Homework 11

(All solution must be supplied with steps and justifications. Numerical answer without justifications will not be graded)

1. Consider $\mathbb{R}^4$. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. and $\mathbf{v} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$. Find the basis for the orthogonal complement of $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$.

2. Find the orthogonal projection of the vector $(1, 1, 1)$ onto the subspace defined by the equations

$$\begin{cases} x + y + z = 0, & ; \\ x - y - 2z = 0, & . \end{cases}$$

3.Find the orthogonal basis of $\mathbb{R}^3$ with $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as one of the vectors. Hint: You can use Gram-Schmidt process on a basis with $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as the first vector. e.g.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \right\}.$$

4 (a). Find an orthonormal basis for the kernel of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & -1 & -1 \end{bmatrix}.$$

(b) Find an orthonormal basis for $(\ker(A))^{\perp}$, the orthogonal complement of $\ker(A)$.

(c) Does the orthonormal basis in (i) combined with the orthonormal basis in (ii) for an orthonormal basis for $\mathbb{R}^4$? Explain.

5. Consider the following subspace of $\mathbb{R}^4$

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(i) What is the dimension of V?

(ii) Using Gram-Schmidt Process, find an orthogonal basis for V.

(iii) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ to V.

6. (i). Find the least square solution of the following system

$$\begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

(ii) Find the orthogonal projection of b onto the image of A using the least square solution.

7. Find the least square fitting straight line $y = C + Dt$ given the following set of data.

t_i	-2	0	1	3
y_i	0	1	2	5

7. Consider the non-standard inner product on $\mathbb{R}^2$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(a) Verify this is an inner product of $\mathbb{R}^2$.

(b) Using Gram-Schmidt process, starting with the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find an orthogonal basis under this inner product.

8. Consider the space of all continuous functions on $[0, 1]$, $C[0, 1]$ with the standard inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

(a) Use Gram-Schmidt process. Find an orthogonal basis using $\text{span}\{x^2, x, 1\}$.

(b) We prove previously that for any $m \neq n$, $\sin 2\pi mx$ and $\sin 2\pi nx$ are always mutually orthogonal. Write down the formula of the orthogonal projection of x^2 onto the subspace

$$\operatorname{span}\left\{1, \sin(2\pi x), \sin(2\pi 2x)\right\}.$$

Compute it. You need to do integration by part to find the coefficient, but you can use an online integration calculator to find it.