Homework 8

1. Find the algebraic and geometric multiplicity of the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Are they diagonalizable? Explain.

For $A = \begin{bmatrix} 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & -1 & 1 & 2 \\ 0 & 2-\lambda & 1 & 1 & 3 \\ 0 & 0 & 3-\lambda & 0 & 1 \\ 0 & 0 & 0 & 4-\lambda & -1 \\ 0 & 0 & 0 & 0 & -1-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)(-1-\lambda) = 0$$
$$\therefore \lambda = -1, 1, 2, 3, 4$$

By observation, we can see that there are no repeated roots in the characteristic polynomial. As such, the algebraic multiplicity of all five eigenvalues are 1.

For $\lambda_1 = -1$,

$$\text{rref}(A - (-1)I) = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{11}{15} \\ 0 & 1 & 0 & 0 & \frac{59}{60} \\ 0 & 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There's one free variable, so the geometric multiplicity for $\lambda_1 = -1$ is 1 (because the nullity i.e., $\dim\ker(A - \lambda)$ is 1).

And similarly for $\lambda_2, \lambda_3, \lambda_4, \lambda_5$, we find that they all have one free variable.

$\lambda_2 = 1$ $\text{rref}(A - I)$	$\lambda_3 = 2$ $\text{rref}(A - 2I)$	$\lambda_4 = 3$ $\text{rref}(A - 3I)$	$\lambda_5 = 4$ $\text{rref}(A - 4I)$
$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

As such, the geometric multiplicity of all five eigenvalues are also 1.

Since the algebraic and geometric multiplicity are the same for all five eigenvalues (all one), A is diagonalizable.

For $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\begin{aligned} B - \lambda I &= \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ \det(B - \lambda I) &= (1-\lambda)(1-\lambda)(2-\lambda) && = 0 \\ &= (1-\lambda)^2(2-\lambda) && = 0 \\ &\therefore \lambda = 1, 2 \end{aligned}$$

Here, we see that 1 is a repeated root. So, the algebraic multiplicity for:

- $\lambda_1 = 1$ is 2 and
- $\lambda_2 = 2$ is 1.

For $\lambda_1 = 1$,

$$\text{rref}(B - I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There's one free variable, so the geometric multiplicity for $\lambda_1 = 1$ is 1.

Similarly for $\lambda_2 = 2$,

$$\text{rref}(B - 2I) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

There's one free variable, so the geometric multiplicity for $\lambda_2 = 2$ is 1.

Since the algebraic and geometric multiplicity for $\lambda_1 = 1$ are not equal (2 and 1, respectively), B is not diagonalizable.

2. Find the conditions on a, b, c so that the following matrix is diagonalizable.

$$\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$. A is diagonalizable if the algebraic and geometric multiplicities for all eigenvalues are equal. By inspection, we have that:

$$\det(A - \lambda I) = (1-\lambda)^2(2-\lambda) = 0$$
$$\therefore \lambda = 1, 2$$

So the eigenvalues are 1 and 2, with algebraic multiplicities of two and one, respectively.

For $\lambda_1 = 1$,

$$A - I = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} \implies \text{rref}(A - I) = \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & b - ac \\ 0 & 0 & 0 \end{bmatrix}$$

A is diagonalizable if there exists two free variables for $\lambda_1 = 1$. The last column will be a non-pivot if $b - ac = 0$.

For $\lambda_2 = 2$,

$$A - 2I = \begin{bmatrix} -1 & a & b \\ 0 & 0 & c \\ 0 & 0 & -1 \end{bmatrix} \implies \text{rref}(A - 2I) = \begin{bmatrix} 1 & -a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Additionally, for $\lambda_2 = 2$, there must be only one free variable. Luckily, the rank is already 2 and $-a$ is already in a non-pivot position, so there's no further restriction on a.

As such, the matrix is diagonalizable if $b - ac = 0$.

3. Consider the set of all 3×3 upper triangular matrices

$$\mathcal{U} = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}.$$

(a) Show that $\mathcal{U}$ is a subspace of $\mathcal{M}_{3\times 3}$.

Checking if $\mathcal{U}$ is closed under addition

Consider two matrices $A, B \in \mathcal{U}$:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix}$$

Sure enough, adding A and B will still produce an upper-triangular matrix.

$$A+B=\begin{bmatrix}a_1 & b_1 & c_1\\0 & d_1 & e_1\\0 & 0 & f_1\end{bmatrix}+\begin{bmatrix}a_2 & b_2 & c_2\\0 & d_2 & e_2\\0 & 0 & f_2\end{bmatrix}=\begin{bmatrix}a_1+a_2 & b_1+b_2 & c_1+c_2\\0 & d_1+d_2 & e_1+e_2\\0 & 0 & f_1+f_2\end{bmatrix}$$

Since $A+B\in\mathcal{U}$, it is closed under addition.

Checking if $\mathcal{U}$ is closed under scalar multiplication

Consider a scalar $\alpha\in\mathbb{R}$ and a matrix $a\in\mathcal{U}$:

$$A=\begin{bmatrix}a_1 & b_1 & c_1\\0 & d_1 & e_1\\0 & 0 & f_1\end{bmatrix}$$

Multiplying a scalar to an upper-triangular matrix will still produce an upper-triangular matrix.

$$\alpha A=\begin{bmatrix}\alpha a_1 & \alpha b_1 & \alpha c_1\\0 & \alpha d_1 & \alpha e_1\\0 & 0 & \alpha f_1\end{bmatrix}$$

Since $\alpha A\in\mathcal{U}$, it is closed under scalar multiplication.

Hence, $\mathcal{U}$ is a subspace of $\mathcal{M}_{3\times 3}$.

(b) What is the dimension of $\mathcal{U}$?

