Homework 7

1. Find the determinant of the matrices using Gaussian elimination.

(a)

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\det \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} &\overset{R_2 - R_3}{=} \begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&\overset{R_1 \leftrightarrow R_2}{=} (-1) \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \\
&\overset{R_2 - 2R_1}{=} (-1) \begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 1 \end{vmatrix} \\
&\overset{R_2 + 5R_1}{=} (-1) \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 8 \\ 0 & 1 & 1 \end{vmatrix} \\
&\overset{R_3 - R_2}{=} (-1) \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & -7 \end{vmatrix} \\
&= (-1)(1 \cdot 1 \cdot -7) \\
&= 7
\end{aligned}$$

(b)

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{aligned}
\det \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & -2 & 0 \end{bmatrix} &\overset{R_1 \leftrightarrow R_2}{=} (-1) \begin{vmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 0 \end{vmatrix} \\
&\overset{R_3 + 2R_2}{=} (-1) \begin{vmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{vmatrix} \\
&= (-1)(-1 \cdot 1 \cdot 6) \\
&= 6
\end{aligned}$$

(c)

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & -2 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned}
\det \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & -2 \\ 2 & 0 & 2 & 1 \end{bmatrix} &\overset{R_2 - R_1}{=} \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 1 & 2 & 0 & -2 \\ 2 & 0 & 2 & 1 \end{vmatrix} \\
&\overset{R_3 - R_1}{=} \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 3 & -1 & -3 \\ 2 & 0 & 2 & 1 \end{vmatrix} \\
&\overset{R_4 - 2R_1}{=} \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 3 & -1 & -3 \\ 0 & 2 & 0 & -1 \end{vmatrix} \\
&\overset{R_2 \leftrightarrow R_3}{=} (-1) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -1 & -3 \\ 0 & 0 & -2 & -1 \\ 0 & 2 & 0 & -1 \end{vmatrix} \\
&\overset{\frac{2}{3}R_2}{=} \left(-\frac{3}{2}\right) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -\frac{2}{3} & -2 \\ 0 & 0 & -2 & -1 \\ 0 & 2 & 0 & -1 \end{vmatrix} \\
&\overset{R_4 - R_2}{=} \left(-\frac{3}{2}\right) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -\frac{2}{3} & -2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & \frac{2}{3} & 1 \end{vmatrix} \\
&\overset{\frac{1}{3}R_3}{=} \left(-\frac{9}{2}\right) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -\frac{2}{3} & -2 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{2}{3} & 1 \end{vmatrix} \\
&\overset{R_4 + R_3}{=} \left(-\frac{9}{2}\right) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -\frac{2}{3} & -2 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} \end{vmatrix} \\
&= \left(-\frac{9}{2}\right) \left(1 \cdot 2 \cdot -\frac{2}{3} \cdot \frac{2}{3}\right) \\
&= 4
\end{aligned}$$

(d)

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\det\begin{bmatrix}2&1&1&1&1\\1&2&1&1&1\\1&1&2&1&1\\1&1&1&2&1\\1&1&1&1&2\end{bmatrix} \overset{R_1\leftrightarrow R_2}{=} (-1)\begin{vmatrix}1&2&1&1&1\\2&1&1&1&1\\1&1&2&1&1\\1&1&1&2&1\\1&1&1&1&2\end{vmatrix}$$

$$\overset{R_2-2R_1}{=} (-1)\begin{vmatrix}1&2&1&1&1\\0&-3&-1&-1&-1\\1&1&2&1&1\\1&1&1&2&1\\1&1&1&1&2\end{vmatrix}$$

$$\overset{\substack{R_3-R_1\\R_4-R_1\\R_5-R_1}}{=} (-1)\begin{vmatrix}1&2&1&1&1\\0&-3&-1&-1&-1\\0&-1&1&0&0\\0&-1&0&1&0\\0&-1&0&0&1\end{vmatrix}$$

$$\overset{R_2\leftrightarrow R_3}{=} \begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&-3&-1&-1&-1\\0&-1&0&1&0\\0&-1&0&0&1\end{vmatrix}$$

