Homework 5

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1. *(Although this question is just copying definitions, this is important to understand the whole concepts and I expect students should remember these definitions)*

Write down the definition of the following:

(a) W is a subspace of a vector space V

W is a subspace if:

- given vectors $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$; and
- given scalar $\alpha \in \mathbb{R}$ and vector $\mathbf{v} \in W$, then $\alpha\mathbf{v} \in W$.

(b) $\operatorname{span}\{\, \mathbf{v}_1, ..., \mathbf{v}_n \,\}$.

The set of all linear combinations $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called their span.

$$\operatorname{span}\{\, \mathbf{v}_1, \ldots, \mathbf{v}_n \,\} = \{\, c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n : c_1, \ldots, c_n \in \mathbb{R} \,\}$$

(c) $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly independent.

...if and only if

$$\{\, x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n : x_i \in \mathbb{R} \,\}$$

are distinct vectors.

(d) $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly dependent.

...if and only if

$$\exists \mathbf{v}_j \in \operatorname{span}\{\, \mathbf{v}_i : i \neq j \,\}$$

(e) $\mathbf{v}_1, ..., \mathbf{v}_n$ forms a basis of V.

...if:

- $\{\, \mathbf{v}_1, \ldots, \mathbf{v}_n \,\}$ are linearly independent; and
- $\operatorname{span}\{\, \mathbf{v}_1, \ldots, \mathbf{v}_n \,\} = V$.

This means that every vector in V is a unique linear combination of $\{\, \mathbf{v}_1, \ldots, \mathbf{v}_n \,\}$.

(f) The dimension of a vector space V.

...is the number of vectors that make up a basis of V.

2. Determine if the following sets are subspaces of $\mathbb{R}^3$. Justify your answer.

(i) $W_1 = \{ (x, y, z) : x = z + 2 \}$

Checking $\mathbf{0} \in W_1$

Clearly, $(0, 0, 0) \notin W_1$. Therefore W_1 is not a subspace of $\mathbb{R}^3$.

$$(x, y, z) = (0, 0, 0) \implies 0 \neq 0 + 2$$
$$\therefore \mathbf{0} \notin W_1$$

(ii) $W_2 = \{ (x, y, z) : x = 3y \text{ and } z = -y \}$

Checking if W_2 is closed under addition

Consider two vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in W_2$.

Then,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \iff \begin{cases} x_1 = 3y_1 \\ z_1 = -y_1 \end{cases}, \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \iff \begin{cases} x_2 = 3y_2 \\ z_2 = -y_2 \end{cases}$$

And so, for $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$, notice that:

$$\begin{cases} x_1 + x_2 = 3y_1 + 3y_2 = 3(y_1 + y_2) \\ z_1 + z_2 = -y_1 - y_2 = -(y_1 + y_2) \end{cases} \implies \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \in W_2$$

As such, W_2 is closed under addition.

Checking if W_2 is closed under scalar multiplication

Consider $\alpha \in \mathbb{R}$ and $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in W_2$. Then, for $\alpha\mathbf{u} = \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{pmatrix}$, notice that:

$$\begin{cases} \alpha x_1 = \alpha(3y_1) = 3(\alpha y_1) \\ \alpha z_1 = \alpha(-y_1) = -(\alpha y_1) \end{cases} \implies \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{pmatrix} \in W_2$$

As such, W_2 is closed under scalar multiplication.

Therefore, we can conclude that W_2 is a subspace of $\mathbb{R}^3$.

(iii) $W_3 = \{ (x, y, z) : z = x^2 + y^2 \}$

Checking if W_3 is closed under addition

Consider two vectors $\mathbf{u}, \mathbf{v} \in W_3$ where: $\mathbf{u} = (1, 1, 2)$ and $\mathbf{v} = (2, 2, 8)$.

$$\begin{aligned} \mathbf{u} = (1,1,2) &\implies 2 = 1^2 + 1^2 \\ \mathbf{v} = (2,2,8) &\implies 8 = 2^2 + 2^2 \end{aligned}$$

Then, $\mathbf{u} + \mathbf{v} = (3, 3, 10)$. But clearly, $\mathbf{u} + \mathbf{v} \notin W_2$:

$$\mathbf{u} + \mathbf{v} = (3, 3, 10) \implies 10 \neq 3^2 + 3^2$$

W_3 is not closed under addition, therefore it is not a subspace of $\mathbb{R}^3$.

3. For the following sets of vectors

(i) $\mathbf{v}_1 = (0, 1, 1)$, $\mathbf{v}_2 = (1, -1, 0)$ and $\mathbf{v}_3 = (3, -1, 2)$.
(ii) $\mathbf{v}_1 = (2, 1, 3)$, $\mathbf{v}_2 = (1, -2, 1)$, $\mathbf{v}_3 = (2, -3, 0)$ and $\mathbf{v}_4 = (0, -1, 4)$.
(iii) $\mathbf{v}_1 = (1, 0, 2, 1)$, $\mathbf{v}_2 = (-2, 3, -1, 1)$ and $\mathbf{v}_3 = (2, -2, 1, -1)$.

(a) Determine if the above set of vectors linearly dependent or linearly independent.

(i) $\mathbf{v}_1 = (0, 1, 1), \mathbf{v}_2 = (1, -1, 0)$ and $\mathbf{v}_3 = (3, -1, 2)$.

