Final exam

Question 1. (10 points)

(a) What is the definition that $\{ v_1, ..., v_n \}$ forms a basis for a vector space V.

$\{ v_1, ..., v_n \}$ forms a basis for a vector space V if and only if they are linearly independent and spans the space V.

(b) What is the definition of the dimension of a vector space V?

The dimension of a vector space V is the number of vectors in a basis of V.

(c) Explain why the vector space $\mathcal{M}_{3,2}$, the set of all 3×2 matrices has dimension 6.

Consider a matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \in \mathcal{M}_{3,2}$. This matrix can be expanded as

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &+ d \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

By inspection,

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

are linearly independent and and every matrix in $\mathcal{M}_{3,2}$ can be expanded by them. Hence, it forms a basis of $\mathcal{M}_{3,2}$ and its dimension is six.

Question 2. (10 points) Find the answer of the following problem. Write a brief solution to explain.

a. Suppose that A is a 8×17 matrix and the kernel of A has dimension 12. What is the dimension of $\mathrm{Im}(A)$?

If $\dim \ker(A) = 12$, then finding $\mathrm{rref}(A)$ will yield five pivot columns because the rest are free variables.

The corresponding pivot column on A will make up a basis of $\mathrm{Im}(A)$. As such, the dimension of $\mathrm{Im}(A)$ is five.

b.Find the inverse of the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

c. Find the dimension of the following subspace on $\mathbb{R}^4$.

$$W = \{ (x, y, z, w) : x + y + z + w = 0 \}.$$

$$x = -y - z - w$$

$$W = \left\{ \begin{pmatrix} -y-z-w \\ y \\ z \\ w \end{pmatrix} : y, z, w \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\therefore \dim W = 3$$

d. Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 100 & 100 & 100 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 100 & 100 & 100 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 100 & 100 & 100 \end{pmatrix} \xrightarrow{R_3 - 100R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \det A = 1 \cdot 1 \cdot 0 = 0$$

Question 3. (15 points) Let

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Using Gram-Schmidt Process, find an orthogonal basis for the $\mathrm{Im}(A)$.

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathrm{Im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Let $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then:

$$\vec{\mathbf{v}}_2 = \begin{pmatrix}4\\2\\0\\0\end{pmatrix} - \frac{\left\langle \begin{pmatrix}4\\2\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\|^2} \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}4\\2\\0\\0\end{pmatrix} - 4\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\2\\0\\0\end{pmatrix}$$

As such, an orthogonal basis for $\mathrm{Im}(A)$ is $\{\,\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\,\} = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\end{pmatrix} \right\}$.

(b) Find the basis for the orthogonal complement for the $\mathrm{Im}(A)$.

$$A^\top = \begin{bmatrix}1&0&0&0\\3&0&0&0\\4&2&0&0\\5&6&0&0\end{bmatrix} \implies \mathrm{rref}(A^\top) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\therefore \mathrm{Im}(A)^\perp = \ker(A^\top) = \mathrm{span}\left\{ \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

As such, the basis for the orthogonal complement of $\mathrm{Im}(A)$ is $\left\{ \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$.

(c) Let $\mathbf{b} = \begin{pmatrix}0\\2\\1\\-1\end{pmatrix}$

(i) Find the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Im}(A)$.

From (a), an orthogonal basis for $\mathrm{Im}(A)$ is $\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\end{pmatrix} \right\}$. As such:

$$\operatorname{proj}_{\operatorname{Im}(A)}(\mathbf{b}) = \frac{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= 0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{4}{4} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

(i) Find the orthogonal projection of b onto the orthogonal complement of $\operatorname{Im}(A)$

From (b), the orthogonal complement of $\operatorname{Im}(A)$ is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ As such:

$$\operatorname{proj}_{\operatorname{Im}(A)^{\perp}}(\mathbf{b}) = \frac{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Question 4. (10 points) Suppose that we want to find the least square best fitting hyperplane $z = Ax + By + C$ for a set of datas $(x_1, y_1, z_1), ..., (x_k, y_k, z_k)$. Explain step by step the procedure we need to do.

