Homework

(All solution must be supplied with steps and justifications. Numerical answer without justifications will not be graded)

1. Consider $\mathbb{R}^4$ with standard inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

(i) Find the norm of the vectors $\mathbf{u} = (1, 2, 3, 2)$ and $\mathbf{v} = (2, 1, -1, 0)$.

(ii) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

2. Consider $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(i) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for $\mathbb{R}^3$.

(ii) Find the orthonormal basis generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(ii) Express $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

3. Consider the space of all continuous functions on $[0, 1]$, $C[0, 1]$ with the standard inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

(i) Find the norm of $f(x) = x^n$, for any positive integer n.

(ii) Find the angle between x^n and x^m.

(iii) Show that for any $m \neq n$, $\sin 2\pi mx$ and $\sin 2\pi nx$ are always mutually orthogonal. (Hint: Check out product-to-sum formula)

4. Prove the identity

$$\langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle = ac\|\mathbf{v}\|^2 + (ad + bc)\langle \mathbf{v}, \mathbf{w} \rangle + bd\|\mathbf{w}\|^2.$$

5. Given an inner product space V.

(i) Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

(This is called the parallelogram identity)

(ii) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2)$$

(This is called the polarization identity)

(iii) Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

(This is Pythagorean Theorem)