Homework 11

1. Find the singular value decomposition of the following two matrices.

(a)

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$.

$$\begin{aligned} A^\top A &= \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -20 \\ -20 & 40 \end{bmatrix} \\ \det(A^\top A - \lambda I) &= \begin{bmatrix} 10-\lambda & -20 \\ -20 & 40-\lambda \end{bmatrix} &= 0 \\ &= (10-\lambda)(40-\lambda) - (-20)(-20) &= 0 \\ &= \lambda^2 - 50\lambda &= 0 \\ &= \lambda(\lambda - 50) &= 0 \\ &\therefore \lambda = 0, 50 \end{aligned}$$

$\lambda_1 = 0$ and $\lambda_2 = 50$ are the eigenvalues of $A^\top A$.

$$\begin{aligned} \lambda_1 = 0 &\implies & \sigma_1 = 0 \\ \lambda_2 = 50 &\implies & \sigma_2 = \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\boxed{\therefore \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}}$$

And so:

$$\begin{aligned} \operatorname{rref}(A^\top A - \lambda_1 I) &= \operatorname{rref} \begin{bmatrix} 10-0 & -20 \\ -20 & 40-0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\ \therefore \ker(A^\top A - \lambda_1 I) &= \ker \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

As such, $\vec{\mathbf{v}}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 0$.

$$\begin{aligned} \operatorname{rref}(A^\top A - \lambda_2 I) &= \operatorname{rref} \begin{bmatrix} 10-50 & -20 \\ -20 & 40-50 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \\ \therefore \ker(A^\top A - \lambda_2 I) &= \ker \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

As such, $\vec{\mathbf{v}}_2 = \frac{2}{\sqrt{5}} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 50$.

$$\boxed{\therefore V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}$$

Then, $\vec{\mathbf{u}}_i = \dfrac{1}{\sigma_i} A\vec{\mathbf{v}}_i$ for $i = 1, 2$:

Note $\sigma_1 = 0$, so we move on to $\vec{\mathbf{u}}_2$.

$$\begin{aligned}\vec{\mathbf{u}}_2 &= \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \frac{2}{\sqrt{5}} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}\end{aligned}$$

Since $\sigma_1 = 0$ and we need one more vector, we need to choose a vector $\vec{\mathbf{u}}_1$ such that it is orthogonal to $\vec{\mathbf{u}}_2$. Since it's just $\mathbb{R}^2$, we can just eyeball and make $\vec{\mathbf{u}}_1 = \dfrac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Sanity check: $\langle \vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2 \rangle = \dfrac{1}{\sqrt{10}}((-1)(3) + 3(1)) = 0$.

$$\boxed{\therefore U = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}}$$

And so, we have that the SVD for $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ is:

$$\begin{aligned} A &= U\Sigma V^\top \\ &= \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$.

$$\begin{aligned} A^\top A &= \begin{bmatrix} 4 & 2 & 0 & -2 \\ 2 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ -2 & 0 & -1 & 2 \end{bmatrix} \\ \det(A^\top A - \lambda I) &= \begin{bmatrix} 4-\lambda & 2 & 0 & -2 \\ 2 & 2-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & -1 \\ -2 & 0 & -1 & 2-\lambda \end{bmatrix} && = 0 \\ &= \lambda^4 - 9\lambda^3 + 18\lambda^2 && = 0 \\ &= \lambda^2(\lambda-3)(\lambda-6) && = 0 \\ &\therefore \lambda = 0, 3, 6 \end{aligned}$$

$\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 6$ are eigenvalues of $A^\top A$.

$$\begin{aligned} \lambda_1 = 0 &\quad\Longrightarrow\quad \sigma_1 = 0 \\ \lambda_2 = 3 &\quad\Longrightarrow\quad \sigma_2 = \sqrt{3} \\ \lambda_3 = 6 &\quad\Longrightarrow\quad \sigma_3 = \sqrt{6} \end{aligned}$$

$$\boxed{\therefore \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \end{bmatrix}}$$

And so:

$$\operatorname{rref}(A^\top A - \lambda_1 I) = \operatorname{rref}\begin{bmatrix} 4-0 & 2 & 0 & -2 \\ 2 & 2-0 & -1 & 0 \\ 0 & -1 & 1-0 & -1 \\ -2 & 0 & -1 & 2-0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \ker(A^\top A - \lambda_1 I) = \ker\begin{bmatrix} 1 & 0 & 1/2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

As such, $\frac{2}{3}\begin{bmatrix} -1/2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to $\lambda_1 = 0$.

$$\operatorname{rref}(A^\top A - \lambda_2 I) = \operatorname{rref}\begin{bmatrix} 4-3 & 2 & 0 & -2 \\ 2 & 2-3 & -1 & 0 \\ 0 & -1 & 1-3 & -1 \\ -2 & 0 & -1 & 2-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \ker(A^\top A - \lambda_2 I) = \ker\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

As such, $\vec{\mathbf{v}}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

$$\operatorname{rref}(A^\top A - \lambda_3 I) = \operatorname{rref}\begin{bmatrix} 4-6 & 2 & 0 & -2 \\ 2 & 2-6 & -1 & 0 \\ 0 & -1 & 1-6 & -1 \\ -2 & 0 & -1 & 2-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \ker(A^\top A - \lambda_3 I) = \ker\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

As such, $\vec{\mathbf{v}}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_3 = 6$.

$$\boxed{\therefore V = \begin{bmatrix} -1/3 & 1/\sqrt{3} & 0 & -\sqrt{2/3} \\ 2/3 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{6} \\ 2/3 & 0 & -1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}}$$

Then, $\vec{\mathbf{u}}_i = \dfrac{1}{\sigma_i} A\vec{\mathbf{v}}_i$ for $i = 2, 3$:

$$\begin{aligned} \vec{\mathbf{u}}_2 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \vec{\mathbf{u}}_3 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\boxed{\therefore U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$$

And so, we have that the SVD for $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ is:

$$\begin{aligned} A &= U\Sigma V^\top \\ &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 & 1/\sqrt{3} & 0 & -\sqrt{2/3} \\ 2/3 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{6} \\ 2/3 & 0 & -1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}^\top \\ &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -\sqrt{2/3} & -1/\sqrt{6} & 0 & 1/\sqrt{6} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \end{aligned}$$

2. A famous application of spectral theorem and SVD is the spectral graph theory. A graph (V, E) is a set of vertices V with edge set E. We say that for $x, y \in V$, $x \sim y$ if x and y are connected by an edge in E. The Graph Laplacian is defined to be a matrix A of size $|V| \times |V|$

$$A_{x,y} = \begin{cases} 1 & \text{if } x \sim y \\ -(\text{number of edges starting from } x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

For example, a triangle with three vertices

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(a) Find the SVD for the Laplacian of the triangle graph.

Using an online calculator, we find $A = U\Sigma V^\top$ for

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -\sqrt{2/3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

(b) Write down the graph Laplacian matrix for the square graph and then find its SVD.

Using the definition above, the Laplacian matrix A for a square graph is:

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Again, sparing the computation. The SVD for the square graph $A = U\Sigma V^\top$ is given by:

$$U = V = \begin{bmatrix} -1/2 & 0 & -1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 1/2 \\ -1/2 & 0 & 1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 0 & 1/2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$