Homework 3

1. Find the matrix representation for the following linear transformations.

(a) $T(x_1, x_2, x_3) = (3x_1 - x_2, x_2 + x_3, x_1 - x_2 - x_3)$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(\vec{x}) = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b) T **maps** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$**, and** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **and** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **respectively to** $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

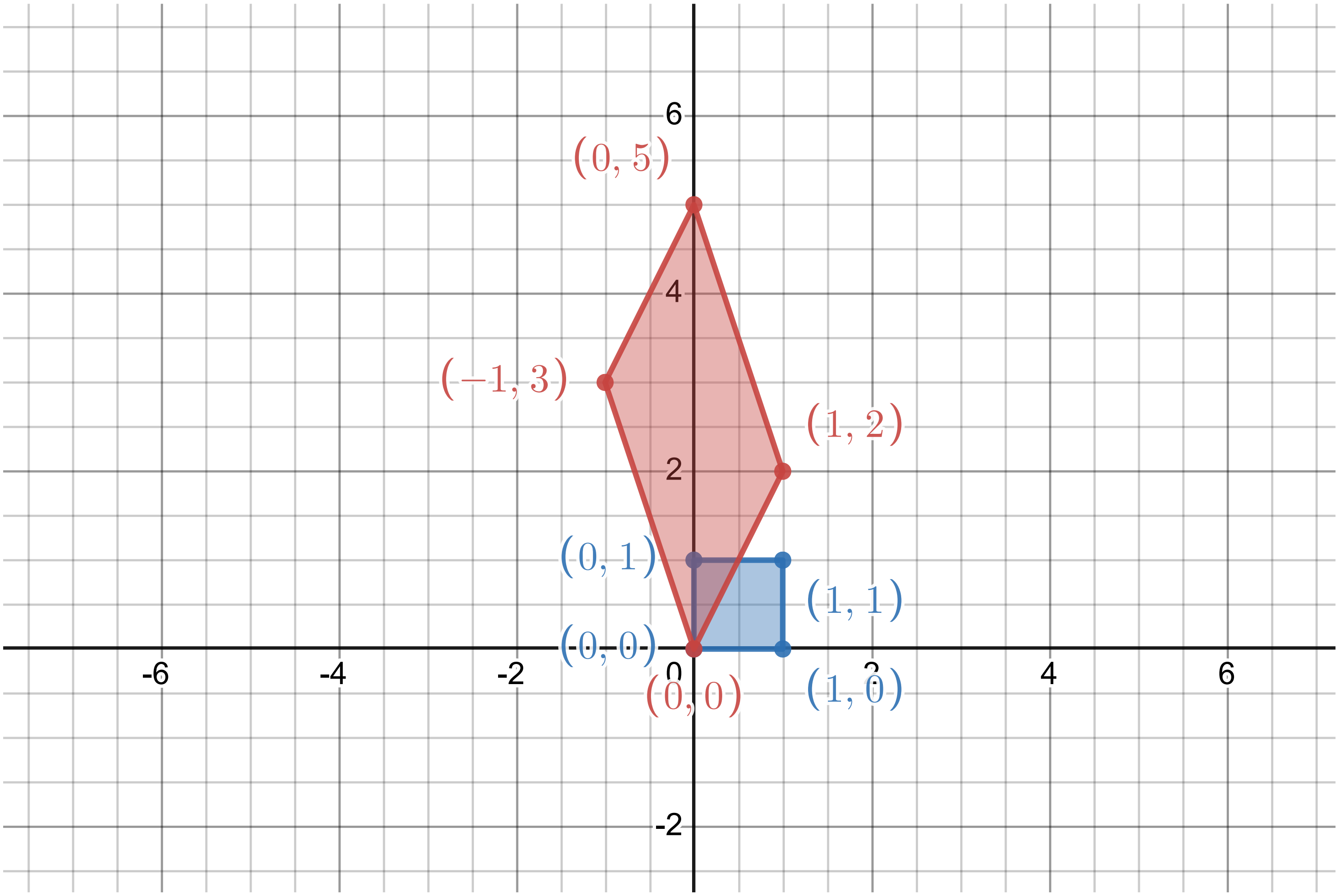
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$T(\vec{x}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) $T(x_1, x_2) = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
$$T(\vec{x}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Sketch the image of the square formed by vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ under the linear transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \mathbf{x}$.