First, interpret the data points $(x_1, y_1, z_1), ..., (x_k, y_k, z_k)$, as a system of equation:

$$\begin{cases} z_1 = Ax_1 + By_1 + C \\ \vdots \\ z_k = Ax_k + By_k + C \end{cases}$$

Then, they can be written in the form $\vec{\mathbf{b}} = A\hat{\mathbf{x}}$.

$$\begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ & \ddots & \\ x_k & y_k & 1 \end{bmatrix} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{pmatrix}$$

Finally, to find $\hat{\mathbf{x}}$, we apply $A^\top$ to both sides.

$$A\hat{\mathbf{x}} = \vec{\mathbf{b}}$$
$$A^\top A\hat{\mathbf{x}} = A^\top\vec{\mathbf{b}}$$
$$\therefore \hat{\mathbf{x}} = (A^\top A)^{-1}A^\top\vec{\mathbf{b}}$$

Thus, $\hat{\mathbf{x}} = \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{pmatrix}$ is the least squared solution, giving us the best fitting hyperplane $z = \hat{A}x + \hat{B}y + \hat{C}$.

Question 5 (15 points)

(a) State the definition of eigenvalue and eigenvectors of a matrix A.

We say that λ is an *eigenvalue* of A if we can find some vector $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ such that $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. Subsequently, $\vec{\mathbf{v}}$ is the corresponding *eigenvectors* associated with λ.

(b) State the definition of geometric multiplicity and algebraic multiplicity of the eigenvalue λ for the matrix A.

The *geometric multiplicity* of an eigenvalue λ is $\dim\ker(A - \lambda I)$.

The *algebraic multiplicity* of an eigenvalue λ_i is the highest power p such that $(\lambda - \lambda_i)^p$ is a factor of $\det(A - \lambda I)$.

(c) Let

$$A = \begin{pmatrix} 3 & -2 & 4 & -4 \\ 1 & 0 & 2 & -2 \\ -1 & 1 & -1 & 2 \\ -1 & 1 & -2 & 3 \end{pmatrix}$$

Find the eigenvalues of A (computer is allowed, but you need to write down the polynomial equation required to solve) and determine if A is diagonalizable.

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -2 & 4 & -4 \\ 1 & 0-\lambda & 2 & -2 \\ -1 & 1 & -1-\lambda & 2 \\ -1 & 1 & -2 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (\lambda - 1)^3(\lambda - 2) = 0$$
$$\therefore \lambda = 1, 2$$

The eigenvalues of A are 1 and 2 with algebraic multiplicities of 3 and 2, respectively.

For A to be diagonalizable, the geometric multiplicities must be equal to the algebraic multiplicities for all corresponding eigenvectors.

For $\lambda = 1$,

$$\operatorname{rref}(A - I) = \operatorname{rref}\begin{pmatrix} 3-1 & -2 & 4 & -4 \\ 1 & 0-1 & 2 & -2 \\ -1 & 1 & -1-1 & 2 \\ -1 & 1 & -2 & 3-1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three free variables, as such $\dim\ker(A - I) = 3$. So the geometric and algebraic multiplicity for $\lambda = 1$ matches.

For $\lambda = 2$,

$$\operatorname{rref}(A - 2I) = \operatorname{rref}\begin{pmatrix} 3-2 & -2 & 4 & -4 \\ 1 & 0-2 & 2 & -2 \\ -1 & 1 & -1-2 & 2 \\ -1 & 1 & -2 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is one free variable, as such $\dim\ker(A - 2I) = 1$. And so, the geometric and algebraic multiplicity for $\lambda = 2$ also matches.

As such, we conclude that A is diagonalizable.

Question 6. (10 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be any 5 vectors in a vector space V of dimension 4. Determine if the following statements are correct. Explain.

(i) These 5 vectors must be linearly dependent.

True. Since these vectors are of dimension four, at least one of them must be the same vector or a multiple or each other.

(ii) We can always extract a basis for V from these 5 vectors.

False. Consider $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = \mathbf{v}_5$.

(iii) We can always extract a basis for the subspace $\operatorname{span}\{\,\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\,\}$ from these 5 vectors.

True. Let $W \subseteq V$ be a subspace where $W = \operatorname{span}\{\,\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\,\}$. A basis of W would just be the set of linearly independent vectors in the span of W. Just to be clear, this only applies to W and may not necessarily span the vector space V.

Question 7. (15 points)

(a) Define rigorously the definition of the least square solution for the system $A\mathbf{x} = b$. Using your definition, explain why if the system $A\mathbf{x} = b$ has a solution x_0, then x_0 must be the least square solution.

The least square solution for the system $A\mathbf{x} = b$ is a vector $\hat{\mathbf{x}}$ such that $||b - A\hat{\mathbf{x}}||$ is minimized. More concretely, it is a solution such that

$$||b - A\hat{\mathbf{x}}|| \leq ||b - A\mathbf{x}||$$

for all other $\mathbf{x}$. In other words, the whole point of finding the least square solution is to satisfy the system $A\mathbf{x} = b$ as closely as possible.

