Homework 1

1. Solve the following systems of linear equations by Gaussian elimination:

(a)

$$\begin{cases} 2y - 8z = 8 \\ x - 2y + z = 0 \\ -4x + 5y + 9z = -9 \end{cases}$$

$$\begin{cases} 2y - 8z = 8 \\ x - 2y + z = 0 \\ -4x + 5y + 9z = -9 \end{cases} \iff \left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\xrightarrow[R_2+R_1]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 0 & 1 & -4 & 4 \\ 1 & 0 & -7 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 0 & 1 & -4 & 4 \\ 1 & 0 & -7 & 8 \\ 0 & 5 & -19 & 23 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 5 & -19 & 23 \end{array}\right]$$

$$\xrightarrow{R_3-5R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2+4R_3]{R_1+7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\therefore x = 29, y = 16, z = 3$$

(b)

$$
\begin{cases}
x_1 - 2x_3 = -1 \\
x_2 - x_4 = 2 \\
-3x_2 + 2x_3 = 0 \\
-4x_1 + 7x_4 = -5
\end{cases}
$$

$$
\begin{cases}
x_1 - 2x_3 = -1 \\
x_2 - x_4 = 2 \\
-3x_2 + 2x_3 = 0 \\
-4x_1 + 7x_4 = -5
\end{cases}
\iff
\left[\begin{array}{cccc|c}
1 & 0 & -2 & 0 & -1 \\
0 & 1 & 0 & -1 & 2 \\
0 & -3 & 2 & 0 & 0 \\
-4 & 0 & 0 & 7 & -5
\end{array}\right]
$$

$$
\xrightarrow[R_3+3R_2]{R_4+4R_1}
\left[\begin{array}{cccc|c}
1 & 0 & -2 & 0 & -1 \\
0 & 1 & 0 & -1 & 2 \\
0 & 0 & 2 & -3 & 6 \\
0 & 0 & -8 & 7 & -9
\end{array}\right]
$$

$$
\xrightarrow{R_4+4R_3}
\left[\begin{array}{cccc|c}
1 & 0 & -2 & 0 & -1 \\
0 & 1 & 0 & -1 & 2 \\
0 & 0 & 2 & -3 & 6 \\
0 & 0 & 0 & -5 & 15
\end{array}\right]
$$

$$
\xrightarrow[-\frac{1}{5}R_4]{\frac{1}{2}R_3}
\left[\begin{array}{cccc|c}
1 & 0 & -2 & 0 & -1 \\
0 & 1 & 0 & -1 & 2 \\
0 & 0 & 1 & -\frac{3}{2} & 3 \\
0 & 0 & 0 & 1 & -3
\end{array}\right]
$$

$$
\xrightarrow[R_2+R_3]{R_3+\frac{3}{2}R_4}
\left[\begin{array}{cccc|c}
1 & 0 & -2 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 1 & -3
\end{array}\right]
$$

$$
\xrightarrow{R_1+2R_3}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & -4 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 1 & -3
\end{array}\right]
$$

$$
\therefore x_1 = -4, x_2 = -1, x_3 = -\frac{3}{2}, x_4 = -3
$$

2. The sum of any two of three real numbers are $24, 28, 30$. Find these three numbers.

Let $x, y, z \in \mathbb{R}$. Then:

$$\begin{cases} x + y = 24 \\ y + z = 28 \\ z + x = 30 \end{cases} \quad \Longleftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 24 \\ 0 & 1 & 1 & 28 \\ 1 & 0 & 1 & 30 \end{array}\right]$$

$$\xrightarrow[R_1 - R_2]{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 28 \\ 0 & -1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 28 \\ 0 & 0 & 2 & 34 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 28 \\ 0 & 0 & 1 & 17 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3]{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 17 \end{array}\right]$$

$$\therefore x = 13, y = 11, z = 17$$

As such, the three real numbers are 13, 11, and 17.

