup_{z\in T}|\langle g, z\rangle|,$$

where $\sqrt{2/\pi}\,||z||_2 = \mathbb{E}|\langle x_j, z\rangle|$. If $T = T_\varepsilon$, then the first term is just $\frac{1}{n}\,||\mathbb{X}z||_1 \leq \varepsilon$. Then

$$\mathbb{E}\sup_{z\in T_\varepsilon}||z||_2 \leq \sqrt{\frac{\pi}{2}}\left(\varepsilon + \frac{4}{\sqrt{n}}\mathbb{E}\sup_{z\in T_\varepsilon}|\langle g, z\rangle|\right).$$

We can just ditch the ε To estimate the LHS which we'll call M, first apply the symmetrization inequality to get

$$M \leq 2\mathbb{E}\sup_{z\in T}|\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j|\langle x_j, z\rangle||.$$

Then we use the contraction inequality with $\varphi(t) = |t|$ to get

$$M \leq 4\mathbb{E}\sup_{z\in T}\left|\frac{1}{n}\sum_{j=1}^{n}\langle\varepsilon_j x_j = x'_j, z\rangle\right| = \frac{4}{\sqrt{n}}\mathbb{E}\sup_{z\in T}\left|\langle\sum_{j=1}^{n}\frac{1}{\sqrt{n}}x'_j, z\rangle\right| = \frac{4}{\sqrt{n}}\mathbb{E}\sup_{z\in T}|\langle g, z\rangle|$$

since $\sum\frac{1}{\sqrt{n}}x'_j \sim \mathcal{N}(0, I_p)$.

Remark 1.

$$\begin{aligned}\mathbb{E}\sup_{z\in T}|\langle g, z\rangle| &= \mathbb{E}\sup_{z\in T}|\langle g, z - z_0\rangle| + \mathbb{E}|\langle g, z_0\rangle| \\ &= \mathbb{E}\sup_{z\in T}\langle g, z\rangle + \mathbb{E}|\langle g, z_0\rangle| \leq \mathbb{E}\sup_{z\in T}\langle g, z\rangle| \\ &\leq \mathbb{E}\sup_{z\in T}\langle g, z\rangle + \sqrt{\mathbb{E}\langle g, z_0\rangle^2}.\end{aligned}$$

Finally

$$\mathbb{E}\sup_{z\in T}|\langle g, z\rangle| \leq \mathbb{E}\sup_{z\in T}\langle g, z\rangle + \inf_{z\in T}||z_0||_2.$$

Assume that K is convex.