Week 7

Subspace

$$\ker(A) = \{\, \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \,\}$$

To find a basis for $\ker(A)$:

1. Find $\operatorname{rref}(A)$.
2. Get the solution set.
3. Write the solution as a span of vectors.
4. Then, the vectors in the span are the basis of W.

$$\begin{aligned} \operatorname{Im}(A) &= \{\, A\vec{x} : \vec{x} \in \mathbb{R}^n \,\} \qquad A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_2 \\ | & & | \end{pmatrix} \\ &= \operatorname{span}\{\, \vec{v}_1, \dots, \vec{v}_2 \,\} \end{aligned}$$

To find a basis for $\operatorname{Im}(A)$:

1. Find $\operatorname{rref}(A)$.
2. Then, $\{\, \vec{v}_i \,\}$ corresponding to the pivot are the basis for $\operatorname{Im}(A)$.

Example

Given a set of a hundred vectors $\{\, \vec{v}_1, \dots, \vec{v}_{100} \,\}$ in $\mathbb{R}^5$. Find its basis.

First, put the vectors in in a vector A:

$$A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_2 \\ | & & | \end{pmatrix}$$

Then, do Gaussian to get its (reduced) row echelon form. Let's say there are pivots in columns one, three, and 37:

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & & 0 & & 0 & \\ 0 & & 1 & & 0 & \\ 0 & \cdots & 0 & \cdots & 1 & \cdots \\ 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & \end{pmatrix}$$

Then, the corresponding vectors in A

$$\{\, \vec{v}_1, \vec{v}_3, \vec{v}_{37} \,\}$$

is a basis of $\{\, \vec{v}_1, \dots, \vec{v}_{100} \,\}$.

Note

$$\ker(A) = \{\, \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \,\}$$
$$\operatorname{span}\{\, \vec{v}_1, \ldots, \vec{v}_n \,\} = \operatorname{Im}\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$
$$A\vec{x} = \sum_{i=1}^{n} x_i \vec{v}_i$$

Example

$$W = \operatorname{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}3\\4\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$
$$= \operatorname{Im}\begin{pmatrix}1 & 2 & 3 & 0\\1 & 3 & 4 & 1\\1 & 1 & 2 & 1\end{pmatrix} \in \mathbb{R}^3$$

Also note here W must be linearly dependent because there are four vectors in $\mathbb{R}^3$.

To find basis of W, do Gaussian on the image:

$$\operatorname{rref}\begin{pmatrix}1 & 2 & 3 & 0\\1 & 3 & 4 & 1\\1 & 1 & 2 & 1\end{pmatrix} = \begin{pmatrix}\boxed{1} & 0 & 1 & 0\\0 & \boxed{1} & 1 & 0\\0 & 0 & 0 & \boxed{1}\end{pmatrix}$$

The pivots are in columns one, two, and four. Then, go back to the image: $\{\, \vec{v}_1, \vec{v}_2, \vec{v}_4 \,\}$ are the basis of W:

$$\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

Which also means that

$$W = \operatorname{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

and also that $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$ are linearly independent.

So, for an $m \times n$ matrix A,

$$\dim(\ker(A)) = \text{nullity}(A) = n - \text{rank}(A)$$
$$\dim(\text{Im}(A)) = \text{rank}(A)$$

A summary of findings from the three chapters

For an $m \times n$ matrix $A = \begin{matrix} \vert & & \vert \\ v_1 & \cdots & v_n \\ \vert & & \vert \end{matrix}$	Chapter 1: Applying algorithms $A \to \text{rref}(A)$	Chapter 2: Linear transformation $T(x) = Ax$	Chapter 3: $\{ v_1, \ldots, v_n \}$
Only trivial solutions $A\vec{x} = \vec{0} \implies \vec{x} = \vec{0}$	No free variables $\text{rank}(A) = n$	T is injective $\ker(A) = \{ \vec{0} \}$	$\{ \vec{v}_1, \ldots, \vec{v}_n \}$ is linearly independent
$A\vec{x} = \vec{b}$ is solvable for all $\vec{b} \in \mathbb{R}^m$	Full row rank $\text{rank}(A) = m$	T is surjective $\text{Im}(A) = R^m$	$\text{span}\{ \vec{v}_1, \ldots, \vec{v}_n \} = \mathbb{R}^m$
If both	A is a square matrix $\text{rank}(A) = m = n$	T is bijective	$\{ \vec{v}_1, \ldots, \vec{v}_n \}$ is a basis

No matter what you are trying to do... bring the vectors into a matrix and Gaussian that bitch.

