Homework 10

1. Consider $\mathbb{R}^4$. Let $\mathbf{u} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$. Find the basis for the orthogonal complement of $W = \text{span}\{\,\mathbf{u},\mathbf{v}\,\}$.

Let $A = \begin{bmatrix}| & | \\ \mathbf{u} & \mathbf{v} \\ | & |\end{bmatrix}$. Then, $W^\perp = \ker A^\top = \ker \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}$.

$$\text{rref}\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix} = \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}$$

$$\begin{gathered}\therefore y = -2z - 3w\\ x = z + 2w\\ z, w \in \mathbb{R}\end{gathered}$$

As such,

$$W^\perp = \left\{ \begin{bmatrix}z+2w\\-2z-3w\\z\\w\end{bmatrix} : z, w \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix} \right\}.$$

2. Find the orthogonal projection of the vector $(1,1,1)$ onto the subspace defined by the equations

$$\begin{cases}x+y+z=0,\\ x-y-2z=0,\end{cases}$$

Let W be the subspace defined by the above equation.

$$\text{rref}\begin{bmatrix}1 & 1 & 1\\1 & -1 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{3}{2}\end{bmatrix}$$

$$\begin{gathered}\therefore y = -\frac{3}{2}z\\ x = \frac{1}{2}z\\ z \in \mathbb{R}\end{gathered}$$

$$\therefore W = \left\{ \begin{bmatrix}\frac{1}{2}z\\-\frac{3}{2}z\\z\end{bmatrix} : z \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} \right\}$$

Let $\vec{\mathbf{x}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix}$. Then, the orthogonal projection of $\vec{\mathbf{x}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is given by

$$\begin{aligned}
\text{proj}_W(\vec{\mathbf{x}}) = \frac{\langle \vec{\mathbf{x}}, \vec{\mathbf{v}} \rangle}{||\vec{\mathbf{v}}||^2}\vec{\mathbf{v}} &= \frac{\left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} \right|\right|^2} \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} \\
&= \frac{\frac{1}{2} - \frac{3}{2} + 1}{\frac{1}{4} + \frac{9}{4} + 1} \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} \\
&= \vec{\mathbf{0}}.
\end{aligned}$$

3. Find the orthogonal basis of $\mathbb{R}^3$ with $\begin{bmatrix}1\\1\\0\end{bmatrix}$ as one of the vectors. Hint: You can use Gram-Schmidt process on a basis with $\begin{bmatrix}1\\1\\0\end{bmatrix}$ as the first vector. e.g.

$$\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

Let $\vec{\mathbf{w}}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{\mathbf{w}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{\mathbf{w}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Let $\vec{\mathbf{v}}_1 = \vec{\mathbf{w}}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$. Then, using the Gram–Schmidt process, the orthogonal vectors $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ are given by the following.

$$\begin{aligned}
\vec{\mathbf{v}}_2 &= \vec{\mathbf{w}}_2 - \text{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{w}}_2) \\
&= \vec{\mathbf{w}}_2 - \frac{\langle \vec{\mathbf{w}}_2, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2}\vec{\mathbf{v}}_1 \\
&= \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{\left\langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix}1\\1\\0\end{bmatrix} \right|\right|^2} \begin{bmatrix}1\\1\\0\end{bmatrix} \\
&= \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{0(1)+1(1)+0(0)}{1^2+1^2+0^2}\begin{bmatrix}1\\1\\0\end{bmatrix} \\
&= \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\vec{\mathbf{v}}_3 &= \vec{\mathbf{w}}_3 - \text{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{w}}_3) - \text{proj}_{\vec{\mathbf{v}}_2}(\vec{\mathbf{w}}_3) \\
&= \vec{\mathbf{w}}_3 - \frac{\langle \vec{\mathbf{w}}_3, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2}\vec{\mathbf{v}}_1 - \frac{\langle \vec{\mathbf{w}}_3, \vec{\mathbf{v}}_2 \rangle}{||\vec{\mathbf{v}}_2||^2}\vec{\mathbf{v}}_2 \\
&= \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{\left\langle \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix}1\\1\\0\end{bmatrix} \right|\right|^2} \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\left\langle \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \right|\right|^2} \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \\
&= \begin{bmatrix}0\\0\\1\end{bmatrix} - 0\begin{bmatrix}1\\1\\0\end{bmatrix} - 0\begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \\
&= \begin{bmatrix}0\\0\\1\end{bmatrix}
\end{aligned}$$