However, if $A\mathbf{x} = b$ has a solution x_0, then $||b - Ax_0|| = 0$. Which means that x_0 is a solution such that the distance is minimized. Hence, x_0 must be the least square solution.

(b). Let A be an $m \times n$ matrix with $\text{rank}(A) = n$. Let also $A = U\Sigma V^T$ be its singular value decomposition. Show that the least square solution of the system $A\mathbf{x} = \mathbf{b}$ is equal to

$$\hat{\mathbf{x}} = \frac{\langle \mathbf{b}, \mathbf{u}_1 \rangle}{\sigma_1}\mathbf{v}_1 + \cdots + \frac{\langle \mathbf{b}, \mathbf{u}_n \rangle}{\sigma_n}\mathbf{v}_n$$

If A is an $m \times n$ matrix with $\text{rank}(A) = n$. Then, $A^\top A$ is an $m \times m$ matrix with $\text{rank}(A^\top A) = m$.

Then, let

$$U = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ | & & | \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_n & 0 \end{bmatrix}, \quad V = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix},$$

where $U^\top U = I$ and $V^\top V = I$.

Since $A^\top A$ is invertible, the least square solution of the system $A\mathbf{x} = \mathbf{b}$ is

$$\hat{\mathbf{x}} = (A^\top A)^{-1} A^\top \mathbf{b}.$$

Now, let's unpack $A^\top A$.

$$A = U\Sigma V^\top \implies A^\top = V\Sigma^\top U^\top$$

$$\begin{aligned} A^\top A &= V\Sigma^\top \underbrace{U^\top U}_{I} \Sigma V^\top \\ &= V \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_n & 0 \end{bmatrix} V^\top \\ &= V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^\top \end{aligned}$$

Then, taking the inverse yields:

$$\begin{aligned} (A^\top A)^{-1} &= \left(V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^\top \right)^{-1} \\ &= V \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix} V^\top \end{aligned}$$

Finally, plugging it into the expression for $\hat{\mathbf{x}}$:

$$
\begin{aligned}
\hat{\mathbf{x}} &= (A^\top A)^{-1}A^\top \mathbf{b} \\
&= V \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix} \underbrace{V^\top V}_{I} \Sigma^\top U^\top \mathbf{b} \\
&= \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_n & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \mathbf{b} \\
&= \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix} \begin{bmatrix} \langle \mathbf{b}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{b}, \mathbf{u}_n \rangle \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \frac{\langle \mathbf{b}, \mathbf{u}_1 \rangle}{\sigma_1} \\ \vdots \\ \frac{\langle \mathbf{b}, \mathbf{u}_n \rangle}{\sigma_n} \end{bmatrix} \\
&= \frac{\langle \mathbf{b}, \mathbf{u}_1 \rangle}{\sigma_1} \mathbf{v}_1 + \cdots + \frac{\langle \mathbf{b}, \mathbf{u}_n \rangle}{\sigma_n} \mathbf{v}_n
\end{aligned}
$$

Question 8. (15 points) Let $\mathcal{P}_n$ be the vector space of polynomials of degree at most n. Let

$$W_1 = \{\, P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n : P(1) = 0 \,\}.$$

(i) Show that W_1 is a subspace of $\mathcal{P}_n$.

Checking if W_1 is closed under addition

Consider $P_1, P_2 \in W_1$. Then, $P_1(1) = 0$ and $P_2(1) = 0$. Subsequently,

$$(P_1 + P_2)(1) = P_1(1) + P_2(1) = 0 + 0 = 0.$$

Hence, $P_1 + P_2 \in W_1$.

Checking if W_1 is closed under scalar multiplication

Consider $P \in W_1$ and $\alpha \in \mathbb{R}$. Then, $P(1) = 0$ and $\alpha P(1) = \alpha(0) = 0$.

Hence, $\alpha P \in W_1$.

Since W_1 is closed under addition and scalar multiplication, it is a subspace of $\mathcal{P}_n$.

(ii) Find a basis for W_1.

