Final Exam: MATH 325

Question 1. (10 points)

(a) What is the definition that $\{v_1, ..., v_n\}$ forms a basis for a vector space V.

(b) What is the definition of the dimension of a vector space V?

(c) Explain why the vector space $\mathcal{M}_{3,2}$, the set of all 3×2 matrices has dimension 6.

Question 2. (10 points) Find the answer of the following problem. Write a brief solution to explain.

a. Suppose that A is a 8×17 matrix and the kernel of A has dimension 12. What is the dimension of $\mathrm{Im}(A)$?

b.Find the inverse of the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

c. Find the dimension of the following subspace on $\mathbb{R}^4$.

$$W = \{(x, y, z, w) : x + y + z + w = 0\}.$$

d. Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 100 & 100 & 100 \end{pmatrix}.$$

Question 3.(15 points) Let

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Using Gram-Schmidt Process, find an orthogonal basis for the $\mathrm{Im}(A)$.

(b) Find the basis for the orthogonal complement for the $\mathrm{Im}(A)$.

(c) Let $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$.

(i) Find the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Im}(A)$.
(i) Find the orthogonal projection of $\mathbf{b}$ onto the orthogonal complement of $\mathrm{Im}(A)$.

Question 4. (10 points) Suppose that we want to find the least square best fitting hyperplane $z = Ax + By + C$ for a set of datas (x_1, y_1, z_1),...,(x_k, y_k, z_k). Explain step by step the procedure we need to do.

Question 5 (15 points) (a) State the definition of eigenvalue and eigenvectors of a matrix A.

(b) State the definition of geometric multiplicity and algebraic multiplicity of the eigenvalue λ for the matrix A.

(c) Let

$$A = \begin{pmatrix} 3 & -2 & 4 & -4 \\ 1 & 0 & 2 & -2 \\ -1 & 1 & -1 & 2 \\ -1 & 1 & -2 & 3 \end{pmatrix}$$

Find the eigenvalues of A (computer is allowed, but you need to write down the polynomial equation required to solve) and determine if A is diagonalizable.

Question 6. (10 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be any 5 vectors in a vector space V of dimension 4. Determine if the following statements are correct. Explain.

(i) These 5 vectors must be linearly dependent.

(ii) We can always extract a basis for V from these 5 vectors.

(iii) We can always extract a basis for the subspace $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ from these 5 vectors.

Question 7. (15 points) (a) Define rigorously the definition of the least square solution for the system $A\mathbf{x} = b$. Using your definition, explain why if the system $A\mathbf{x} = b$ has a solution $\mathbf{x}_0$, then $\mathbf{x}_0$ must be the least square solution.

(b). Let A be an $m \times n$ matrix with $\text{rank}(A) = n$. Let also $A = U\Sigma V^T$ be its singular value decomposition. Show that the least square solution of the system $A\mathbf{x} = \mathbf{b}$ is equal to

$$\widehat{\mathbf{x}} = \frac{\langle \mathbf{b}, \mathbf{u}_1 \rangle}{\sigma_1}\mathbf{v}_1 + \cdots \frac{\langle \mathbf{b}, \mathbf{u}_n \rangle}{\sigma_n}\mathbf{v}_n.$$

Question 8. (15 points) Let $\mathcal{P}_n$ be the vector space of polynomials of degree at most n. Let

$$W_1 = \{P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n : P(1) = 0\}.$$

(i) Show that W_1 is a subspace of $\mathcal{P}_n$.

(ii) Find a basis for W_1.

We now let

$W = \{P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n : P(i) = 0, \text{ for all } i = 1, 2, \cdots, n.\}$.

(iii) Google online the definition of the "Vandermonde matrix" and write down the determinant of the Vandermonde matrix.

(iv) Use Vandermonde matrix, show that $W = \{0\}$.

*** **THE END** ***