As such, an orthogonal basis of $\mathbb{R}^3$ is $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

4.

(a) Find an orthonormal basis for the kernel of the following matrix.

$$A = \begin{bmatrix}2 & 1 & 0 & -1\\ 3 & 2 & -1 & -1\end{bmatrix}.$$

$$\text{rref}(A) = \begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & -2 & 1\end{bmatrix}$$

$$\therefore y = 2z - w$$
$$x = -z + w$$

$$\ker(A) = \left\{ \begin{bmatrix} -z+w \\ 2z-w \\ z \\ w \end{bmatrix} : z, w \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let $\vec{\mathbf{u}}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Then, $\{ \vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2 \}$ is a basis of $\ker(A)$ (and are linearly independent).

To produce an orthogonal basis, we apply the Gram–Schmidt process. Let $\vec{\mathbf{v}}_1 = \vec{\mathbf{u}}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Then,

$$\begin{aligned} \vec{\mathbf{v}}_2 &= \vec{\mathbf{u}}_2 - \operatorname{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{u}}_2) \\ &= \vec{\mathbf{u}}_2 - \frac{\langle \vec{\mathbf{u}}_2, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2} \vec{\mathbf{v}}_1 \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right|\right|^2} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1(-1) + (-1)2 + 0(1) + 1(0)}{(-1)^2 + 2^2 + 1^2 + 0^2} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{aligned}.$$

Hence, $\{ \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \}$ is an **orthogonal** basis of $\ker(A)$. Finally, an **orthonormal** basis of $\ker(A)$ is

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\}.$$

(b) Find an orthonormal basis for $(\ker(A))^{\perp}$, the orthogonal complement of $\ker(A)$.

Since $(\ker(A))^{\perp} = \operatorname{Im}(A^{\top}) = \operatorname{Im} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}$.

$$\text{rref}(A^\top) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \operatorname{Im}(A^\top) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Let $\vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$. $\{ \vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2 \}$ is a basis of $\operatorname{Im}(A^\top)$. Then, we apply Gram–Schmidt to orthogonalize them.

Let $\vec{\mathbf{v}}_1 = \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Then,

$$\begin{aligned}
\vec{\mathbf{v}}_2 &= \vec{\mathbf{u}}_2 - \operatorname{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{u}}_2) \\
&= \vec{\mathbf{u}}_2 - \frac{\langle \vec{\mathbf{u}}_2, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2} \vec{\mathbf{v}}_1 \\
&= \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right|\right|^2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix} - \frac{3(2) + 2(1) + (-1)(0) + (-1)(-1)}{2^2 + 1^2 + 0^2 + (-1)^2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

Then, $\{ \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \}$ is an **orthogonal** basis of $(\ker(A))^\perp$. Finally, an **orthonormal** basis of $(\ker(A))^\perp$ is

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\}.$$

(c) Does the orthonormal basis in (i) combined with the orthonormal basis in (ii)for an orthonormal basis for $\mathbb{R}^4$? Explain.

The union of the orthonormal bases for A and $A^\top$ found in parts (i) and (ii) is

$$\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}, \frac{1}{\sqrt{\frac{3}{2}}}\begin{bmatrix}\frac{1}{2}\\0\\\frac{1}{2}\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{\frac{3}{2}}}\begin{bmatrix}0\\\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix} \right\}.$$

Placing them as column vectors in a matrix:

$$\operatorname{rref}\begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

We find that the reduced-row echelon form is full rank. As such, these vectors span $\mathbb{R}^4$.

