

Homework 4

1. Compute the inverse of the following matrices.

(a)

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

Using the fact that $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ is a 2×2 matrix such that:

$$\det \left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \right) = 4(-1) - 3(2) = -10 \neq 0$$

Then:

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} = -\frac{1}{10} \begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 30 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 2 & 30 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right] & \xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 30 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right] \\ & \xrightarrow[\frac{R_1-30R_3}{R_2-R_3}]{\frac{R_2-R_3}{R_1-30R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -15 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right] \\ & \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{5} & -14 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array} \right] \end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 2 & 30 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -14 \\ 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 & -4 & -140 \\ 0 & 2 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 6 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 3 & 1 & 6 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\begin{array}{c} R_2+3R_3 \\ R_3+R_1 \end{array}]{\begin{array}{c} R_2-R_3 \\ \frac{1}{5}R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 7 & 12 & 0 & 1 & 3 \\ 0 & 5 & 5 & 1 & 0 & 1 \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_2-R_3 \\ \frac{1}{5}R_3 \end{array}]{\begin{array}{c} R_2-R_3 \\ \frac{1}{5}R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 7 & -1 & 1 & 2 \\ 0 & 1 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_2-R_3 \\ \frac{1}{5}R_3 \end{array}]{\begin{array}{c} R_2-R_3 \\ \frac{1}{5}R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 1 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_3-R_2 \\ \frac{1}{5}R_3 \end{array}]{\begin{array}{c} R_3-R_2 \\ \frac{1}{5}R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 0 & -5 & \frac{7}{5} & -1 & -\frac{8}{5} \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_3-R_2 \\ \frac{1}{5}R_3 \end{array}]{\begin{array}{c} R_3-R_2 \\ \frac{1}{5}R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_1-3R_3 \\ R_2-6R_3 \end{array}]{\begin{array}{c} R_1-3R_3 \\ R_2-6R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{46}{25} & -\frac{3}{5} & -\frac{24}{25} \\ 0 & 1 & 0 & \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array} \right] \\ & \xrightarrow[\begin{array}{c} R_1-3R_2 \\ R_2-6R_3 \end{array}]{\begin{array}{c} R_1-3R_2 \\ R_2-6R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 & \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array} \right] \end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 6 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{5} & 0 & -\frac{3}{5} \\ \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 10 & 0 & -15 \\ 12 & -5 & -3 \\ -7 & 5 & 8 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} 1 & 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\substack{R_2-R_3 \\ R_3-R_4}]{} \left[\begin{array}{cccc|cccc} 1 & 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_1-3R_3} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 3 & 1 & 0 & -3 & 3 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_1-3R_4} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_1-R_2} \left[\begin{array}{cccc|cccc} 0 & 3 & 0 & 0 & 1 & -1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow[\substack{R_1 \leftrightarrow R_2 \\ \frac{1}{3}R_2}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 & 3 & -3 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(e)

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Using the fact that $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is a 2×2 matrix such that:

$$\det \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \right) = \cos^2 \theta - (-\sin \theta \sin \theta) = \cos^2 \theta + \sin^2 \theta = 1 \neq 0$$

Then:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation that maps the standard vector $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Is T invertible? Explain.

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then, T^{-1} exists if A^{-1} exists.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A has a full rank. Therefore, T is invertible.

3.

(a) Find invertible matrix A, B such that $A + B$ is not invertible.

An $n \times n$ matrix M is invertible if and only if $\text{rank}(M) = n$.

Let A and B be 1×1 matrices.

$$\begin{array}{llll} A = [1] & \implies & \text{rank}(A) = 1 & \implies & A^{-1} = [1] \\ B = [-1] & \implies & \text{rank}(B) = 1 & \implies & B^{-1} = [-1] \end{array}$$

Hence, A and B are invertible.

$$A + B = [1] + [-1] = [0] \implies \text{rank}(A + B) = 0 \neq 1$$

Hence, $A + B$ is not invertible.

(b) Find non-invertible matrix A, B such that $A + B$ is invertible.

Let A and B be 2×2 matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies \text{rank}(A) = 1 \neq 2$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \implies \text{rank}(B) = 1 \neq 2$$

Hence, A and B are not invertible.

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \text{rank}(A + B) = 2$$

$$\implies (A + B)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $A + B$ is invertible.