Change of basis

For a standard basis:

- $\mathbb{R}^n = \text{span}\{ \vec{e}_1, \ldots, \vec{e}_n \}$
- $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$

If we have a basis $\mathfrak{B} = \{ \vec{v}_1, \ldots, \vec{v}_n \}$ of $\mathbb{R}^n$, then

$$\forall \vec{x} \in \mathbb{R}^n \quad \vec{x} = \tilde{x}_1\vec{v}_1 + \cdots + \tilde{x}_n\vec{v}_n$$
$$[\vec{x}]_{\mathfrak{B}} \triangleq \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix}_{\mathfrak{B}}$$

Example

Let $P = \begin{pmatrix} \vert & & \vert \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vert & & \vert \end{pmatrix}$. Then,

$$\begin{aligned}\vec{x} &= \tilde{x}\vec{v}_1 + \cdots + \tilde{x}\vec{v}_n \\ &= P\begin{pmatrix}\tilde{x}_1 \\ \vdots \\ \tilde{x}_n\end{pmatrix} \\ &= P[\vec{x}]_{\mathfrak{B}}\end{aligned}$$

So, $\boxed{\vec{x} = P[\vec{x}]_{\mathfrak{B}}}$.

Example

Let $\mathfrak{B} = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\}$. What is $\vec{x}$ if $[\vec{x}]_{\mathfrak{B}} = \begin{pmatrix}3\\-1\end{pmatrix}$?

$$\begin{aligned}\vec{x} &= 3\begin{pmatrix}1\\2\end{pmatrix} + (-1)\begin{pmatrix}2\\1\end{pmatrix} \\ &= \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix} \\ &= \begin{pmatrix}1\\5\end{pmatrix}\end{aligned}$$

Homework 6 hint — Book question #55

We want a P such that $[\vec{x}]_{\mathfrak{R}} = [\vec{x}]_{\mathfrak{B}}$.

Let $U = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$ and $V = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$.

By definition:

$$\begin{aligned}\vec{x} &= U[\vec{x}]_{\mathfrak{B}} \\ \vec{x} &= V[\vec{x}]_{\mathfrak{R}}\end{aligned}$$

Rewrite the second one by applying inverse on both sides:

$$\vec{x} = V[\vec{x}]_{\mathfrak{R}} \implies V^{-1}\vec{x} = [\vec{x}]_{\mathfrak{R}}$$

Then plug in the expression for $\vec{x}$ in terms of the U transformation.

$$\begin{aligned}[\vec{x}]_{\mathfrak{R}} &= V^{-1}\vec{x} \\ &= V^{-1}U[\vec{x}]_{\mathfrak{B}}\end{aligned}$$

By comparison, $V^{-1}U$ is the transformation P.

Linear transformation

Now, we look at the change of basis as a linear transformation. Using the definition and applying the inverse, we can find the transformation matrix A and B.

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & T(\vec{x}) \\ {\scriptstyle P}\Big\downarrow & & \Big\downarrow{\scriptstyle P} \\ [\vec{x}]_{\mathfrak{B}} & \xrightarrow{B} & [T(\vec{x})]_{\mathfrak{B}} \end{array}$$

Definition

$$\begin{aligned} \vec{x} &= P[\vec{x}]_{\mathfrak{B}} \\ [\vec{x}]_{\mathfrak{B}} &= P^{-1}\vec{x} \end{aligned}$$

And so:

$$\begin{aligned} B[\vec{x}]_{\mathfrak{B}} &= [T(\vec{x})]_{\mathfrak{B}} \\ BP^{-1}(\vec{x}) &= P^{-1}T(\vec{x}) \\ &= P^{-1}A(\vec{x}) \end{aligned}$$

We have that the compositions

$$BP^{-1} = P^{-1}A,$$

from which we can derive:

$$\begin{aligned} A &= PBP^{-1} \\ B &= P^{-1}AP \end{aligned}$$

Example

Let $\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. What is the linear transformation under standard basis if $[T(\vec{x})]_{\mathfrak{B}} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} [\vec{x}]_{\mathfrak{B}}$?