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The linear combination of the vectors has free variables. As such, they are linearly dependent.

(ii) $\mathbf{v}_1 = (2, 1, 3), \mathbf{v}_2 = (1, -2, 1), \mathbf{v}_3 = (2, -3, 0)$ and $\mathbf{v}_4 = (0, -1, 4)$.

The number of vectors (4) is greater than their dimensions (3). As such, they are linearly dependent.

(iii) $\mathbf{v}_1 = (1, 0, 2, 1), \mathbf{v}_2 = (-2, 3, -1, 1)$ and $\mathbf{v}_3 = (2, -2, 1, -1)$.

$$\operatorname{rref} \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \operatorname{rref} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The linear combination of the vectors do not have free variables. As such, they are linearly independent.

(b) For (i), determine if $\mathbf{w} = (1, 1, 1)$ lies in the span.

Where $\mathbf{v}_1 = (0, 1, 1)$, $\mathbf{v}_2 = (1, -1, 0)$, and $\mathbf{v}_3 = (3, -1, 2)$.

$$\operatorname{rref} \left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{w} \\ | & | & | & | \end{array} \right] = \operatorname{rref} \left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 1 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The system has no solution. Therefore, $\mathbf{w} = (1, 1, 1)$ is not in the span.

(c) For (ii), express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

Where $\mathbf{v}_1 = (2, 1, 3)$, $\mathbf{v}_2 = (1, -2, 1)$, $\mathbf{v}_3 = (2, -3, 0)$, and $\mathbf{v}_4 = (0, -1, 4)$.

$$\text{rref}\left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{array}\right] = \text{rref}\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & -2 & -3 & -1 \\ 3 & 1 & 0 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{11} \\ 0 & 1 & 0 & \frac{38}{11} \\ 0 & 0 & 1 & -\frac{21}{11} \end{array}\right]$$

$$\therefore \mathbf{v}_4 = \frac{2}{11}\mathbf{v}_1 + \frac{38}{11}\mathbf{v}_2 - \frac{21}{11}\mathbf{v}_3$$

4. Expand the kernel of the following matrices as span of vectors and then compute the dimension.

> Assuming the question is asking for the dimension of the kernel i.e., the *nullity*.

(a)

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\text{rref}\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \ker \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

The nullity is zero.

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x_1 = -x_2 - x_3$$

$$x_2, x_3 \in \mathbb{R}$$

$$
\begin{aligned}
\therefore \ker \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} &= \left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\} \\
&= \left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}
\end{aligned}
$$

The nullity is two.

(c)

$$
\begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}
$$

$$
\operatorname{rref} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 1 & -1 & -2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array} \right] = \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]
$$

$$
\begin{gathered}
\therefore x_5 = 0 \\
x_3 = -x_4 \\
x_1 = x_2 - 2x_4 \\
x_2, x_4 \in \mathbb{R}
\end{gathered}
$$

$$
\begin{aligned}
\therefore \ker \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix} &= \left\{ \begin{pmatrix} x_2 - 2x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\} \\
&= \left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

The nullity is two.

5. Let $W = \{ (x_1, x_2, x_3, x_4) : x_1 - x_2 + 2x_3 - x_4 = 0 \}$. Find a basis for the subspace W.

$$
x_1 - x_2 + 2x_3 - x_4 = 0 \implies x_1 = x_2 - 2x_3 + x_4
$$

$$W=\left\{\begin{pmatrix}x_2-2x_3+x_4\\x_2\\x_3\\x_4\end{pmatrix}: x_2,x_3,x_4\in\mathbb{R}\right\}$$

$$=\left\{x_2\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+x_3\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}+x_4\begin{pmatrix}1\\0\\0\\1\end{pmatrix}: x_2,x_3,x_4\in\mathbb{R}\right\}$$

$$\therefore \operatorname{basis}(W)=\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}-2\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$$

6. Let $W=\left\{(x_1,x_2,x_3,x_4):\begin{cases}x_1+x_2-x_3+x_4=0,\\2x_1+2x_2-2x_3+x_4=0,\end{cases}\right\}$. Find a basis for the subspace W and what is its dimension?

$$\begin{cases}x_1+x_2-x_3+x_4=0\\2x_1+2x_2-2x_3+x_4=0\end{cases}\iff\left[\begin{array}{cccc|c}1&1&-1&1&0\\2&2&-2&1&0\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1&1&-1&1&0\\2&2&-2&1&0\end{array}\right]\xrightarrow{R_2-2R_1}\left[\begin{array}{cccc|c}1&1&-1&1&0\\0&0&0&-1&0\end{array}\right]\xrightarrow[-R_2]{R_1+R_2}\left[\begin{array}{cccc|c}1&1&-1&0&0\\0&0&0&1&0\end{array}\right]$$

$$\therefore x_4=0$$
$$x_1=-x_2+x_3$$
$$x_2,x_3\in\mathbb{R}$$

$$W=\left\{\begin{pmatrix}-x_2+x_3\\x_2\\x_3\\0\end{pmatrix}: x_2,x_3\in\mathbb{R}\right\}$$

$$=\left\{x_2\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+x_3\begin{pmatrix}1\\0\\1\\0\end{pmatrix}: x_2,x_3\in\mathbb{R}\right\}$$

$$\therefore \operatorname{basis}(W)=\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right\}$$

The nullity is two.