Then, we need to check if they are mutually orthogonal to determine if they are an *orthonormal* basis of $\mathbb{R}^4$. By checking all pairs in the set, we find that they are orthogonal.

As such, the abovementioned set is an orthonormal basis of $\mathbb{R}^4$.

5. Consider the following subspace of $\mathbb{R}^4$

$$V = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\\-1\end{bmatrix} \right\}$$

(i) What is the dimension of V?

$$\operatorname{rref}\begin{bmatrix}1&1&0\\1&0&2\\1&0&1\\1&1&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

A matrix composed of the three vectors has a full column rank. As such, they form a basis of V and thus $\dim V = 3$.

(ii) Using Gram-Schmidt Process, find an orthogonal basis for V.

Let $\vec{\mathbf{u}}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, and $\vec{\mathbf{u}}_3 = \begin{bmatrix}0\\2\\1\\-1\end{bmatrix}$. As shown in (i), $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ is a basis of V.

Now let $\vec{\mathbf{v}}_1 = \vec{\mathbf{u}}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Then, applying Gram–Schmidt:

$$
\begin{aligned}
\vec{\mathbf{v}}_2 &= \vec{\mathbf{u}}_2 - \text{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{u}}_2) \\
&= \vec{\mathbf{u}}_2 - \frac{\langle \vec{\mathbf{u}}_2, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2} \vec{\mathbf{v}}_1 \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right|\right|^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1(1) + 0(1) + 0(1) + 0(1) + 1(1)}{1^2 + 1^2 + 1^2 + 1^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{v}}_3 &= \vec{\mathbf{u}}_3 - \text{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{u}}_3) - \text{proj}_{\vec{\mathbf{v}}_2}(\vec{\mathbf{u}}_3) \\
&= \vec{\mathbf{u}}_3 - \frac{\langle \vec{\mathbf{u}}_3, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2} \vec{\mathbf{v}}_1 - \frac{\langle \vec{\mathbf{u}}_3, \vec{\mathbf{v}}_2 \rangle}{||\vec{\mathbf{v}}_2||^2} \vec{\mathbf{v}}_2 \\
&= \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right|\right|^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right|\right|^2} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \frac{0(1) + 2(1) + 1(1) + (-1)(1)}{1^2 + 1^2 + 1^2 + 1^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0(\frac{1}{2}) + 2(-\frac{1}{2}) + 1(-\frac{1}{2}) + (-1)(\frac{1}{2})}{(\frac{1}{2})^2 + (-\frac{1}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}
$$

And so, an orthogonal basis of V is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$.

(iii) Find the orthogonal projection of $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ to V.

From (ii), $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\right\}$ is an orthogonal basis of V. As such,

$$
\begin{aligned}
\operatorname{proj}_V \begin{bmatrix}1\\2\\3\\4\end{bmatrix} &= \frac{\left\langle \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\rangle}{\left\|\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\|^2} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\left\langle \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} \right\rangle}{\left\|\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right\|^2} \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} + \frac{\left\langle \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \right\rangle}{\left\|\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\right\|^2} \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \\
&= \frac{1(1)+2(1)+3(1)+4(1)}{1^2+1^2+1^2+1^2} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0 \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} + \frac{1(\frac{1}{2})+2(\frac{1}{2})+3(-\frac{1}{2})+4(-\frac{1}{2})}{(\frac{1}{2})^2+(\frac{1}{2})^2+(-\frac{1}{2})^2+(-\frac{1}{2})^2} \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}3\\3\\7\\7\end{bmatrix}.
\end{aligned}
$$

6.

