Homework 7

1. Consider the linear transformation considered in the previous homework,

(i) $T(x_1, x_2, x_3) = (3x_1 - x_2, x_2 + x_3, x_1 - x_2 - x_3)$.

(ii) T maps $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$.

(iii) $T(x_1, x_2) = x_1 \begin{bmatrix}1\\1\\2\end{bmatrix} + x_2 \begin{bmatrix}-1\\1\\5\end{bmatrix}$,

Determine if they are surjective or injective.

2. Determine if the following statements are true or false. Give explanation.
(a) Suppose that there are 6 vectors in $\mathbb{R}^4$, it must be linearly decpendent.

(b) Suppose that there are 6 vectors in $\mathbb{R}^4$, it must span $\mathbb{R}^4$.

(c) Suppose that there are 4 vectors in $\mathbb{R}^6$, it must be linearly independent.

(d) Suppose that there are 4 vectors in $\mathbb{R}^6$, it cannot span $\mathbb{R}^6$.

3. Find a basis for the kernel and image of the following matrices and compute its dimensions.

$$A = \begin{bmatrix} 1 & 2 & 4 & -2 & 2 \\ 2 & 4 & 6 & 1 & 1 \\ 2 & 3 & 4 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ -2 & 4 & -6 \\ 3 & 0 & -1 \end{bmatrix}.$$

4. Let $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}, \begin{bmatrix}10\\11\\12\end{bmatrix}, \begin{bmatrix}13\\14\\15\end{bmatrix} \right\}$. Is it possible to extract a basis for $\mathbb{R}^3$ from the set S? Explain

5. Let A be the 6×4 matrix with $A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix}$.Suppose that after Gaussian elimination, the row echelon form of A is given by

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Find the rank of A.

(ii) Find a basis for the $\ker(A)$. What is its dimension?

(iii) Find the subset of the columns of A so that it forms a basis for the $\mathrm{Im}(A)$. What is the dimension of $\mathrm{Im}(A)$?

6. Book Question 26, 27, 53, 55, 56.

EXERCISES 3.4

GOAL *Use the concept of coordinates. Apply the definition of the matrix of a linear transformation with respect to a basis. Relate this matrix to the standard matrix of the transformation. Find the matrix of a linear transformation (with respect to any basis) column by column. Use the concept of similarity.*

In Exercises 1 through 18, determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v}_1, \dots, \vec{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\vec{x}$ is in V, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$.

1. $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2. $\vec{x} = \begin{bmatrix} 23 \\ 29 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 46 \\ 58 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 61 \\ 67 \end{bmatrix}$

3. $\vec{x} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 23 \\ 29 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 31 \\ 37 \end{bmatrix}$

4. $\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $\vec{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

6. $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

7. $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

8. $\vec{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

9. $\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

10. $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

11. $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$

12. $\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

13. $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

14. $\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

15. $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$

16. $\vec{x} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

17. $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$

18. $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 19 through 24, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. For practice, solve each problem in three ways: (a) Use the formula $B = S^{-1}AS$, (b) use a commutative diagram (as in Examples 3 and 4), and (c) construct B "column by column."

19. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

20. $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

23. $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

In Exercises 25 through 30, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

25. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

27. $A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}$;

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$;

$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

29. $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}$;

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

30. $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix}$;

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be any basis of $\mathbb{R}^3$ consisting of perpendicular unit vectors, such that $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. In Exercises 31 through 36, find the $\mathfrak{B}$-matrix B of the given linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$. Interpret T geometrically.

31. $T(\vec{x}) = \vec{v}_2 \times \vec{x}$

32. $T(\vec{x}) = \vec{x} \times \vec{v}_3$

33. $T(\vec{x}) = (\vec{v}_2 \cdot \vec{x})\vec{v}_2$

34. $T(\vec{x}) = \vec{x} - 2(\vec{v}_3 \cdot \vec{x})\vec{v}_3$

35. $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_2$

36. $T(\vec{x}) = \vec{v}_1 \times \vec{x} + (\vec{v}_1 \cdot \vec{x})\vec{v}_1$

In Exercises 37 through 42, find a basis $\mathfrak{B}$ of $\mathbb{R}^n$ such that the $\mathfrak{B}$-matrix B of the given linear transformation T is diagonal.

