

Homework 3

1. Find the matrix representation for the following linear transformations.

(a) $T(x_1, x_2, x_3) = (3x_1 - x_2, x_2 + x_3, x_1 - x_2 - x_3)$

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$
$$T(\vec{x}) = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b) T maps $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively to $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

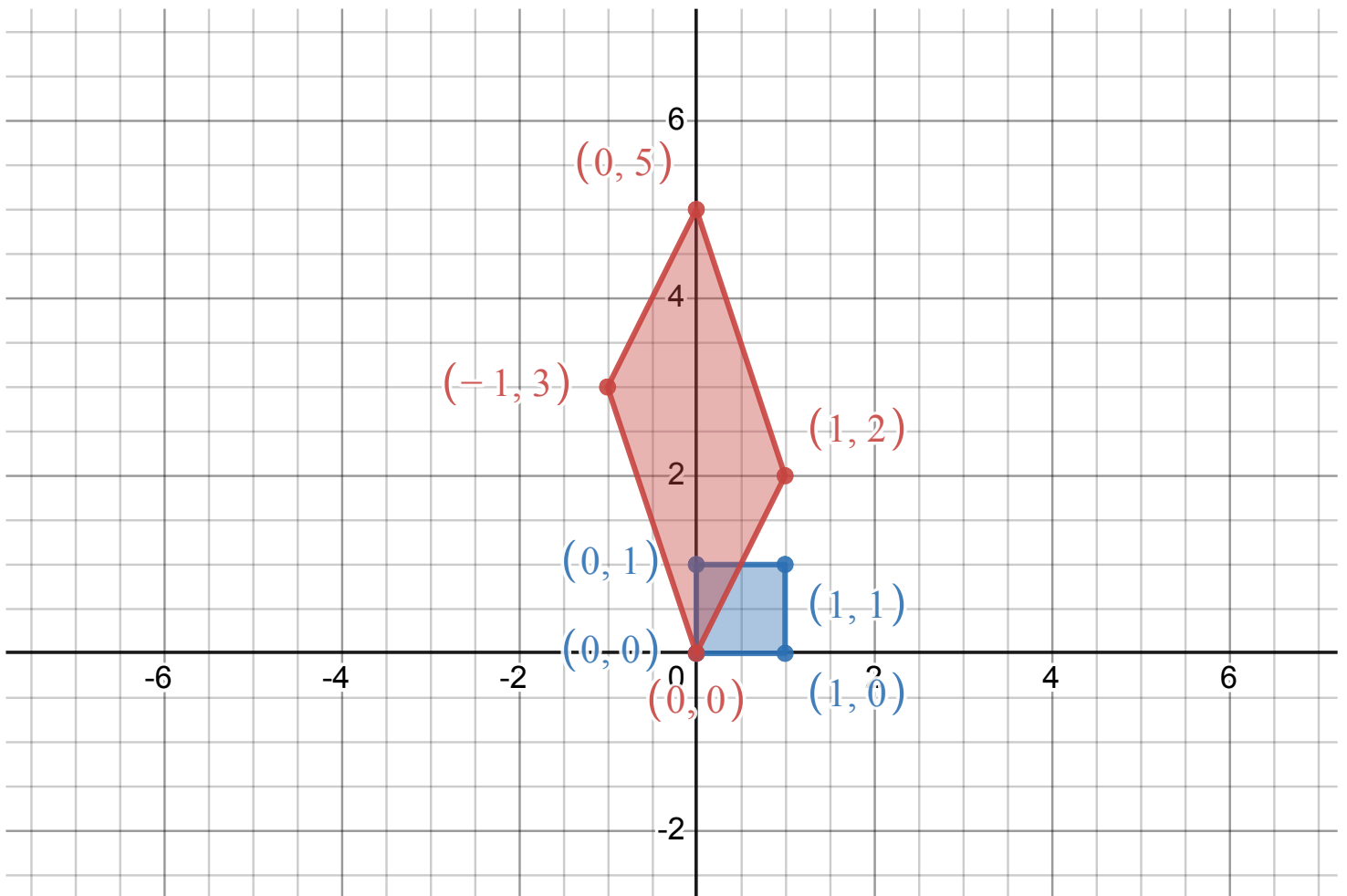
$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$
$$T(\vec{x}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) $T(x_1, x_2) = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$
$$T(\vec{x}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Sketch the image of the square formed by vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ under the linear transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \mathbf{x}$.

