

Homework 9

1. Consider \mathbb{R}^4 with standard inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

(i) Find the norm of the vectors $\mathbf{u} = (1, 2, 3, 2)$ and $\mathbf{v} = (2, 1, -1, 0)$.

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{1^2 + 2^2 + 3^2 + 2^2} = 3\sqrt{2} \\ \|\mathbf{v}\| &= \sqrt{2^2 + 1^2 + (-1)^2 + 0^2} = \sqrt{6}\end{aligned}$$

(ii) What is the angle between \mathbf{u} and \mathbf{v} ?

$$\theta = \cos^{-1} \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \cos^{-1} \frac{1(2) + 2(1) + 3(-1) + 2(0)}{3\sqrt{2}\sqrt{6}} = \cos^{-1} \frac{1}{6\sqrt{3}} \approx 84.4782^\circ$$

2. Consider $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for \mathbb{R}^3 .

The inner product for all pairs of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are zero.

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= (-1)(1) + 2(0) + 1(1) = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1(-1) + 0(-1) + 1(1) = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= (-1)(-1) + 2(-1) + 1(1) = 0\end{aligned}$$

And a matrix composed of these vectors is full rank.

$$\text{rref} \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \text{rref} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for \mathbb{R}^3 .

(ii) Find the orthonormal basis generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{(-1)^2 + 2^2 + 1^2}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{1^2 + 0^2 + 1^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{(-1)^2 + (-1)^2 + 1^2}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The orthonormal basis $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\}$ generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}.$$

(iii) Express $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\text{rref} \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & & \\ & & & & & \end{array} \right] = \text{rref} \left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & & \\ 2 & 0 & -1 & 2 & & \\ 1 & 1 & 1 & 0 & & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & & \\ 0 & 1 & 0 & \frac{1}{2} & & \\ 0 & 0 & 1 & -1 & & \end{array} \right]$$

$$\therefore \mathbf{v}_4 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3$$

3. Consider the space of all continuous functions on $[0, 1], C[0, 1]$ with the standard inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$$

(i) Find the norm of $f(x) = x^n$, for any positive integer n .

For $n \in \mathbb{Z}^+$:

$$\langle x^n, x^n \rangle = \int_0^1 x^{2n} \, dx = \frac{x^{2n+1}}{2n+1} \Big|_0^1 = \frac{1^{2n+1}}{2n+1} = \frac{1}{2n+1}$$

$$\|x^n\| = \sqrt{\langle x^n, x^n \rangle} = \frac{1}{\sqrt{2n+1}}$$

(ii) Find the angle between x^n and x^m .

Assuming $n, m \in \mathbb{Z}^+$.

$$\begin{aligned} \theta &= \frac{\langle x^n, x^m \rangle}{\|x^n\| \cdot \|x^m\|} = \cos^{-1} \frac{\int_0^1 x^{nm} \, dx}{\int_0^1 x^{2n} \, dx \int_0^1 x^{2m} \, dx} \\ &= \cos^{-1} \frac{1^{nm}}{\frac{nm+1}{\sqrt{\frac{1^{2n}}{2n+1} \frac{1^{2m}}{2m+1}}}} \\ &= \cos^{-1} \frac{1}{(nm+1) \frac{1}{\sqrt{(2n+1)(2m+1)}}} \\ &= \cos^{-1} \frac{\sqrt{(2n+1)(2m+1)}}{nm+1} \end{aligned}$$

Since it wasn't specified in the question, if n and m are not restricted to positive integers, then this solution is valid for all n and m such that $nm \neq -1 \wedge (2n+1)(2m+1) \geq 0$.

(iii) Show that for any $m \neq n$, $\sin 2\pi mx$ and $\sin 2\pi nx$ are always mutually orthogonal. (Hint: Check out product-to-sum formula)

Again, assuming $m, n \in \mathbb{Z}^+$. Suppose $\langle \sin 2\pi mx, \sin 2\pi nx \rangle = 0 \ \forall n \neq m$.

Note

Again, since it wasn't specified in the question, we assume $m, n \in \mathbb{Z}^+$. Note that the assumption do not hold if either m or n are not positive integers.