(i) Find the least square solution of the following system

$$
\begin{bmatrix}2 & 3\\4 & -2\\1 & 5\\2 & 0\end{bmatrix}\mathbf{x} = \begin{bmatrix}2\\-1\\1\\3\end{bmatrix}
$$

Let $A = \begin{bmatrix}2 & 3\\4 & -2\\1 & 5\\2 & 0\end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix}2\\-1\\1\\3\end{bmatrix}$. Then, $A^\top = \begin{bmatrix}2 & 4 & 1 & 2\\3 & -2 & 5 & 0\end{bmatrix}$.

Since $A^\top A = \begin{bmatrix}2 & 4 & 1 & 2\\3 & -2 & 5 & 0\end{bmatrix}\begin{bmatrix}2 & 3\\4 & -2\\1 & 5\\2 & 0\end{bmatrix} = \begin{bmatrix}25 & 3\\3 & 38\end{bmatrix}$ and its inverse exists. Then, the least square solution $\hat{\mathbf{x}}$ can be derived by applying $A^\top$ to both sides.

$$
A^\top A\hat{\mathbf{x}} = A^\top\vec{\mathbf{b}}
$$

$$\begin{aligned}
\therefore \hat{\mathbf{x}} &= (A^\top A)^{-1} A^\top \vec{\mathbf{b}} \\
&= \begin{bmatrix} 25 & 3 \\ 3 & 38 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 & 1 & 2 \\ 3 & -2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} \\
&= \frac{1}{941} \begin{bmatrix} 227 \\ 304 \end{bmatrix}
\end{aligned}$$

(ii) Find the orthogonal projection of b onto the image of A using the least square solution.

From (i), where $A = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \\ 2 & 0 \end{bmatrix}$ and $\hat{\mathbf{x}} = \frac{1}{941} \begin{bmatrix} 227 \\ 304 \end{bmatrix}$. Then,

$$\vec{\mathbf{b}} = A\hat{\mathbf{x}} = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} \frac{1}{941} \begin{bmatrix} 227 \\ 304 \end{bmatrix} = \frac{1}{941} \begin{bmatrix} 1366 \\ 300 \\ 1747 \\ 454 \end{bmatrix}.$$

7. Find the least square fitting straight line $y = C + Dt$ given the following set of data.

t_i	-2	0	1	3
y_i	0	1	2	5

Using the equation of a straight line, we have the following system of equation:

$$\begin{cases} 0 &= C + D(-2) \\ 1 &= C + D(0) \\ 2 &= C + D(1) \\ 5 &= C + D(3) \end{cases} \iff \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

Let $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$, and $\vec{\mathbf{x}} = \begin{bmatrix} C \\ D \end{bmatrix}$. Here, we want to find $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, will produce an inconsistent solution. Instead, we find a least square solution for $\hat{\mathbf{x}} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$ by applying $A^\top$ to both sides, such that

$$A^\top A\hat{\mathbf{x}} = A^\top \vec{\mathbf{b}}.$$

As such, we have:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 2 \\ 2 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 17 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Therefore, our line of best fit is given by the equation $y = \frac{3}{2} + t$.

8. Consider the non-standard inner product on $\mathbb{R}^2$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(a) Verify this is an inner product of $\mathbb{R}^2$.

First, notice that this definition results in a 1×1 matrix. For $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^2$,

$$\begin{aligned} \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2) \end{bmatrix} \end{aligned}$$

Symmetry and bilinearity should be quiet obvious since we can just apply commutative, associative, and distributive properties of addition and multiplication here.

But for the sake of completion, consider $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^2$ and $c, d \in \mathbb{R}$.

Symmetry

$$\begin{aligned} \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2) \end{bmatrix} \\ \langle \vec{\mathbf{v}}, \vec{\mathbf{u}} \rangle &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1(v_1 + 2v_2) + u_2(2v_1 + 5v_2) \end{bmatrix} \end{aligned}$$

And indeed,

$$v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2) = u_1(v_1 + 2v_2) + u_2(2v_1 + 5v_2)$$

if you expand each of the term. Hence, $\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{v}}, \vec{\mathbf{u}} \rangle$.

