Homework 9

1. Consider $\mathbb{R}^4$ with standard inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

(i) Find the norm of the vectors $\mathbf{u} = (1, 2, 3, 2)$ and $\mathbf{v} = (2, 1, -1, 0)$.

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{1^2 + 2^2 + 3^2 + 2^2} = 3\sqrt{2} \\ ||\mathbf{v}|| &= \sqrt{2^2 + 1^2 + (-1)^2 + 0^2} = \sqrt{6} \end{aligned}$$

(ii) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

$$\theta = \cos^{-1} \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}|| \cdot ||\mathbf{v}||} = \cos^{-1} \frac{1(2) + 2(1) + 3(-1) + 2(0)}{3\sqrt{2}\sqrt{6}} = \cos^{-1} \frac{1}{6\sqrt{3}} \approx 84.4782^\circ$$

2. Consider $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for $\mathbb{R}^3$.

The inner product for all pairs of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are zero.

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (-1)(1) + 2(0) + 1(1) &= 0 \\ = \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1(-1) + 0(-1) + 1(1) &= 0 \\ = \mathbf{v}_1 \cdot \mathbf{v}_3 &= (-1)(-1) + 2(-1) + 1(1) &= 0 \end{aligned}$$

And a matrix composed of these vectors is full rank.

$$\text{rref} \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \text{rref} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for $\mathbb{R}^3$.

(ii) Find the orthonormal basis generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \frac{1}{\sqrt{(-1)^2 + 2^2 + 1^2}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{||\mathbf{v}_2||} = \frac{1}{\sqrt{1^2 + 0^2 + 1^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3}{||\mathbf{v}_3||} = \frac{1}{\sqrt{(-1)^2 + (-1)^2 + 1^2}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The orthonormal basis $\{ \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3 \}$ genetrated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}.$$

(iii) Express $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\text{rref} \left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{array} \right] = \text{rref} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\therefore \mathbf{v}_4 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3$$

3. Consider the space of all continuous functions on $[0, 1], C[0, 1]$ with the standard inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$$

(i) Find the norm of $f(x) = x^n$, for any positive integer n.

For $n \in \mathbb{Z}^+$:

$$\langle x^n, x^n \rangle = \int_0^1 x^{2n}\,\mathrm{d}x = \left.\frac{x^{2n}}{2n+1}\right|_0^1 = \frac{1^{2n}}{2n+1} = \frac{1}{2n+1}$$

$$||x^n|| = \sqrt{\langle x^n, x^n \rangle} = \frac{1}{\sqrt{2n+1}}$$

(ii) Find the angle between x^n and x^m.

Assuming $n, m \in \mathbb{Z}^+$.

$$\begin{aligned} \theta = \frac{\langle x^n, x^m \rangle}{||x^n|| \cdot ||x^m||} &= \cos^{-1} \frac{\displaystyle\int_0^1 x^{nm}\,\mathrm{d}x}{\displaystyle\int_0^1 x^{2n}\,\mathrm{d}x \int_0^1 x^{2m}\,\mathrm{d}x} \\ &= \cos^{-1} \frac{\frac{1^{nm}}{nm+1}}{\sqrt{\frac{1^{2n}}{2n+1}\frac{1^{2m}}{2m+1}}} \\ &= \cos^{-1} \frac{1}{(nm+1)\frac{1}{\sqrt{(2n+1)(2m+1)}}} \\ &= \cos^{-1} \frac{\sqrt{(2n+1)(2m+1)}}{nm+1} \end{aligned}$$

Since it wasn't specified in the question, if n and m are not restricted to positive integers, then this solution is valid for all n and m such that $nm \neq -1 \land (2n+1)(2m+1) \geq 0$.

(iii) Show that for any $m \neq n$, $\sin 2\pi mx$ and $\sin 2\pi nx$ are always mutually orthogonal. (Hint: Check out product-to-sum formula)

Again, assuming $m, n \in \mathbb{Z}^+$. Suppose $\langle \sin 2\pi mx, \sin 2\pi nx \rangle = 0 \ \forall n \neq m$.

> **Note**
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> Again, since it wasn't specified in the question, we assume $m, n \in \mathbb{Z}^+$. Note that the assumption do not hold if either m or n are not positive integers.
>
> For example, take $m = -1$ and $n = 1$:
>
> $$\begin{aligned} \langle \sin 2\pi mx, \sin 2\pi nx \rangle &= \langle \sin(-2\pi x), \sin 2\pi x \rangle \\ &= \int_0^1 \sin(-2\pi x) \sin 2\pi x \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \cos(-4\pi x) - \cos(0) \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \cos(-4\pi x) - \frac{1}{2} \int_0^1 \cos(0) \, \mathrm{d}x \\ &= -\frac{1}{2} \end{aligned}$$

Then, using the product-to-sum formula

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)),$$

the inner product can be written as follows:

$$\begin{aligned} \langle \sin 2\pi mx, \sin 2\pi nx \rangle &= \int_0^1 (\sin 2\pi mx)(\sin 2\pi nx) \, \mathrm{d}x \\ &= \int_0^1 \frac{1}{2}(\cos(2\pi mx - 2\pi nx) - \cos(2\pi mx + 2\pi nx)) \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \cos 2\pi x(m-n) - \cos 2\pi x(m+n) \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \cos 2\pi x(m-n) \, \mathrm{d}x - \frac{1}{2} \int_0^1 \cos 2\pi x(m+n) \, \mathrm{d}x \end{aligned}$$

Notice that if m and n are positive integers such that $m \neq n$, then x must always be a factor of 2π in both terms.