For $a,b,c,d,e,f\in\mathbb{R}$, the matrix $\begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}$ can be decomposed as:

$$\begin{aligned}\begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}&=a\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}+b\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\\&+d\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}+e\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}+f\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\end{aligned}$$

Note that these six 3×3 matrices are linearly independent. By observation, we can see that

$$\vec{\mathbf{0}}=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\iff a=b=c=d=e=f=0.$$

As such, by definition, it follows that these six vectors forms a basis of $\mathcal{U}$. Which subsequently means that $\dim\mathcal{U}=6$.

(c)

(i) Is it true that any 7 matrices taken from $\mathcal{U}$ must be linearly dependent? Explain

Yes. As shown in (b), its dimension is six. Choosing more than six means that at least one matrix must be the same or is a multiple of the other.

(ii) Is it true that any 6 matrices taken from $\mathcal{U}$ must be a basis for $\mathcal{U}$? Explain

No. If all six are identical or are multiple of each other, then they may not be a basis of $\mathcal{U}$.

4. Consider the set of all continuous functions on the interval $[a, b]$, denoted by $C([a, b])$. Show that the set of all functions with mean value zero, i.e.

$$M = \left\{ f : \frac{1}{b-a} \int_a^b f(x)dx = 0 \right\}$$

is a subspace of $C([a, b])$.

Checking if M is closed under addition

Consider two functions $f, g \in M$.

By linearity of integrals,

$$\int (f+g) = \int f + \int g = 0 + 0.$$

Since $f + g \in M$, it is closed under addition.

Checking if M is closed under scalar multiplication

Similarly, for a scalar $\alpha \in \mathbb{R}$ and a function $f \in M$.

By linearity, we know that

$$\int \alpha f = \alpha \int f = \alpha(0).$$

As such, M is closed under scalar multiplication.

Hence, M is a subspace of $C([a, b])$.

5. Determine if the following sets of vectors linearly independent in their own vector space.

(i) $x^2 - 3, 2 - x, (x-1)^2$ on $\mathcal{P}_2$.

For $a, b, c \in \mathbb{R}$, suppose that

$$a(x^2 - 3) + b(2 - x) + c(x-1)^2 = 0 \quad \forall x \in \mathbb{R}.$$

If $x = 0$, then:

$$\begin{gathered} a(0^2 - 3) + b(2 - 0) + c(0 - 1)^2 = 0 \\ -3a + 2b + c = 0 \end{gathered}$$

If $x = 1$, then:

$$\begin{gathered} a(1^2 - 3) + b(2 - 1) + c(1 - 1)^2 = 0 \\ -2a + b = 0 \end{gathered}$$

If $x = 2$, then:

$$a(2^2 - 3) + b(2 - 2) + c(2 - 1)^2 = 0$$
$$a + c = 0$$

And so, we have a homogenous system of equations:

$$\begin{cases} -3a + 2b + c & = 0 \\ -2a + b & = 0 \\ a + c & = 0 \end{cases}$$

$$\text{rref} \left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since there are free variables, they are linearly dependent.

(ii) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix}$ **on** $\mathcal{M}_{2\times 2}$

For $a, b, c \in \mathbb{R}$, suppose that:

$$a \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + c \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \forall a, b, c \in \mathbb{R}.$$

Then:

$$\begin{aligned} a \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + c \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} &= \begin{bmatrix} 2a + b + c & a + 2b + 5c \\ 3a + 2c & 2a + 3b \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Putting it into a homogenous system, we get:

$$\text{rref} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 2 & 5 & 0 \\ 3 & 0 & 2 & 0 \\ 2 & 3 & 0 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since there are no free variables, they are linearly independent.

(iii) e^x, e^{3x} **on** $C([0, 1])$**.**

For $a, b \in \mathbb{R}$, suppose that:

$$ae^x + be^{3x} = 0 \quad \forall a, b \in \mathbb{R}.$$

Let $x = 0$, then:

$$ae^0 + be^0 = 0$$
$$a + b = 0$$

Let $x = 1$, then:

$$\begin{gathered} ae^1 + be^3 = 0 \\ ae + be^3 = 0 \end{gathered}$$

Again, putting them into a system, we get:

$$\text{rref}\left[\begin{array}{cc|c} 1 & 1 & 0 \\ e & e^3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

As such, they are linearly independent.

6. Let

$$M = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_{11} + a_{22} + a_{33} = 0 \right\}.$$

(i) Show that M is a subspace for $\mathcal{M}_{3\times 3}$.

Checking if M is closed under addition

Consider two matrices $A, B \in M$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Then,

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}.$$

And so the trace of $A + B$ is:

$$\begin{aligned} (a_{11}+b_{11}) + (a_{22}+b_{22}) + (a_{33}+b_{33}) &= (a_{11}+a_{22}+a_{33}) + (b_{11}+b_{22}+b_{33}) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, M is closed under addition.

Checking if M is closed under scalar multiplication

Consider a scalar $\alpha \in \mathbb{R}$ and a matrix $A \in M$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then,

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix}.$$

And so,

$$\alpha a_{11} + \alpha a_{22} + \alpha a_{33} = \alpha(a_{11} + a_{22} + a_{33}) = \alpha(0) = 0.$$

Since $\alpha A \in M$, it is closed under scalar multiplication.

Hence, M is a subspace of $\mathcal{M}_{3\times3}$.

(ii) Find a basis for M and what is the dimension of M?

For $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in M$. Since $a_{11} + a_{22} + a_{33} = 0$, then $a_{11} = -a_{22} - a_{33}$.

And so,

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= \begin{bmatrix} -a_{22} - a_{33} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= a_{22} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{33} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad + a_{21} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{32} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

A basis of M is

$$\left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

and its dimension is eight.