$\widehat{\pi}$

Classical result: Let $(X, x) \in \mathbf{Top}_*$. There is a functor

$$\{\text{coverings } p : Y \to X\} \to \{\pi_1(X)\text{-sets}\}$$
$$p \mapsto p^{-1}(x)$$

In addition, if X has a universal cover, then the functor above is an equivalence of categories, where the map going to the right is given by $S \times_\pi \tilde{X} \leftarrow\!\shortmid S$. We also have, for any pointed covering $p : (Y, y) \to (X, x)$, isomorphisms

$$p^{-1}(x) \cong \pi_1(X, x)/p_*\pi_1(Y, y) \left(\cong Hom_X(\tilde{X}, Y)\right)$$

.

In particular, if X has a universal cover $\tilde{p} : \tilde{X} \to X$, then $\pi_1(X)$ can be recovered easily from the fiber $\tilde{p}^{-1}(x)$. We say in this case that the fiber functor is (co?)represented by the universal cover. We will use the classical result

$$\pi_1(X) = \mathrm{Aut}_X(\tilde{X}/X) = \text{"Gal}(\tilde{X}/X)\text{"}$$

as our intuition for this new π_1; i.e. we want to find some object that behaves like a universal cover and look at its deck transformations.

Problem: No universal cover for schemes and other ill-behaved topological spaces, e.g. $\mathbb{R} \to S^1$ is given by the transcendental map $e^{2\pi i t}$, which doesn't exist in algebraic categories.

Theorem: If X is connected, then there exists a *profinite* group $\widehat{\pi}_1()$, unique up to isomorphism, such that there exists an equivalence of categories

$$\{\textit{finite} \text{ coverings } p : Y \to X\} \leftrightarrow \{\textit{finite } \widehat{\pi}_1() - sets\}$$

where the action of a profinite group is, by definition, continuous. The functor is still given by taking the fiber of the covering.

This can be proven using Grothendieck's Galois theory, which sets up axioms for a category to admit a Galois-type symmetry. DO 3.8. This is the theorem we would like to apply to define the fundamental group of schemes. We would still like a good description of something which behaves like a "universal cover," but in light of the above theorem we can simply work out a good notion of *finite* cover and then take a limit over all those, so that our "universal cover" is a pro-object and the fiber functor is *pro-represented.* What remains is to work out a description of covering maps in the algebraic category.

First, a discussion of basepoints. If we have a scheme X/k, then we have an inclusion $\mathrm{Spec}(k) \to X$. However, any extension $k \hookrightarrow k'$ induces a map $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, suggesting that even though $\mathrm{Spec}(k)$ is a point, it is perhaps not as fine a point as we would like, since it contains another "point" for every extension. The correct notion of basepoint, then, is a *geometric point*, i.e. $\mathrm{Spec}(\overline{k})$ for some algebraic closure $\overline{k}$ of k. This agrees with our usual intuitive notion of well-behaved point, because reasons.

We would like our Galois theory for schemes to wed both classical covering space theory and classical Galois theory; in particular, we would like to regard $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^n}$, or possibly rather $\mathrm{Spec}(\mathbb{F}_{q^n}) \to \mathrm{Spec}(\mathbb{F}_q)$, as an n-sheeted covering. We may as well take this as a starting point. What about arbitrary Galois extensions?

Proposition: Let $K \hookrightarrow L$ be Galois. Then if I is an index category indexing the finite Galois extensions of K, i.e. $K \hookrightarrow K_i$ is finite Galois for all $i \in I$, then I is a directed system and

$$L = \varprojlim_{i \in I} K_i \qquad \mathrm{Gal}(L/K) \cong \varprojlim_{i \in I} \mathrm{Gal}(K_i/K)$$

Proof: That the set of finite Galois extensions forms a directed system is obvious. Each element $\alpha \in L$ is algebraic over K and therefore contained in $K(\alpha)$, which is finite over K, and $\alpha \mapsto \alpha|_{K_i}$ is clearly an isomorphism since the restriction maps are compatible with the arrows in I. $\square$

Now notice that for any field k, $\overline{k}$ contains all finite Galois extensions of k. In fact, if we fix $\overline{k}$ then the separable closure of k inside $\overline{k}$, k^{sep} is equal to the inverse limit of *all* finite Galois extensions. In this sense, once we fix an algebraic closure of k, we can take our "universal cover" to be k^{sep} inside the closure. This suggests we define for fields

$$\widehat{\pi}_1(\mathrm{Spec}(k), \mathrm{Spec}(\overline{k})) = \mathrm{Gal}(k^{sep}/k)$$

the absolute Galois group of k.

