MATH 650 HW 1 Additional Exercise

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1.a) $$\frac{\partial^2}{\partial x^2}(-\log p) = \frac{\partial^2}{\partial x^2}\left(\log \prod \left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{2\left(\frac{x_i-\mu}{\sigma}\right)\frac{1}{\sigma}}{1+\left(\frac{x_i-\mu}{\sigma}\right)^2}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{2(x_i-\mu)}{\sigma^2+(x-\mu)^2}\right)$$

$$= \sum \frac{2(\sigma^2+(x_i-\mu)^2) - 2(x_i-\mu)\cdot 2(x_i-\mu)}{(\sigma^2+(x_i-\mu)^2)^2}$$

$$= \sum 2\frac{\sigma^2-(x_i-\mu)^2}{(\sigma^2+(x-\mu)^2)^2}$$

$$= \sum 2\frac{\frac{1}{\sigma^2} - \frac{1}{\sigma^2}\left(\frac{x_i-\mu}{\sigma}\right)^2}{\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)^2}$$

$$= \sum \frac{2}{\sigma^2}\left(\frac{1-\left(\frac{x_i-\mu}{\sigma}\right)^2}{\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)^2}\right)$$

$$I_x(\mu) = \sum\int_{-\infty}^{\infty} \frac{2}{\sigma^2}\frac{1-\left(\frac{x_i-\mu}{\sigma}\right)^2}{\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)^2}\cdot\frac{1}{\pi\sigma\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)}\,dx_i$$

$$= \sum \frac{2}{\pi\sigma^3}\int_{-\infty}^{\infty}\frac{1-y_i^2}{(1+y_i^2)^2}\cdot\frac{1}{1+y_i^2}(\sigma\, dy_i)$$

$$= \sum \frac{2}{\pi\sigma^2}\int_{-\infty}^{\infty}\frac{1-y_i^2}{(1+y_i^2)^3}\,dy_i$$

$$= \sum \frac{2}{\pi\sigma^2}\int_{-\infty}^{\infty}\frac{2}{(1+y_i^2)^3} - \frac{1}{(1+y_i^2)^2}\,dy_i$$

$$y_i = \tan u_i \quad dy_i = \frac{du_i}{y_i^2+1}$$

$$= \sum \frac{2}{\pi\sigma^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2}{\sec^4 u_i} - \frac{1}{\sec^2 u_i}\,du_i$$

$$= \sum \frac{2}{\pi\sigma^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^4 u_i - \cos^2 u_i\,du_i$$

$$= \sum \frac{2}{\pi\sigma^2}\left[\frac{1}{2}\cos^3 u_i \sin u_i\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u_i\,du_i - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u_i\,du_i\right]$$

$$= \sum \frac{1}{\pi\sigma^2}\left[\frac{1}{2}\pi\right] = \frac{n}{2\sigma^2}$$

~~Because the draws are iid, the Fisher information is~~ $\cancel{\frac{n}{2\sigma^2}}$. Since MLEs are consistent and asymptotically efficient, $\lim_{n\to\infty} \operatorname{var} \hat{\theta} = \left[\frac{n}{2\sigma^2}\right]^{-1}$, so

$$\hat{\theta} \longrightarrow N\left(\mu, \frac{2\sigma^2}{n}\right)$$

b) From Example 10.2.3 of Casella & Berger,
$\sqrt{n}\,(\hat{m} - \mu) \rightarrow N\left(0, \left[\frac{2}{\pi\sigma}\right]^{-1}\right) = N\left(0, \frac{\pi\sigma}{2}\right)$

c) $\dfrac{\sigma_{\hat{m}}^2}{\sigma_{\hat{\theta}}^2} = \dfrac{\frac{\pi\sigma}{2}}{\frac{2\sigma^2}{n}} = \dfrac{n\pi}{4\sigma}$ $\quad\therefore$ the ARE of $\hat{\theta}$ wrt $\hat{m}$ approaches $+\infty$ as $n \to \infty$.