For example, take $m = -1$ and $n = 1$:

$$\begin{aligned} \langle \sin 2\pi mx, \sin 2\pi nx \rangle &= \langle \sin(-2\pi x), \sin 2\pi x \rangle \\ &= \int_0^1 \sin(-2\pi x) \sin 2\pi x \, dx \\ &= \frac{1}{2} \int_0^1 \cos(-4\pi x) - \cos(0) \, dx \\ &= \frac{1}{2} \int_0^1 \cos(-4\pi x) - \frac{1}{2} \int_0^1 \cos(0) \, dx \\ &= -\frac{1}{2} \end{aligned}$$

Then, using the product-to-sum formula

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)),$$

the inner product can be written as follows:

$$\begin{aligned} \langle \sin 2\pi mx, \sin 2\pi nx \rangle &= \int_0^1 (\sin 2\pi mx)(\sin 2\pi nx) \, dx \\ &= \int_0^1 \frac{1}{2}(\cos(2\pi mx - 2\pi nx) - \cos(2\pi mx + 2\pi nx)) \, dx \\ &= \frac{1}{2} \int_0^1 \cos 2\pi x(m - n) - \cos 2\pi x(m + n) \, dx \\ &= \frac{1}{2} \int_0^1 \cos 2\pi x(m - n) \, dx - \frac{1}{2} \int_0^1 \cos 2\pi x(m + n) \, dx \end{aligned}$$

Notice that if m and n are positive integers such that $m \neq n$, then x must always be a factor of 2π in both terms.

Since

$$\int_0^1 \cos 2\pi x \, dx = 0 \quad \forall x \in \mathbb{Z}^+,$$

then the inner product must be zero for all positive integers $m \neq n$.

Or more clearly, if we recall our Calculus II nightmare by performing u -substitution:

$$\begin{aligned}\langle \sin 2\pi m x, \sin 2\pi n x \rangle &= \frac{1}{2} \int_0^1 \cos 2\pi x(m-n) dx - \frac{1}{2} \int_0^1 \cos 2\pi x(m+n) dx \\ &= \text{careful calculations} \\ &= \frac{1}{4\pi} \left(\frac{\sin 2\pi(m-n)}{m-n} - \frac{\sin 2\pi(m+n)}{m+n} \right)\end{aligned}$$

We can see that the argument of \sin will be always be a multiple of 2π (and hence is always zero). Additionally, the inner product will not be defined for $m = n$.

4. Prove the identity

$$\langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle = ac\|\mathbf{v}\|^2 + (ad + bc)\langle \mathbf{v}, \mathbf{w} \rangle + bd\|\mathbf{w}\|^2.$$

$$\begin{aligned}\langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle &= \langle a\mathbf{v}, c\mathbf{v} + d\mathbf{w} \rangle + \langle b\mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle \\ &= \langle a\mathbf{v}, c\mathbf{v} \rangle + \langle a\mathbf{v}, d\mathbf{w} \rangle + \langle b\mathbf{w}, c\mathbf{v} \rangle + \langle b\mathbf{w}, d\mathbf{w} \rangle \\ &= ac\langle \mathbf{v}, \mathbf{v} \rangle + ad\langle \mathbf{v}, \mathbf{w} \rangle + bc\langle \mathbf{w}, \mathbf{v} \rangle + bd\langle \mathbf{w}, \mathbf{w} \rangle \\ &= ac\|\mathbf{v}\|^2 + (ad + bc)\langle \mathbf{v}, \mathbf{w} \rangle + bd\|\mathbf{w}\|^2\end{aligned}$$

5. Given an inner product space V .

(i) Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

(This is called the parallelogram identity)

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| \\ \|\mathbf{x} - \mathbf{y}\|^2 &= (\|\mathbf{x}\| - \|\mathbf{y}\|)^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\| \\ \therefore \|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\| \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \\ &= 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)\end{aligned}$$

(ii) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2)$$

(This is called the polarization identity)

Note

Assuming the left-hand side is meant to be $\langle \mathbf{x}, \mathbf{y} \rangle$ i.e., proving

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2).$$

$$\begin{aligned}
\frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2) &= \frac{1}{4}(\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle - \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle) \\
&= \frac{1}{4}(\langle \mathbf{x} + \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{x} + \mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x} - \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{x} - \mathbf{y}, \mathbf{y} \rangle)) \\
&= \frac{1}{4}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - (\langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle))) \\
&= \frac{1}{4}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle)) \\
&= \frac{1}{4}(\cancel{\langle \mathbf{x}, \mathbf{x} \rangle} + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \cancel{\langle \mathbf{y}, \mathbf{y} \rangle} - \cancel{\langle \mathbf{x}, \mathbf{x} \rangle} + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - \cancel{\langle \mathbf{y}, \mathbf{y} \rangle}) \\
&= \frac{1}{4}(4\langle \mathbf{x}, \mathbf{y} \rangle) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

(iii) Show that if \mathbf{u} and \mathbf{v} are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

(This is Pythagorean Theorem)

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u} + \mathbf{v}, \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \cancel{\langle \mathbf{v}, \mathbf{u} \rangle} + \cancel{\langle \mathbf{u}, \mathbf{v} \rangle} + \langle \mathbf{v}, \mathbf{v} \rangle \quad \boxed{\because \mathbf{u} \perp \mathbf{v}} \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2
\end{aligned}$$