Homework 6

1. Consider the linear transformation considered in the previous homework. Determine if they are surjective or injective.

(i) $T(x_1, x_2, x_3) = (3x_1 - x_2, x_2 + x_3, x_1 - x_2 - x_3)$

$$\text{rref}\begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Full rank. Therefore, bijective.

(ii) T **maps** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$**, and** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **and** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **respectively to** $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\text{rref}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

No free variables. Nullity is not zero. Therefore, not injective.

Rank two, which is equal to the dimension of the codomain. Therefore, surjective.

(iii) $T(x_1, x_2) = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

$$\text{rref}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Nullity is zero. Therefore, injective.

Rank two but codomain is in $\mathbb{R}^3$. Therefore, not surjective.

2. Determine if the following statements are true or false. Give explanation.

(a) Suppose that there are 6 vectors in $\mathbb{R}^4$, it must be linearly decpendent.

True. Only four linearly independent vectors are needed to span $\mathbb{R}^4$.

(b) Suppose that there are 6 vectors in $\mathbb{R}^4$, it must span $\mathbb{R}^4$.

False. At least four linearly independent vectors are needed to span $\mathbb{R}^4$. If at least four of the six are the same or are multiples of each other, then they cannot $\mathbb{R}^4$.

(c) Suppose that there are 4 vectors in $\mathbb{R}^6$, it must be linearly independent.

False. If they are not distinct vectors or are multiples of each other, then they would be linearly dependent.

(d) Suppose that there are 4 vectors in $\mathbb{R}^6$, it cannot span $\mathbb{R}^6$.

True. At least six linearly independent vectors are needed to span $\mathbb{R}^6$.

3. Find a basis for the kernel and image of the following matrices and compute its dimensions.

$$A = \begin{bmatrix} 1 & 2 & 4 & -2 & 2 \\ 2 & 4 & 6 & 1 & 1 \\ 2 & 3 & 4 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ -2 & 4 & -6 \\ 3 & 0 & -1 \end{bmatrix}$$

Using our brainpower, we find that the RREF of A and B are given by:

$$\begin{gathered} \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \\ \therefore x_3 = \frac{5}{2}x_4 - \frac{3}{2}x_5 \\ x_2 = -5x_4 + 3x_5 \\ x_1 = 2x_4 - 2x_5 \\ x_4, x_5 \in \mathbb{R} \end{gathered}$$

$$\therefore \ker(A) = \left\{ \begin{pmatrix} 2x_4 - 2x_5 \\ -5x_4 + 3x_5 \\ \frac{5}{2}x_4 - \frac{3}{2}x_5 \\ x_4 \\ x_5 \end{pmatrix} : x_4, x_5 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} 2 \\ -5 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 3 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

A basis of $\ker(A)$ is $\left\{ \begin{pmatrix} 2 \\ -5 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$ and its dimension is 2.

Note that the pivots in $\operatorname{rref}(A)$ are located in columns one, two, and three. As such, the basis of $\operatorname{Im}(A)$ are the corresponding column vectors in A.

A basis of $\operatorname{Im}(A)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} \right\}$ and its dimension is 3.

$$\begin{gathered} \operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \therefore x_2 = \frac{5}{3}x_3 \\ x_1 = \frac{1}{3}x_3 \\ x_3 \in \mathbb{R} \end{gathered}$$

$$\therefore \ker(B) = \left\{ \begin{pmatrix} \frac{1}{3}x_3 \\ \frac{5}{3}x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

A basis of $\ker(B)$ is $\left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix} \right\}$ and its dimension is 1.

Note that the pivots in $\text{rref}(B)$ are located in columns one and two. As such, the basis of $\text{Im}(B)$ are the corresponding column vectors in B.

A basis of $\text{Im}(B)$ is $\left\{ \begin{pmatrix} 1 \\ -3 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ 4 \\ 0 \end{pmatrix} \right\}$ and its dimension is 2.

4. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ 4 \\ 5 \end{bmatrix} \right\}$. Is it possible to extract a basis for $\mathbb{R}^3$ from the set S? Explain.

Let P be a matrix composed of the vectors in S such that

$$P = \begin{pmatrix} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 1 & 4 \\ 3 & 6 & 9 & 2 & 5 \end{pmatrix}.$$

Then,

$$\text{rref}(P) = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Notice $\text{rank}(P) = 3$. As such, there exists three linearly independent vectors in S (which are the basis of $\text{Im}(P)$), namely:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix} \right\}$$

which are sufficient to span $\mathbb{R}^3$.