$$\overset{R_3-3R_2}{=} \begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-4&-1&-1\\0&-1&0&1&0\\0&-1&0&0&1\end{vmatrix}$$

$$\overset{\substack{R_4-R_2\\R_5-R_2}}{=} \begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-4&-1&-1\\0&0&-1&1&0\\0&0&-1&0&1\end{vmatrix}$$

$$\overset{R_3\leftrightarrow R_4}{=} (-1)\begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-1&1&0\\0&0&-4&-1&-1\\0&0&-1&0&1\end{vmatrix}$$

$$\overset{R_4-4R_3}{=} (-1)\begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-1&1&0\\0&0&0&-5&-1\\0&0&-1&0&1\end{vmatrix}$$

$$\overset{R_5-R_3}{=} (-1)\begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-1&1&0\\0&0&0&-5&-1\\0&0&0&-1&1\end{vmatrix}$$

$$\overset{R_4\leftrightarrow R_5}{=} \begin{vmatrix}1&2&1&1&1\\0&-1&1&0&0\\0&0&-1&1&0\\0&0&0&-1&1\\0&0&0&-5&-1\end{vmatrix}$$

$$\overset{R_5-5R_4}{=} \begin{vmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{vmatrix}$$
$$= 1 \cdot -1 \cdot -1 \cdot -1 \cdot -6$$
$$= 6$$

2. Given 5×5 matrices A, B, Q. Suppose that $\det A = 3, \det B = 2$ and Q is an invertible matrix. Find the determinant of $A^T B, A^3, 2A, ABA$ and $Q^{-1}AQ$.

$\det A^T B$

$$\begin{aligned} \det A^T B &= \det A^T \det B \\ &= \det A \det B \\ &= 3 \cdot 2 \\ &= 6 \end{aligned}$$

$\det A^3$

$$\begin{aligned} \det A^3 &= \det AAA \\ &= \det A \det A \det A \\ &= 3 \cdot 3 \cdot 3 \\ &= 27 \end{aligned}$$

$\det 2A$

$$\begin{aligned} \det 2A &= \det 2 \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{55} \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \\ &= \det \begin{pmatrix} 2a_{11} & & \\ & \ddots & \\ & & 2a_{55} \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \\ &= 2^5 \det \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{55} \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \\ &= 2^5 \det A \\ &= 32 \cdot 3 \\ &= 96 \end{aligned}$$

$\det ABA$

$$\begin{aligned} \det ABA &= \det A \det B \det A \\ &= 3 \cdot 2 \cdot 3 \\ &= 18 \end{aligned}$$

$\det Q^{-1}AQ$

$$\begin{aligned}\det Q^{-1}AQ &= \det Q^{-1} \det A \det Q \\ &= \frac{1}{\det Q} \det A \det Q \\ &= \det A \\ &= 3\end{aligned}$$

3. Consider the following system of linear equations:

$$\begin{cases} px + y + z = 6, \\ 3x - y + 11z = 6, \\ 2x + y + 4z = q, \end{cases}$$

(a) Find the condition on p so that the system has unique solution (Hint: $\det(A) \neq 0$).

$$\begin{cases} px + y + z \\ 3x - y + 11z \\ 2x + y + 4z \end{cases} \iff \begin{pmatrix} p & 1 & 1 \\ 3 & -1 & 11 \\ 2 & 1 & 4 \end{pmatrix}$$

$$\begin{aligned}\det \begin{pmatrix} p & 1 & 1 \\ 3 & -1 & 11 \\ 2 & 1 & 4 \end{pmatrix} &= p \begin{vmatrix} -1 & 11 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 11 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \\ &= p(-4 - 11) - (12 - 22) + (3 - (-2)) \\ &= -15p + 15 \\ \therefore \det \begin{pmatrix} p & 1 & 1 \\ 3 & -1 & 11 \\ 2 & 1 & 4 \end{pmatrix} &\neq 0 \iff p \neq 1\end{aligned}$$

As such, the system has unique solution for all $p \neq 1$.