Since

$$\int_0^1 \cos 2\pi x \, \mathrm{d}x = 0 \quad \forall x \in \mathbb{Z}^+,$$

then the inner product must be zero for all positive integers $m \neq n$.

Or more clearly, if we recall our Calculus II nightmare by performing u-substitution:

$$\begin{aligned}\langle \sin 2\pi m x, \sin 2\pi n x\rangle &= \frac{1}{2}\int_0^1 \cos 2\pi x(m-n)\,\mathrm{d}x - \frac{1}{2}\int_0^1 \cos 2\pi x(m+n)\,\mathrm{d}x \\ &= \text{careful calculations} \\ &= \frac{1}{4\pi}\left(\frac{\sin 2\pi(m-n)}{m-n} - \frac{\sin 2\pi(m+n)}{m+n}\right)\end{aligned}$$

We can see that the argument of $\sin$ will be always be a multiple of 2π (and hence is always zero). Additionally, the inner product will not be defined for $m = n$.

4. Prove the identity

$$\langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w}\rangle = ac||\mathbf{v}||^2 + (ad+bc)\langle \mathbf{v}, \mathbf{w}\rangle + bd||\mathbf{w}||^2.$$

$$\begin{aligned}\langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w}\rangle &= \langle a\mathbf{v}, c\mathbf{v} + d\mathbf{w}\rangle + \langle b\mathbf{w}, c\mathbf{v} + d\mathbf{w}\rangle \\ &= \langle a\mathbf{v}, c\mathbf{v}\rangle + \langle a\mathbf{v}, d\mathbf{w}\rangle + \langle b\mathbf{w}, c\mathbf{v}\rangle + \langle b\mathbf{w}, d\mathbf{w}\rangle \\ &= ac\langle \mathbf{v}, \mathbf{v}\rangle + ad\langle \mathbf{v}, \mathbf{w}\rangle + bc\langle \mathbf{w}, \mathbf{v}\rangle + bd\langle \mathbf{w}, \mathbf{w}\rangle \\ &= ac||\mathbf{v}||^2 + (ad+bc)\langle \mathbf{v}, \mathbf{w}\rangle + bd||\mathbf{w}||^2\end{aligned}$$

5. Given an inner product space V.

(i) Show that

$$||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2).$$

(This is called the parallelogram identity)

$$\begin{aligned}||\mathbf{x}+\mathbf{y}||^2 = (||\mathbf{x}|| + ||\mathbf{y}||)^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}||||\mathbf{y}|| \\ ||\mathbf{x}-\mathbf{y}||^2 = (||\mathbf{x}|| + ||\mathbf{y}||)^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||||\mathbf{y}||\end{aligned}$$

$$\begin{aligned}\therefore ||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \cancel{+ 2||\mathbf{x}||||\mathbf{y}||} + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \cancel{- 2||\mathbf{x}||||\mathbf{y}||} \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \\ &= 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2)\end{aligned}$$

(ii) Show that

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2)$$

(This is called the polarization identity)

> **Note**
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> Assuming the left-hand side is meant to be $\langle \mathbf{x}, \mathbf{y}\rangle$ i.e., proving
>
> $$\langle \mathbf{x}, \mathbf{y}\rangle = \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2).$$

$$
\begin{aligned}
\frac{1}{4}(||x+y||^2 - ||x-y||^2) &= \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle) \\
&= \frac{1}{4}(\langle x+y, x\rangle + \langle x+y, y\rangle - (\langle x-y, x\rangle - \langle x-y, y\rangle)) \\
&= \frac{1}{4}(\langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle - (\langle x, x\rangle - \langle x, y\rangle - (\langle x, y\rangle - \langle y, y\rangle))) \\
&= \frac{1}{4}(\langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle - (\langle x, x\rangle - \langle x, y\rangle - \langle x, y\rangle + \langle y, y\rangle)) \\
&= \frac{1}{4}(\cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle x, y\rangle + \cancel{\langle y, y\rangle} - \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle x, y\rangle - \cancel{\langle y, y\rangle}) \\
&= \frac{1}{4}(4\langle x, y\rangle) \\
&= \langle x, y\rangle
\end{aligned}
$$

(iii) Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

(This is Pythagorean Theorem)

$$
\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}\rangle\langle \mathbf{u}+\mathbf{v}, \mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle \cancel{+ \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle} + \langle \mathbf{v}, \mathbf{v}\rangle \qquad \boxed{\because \mathbf{u} \perp \mathbf{v}} \\
&= ||\mathbf{u}||^2 + ||\mathbf{v}||^2
\end{aligned}
$$