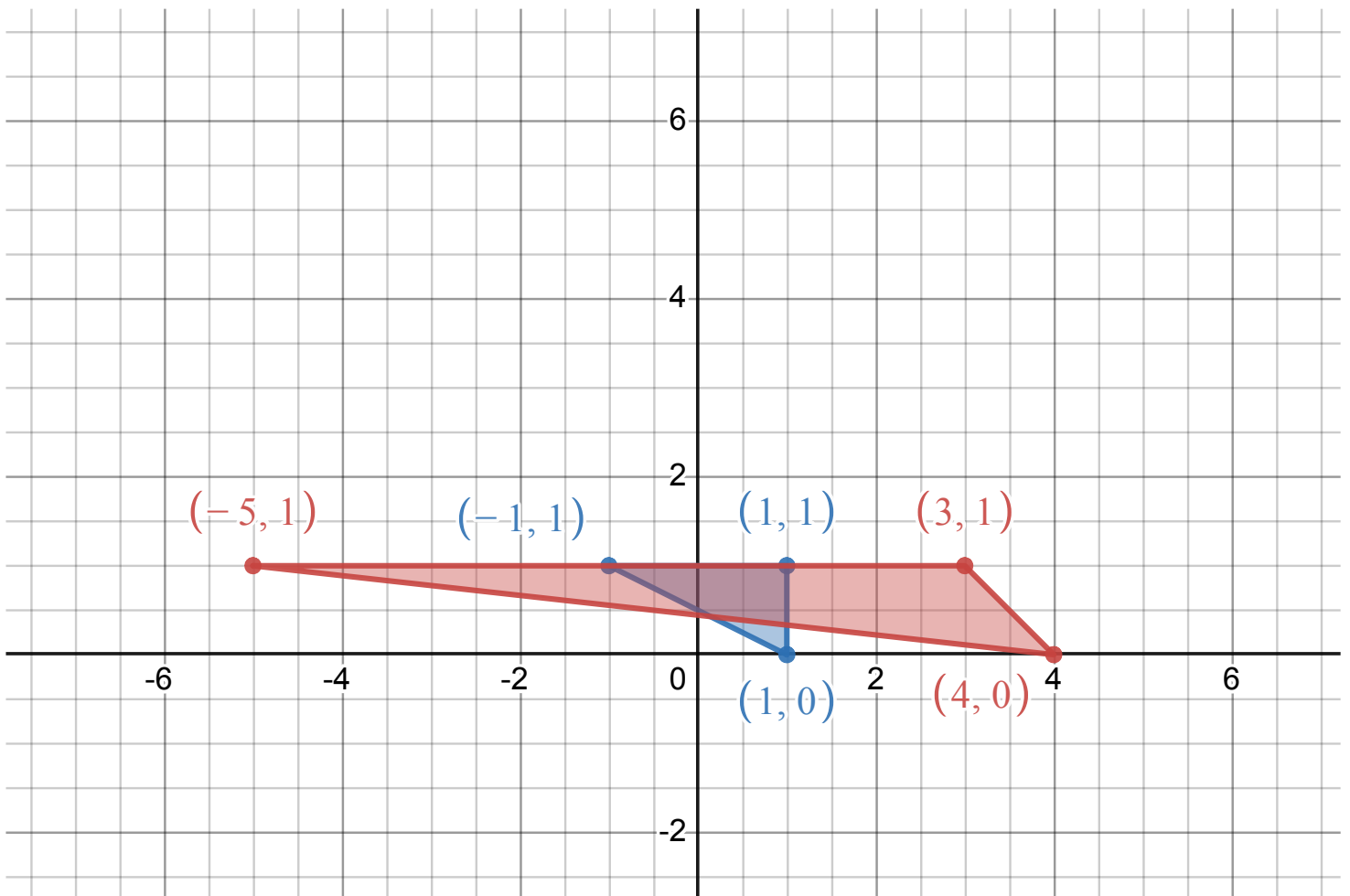
The transformed vertices and its image are in red.