Consider a set of polynomial $P_1, P_2, \ldots, P_n \in W_1$, where:

$$
\begin{aligned}
P_1(x) &= a_0 + a_1 x \\
P_2(x) &= a_0 + a_1 x + a_2 x^2 \\
&\vdots \\
P_n(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n
\end{aligned}
$$

For some $P \in W_1$, we need $P(1) = 0$.

$$
\begin{aligned}
P(1) = a_0 + a_1(1) + a_2(1)^2 + \cdots + a_n(1)^n &= 0 \\
= a_0 + a_1 + a_2 + \cdots + a_n &= 0 \\
\therefore a_0 + a_1 + a_2 + \cdots + a_n = 0
\end{aligned}
$$

So, we need the sum of the coefficients to be zero. An easy way to satisfy this is to:

- let the coefficient of the degree zero term, $a_0 = 1$,
- let the coefficient of the highest degree term $a_n = -1$,
- and set the coefficient of all other terms $a_1 = a_2 = \cdots = a_{n-1} = 0$.

$$
\begin{gathered}
a_0 + \cancel{a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}} + a_nx^n \\
a_0 = 1 \quad \Big\downarrow \quad a_n = -1 \\
1 - x^n
\end{gathered}
$$

Simply put, we want to kill off the middle terms so that we get polynomials that will give us $1 - 1^n = 0$.

And so, we have:

$$
\begin{aligned}
P_1(x) &= 1 - x \\
P_2(x) &= 1 - x^2 \\
&\vdots \\
P_n(x) &= 1 - x^n
\end{aligned}
$$

By observation, we can see that $P_i(1) = 0$ for all $i \in \mathbb{Z}^+$.

Let $c_1, c_2, \ldots, c_n \in \mathbb{R}$ be some scalars. Then, using the abovementioned, we have that

$$
\begin{aligned}
&c_1P_1(x) + c_2P_2(x) + \cdots + c_nP_n(x) \\
&= c_1(0) + c_2(0) + \cdots + c_n(0) \\
&= 0.
\end{aligned}
$$

As such, a basis of W_1 is $\{\, 1 - x, 1 - x^2, \cdots, 1 - x^n \,\}$.

We now let

$$W = \{\, P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n : P(i) = 0, \text{for all } i = 1, 2, \cdots, n \,\}.$$

(iii) Google online the definition of the "Vandermonde matrix" and write down the determinant of the Vandermonde matrix.

From Wikipedia:

In linear algebra, a *Vandermonde matrix*, named after Alexandre-Théophile Vandermonde, is a matrix with the terms of a geometric progression in each row: an $(m + 1) \times (n + 1)$ matrix

$$V = V(x_0, x_1, \cdots, x_m) = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}$$

with entries $V_{i,j} = x_i^j$, the j^{th} power of the number x_i, for all zero-based indices i and j.

The determinant of a square Vandermonde matrix (when $n = m$) is called a *Vandermonde determinant* or Vandermonde polynomial. Its value is:

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i).$$

This is non-zero if and only if all x_i are distinct (no two are equal), making the Vandermonde matrix invertible.

(iv) Use Vandermonde matrix, show that $W = \{0\}$.

If $P \in W$, then it is a polynomial in the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

such that it satisfies

$$P(x_0) = y_0, P(x_1) = y_1, \ldots, P(x_m) = y_m.$$

In our case, we have that

$$\vec{\mathbf{x}} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{y}} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} P(x_0) \\ P(x_1) \\ P(x_2) \\ \vdots \\ P(x_m) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Here, we note that $\vec{\mathbf{y}} = \vec{\mathbf{0}}$ because

$$\begin{aligned} P \in W &\iff P(x_0) = P(x_1) = P(x_2) = \cdots = P(x_m) = 0 \\ &\iff P(1) = P(2) = P(3) = \cdots = P(m) = 0. \end{aligned}$$

Then, our Vandermonde matrix V is given by:

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ 1 & 3 & 3^2 & \cdots & 3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & m^2 & \cdots & m^n \end{bmatrix}$$

Let $\vec{\mathbf{a}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ be a column vector representing the coefficients. Then, we can construct a system $V\vec{\mathbf{a}} = \vec{\mathbf{y}}$. Since $\vec{\mathbf{y}} = \vec{\mathbf{0}}$, we just have a homogenous system.

$$V\vec{\mathbf{a}} = \vec{\mathbf{y}}$$
$$\begin{bmatrix} 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ 1 & 3 & 3^2 & \cdots & 3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & m^2 & \cdots & m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

As such, we know there exists a trivial solution for $\vec{\mathbf{a}}$.

Finally, since our x_i terms are distinct (ascending natural numbers), $\det(V) \neq 0$. Therefore, this system contains **only trivial solution**.

Further, given that $m = n$, V is also invertible. As such, P can be obtained by finding that $\vec{\mathbf{a}} = V^{-1}\vec{\mathbf{y}} = \vec{\mathbf{0}}$.

Since

$$\vec{\mathbf{a}} = \vec{\mathbf{0}} \implies a_0 = \cdots = a_n = 0,$$

then such a polynomial $P \in W$ must be zero. Hence, W is a trivial subspace.