5. Let A be the 6×4 matrix with $A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix}$. Suppose that after Gaussian elimination, the row echelon form of A is given by

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Find the rank of A.

The three pivots in the row echelon form tells us that $\operatorname{rank}(A) = 3$.

(ii) Find a basis for the $\ker(A)$. What is its dimension?

$$\operatorname{rref}\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\therefore \ker(A) = \left\{ \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$$

A basis for $\ker(A)$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and its dimension is 1.

(iii) Find the subset of the columns of A so that it forms a basis for the $\operatorname{Im}(A)$. What is the dimension of $\operatorname{Im}(A)$?

The pivots in the row echelon form of A are at columns one, three, and four. Correspondingly, the basis for $\operatorname{Im}(A)$ is $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ and its dimension is 3.

6. Book Question 26, 27, 53, 55, 56.

In Exercises 25 through 30, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$.

26. $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let P be a matrix composed of column vectors $\{\vec{v}_1, \vec{v}_2\}$ such that $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

Then:

$$P^{-1}AP = B$$

$$\begin{aligned}\therefore B &= \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\\ &= \begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\\ &= \begin{pmatrix}6 & 4\\ -4 & 3\end{pmatrix}\end{aligned}$$

27. $A = \begin{bmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 4\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}2\\ 1\\ -2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

Let P be a matrix composed of column vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ such that $P = \begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 0\\ -2 & 1 & 1\end{pmatrix}$.

Then:

$$\begin{aligned}P^{-1}AP &= B\\ \therefore B &= \begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 0\\ -2 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 4\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 0\\ -2 & 1 & 1\end{pmatrix}\\ &= \frac{1}{9}\begin{pmatrix}2 & 1 & -2\\ -1 & 4 & 1\\ 5 & -2 & 4\end{pmatrix}\begin{pmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 4\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 0\\ -2 & 1 & 1\end{pmatrix}\\ &= \begin{pmatrix}9 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

53. Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\begin{bmatrix}3\\ 4\end{bmatrix}$. We are told that $\left[\vec{x}\right]_{\mathfrak{B}} = \begin{bmatrix}7\\ 11\end{bmatrix}$ for a certain vector $\vec{x}$ in $\mathbb{R}^2$. Find $\vec{x}$.

Let $P = \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$. Then:

$$\begin{aligned}\vec{x} &= P[\vec{x}]_{\mathfrak{B}}\\ &= \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}7\\ 11\end{pmatrix}\\ &= \begin{pmatrix}40\\ 58\end{pmatrix}\end{aligned}$$

55. Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\begin{bmatrix}1\\ 2\end{bmatrix}$ and let $\mathfrak{R}$ be the basis consisting of $\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\begin{bmatrix}3\\ 4\end{bmatrix}$. Find a matrix P such that $\left[\vec{x}\right]_{\mathfrak{R}} = P\left[\vec{x}\right]_{\mathfrak{B}}$, for all $\vec{x}$ in $\mathbb{R}^2$.

Let U and V be a matrix composed of the basis vectors of $\mathfrak{B}$ and $\mathfrak{R}$ where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

By definition:

$$\begin{aligned} \vec{x} &= U[\vec{x}]_{\mathfrak{B}} \\ \vec{x} &= V[\vec{x}]_{\mathfrak{R}} \end{aligned}$$

Applying the inverse for one of them yields:

$$V^{-1}\vec{x} = [\vec{x}]_{\mathfrak{R}}$$

Then, writing $\vec{x}$ as a U transformation:

$$\begin{aligned} [\vec{x}]_{\mathfrak{R}} &= V^{-1}\vec{x} \\ &= \underbrace{V^{-1}U}_{P}[\vec{x}]_{\mathfrak{B}} \end{aligned}$$

By comparison, $P = V^{-1}U$.

$$\begin{aligned} \therefore P &= V^{-1}U \\ &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

56. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Let $P = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$ for some vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.

Then, by $\vec{x} \stackrel{\Delta}{=} P[\vec{x}]_{\mathfrak{B}}$, we have:

$$\begin{cases} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = P \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \end{pmatrix} = P \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{cases} \implies P \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$
\begin{aligned}
\therefore P &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 12 & -7 \\ 14 & -8 \end{pmatrix}
\end{aligned}
$$

As such, $\mathfrak{B} = \left\{ \begin{pmatrix} 12 \\ 14 \end{pmatrix}, \begin{pmatrix} -7 \\ -8 \end{pmatrix} \right\}$.