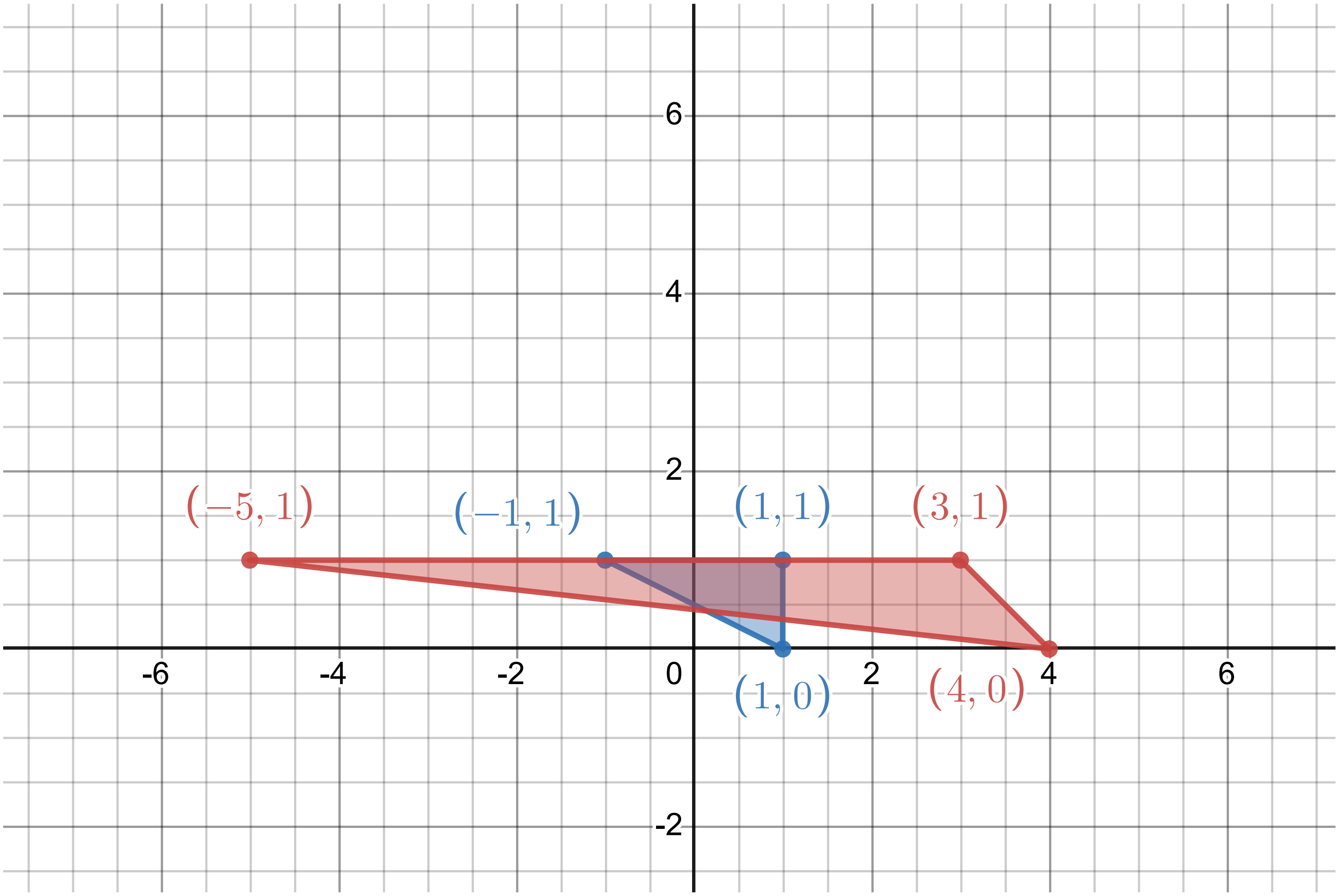
The transformed vertices and its image are in red.



$$T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1(0)+(-1)(0)\\2(0)+3(0)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1(0)+(-1)(1)\\2(0)+3(1)\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$
$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1(1)+(-1)(0)\\2(1)+3(0)\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$
$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1(1)+(-1)(1)\\2(1)+3(1)\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$$

3. Sketch the image of the triangle formed by vertices $(-1,1)$, $(1,0)$ and $(1,1)$ under the linear transformation $T(\mathbf{x}) = \begin{bmatrix}4 & -1\\0 & 1\end{bmatrix}\mathbf{x}$.