37. Orthogonal projection T onto the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

38. Reflection T about the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

39. Reflection T about the line in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

40. Orthogonal projection T onto the line in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

41. Orthogonal projection T onto the plane $3x_1 + x_2 + 2x_3 = 0$ in $\mathbb{R}^3$

42. Reflection T about the plane $x_1 - 2x_2 + 2x_3 = 0$ in $\mathbb{R}^3$

43. Consider the plane $x_1 + 2x_2 + x_3 = 0$ with basis $\mathfrak{B}$ consisting of vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. If $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\vec{x}$.

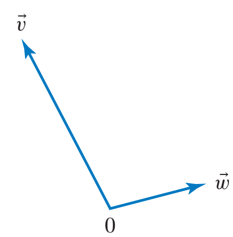
44. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ with basis $\mathfrak{B}$ consisting of vectors $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. If $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{x}$.

45. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis $\mathfrak{B}$ of this plane such that $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

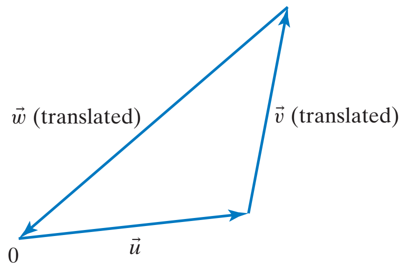
46. Consider the plane $x_1 + 2x_2 + x_3 = 0$. Find a basis $\mathfrak{B}$ of this plane such that $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

47. Consider a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. We are told that the matrix of T with respect to the basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find the standard matrix of T in terms of a, b, c, and d.

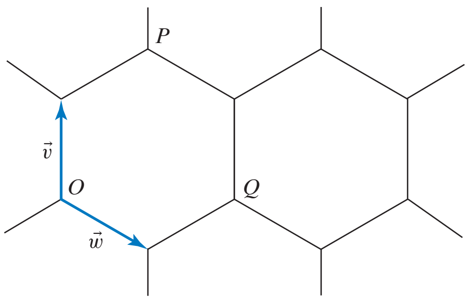
48. In the accompanying figure, sketch the vector $\vec{x}$ with $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where $\mathfrak{B}$ is the basis of $\mathbb{R}^2$ consisting of the vectors $\vec{v}$, $\vec{w}$.



49. Consider the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ sketched in the accompanying figure. Find the coordinate vector of $\vec{w}$ with respect to the basis $\vec{u}$, $\vec{v}$.



50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\vec{v}$, $\vec{w}$ in the following sketch:



a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathfrak{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathfrak{B}}$. *Hint*: Sketch the coordinate grid defined by the basis $\mathfrak{B} = (\vec{v}, \vec{w})$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R. Is R a vertex or a center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathfrak{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is S a center or a vertex of a tile?

51. Prove part (a) of Theorem 3.4.2.

52. If $\mathfrak{B}$ is a basis of $\mathbb{R}^n$, is the transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by

$$T(\vec{x}) = \left[\vec{x}\right]_{\mathfrak{B}}$$

linear? Justify your answer.

53. Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We are told that $\left[\vec{x}\right]_{\mathfrak{B}} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ for a certain vector $\vec{x}$ in $\mathbb{R}^2$. Find $\vec{x}$.

54. Let $\mathfrak{B}$ be the basis of $\mathbb{R}^n$ consisting of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and let $\mathfrak{T}$ be some other basis of $\mathbb{R}^n$. Is

$$\left[\vec{v}_1\right]_{\mathfrak{T}}, \quad \left[\vec{v}_2\right]_{\mathfrak{T}}, \quad \ldots, \quad \left[\vec{v}_n\right]_{\mathfrak{T}}$$

a basis of $\mathbb{R}^n$ as well? Explain.

55. Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\mathfrak{R}$ be the basis consisting of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix P such that

$$\left[\vec{x}\right]_{\mathfrak{R}} = P \left[\vec{x}\right]_{\mathfrak{B}},$$

for all $\vec{x}$ in $\mathbb{R}^2$.

56. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

57. Show that if a 3×3 matrix A represents the reflection about a plane, then A is similar to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

58. Consider a 3×3 matrix A and a vector $\vec{v}$ in $\mathbb{R}^3$ such that $A^3\vec{v} = \vec{0}$, but $A^2\vec{v} \neq \vec{0}$.

a. Show that the vectors $A^2\vec{v}$, $A\vec{v}$, $\vec{v}$ form a basis of $\mathbb{R}^3$. *Hint*: It suffices to show linear independence. Consider a relation $c_1A^2\vec{v} + c_2A\vec{v} + c_3\vec{v} = \vec{0}$ and multiply by A^2 to show that $c_3 = 0$.

b. Find the matrix of the transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $A^2\vec{v}$, $A\vec{v}$, $\vec{v}$.

59. Is matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ similar to matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$?

60. Is matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

61. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{x} \quad \text{is} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

62. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x} \quad \text{is} \quad B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

63. Is matrix $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ similar to matrix $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$ for all p and q?

64. Is matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ similar to matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ for all a, b, c, d?

65. Prove parts (a) and (b) of Theorem 3.4.6.

66. Consider a matrix A of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$ and $a \neq 1$. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}$. Interpret the answer geometrically.

67. If $c \neq 0$, find the matrix of the linear transformation $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$ with respect to basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}$.

68. Find an invertible 2×2 matrix S such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form $\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$. See Exercise 67.

69. If A is a 2×2 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that A is similar to a diagonal matrix D. Find an invertible S such that $S^{-1}AS = D$.

70. Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\mathfrak{B}$-matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular? *Hint*: Think about the first column of B.

71. Suppose that matrix A is similar to B, with $B = S^{-1}AS$.

a. Show that if $\vec{x}$ is in $\ker(B)$, then $S\vec{x}$ is in $\ker(A)$.

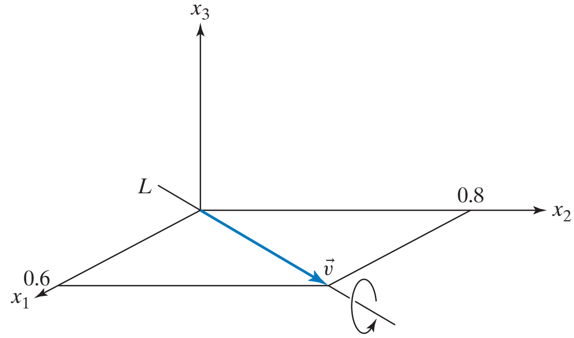
b. Show that $\text{nullity}(A) = \text{nullity}(B)$. *Hint*: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is a basis of $\ker(B)$, then the vectors $S\vec{v}_1, S\vec{v}_2, \ldots, S\vec{v}_p$ in $\ker(A)$ are linearly independent. Now reverse the roles of A and B.

72. If A is similar to B, what is the relationship between $\text{rank}(A)$ and $\text{rank}(B)$? See Exercise 71.

73. Let L be the line in $\mathbb{R}^3$ spanned by the vector

$$\vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}.$$

Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the rotation about this line through an angle of $\pi/2$, in the direction indicated in the accompanying sketch. Find the matrix A such that $T(\vec{x}) = A\vec{x}$.