(b) Find the condition on p and q so that the system has infinitely many solutions (Hint: $\det(A) = 0$ and no inconsistent equations). Describe the solution set.

$$\begin{cases} px + y + z = 6 \\ 3x - y + 11z = 6 \\ 2x + y + 4z = q \end{cases} \iff \left(\begin{array}{ccc|c} p & 1 & 1 & 6 \\ 3 & -1 & 11 & 6 \\ 2 & 1 & 4 & q \end{array}\right)$$

From (a), we know that $\det \begin{pmatrix} p & 1 & 1 \\ 3 & -1 & 11 \\ 2 & 1 & 4 \end{pmatrix} = 0 \iff p = 1$. So, we can just Gaussian with $p = 1$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & -1 & 11 & 6 \\ 2 & 1 & 4 & q \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & 8 & -12 \\ 0 & -1 & 2 & q-12 \end{array}\right)$$
$$\xrightarrow{-\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & q-12 \end{array}\right)$$
$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & q-9 \end{array}\right)$$

Here, we see that the last row is $0 = q - 9$. As such, the system will have a consistent solution if and only if $q = 9$.

To get the solution set, we continue to Gaussian to obtain the reduced-row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & q-9 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & q-9 \end{array}\right)$$
$$\therefore y = 2z + 3$$
$$x = -3z + 3$$

Therefore, the system

$$\begin{cases} px + y + z = 6 \\ 3x - y + 11z = 6 \\ 2x + y + 4z = q \end{cases}$$

contains infinitely many solutions if and only if $p = 1$ and $q = 9$, for which its solution set is

$$\left\{ (x, y, z) : \begin{pmatrix} -3z + 3 \\ 2z + 3 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}.$$

4. Show that if A is an $n \times n$ skew-symmetric matrix (i.e. $A^T = -A$) and n is an odd number, then $\det A = 0$.

Given that $\det A^T = \det A$. Then if $A^T = -A$ (A is skew-symmetric), we have that:

$$\det A^T = \det A = \det(-A)$$

If n is odd, then $n = k + 1$ for $k \in \mathbb{N}$. As such:

$$
\begin{aligned}
\det(-A) &= \det -1 \cdot \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}, && a_{ij} \in \mathbb{R} \\
&= \det \begin{pmatrix} (-1)a_{11} & & \\ & \ddots & \\ & & (-1)a_{nn} \end{pmatrix}, && a_{ij} \in \mathbb{R} \\
&= (-1)^n \det \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}, && a_{ij} \in \mathbb{R} \\
&= (-1)^{k+1} \det A && \boxed{n = k+1} \\
&= -\det A && \boxed{(-1)^{k+1} = -1 \quad \forall k \in \mathbb{N}}
\end{aligned}
$$

Since $\det A^T = \det A = \det(-A) = -\det A$. Then, $\det A = -\det A \iff \det A = 0$.

5. Let

$$
A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}.
$$

(i) Find the eigenvalues and eigenvectors of both A and B.

Eigenvalues and eigenvectors for A

$$
\begin{aligned}
\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} && = 0 \\
= (1-\lambda)(2-\lambda) - 12 && = 0 \\
= \lambda^2 - 3\lambda - 10 && = 0 \\
= (\lambda+2)(\lambda-5) && = 0 \\
\therefore \lambda = -2, 5 &&
\end{aligned}
$$

The eigenvalues of A are -2 and 5.

$$
\begin{gathered}
\lambda = -2 \implies A + 2I \\
A + 2I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \implies \operatorname{rref}(A+2I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
\therefore x_1 = -x_2 \\
x_2 \in \mathbb{R} \\
\therefore \ker(A+2I) = \left\{ \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} -1 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}
\end{gathered}
$$

An eigenvector corresponding to $\lambda = -2$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$
\begin{gathered}
\lambda = 5 \implies A - 5I \\
A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \implies \operatorname{rref}(A-5I) = \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix} \\
\therefore x_1 = \frac{3}{4} x_2 \\
x_2 \in \mathbb{R}
\end{gathered}
$$

$$\therefore \ker(A-5I) = \left\{ \begin{pmatrix} \frac{3}{4}x_2 \\ x_2 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

An eigenvector corresponding to $\lambda = 5$ is $\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$.