3. Find the polynomial of degree 2 $f(t) = a + bt + ct^2$ whose graph passes through $(1, -1), (2, 3)$ and $(3, 13)$.

$$
\begin{cases} a + b(1) + c(1)^2 & = -1 \\ a + b(2) + c(2)^2 & = 3 \\ a + b(3) + c(3)^2 & = 13 \end{cases} \iff \begin{cases} a + b + c & = -1 \\ a + 2b + 4c & = 3 \\ a + 3b + 9c & = 13 \end{cases}
$$

$$
\iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 13 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 14 \end{array}\right]
$$

$$
\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 6 \end{array}\right]
$$

$$
\xrightarrow{\frac{1}{2} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$
\xrightarrow[R_2 - 3R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$
\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

As such, the function

$$
f(t) = 1 - 5t + 3t^2
$$

is a polynomial of degree two that passes through the points $(1, -1)$, $(2, 3)$, and $(3, 13)$.

4. Use some online program, write down the echelon form of the following system and solve the system as well.

$$\begin{cases} x - 2y + 3z - 4w + 5v = -1 \\ 2x + 3y + 4z + 5w - 6v = 2 \\ 2x - 2y + 3z - 3w + 6v = 0 \\ x + y - z - w + 3v = 2 \\ 3x + 4y + 5z - 6w - 4v = 0 \end{cases}$$

$$\begin{cases} x - 2y + 3z - 4w + 5v = -1 \\ 2x + 3y + 4z + 5w - 6v = 2 \\ 2x - 2y + 3z - 3w + 6v = 0 \\ x + y - z - w + 3v = 2 \\ 3x + 4y + 5z - 6w - 4v = 0 \end{cases} \iff \left[\begin{array}{ccccc|c} 1 & -2 & 3 & -4 & 5 & -1 \\ 2 & 3 & 4 & 5 & -6 & 2 \\ 2 & -2 & 3 & -3 & 6 & 0 \\ 1 & 1 & -1 & -1 & 3 & 2 \\ 3 & 4 & 5 & -6 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{\text{careful calculations}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{171}{664} \\ 0 & 1 & 0 & 0 & 0 & \frac{543}{664} \\ 0 & 0 & 1 & 0 & 0 & -\frac{51}{664} \\ 0 & 0 & 0 & 1 & 0 & \frac{229}{664} \\ 0 & 0 & 0 & 0 & 1 & \frac{33}{83} \end{array}\right]$$

$$\therefore x = \frac{171}{664}, y = \frac{543}{664}, z = -\frac{51}{664}, w = \frac{229}{664}, v = \frac{33}{83}$$

5. Find the following products. Explain why if it is undefined.

(a)

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(-3) \\ 3(2) + 2(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ is undefined because the number of columns in the first matrix does not match the number of rows in the second.

(c)

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(1) \\ 3(2) + 2(1) \\ 5(2) + 6(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 16 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(1) + 3(-1) + 4(4) \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}$ is undefined because the number of columns in the first matrix does not match the number of rows in the second.

6. Express the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ as a linear combination of $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ -1 & 2 & 3 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 5 & 3 \\ 2 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 2 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & -13 & -6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & \frac{6}{13} \end{array}\right]$$

$$\xrightarrow{R_2-5R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 0 & \frac{9}{13} \\ 0 & 0 & 1 & \frac{6}{13} \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{35}{13} \\ 0 & 1 & 0 & \frac{9}{13} \\ 0 & 0 & 1 & \frac{6}{13} \end{array}\right]$$

$$\xrightarrow{R_1-2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{23}{13} \\ 0 & 1 & 0 & \frac{9}{13} \\ 0 & 0 & 1 & \frac{6}{13} \end{array}\right]$$

7. Can the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ be expressed as a linear combination of $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$? Explain.

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

No. The vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ cannot be expressed as a linear combination of $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ because the system does not have a solution.