Homework 4

1. Compute the inverse of the following matrices.

(a)

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

Using the fact that $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ is a 2×2 matrix such that:

$$\det\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\right) = 4(-1) - 3(2) = -10 \neq 0$$

Then:

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} = -\frac{1}{10}\begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 30 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 30 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 30 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$
$$\xrightarrow[R_1-30R_3]{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -15 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$
$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{5} & -14 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & 2 & 30 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -14 \\ 0 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 & -4 & -140 \\ 0 & 2 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 6 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 3 & 1 & 6 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+3R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 7 & 12 & 0 & 1 & 3 \\ 0 & 5 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_3]{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 7 & -1 & 1 & 2 \\ 0 & 1 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 1 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 0 & -5 & \frac{7}{5} & -1 & -\frac{8}{5} \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & -\frac{6}{5} & 1 & \frac{9}{5} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array}\right]$$

$$\xrightarrow[R_2-6R_3]{R_1-3R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{46}{25} & -\frac{3}{5} & -\frac{24}{25} \\ 0 & 1 & 0 & \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array}\right]$$

$$\xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 & \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ 0 & 0 & 1 & -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 6 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{5} & 0 & -\frac{3}{5} \\ \frac{12}{25} & -\frac{1}{5} & -\frac{3}{25} \\ -\frac{7}{25} & \frac{1}{5} & \frac{8}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -15 \\ 12 & -5 & -3 \\ -7 & 5 & 8 \end{bmatrix}$$

(d)

$$\begin{bmatrix}1 & 3 & 3 & 3\\ 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc}1 & 3 & 3 & 3 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[R_3-R_4]{R_2-R_3} \left[\begin{array}{cccc|cccc}1 & 3 & 3 & 3 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_1-3R_3} \left[\begin{array}{cccc|cccc}1 & 3 & 0 & 3 & 1 & 0 & -3 & 3\\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_1-3R_4} \left[\begin{array}{cccc|cccc}1 & 3 & 0 & 0 & 1 & 0 & -3 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{cccc|cccc}0 & 3 & 0 & 0 & 1 & -1 & -2 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_2]{R_1\leftrightarrow R_2} \left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\therefore \begin{bmatrix}1 & 3 & 3 & 3\\ 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1 & -1 & 0\\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}0 & 3 & -3 & 0\\ 1 & -1 & -2 & 0\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 3\end{bmatrix}$$

(e)

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Using the fact that $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ is a 2×2 matrix such that:

$$\det\left(\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right) = \cos^2\theta - (-\sin\theta\sin\theta) = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Then:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that maps the standard vector $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Is T invertible? Explain.

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then, T^{-1} exists if A^{-1} exists.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A has a full rank. Therefore, T is invertible.

3.

(a) Find invertible matrix A, B such that $A + B$ is not invertible.

An $n \times n$ matrix M is invertible if and only if $\text{rank}(M) = n$.

Let A and B be 1×1 matrices.

$$\begin{aligned} A = \begin{bmatrix} 1 \end{bmatrix} &\implies \text{rank}(A) = 1 \implies A^{-1} = \begin{bmatrix} 1 \end{bmatrix} \\ B = \begin{bmatrix} -1 \end{bmatrix} &\implies \text{rank}(B) = 1 \implies B^{-1} = \begin{bmatrix} -1 \end{bmatrix} \end{aligned}$$

Hence, A and B are invertible.

$$A + B = \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \implies \text{rank}(A + B) = 0 \neq 1$$

Hence, $A + B$ is not invertible.

(b) Find non-invertible matrix A, B such that $A + B$ is invertible.

Let A and B be 2×2 matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies \operatorname{rank}(A) = 1 \neq 2$$
$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \implies \operatorname{rank}(B) = 1 \neq 2$$

Hence, A and B are not invertible.

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \operatorname{rank}(A+B) = 2$$
$$\implies (A+B)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $A + B$ is invertible.

4. For which values of the constant a, b is the following matrix not invertible?

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

(Hint: find its row echelon form)

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \xrightarrow[R_3 - R_2]{R_2 - R_1} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

The matrix is not invertible if $a = b$ or $a = 0$.