$$\begin{aligned}
 T \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(0) + (-1)(0) \\ 2(0) + 3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 T \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(0) + (-1)(1) \\ 2(0) + 3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
 T \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(0) \\ 2(1) + 3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 T \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(1) \\ 2(1) + 3(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}
 \end{aligned}$$

3. Sketch the image of the triangle formed by vertices $(-1, 1)$, $(1, 0)$ and $(1, 1)$ under the linear transformation $T(\mathbf{x}) = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

The transformed vertices and its image are in red.



$$\begin{aligned} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(-1) + (-1)(1) \\ 0(-1) + 1(1) \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \\ T \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4(1) + (-1)(0) \\ 0(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ T \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + (-1)(1) \\ 0(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

4. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$.

(a) Find A^2 and A^3 .

Since A is a 2×2 square matrix, A^k is well defined for all natural k .

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 2(4) & 1(2) + 2(-3) \\ 4(1) + (-3)(4) & 4(2) + (-3)(-3) \end{bmatrix} \\ &= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 A^3 &= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \\
 &= \begin{bmatrix} 9(1) + (-4)(4) & 9(2) + (-4)(-3) \\ -8(1) + 17(4) & -8(2) + 17(-3) \end{bmatrix} \\
 &= \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}
 \end{aligned}$$

(b) Find $2A^3 - 4A + 5I_2$ and $A^2 + 2A - 11I_2$.

Assuming $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{aligned}
 2A^3 - 4A + 5I_2 &= 2 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3 - 4 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -14 & 60 \\ 120 & -134 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A^2 + 2A - 11I_2 &= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2 + 2 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 11 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

5. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\mathbf{v}^T = [v_1 \quad v_2 \quad v_3]$, and $v_1 \neq 0$.

(a) Find $\mathbf{v}^T \mathbf{v}$ and $\mathbf{v} \mathbf{v}^T$.

$$\begin{aligned}
 \mathbf{v}^T \mathbf{v} &= [v_1 \quad v_2 \quad v_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
 &= [v_1^2 + v_2^2 + v_3^2]
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v} \mathbf{v}^T &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} [v_1 \quad v_2 \quad v_3] \\
 &= \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{bmatrix}
 \end{aligned}$$

(b) If $\mathbf{v} \neq \mathbf{0}$, verify that the rank of $\mathbf{v}^T \mathbf{v}$ and $\mathbf{v} \mathbf{v}^T$ are 1.

$$v_1 \neq 0 \implies v_1^2 > 0$$

$$\iff v_1^2 + v_2^2 + v_3^2 > 0$$

$$\therefore \mathbf{v} \neq \mathbf{0} \wedge v_1 \neq 0 \implies \mathbf{v}^T \mathbf{v} \neq 0 \iff \text{rank}(\mathbf{v}^T \mathbf{v}) = 1$$

$$\mathbf{v} \mathbf{v}^T = \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{bmatrix} \xrightarrow[\frac{1}{v_3} R_3]{\frac{1}{v_2} R_2} \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \xrightarrow[\frac{R_3 - R_2}{R_3 - R_2}]{\frac{R_2 - R_3}{R_3 - R_2}} \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \mathbf{v} \neq \mathbf{0} \wedge v_1 \neq 0 \implies \mathbf{v} \mathbf{v}^T \neq \mathbf{0} \iff \text{rank}(\mathbf{v} \mathbf{v}^T) = 1$$