Homework 2

1. Find the reduced row echelon form of the following matrices and compute the rank.

(a)

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} & \xrightarrow[R_3+R_2]{R_2-R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ & \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore \operatorname{rank}\left(\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix}\right) = 2$$

(b)

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -5 & -2 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_4]{\substack{R_2-R_3 \\ R_3-R_2 \\ R_5-R_2}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -5 & -2 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 4 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2-4R_1} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \operatorname{rank}\left(\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \\ 4 & -1 & 3 & 2 \\ 0 & 3 & -5 & -2 \end{bmatrix}\right) = 2$$

(c)

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix} \xrightarrow[R_2-R_1]{R_3-R_2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
$$\xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow[R_3-R_2]{R_1+R_2} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \operatorname{rank}\left(\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}\right) = 2$$

(d)

$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_3]{R_1+\frac{3}{2}R_3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \operatorname{rank}\left(\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}\right) = 1$$

2. For which values of a, b, c, d, e is the following matrix in reduced row echelon form?

$$\begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & 0 & e & 0 & 1 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & 0 & e & 0 & 1 & 1 \end{bmatrix}$ is in reduced row echelon form if $c = 1$ and $b = d = e = 0$.

3. If the rank of a 4×4 matrix A is 4, what is its $\operatorname{rref}(A)$?

A must have a full row rank, therefore $\operatorname{rref}(A)$ must be:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Find all the possible solutions of the following systems.

(i)

$$\begin{cases} x - 2y + 2z - w = 3 \\ 3x + y + 6z + 11w = 16 \\ 2x - y + 4z + w = 9 \end{cases}$$

$$\begin{cases} x - 2y + 2z - w = 3 \\ 3x + y + 6z + 11w = 16 \\ 2x - y + 4z + w = 9 \end{cases} \iff \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 1 & 9 \end{array}\right]$$

$$\xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 7 & 0 & 14 & 7 \\ 0 & 3 & 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{7}R_2]{\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{R_2-R_3} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-R_2]{R_1+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x + 2z &= 5 \\ y &= 1 \\ w &= 0 \end{cases}$$

Let $z \in \mathbb{R}$. Then, $x = 5 - 2z$. As such, the solution set is:

$$\left\{ \begin{pmatrix} 5-2z \\ 1 \\ z \\ 0 \end{pmatrix} : z \in \mathbb{R} \right\}$$

(ii)

$$\begin{cases} x + y - 2z = -3 \\ 2x - y + 3z = 7 \\ x - 2y + 5z = 1 \end{cases}$$

$$\begin{aligned} \begin{cases} x + y - 2z = -3 \\ 2x - y + 3z = 7 \\ x - 2y + 5z = 1 \end{cases} &\iff \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 2 & -1 & 3 & 7 \\ 1 & -2 & 5 & 1 \end{array}\right] \\ &\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & -3 & 7 & 13 \\ 0 & -3 & 7 & 4 \end{array}\right] \\ &\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & -3 & 7 & 13 \\ 0 & 0 & 0 & -9 \end{array}\right] \end{aligned}$$

No solutions.

(iii)

$$\begin{cases} x + 2y = 1 \\ 2x + 5y = 2 \\ 3x + 6y = 3 \end{cases}$$

$$\begin{cases} x + 2y = 1 \\ 2x + 5y = 2 \\ 3x + 6y = 3 \end{cases} \iff \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore x = 1, y = 0$$

(iv)

$$\begin{cases} x_1 + x_2 + x_3 + 9x_4 = 8 \\ x_2 + 2x_3 + 8x_4 = 7 \\ -3x_1 + x_3 - 7x_4 = 9 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 + 9x_4 = 8 \\ x_2 + 2x_3 + 8x_4 = 7 \\ -3x_1 + x_3 - 7x_4 = 9 \end{cases} \iff \left[\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ -3 & 0 & 1 & -7 & 9 \end{array}\right]$$

$$\xrightarrow{R_3+3R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ 0 & 3 & 4 & 20 & 33 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ 0 & 0 & -2 & -4 & 12 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ 0 & 0 & 1 & 2 & -6 \end{array}\right]$$

$$\xrightarrow[R_1-R_3]{R_2-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 7 & 14 \\ 0 & 1 & 0 & 4 & 19 \\ 0 & 0 & 1 & 2 & -6 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -5 \\ 0 & 1 & 0 & 4 & 19 \\ 0 & 0 & 1 & 2 & -6 \end{array}\right]$$

$$\therefore \begin{cases} x_1 + 3x_4 & = -5 \\ x_2 + 4x_4 & = 19 \\ x_3 + 2x_4 & = -6 \end{cases}$$

Let $x_4 \in \mathbb{R}$. Then:

$$\begin{gathered} x_1 = -5 - 3x_4 \\ x_2 = 19 - 4x_4 \\ x_3 = -6 - 2x_4 \end{gathered}$$

As such, the solution set is:

$$\left\{\begin{pmatrix}-5-3x_4\\19-4x_4\\-6-2x_4\\x_4\end{pmatrix} : x_4 \in \mathbb{R}\right\} = \left\{\begin{pmatrix}-5\\19\\-6\\0\end{pmatrix} + \begin{pmatrix}-3\\-4\\-2\\1\end{pmatrix} x_4 : x_4 \in \mathbb{R}\right\}$$

5. Determine k for which the following system has infinitely many solutions.

$$\begin{cases} x+y=0 \\ 2y+2kz=1 \\ y+kz=2k \end{cases}$$

$$\begin{cases} x+y=0 \\ 2y+2kz=1 \\ y+kz=2k \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2k & 1 \\ 0 & 1 & k & 2k \end{array}\right]$$

$$\xrightarrow{R_2-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-4k \\ 0 & 1 & k & 2k \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & k & 2k \\ 0 & 0 & 0 & 1-4k \end{array}\right]$$

$$\therefore \operatorname{rank}\left(\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & k & 2k \\ 0 & 0 & 0 & 1-4k \end{array}\right]\right) < 3 \iff k = \frac{1}{4}$$

The system will have infinitely many solutions for $k = \frac{1}{4}$.

6. (True or False) Determine if the following statements are true or false. If it is true, explain and prove it. If it is false, give a counterexample.

Let A be an 3×5 matrix, then:

(i) $Ax = b$ always has a solution.

False. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Clearly, $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$ has no solutions.

(ii) $Ax = 0$ always has a solution.

True. $A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$.

(iii) If a system $Ax = b$ has no solution, then $\operatorname{rank}(A) < 3$.

True. Consider the inverse, where $\operatorname{rank}(A) = 3$ (which means it has a full row rank). Then, there must be at least one solution.

(iv) There are always infinitely many solutions to the system $Ax = 0$.

True. $\operatorname{rank}(A) \leq 3$ and the number of columns $n = 5$. By definition, for a matrix A with n columns where $\operatorname{rank}(A) < n$, $A\vec{x} = \vec{0}$ will have infinitely many solutions.

(v) It is possible that the system $Ax = b$ has a unique solution.

True. Sure, it's *possible*. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then clearly,

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ has a unique solution.