Example: Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$ and let $\mathbb{F}_q^{sep}$ be the separable closure of $\mathbb{F}_q$ inside $\overline{\mathbb{F}}_q$. Then

$$\mathrm{Gal}(\mathbb{F}_q^{sep}/\mathbb{F}_q) = \varprojlim_{i \in I} \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}$$

so that

$$\widehat{\pi}_1(\mathrm{Spec}(\mathbb{F}_q), \mathrm{Spec}(\overline{\mathbb{F}}_q)) = \widehat{\mathbb{Z}}$$

We want to work in the category of schemes, so we need a nice category of finite covering spaces.

The correct notion of covering is given by *finite etale* maps. There are many equivalent definitions for an etale map of schemes.

Definition: Let A be a (commutative) ring. A separable A-algebra B is one which is projective as a module over $A \otimes A^{op}$.

Definition: A map of schemes $f : Y \to X$ is finite etale if X every affine covering $U_i \cong \mathrm{Spec}(B_i)$ such that $f^{-1}(U_i) \cong \mathrm{Spec}(A_i)$ with $A_i \to B_i$ makes B_i into a projective, finite (as a module), separable, A_i-algebra.

There are (many) other definitions of separable and finite etale.

A finite etale morphism $V \to X$ is our notion of finite cover. The fiber product of two etale morphisms is etale, so that we can take the "universal cover" of X to be the inverse limit of the finite etale covers. Note also that the geometric fiber of an etale map is $V \times_X \operatorname{Spec}(\overline{k})$ so that our choice of basepoint passes to the inverse limit. The fundamental group acts on the geometric fibers $V \times_X \operatorname{Spec}(\overline{k})$, and the stabilizer is the fundamental group of V.

One necessary property is that if we have a map $f : Y \to X$ of schemes, it should induce a finite separable extension on residue fields:

$$\mathcal{O}_{X,f(y)}/\mathfrak{m}_y\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$$

Necessary for this is that the map f is *locally of finite type*, i.e. for all affine covers, the maps $A_i \to B_i$ on coordinate rings make B_i into a finitely generated A-module. If f additionally satisfies the above property then it is called *unramified*.

Definition: A map of schemes $f : Y \to X$ is flat if for all $y \in Y$, $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.

Definition: A map of schemes $f : Y \to X$ is etale if it is flat and unramified.
In particular, this is a local property.

Definition: $f : Y \to X$ is etale if it is flat, locally of finite presentation, and for every x in X, the fiber $f^{-1}(x)$ is the disjoint union of points, each of which is the spectrum of a finite separable field extension of the residue field $\kappa(x)$.

Examples

- Open immersions, since they are locally isomorphisms.
- Morphisms induced by finite separable field extensions.

For any finite and locally free morphism of schemes $f : Y \to X$ there is a continuous function $X \to \mathbb{Z}$ given locally by the rank of Y over X $[Y : X]$. Clearly agrees on overlap.

Definition: A morphism $f : Y \to X$ is called totally split if X can be written as a disjoint union of schemes X_n, $n \in \mathbb{Z}_{\geq 0}$ such that for each n, $f^{-1}(X_n)$ is isomorphic to the disjoint union of n copies of X_n with the natural morphism to X_n.

Proposition: A map $Y \to X$ is finite etale iff $\exists W \to X$ surjective, finite, and locally free such that $Y \times_X W \to W$ is totally split.

- *Standard etale maps:* A ring map $R \to S$ makes S an R-algebra. It is standard etale if there exist $f, g \in R[t]$ s.t. $S \cong R[t, 1/g]/(f)$, f is monic, and $f' \in S^\times$. EVERY etale map of schemes happens to be locally of finite type and locally standard etale.

Definition: A map of schemes $f : Y \to X$ is etale if it is locally of finite presentation, i.e. B_i is a finitely presented A_i algebra, and it is locally standard etale.

Suppose $\operatorname{Spec}(A)$ is normal, i.e. A is an integrally closed domain. Then for any finite extension K_i of the fraction field of A, if B_i is the integral closure of A in K_i, then the universal covering of A is $\varprojlim_{i \in I} \operatorname{Spec}(B_i)$, where $\operatorname{Spec}(B_i) \to \operatorname{Spec}(A)$ is finite etale. Applying to $A = \mathbb{Z}$, we see that to determine $\widehat{\pi}_1(\mathbb{Z})$, it suffices to find the number rings $\mathcal{O}_K$ that are finite etale over $\mathbb{Z}$. A finite $\mathbb{Z}$-module (finite abelian group) flat iff it is torsion free, and unramified at (p) if for every prime $(p) \in \operatorname{Spec}(\mathbb{Z})$,