The transformed vertices and its image are in red.



$$
\begin{aligned}
T\begin{bmatrix}-1\\1\end{bmatrix} &= \begin{bmatrix}4 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} &= \begin{bmatrix}4(-1)+(-1)(1)\\0(-1)+1(1)\end{bmatrix} &= \begin{bmatrix}-5\\1\end{bmatrix}\\
T\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}4 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}4(1)+(-1)(0)\\0(1)+1(0)\end{bmatrix} &= \begin{bmatrix}4\\0\end{bmatrix}\\
T\begin{bmatrix}1\\1\end{bmatrix} &= \begin{bmatrix}4 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} &= \begin{bmatrix}4(1)+(-1)(1)\\0(1)+1(1)\end{bmatrix} &= \begin{bmatrix}3\\1\end{bmatrix}
\end{aligned}
$$

4. Let $A = \begin{bmatrix}1 & 2\\4 & -3\end{bmatrix}$.

(a) Find A^2 and A^3.

Since A is a 2×2 square matrix, A^k is well defined for all natural k.

$$
\begin{aligned}
A^2 &= \begin{bmatrix}1 & 2\\4 & -3\end{bmatrix}\begin{bmatrix}1 & 2\\4 & -3\end{bmatrix}\\
&= \begin{bmatrix}1(1)+2(4) & 1(2)+2(-3)\\4(1)+(-3)(4) & 4(2)+(-3)(-3)\end{bmatrix}\\
&= \begin{bmatrix}9 & -4\\-8 & 17\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^3 &= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 9(1) + (-4)(4) & 9(2) + (-4)(-3) \\ -8(1) + 17(4) & -8(2) + 17(-3) \end{bmatrix} \\
&= \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}
\end{aligned}
$$

(b) Find $2A^3 - 4A + 5I_2$ and $A^2 + 2A - 11I_2$.

Assuming $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$
\begin{aligned}
2A^3 - 4A + 5I_2 &= 2\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3 - 4\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -14 & 60 \\ 120 & -134 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^2 + 2A - 11I_2 &= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2 + 2\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

5. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, and $v_1 \neq 0$.

(a) Find $\mathbf{v}^T\mathbf{v}$ and $\mathbf{v}\mathbf{v}^T$.

$$
\begin{aligned}
\mathbf{v}^T\mathbf{v} &= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= \begin{bmatrix} v_1^2 + v_2^2 + v_3^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}\mathbf{v}^T &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \\
&= \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{bmatrix}
\end{aligned}
$$

(b) If $\mathbf{v} \neq \mathbf{0}$, verify that the rank of $\mathbf{v}^T\mathbf{v}$ and $\mathbf{v}\mathbf{v}^T$ are 1.

$$\begin{aligned} v_1 \neq 0 &\implies v_1^2 > 0 \\ &\iff v_1^2 + v_2^2 + v_3^2 > 0 \\ \therefore \mathbf{v} \neq \mathbf{0} \wedge v_1 \neq 0 &\implies \mathbf{v}^T\mathbf{v} \neq \mathbf{0} \iff \operatorname{rank}(\mathbf{v}^T\mathbf{v}) = 1 \end{aligned}$$

$$\begin{aligned} \mathbf{v}\mathbf{v}^T = \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{bmatrix} &\xrightarrow[\frac{1}{v_3}R_3]{\frac{1}{v_2}R_2} \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &\xrightarrow[R_3 - R_2]{R_2 - R_3} \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \therefore \mathbf{v} \neq \mathbf{0} \wedge v_1 \neq 0 &\implies \mathbf{v}\mathbf{v}^T \neq \mathbf{0} \iff \operatorname{rank}(\mathbf{v}\mathbf{v}^T) = 1 \end{aligned}$$