5. Determine if the following statement true or false.

(a) If a matrix A has a completely zero row, then it is not invertible.

True. If an $n \times n$ matrix A contains a zero row, then $\operatorname{rank}(A) < n$.

(b) Upper triangular matrices are always invertible.

False. If there is a zero on the major diagonal, then they will not have a full rank.

(c) If A is invertible, then $Ax = 0$ may have non-trivial solution.

False. $\operatorname{rref}(A)$ would be the identity matrix and therefore $A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$.

(d) If AB is invertible, then A is invertible.

False. Let A be an $m \times n$ matrix and B be an $n \times m$ for nonzero $m \neq n$.

A and B are not invertible because they are not square matrices. AB is an $m \times m$ square matrix and *could* be invertible if $\operatorname{rank}(AB) = n$.

For example: let $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 2 \end{bmatrix} \implies (AB)^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$.

6. For the matrices A, B are invertible, Is the following true? If it is true, verify it. If it is false, give an example to explain why it is false.

(i) $(A^2)^{-1} = (A^{-1})^2$

True.

$$\begin{aligned} (A^2)^{-1} &= (AA)^{-1} \\ &= A^{-1}A^{-1} \\ &= (A^{-1})^2 \end{aligned}$$

(ii) $(A+B)^{-1} = A^{-1} + B^{-1}$

False. Let $A = B = \begin{bmatrix} 1 \end{bmatrix}$. Then:

$$(A+B)^{-1} = \begin{bmatrix} 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$$
$$A^{-1} + B^{-1} = \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\therefore (A+B)^{-1} \neq A^{-1} + B^{-1}$$

7.

(i) Let A be an $n \times n$ matrix. Expand $(I-A)(I+A+A^2)$.

$$\begin{aligned} (I-A)(I+A+A^2) &= I(I-A) + A(I-A) + A^2(I-A) \\ &= I^2 \cancel{- AI + AI - A^2 + A^2 I} - A^3 \qquad \boxed{A^2 I = A^2} \\ &= I^2 - A^3 \\ &= I - A^3 \end{aligned}$$

(ii) Suppose that $A^3 = O$, the zero matrix. Use (i), find $(I-A)^{-1}$ in terms of A.

$$\begin{aligned} (I-A)(I+A+A^2) = I &\iff (I-A)^{-1}(I+A+A^2) = I \\ &\iff (I-A)(I+A+A^2)^{-1} = I \end{aligned}$$

$$\therefore (I-A)^{-1} = (I+A+A^2)$$

(iii) (Bonus 1 point) If $A^k = O$, find $(I-A)^{-1}$.

From (i) and (ii), we use

$$(I-A)(I+A+A^2)$$

to find $(I-A)^{-1}$ for $A^3 = O$. Then, for $A^k = O$, we can use:

$$(I - A)(I + A + A^2 + \cdots + A^{k-1})$$

We note that the second term is in a geometric form, something that I definitely remembered from Calculus II without Dr. Lai pointing it out.

So, let the identity matrix I be the coefficient and the $n \times n$ matrix A be the common factor. As such, we have:

$$\begin{aligned}
(I - A)\sum_{n=0}^{k-1} IA^n &= (I - A)(IA^0 + IA^1 + IA^2 + \cdots + IA^{k-1}) \\
&= (I - A)(I + IA + IA^2 + \cdots + IA^{k-1}) \\
&= (I - A)(I + A + A^2 + \cdots + A^{k-1}) \\
&= I(I + A + A^2 + \cdots + A^{k-1}) - A(I + A + A^2 + \cdots + A^{k-1}) \\
&= (I + A + A^2 + \cdots + A^{k-1}) - (A + A^2 + \cdots + A^{k-1}) \\
&= I + \cancel{(A + A^2 + \cdots + A^{k-1}) - (A + A^2 + \cdots + A^{k-1})} \\
&= I
\end{aligned}$$

As such, we have that

$$(I - A)\sum_{n=0}^{k-1} IA^n = I$$

which implies

$$(I - A)^{-1} = \sum_{n=0}^{k-1} IA^n = I + A + A^2 + \cdots + A^{k-1}.$$