Eigenvalues and eigenvectors for B

$$\begin{aligned} \det(B-\lambda I) = \begin{vmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{vmatrix} && = 0 \\ = -\lambda \begin{vmatrix} 1-\lambda & -1 \\ 2 & 5-\lambda \end{vmatrix} - (-2) \begin{vmatrix} -1 & -1 \\ 2 & 5-\lambda \end{vmatrix} + (-3) \begin{vmatrix} -1 & 1-\lambda \\ 2 & 2 \end{vmatrix} && = 0 \\ = -\lambda(\lambda^2 - 6\lambda + 7) - (-2)(\lambda - 3) + (-3)(2\lambda - 4) && = 0 \\ = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 && = 0 \\ = -(\lambda-3)(\lambda-2)(\lambda-1) && = 0 \\ \therefore \lambda = 1, 2, 3 && \end{aligned}$$

The eigenvalues of B are 1, 2, and 3.

$$\lambda = 1 \implies B - I$$

$$B - I = \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \implies \operatorname{rref}(B-I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \therefore x_2 &= -x_3 \\ x_1 &= -x_3 \\ x_3 &\in \mathbb{R} \end{aligned}$$

$$\therefore \ker(B-I) = \left\{ \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

An eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

$$\lambda = 2 \implies B - 2I$$

$$B - 2I = \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \implies \operatorname{rref}(B-2I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \therefore x_3 &= 0 \\ x_1 &= -x_2 \\ x_2 &\in \mathbb{R} \end{aligned}$$

$$\therefore \ker(B-2I) = \left\{ \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$$

An eigenvector corresponding to $\lambda = 2$ is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

$$\lambda = 3 \implies B - 3I$$

$$B - 3I = \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix} \implies \text{rref}(B - 3I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore x_2 = 0$$
$$x_1 = -x_3$$
$$x_3 \in \mathbb{R}$$

$$\therefore \ker(B - 3I) = \left\{ \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

An eigenvector corresponding to $\lambda = 3$ is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(ii) Diagonalize A and B.

Diagonalizing A

For eigenvalues $\lambda_{A1} = -2$ and $\lambda_{A2} = 5$, the diagonalization of $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ is

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

where P is a matrix composed of the corresponding eigenvectors of A such that

$$P = \begin{pmatrix} -1 & \frac{3}{4} \\ 1 & 1 \end{pmatrix}.$$

Diagonalizing B

For eigenvalues $\lambda_{B1} = 1$, $\lambda_{B2} = 2$, and $\lambda_{B3} = 3$, the diagonalization of $B = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$ is

$$Q^{-1}BQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where Q is a matrix composed of the corresponding eigenvectors of B such that

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(iii) Find A^{10} and B^3.

Finding A^{10}

$$P = \begin{pmatrix} -1 & \frac{3}{4} \\ 1 & 1 \end{pmatrix} \implies P^{-1} = \frac{1}{7} \begin{pmatrix} -4 & 3 \\ 4 & 4 \end{pmatrix}$$
$$P^{-1}A^{10}P = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}^{10}$$

$$
\begin{aligned}
\therefore A^{10} &= P \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}^{10} P^{-1} \\
&= \begin{pmatrix} -1 & \frac{3}{4} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}^{10} \frac{1}{7} \begin{pmatrix} -4 & 3 \\ 4 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 4\,185\,853 & 4\,184\,829 \\ 5\,579\,772 & 5\,580\,796 \end{pmatrix}
\end{aligned}
$$

Finding B^3

$$
Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \implies Q^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}
$$

$$
Q^{-1}B^3Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^3
$$

$$
\begin{aligned}
\therefore B^3 &= Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^3 Q^{-1} \\
&= \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^3 \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -18 & -26 & -45 \\ -7 & 1 & -7 \\ 26 & 26 & 53 \end{pmatrix}
\end{aligned}
$$