Here, we want to find the matrix A. because it is the transformation under standard basis that makes: $\vec{x} \xrightarrow{A} T(\vec{x})$.

So, we can use the derived equation $A = PBP^{-1}$ where:

$$\begin{aligned} \mathfrak{B} &= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \implies P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Then, just plug in:

$$\begin{aligned} A &= PBP^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{7}{3} & \frac{8}{3} \\ -\frac{8}{3} & \frac{13}{3} \end{pmatrix} \end{aligned}$$

Why do we need to change basis?

Just because we can, we will jump ahead to Chapter 7: Diagonalization to explore why we do this.

Say, if we are given a square matrix A, how do we find A^n?

$$A^n = \underbrace{A \cdot A \cdots A \cdot A}_{n}$$

Take an easy example, where $A = I$. Then,

$$A^n = I^n = I$$

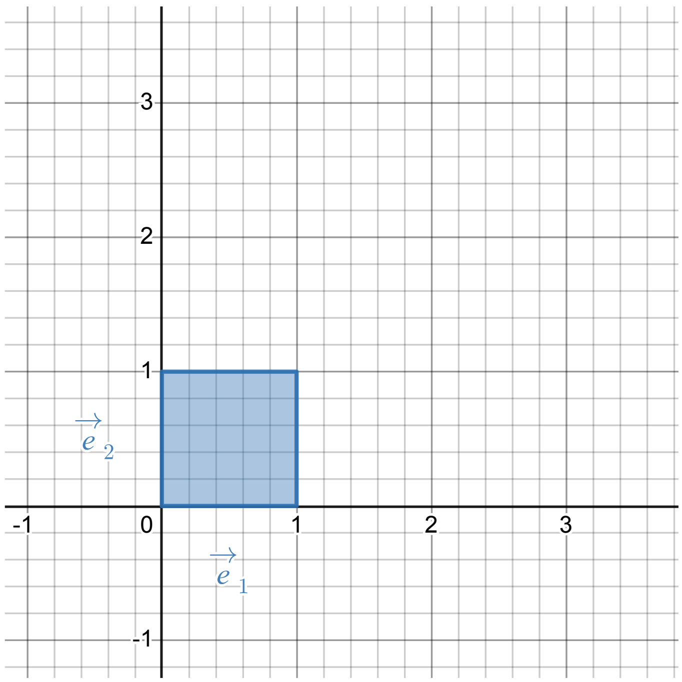
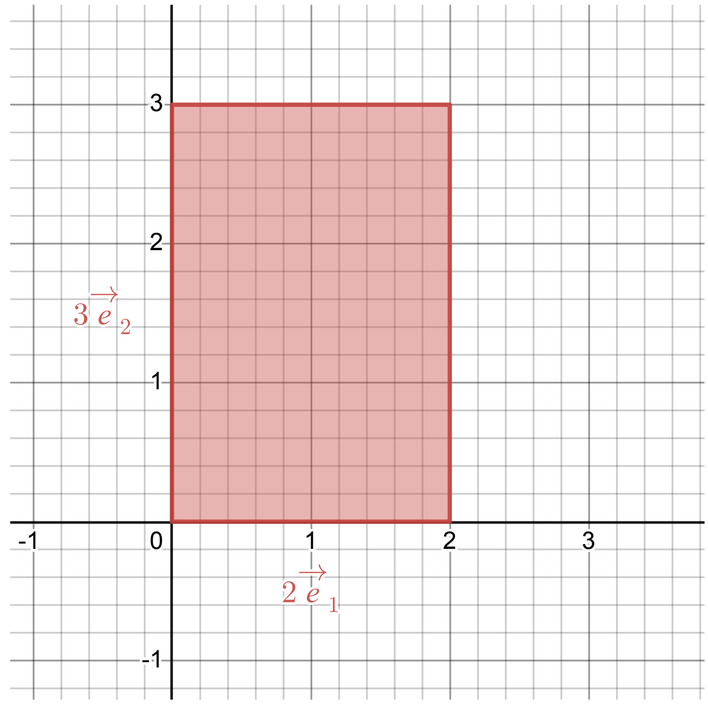
Or, how about diagonal matrices? For example: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