Bilinearity

$$\begin{aligned} \langle c\vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} cu_1 & cu_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &&= \begin{bmatrix} v_1(cu_1 + 2cu_2) + v_2(2cu_1 + 5cu_2) \end{bmatrix} \\ &= c \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &&= \begin{bmatrix} cv_1(u_1 + 2u_2) + cv_2(2u_1 + 5u_2) \end{bmatrix} \\ &= c\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &&= \begin{bmatrix} c(v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2)) \end{bmatrix} \end{aligned}$$

Hence, $\langle c\vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = c\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle$.

$$
\begin{aligned}
\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2) \end{bmatrix} \\
\langle \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1(w_1 + 2w_2) + v_2(2w_1 + 5w_2) \end{bmatrix} \\
\langle \vec{\mathbf{u}} + \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle &= \begin{bmatrix} u_1 + w_1 & u_2 + w_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1(u_1 + w_1 + 2(u_2 + w_2)) + v_2(2(u_1 + w_1) + 5(u_2 + w_2)) \end{bmatrix}
\end{aligned}
$$

By inspection, combining the first term in $\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle$ and $\langle \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle$ together produces the first term in $\langle \vec{\mathbf{u}} + \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle$. And the same applies for the second term.

$$
\begin{aligned}
\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle + \langle \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle = \quad & \begin{bmatrix} v_1(u_1 + 2u_2) + v_2(2u_1 + 5u_2) + v_1(w_1 + 2w_2) + v_2(2w_1 + 5w_2) \end{bmatrix} \\
= \quad & \begin{bmatrix} v_1(u_1 + 2u_2) + v_1(w_1 + 2w_2) + v_2(2u_1 + 5u_2) + v_2(2w_1 + 5w_2) \end{bmatrix} \\
= \quad & \begin{bmatrix} v_1(u_1 + w_1 + 2(u_2 + w_2)) + v_2(2(u_1 + w_1) + 5(u_2 + w_2)) \end{bmatrix} \\
= \quad & \langle \vec{\mathbf{u}} + \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle
\end{aligned}
$$

Hence, $\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle + \langle \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{u}} + \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle$. As such, this inner product satisfies bilinearity.

Positive-definite

$$
\begin{aligned}
\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1(u_1 + 2u_2) + u_2(2u_1 + 5u_2) \end{bmatrix} \\
&= \begin{bmatrix} u_1^2 + 5u_2^2 + 4u_1 u_2 \end{bmatrix}
\end{aligned}
$$

Notice that $\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle$ is positive for all real u_1, u_2 and that $\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle = 0 \iff u_1 = u_2 = 0$. As such, $\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle = 0 \iff \vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle \geq 0$ for all $\vec{\mathbf{u}} \in \mathbb{R}^2$.

Thus, satisfying the properties of an inner product space.

(b) Using Gram-Schmidt process, starting with the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find an orthogonal basis under this inner product.

> **Note**
>
> For this part, we will interpret the output of the inner product (the 1×1 matrix) as a scalar. Otherwise, we will not be able to define an orthogonal basis because division is not generally defined as a matrix operation.

Let $\vec{\mathbf{v}}_1 = \vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then,

$$
\begin{aligned}
\vec{\mathbf{v}}_2 &= \vec{\mathbf{u}}_2 - \operatorname{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{e}}_2) \\
&= \vec{\mathbf{e}}_2 - \frac{\langle \vec{\mathbf{e}}_2, \vec{\mathbf{v}}_1 \rangle}{||\vec{\mathbf{v}}_1||^2} \vec{\mathbf{v}}_1 \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle}{\left|\left| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|\right|^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\end{aligned}
$$

And so, an orthogonal basis under this inner product is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

9. Consider the space of all continuous functions on $[0, 1]$, $C[0, 1]$ with the standard inner product.

$$
\langle f, g \rangle = \int_0^1 f(x) g(x) \, \mathrm{d}x
$$

(a) Use Gram-Schmidt process. Find an orthogonal basis using $\operatorname{span}\{ x^2, x, 1 \}$.