4. For which values of the constant a, b is the following matrix not invertible?

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

(Hint: find its row echelon form)

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \xrightarrow[R_3 - R_2]{R_2 - R_1} \begin{bmatrix} a & b & b \\ 0 & a - b & 0 \\ 0 & 0 & a - b \end{bmatrix}$$

The matrix is not invertible if $a = b$ or $a = 0$.

5. Determine if the following statement true or false.

(a) If a matrix A has a completely zero row, then it is not invertible.

True. If an $n \times n$ matrix A contains a zero row, then $\text{rank}(A) < n$.

(b) Upper triangular matrices are always invertible.

False. If there is a zero on the major diagonal, then they will not have a full rank.

(c) If A is invertible, then $Ax = 0$ may have non-trivial solution.

False. $\text{rref}(A)$ would be the identity matrix and therefore $A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$.

(d) If AB is invertible, then A is invertible.

False. Let A be an $m \times n$ matrix and B be an $n \times m$ for nonzero $m \neq n$.

A and B are not invertible because they are not square matrices. AB is an $m \times m$ square matrix and *could* be invertible if $\text{rank}(AB) = m$.

For example: let $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 2 \end{bmatrix} \implies (AB)^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$.

6. For the matrices A, B are invertible, Is the following true? If it is true, verify it. If it is false, give an example to explain why it is false.

(i) $(A^2)^{-1} = (A^{-1})^2$

True.

$$\begin{aligned}(A^2)^{-1} &= (AA)^{-1} \\ &= A^{-1}A^{-1} \\ &= (A^{-1})^2\end{aligned}$$

(ii) $(A + B)^{-1} = A^{-1} + B^{-1}$

False. Let $A = B = \begin{bmatrix} 1 \end{bmatrix}$. Then:

$$\begin{aligned}(A + B)^{-1} &= [2]^{-1} = \left[\frac{1}{2}\right] \\ A^{-1} + B^{-1} &= [1] + [1] = [2] \\ \therefore (A + B)^{-1} &\neq A^{-1} + B^{-1}\end{aligned}$$

7.

(i) Let A be an $n \times n$ matrix. Expand $(I - A)(I + A + A^2)$.

$$\begin{aligned}(I - A)(I + A + A^2) &= I(I - A) + A(I - A) + A^2(I - A) \\ &= \cancel{I^2 - AI} + \cancel{AI - A^2} + \cancel{A^2I - A^3} \\ &= I^2 - A^3 \\ &= I - A^3\end{aligned}$$

$A^2I = A^2$

(ii) Suppose that $A^3 = O$, the zero matrix. Use (i), find $(I - A)^{-1}$ in terms of A .

$$\begin{aligned}(I - A)(I + A + A^2) &= I \iff (I - A)^{-1}(I + A + A^2) = I \\ &\iff (I - A)(I + A + A^2)^{-1} = I\end{aligned}$$

$$\therefore (I - A)^{-1} = (I + A + A^2)$$

(iii) (Bonus 1 point) If $A^k = O$, find $(I - A)^{-1}$.

From (i) and (ii), we use

$$(I - A)(I + A + A^2)$$

to find $(I - A)^{-1}$ for $A^3 = O$. Then, for $A^k = O$, we can use:

$$(I - A)(I + A + A^2 + \dots + A^{k-1})$$

We note that the second term is in a geometric form, something that I definitely remembered from Calculus II without Dr. Lai pointing it out.

So, let the identity matrix I be the coefficient and the $n \times n$ matrix A be the common factor. As such, we have:

$$\begin{aligned}(I - A) \sum_{n=0}^{k-1} IA^n &= (I - A)(IA^0 + IA^1 + IA^2 + \cdots + IA^{k-1}) \\ &= (I - A)(I + IA + IA^2 + \cdots + IA^{k-1}) \\ &= (I - A)(I + A + A^2 + \cdots + A^{k-1})\end{aligned}$$

Then, use the closed-form formula for a geometric sum:

$$\begin{aligned}(I - A) \sum_{n=0}^{k-1} IA^n &= (I - A) \cdot I \left(\frac{1 - A^k}{1 - A} \right) \\ &= (I - A) \left(\frac{1 - A^k}{1 - A} \right) \\ &= (I - A) \left(\frac{1}{1 - A} \right)\end{aligned}$$

As such, we have that

$$(I - A)^{-1} = \sum_{n=0}^{k-1} IA^n = \frac{1}{1 - A}$$

for $A^k = O$.