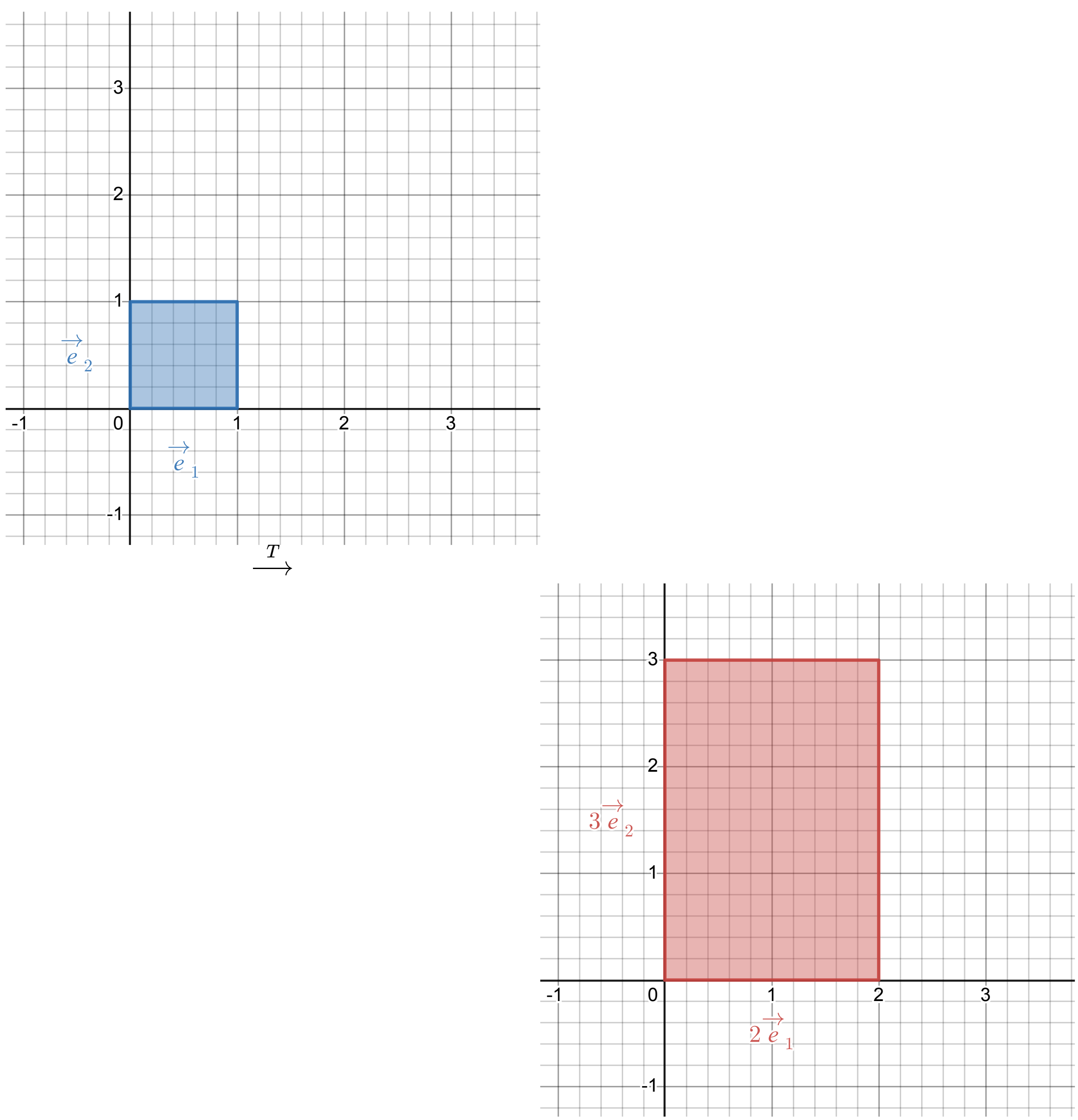
We can just multiply the diagonal $-$ easy!

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \implies A^2 = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} \implies A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

But... how would we deal with non-diagonal matrices? We simply cannot do proof by induction to find a “general” form for which we can multiply matrix. And so, this is where *change of basis* is needed.

Under standard basis, given two vectors:

x	$T(x)$
	


$\vec{e}_2$
$\vec{e}_1$
T
$3\vec{e}_2$
$2\vec{e}_1$