Let $U_1 = x^2$, $U_2 = x$, and $U_3 = 1$.

Take $V_1 = U_1 = x^2$. Then,

$$
\begin{aligned}
V_2 &= U_2 - \text{proj}_{V_1}(U_2) \\
&= U_2 - \frac{\langle U_2, V_1 \rangle}{||V_1||^2} V_1 \\
&= x - \frac{\langle x, x^2 \rangle}{||x^2||^2} x^2 \\
&= x - \frac{\int_0^1 x \cdot x^2 \, \mathrm{d}x}{\int_0^1 x^2 \cdot x^2 \, \mathrm{d}x} x^2 \\
&= x - \frac{5}{4} x^2
\end{aligned}
$$

$$
\begin{aligned}
V_3 &= U_3 - \text{proj}_{V_1}(U_3) - \text{proj}_{V_2}(U_3) \\
&= U_3 - \frac{\langle U_3, V_1 \rangle}{||V_1||^2} V_1 - \frac{\langle U_3, V_2 \rangle}{||V_2||^2} V_2 \\
&= 1 - \frac{\langle 1, x^2 \rangle}{||x^2||^2} x^2 - \frac{\langle 1, x - \frac{5}{4}x^2 \rangle}{||x - \frac{5}{4}x^2||^2} \left(x - \frac{5}{4} x^2 \right) \\
&= 1 - \frac{\int_0^1 1 \cdot x^2 \, \mathrm{d}x}{\int_0^1 (x^2)^2 \, \mathrm{d}x} x^2 - \frac{\int_0^1 1 \cdot (x - \frac{5}{4}x^2) \, \mathrm{d}x}{\int_0^1 (x - \frac{5}{4}x^2)^2 \, \mathrm{d}x} \left(x - \frac{5}{4} x^2 \right) \\
&= \frac{10x^2 - 12x}{3} + 1
\end{aligned}
$$

And so, we have that $\left\{ x^2, x - \frac{5}{4}x^2, \frac{10x^2 - 12x}{3} + 1 \right\}$ is an orthogonal basis of this inner product space.

(b) We prove previously that for any $m \neq n, \sin 2\pi m x$ and $\sin 2\pi n x$ are always mutually orthogonal. Write down the formula of the orthogonal projection of x^2 onto the subspace

$$\text{span}\{\, 1, \sin(2\pi x), \sin(2\pi 2x) \,\}.$$

Compute it. You need to do integration by part to find the coefficient, but you can use an online integration calculator to find it.

Let $W \subset C[0,1]$ be a subspace where $\{\, 1, \sin(2\pi x), \sin(2\pi 2x) \,\}$ is an orthonormal basis of W, as shown in the previous homework. Then, the orthogonal projection of x^2 on to W is given by:

$$
\begin{aligned}
\text{proj}_W(x^2) &= \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle x^2, \sin(2\pi x) \rangle}{\langle \sin(2\pi x), \sin(2\pi x) \rangle} \sin(2\pi x) + \frac{\langle x^2, \sin(2\pi 2x) \rangle}{\langle \sin(2\pi 2x), \sin(2\pi 2x) \rangle} \sin(2\pi 2x) \\
&= \frac{\int_0^1 x^2 \cdot 1 \, \mathrm{d}x}{\int_0^1 1^2 \, \mathrm{d}x} 1 + \frac{\int_0^1 x^2 \cdot \sin(2\pi x) \, \mathrm{d}x}{\int_0^1 \sin^2(2\pi x) \, \mathrm{d}x} \sin(2\pi x) + \frac{\int_0^1 x^2 \cdot \sin(2\pi 2x) \, \mathrm{d}x}{\int_0^1 \sin^2(2\pi 2x) \, \mathrm{d}x} \sin(2\pi 2x) \\
&= \frac{1}{3} - \frac{\sin(2\pi x)}{\pi} - \frac{\sin(4\pi x)}{2\pi}
\end{aligned}
$$