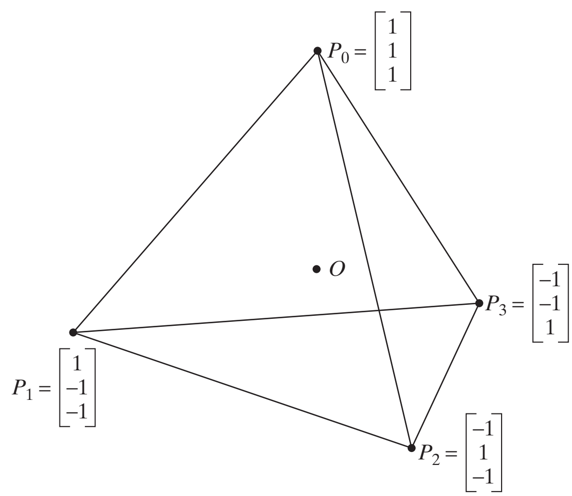
74. Consider the regular tetrahedron in the accompanying sketch whose center is at the origin. Let $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$ be the position vectors of the four vertices of the tetrahedron:

$$\vec{v}_0 = \overrightarrow{OP}_0, \ldots, \quad \vec{v}_3 = \overrightarrow{OP}_3.$$

a. Find the sum $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3$.

b. Find the coordinate vector of $\vec{v}_0$ with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

c. Let T be the linear transformation with $T(\vec{v}_0) = \vec{v}_3$, $T(\vec{v}_3) = \vec{v}_1$, and $T(\vec{v}_1) = \vec{v}_0$. What is $T(\vec{v}_2)$? Describe the transformation T geometrically (as a reflection, rotation, projection, or whatever). Find the matrix B of T with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$. What is B^3? Explain.



75. Find the matrix B of the rotation $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$ with respect to the basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Interpret your answer geometrically.

76. If t is any real number, what is the matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \vec{x}$$

with respect to basis $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$? Interpret your answer geometrically.

77. Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let B be the matrix of T with respect to the basis $\vec{e}_n, \vec{e}_{n-1}, \ldots, \vec{e}_2, \vec{e}_1$ of $\mathbb{R}^n$. Describe the entries of B in terms of the entries of A.

78. This problem refers to Leontief's input–output model, first discussed in the Exercises 1.1.24 and 1.2.39. Consider three industries I_1, I_2, I_3, each of which produces only one good, with unit prices $p_1 = 2$, $p_2 = 5$, $p_3 = 10$ (in U.S. dollars), respectively. Let the three products be labeled good 1, good 2, and good 3. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}$$

be the matrix that lists the interindustry demand in terms of dollar amounts. The entry a_{ij} tells us how many dollars' worth of good i are required to produce one dollar's worth of good j. Alternatively, the interindustry demand can be measured in units of goods by means of the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

where b_{ij} tells us how many units of good i are required to produce one unit of good j. Find the matrix B for the economy discussed here. Also, write an equation relating the three matrices A, B, and S, where

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

is the diagonal matrix listing the unit prices on the diagonal. Justify your answer carefully.

79. Consider the matrix $A = \begin{bmatrix} 11 & -30 \\ 4 & -11 \end{bmatrix}$. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix B of $T(\vec{x}) = A\vec{x}$ is $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

80. Consider the matrix $A = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix B of $T(\vec{x}) = A\vec{x}$ is $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

81. Consider the linear transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_2 + x_3 \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$.

a. Find all vectors of the form $\vec{x} = \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $T(\vec{x})$ is a scalar multiple of $\vec{x}$. Be prepared to deal with irrational numbers.

b. Find a basis $\mathfrak{B}$ of $\mathbb{R}^3$ such that the $\mathfrak{B}$-matrix B of T is diagonal.

82. Consider the linear transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 3x_3 - 2x_2 \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$.

a. Find all vectors of the form $\vec{x} = \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $T(\vec{x})$ is a scalar multiple of $\vec{x}$.

b. Find a basis $\mathfrak{B}$ of $\mathbb{R}^3$ such that the $\mathfrak{B}$-matrix B of T is diagonal.

Chapter Three Exercises

TRUE OR FALSE?

1. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are any two bases of a subspace V of $\mathbb{R}^{10}$, then n must equal m.

2. If A is a 5×6 matrix of rank 4, then the nullity of A is 1.

3. The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

4. The span of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ consists of all linear combinations of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

5. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$, then they must form a basis of $\mathbb{R}^n$.

6. There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.

7. The kernel of any invertible matrix consists of the zero vector only.

8. The identity matrix I_n is similar to all invertible $n \times n$ matrices.